

COLOURING SIMPLICIAL COMPLEXES: ON THE LECHUGA-MURILLO'S MODEL

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ABSTRACT. L. Lechuga and A. Murillo showed that a non-oriented, simple and connected finite graph G is k -colourable if and only if a certain pure Sullivan algebra, constructed from G and k , is non-elliptic. In this paper, we settle us in the framework of finite simplicial complexes where no standard definition of colourability exists. Therefore, we introduce different colourings for simplicial complexes and we extend Lechuga-Murillo's result for them. We also prove that determining whether a simplicial complex admits most of the considered colourings is a NP -hard problem.

1. INTRODUCTION

The first to relate graph theory and rational homotopy theory were L. Lechuga and A. Murillo in a celebrated paper [13]. They showed that the (vertex) k -colourability of a non-oriented, simple, connected, finite graph $G = (V, E)$ can be codified through the ellipticity of a pure Sullivan algebra derived from it. This interplay between graph theory and rational homotopy theory has been proven fruitful: recently, C. Costoya and A. Viruel were able to use this interaction to solve a question of realisability of groups [2,3].

This relation between graph theory and rational homotopy theory was not the only interesting connection between widely different theories that the authors were able to obtain in [13]. They were also able to provide a link between rational homotopy theory and algorithmic complexity by proving that the problem of graph colourability can be reduced in polynomial time to the problem of determining whether a certain Sullivan algebra is elliptic or not. Hence, since the former is a NP -complete problem, the latter is a NP -hard problem. In [14] they go further by reducing the problem of colouring a graph to deciding whether a certain cohomology class of a Sullivan algebra vanishes. As a consequence, they show that determining whether certain cohomology classes of a Sullivan algebra vanish or not is also a NP -hard problem.

The aim of this work is to deep into this relation and to extend Lechuga and Murillo result to the framework of connected simplicial complexes. The difficulty arises when working with the notion of colourability of a simplicial complex: there is no definition of colourability that may be considered standard. In fact, in literature there exist many definitions that could be suitable in different situations. In this

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paper, we will introduce quite a large number of different notions of colourability for simplicial complexes, which we will divide in three different groups. Namely, in Section 3.1 we study the vertex, the face, and the total colourings. This family of colourings is inherited from hypergraph colourings so, although they are very natural, they have the disadvantage of not taking in consideration the special structure of simplicial complexes. For that reason, in Section 3.2 we introduce colourings that are specially thought for simplicial complexes: the complete ascending, the complete descending, the full and the (P, s) -colourings. Finally, Section 4 deals with colourings that are also conceived for simplicial complexes but for technical reasons, they need the simplicial complex to be strongly connected and homogeneous: the maximal ascending, the maximal descending, the minimal ascending and the minimal descending colourings. For ease of notation, we will refer to all the colourings above as \mathfrak{C}_i -colouring, for $i = 1, \dots, 11$ respectively.

In this paper, we prove the following:

Theorem 1.1. *Let X be a connected simplicial complex and \mathfrak{C}_i , $i = 1, \dots, 7$, be one of the colourings above mentioned. Then, there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_i}(X)$ such that X is \mathfrak{C}_i - k -colourable if and only if $\mathcal{M}_k^{\mathfrak{C}_i}(X)$ is non-elliptic. Moreover, if X is strongly connected and homogeneous, for \mathfrak{C}_i , $i = 8, \dots, 11$, there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_i}(X)$ such that X is \mathfrak{C}_i - k -colourable if and only if $\mathcal{M}_k^{\mathfrak{C}_i}(X)$ is non-elliptic.*

We point out here that the same problem has also been tackled in [4] for the \mathfrak{C}_7 -colouring [5]. Nevertheless, thanks to the work developed in [15] where the authors computed the chromatic number related to these colourings, we are able to provide a different proof of the aforementioned result.

We are also able to prove the following regarding the algorithmic complexity of the studied colourings:

Theorem 1.2. *Let X be a connected simplicial complex. For $i \in \{1, 7, 8, 9, 10, 11\}$ and $k \geq 3$ or for $i \in \{4, 5, 6\}$ and $k \geq 4$, determining whether a simplicial complex X is \mathfrak{C}_i - k -colourable is a NP-hard problem.*

2. BACKGROUND

In this section, we introduce some of the concepts and theories which will be needed later, along with the notation to be used in the sequel. Namely, we give a brief introduction to algorithmic complexity and rational homotopy theory, to then recall the construction of Lechuga and Murillo. We also fix our terminology for graphs and simplicial complexes.

We start with a brief introduction to algorithmic complexity, based on the one provided by the authors in [13]. A decision problem is a function $\Pi \rightarrow \{0, 1\}$ where $\Pi = \{I_\alpha\}_{\alpha \in \Gamma}$ is a family of subsets of non-negative integers, each I_α being an instance of the problem. If $I \in \Pi$, then $f(I)$ is the solution of I , the 0 codifying *No* and the 1 codifying *Yes*. The language of a decision problem is the set of instances for which the answer is *Yes*, that is, the $I \in \Pi$ such that $f(I) = 1$.

A decision problem $f: \Pi \rightarrow \{0, 1\}$ belongs to the polynomial class P if there is an algorithm that solves it in polynomial time. This means that for an instance $I \in \Pi$ of length n , the considered algorithm obtains a solution in a number of steps bounded by $p(n)$ for a certain polynomial p .

On the other hand, a problem $f: \Pi \rightarrow \{0, 1\}$ belongs to the non-deterministic polynomial class NP if there is an algorithm whose input data are pairs (C, I) formed by the instance I plus a certificate C verifying the following: for each instance I for which $f(I) = 1$, there is a certificate $C(I)$ such that when having $(I, C(I))$ as input, the considered algorithm is able to obtain that $f(I) = 1$ in a number of steps bounded by a certain polynomial on the length of the instance; whereas for those instances where $f(I) = 0$, any input pair (I, C) will allow the algorithm to determine that the answer is 0 in a number of steps bounded by a certain polynomial on the length of I .

It is obvious that $P \subset NP$, since for a problem in class P one can give an empty certificate and obtain a solution in a number of steps bounded by a polynomial on the length of the instance. The class P is usually referred as the class of *easy* problems, while the NP class is referred as the class of problems that are not necessarily *easy*, but for which it is *easy* to validate a certain solution.

We now need to recall the concept of reducibility: a map $T: \Pi \rightarrow \Pi'$ between two problems is a polynomial or Turing reduction if $T(I)$ belongs to the language of Π' if and only if I belongs to the language of Π , thus solving I is equivalent to solving $T(I)$; and there exists a polynomial p such that if $I \in \Pi$ is an instance of Π of length n , $T(I)$ is an instance of Π' of length bounded above by $p(n)$. Two problems that have such a map between them are said to be polynomially or Turing equivalent.

