A PERTURBATION THEORY OF UNSTEADY HYPERSONIC AND SUPERSONIC FLOWS

by

W.H. Hui, B.Sc.

of the Department of Aeronautics and Astronautics

University of Southampton

Southampton, England

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ABSTRACT

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A general perturbation theory for hypersonic and supersonic flows past a wedge-like body is developed, which can be applied to both unsteady and steady flows for which the bow shock is attached to the body. It may be used to describe inviscid and viscous flows over both slender and thick, rigid or flexible bodies performing either periodic or aperiodic motions.

The exact (linearized) perturbation equations and boundary conditions are first derived, and the problem of finding the flow field reduced to that of solving a wave equation containing only one unknown function.

Approximate formulae for the aerodynamic derivatives of a pitching wedge in inviscid flow are obtained in two forms of power series in the frequency parameter and in the reflection coefficient which include McIntosh's theory and Appleton's theory as special cases. Two sets of waves are shown to exist, the first is due to the disturbance at the body surface, the second is due to the motion of the bow shock and is found to be a factor strongly destablizing the motion of thick bodies.

Exact formulae in closed form are obtained for the stability derivatives of a pitching wedge of any thickness in inviscid hypersonic and supersonic flows. Also obtained is an exact general cirterion for stability. It includes the approximate theory and the theory of Carrier & Van Dyke as special cases.

The effect of viscosity is included and closed form formulae for the stability derivatives of a wedge obtained which include the effects of wave reflection and thickness, and which appears to include Orlik-Ruchemann's theory as a special case.

Finally, by extending the perturbation method previously mentioned, exact formulae for the stability derivatives of a pitching Nonweiler (caret) wing in hypersonic flow are obtained and shown to be independent of its aspect ratio. The three-dimensional effect of the flow is shown to be dominant for the damping derivative.

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CONTENTS

CHAPTER III. EXACT THEORY

CHAPTER IV. EFFECTS OF VISCOSITY

CHAPTER V. STABILITY OF NONWEILER WINGS

CHAPTER VI. GENERAL DISCUSSIONS AND CONCLUSIONS

NOMENCLATURE

A, B, C	=	constants defined in (I.17) and (I.19)
$\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{E}, \bar{F}, \bar{G}, \bar{J}$	=	constants defined in (III.4)
a, b, c, d	=	constants defined in (II.43)
a	=	speed of sound
A_1, B_2, C_3, D_1	=	constants defined in (V.43)
$A(Pr)$, $B(Pr)$	=	constants tabulated in Section IV. 3
b_n	=	coefficients of power series of $F_5(x)$, (II.52)
C	=	constant of proportionality in the linear viscosity-temperature law, (IV.21)
C_o	=	arbitrary constant
C_1	=	constant defined in (III.15)
c_m	=	pitching moment coefficient
$-c_{m_\theta}$	=	stiffness derivative
$-c_{m_{\dot{\theta}}}$	=	damping derivative
C_p	=	pressure coefficient
d	=	parameter defined in (IV.23)
E	=	parameter, $\bar{C}/\bar{A}$
E_o	=	parameter, $\bar{C}_o/\bar{A}_o$
E_a	=	constant defined in (V.19)
e_o	=	constant defined in (V.19)
$F(x,y,t)$	=	function defined by (I.8)
F_1, F_2, F_3, F_4, F_5	=	unknown functions
f_1, f_2, f_3, f_4	=	arbitrary functions
$f_5(x,t)$	=	function describing the shape of the bow shock, (I.9)
$f_b(x,y,z,t)$	=	function describing the caret wing surface, (V.21)
f_1^*, f_2^*	=	arbitrary functions

$G(x,y,z,t)$	=	function describing the bow shock
G	=	parameter defined in (III.23)
G_o	=	parameter definded in (IV.21)
G_a	=	parameter defined in (II.70)
G_2, G_3, G_4	=	parameter defined in (V.45)
g	=	constant defined in (V.40)
H	=	parameter, $(M_o{}^2-1)^{\frac{1}{2}}\tan\varphi$
H_a	=	parameter, $M_o \tan\varphi$
H_1	=	$(M_1{}^2-1)^{\frac{1}{2}}\tan^2\varphi$
H_o	=	$M_o \tan\varphi_o$
h	=	non-dimensionalized pivot position measured from the wedge apex; specific enthalpy
I	=	parameter defined in (III.23)
I_a	=	parameter defined in (II.70)
i	=	$(-1)^{\frac{1}{2}}$
j	=	constant defined in (V.40)
$\vec{i}, \vec{j}$	=	unit vectors in the x,y directions, respectively
K	=	hypersonic similarity parameter, $M_\infty\beta$
$\bar{K}$	=	constant defined in (V.31)
k	=	frequency parameter, $\omega\bar{\ell}/u_o$
$\bar{K}_w, \bar{K}_s$	=	constants defined in (IV.46) and (IV.58), respectively
$\bar{\ell}$	=	chord length of the body
L	=	function defined by (I.22)
$\bar{L}_1, \bar{L}_2$	=	constants defined in (II.42)
M	=	Mach number; also pitching moment
m	=	parameter, $(1-H)/(1+H)$
m_a	=	parameter, $(1-H_a)/(1+H_a)$
$-m_\theta$	=	in-phase component of pitching moment
$-m_{\dot{\theta}}$	=	out-of-phase component of pitching moment

