Approximating optimal finite horizon feedback by model predictive control

A. L. Dontchev[1]

American Mathematical Society and the University of Michigan, dontchev@umich.edu

I. V. Kolmanovsky[2]

Department of Aerospace Engineering, University of Michigan, ilya@umich.edu.

M. I. Krastanov[3]

Faculty of Mathematics and Informatics, University of Sofia, and Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria, krastanov@fmi.uni-sofia.bg

V. M. Veliov[4]

Institute of Statistics and Mathematical Methods in Economics, Vienna University of Technology, Austria, vladimir.veliov@tuwien.ac.at

P. T. Vuong[5]

Institute of Statistics and Mathematical Methods in Economics, Vienna University of Technology, Austria, vuong.phan@tuwien.ac.at

Abstract

We consider a finite-horizon continuous-time optimal control problem with nonlinear dynamics, an integral cost, control constraints and a time-varying parameter which represents perturbations or uncertainty. After discretizing the problem we employ a Model Predictive Control (MPC) approach by first solving the problem over the entire remaining time horizon and then applying the

*Corresponding author: V. M. Veliov

[1]Supported by the National Science Foundation Award Number CMMI 1562209, the Austrian Science Foundation (FWF) Grant P26640-N25, and the Australian Research Council (ARC) Project DP160100854.

[2]Supported by the National Science Foundation Award Number CMMI 1562209.

[3]Supported by the Sofia University "St. Kliment Ohridski" under contract No. 80-10-133/25.04.2018.

[4]Supported by Austrian Science Foundation (FWF) Grant P31400-N32.

[5]Supported by Austrian Science Foundation (FWF) Grant P26640-N25.

first element of the optimal discrete-time control sequence, as a constant in time function, to the continuous-time system over the sampling interval. Then the state at the end of the sampling interval is measured (estimated) with certain error, and the process is repeated at each step over the remaining horizon. As a result, we obtain a piecewise constant function of time representing MPC-generated control signal. Hence MPC turns out to be an approximation to the optimal feedback control for the continuous-time system. In our main result we derive an estimate of the difference between the MPC-generated state and control trajectories and the optimal feedback generated state and control trajectories, both obtained for the same value of the perturbation parameter, in terms of the step-size of the discretization and the measurement error. Numerical results illustrating our estimate are reported.

Keywords: model predictive control, optimal feedback control, discrete approximations, parameter uncertainty, error estimate.

2010 MSC: 49N35, 49K40, 49M25, 93B52.

---

1. Introduction

In this paper we consider an optimal control problem over a fixed finite time interval, which involves an integral cost functional, a control system described by a nonlinear ordinary differential equation, and control constraints given by a closed and convex set. Both the dynamics and the cost depend on a time-varying parameter which represents perturbations or uncertainty. We assume that only a reference trajectory of the time evolution of this parameter is available in advance, which is interpreted as a reference or a prediction for the true time-evolution of the parameter.

The optimal feedback synthesis is a basic problem in control theory. It assumes that the current state can be measured at each instance of time and consists in finding a device (mathematically, a mapping) to automatically generate an optimal control, which is then applied to the input of the system. The advantages of using optimal feedback control, when compared with open-loop control, are well known. However, finding an exact optimal feedback law for a nonlinear system with control constraints could be quite challenging.

In this paper we consider a Model Predictive Control (MPC) algorithm aiming at generating an approximate optimal feedback. It uses an initial time-discretization of the optimal control problem over a uniform mesh $\{t_i\}_0^N$, and assumes that measurements of the state are taken at every t_i from the mesh. Based on this, a control on the current time interval $[t_i, t_{i+1}]$ is computed by solving the corresponding discretized optimal control problem on the remaining time horizon $[t_i, t_N]$, which is then applied to the system input. Note that we focus on a specific MPC algorithm applied to a continuous-time problem, which, however, involves discretization of the problem.

The main contribution of this paper consists of a quantitative comparison of the MPC performance in relation to the optimal feedback control. Specifically, we derive an estimate of the norm of the difference between the MPC-generated state and control and those obtained by applying the optimal feedback control law. We note that there are various ways to compare the performance; here we focus on evaluating the difference between the respective solutions. To the best of authors' knowledge, this problem has not been treated in the literature, at least in the current, or even in a similar, setting.

The MPC has been extensively explored in the last decades; see, e.g., the books [9] and [13] for a broad coverage of the basic aspects of it. MPC has also found numerous applications in various industries. More common variants of MPC involve receding or moving horizon implementations, with the goal to design a time-invariant feedback law for stabilization and tracking. There are however a number of other problems, where the control is performed over a finite time interval. Examples of such problems problems involve spacecraft landing or docking, aircraft landing on runways, helicopter landing on ships, missile guidance, way point following, control of chemical batch processes etc.. The application of MPC in this setting leads to shrinking horizon formulations. Shrinking horizon MPC has been considered, for instance, in [14, 15], but compared to the receding horizon MPC, it has been significantly less studied. As for the receding horizon MPC, the shrinking horizon MPC provides an approximation of the finite horizon optimal feedback control law, that can be used for instance to handle systems whose dimension is too high for using dynamic programming. At the same time, since shrinking horizon MPC is a form of a feedback law, it is expected that it will improve robustness to uncertainty when compared with open-loop finite horizon control. The latter expectation is supported by the results in this paper. Extending the approach presented in this paper to receding horizon MPC algorithms applied over an infinite time duration and the study of robust stability and recursive feasibility represent directions for continuing research. However, these issues are beyond the scope of the present paper. The effect of discretization has been considered for sampled-data MPC in a receding horizon setting, e.g. in [7, 8, 12], but in a different context not involving the optimal feedback control of a finite-horizon optimal control problem.

To put the stage, in this paper we consider the following optimal control problem, which we call problem $\mathcal{P}_p$:

$$\min \left\{ J_p(u) := g(x(T)) + \int_0^T \varphi(p(t), x(t), u(t)) \,\mathrm{d}t \right\}, \tag{1}$$

subject to

$$\dot{x}(t) = f(p(t), x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \in [0, T], \quad x(0) = x_0, \tag{2}$$

where the time $t \in [0, T]$, the state $x(t)$ is a vector in $\mathbb{R}^n$, the control u has values $u(t)$ that belong to a convex and closed set U in $\mathbb{R}^m$ for almost every (a.e.) $t \in [0, T]$, and $p(t)$ is the value of a parameter p which is a function of time on $[0, T]$ representing uncertainty. When we say a "value" of the parameter, we mean a specific function of time $t \in [0, T]$ representing the parameter evolution in time. The initial state x_0 and the final time $T > 0$ are fixed. The set of feasible control functions u, denoted in the sequel by $\mathcal{U}$, consists of all Lebesgue measurable and bounded functions $u : [0, T] \to U$. The parameter p is a Lebesgue measurable and bounded function on $[0, T]$ with values in $\mathbb{R}^l$. A state trajectory, denoted by $x[u, p]$, is a solution of (2) for a feasible control u

and a value p of the parameter; accordingly, the state trajectories are Lipschitz continuous functions of time $t \in [0,T]$. For a specific choice of p, the state equation (2) is thought of as representing the "true" dynamics of a "real" system. Although the "true" representation of the parameter p is considered unknown, some reference (prediction) $\bar{p}$ for it is assumed to be available. In applications, $\bar{p}$ represents a preview/forecast of p which could be a disturbance (e.g., wind gust for an aircraft or road grade for a car) or a load in an electrical power system. In this paper we do not assume that the values of the parameter p are updated throughout the iterations of the MPC algorithm; taking into account such updates is of considerable interest and will be addressed in further research.