This concept allows us to introduce some interesting classes of decision problems, and among the *hardest* of the NP -problems are the NP -complete problems: a NP problem Π is NP -complete if any other NP problem admits a polynomial reduction to Π . Hence any algorithm that solves a NP -complete problem would solve any NP problem in the same range of time. In particular, obtaining an algorithm that can solve a NP -complete problem in polynomial time would immediately lead to a proof that $P=NP$, this being the most important open problem in algorithmic complexity.

Finally we have the class of NP -hard problems. A problem Π is said to be NP -hard if any NP problem can be reduced to it in polynomial time, but Π does not need to be in the NP class itself. Hence, any problem such that there is a NP -complete problem that can be reduced to it is NP -hard.

An extensive list of NP -complete problems can be found in [8]. One of the listed problems is determining whether a certain graph admits a k -colouring, for $k \geq 3$. In [13], the authors show that said problem is Turing-reducible to the problem of determining whether a certain Sullivan algebra they model from the graph is non-elliptic, and hence they show that determining whether certain Sullivan algebras are elliptic or not is a NP -hard problem. For our purposes, we will also make use of the fact that the problem of edge k -colourability of a graph is NP -complete for $k \geq 3$, [7, 11], and that the problem of determining whether a certain graph is total- k -colourable is NP -complete for $k \geq 4$, [16]. Our aim is, on the one hand, to show that the problem of (total, edge or vertex) graph colourability can be reduced to the problem of determining whether a certain simplicial complex admits one of the notions of colourability we consider, thus proving that the problem of \mathfrak{C}_i - k -colourability is NP -hard; and on the other, to extend Lechuga-Murillo's model to all of them, thus obtaining, for each of the considered colourings, a Sullivan algebra whose ellipticity codifies the corresponding simplicial complex colourability.

With that objective in mind, we first recall the construction in [13]. But before we should introduce some fundamental aspects of rational homotopy theory and we refer to [6] for the basics. By Sullivan algebra $(\Lambda V, d)$ we mean the free commutative graded algebra generated by the graded rational vector space V , i.e., $\Lambda V = TV/I$, where TV denotes the tensor algebra over V and I is the ideal generated by $v \otimes w - (-1)^{|v||w|} w \otimes v$, $v, w \in V$. It is therefore a symmetric algebra on V^{even} tensored with an exterior algebra on V^{odd} . Recall that a Sullivan algebra is called elliptic when both V and $H^*(\Lambda V, d)$ are finite dimensional as graded vector spaces.

Now, for a non-oriented, simple, connected, finite graph $G = (V, E)$, and for every integer $k \geq 2$, the Lechuga-Murillo's model is defined as the Sullivan algebra $S_k(G) = (\Lambda V_{G,k}, d)$, where $V_{G,k}$ is the graduated vector space over the field of rational numbers given by

$$\begin{aligned} V_{G,k}^{\text{even}} &= \langle x_v \mid v \in V \rangle, & |x_v| &= 2, & d(x_v) &= 0, \\ V_{G,k}^{\text{odd}} &= \langle y_{v_1 v_2} \mid v_1 v_2 \in E \rangle, & |y_{v_1 v_2}| &= 2k - 3, & d(y_{v_1 v_2}) &= \sum_{l=1}^k x_{v_1}^{k-l} x_{v_2}^{l-1}. \end{aligned}$$

They proved the following result.

Theorem 2.1 (Lechuga-Murillo [13]). *The graph G is k -colourable if and only if the Sullivan algebra $S_k(G)$ is non-elliptic.*

Throughout this paper, we will consider only abstract simplicial complexes with a finite number of simplices, which will also be finite themselves. Thus, a simplicial complex X on a finite vertex set V is a finite and nonempty collection of nonempty finite subsets of V , $X = \{\sigma_1, \dots, \sigma_r\}$, such that if $\sigma \in X$ and there exists $\sigma' \neq \emptyset$ verifying that $\sigma' \subset \sigma$, then σ' is also in X , and such that $V = \bigcup_{i=1}^r \sigma_i$. We use the following notation. The elements of X are called faces or simplices of the complex. The dimension of a face $\sigma_i \in X$ is $\dim(\sigma_i) = |\sigma_i| - 1$, i.e. its cardinality minus 1. A face of dimension s is called a s -face or s -simplex. The set of all the s -faces of X is denoted X^s , in particular, $X^0 = V$. The subcomplex of X spanned by all the r -faces of X for $r \leq s$ is denoted by $X^{(s)}$, and is called the s -skeleton. The dimension of a complex, $\dim X$, is the largest of the dimensions of all its faces. A complex X on a vertex set V is called connected if for every pair of vertices $v, w \in V$ there exists a collection of faces $\{\sigma_{i_0}, \dots, \sigma_{i_m}\}$ such that $v \in \sigma_{i_0}$, $w \in \sigma_{i_m}$ and $\sigma_{i_j} \cap \sigma_{i_{j+1}} \neq \emptyset$ for $j = 0, \dots, m - 1$. Observe that X is connected when its underlying graph (V, X^1) is so. Recall that a hypergraph H is a pair $H = (V(H), E(H))$ formed by a nonempty set of vertices $V(H) = \{v_1, v_2, \dots, v_n\}$ and a set of hyperedges $E(H) = \{e_1, e_2, \dots, e_m\}$, each of them being a nonempty subset of $V(H)$. Then, it is immediate to see that a simplicial complex X can be regarded as a hypergraph $H = (V, X)$.

Finally, to be able to show that the problem of (total, edge or vertex) graph colourability is polynomially reducible to the problem of whether a simplicial complex admits one of the considered colourings it would be useful to know the length of an instance of these problems. A graph is usually represented through its adjacency matrix $A = (a_{ij})$: a square matrix of order n where n is the number of vertices, say $\{v_1, v_2, \dots, v_n\}$, of the graph such that $a_{ij} = 1$ if $v_i v_j$ is an edge in the graph and $a_{ij} = 0$ otherwise. In binary, it would have length $\log_2 n + n^2$, where $\log_2 n$ is the number of binary digits needed to represent the number of vertices n and n^2 is the number of elements in the adjacency matrix. In particular, we can represent a one dimensional simplicial complex in the same manner.

3. MODELS FOR CONNECTED SIMPLICIAL COMPLEXES

Through this section, X will denote a connected simplicial complex with vertex set V and $H = (V, X)$ its associated hypergraph.

3.1. Models for colourings from hypergraphs. Since we can regard a simplicial complex X over a set V as a hypergraph $H = (V, X)$, one source of simplicial complex colourings is hypergraph colourings. In this section we take some of the most common hypergraph colourings and adapt them to simplicial complexes: vertex colouring (Definition 3.1), face colouring (Definition 3.5), and total colouring (Definition 3.7).

For each of those colourings, \mathfrak{C}_i , $i = 1, 2, 3$, we will find a connected graph G_i , $i = 1, 2, 3$, such that the k -colourability of the graph G_i is equivalent to the \mathfrak{C}_i - k -colourability of the simplicial complex X . Then, it suffices to apply Theorem 2.1 to conclude.