N	=	function defined by (I.22)
N_o	=	constant defined in (IV.16)
$\bar{N}_1,\bar{N}_2$	=	constants defined in (II.42)
P	=	time-independent perturbation pressure
p	=	perturbation pressure
$\bar{p}$	=	gas pressure
p_∞	=	gas pressure in the free stream
Pr	=	Prandtl number
Q	=	time-independent function describing the shape of the bow shock
$Q(K,\gamma)$	=	function defined in (II.85)
q_w,q_s	=	constents defined by (IV.46) and (IV.58), respectively
R	=	time-independent perturbation density
Re	=	Reynolds number
r_w,r_s	=	constants defined by (IV.46) and (IV.58), respectively
$\bar{S}$	=	plan area of the caret wing
$\bar{s}$	=	semi-span
s	=	$\bar{s}/\bar{\ell}$
S_q,S_r	=	constants defined by (IV.58)
S_n	=	coefficients defined by (II.62)
T	=	gas temperature
T_n	=	coefficients defined by (II.56)
$\bar{T}_w,\bar{T}_s$	=	constants defined by (IV.46) and IV.58), respectively
$\bar{t}$	=	time variable
t	=	non-dimensional form of $\bar{t}$
$\vec{t}_1,\vec{t}_2$	=	vectors tangential to the shock wave surface
$\bar{u},\bar{v},\bar{w}$	=	velocity components of gas in $\bar{x},\bar{y},\bar{z}$ direction, respectively
u,v,w	=	perturbation velocity components of gas defined in (I.2) and (V.7)

$\bar{u}', \bar{v}'$ = velocity components of inviscid gas behind the bow shock in the x', y' directions, respectively

U, V, W = time-independent perturbation velocity components

$U_\infty, p_\infty, \rho_\infty$ = velocity, pressure and density, respectively, of gas in the free stream

u_o, p_o, ρ_o = velocity, pressure and density, respectively, of gas in the reference steady flow past the original wedge

u_1, p_1, ρ_1 = velocity, pressure and density, respectively, of gas in the reference steady flow past the effective wedge

W_o = constant defined in (I.23)

W_q, W_r = constants defined in (IV.46)

x_m = non-dimensionalized matching point position

X_m = $x_m \cos \theta_1$

$\bar{x}, \bar{y}, \bar{z}$ = cartesian coordinates

x, y, z = non-dimensionalized form of $\bar{x}, \bar{y}, \bar{z}$ respectively

x', y' = non-dimenionalized form cartesian coordinates along and perpendicualr to the original wedge surface in its average position, respectively

x_o, y_o = non-dimensionalized coordinates determining the pivot axis of the caret wing

Y_w, Y_s = parameters defined in (IV.31) and (IV.52), respectively

Z_w, Z_s = parameters defined in (IV.33) and (IV.56), respectively

θ = flow deflection angle in the reference steady flow, also angle of attack of the lower ridge of the caret wing at its design condition

ϵ = samll parameter, maximum amplitude of oscillation

ψ = semivertex angle of the wedge

α = angle of attack of a wedge

β = shock wave angle in the reference steady flow; also incidence of the plane of leading edges of a caret wing at its design condition

φ = $\beta - \theta$

Γ = angle between the plane of symmetry of a caret wing and the wing surface

γ = ratio of specific heats of gas

ω = circular frequency of oscillation

Ω = index in the power law of viscosity-temperature law

$\bar{\rho}$ = gas density

ρ = perturbation gas density

$\delta(x,t)$ = disturbance function at the body surface

$\delta^*(x',t)$ = displacement thickness

$\delta_o^*(1)$ = $\theta_1-\theta_o$, approximately the average inclination of the displacement boundary layer

$\Delta(x)$ = "amplitude function" of the disturbance at the body surface

$\Delta_1(x)$ = $\Delta(x)e^{ikx}$

Δ^* = discriminant defined in (II.74)

λ = reflection coefficient, $(\bar{C} - \varkappa\bar{A})/(\bar{C} + \varkappa\bar{A})$

λ_a = approximate reflection coefficient, $-b/a$

μ,ν = parameters defined in (II.47)

μ_o,ν_o = parameters defined in (IV.18)

μ = viscosity

$\varkappa$ = $M_o/(M_o^2-1)^{\frac{1}{2}}$

τ = characteristic volume defined in (V.38)

$\vec{\tau}$ = vector in the tangential direction of the bow shock

σ = constant tabulated in Section IV.4

ξ,η,ζ,χ = variables defined in (V.10) and (I.5)

ζ' = $M_o\zeta$

$\bar{\chi}$ = hypersonic interaction parameter

SUBSCRIPTS

∞ = free stream

o = reference steady flow past the orignal wedge or past a caret wing at design condition

1	=	reference steady flow past the effective wedge
u	=	upper surface
ℓ	=	lower surface
b	=	body surface
cr	=	critical value
eff	=	effective wedge in inviscid flow
inv	=	time-independent quantities in the unsteady inviscid flow past the original wedge
orig	=	quantities behind the bow shock in the unsteady inviscid flow past the original wedge
v	=	quantities arising from the deformation of the effective wedge due to the change in the displacement thickness only
w	=	weak interaction
s	=	strong interaction
m	=	cases of mixing of a weak interaction and a strong interaction

SUPERSCRIPTS

(o), (1),... (n)	=	the zeroth, the first and the nth order solution for the time-independent perturbation quantities, respectively

LIST OF FIGURES

CHAPTER I

PERTURBATION METHOD

1 INTRODUCTION

Steady supersonic flow has been extensively studied using potential theory, whereas steady hypersonic flow has also been studied in great detail[1,2].