We consider the following MPC algorithm. First, the problem is discretized; we use for that the simplest Euler scheme over a uniform mesh. Given a natural number N, let $\{t_k\}_0^N$ be a grid on $[0,T]$ with equally spaced nodes t_k and a step-size $h = T/N$. To describe the MPC iteration, fix $k \in \{0,1,\dots,N-1\}$ and assume that a control u^N with $u^N(t) \in U$ is already determined on $[0,t_k)$. This control is applied to the real system (that with the value p of the parameter). Assume that the corresponding state at time t_k, $x[u^N,p](t_k)$, is measured (or estimated) with an additive error ξ_k, that is, the vector $x_k^0 := x[u^N,p](t_k) + \xi_k$ becomes available at time t_k. The next step is to solve the discrete-time optimal control problem

$$\min \left\{ g(x_N) + h \sum_{i=k}^{N-1} \varphi(\bar{p}(t_i), x_i, u_i) \right\},$$

subject to

$$x_{i+1} = x_i + hf(\bar{p}(t_i), x_i, u_i), \quad u_i \in U, \quad i = k, \dots, N-1, \quad x_k := x_k^0. \tag{3}$$

Note that this problem is solved for the reference value $\bar{p}$ of the parameter. For $k = 0$ we have $x_0^0 = x_0 + \xi_0$. Suppose that a locally optimal discrete-time control $(\tilde{u}_k, \dots, \tilde{u}_{N-1})$ is obtained as a solution of this problem. Define the constant in time function

$$u^N(t) = \tilde{u}_k \quad \text{ for } t \in [t_k, t_{k+1}),$$

change k to $k+1$ and continue the iterations as long as $k < N$, obtaining at the end a control u^N which is a piecewise constant function and which we call the *MPC-generated control*. Note that the MPC-generated control u^N may not be uniquely determined, e.g., because the discrete-time optimal control problem appearing at some stage does not have a unique solution. Also note that we keep the final time T fixed, so that the time horizon shrinks at each iteration.

Assume that there exists an (exact) optimal feedback $u^*(t,x)$ for problem $\mathcal{P}_{\bar{p}}$ with the reference value $\bar{p}$ of the parameter, provided that the state x entering the optimal feedback $u^*(t,x)$ is measured exactly. In this paper, we give an answer to the following question: what is the impact on state and control trajectories (and consequent loss of performance) if the MPC-generated control u^N (possibly in presence of measurement errors) is used in the system (2) with the value p of the parameter instead of the exact optimal feedback u^*. Our quantitative measure of the deviation for the state trajectory is based on an appropriate norm of the difference between $x[u^N,p]$, the solution of (2) generated by the MPC-generated control, and $x[u^*,p]$, the solution of (2) generated by the optimal feedback control u^* for problem $\mathcal{P}_{\bar{p}}$, both solutions obtained for the system with value p of the parameter. Let $\hat{u}(t) = u^*(t, x[u^*,p](t))$. As a quantitative measure for the deviation of the control input we use a norm of the difference between u^N and $\hat{u}$. The main result of the paper stated in Theorem 2.3 is that, under certain conditions, there exists a constant c such that, for all sufficiently large N, for every p sufficiently close to $\bar{p}$, and for every ξ sufficiently close to zero, one has

$$\|u^N - \hat{u}\|_1 + \|x[u^N,p] - x[u^*,p]\|_{W^{1,1}} \le c\Big(h + h\sum_{i=1}^{N-1} |\xi_i|\Big), \qquad (4)$$

where $\|\cdot\|_1$ and $\|\cdot\|_{W^{1,1}}$ are the standard norms of, respectively, the space of Lebesgue integrable functions and the space of absolutely continuous functions whose first derivatives are Lebesgue integrable.

Clearly, the first term in the parentheses in the right-hand side of (4) comes from the discretization, while the second term is due to the measurement errors. Remarkably, the bound (4) does not depend on the difference between the reference $\bar{p}$ and the real p; the only condition involved is that p must be sufficiently close to $\bar{p}$ for the estimate to hold. That is, a possible change of the parameter p affects in the same way (modulo $O(h)$) both the state-control pair corresponding to the optimal feedback control and the state-control pair obtained by applying the MPC algorithm.

The proof of the error estimate (4) uses results obtained in the recent papers [4] and [5]. In both papers the optimal control problem (1)–(2) is considered under basically the same assumptions. In [4] it is shown that the solution mapping of the discrete-time problem has a Lipschitz continuous single-valued localization with respect to the parameter, whose Lipschitz constant and the sizes of the neighborhoods do not depend on the number N of mesh points, for all sufficiently large N. In [5] it is proved that there exists an optimal feedback control which is a Lipschitz continuous function of the state and time. None of those cited papers considers MPC.

In this setting, a natural question to ask is whether one can obtain a better order of approximation than (4) by using higher-order discretization schemes. The answer is, "yes, but conditionally". Second-order approximations to control-constrained optimal control problems in the form of $\mathcal{P}_p$ (with $p = 0$) by Runge-Kutta discretizations are obtained in [3] under conditions that are similar to those in the present paper. Results for even higher order approximations (essentially for problems without control constraints) are presented in [10]. Using these results, however, would improve only the first term $O(h)$ in the right-hand side of (4). Utilizing higher order schemes would be justified only if the total l_1-error in the measurements is consistent with the discretization error.

In the following Section 2 we state the main result of the paper and in Section 3 we give a proof of this result. In Section 4 we present computational examples which illustrate the theoretical findings.

2. Main result

In this paper we use fairly standard notations. The euclidean norm and the scalar product in $\mathbb{R}^n$ (the elements of which are regarded as vector-columns) are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. The transpose of a matrix (or vector) E is denoted as $E^\top$. For a function $\psi:\mathbb{R}^p\to\mathbb{R}^r$ of the variable z, we denote by gph (ψ) its graph and by $\psi_z(z)$ its derivative (Jacobian) at z, represented by an $(r\times p)$-matrix. If $r=1$, $\nabla_z\psi(z)=\psi_z(z)^\top$ denotes its gradient (a vector-column of dimension p). Also for $r=1$, $\psi_{zz}(z)$ denotes the second derivative (Hessian) at z, represented by a $(p\times p)$-matrix. For a function $\psi:\mathbb{R}^{p\times q}\to\mathbb{R}$ of the variables (z,v), $\psi_{zv}(z,v)$ denotes its mixed second derivative at (z,v), represented by a $(p\times q)$-matrix. The space L^k, with $k=1,2$ or $k=\infty$, consists of all (classes of equivalent) Lebesgue measurable vector-functions defined on an interval of real numbers, for which the standard norm $\|\cdot\|_k$ is finite (the dimension and the interval will be clear from the context). As usual, $W^{1,k}$ denotes the space of absolutely continuous functions on a scalar interval for which the first derivative belongs to L^k. In any metric space we denote by $I\!B_a(x)$ the closed ball of radius a centered at x.

We begin by stating the assumptions under which problem $\mathcal{P}_p$ is considered.

Assumption (A1). The set U is closed and convex, the functions $f:\mathbb{R}^l\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$, $\varphi:\mathbb{R}^l\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ and $g:\mathbb{R}^n\to\mathbb{R}$ are two times continuously differentiable in (x,u) and these functions together with their derivatives in (x,u) up to second order are locally Lipschitz continuous in (p,x,u).

Assumption (A2). The time-varying parameter p representing uncertainty belongs to the following set of functions:

$$\Pi=\{p:[0,T]\to\mathbb{R}^l\ :\ p\in L_\infty(0,T),\ \|p\|_\infty\le\bar{M},\ \|p-\bar{p}\|_1\le\delta\},\qquad(5)$$

where $\bar{M}$ and δ are positive constants. In addition, the reference parameter $\bar{p}\in\Pi$ is Lipschitz continuous in $[0,T]$. Moreover, problem $\mathcal{P}_{\bar{p}}$ has a locally optimal solution $(\bar{x},\bar{u})$.

The local optimality is understood in the following (weak) sense: there exists a number $e_0>0$ such that for every $u\in\mathcal{U}$ with $\|u-\bar{u}\|_\infty\le e_0$, either the differential equation (2) has no solution on $[0,T]$ for u and $\bar p$, or $J_{\bar p}(u) \ge J_{\bar p}(\bar u)$. Note that from Assumption (A2) we have $\|\bar p\| \le \bar M$.