The first idea we may come with when thinking of a colouring for a hypergraph is to extend the notion of vertex colouring from a graph (recall that a graph is k -colourable if there exists a map from its set of vertices to a set of k different elements such that adjacent vertices do not share the same image). There are mainly two different ways of doing this: one is to ask every pair of adjacent vertices of the hypergraph to have different colours, which we call a *strong vertex colouring*. The other possibility is to ask non-unitary hyperedges to have more than one colour, which we will call a *weak vertex colouring*. We are going to see that these ideas are actually the same when working with hypergraphs that come from simplicial complexes, thus we may choose any of the two formulations. As a consequence, we have the following definition.

Definition 3.1. A *vertex k -colouring* of X (\mathfrak{C}_1 - k -colouring), is a map $\varphi: V \rightarrow \{1, 2, \dots, k\}$ such that if $\sigma \in X \setminus X^{(0)}$, then $\#(\varphi(\sigma)) > 1$, that is, X does not have a monochromatic face with two or more different vertices.

This definition for X coincides with the definition of weak vertex colouring for H , and we have:

Proposition 3.2. *Every vertex k -colouring of X is a strong vertex k -colouring of (V, X) .*

Proof. Consider $\varphi: V \rightarrow \{1, 2, \dots, k\}$ a vertex k -colouring of X and take $u, v \in V$ two different vertices such that there exist a simplex σ containing both of them. Then $\{u, v\} \subset \sigma$, and thus, $\{u, v\}$ must also be a simplex in X . Therefore, since $\#(\varphi(\{u, v\})) > 1$, $\varphi(u) \neq \varphi(v)$. \square

Notice that if we have a graph G and consider it as a one-dimensional simplicial complex X with 0-simplices the vertices of G and 1-simplices the edges of G , a k -colouring of G is precisely a vertex k -colouring of X . Since a graph and a one-dimensional simplicial complex have the same length as instances of the corresponding problems, we immediately obtain the following result.

Lemma 3.3. *The problem of graph k -colourability is Turing-reducible to the problem of \mathfrak{C}_1 - k -colourability and hence the latter is a NP-hard problem, for $k \geq 3$.*

We now need to find a graph codifying the strong or weak vertex colourability of a hypergraph. To that purpose, consider for a hypergraph H the corresponding

2-section graph H_2 [1, p. 25] defined as the graph with vertices $V(H_2) = V(H)$ and edges $E(H_2) = \{uv \mid \exists e \in E(H), u, v \in e\}$. Then, the following result holds.

Theorem 3.4. *Let X be a connected simplicial complex. Then there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}^1}(X)$ such that X is vertex k -colourable if and only if $\mathcal{M}_k^{\mathfrak{C}^1}(X)$ is non-elliptic.*

Proof. First, we regard X as a hypergraph $H = (V, X)$. Now, as H is connected we claim that its corresponding 2-section graph, H_2 , is also connected. Indeed for H connected, if $u, v \in V(H) = V(H_2)$, there is a collection of hyperedges $e_1, e_2, \dots, e_r \in E(H)$ such that $u \in e_1, v \in e_r$ and $e_i \cap e_{i+1} \neq \emptyset$, for $i \in \{1, 2, \dots, r-1\}$. If we take $v_i \in e_i \cap e_{i+1}$ any vertex, then $uv_1v_2 \dots v_{r-1}v$ is a path in H_2 joining u and v . Therefore, H_2 is connected. Moreover, an application $\varphi: V(H) = V(H_2) \rightarrow \{1, 2, \dots, k\}$ is a strong vertex k -colouring for H if and only if it is a (vertex) colouring for H_2 . We only need to realise that two vertices are adjoint in a hypergraph if and only if they are adjoint on its 2-section graph.

Notice that when regarding X as a hypergraph, two vertices are adjoint if and only if the set of those two vertices is a simplex in X . Thus, when taking the 2-section graph of a simplicial complex, its edges are precisely the 1-simplices of X . Therefore, the 2-section graph of X is $X_2 = G_1 = (V, X^1)$, and the desired algebra is $\mathcal{M}_k^{\mathfrak{C}^1}(X) = S_k(G_1)$, where $S_k(G_1)$ is the Sullivan algebra introduced in Section 2. \square

We now move on to a different notion of colouring for a hypergraph: a hyperedge colouring. A hyperedge colouring for a hypergraph H is a map $\varphi': E(H) \rightarrow \{1, 2, \dots, k\}$ such that adjoint hyperedges have different colours, that is, different images through φ' . This yields the following definition for simplicial complexes.

Definition 3.5. A *face k -colouring* of X (\mathfrak{C}_2 - k -colouring) is a map $\varphi': X \rightarrow \{1, 2, \dots, k\}$ such that if $\sigma, \tau \in X$ are two different simplices and $\sigma \cap \tau \neq \emptyset$, then $\varphi'(\sigma) \neq \varphi'(\tau)$.

Notice that this kind of colouring only takes in consideration intersecting faces, so when restricted to $X^0 = V$ it may not be a vertex colouring of X .

We now need to find a graph codifying this colourability. To that purpose we recall that for a hypergraph H , we can consider its *intersection graph* (also called *line graph* or *representation graph*) $L(H) = (E(H), \{de \mid d \cap e \neq \emptyset\})$, defined in [1, p. 24]. In this graph, the set of vertices is the set of edges of the hypergraph, and two of them will be connected by an edge if and only if the edges are adjacent in H . So it is immediate that an edge colouring of the hypergraph H corresponds to a colouring of the graph $L(H)$, and vice-versa. Thus, we can prove the following result.

Theorem 3.6. *Let X be a connected simplicial complex. Then there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}^2}(X)$ such that X is face k -colourable if and only if $\mathcal{M}_k^{\mathfrak{C}^2}(X)$ is non-elliptic.*

Proof. We only need to show that if the hypergraph $H = (V, X)$ is connected, then its intersection graph $L(H)$ is also connected so we can apply Lechuga-Murillo's result. In order to do that, take $d, e \in E(H)$ two vertices on the intersection graph and take $u \in d, v \in e$. Since H is connected, we can find a collection of hyperedges e_1, e_2, \dots, e_r such that $u \in e_1, v \in e_r$ and $e_i \cap e_{i+1} \neq \emptyset$ for $i \in \{1, 2, \dots, r-1\}$. In

that case, $d \cap e_1 \neq \emptyset \neq e \cap e_r$, and $de_1e_2 \dots e_re$ is a path connecting d and e . As we mentioned above, an edge colouring of H corresponds to a colouring of the graph $G_2 = L(H)$. Therefore, by using Theorem 2.1, X is face k -colourable if and only if $\mathcal{M}_k^{\mathfrak{C}_2}(X) = S_k(G_2)$ is non-elliptic, where $S_k(G_2)$ is the Sullivan algebra introduced in Section 2. \square

To finish with the first family of colourings, we are going to give a model for the total colouring. A total colouring of a hypergraph is a map $\psi: V(H) \cup E(H) \rightarrow \{1, 2, \dots, k\}$ such that any pair formed by either two adjacent vertices, two adjacent faces or an incident vertex and edge, have different images. Thus, this raises the following definition in the scenario of simplicial complexes.