The purpose of studying unsteady hypersonic and supersonic flows is mainly to predict the aerodynamic forces that act on vehicles as a result of unsteady motions relative to uniform hypersonic or supersonic flight. This study is of special importance for aerodynamic stability and control of vehicles.

Unsteady supersonic flow has been systematically studied using potential theory[3], which is a linearized theory. For hypersonic flow the assumptions used in linearization of the flow equations are no longer valid even for slender bodies, and the problems are essentially non-linear. Also the entropy gradients produced by curved shock waves make the classical isentropic irrotational approach inapplicable.

Many experimental results for unsteady supersonic flow have been reported[3,4]. In contrast to this,for hypersonic flow, those which can be used to compare with theoretical prediction appear to be quite limited, this is due to the difficulties involved in experiments. East[5] has obtained experimental results on oscillating wedge-shaped airfoils with and without nose blunting, and has compared these with various predictions. He also gives a study of available theories.

Unsteady hypersonic flow has been studied using Newtonian impact theory[1] which assumes that there are no interactions between the fluid particles, and that when such particles collide with the surface of the body, their component of momentum normal to the surface is altered, with the result that a pressure force is exerted on the body. This simple theory is valid only in the double limit that the flight Mach number $M_\infty \to \infty$ and the ratio of specific heats $\gamma \to 1$, but otherwise gives poor results. It is usually used for the estimation of pressures and overall forces on bodies in both steady and unsteady flows[6].

Lightbill's piston theory[7] appears to be the first theory which can be used to predict aerodynamic forces acting on a supersonic or hypersonic oscillating airfoil. It gives a simple, explicit relationship between the pressure on a surface and the downwash. However, it is based on the hypersonic small-disturbance theory and ignores the existence of the bow shock, and therefore can only be applied for $M_\infty^2 >> 1$ and $M_\infty\theta < 1$, where θ is a measure of the maximum surface slope. This limits its range of applicability. Thus for $M_\infty\theta > 1$, Miles' strong-shock piston theory[8] may be used instead, in which the simple wave relationship in the piston theory is replaced by the shock wave relationship for compression surfaces. Although this is on a semi-empirical basis it gives better results for $M_\infty\theta > 1$ and is shown by East[5] to be coincident with Newtonian impact theory in the double limit $M_\infty \to \infty$ and $\gamma \to 1$.

In the unsteady shock-expansion theory[9,10] which is analogous to the strong-shock piston theory, an unsteady flow problem is, by suitably interpreting the results of the hypersonic small-disturbance theory, reduced to a steady problem with the body shape slightly distorted to account for the unsteady motion, and this steady problem is then solved by the shock-expansion method.

Piston theory, strong-shock piston theory and unsteady shock-expansion theory are all closly related to the hypersonic small-disturbance theory and subject to the same approximations. These theories can therefore be applied only for hypersonic flow past slender bodies. Besides, in these theories the bow shock is either ignored or assumed to be able to completely attenuate waves coming from the body surface and hence the effects of the secondary waves coming from the body and reflected from the bow shock are neglected.

For an oscillating wedge in supersonic flow, exact potential theory has been developed[11,12]. On the other hand for an oscillating wedge in hypersonic flow, theoretical studies have been made by Appleton[13] (see also Ref. 22 for some correction to Ref. 13) and by McIntosh[14] with the aim to include the effect of the reflected waves from the bow shock. These two investigators both assume that the perturbation from a pure wedge flow is small and thus linearize the equations of motion of the fluid behind the bow shock. Besides they both made the piston theory approximation, or the hypersonic small-disturbance approximation without discussion of its validity. (see also next paragraph) It is shown independently that the strong-shock piston theory is a special case in their theories when the reflected waves are neglected. McIntosh also points out that an important effect of the waves reflected from the bow shock is a phase shift in the unsteady pressure distribution. In fact, Appleton's theory, being accurate up to the first order in the frequency parameter, is also a special case in McIntosh's theory which was therefore, before the present theory, the most up-to-date theory for hypersonic flow past an oscillating slender wedge.

However, McIntosh's theory is based on the hypersonic small-disturbance approximation without discussion of the validity of approximations, and the effect of thickness of a wedge is not known. Although hypersonic small-disturbance theory has been shown to give good results for steady hypersonic flow past slender bodies[1,2], as compared with experiments and with some exact theories, Miles has remarked, 'that the success of the linearized perfect fluid solution for a specific configuration in steady flow, as determined by a comparison with experiment, offers no guarantee of the practical validity of the corresponding unsteady flow solution'[3].

As for Mach wave interaction with and reflection from a shock wave, the problem has been solved by Lighthill[15], Chu[16], and Chernyi[2] for wedge-like bodies of any thickness in steady supersonic and hypersonic flows providing the bow shock is attached. In contrast to this, in unsteady flow the problem has only been studied[13,14] using hypersonic small-disturbance approximations.