In terms of the Hamiltonian

$$H(t,x,u,\lambda) = \varphi(\bar p(t),x,u) + \lambda^\top f(\bar p(t),x,u)$$

for problem $\mathcal{P}_{\bar p}$, the Pontryagin maximum (here minimum) principle claims that there exists an absolutely continuous (here Lipschitz continuous) function $\bar\lambda : [0,T] \to \mathbb{R}^n$ such that the triple $(\bar x, \bar u, \bar\lambda)$ satisfies for a.e. $t \in [0,T]$ the following optimality system:

$$\begin{aligned}
0 &= -\dot x(t) + f(\bar p(t), x(t), u(t)), \quad x(0) - x_0 = 0,\\
0 &= \dot\lambda(t) + \nabla_x H(t, x(t), u(t), \lambda(t)),\\
0 &= \lambda(T) - \nabla_x g(x(T)),\\
0 &\in \nabla_u H(t, x(t), u(t), \lambda(t)) + N_U(u(t)),
\end{aligned} \tag{6}$$

where the normal cone mapping N_U to the set U is defined as

$$\mathbb{R}^m \ni u \mapsto N_U(u) = \begin{cases} \{y \in \mathbb{R}^n \mid \langle y, v-u\rangle \le 0 \text{ for all } v \in U\} & \text{if } u \in U,\\ \emptyset & \text{otherwise.}\end{cases}$$

To shorten the notations we skip arguments with "bar" in functions, shifting the "bar" to the function; that is, $\bar H(t) := H(t, \bar x(t), \bar u(t), \bar\lambda(t))$, $\bar H(t,u) := H(t, \bar x(t), u, \bar\lambda(t))$, $\bar f(t) := f(\bar p(t), \bar x(t), \bar u(t))$, etc. Define the matrices

$$A(t) = \bar f_x(t),\ B(t) = \bar f_u(t),\ F = g_{xx}(\bar x(T)), \tag{7}$$

$$Q(t) = \bar H_{xx}(t),\ S(t) = \bar H_{xu}(t),\ R(t) = \bar H_{uu}(t). \tag{8}$$

Assumption (A3) – Coercivity. There exists a constant $\rho > 0$ such that

$$\begin{aligned} y(T)^\top F y(T) + \int_0^T \Big(y(t)^\top Q(t) y(t) + w(t)^\top R(t) w(t) + 2 y(t)^\top S(t) w(t) \Big)\, \mathrm{d}t \\ \ge \rho \int_0^T |w(t)|^2 \, \mathrm{d}t \end{aligned}$$

for all $y \in W^{1,2}$ with $y(0) = 0$, and $w \in L^2$ with $w(t) \in U - U$ for a.e. $t \in [0,T]$, such that $\dot y(t) = A(t) y(t) + B(t) w(t)$.

Coercivity condition (A3) first appeared in [11] (if not earlier); after the publication of [2] it has been widely used in studies of regularity and approximations for problems like $\mathcal{P}_{\bar p}$. In particular, it plays a key role for Lipschitz continuity of the open-loop optimal control; also see Theorem 2.2 below. Later in the paper we show that Assumption (A3) implies a strong form of the Legendre condition regarding the positive definiteness of $R(t)$.

Next, we describe what we mean by optimal feedback control for problem $\mathcal{P}_{\bar p}$. For that we use the definition introduced in [5]. For any $\tau \in [0,T)$ and $y \in \mathbb{R}^n$ consider the following problem, denoted $\mathcal{P}_{\bar p}(\tau, y)$:

$$\min_{\mathcal{U}_\tau} \left\{ J_{\bar p}(\tau, y; u) := g(x(T)) + \int_\tau^T \varphi(\bar p(t), x(t), u(t))\,\mathrm{d}t \right\},$$

where x is the solution of the initial-value problem

$$\dot x(t) = f(\bar p(t), x(t), u(t)) \quad \text{for a.e. } t \in [\tau, T], \quad x(\tau) = y, \tag{9}$$

and $\mathcal{U}_\tau$ is the set of feasible controls $u \in \mathcal{U}$ restricted to the interval $[\tau, T]$.

Definition 2.1. A function $u^* : [0,T] \times \mathbb{R}^n \to U$ is said to be a locally optimal feedback control around a reference optimal solution pair $(\bar x, \bar u)$ of problem $\mathcal{P}_{\bar p}$ if there exist positive numbers η and a, and a set $\Gamma \subset [0,T] \times \mathbb{R}^n$ such that

(i) $\operatorname{gph}(\bar x) + \{0\} \times I\!\!B_\eta(0) \subset \Gamma$;

(ii) for every $(\tau, y) \in \Gamma$ the equation

$$\dot x(t) = f(\bar p(t), x(t), u^*(t, x(t))), \quad \text{for a.e. } t \in [\tau, T], \quad x(\tau) = y, \tag{10}$$

has a unique absolutely continuous solution $\bar x[\tau, y]$ on $[\tau, T]$ which satisfies $\operatorname{gph}(\bar x[\tau, y]) \subset \Gamma$;

(iii) for every $(\tau, y) \in \Gamma$ the function $\bar u[\tau, y](\cdot) := u^*(\cdot, \bar x[\tau, y](\cdot))$ is measurable, bounded, and satisfies

$$\|\bar u[\tau, y] - \bar u\|_\infty \le a \quad \text{and} \quad J(\tau, y; \bar u[\tau, y]) \le J(\tau, y; u),$$

where u is any admissible control on $[\tau, T]$ with $\|u - \bar u\|_\infty \le a$, for which a corresponding solution x of (10) exists on $[\tau, T]$ and is such that $\operatorname{gph}(x) \subset \Gamma$;

(iv) $u^*(\cdot, \bar x(\cdot)) = \bar u(\cdot)$.

In particular, property (iv) yields that $\bar{x}$ is a solution of (10) for $\tau = 0$ and $y = x_0$. Then the uniqueness requirement in (ii) implies that $\bar{x}[0, x_0] = \bar{x}$. In the sequel we call the function $t \mapsto \hat{u}(t) := u^*(t, x[u^*, p](t))$ a *realization* of the feedback control u^* when u^* is applied to (2) with value p of the parameter and with exact measurements. As before, here $x[u, p]$ denotes a solution of equation (2) for control u and parameter p. Note that, under our assumptions for the system, the solution is unique if u is feedback control which is a Lipschitz continuous function.

Recall that the admissible controls are elements of the space L^∞, that is, every admissible u is actually a class of functions $u : [0, T] \to U$ that differ from each other on a set of zero Lebesgue measure. Any of the members of this class (call it "representative") generates the same trajectory of (2) and the same value of the objective functional (1). Lemma 4.1 in [5] claims that under (A1)–(A3) there exists a "special" representative of the optimal control $\bar{u}$ which satisfies for *all* $t \in [0, T]$ the inclusion (6) in the maximum principle (with $\bar{x}$ and $\bar{\lambda}$ at the place of x and λ) and the pointwise coercivity condition

$$w^\top R(t) w \geq \rho |w|^2 \quad \text{for every } w \in U - U. \tag{11}$$

The following condition is introduced in [1] and used in [5] to prove existence of a Lipschitz continuous locally optimal feedback control.

Assumption (A4) – Isolatedness. The representative of the optimal control $\bar{u}$ described in the preceding lines is an isolated solution of the inclusion $\nabla_u \bar{H}(t, u) + N_U(u) \ni 0$ for all $t \in [0, T]$, meaning that there exists a (relatively) open set $\mathcal{O} \subset [0, T] \times \mathbb{R}^m$ such that

$$\{(t, u) \in [0, T] \times \mathbb{R}^m \ : \ \nabla_u \bar{H}(t, u) + N_U(u) \ni 0\} \cap \mathcal{O} = \mathrm{gph}\,(\bar{u}).$$

For example, the isolatedness assumption holds if for every $t \in [0, T]$ the inclusion $\nabla_u \bar{H}(t, u) + N_U(u) \ni 0$ has a unique solution (which has to be $\bar{u}(t)$). In this case, one can verify the isolatedness condition taking any (relatively) open set $\mathcal{O} \subset [0, T] \times \mathbb{R}^m$ containing $\mathrm{gph}\,(\bar{u})$. This will happen, for example, in the case when $\bar{H}$ is strongly convex in u for each t.

The main result of the paper is presented in Theorem 2.3, where we use the following theorem obtained previously:

Theorem 2.2. *([5, Theorem 5.2]) Under conditions (A1)–(A4) there exists a locally optimal feedback control* $u^* : [0,T] \times \mathbb{R}^n \to U$ *around* $(\bar{x}, \bar{u})$ *which is Lipschitz continuous on a set* Γ *appearing (together with the positive numbers* η *and* a*) in Definition 2.1.*

From Theorem 2.2 it follows that, if $\|p - \bar{p}\|_1$ is sufficiently small, the feedback control u^* when plugged in (2) generates a unique trajectory on $[0,T]$, which we denote in the sequel by $x[u^*, p]$. For completeness, a proof of this standard observation is given in the beginning of the proof of Theorem 2.3.