Definition 3.7. A *total k -colouring* of X (\mathfrak{C}_3 - k -colouring) is a map $\psi: V \cup X \rightarrow \{1, 2, \dots, k\}$ such that

- if $u, v \in V$ and $\{u, v\} \in X$, then $\psi(u) \neq \psi(v)$.
- if $\sigma, \tau \in X$ and $\sigma \cap \tau \neq \emptyset$, then $\psi(\sigma) \neq \psi(\tau)$.
- if $u \in V$, $\sigma \in X$ and $u \in \sigma$, then $\psi(u) \neq \psi(\sigma)$.

Notice that when restricted to proper sets, a total k -colouring yields a vertex k -colouring and a face k -colouring.

The graph which is going to codify the total colouring is the *total graph*, which is, for a given hypergraph H , a graph $T(H)$ with vertices $V(H) \cup E(H)$ and edges any pair formed by two adjacent vertices, two adjacent edges or an incident vertex and edge. Being defined like this, it is clear that a total colouring of a hypergraph is precisely a colouring of this graph, so we obtain the following result:

Theorem 3.8. *Let X be a connected simplicial complex. Then there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_3}(X)$ such that X is total k -colourable if and only if $\mathcal{M}_k^{\mathfrak{C}_3}(X)$ is non-elliptic.*

Proof. We associate to X a connected hypergraph H and we have to show that a connected hypergraph yields a connected total graph $T(H)$. To do so, we need to consider three different ways of choosing vertices.

First, choose, $u, v \in V(H)$. Since H is connected, there exists a collection of hyperedges e_1, e_2, \dots, e_r such that $u \in e_1$, $v \in e_r$ and $e_i \cap e_{i+1} \neq \emptyset$, $i \in \{1, 2, \dots, r-1\}$. But then, $ue_1e_2 \dots e_re$ is a path from u to v .

Now take $u \in V(H)$ and $d \in E(H)$. If we take $v \in d$, then we have shown that there is a path $ue_1e_2 \dots e_re$ joining u and v . Now, since $v \in d$, there is an edge joining v and d , so $ue_1e_2 \dots e_revd$ is also a path in $T(H)$, which connects u and d .

Finally, if $d, e \in E(H)$, we can get a path connecting them by taking $u \in d, v \in e$, finding a path from u to v and adding d and e to each end of the path. Therefore $G_3 = T(H)$ is connected. As we mentioned above, the colouring of G_3 defines the total colouring of H and hence of X so it suffices to consider $\mathcal{M}_k^{\mathfrak{C}_3}(X) = S_k(G_3)$, where $S_k(G_3)$ is the Sullivan algebra introduced in Section 2. \square

3.2. Proper simplicial complex colourings. All the previous colourings were obtained from hypergraph colourings. As a matter of fact, they did not take advantage of the special structure of simplicial complexes. For instance, a total colouring on a simplicial complex colours the vertices twice: once as a vertex and then as a 0-simplex. Similarly, a vertex colouring only uses the information of the 1-skeleton

of the simplicial complex, and face colourings do not take into account the dimension of the intersection of the faces. For this reason, we now introduce the following definitions.

Definition 3.9. An *ascending k -colouring of X in dim r* is a map $\varphi: X^r \rightarrow \{1, 2, \dots, k\}$ such that if $\sigma, \tau \in X^r$ join in a $(r+1)$ -simplex, then $\varphi(\sigma) \neq \varphi(\tau)$. We will denote by $\chi_r(X)$ the chromatic number associated to this colouring (that is, the minimum k for which an ascending k -colouring of X in dimension r exists).

Definition 3.10. A *descending k -colouring of X in dim r* is a map $\varphi': X^r \rightarrow \{1, 2, \dots, k\}$ such that if $\sigma, \tau \in X^r$ intersect in a $(r-1)$ -simplex, then $\varphi'(\sigma) \neq \varphi'(\tau)$. We will denote by $\chi'_r(X)$ the corresponding chromatic number.

Thus, φ distinguishes faces of $(r+1)$ -simplices, while φ' distinguishes r -faces that intersect in a $(r-1)$ -face. We will now give some observations corresponding to these colourings.

- An ascending colouring in dim 0 is a vertex colouring of the 2-section graph.
- A descending colouring in dim 1 is an edge colouring of the same graph.
- Any map $X^{\dim X} \rightarrow \{1, 2, \dots, k\}$ is an ascending colouring in dimension $\dim X$.
- Similarly, any map $X^0 \rightarrow \{1, 2, \dots, k\}$ is a descending colouring in dim 0.
- Any descending colouring is also an ascending colouring in the same dimension.

It is immediate to observe that an ascending k -colouring on dim r is a colouring of the r -th exchange graph, [9],

$$(1) \quad G_r(X) = (X^r, \{\sigma\tau \mid \sigma \cup \tau \in X^{r+1}\}),$$

which is a generalisation of the 2-section graph $G_0(X)$. Similarly, a descending k -colouring in dim r is a colouring of the graph

$$(2) \quad G'_r(X) = (X^r, \{\sigma\tau \mid \sigma \cap \tau \in X^{r-1}\}).$$

We remark that, in this case, the connectivity of the simplicial complex does not imply the connectivity of the above graphs except for the particular cases of $G_0(X)$ and $G'_1(X)$. This issue will be treated in Section 4.

We now introduce some definitions that make use of the previous ones and which we will be able to model in this section.

Definition 3.11. A *complete ascending k -colouring of X (\mathfrak{C}_4 - k -colouring)* is a map $\varphi: X \rightarrow \{1, 2, \dots, k\}$ such that, for any given $r, s \in \{0, 1, \dots, \dim X\}$,

- $\sigma, \tau \in X^r, \sigma \cup \tau \in X^{r+1} \Rightarrow \varphi(\sigma) \neq \varphi(\tau)$.
- $\sigma \in X^r, \tau \in X^s, r \neq s \Rightarrow \varphi(\sigma) \neq \varphi(\tau)$.

Thus, φ is, when restricted to X^r , an ascending k -colouring of X in dim r , for every r . Also, simplices of different dimensions will have different colours. Let $\chi_c(X)$ be the corresponding chromatic number.

We can easily prove that the problem of graph colourability can be reduced to the problem of \mathfrak{C}_4 -colourability of a certain simplicial complex.

Lemma 3.12. *The problem of k -colourability of a graph G is reducible to the problem of \mathfrak{C}_4 - $(k+1)$ -colourability of the same graph considered as a simplicial complex X . Hence the problem of \mathfrak{C}_4 - k -colourability is NP-hard, for $k \geq 4$.*

Proof. Since the graph G and the simplicial complex X have the same length as instances of the corresponding problems, we only have to show that it is equivalent to give a k -colouring of G and to give a \mathfrak{C}_4 - $(k+1)$ -colouring of X .