In this thesis it is proposed to find a solution of an unsteady inviscid or viscous flow over rigid or flexible wedge-like bodies of any thickness performing either periodic or aperiodic motions, providing the bow shock is attached. From this solution one is then in a position to find out what was neglected and to give comments on the existing theories.

The same assumption is made that the perturbation from a pure wedge flow is small, but the perturbation method is applied to the original equations which describe the motion of the inviscid gas rather than to the hypersonic small-disturbance equations. An exact system of linearized perturbation equations is thus obtained in Chapter I, and the problem of finding the flow field reduced to that of solving a wave equation containing only one unknown function. These perturbation equations include those obtained by Chu[16] and by

Chernyi[2] as a special case when the flow is steady. It will be shown that McIntosh's theory and Appleton's theory are special cases in the approximate theory (when the flow is hypersonic and the wedge is slender) developed in Chapter II of this thesis which, in turn, is a special case of the exact theory developed in Chapter III of the thesis. The exact theory also includes the theory of Carrier and Van Dyke[11,12] as a special case when the flow is supersonic, and gives a general criterion for stability of a pitching wedge.

It will be shown that in addition to the set of waves due to the disturbance at the body surface which has been discussed previously, another set of waves is discovered which is due to the motion of the bow shock and which exists for relatively thick wedges only. This new set of waves is found to be a factor which tends to strongly destablize the motion of the body.

The extention of the inviscid perturbation theory in Chapters II and III to viscous flow is made by suitably modifying the body shape to account for the displacement boundary layer. This produces formulae in closed form for the stability derivatives of sharp, oscillating wedges in viscous hypersonic flow. This viscous perturbation theory appears to include Orlik-Ruckemann's theory[17] as a special case when the wedge is very thin.

Application of the exact perturbation theory to a special type of hypersonic lifting vehicles - the Nonweiler wing or Caret wing - is made possible by extending the two-dimensional perturbation method to a three-dimensional one. The stability of a pitching Nonweiler wing is thus studied in detail and a criterion for stability given in Chapter V. The three-dimensional effect of the flow on the stability derivatives has also been discussed.

2 THE PERTURBATION EQUATIONS

As stated in the introduction, the unsteady hypersonic or supersonic flow past a wedge-like body to be studied in this thesis is assumed to be a small perturbation to some reference steady flow past a wedge — the pure wedge flow.

The bow shock is assumed to be attached to the wedge apex, hence the flow field between the bow shock and the upper surface of the wedge and that between the bow shock and the lower sureface are independent of each other and can be treated using identical analyses. We shall therefore consider only the flow field between the bow shock and the upper surface.

Let the system of coordinates $\bar{x}O\bar{y}$ be such that $O\bar{x}$ coincides with the wedge surface in the reference steady flow case, and O is the wedge apex (Fig.1). Denote by $\bar{u}$ and $\bar{v}$, respectively, the velocity components of the gas in the $\bar{x},\bar{y}$ direction, and by $\bar{p}$ and $\bar{\rho}$ the pressure and density of the gas; also by u_o, v_o, p_o and ρ_o the corresponding quantities in the pure wedge flow. Obiously, $v_o = o$, under the system of coordinates chosen. The time variable is denoted by $\bar{t}$. The basic equations which describe the inviscid gas flow between the bow shock and the upper wedge surface may be expressed in the following form.

$$\begin{aligned}
&\frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial(\bar{\rho}\bar{u})}{\partial \bar{x}} + \frac{\partial(\bar{\rho}\bar{v})}{\partial \bar{y}} = 0 \\
&\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{x}} \\
&\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial \bar{y}} \\
&\frac{\partial(\bar{p}/\bar{\rho}^{\gamma})}{\partial \bar{t}} + \bar{u}\frac{\partial(\bar{p}/\bar{\rho}^{\gamma})}{\partial \bar{x}} + \bar{v}\frac{\partial(\bar{p}/\bar{\rho}^{\gamma})}{\partial \bar{y}} = 0
\end{aligned} \qquad (1)$$

where γ is the ratio of the specific heat of the gas at constant pressure to the specific heat at constant volume.

For small perturbation, we may express

$$\begin{aligned} \bar{u} &= u_o + \epsilon u + \ldots \\ \bar{v} &= \epsilon v + \ldots \\ \bar{p} &= p_o + \epsilon p + \ldots \\ \bar{\rho} &= \rho_o + \epsilon \rho + \ldots \end{aligned} \qquad (2)$$

where ϵ is a small quantity which characterized the deviation of the unsteady flow from the pure wedge flow. The quantities u/u_o, p/p_o and ρ/ρ_o, together with their derivatives are assumed to be of order unity in the flow region being considered. In this thesis we shall limit ourselves to finding only the terms of the first degree in the small parameter ϵ.

The non-dimensionalized independent variables x,y and t are introduced as follows

$$x = \frac{\bar{x}}{\bar{\ell}}, \quad y = \frac{\bar{y}}{\bar{\ell}}, \quad t = \frac{u_o t}{\bar{\ell}}, \qquad (3)$$

where $\bar{\ell}$ is some characteristic length for which the chord length of the wedge is taken throughout the thesis.