Recall that for any $p \in \Pi$ and measurement errors $\xi_0, \ldots, \xi_{N-1}$, the MPC method, as described in the introduction, generates a control u^N (possibly not uniquely). In order to indicate the dependence of u^N on p and ξ we sometimes use the extended notation $u^N[p, \xi]$.

The main result of this paper follows.

Theorem 2.3. *Suppose that assumptions (A1)–(A4) hold with constants* $\bar{M}$ *and* ρ*. Then there exist constants* N_0*,* c *and* $\delta > 0$ *such that for every* $N \geq N_0$*, for every* $p \in \Pi$*, where the set* Π *defined in (5) depends on* $\bar{M}$ *and* δ*, and for every* $\xi = (\xi_0, \ldots \xi_{N-1})$ *with* $\max_{k=0,\ldots,N-1} |\xi_k| \leq \delta$ *there exists a control* u^N *generated by the MPC algorithm for the system (2) with disturbance parameter* p *and measurement error* ξ *such that,*

$$|u^N(t_i) - \hat{u}(t_i)| \leq c\Big(h + |\xi_i| + h \sum_{k=0}^{N-1} |\xi_k|\Big), \quad i = 0, \ldots, N-1. \tag{12}$$

where $\hat{u}(t) := u^*(t, x[u^*, p](t))$ *is the realization of the feedback control* u^**. Furthermore, if* $\hat{x} = x[\hat{u}, p] = x[u^*, p]$ *is the trajectory of (2) for* $\hat{u}$ *and* p *and* $x^N := x[u^N, p]$ *is the trajectory of (2) for* u^N *and* p*, then*

$$\|u^N - \hat{u}\|_1 + \|x^N - \hat{x}\|_{W^{1,1}} \leq c\Big(h + h \sum_{k=0}^{N-1} |\xi_k|\Big). \tag{13}$$

We complement the statement of the theorem with two remarks.

Remark 2.4. As mentioned in the introduction, the MPC algorithm considered does not necessarily generate a unique control u^N, since the optimal control sequence $(\tilde{u}_k, \dots, \tilde{u}_{N-1})$ appearing at each stage k of the MPC does not need to be unique. Of course, the MPC-solution u^N will be uniquely determined if each of the discrete optimal control problems which is solved at each stage of the MPC algorithm has a unique solution.

Remark 2.5. Note that the bound in (13) for the difference between the MPC-generated control and the realization of the optimal feedback control does not depend on the uncertainty parameter p and its distance to the reference $\bar{p}$. In particular, this applies to the constant c, which however, depends on the "size" of the uncertainty set Π, characterized by the numbers $\bar{M}$ and δ. The term h in the estimate (13) comes from the discretization and due to the fact that measurements are taken on the mesh of size h only. The term involving $|\xi_k|$ represents the effect of the measurement error.

3. Proof of Theorem 2.3

Before presenting the proof we give some preparatory material and also explain the main ideas utilized in the proof. Let Γ, η, a, and the optimal feedback control u^* be as in Theorem 2.2. Without loss of generality we may assume that the set Γ is bounded. Indeed, one can redefine Γ as

$$\Gamma := \{(t,x) \in \Gamma \,:\, \exists(\tau,y) \in \Gamma \text{ such that } \tau \le t,\, y \in I\!B_\eta(\bar{x}(\tau)),\text{ and } x = \bar{x}[\tau,y](t)\}\,.$$

It is straightforward to show that the requirements of Definition 2.1 are satisfied by this new set Γ (which is bounded) and the same constants a and η.

Denote by $\hat{L}$ a Lipschitz constant of u^* on Γ. In further lines M and L denote respectively, a bound on the values and a Lipschitz constant of the functions f, f_x and φ_x on the following bounded set:

$$\left\{(s,x,u) \in \mathbb{R}^l \times \mathbb{R}^n \times \mathbb{R}^m :\ |s| \le \bar{M},\, x \in P(\Gamma),\, u \in I\!B_a(\bar{u}([0,T]))\right\},$$

where $P(\Gamma) := \{x \in \mathbb{R}^n \,:\, (t,x) \in \Gamma \text{ for some } t \in [0,T]\}$ is the projection of Γ on $\mathbb{R}^n$, and we abbreviate $I\!B_a(\bar{u}([0,T])) := \cup\{I\!B_a(\bar{u}(t)) \,:\, t \in [0,T]\}$.

Let $\bar{x}[\tau, y]$ and $\bar{u}[\tau, y](t) := u^*(t, \bar{x}[\tau, y](t))$ be as in Definition 2.1. Then $\bar{u}[\tau, y]$ is a locally optimal solution of problem $\mathcal{P}_{\bar{p}}(\tau, y)$. In [5, Remark 5.6] it is shown that in fact, $\bar{u}[\tau, y]$ is the *unique* locally optimal control in problem $\mathcal{P}_{\bar{p}}(\tau, y)$ in the set $\mathbb{B}_a(\bar{u}) \subset L^\infty(\tau, T)$ and, moreover, the function $t \mapsto \bar{u}[\tau, y](t)$ is Lipschitz continuous, uniformly in $(\tau, y) \in \Gamma$. Namely, for any $t, s \in [0, T]$ we have

$$\begin{aligned} |\bar{u}[\tau, y](t) - \bar{u}[\tau, y](s)| &= |u^*(t, \bar{x}[\tau, y](t)) - u^*(s, \bar{x}[\tau, y](s))| \\ &\leq \hat{L}(|t - s| + |\bar{x}[\tau, y](t) - \bar{x}[\tau, y](s)|) \leq \hat{L}(1 + M)|t - s|. \end{aligned} \tag{14}$$

By applying the Grönwall inequality to (10), for any y_1, y_2 with (τ, y_1), $(\tau, y_2) \in \Gamma$ and $t \in [\tau, T]$ we obtain that

$$|\bar{x}[\tau, y_1](t) - \bar{x}[\tau, y_2](t)| \leq e^{L(1+\hat{L})t}|y_1 - y_2|.$$

Hence, for any such y_1, y_2 and t,

$$\begin{aligned} |\bar{u}[\tau, y_1](t) - \bar{u}[\tau, y_2](t)| &= |u^*(t, \bar{x}[\tau, y_1](t)) - u^*(t, \bar{x}[\tau, y_2](t))| \\ &\leq \hat{L}|\bar{x}[\tau, y_1](t) - \bar{x}[\tau, y_2](t)| \leq \hat{L}e^{L(1+\hat{L})t}|y_1 - y_2|. \end{aligned}$$

In the proof of Theorem 2.3 we also utilize the following result, which we state below as Theorem 3.1. It concerns the discrete-time problem

$$\min \Big\{ g(x_N) + h \sum_{i=k}^{N-1} \varphi(\bar{p}(t_i), x_i, u_i) \Big\}, \tag{15}$$

subject to

$$x_{i+1} = x_i + h f(\bar{p}(t_i), x_i, u_i), \quad u_i \in U, \quad i = k, \dots, N-1, \quad x_k = y. \tag{16}$$

Theorem 3.1. *Suppose that assumptions (A1)–(A4) hold. Then there exist numbers N_0 and c_0 such that for every $N \geq N_0$, every $k \in \{0, \dots, N-1\}$ and every $y \in \mathbb{B}_\eta(\bar{x}(t_k))$, problem (15)–(16) has a locally optimal control $\tilde{u}^N[k, y] = (\tilde{u}^N_k[k, y], \dots, \tilde{u}^N_{N-1}[k, y])$ that satisfies the bound*

$$\max_{i=k,\dots N-1} |\tilde{u}^N_i[k, y] - \bar{u}[t_k, y](t_i)| \leq c_0 h.$$

Theorem 3.1 is similar to [2, Theorem 6], but there are some important differences; namely, it is about a time-dependent problem and, more importantly, it establishes an error estimate which is uniform with respect to the initial state. Therefore we present a complete proof which details the differences as well as the similarities.