Suppose first that V is the set of vertices of G (and X) and we have $\varphi: V \rightarrow \{1, 2, \dots, k\}$ a k -colouring of G . We can then define a map $\psi: X \rightarrow \{1, 2, \dots, k+1\}$ as

$$\psi(\sigma) = \begin{cases} \varphi(\sigma), & \text{if } \sigma \in X^0, \\ k+1, & \text{if } \sigma \in X^1. \end{cases}$$

It is immediate that ψ is a \mathfrak{C}_4 - $(k+1)$ -colouring of X .

Reciprocally, suppose that $\psi: X \rightarrow \{1, 2, \dots, k+1\}$ is a \mathfrak{C}_4 - $(k+1)$ -colouring of X . Then we know that the 1-simplices and 0-simplices of X receive different colours. We may assume that there is at least a 1-simplex receiving the colour $k+1$, so no 0-simplex may receive that colour. We can then define a map $\varphi: V \rightarrow \{1, 2, \dots, k\}$ by $\varphi(v) = \psi(\{v\})$, for $v \in V$. Since two vertices are adjacent if and only their union is a simplex in X , adjacent vertices must receive different colours by φ , so φ is a k -colouring for X . \square

Similarly, we can make the restrictions of φ to X^r descending colourings instead of ascending, which yields the following definition.

Definition 3.13. A *complete descending k -colouring of X* (\mathfrak{C}_5 - k -colouring) is a map $\varphi': X \rightarrow \{1, 2, \dots, k\}$ such that, for any given $r, s \in \{0, 1, \dots, \dim X\}$,

- $\sigma, \tau \in X^r, \sigma \cap \tau \in X^{r-1} \Rightarrow \varphi'(\sigma) \neq \varphi'(\tau)$.
- $\sigma \in X^r, \tau \in X^s, r \neq s \Rightarrow \varphi'(\sigma) \neq \varphi'(\tau)$.

We will call the corresponding chromatic number $\chi'_c(X)$.

We can prove that this problem is *NP*-hard by using techniques similar to those in the proof of Lemma 3.12.

Lemma 3.14. *The problem of edge k -colourability of a graph G is reducible to the problem of \mathfrak{C}_5 - $(k+1)$ -colourability of the same graph considered as a simplicial complex X . Hence the problem of \mathfrak{C}_5 - k -colourability is *NP*-hard, for $k \geq 4$.*

Proof. We have to show that it is equivalent to give an edge k -colouring of $G = (V, E)$ and to give a \mathfrak{C}_5 - $(k+1)$ -colouring of X .

Suppose first that $\varphi: E \rightarrow \{1, 2, \dots, k\}$ is an edge k -colouring for G and define a map $\varphi': X \rightarrow \{1, 2, \dots, k+1\}$ by

$$\varphi'(\sigma) = \begin{cases} \varphi(\sigma), & \text{if } \sigma \in X^1 \equiv E, \\ k+1, & \text{if } \sigma \in X^0 \equiv V. \end{cases}$$

If two 1-simplices intersect in a 0-simplex, that would mean that the edges they represent have a vertex in common, so they would receive different images through φ and hence through φ' . Since the image of the 0-simplices is not restricted other than it being different from the image of any simplex of a different dimension, it is clear that φ' is a complete descending $(k+1)$ -colouring for X .

Reciprocally, if $\varphi': X \rightarrow \{1, 2, \dots, k+1\}$ is a complete descending $(k+1)$ -colouring for X , we may suppose that at least one 0-simplex receives image $k+1$, so $k+1$ does not fall in the image of X^1 through φ' . Hence we may define $\varphi = \varphi'_{|X^1}: E \equiv X^1 \rightarrow \{1, 2, \dots, k\}$, and it is clear that this map is an edge k -colouring for G . \square

Since in both cases, the colours of different dimensions are distinct, and since restricted to each dimension these colourings are ascending or descending respectively, the next equalities follow:

$$\begin{aligned}\chi_c(X) &= \chi_0(X) + \chi_1(X) + \cdots + \chi_{\dim X}(X), \\ \chi'_c(X) &= \chi'_0(X) + \chi'_1(X) + \cdots + \chi'_{\dim X}(X).\end{aligned}$$

Now, to model them through graphs, one possible idea would be to get the graphs modelling each of the ascending or descending colourings and join every vertex of two different graphs with an edge. Then a colouring of the resulting graph will be a colouring when restricted to each of the considered graphs, and also, since every vertex is connected to all the vertices of the rest of the graphs, no two different graphs will have vertices with the same colour. This is the idea of the *sum of graphs* defined inductively from the sum of two graphs G_1 and G_2 , that is the graph $G = G_1 + G_2$ with vertices $V(G) = V(G_1) \sqcup V(G_2)$ and edges $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. It is easily shown that $\chi(G) = \chi(G_1) + \chi(G_2)$, which will extend inductively to the sum of any finite number of graphs.

Hence, we have the following result.

Theorem 3.15. *Let X be a simplicial complex. Then there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_4}(X)$ (respectively $\mathcal{M}_k^{\mathfrak{C}_5}(X)$) such that X admits a complete ascending (respectively descending) k -colouring if and only if $\mathcal{M}_k^{\mathfrak{C}_4}(X)$ (respectively $\mathcal{M}_k^{\mathfrak{C}_5}(X)$) is non-elliptic.*

Proof. As we have mentioned above, complete ascending k -colourings and complete descending k -colourings are respectively modelled by the colourings of the following graphs:

$$\begin{aligned}G_c(X) &= G_0(X) + G_1(X) + \cdots + G_{\dim X}(X), \\ G'_c(X) &= G'_0(X) + G'_1(X) + \cdots + G'_{\dim X}(X),\end{aligned}$$

where $G_r(X)$ and $G'_s(X)$ are defined in (1) and (2) respectively. Now, one can easily see that the sum of two graphs is always connected. Indeed, if we choose one vertex for each of the graphs, they are by definition connected with an edge, and if they are in the same graph, we can choose any vertex on the other graph and we have a path joining the two vertices through this one. As a consequence, for the graph $G_4 = G_c(X)$ we get that $\mathcal{M}_k^{\mathfrak{C}_4}(X) = S_k(G_4)$ (respectively for the graph $G_5 = G'_c(X)$ we get that $\mathcal{M}_k^{\mathfrak{C}_5}(X) = S_k(G_5)$), where $S_k(G_i)$ is the Sullivan model introduced in Section 2. \square

The following colouring is also formulated by combining colourings in different dimensions. It was introduced for dimension two in [12], and it has been recently studied, also for dimension two, under the name of VEF-colouring in [17].

Definition 3.16. A map $\psi: X \rightarrow \{1, 2, \dots, k\}$ is a *full k -colouring* of X (\mathfrak{C}_6 - k -colouring) if it satisfies:

- $\sigma \subset \tau \Rightarrow \psi(\sigma) \neq \psi(\tau)$.
- $\sigma, \tau \in X^0, \sigma \cup \tau \in X^1 \Rightarrow \psi(\sigma) \neq \psi(\tau)$.
- $1 \leq r \leq \dim X, \sigma, \tau \in X^r, \sigma \cap \tau \in X^{r-1} \Rightarrow \psi(\sigma) \neq \psi(\tau)$.