Putting (2) and (3) into (1) and on dropping quardratic and higher order terms in ϵ, we obtain the following system of linear equations for the determination of the perturbation flow quantities u,v,p and ρ:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} + \frac{\rho_o}{u_o}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = - \frac{1}{\rho_o u_o} \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = - \frac{1}{\rho_o u_o} \frac{\partial p}{\partial y} \qquad (4)$$

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} = a_o^2\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \right)$$

where $a_o = (\gamma p_o/\rho_o)^{\frac{1}{2}}$ is the speed of sound in the pure wedge flow.

Making a transformation

$$\xi = x + t$$

$$\eta = x - t \qquad (5)$$

$$\zeta = 2y$$

we obtain,

$$\frac{\partial \rho}{\partial \xi} + \frac{\rho_o}{u_o}\left[\tfrac{1}{2}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) + \frac{\partial v}{\partial \zeta} \right] = 0$$

$$\frac{\partial u}{\partial \xi} = - \frac{1}{2\rho_o u_o}\left(\frac{\partial p}{\partial \xi} + \frac{\partial p}{\partial \eta} \right)$$

$$\frac{\partial v}{\partial \xi} = - \frac{1}{\rho_o u_o} \frac{\partial p}{\partial \zeta} \qquad (6)$$

$$\frac{\partial p}{\partial \xi} = a_o^2 \frac{\partial \rho}{\partial \xi} .$$

Differentiating the last equation with respect to ξ of (6) and then successively making use of the first, the second and the third equations of (6), we obtain

$$\frac{u_o^2}{a_o^2}\frac{\partial^2 p}{\partial\xi^2} = -\rho_o u_o \left[\frac{1}{2}\left(\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\xi\partial\eta} \right) + \frac{\partial^2 v}{\partial\xi\partial\zeta} \right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2 p}{\partial\xi^2} + \frac{\partial^2 p}{\partial\xi\partial\eta} \right) + \left(\frac{\partial^2 p}{\partial\xi\partial\eta} + \frac{\partial^2 p}{\partial\eta^2} \right)\right] + \frac{\partial^2 p}{\partial\zeta^2} ,$$

or

$$M_o^2 \frac{\partial^2 p}{\partial\xi^2} - \frac{\partial^2 p}{\partial\zeta^2} = \frac{1}{4}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} \right)^2 p ,$$

where $M_o = u_o/a_o$ is the Mach number behind the bow shock in the pure wedge flow. Returning from variables ξ, η and ζ to x,y and t, we have

$$(M_o^2 - 1)\frac{\partial^2 p}{\partial x^2} + 2M_o^2 \frac{\partial^2 p}{\partial x\partial t} + M_o^2 \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial y^2} = 0 \qquad (7)$$

which includes only one unknown function p.

The required flow field can now be found by first solving the wave equation (7) for p, then the second equation of (4) for u, the third for v and the fourth for ρ.

Equation (7), the key equation in the problem, should be recognized to be the same equation as that satisfied by the perturbation velocity potential in subsonic and moderate supersonic flows past slender bodies. In those cases the perturbation pressure is a linear function of the first derivatives of the perturbation velocity potential[3].

For steady flow we have $\partial/\partial t = 0$, and equations (4) and (7) reduce to those obtained by Chu[16] and by Chernyi[2].

3 THE BOUNDARY CONDITIONS

We shall now derive and linearize the required boundary conditions for determining the flow field. Denote by θ and β , respectively the flow deflection angle and the shock wave angle in the pure wedge flow (Fig.1), and for brevity, let $\varphi = \beta - \theta$. If in the reference steady flow, the wedge is at zero incidence, θ is equal to the semi-vertex angle ψ of the wedge otherwise it is equal to the sum (for the lower surface) or the difference (for the upper surface) of the semivertex angle and the angle of attack α. Let the wedge surface be given by

$$F(x,y,t) \equiv \epsilon\delta(x,t) - y = 0 \qquad (8)$$

with $\delta(x,t)$ known. δ represents the disturbance at the body surface in most general cases. Such a disturbance may be due to any motion — periodic or aperiodic — of a rigid wedge, or due to a steady or unsteady deformation of a wedge-like body, or due to a changing of body shape or of incidence. The bow shock may be described by

$$G(x,y,t) \equiv \epsilon f_5(x,t) + x \tan\varphi - y = 0 \qquad (9)$$

where $f_5(x,t)$ is unknown and to be determined as part of the solution. The two functions δ and f_5 together with their derivatives are assumed to be of order unity.

The boundary condition at the body surface, written in terms of x,y and t, is simply

$$u_o \frac{\partial F}{\partial t} + \vec{v}.\nabla F = 0$$

where

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$$

and $\vec{v}$ is the velocity vector of the gas, $\vec{v} = \vec{i}\tilde{u} + \vec{j}\tilde{v}$, where $\vec{i},\vec{j}$ are the unit vectors in $\bar{x},\bar{y}$ direction, respectively. After linearization, this boundary condition becomes

$$\text{at } y = 0, \qquad \frac{v}{u_o} = \frac{\partial\delta}{\partial t} + \frac{\partial\delta}{\partial x} \tag{10}$$

The boundary conditions at the bow shock, written in terms of x,y and t, are as follows[1]

Continuity condition: $$\left[\rho(u_o \frac{\partial G}{\partial t} + \vec{v}\cdot\nabla G)\right]_{(\infty)}^{(s)} = 0 \tag{11}$$

Momentum condition in the tangential direction:

$$\left[\vec{v}\cdot\vec{\tau}\right]_{(\infty)}^{(s)} = 0 \tag{12}$$

where $\vec{\tau}$ is any vector in the tangential direction of the shock.