Proof of Theorem 3.1. The first step of the proof is to estimate the residual when the continuous-time solution is plugged into the optimality system of the discretized problem. Then we apply a version of Robinson's implicit function theorem to the resulting variational inequality, which is stated as Theorem 3.2 in the sequel. For that purpose we show that the assumptions of that theorem are satisfied in the context considered.

The optimality system representing the first-order necessary optimality conditions for problem (15)–(16) has the form

$$\left\{\begin{array}{lcl} x_{i+1} & = & x_i + hf(\bar{p}(t_i), x_i, u_i), \quad x_k = y, \\ \lambda_{i-1} & = & \lambda_i + h\nabla_x H(t_i, x_i, u_i, \lambda_i), \\ \lambda_{N-1} & = & \nabla_x g(x_N), \\ 0 & \in & \nabla_u H(t_i, x_i, u_i, \lambda_i) + N_U(u_i), \end{array}\right. \tag{17}$$

where $i = k, k+1, \ldots, N-1$ in the first and in the last relations, $i = k+1, k+2, \ldots, N-1$ in the second equation. Here λ_i is the co-state for the discretized problem and H is, as before, the corresponding Hamiltonian. In the proof we consider k and y fixed but monitor how the constants involved depend on them. For short, denote $\bar{u}_i = \bar{u}[t_k, y](t_i)$, $\bar{x}_i = \bar{x}[t_k, y](t_i)$, and $\bar{\lambda}_i = \bar{\lambda}[t_k, y](t_i)$, skipping the argument $[t_k, y]$.

First, observe that the sequence $\{(\bar{u}_i, \bar{x}_i, \bar{\lambda}_i)\}_i$ satisfies the system

$$\left\{\begin{array}{lcl} 0 & = & -x_{i+1} + x_i + hf(\bar{p}(t_i), x_i, u_i) + b_i, \quad x_k = y, \\ 0 & = & -\lambda_{i-1} + \lambda_i + h\nabla_x H(t_i, x_i, u_i, \lambda_i) + d_i \\ 0 & = & -\lambda_{N-1} + \nabla_x g(x_N) + d_N, \\ 0 & \in & \nabla_u H(t_i, x_i, u_i, \lambda_i) + e_i + N_U(u_i), \end{array}\right. \tag{18}$$

with $b_i = \bar b_i(h)$, $d_i = \bar d_i(h)$, $e_i = \bar e_i(h) = 0$, where

$$\bar b_i(h) = \int_{t_i}^{t_{i+1}} f(\bar p(s), \bar x[t_k,y](s), \bar u[t_k,y](s))\,\mathrm{d}s - h f(\bar p(t_i), \bar x_i, \bar u_i),$$

$$\begin{aligned}\bar d_i(h) \quad &= \quad \int_{t_i-1}^{t_i} \nabla_x H(s, \bar x[t_k,y](s), \bar u[t_k,y](s), \bar\lambda[t_k,y](s))\,\mathrm{d}s \\ &\qquad -h\nabla_x H(t_i, \bar x_i, \bar u_i, \bar\lambda_i), \qquad i = k+1,\dots,N-1,\end{aligned}$$

$$\bar d_N(h) = \bar\lambda_{N-1} - \bar\lambda_N = \int_{t_{i-1}}^{t_i} \nabla_x H(t, \bar x(t), \bar u(t), \bar\lambda(t))\,\mathrm{d}t. \tag{19}$$

Recall that, according to Assumption (A2), $\bar p$ is Lipschitz continuous on $[0,T]$. According to (14), $\bar u[t_k,y]$ is Lipschitz continuous with Lipschitz constant $\hat L(1+M)$. Moreover, $(t, \bar x[t_k,y](t)) \in \Gamma$ and $\bar u[t_k,y](t) \in \mathbb{B}_a(\bar u([0,T]))$. Hence, $\bar x[t_k,y]$ is Lipschitz continuous with Lipschitz constant M. Then $\bar\lambda[t_k,y]$ is also Lipschitz continuous with a Lipschitz constant depending only on M. Hence, there exists a constant c_1, independent of k and y and N, such that

$$\max_{k\le i\le N-1} |\bar b_i(h)| + \max_{k+1\le i\le N-1} |\bar d_i(h)|) \le c_1 h. \tag{20}$$

In fact, the constant c_1, as well as c_2 and c_3 that appear below, depend on the numbers M, L, $\hat L$ and the Lipschitz constant of $\bar p$ only. From (19) we get that

$$|\bar d_N(h)| \le c_2 h. \tag{21}$$

Now we consider the right-hand side of (18) as a mapping acting on $v = \{(x_i, u_i, \lambda_i)\}_i \in \mathbb{R}^K$, where $K = (2n+m)(N-k)$. For the vector $q = (b,d,e)$ we have $b \in \mathbb{R}^{(N-k)n}$, $d \in \mathbb{R}^{(N-k)n}$, $e \in \mathbb{R}^{(N-k)m}$. In both the domain and the range spaces of the optimality mapping we use the l_∞-norm. This means that in this proof all balls and Lipschitz constants are with respect to the l_∞-norm. This choice of norms is important since the dimension K depends on N.

In the remainder of the proof we show that the solution mapping of the system (18) has a Lipschitz continuous localization around $\bar v := \{(\bar x_i, \bar u_i, \bar\lambda_i)\}_i$ for $\bar q = (\bar b(h), \bar d(h), \bar e(h))$ which is uniform in k, y and N; that is, there are constants α, β and γ independent of k, y and N such that for each $q := (b,d,e) \in \mathbb{B}_\beta(\bar q)$

there exists a unique solution $v(q) := \{(x_i, u_i, \lambda_i)\}_i(q)$ of the system (18) in $\mathbb{B}_\alpha(\bar{v})$ and the mapping $\mathbb{B}_\beta(0) \ni q \mapsto v(q)$ is a Lipschitz continuous function with Lipschitz constant γ.

First, observe that system (18) is a special case of the following variational inequality

$$E(v) + q + N_C(v) \ni 0, \tag{22}$$

where $v \in \mathbb{R}^K$, and $C = (\mathbb{R}^n)^{N-k} \times U^{N-k} \times (\mathbb{R}^n)^{N-k}$ and $U^{N-k} \subset (\mathbb{R}^m)^{N-k}$ is the product of $N-k$ copies of the set U. Here

$$v = (x, u, \lambda) = (x_{k+1}, \dots, x_N, u_k, \dots, u_{N-1}, \lambda_k, \dots \lambda_{N-1})$$

is considered as a vector (column) of dimension $n(N-k)+m(N-k)+n(N-k)$. The components of the vector-function E are as follows: First come the $N-k$ vectors $-x_{i+1} + x_i + hf(\bar{p}(t_i), x_i, u_i)$ each of dimension n. They are followed by the $(N-k-1)$ vectors $-\lambda_{i-1} + \lambda_i + hH_x(t_i, x_i, u_i, \lambda_i)$ each of dimension n. Then comes the n-dimensional vector $\lambda_{N-1} - g_x(x_N)$, followed by the $N-k$ vectors $H_u(t_i, x_i, u_i, \lambda_i)$ each of dimension m.

Note that the function E is continuously differentiable and the derivative $E_v(v)$ has a specific sparse structure. Namely, the rows corresponding to the first group of variables involved in the equation describing the control system contain at most $2n + m$ non-zero elements, the rows from the second group of variables involved in the adjoint equation contain at most $3n + m$ nonzero elements, the rows from the third group – $2n$, and those from the fourth group – $2n + m$. If we consider E_v on the unit ball $\mathbb{B}_1(\bar{v})$ (in the l_∞-norm), then each of the non-zero components of E_v is a Lipschitz function of its variables with a Lipschitz constant L_1, where L_1 depends on the data f, g and φ. It is straightforward to verify that E_v is Lipschitz continuous in $\mathbb{B}_1(\bar{s})$ with respect the operator norm of $E_v : \mathbb{R}^K \mapsto \mathbb{R}^K$ and the l_∞-norms in the two spaces. Thanks to the sparseness, the Lipschitz constant, $\bar{L}$, depends on L_1, n and m only, thus, it is independent of N.

We employ next the following version of Robinson's implicit function theorem proved in [4], the the corresponding assumptions for which regarding E_v

were shown to be satisfied in the preceding lines.