Thus, ψ distinguishes incident faces, it is an ascending colouring when restricted to X^0 , and a descending colouring when restricted to X^r , on all the other dimensions. Particularly, any complete ascending k -colouring is a full k -colouring.

Notice that if we take a graph G and consider it as a one-dimensional complex X , then a \mathfrak{C}_6 - k -colouring of X is precisely a total k -colouring of G . Therefore:

Lemma 3.17. *The problem of total graph k -colourability is Turing-reducible to the problem of \mathfrak{C}_6 - k -colourability of a simplicial complex, and hence, the later is a NP-hard problem, for $k \geq 4$.*

On the other hand, the full k -colouring is equivalent to a colouring of the full graph of X ,

$$(3) \quad G^e(X) = I \cup (G_0(X) \sqcup G'_1(X) \sqcup \cdots \sqcup G'_{\dim X}(X)),$$

where the union of two graphs is the graph with vertices the union of the vertices of the two, and edges the union of the edges of the two, and I is the strict inclusion graph, that is, a graph whose vertices are the simplices of the complex, and they are joined by an edge if and only if the simplices they represent are one contained in the other. This graph is connected when X is also connected, so the next result holds.

Theorem 3.18. *Let X be a connected simplicial complex. Then, there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_6}(X)$ such that X is full k -colourable if and only if $\mathcal{M}_k^{\mathfrak{C}_6}(X)$ is non-elliptic.*

Proof. It suffices to consider $\mathcal{M}_k^{\mathfrak{C}_6}(X) = S_k(G^e(X))$, where $S_k(G^e(X))$ is the Sullivan algebra introduced in Section 2. \square

We are now going to introduce the last colouring for this section. For a regular vertex colouring of a simplicial complex X with colours from a palette P , we assign to each vertex a colour from P such that for every face $\sigma \in X$, the vertices of σ have different colours. That is we look at maps $f : V \rightarrow P$ such that its restriction $f|_{\sigma}$ is injective. In [5] a more relaxed definition of vertex colouring is introduced.

Definition 3.19. Let P be a finite set (palette of colours) and $s \geq 1$ be a natural number.

- (1) A (P, s) -colouring of X (or a \mathfrak{C}_7 - (P, s) -colouring) is a map $f : V \rightarrow P$ such that, for all $\sigma \in X$ and all $p \in P$, we have $|\sigma \cap f^{-1}(p)| \leq s$.
- (2) X is (k, s) -colourable if X admits a (P, s) -colouring from a palette P of $|P| = k$ colours.
- (3) The s -chromatic number of X , $\text{chr}^s(X)$, is the least k so that X is (k, s) -colourable.

Notice that a $(k, 1)$ -colouring of X is a traditional k -colouring of X with respect to the 1-skeleton, and $\text{chr}^1(X)$ is the usual chromatic number of $X^{(1)}$. This means that a k -colouring for a graph G is precisely a $(k, 1)$ -colouring for the one-dimensional simplicial complex it induces, and hence:

Lemma 3.20. *The problem of graph k -colourability can be polynomially reduced to the problem of \mathfrak{C}_7 - (P, s) -colourability of a certain simplicial complex, and hence, the latter is a NP-hard problem, for $k \geq 3$.*

Observe also that $f : V \rightarrow P$ is a (P, s) -colouring if and only if $|f(\sigma)| \geq 2$ for all s -dimensional faces.

Definition 3.21. Let $B \subset V$ be a set of vertices of X and denote $D[B]$ the complete simplicial complex of all subsets of the finite set B . Then

- B is *s-independent* if B contains no s -simplex of X ;
- B is *connected* if $X \cap D[B]$ is a connected simplicial complex.

Definition 3.22. Let P be a partition of V . The graph $G_0(P)$ of P is the simple graph whose vertices are the blocks of P and where two blocks are joined by an edge whenever their union is connected.

Following the notation in [15], the set of all block-connected s -independent partitions of V will be denoted as $\text{BCP}^s(X)$. We are then able to prove the following result.

Theorem 3.23. *Let X be a connected simplicial complex. Then, there exists a pure Sullivan algebra $\mathcal{M}_{r,s}^{\mathfrak{C}_7}(X)$ such that X is \mathfrak{C}_7 - (r, s) -colourable if and only if $\mathcal{M}_{r,s}^{\mathfrak{C}_7}(X)$ is not elliptic.*

Proof. In [15, Theorem 2.5] the authors show that the s -chromatic number of X is the minimum

$$\text{chr}^s(X) = \min_{P \in \text{BCP}^s(X)} \text{chr}^1(G_0(P))$$

of the 1-chromatic numbers of the graphs of all s -independent and block-connected partitions of V . Moreover, it is easy to show that if X is connected, then $G_0(P)$ is connected for every $P \in \text{BCP}^s(X)$. Hence, we may consider

$$\mathcal{M}_{r,s}^{\mathfrak{C}_7}(X) = \bigotimes_{P \in \text{BCP}^s(X)} S_r(G_0(P)),$$

where $S_r(G_0(P))$ is the Sullivan algebra introduced in Section 2. Since the tensor product of Sullivan algebras is non-elliptic if and only if at least one of the factors is non-elliptic, when this algebra is non-elliptic at least one of the graphs $G_0(P)$ is r -colourable, so the s -chromatic number is, at most, r , and X is (r, s) -colourable, and reciprocally. \square

It is worth noting that the Sullivan algebras obtained in Theorem 3.23 are different from those obtained by the authors in [4] to codify the (P, s) -colourings.

4. MODELS FOR STRONGLY CONNECTED HOMOGENEOUS SIMPLICIAL COMPLEXES

A simplicial complex is strongly connected when any two simplices of maximum dimension can be joined via a finite list of simplices of maximum dimension verifying that the intersection of one simplex of the list with the next one is a simplex on the previous dimension.

This notion of connectivity, though apparently stronger than the previous one, does not necessarily imply connectivity. Indeed, any simplicial complex with only one simplex in maximum dimension will be strongly connected, but it may not be connected. However, if we consider homogeneous simplicial complexes, then strong connectivity implies connectivity. We recall that homogeneous simplicial complexes satisfy that for every given vertex there is a face of maximum dimension containing it. It is clear that if X is homogeneous and strongly connected, then $X^{(k)}$ is also homogeneous and strongly connected for $0 \leq k \leq \dim X$.

We start showing that, under these restrictions, graphs (1) and (2), which model ascending and descending k -colourings respectively, are connected, and thus we can

use Lechuga-Murillo's result. First, notice that X is strongly connected if and only if $G'_{\dim X}(X)$ is connected. Indeed, since the vertices of this graph are simplices of X of maximum dimension and they are connected through an edge, when they intersect in a simplex of the previous dimension, a chain like the one in the definition of strong connectivity is a path in this graph. For the rest of the dimensions, we have the following result.