Momentum condition in the normal direction:

$$\left[\rho(u_o \frac{\partial G}{\partial t} + \vec{v}\cdot\nabla G)^2 + (\nabla G)^2 p\right]_{(\infty)}^{(s)} = 0 \tag{13}$$

Energy condition:

$$\left[\frac{1}{2}(u_o \frac{\partial G}{\partial t} + \vec{v}\cdot\nabla G)^2 + (\nabla G)^2 h\right]_{(\infty)}^{(s)} = 0 \tag{14}$$

where h denotes the specific enthalpy of the gas.

In (11) to (14) the subscript s refers to the flow quantities just behind the bow shock, while the subscript ∞ refers to the flow quantities in the free stream. Bracket means the jump of the quantity inside the bracket from ∞ to s.

Setting $\epsilon = 0$ in (11) to (14), we obtain the following relations, which the flow quantities in the pure wedge flow should satisfied, and which are useful in deriving the boundary conditions for unsteady flow,

$$\begin{aligned} \rho_o u_o \sin\varphi &= \rho_\infty U_\infty \sin\beta \\ u_o \cos\varphi &= U_\infty \cos\beta \\ p_o + \rho_o u_o^2 \sin^2\varphi &= p_\infty + \rho_\infty U_\infty^2 \sin^2\beta \\ h_o + \tfrac{1}{2} u_o^2 &= h_\infty + \tfrac{1}{2} U_\infty^2 \end{aligned} \tag{15}$$

where U_∞ is the speed of the gas in the free stream.

From (11) we have

$$(\rho_o + \epsilon\rho)\left[\epsilon u_o \frac{\partial f_5}{\partial t} + (u_o + \epsilon u)(\tan\varphi + \frac{\partial f_5}{\partial x}) - \epsilon v \right]$$

$$= \rho_\infty\left[\epsilon u_o \frac{\partial f_5}{\partial t} + U_\infty \cos\theta\,(\tan\varphi + \epsilon \frac{\partial f_5}{\partial x}) + U_\infty \sin\theta \right],$$

or,

$$\rho_o u_o \tan\varphi + \epsilon\left[\rho u_o \tan\varphi + \rho_o(u_o \frac{\partial f_5}{\partial t} + u \tan\varphi + u_o \frac{\partial f_5}{\partial x} - v) \right]$$

$$= \rho_\infty U_\infty \frac{\sin\beta}{\cos\varphi} + \epsilon\rho_\infty(u_o \frac{\partial f_5}{\partial t} + U_\infty \cos\theta \frac{\partial f_5}{\partial x})$$

Upon using the first and the second relations of (15) we obtain,

at $y = x\tan\varphi$,

$$\frac{u}{u_o}\tan\varphi - \frac{v}{u_o} + \frac{\rho}{\rho_o}\tan\varphi = -A\frac{\partial f_5}{\partial t} - B\frac{\partial f_5}{\partial x} \tag{16}$$

where

$$A = 1 - \frac{\rho_\infty}{\rho_o}, \qquad B = A\cos^2\varphi. \tag{17}$$

As the tangential vector $\vec{\tau} \perp \nabla G$

$$\vec{\tau} = \vec{i} + \vec{j}(\tan\varphi + \epsilon \frac{\partial f_5}{\partial x}),$$

and equation (12) becomes

$$(u_o + \epsilon u - U_\infty \cos\theta) + (\epsilon v + U_\infty \sin\theta)(\tan\varphi + \epsilon \frac{\partial f_5}{\partial x}) = 0$$

which can be further simplified by using the first two relations of (15) to obtain,

at $y = x \tan\varphi$,
$$\frac{u}{u_o} + \frac{v}{u_o}\tan\varphi = -C\frac{\partial f_5}{\partial x} \tag{18}$$

where

$$C = \left(\frac{\rho_o}{\rho_\infty} - 1\right)\sin\varphi\cos\varphi \tag{19}$$

The linearized expression for ∇G^2 is

$$\nabla G^2 = \left(\epsilon\frac{\partial f_5}{\partial x} + \tan\varphi\right)^2 + 1^2 = \sec^2\varphi + \epsilon\, 2\tan\varphi\frac{\partial f_5}{\partial x},$$

and hence equation (13) becomes

$$(\rho_o + \epsilon\rho)\left[\epsilon u_o\frac{\partial f_5}{\partial t} + (u_o + \epsilon u)\left(\tan\varphi + \epsilon\frac{\partial f_5}{\partial x}\right) - \epsilon v\right]^2 + (p_o + \epsilon p)\left(\sec^2\varphi + \epsilon 2\tan\varphi\frac{\partial f_5}{\partial x}\right)$$
$$= \rho_\infty\left[\epsilon u_o\frac{\partial f_5}{\partial t} + U_\infty\cos\theta\left(\tan\varphi + \epsilon\frac{\partial f_5}{\partial x}\right) + U_\infty\sin\theta\right]^2 + p_\infty\left(\sec^2\varphi + \epsilon\, 2\tan\varphi\frac{\partial f_5}{\partial x}\right),$$