Theorem 3.2. *Let $\bar{v}$ be a solution of (22) for $\bar{q}$. Suppose that the derivative E_v exists and is Lipschitz continuous on $\mathbb{B}_1(\bar{v})$ with Lipschitz constant $\bar{L}$. Moreover, suppose that the following condition holds: for each $q \in \mathbb{R}^K$ there is a unique solution $w(q) \in \mathbb{R}^K$ of the linear variational inequality*

$$E_v(\bar{v})w + N_C(w) \ni q, \tag{23}$$

and the solution function $q \mapsto w(q)$ is Lipschitz continuous with a Lipschitz constant ℓ. Then there exist positive numbers α, β and γ which depend on ℓ and $\bar{L}$ only, such that for each $q \in \mathbb{B}_\beta(\bar{q})$ system (22) has a uniques solution $v \in \mathbb{B}_\alpha(\bar{v})$, and the function $q \mapsto v(q)$ is Lipschitz continuous on $\mathbb{B}_\beta(\bar{q})$ with Lipschitz constant γ.

Thus, we have to show that the condition involving (23) holds for the particular optimality system (18). Let A_i, B_i, Q_i, S_i, R_i denote the matrices in (7)–(8) evaluated at $t = t_i$. Also denote $\bar{H}_u^i = H(t_i, \bar{x}_i, \bar{u}_i, \bar{\lambda}_i)$. Then for $v = \{(x_i, u_i, \lambda_i)\}_i$ we have

$$E_v(\bar{v})(v) = \begin{pmatrix} -x_{i+1} + x_i + hA_i x_i + hB_i u_i \\ -\lambda_{i-1} + \lambda_i + hA_i^T \lambda_i + hQ_i x_i + hS_i u_i \\ -\lambda_{N-1} + Fx_N \\ \bar{H}_u^i + R_i u_i + S_i^T x_i + B_i^T \lambda_i \end{pmatrix}.$$

The matrices A, B, Q, S, R are continuous in t due to continuity of $\bar{u}$, $\bar{x}$ and $\bar{p}$. Then, from [2, Lemma 11], we have that there is a natural number N_1 and a positive constant $\rho_1 \leq \rho$ such that for all $N > N_1$ the quadratic form

$$\mathcal{B}(y, \nu) = y_N^T F y_N + \sum_{i=0}^{N-1} (y_i^T Q_i y_i + 2y_i^T S_i \nu_i + \nu_i^T R_i \nu_i) \tag{24}$$

satisfies the discretized coercivity condition:

$$\mathcal{B}(y, \nu) \geq \rho_1 \sum_{i=k}^{N-1} |\nu_i|^2 \tag{25}$$

for all (y, ν) from the set

$$\mathcal{C} = \{(y, \nu) \mid y_{i+1} = y_i + hA_i y_i + hB_i \nu_i,\ \nu_i \in U - U,\ i = k, \ldots, N-1,\ y_0 = 0\}.$$

According to (11), for every $i = 0, 1, \ldots, N-1$ we have

$$\nu^T R_i \nu \geq \rho |\nu|^2 \quad \text{for all } \nu \in U - U. \tag{26}$$

The remainder of the proof of the Lipschitz continuity of the function $I\!B_\beta(\bar{q}) \ni q \mapsto v(q) \in I\!B_\alpha(\bar{v})$ is just a repetition of the corresponding part of the proof of Theorem 1.2 in [4].

Finally, from (20) and (21) we have that $\|\bar{q}\|_\infty \leq c_3 h$. By taking N sufficiently large we can ensure that $c_3 h \leq \beta$, thus $0 \in I\!B_\beta(\bar{q})$. Then, the value $v(0)$ exists and

$$\|v(0) - \bar{v}\|_\infty = \|v(0) - v(\bar{q})\|_\infty \leq \gamma \|\bar{q}\|_\infty \leq \gamma c_3 h.$$

This implies the claim of the theorem with $c_0 := \gamma c_3$.

To complete the proof it remains to observe that, for the state-control pair $(\bar{x}^N, \bar{u}^N)$ obtained from solving the optimality system (17), the coercivity condition (25) is a sufficient condition for a (strict) local minimum, see [2, Appendix 1]. $\square$

Proof of Theorem 2.3. The main steps of the proof are as follows. First we show the existense of an appropriate solution of the state equation. Then, by using induction, we show that the MPC algorithm considered generates a control with the desired properties.

Choose δ to satisfy

$$0 < \delta < \frac{\eta}{4} e^{-L(1+\hat{L})T} \tag{27}$$

and define Π as in (5). We will prove next that for every $p \in \Pi$ a solution $\hat{x} := x[u^*, p]$ of (2) for u^* and p exists on $[0, T]$ and $\hat{x}(t) \in I\!B_{\eta/4}(\bar{x}(t))$, $t \in [0, T]$. Since $\hat{x}(0) = \bar{x}(0)$, the smoothness of f in assumption (A1) implies that $\hat{x}$ exists locally and can be extended to a maximal interval $[0, \theta] \subset [0, T]$ in which $\hat{x}(t) \in I\!B_{\eta/4}(\bar{x}(t))$ holds. Note that $|\hat{x}(\theta) - \bar{x}(\theta)| = \eta/4$ if $\theta < T$. Then

$\Delta(t) := |\hat{x}(t) - \bar{x}(t)|$, $t \in [0, \theta]$, satisfies

$$\begin{aligned}\Delta(t) &\le \int_0^t |f(p(s), \hat{x}(s), \hat{u}(s)) - f(\bar{p}(s), \bar{x}(s), \bar{u}(s))| \,\mathrm{d}s \\ &= \int_0^t |f(p(s), \hat{x}(s), u^*(s, \hat{x}(s))) - f(\bar{p}(s), \bar{x}(s), u^*(s, \bar{x}(s)))| \,\mathrm{d}s \\ &\le \int_0^t L \left[|p(s) - \bar{p}(s)| + \Delta(s) + \hat{L}\Delta(s) \right] \,\mathrm{d}s,\end{aligned}$$

where we use the identities $\bar{u}(s) = u^*(s, \bar{x}(s))$ (see property (iv) in Definition 2.1) and $\hat{u}(s) = u^*(s, x[u^*, p](s)) = u^*(s, x[\hat{u}, p](s)) = u^*(s, \hat{x}(s))$. Applying the Grönwall inequality and taking into account (27), we obtain

$$|\Delta(t)| \le e^{L(1+\hat{L})t} \|p - \bar{p}\|_1 \le e^{L(1+\hat{L})t} \delta < \eta/4, \quad t \in [0, \theta]. \tag{28}$$

Since the last inequality is strict, it follows that $\theta = T$ and $x[\hat{u}, p](t) \in I\!B_{\eta/4}(\bar{x}(t))$, $t \in [0, T]$.

Define recursively the sequence

$$d_{k+1} = e^{L(1+\hat{L})h} \left[d_k + L(\hat{L}(1 + M) + c_0)h^2 + L\hat{L}|\xi_k|h \right], \tag{29}$$

$k = 0, \ldots, N - 1$, starting with $d_0 = 0$. By using induction we obtain

$$d_k = \sum_{i=0}^{k-1} e^{L(1+\hat{L})(k-i)h} L \left[(\hat{L}(1 + M) + c_0)h^2 + \hat{L}|\xi_i|h \right],$$

which implies the estimation

$$\begin{aligned}d_k &\le e^{L(1+\hat{L})T} L \left(T(\hat{L}(1 + M) + c_0)h + \hat{L}h \sum_{i=0}^{N-1} |\xi_i| \right) \\ &\le e^{L(1+\hat{L})T} L \left(T(\hat{L}(1 + M) + c_0)h + \hat{L}T\delta \right) =: \bar{d}(\delta, h).\end{aligned} \tag{30}$$

Make $\delta > 0$ and $h = T/N$ smaller if necessary so that

$$\bar{d}(\delta, h) < \frac{\eta}{4} \tag{31}$$

and adjust Π according to (5). We will prove next that the MPC algorithm (applied for a parameter p and a vector ξ of measurement errors) generates a control $u^N := u^N[p, \xi]$ with the properties claimed in the statement of the theorem. To do that, we use induction in k.