Proposition 4.1. *Let X be an n -dimensional strongly connected homogeneous simplicial complex. Then $G_r(X)$ and $G'_s(X)$ are connected, for $0 \leq r < n$ and $0 < s \leq n$.*

Proof. We will start by proving the connectivity of $G_r(X)$. Choose $\sigma, \tau \in X^r$. If $0 \leq r < n$, since X is homogeneous, there is a simplex of maximum dimension containing σ , and if we choose a vertex in this simplex that is not in σ , we may add it to σ to get a $(r+1)$ -simplex $\tilde{\sigma}$ of which σ is a face. In a similar way, we can choose a $(r+1)$ -simplex $\tilde{\tau}$ containing τ . Now $\tilde{\sigma}, \tilde{\tau} \in X^{(r+1)}$, and $X^{(r+1)}$ is homogeneous and strongly connected. Then we have a list of $(r+1)$ -simplices in $X^{(r+1)}$, $\tilde{\sigma} = \tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_t = \tilde{\tau}$ such that $\sigma_i = \tilde{\sigma}_i \cap \tilde{\sigma}_{i+1}$ is a r -simplex, $i \in \{1, 2, \dots, t-1\}$. It is easily seen that $\sigma_i \cup \sigma_{i+1}$ is a $(r+1)$ -simplex, and also, $\sigma = \sigma_1$ or they join in a $(r+1)$ -simplex. The same happens with τ and σ_{t-1} . In any case, $\sigma\sigma_1 \dots \sigma_{t-1}\tau$ is a path in $G_r(X)$ between σ and τ , so $G_r(X)$ is connected. The connectivity of $G'_s(X)$, for $0 < s \leq n$ is an immediate consequence of the strong connectivity of $X^{(s)}$. \square

We thus have the next result.

Theorem 4.2. *Let X be an n -dimensional strongly connected homogeneous simplicial complex. Then, for every $0 \leq r < n$ (respectively for every $0 < s \leq n$), there exists a Sullivan algebra $\mathcal{M}_k(X, r)$ (respectively $\mathcal{M}'_k(X, s)$) such that X admits an ascending k -colouring in dim r (respectively a descending k -colouring in dim s) if and only if $\mathcal{M}_k(X, r)$ (respectively $\mathcal{M}'_k(X, s)$) is non-elliptic.*

Proof. By the previous proposition, under these hypothesis the graph $G_r(X)$ is connected, for $0 \leq r < n$, so X admits an ascending k -colouring in dim r if and only if the Sullivan algebra $\mathcal{M}_k(X, r) = S_k(G_r(X))$ is non-elliptic. Similarly, for $0 < s \leq n$, X will admit a descending k -colouring in dim s if and only if the Sullivan algebra $\mathcal{M}'_k(X, s) = S_k(G'_s(X))$ is non-elliptic. \square

We now introduce the last collection of colourings.

Definition 4.3. We say that a map $\varphi: X \rightarrow \{1, 2, \dots, k\}$ is:

- a *maximal ascending k -colouring* (\mathfrak{C}_8 - k -colouring) if, for $0 \leq r \leq \dim X$, the restriction $\varphi|_{X^r}$ is an ascending k -colouring in dim r for X .
- a *maximal descending k -colouring* (\mathfrak{C}_9 - k -colouring) if, for $0 \leq s \leq \dim X$, the restriction $\varphi|_{X^s}$ is a descending k -colouring in dim s for X .
- a *minimal ascending k -colouring* (\mathfrak{C}_{10} - k -colouring) if there exists $0 \leq r < \dim X$ such that the restriction $\varphi|_{X^r}$ is an ascending k -colouring in dim r for X .
- a *minimal descending k -colouring* (\mathfrak{C}_{11} - k -colouring) if there exists $0 < s \leq \dim X$ such that the restriction $\varphi|_{X^s}$ is a descending k -colouring in dim s for X .

The corresponding chromatic numbers are respectively denoted by $\chi_{\max}(X)$, $\chi'_{\max}(X)$, $\chi_{\min}(X)$ and $\chi'_{\min}(X)$. We start by proving that both ascending colourings are *NP*-hard problems.

Lemma 4.4. *The k -colourability of a graph G is polynomially reducible to both the \mathfrak{C}_8 - k -colourability and the \mathfrak{C}_{10} - k -colourability of the one-dimensional simplicial complex it induces, X , and hence both are *NP*-hard problems, for $k \geq 3$.*

Proof. First notice that for a one-dimensional simplicial complex, the concepts of maximal and minimal descending colourings are actually the same, since any map $\varphi: X^1 \rightarrow \{1, 2, \dots, k\}$ will be an ascending colouring in dimension 1. Hence we only have to prove that having a k -colouring of G is equivalent to having a \mathfrak{C}_8 - k -colouring of X .

Suppose then that V is the set of vertices of G (and X) and that we have a k -colouring $\varphi: V \rightarrow \{1, 2, \dots, k\}$ for G . We define a map $\psi: X \rightarrow \{1, 2, \dots, k\}$ by

$$\psi(\sigma) = \begin{cases} \varphi(\sigma), & \text{if } \sigma \in X^0 \equiv V, \\ k, & \text{if } \sigma \in X^1. \end{cases}$$

Since X is one-dimensional, we know that any map $X^1 \rightarrow \{1, 2, \dots, k\}$, and in particular $\psi|_{X^1}$, is an ascending k -colouring in dimension one. In dimension zero, if $\sigma, \tau \in X^0$ are such that $\{\sigma, \tau\} \in X^1$, that would mean that the corresponding vertices in G are joined by an edge, so they receive different colours through φ and hence through ψ , so ψ is a maximal ascending k -colouring.

Reciprocally, suppose that $\psi: X \rightarrow \{1, 2, \dots, k\}$ is a maximal ascending k -colouring for X . We know in particular that $\varphi = \psi|_{X^0}: X^0 \equiv V \rightarrow \{1, 2, \dots, k\}$ is an ascending k -colouring in dimension 0. Hence two vertices that form a 1-simplex, or equivalently, that are joined through an edge in G , must receive different colours through φ , and φ is, indeed, a k -colouring for G . \square

We can show in a similar manner that descending k -colourings are both *NP*-hard problems.

Lemma 4.5. *The edge k -colourability of a graph G is equivalent to both the \mathfrak{C}_9 - k -colourability and the \mathfrak{C}_{11} - k -colourability of the one-dimensional simplicial complex it induces, X , and hence both are *NP*-hard problems, for $k \geq 3$.*

Proof. In a similar way as what we have shown in Lemma 4.4, maximal and minimal descending colourings are the same concept when applied to one dimensional simplicial complexes, since they both must be descending k -colourings in dimension one while they do not have any restriction on the image of 0-simplices. Hence we may only prove that having an edge k -colouring of G is equivalent to having a \mathfrak{C}_9 - k -colouring of X .