or

$$\rho_o u_o^2\tan^2\varphi + p_o\sec^2\varphi + \epsilon\Big[\rho u_o^2\tan^2\varphi + p\sec^2\varphi + 2p_o\tan\varphi\frac{\partial f_5}{\partial x}$$
$$+ 2\rho_o u_o\tan\varphi\left(u_o\frac{\partial f_5}{\partial t} + u_o\frac{\partial f_5}{\partial x} + u\tan\varphi - v\right)\Big] =$$
$$\frac{\rho_\infty U_\infty^2\sin^2\beta}{\cos^2\varphi} + p_\infty\sec^2\varphi + \epsilon\left[2\rho_\infty U_\infty\frac{\sin\beta}{\cos\varphi}\left(u_o\frac{\partial f_5}{\partial t} + U_\infty\cos\theta\frac{\partial f_5}{\partial x}\right) + 2p_\infty\tan\varphi\frac{\partial f_5}{\partial x}\right]$$

Using the first three relations of (15) we obtain,

at $y = x\tan\varphi$,

$$\frac{2u}{u_o}\tan\varphi - \frac{2v}{u_o} + \frac{\rho}{\rho_o}\tan\varphi + \frac{1}{M_o\sin\varphi\cos\varphi}\frac{p}{\rho_o a_o u_o} = 0 \tag{20}$$

Finally, from (14) we get

$$\frac{1}{2}\Big[\ \epsilon u_o \frac{\partial f_5}{\partial t} + (u_o + \epsilon u)(\tan\varphi + \epsilon \frac{\partial f_5}{\partial x}) - \epsilon v\ \Big]^2$$

$$+ h_o(\sec^2\varphi + \epsilon\, 2\tan\varphi \frac{\partial f_5}{\partial x}) \frac{1 + \epsilon p/p_o}{1 + \epsilon\rho/\rho_o}$$

$$= \frac{1}{2}\Big[\ \epsilon u_o \frac{\partial f_5}{\partial t} + U_\infty \cos\theta(\tan\varphi + \epsilon \frac{\partial f_5}{\partial x}) - \epsilon v\ \Big]^2 + h_\infty(\sec^2\varphi + \epsilon\, 2\tan\varphi \frac{\partial f_5}{\partial x}).$$

Expanding and linearising the last equation and then simplifying the resulting equation with the fourth and the second relations of (15), we obtain, after some algebraic manipulations,

at $y = x\tan\varphi$,

$$\frac{u}{u_o}\tan\varphi - \frac{v}{u_o} - \frac{1}{(\gamma - 1)M_o^2 \sin\varphi\cos\varphi}\frac{\rho}{\rho_o} + \frac{\gamma}{(\gamma - 1)M_o \sin\varphi\cos\varphi}\frac{p}{\rho_o a_o u_o}$$

$$= (\frac{\rho_o}{\rho_\infty} - 1)\frac{\partial f_5}{\partial t} + \Big[\ \frac{\rho_o U_\infty \cos\theta}{\rho_\infty u_o} - 1 - \frac{2(h_o - h_\infty)}{u_o^2}\ \Big]\frac{\partial f_5}{\partial x}.$$

Eliminating v from the last equation and equation (18) gives,

$$\text{at } y = x\tan\varphi,\quad \frac{u}{u_o} - \frac{1}{(\gamma - 1)M_o^2}\frac{\rho}{\rho_o} + \frac{\gamma}{(\gamma - 1)M_o}\frac{p}{\rho_o a_o u_o} = C\frac{\partial f_5}{\partial t} \qquad (21)$$

Equations (16), (18), (20) and (21) are the required boundary conditions at the bow shock which may be solved to give

at $y = x\tan\varphi$:

$$\frac{v}{u_o} = \frac{\cot\varphi}{1 - W}\Big[\Big\{ A(1 + \gamma W)\sin\varphi\cos\varphi - C(\gamma - 1)W \Big\}\frac{\partial f_5}{\partial t}$$

$$+ \Big\{ B(1 + \gamma W)\sin\varphi\cos\varphi + C(W - \gamma W\cos^2\varphi - \sin^2\varphi) \Big\}\frac{\partial f_5}{\partial x}\Big] \equiv N$$

$$\frac{p}{\rho_o a_o u_o} = \frac{H_a \cot\varphi}{1 - W}\Big[\Big\{ A(2 + (\gamma - 1)W)\sin\varphi\cos\varphi - C(\gamma - 1)W \Big\}\frac{\partial f_5}{\partial t}$$

$$+ \Big\{ B(2 + (\gamma - 1)W)\sin\varphi\cos\varphi - C(\gamma - 1)W\cos^2\varphi \Big\}\frac{\partial f_5}{\partial x}\Big] \equiv L$$

$$\frac{\rho}{\rho_o} = \frac{\cot\varphi}{1 - W}\Big[\Big\{ A(\gamma + 1)W - \frac{C(\gamma - 1)W}{\sin\varphi\cos\varphi} \Big\}\frac{\partial f_5}{\partial t} \qquad (22)$$

$$+ \Big\{ B(\gamma + 1)W - C(\gamma - 1)W\cot\varphi \Big\}\frac{\partial f_5}{\partial x}\Big]$$

$$\frac{u}{u_o} = \frac{-1}{1 - W}\left[\left\{ A(1 + \gamma W) \sin\varphi \cos\varphi - C(\gamma - 1)W \right\} \frac{\partial f_5}{\partial t} + \left\{ B(1 + \gamma W) \sin\varphi \cos\varphi + C(1 - \gamma W) \cos^2\varphi \right\} \frac{\partial f_5}{\partial x} \right]$$

where

$$H_a = M_o \tan\varphi, \qquad W = M_o^2 \sin^2\varphi. \tag{23}$$

If the motion of the gas is periodic or even steady, the above five boundary conditions (10) and (22) are sufficient for the determination of the five unknown functions u, v, p, ρ and f_5. For cases of aperiodic motions, we need initial conditions as well.