For $k = 0$ we have $|x^N(0) - \hat{x}(0)| = 0 = d_0$. Assume that $0 \le k < N$ and that the MPC-generated control u^N and corresponding trajectory x^N are defined on $[t_0, t_k]$ in such a way that the inequality $|x^N(t) - \hat{x}(t)| \le d_k$ holds for all $t \in [0, t_k]$ and, moreover,

$$|x^N(t_k) - \bar{x}(t_k)| \le \eta/2. \tag{32}$$

Then, from (27) and (32), the quantity $y := x^N(t_k) + \xi_k$ satisfies the inequality

$$|y - \bar{x}(t_k)| \le |x^N(t_k) - \bar{x}(t_k)| + |\xi_k| \le \frac{\eta}{2} + \delta < \frac{\eta}{2} + \frac{\eta}{4} < \eta.$$

Let $\tilde{u} := \bar{u}[t_k, y]$ be the (unique) locally optimal control in problem $\mathcal{P}_{\bar{p}}(t_k, y)$ in the set $\mathbb{B}_a(\bar{u})$ (whose existence follows from Theorem 2.2 and the condition $y \in \mathbb{B}_\eta(\bar{x}(\tau))$). Let $\tilde{x}$ be the corresponding trajectory of (2) for $p = \bar{p}$ starting from y at t_k. According to Theorem 3.1, there exists a locally optimal control $\tilde{u}^N[k, y] = (\tilde{u}^N_k[k, y], \ldots, \tilde{u}^N_{N-1}[k, y])$ for problem (15)–(16), which satisfies the inequality

$$|\tilde{u}^N_k[k, y] - \tilde{u}(t_k)| \le c_0 h. \tag{33}$$

The constant function $u^N(t) = \tilde{u}^N_k[k, y]$ for all $t \in [t_k, t_{k+1})$ is generated by the MPC algorithm and satisfies $|u^N(t) - \tilde{u}(t_k)| \le c_0 h$ for $t \in [t_k, t_{k+1})$.

Let $s \in [t_k, t_{k+1})$; then, utilizing (33), Theorem 2.2, and the definition of $\hat{x}$, we have

$$\begin{aligned}
|u^N(s) - \hat{u}(s)| &\le |u^N(s) - \tilde{u}(t_k)| + |\tilde{u}(t_k) - \hat{u}(s)| \\
&\le |\tilde{u}^N_k[k, y] - \tilde{u}(t_k)| + |\tilde{u}(t_k) - \hat{u}(s)| \\
&\le c_0 h + |u^*(t_k, y) - u^*(s, \hat{x}(s)| \\
&= c_0 h + \hat{L}\,(|t_k - s| + |y - \hat{x}(s)|) \\
&\le c_0 h + \hat{L}\left(h + |x^N(t_k) - \hat{x}(s)| + |\xi_k|\right) \\
&\le c_0 h + \hat{L}\left(|x^N(s) - \hat{x}(s)| + |\xi_k| + (1 + M)h\right). \qquad (34)
\end{aligned}$$

Using (34) in the state equation, we obtain the following estimate for $t \in$

$[t_k, t_{k+1}]$:

$$\begin{aligned}
|x^N(t) - \hat{x}(t)| &\leq |x^N(t_k) - \hat{x}(t_k)| \\
&\qquad + \int_{t_k}^{t} \big| f(p(s), x^N(s), u^N(s)) - f(p(s), \hat{x}(s), \hat{u}(s)) \big| \, \mathrm{d}s \\
&\leq d_k + L \int_{t_k}^{t} \big(|x^N(s) - \hat{x}(s)| + |u^N(s) - \hat{u}(s)| \big) \, \mathrm{d}s \\
&\leq d_k + L \int_{t_k}^{t} \Big((1 + \hat{L}) |x^N(s) - \hat{x}(s)| + \hat{L} |\xi_k| + \hat{L}(1 + M) h + c_0 h \Big) \, \mathrm{d}s.
\end{aligned}$$

The Grönwall inequality yields

$$|x^N(t) - \hat{x}(t)| \leq e^{L(1+\hat{L})h} \Big(d_k + L\hat{L} |\xi_k| h + (L\hat{L}(1 + M) + L c_0) h^2 \Big); \qquad (35)$$

that is,

$$|x^N(t) - \hat{x}(t)| \leq d_{k+1}.$$

Using (28) and (31), we get that for all $t \in [t_k, t_{k+1}]$ one has

$$|x^N(t) - \bar{x}(t)| \leq |x^N(t) - \hat{x}(t)| + |\hat{x}(t) - \bar{x}(t)| \leq d_{k+1} + \frac{\eta}{4} < \frac{\eta}{2}.$$

The induction step is complete. As a result, from (35) combined with (30), we conclude that there exists a constant c_4 such that the MPC-generated control u^N and the corresponding trajectory x^N satisfy

$$\|x^N - \hat{x}\|_{C[0,T]} \leq c_4 \Big(h + h \sum_{k=0}^{N-1} |\xi_k| \Big). \qquad (36)$$

This last estimate, combined with (34), gives us the estimate (12) in the statement of the theorem with a constant c which depends on L, $\hat{L}$, M, c_0 and c_4

only. By integrating in (34) and utilizing (36), we get

$$
\begin{aligned}
\|u^N - \hat{u}\|_1 &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} |u^N(t) - \hat{u}(t)| dt \\
\leq & \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \Big(c_0 h + \hat{L}(|x^N(t) - \hat{x}(t)| + |\xi_k| + (1+M)h \Big) \,\mathrm{d}t \\
\leq & \sum_{k=0}^{N-1} \left(c_0 h^2 + \hat{L} h |\xi_k| + \hat{L}(1+M)h^2 + \hat{L} \int_{t_k}^{t_{k+1}} c_4 \left(h + h \sum_{k=0}^{N-1} |\xi_k| \right) dt \right) \\
\leq & \; T(c_0 + \hat{L}(1+M))h + \hat{L} h \sum_{k=0}^{N-1} |\xi_k| + \hat{L} T c_4 \left(h + h \sum_{k=0}^{N-1} |\xi_k| \right) \\
\leq & \; c \left(h + h \sum_{k=0}^{N-1} |\xi_k| \right),
\end{aligned}
$$

where c depends on L, $\hat{L}$, M, c_0 and c_4 only. Using this last estimate and the smoothness of f, and taking the constant c larger if needed, we obtain (13). This completes the proof. $\square$

4. Numerical examples

In this section we illustrate the result obtained in Theorem 2.3 by considering a problem of axisymmetric spacecraft spin stabilization (nutation damping) from [16] (p. 353). The transversal angular velocity components ω_1 and ω_2 of the spacecraft satisfy

$$
\begin{aligned}
\dot{\omega}_1 &= \lambda \omega_2 + \frac{M_d}{J_t}, \\
\dot{\omega}_2 &= -\lambda \omega_1 + \frac{M_c}{J_t},
\end{aligned}
$$

where $\lambda = \frac{J_t - J_3}{J_t} n$, J_t is the spacecraft transversal moment of inertia, J_3 is the spacecraft moment of inertia about the spin axis, n is the spin rate, M_d is the disturbance torque which can for instance be caused by thruster misalignment, and M_c is the control moment. Rescaling the time ($t \to \lambda t$), and letting $x_1 = \omega_1$, $x_2 = \omega_2$, $p = \frac{M_d}{J_t}$, $u = \frac{M_c}{J_t}$, we come to the following optimal control problem:

$$
\min \left\{ |x(T)|^2 + \frac{\alpha}{2} \int_0^T |u(t)|^2 \,\mathrm{d}t \right\},
$$

subject to

$$\dot{x}_1 = x_2 + p,$$
$$\dot{x}_2 = -x_1 + u,$$
$$x_1(0) = x_2(0) = 1, \quad |u| \le a,$$

where $p(\cdot)$ is as before, a time-dependent parameter representing uncertainty. We consider two examples, in both of which we use the specifications $T = 4\pi$, $\alpha = 0.5$, $a = 0.2$ and choose as a reference (predicted) parameter $\bar{p}(t) \equiv 0$. In the simulations, the measurement error ξ_k is sampled randomly from a uniform distribution, with $|\xi_k| \le 0.01$. The difference between the two examples below is in the choice of $p(\cdot)$.