Suppose then that $G = (V, E)$ and that $\varphi': E \rightarrow \{1, 2, \dots, k\}$ is an edge k -colouring of G , and consider the map $\varphi: X \rightarrow \{1, 2, \dots, k\}$ defined by

$$\varphi(\sigma) = \begin{cases} \varphi'(\sigma), & \text{if } \sigma \in X^1 \equiv E, \\ k, & \text{if } \sigma \in X^0 \equiv V. \end{cases}$$

We know that $\varphi|_{X^0}$ is a descending k -colouring in dimension 0. In dimension one, if $\sigma, \tau \in X^1$ are such that $\sigma \cap \tau \in X^0$, that means that the edges that σ and τ represent in X have a vertex in common, so they would receive different images through φ' and hence through φ , so φ is a maximal descending k -colouring.

Reciprocally, suppose that $\varphi: X \rightarrow \{1, 2, \dots, k\}$ is a maximal descending k -colouring for X . We know in particular that $\varphi' = \varphi|_{X^1}: X^1 \equiv E \rightarrow \{1, 2, \dots, k\}$ is a descending k -colouring in dimension one. Hence two 1-simplices that intersect in a 0-simplex receive different images through φ' . In terms of the graph G , what we are saying is that two edges that intersect in a vertex receive different images through φ' , and hence, φ' is an edge colouring for G . \square

We continue by showing that maximal colourings can be modelled using the Lechuga-Murillo's result.

Theorem 4.6. *Let X be a n -dimensional strongly connected homogeneous simplicial complex. Then there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{e}^s}(X)$ such that X has a maximal ascending k -colouring (that is, $\chi_{\max}(X) \leq k$) if and only if the pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{e}^s}(X)$ is non-elliptic. Similarly, there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{e}^o}(X)$ such that X has a maximal descending k -colouring (or $\chi'_{\max}(X) \leq k$) if and only if $\mathcal{M}_k^{\mathfrak{e}^o}(X)$ is non-elliptic.*

Proof. First, since in maximal ascending k -colourings we allow colours to be repeated between different dimensions, $\chi_{\max}(X)$ will be the least integer k such that, for every $r \leq n$, X admits an ascending k -colouring in $\dim r$. Also, since any map $X^n \rightarrow \{1, 2, \dots, k\}$ is an ascending colouring in dimension n , $\chi_n(X) = 1$. Thus

$$\chi_{\max}(X) = \max\{\chi_0(X), \chi_1(X), \dots, \chi_{n-1}(X)\}.$$

The same reasoning applies to the maximal descending k -colouring. In this case, we know that any map $X^0 \rightarrow \{1, 2, \dots, k\}$ is a descending k -colouring, so $\chi'_0(X) = 1$, and thus,

$$\chi'_{\max}(X) = \max\{\chi'_1(X), \chi_2(X), \dots, \chi'_n(X)\}$$

Consider now the Cartesian product of graphs, $G \square G'$. Since we know that $\chi(G \square G') = \max\{\chi(G), \chi(G')\}$ [10, Theorem 26.1], we inductively obtain that

$$\begin{aligned} \chi_{\max}(X) &= \max\{\chi_0(X), \chi_1(X), \dots, \chi_{n-1}(X)\} \\ &= \max\{\chi(G_0(X)), \chi(G_1(X)), \dots, \chi(G_{n-1}(X))\} \\ &= \chi(G_0(X) \square G_1(X) \square \dots \square G_{n-1}(X)), \end{aligned}$$

therefore having a maximal ascending k -colouring of X is equivalent to having a k -colouring of the graph

$$G_{\square}(X) = G_0(X) \square G_1(X) \square \dots \square G_{n-1}(X).$$

Also, the Cartesian product of connected graphs is itself connected, [10, Corollary 5.3], and since by Proposition 4.1 we already know that graphs $G_r(X)$ are connected for $0 \leq r < n$, we deduce that $G_{\square}(X)$ is connected. Then X admits a maximal ascending k -colouring if and only if the Sullivan algebra $\mathcal{M}_k^{\mathfrak{e}^s}(X) = S_k(G_{\square}(X))$ is non-elliptic.

Similarly, graphs $G'_s(X)$ are connected for $0 < s \leq n$, so

$$G'_{\square}(X) = G'_1(X) \square G'_2(X) \square \dots \square G'_n(X)$$

is connected, and X will admit a maximal descending k -colouring if and only if the pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{e}^o}(X) = S_k(G'_{\square}(X))$ is non-elliptic. \square

In the previous theorem we have seen that the chromatic number for the maximal k -colourings is the maximum of the chromatic numbers for the corresponding colourings, by using that the Cartesian product of graphs models this colourability. In a similar way, it is easy to see that the chromatic number for the minimal colourings is the minimum of the chromatic numbers for the corresponding colourings, so we might be tempted to search for a graph operation such that the chromatic number of the resulting graph is the minimum of the chromatic number of the factors. Though a good candidate seems to be the direct product of graphs, this is an open problem known under the name of Hedetniemi's conjecture [10, Conjecture 26.25]. Since we do not know any suitable graph operation, we may proceed in a similar way as we did in Theorem 3.23 to obtain an algebra that codified the \mathfrak{C}_7 -colourings.

Theorem 4.7. *Let X be a n -dimensional strongly connected homogeneous simplicial complex. Then there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_{10}}(X)$ such that X has a minimal ascending k -colouring (that is, $\chi_{\min}(X) \leq k$) if and only if the pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_{10}}(X)$ is non-elliptic. Similarly, there exists a pure Sullivan algebra $\mathcal{M}_k^{\mathfrak{C}_{11}}(X)$ such that X has a minimal descending k -colouring (or $\chi'_{\min}(X) \leq k$) if and only if $\mathcal{M}_k^{\mathfrak{C}_{11}}(X)$ is non-elliptic.*

Proof. It is easily seen that a simplicial complex X admits a minimal ascending k -colouring if and only if there exists an integer $0 \leq r < n$ such that X admits an ascending k -colouring in dimension r , which is also equivalent to the non-ellipticity of the Sullivan algebra $S_k(G_r(X))$. Since the tensor product of pure Sullivan algebras is elliptic if and only if all factors are elliptic, such r exists if and only if the pure Sullivan algebra

$$\mathcal{M}_k^{\mathfrak{C}_{10}}(X) = S_k(G_0(X)) \otimes S_k(G_1(X)) \otimes \cdots \otimes S_k(G_{n-1}(X))$$

is non-elliptic. Similarly, a minimal descending k -colouring exists if and only if the pure Sullivan algebra

$$\mathcal{M}_k^{\mathfrak{C}_{11}}(X) = S_k(G'_1(X)) \otimes S_k(G'_2(X)) \otimes \cdots \otimes S_k(G'_n(X)).$$

is non-elliptic. \square

To conclude this work, we just need to gather Theorems 3.4 – 4.7 to obtain the proof of Theorem 1.1 and Lemmas 3.3 – 4.5 to obtain the proof of Theorem 1.2, so we have proven both of our main results.

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