Example 1. Here $p(\cdot)$ is a piecewise constant function on the uniform mesh with 2560 points in $[0, T]$. The values of $p(t)$ in every subinterval are chosen randomly in the interval $[-\bar{M}, \bar{M}] = [-\delta/T, \delta/T]$ with $\delta = 0.1$. Then any such p belongs to the set Π defined with the above values of $\bar{M}$ and δ.

In the notation of Theorem 2.3, the following two quantities,

$$RE_x := \frac{\|x^N - \hat{x}\|_{W^{1,1}}}{h + h\sum_{k=1}^{N-1}|\xi_k|} \qquad \text{and} \qquad RE_{ux} := \frac{\|u^N - \hat{u}\|_1 + \|x^N - \hat{x}\|_{W^{1,1}}}{h + h\sum_{k=1}^{N-1}|\xi_k|},$$

represent the relative errors with respect to x and with respect to u and x. According to the estimate (13) in Theorem 2.3 these quantities are bounded. The numerical results obtained confirm this result. Indeed, Table 4 presents the numerically obtained values of these ratios for several values of N. The results are consistent with the theoretical estimate (13).

Table 1: Relative error for perturbation p generated randomly

N	80	160	240	320	400	480	560	640
RE_x	5.140	1.484	0.914	0.653	0.575	0.442	0.448	0.354
RE_{ux}	23.609	40.765	47.809	48.210	47.519	46.353	45.162	43.448

Observe that, despite the presence of perturbations, the performance of the open-loop optimal control is not significantly worse than that of the optimal feedback control law. Indeed, denoting by $\hat{J}^{\rm ol}(p,\xi)$ the value of the objective functional when the optimal open-loop control (for the reference problem with $\bar{p}=0$) is implemented, by $\hat{J}^{\rm fb}(p,\xi)$ the objective value when the "exact" feedback u^* is implemented, and by $\hat{J}^{\rm mpc}_N(p,\xi)$ the objective value when the MPC-generated control with mesh size N is implemented, we obtain

$$\hat{J}^{\rm ol}(p,\xi) \approx \hat{J}^{\rm fb}(p,\xi) = 0.0766, \quad \hat{J}^{\rm mpc}_{160}(p,\xi) = 0.1617, \quad \hat{J}^{\rm mpc}_{640}(p,\xi) = 0.0858,$$

The reason for this effect is that the high-frequency random (uniformly distributed around the predicted value) perturbations "neutralize" each other; then the measurements in the feedback case do not bring a considerable advantage.

Example 2. Here we assume that p is a "systematic" error caused by a model imperfection, namely, $p(t) = 0.1x_2(t)$.

Table 2: Relative error estimation for perturbation $p(t) = 0.1x_2^*(t)$

N	80	160	240	320	400	480	560	640
RE_x	5.667	4.221	4.616	4.995	5.265	5.470	5.624	5.743
RE_{ux}	23.826	43.325	55.730	61.211	55.996	55.170	54.211	52.822

This time the performance of the optimal open-loop control, on one hand, and those of the optimal feedback control and the MPC-generated controls, on the other hand, differ substantially:

$$\hat{J}^{\rm ol}(p,\xi) = 0.1929,\ \hat{J}^{\rm fb}(p,\xi) = 0.0898,\ \hat{J}^{\rm mpc}_{160}(p,\xi) = 0.1879,\ \hat{J}^{\rm mpc}_{640}(p,\xi) = 0.1002.$$

Figure 1 shows the controls in Example 2 generated by the "true" feedback control law u^* and by the MPC algorithm with $N = 160$ and $N = 640$. The controls in Example 1 look similarly.

Fig. 2 shows the trajectories in Example 2 generated by the feedback control law u^* and by the MPC algorithm with $N = 160$ and $N = 640$. As seen, the

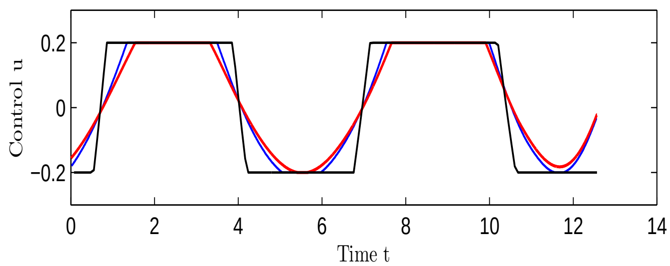

Figure 1: Example 2: Control functions generated by the optimal feedback u^* (red line, with lowest steepness) and by the MPC algorithm with $N = 160$ (black line, the steepest one) and with $N = 640$ (blue line).

MPC algorithm with $N = 640$ discretization points steers the system to the origin almost as well as the optimal feedback control law.

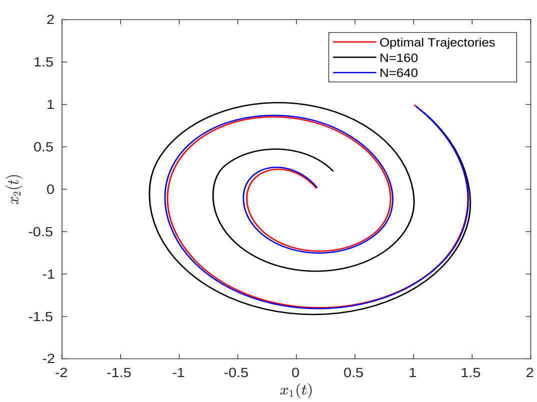

Figure 2: Example 2: Trajectories generated by the optimal feedback u^* (red line) and by the MPC algorithm with $N = 160$ (black line) and with $N = 640$ (blue line).

References

[1] R. Cibulka, A. L. Dontchev, M. I. Krastanov, V. M. Veliov, Metrically Regular Differential Generalized Equations, SIAM J. Control Optim. **56** (2018) 316–342, 2018.

[2] A. L. Dontchev, W. W. Hager, Lipschitzian stability in nonlinear control and optimization, SIAM J. Control Optim. **31** (1993) 569–603.

[3] A. L. Dontchev, W. W. Hager, and V. M. Veliov, Second-order Runge-Kutta approximations in control constrained optimal control, SIAM J. Numerical Anal. **38** (2000) 202–226.

[4] A. L. Dontchev, M. I. Krastanov, I. V. Kolmanovsky, M. M. Nicotra, V. M. Veliov, Lipschitz stability in discretized optimal control, SIAM J. Control and Optim., **57** (2019) 468–489.

[5] A. L. Dontchev, M. I. Krastanov, V. M. Veliov, On the existence of Lipschitz continuous optimal feedback control in ODE control problems, Vietnam J. Math. (2019) https://doi.org/10.1007/s10013-019-00347-5.

[6] A. L. Dontchev, R. T. Rockafellar, *Implicit functions and solution mappings*, Second Edition, Springer 2014.

[7] R. Findeisen, L. Imsland, F. Allgower, B. Foss, Towards a sampled-data theory for nonlinear model predictive control, In New trends in nonlinear dynamics and control and their applications, Springer (2003) 295–311.

[8] F. Fontes, L. Magni, E. Gyurkovics, Sampled-data model predictive control for nonlinear time-varying systems: Stability and robustness. In Assessment and Future Directions of Nonlinear Model Predictive Control, Springer (2017) 115–129.

[9] L. Grüne, J. Panek, *Nonlinear model predictive control*, Second Edition, Springer 2017.

[10] W. W. Hager, Runge-Kutta methods in optimal control and the transformed adjoint system, Numerische Mathematik **87** (2000) 247–282.

[11] W. W. Hager, Multiplier methods for nonlinear optimal control, SIAM J. Numer. Anal. **27** (1990) 1061–1080.

[12] L. Paiva, F. Fontes, Sampled–data model predictive control using adaptive time–mesh refinement algorithms, CONTROLO 2016, Springer (2017) 143–153.

[13] Handbook of Model Predictive Control, edited by S. V. Raković, W. S. Levine, Birkhäuser 2018.

[14] J. Skaf, S. Boyd, A. Zeevi, Shrinking horizon dynamic programming, International Journal of Robust and Nonlinear Control **20**(17) (2010) 1993–2002.

[15] A. Sony, R. Parker, Closed-loop control of fed-batch bioreactors: A shrinking-horizon approach, Industrial and Engineering Chemistry Research **43**(13) (2004) 3381–3393.

[16] B. Wie, *Space vehicle dynamics and control*, AIAA, 1998.