This thesis is mainly concerned with periodic motions of a wedge in inviscid and in viscous flows, and also with steady flow. But a method of applying the approximate general solution given in Chapter II of the thesis to aperiodic motions of a wedge has also been developed[18] which, to some extent, may be useful in determining the effect of ablation of a wedge-like body on its stability derivatives.

CHAPTER II

APPROXIMATE THEORY [20, 21]

1 INTRODUCTION

In this Chapter, the perturbation equations derived in Chapter I are first approximated and then solved to obtain a general solution, which can be applied to both periodic and aperiodic motions of a rigid or a flexible wedge with or without angle of attack, no matter whether the wedge is thick or thin. The application of this general solution to a pitching wedge gives an exact formula for the stiffness derivative for the most general cases and an approximate formula for the damping derivative. A new set of reflected waves due to the motion of the bow shock is found to be important for relatively thicker wedges and to be a factor which tends to strongly destabilize the motion of the body. Also it becomes dominant for very thick wedges.

McIntosh's result[14] and Appleton's result[13] (see also Ref.22) are reproduced here as special cases when the flow is hypersonic and the wedge is slender. Also for supersonic flow, the approximate theory agrees very well with Van Dyke's theory[19], and with the experimental results by Pugh and Woodgate[4].

The validity of the approximation is discussed in this Chapter, and will be demonstrated by a comparison with the exact theory in Chapter III. In the light of the exact theory it is seen that the approximate theory is valid for flow Mach numbers behind the bow shock as low as 2.0.

For stability analysis of an oscillating wedge, this approximate theory will be shown to be a special case in the exact theory. However, the former is exact with respect to the frequency parameter and can be applied to cases of aperiodic motions, whilst the exact theory can only be applied to periodic motions. Also the physical meanings of wave interaction and reflection are easier to discuss using the approximate theory.

2 GENERAL SOLUTION

As stated in Chapter I, the key equation to the problem of finding the flow field is

$$(M_o^2 - 1)\frac{\partial^2 p}{\partial x^2} + 2M_o^2\frac{\partial^2 p}{\partial x\partial t} + M_o^2\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial y^2} = 0 \qquad (1)$$

Approximation is made here that the coefficient $M_o^2 - 1$ of the first term of equation (1) is to be replaced by M_o^2, thus

$$M_o^2\left(\frac{\partial^2 p}{\partial x^2} + 2\frac{\partial^2 p}{\partial x\partial t} + \frac{\partial^2 p}{\partial t^2}\right) - \frac{\partial^2 p}{\partial y^2} = 0, \qquad (2)$$

which, after transforming to variables ξ, η and ζ by the transformation (I.5), becomes

$$M_o^2\frac{\partial^2 p}{\partial \xi^2} - \frac{\partial^2 p}{\partial \zeta^2} = 0 \qquad (3)$$

This approximation is obviously valid for large values of M_o, e.g. for the case of hypersonic flow past a wedge which is not too thick.

In the hypersonic small-disturbance theory both the equations of motion and the boundary conditions are approximate and the error of the theory is of order of β^2. Whereas in the present theory the boundary conditions are exact (see Chapter I) and the only approximation introduced is to neglect, in the coefficient of one term in an equation, M_o^{-2} compared with unity. Thus the error in the present theory is of order of M_o^{-2}. For hypersonic flow past a very thin body β^2 and M_o^{-2} are of the same order of magnitude, and the present theory should reduce to McIntosh's theory which is based on the hypersonic small-disturbance theory, and this will be shown in Section 3.2.6. However, for relatively thick wedges β^2 and M_o^{-2} are of different orders of magnitude, e.g. for the free stream

Mach number $M_\infty = 17$, $\theta = 20^o$, we have $\beta^2 = 19\%$, $M_o^{-2} = 4.5\%$, and for $M_\infty = 17$, $\theta = 30^o$, we have $\beta^2 = 45\%$, $M_o^{-2} = 11.7\%$. Therefore the present theory can give better results than either McIntoch's or Appleton's theories and can be applied to relatively thick wedges for which the other two theories cannot apply. Besides, the present theory is even a very good approximate theory for both hypersonic and supersonic flows providing M_o is larger than 2.0, as may be deduced by comparison with the exact theory in Chapter III and with experimental results (see Section 3.2.4).

From the derivation of equation (I.7), it is clear that this approximation is equivalent to dropping the term $\frac{\rho_o}{u_o}\frac{\partial u}{\partial x}$ in the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} + \frac{\rho_o}{u_o}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (4)$$

which now becomes

$$\frac{\partial \rho}{\partial \xi} + \frac{\rho_o}{u_o}\frac{\partial v}{\partial \zeta} = 0 \qquad (5)$$

Using the fourth equation of (I.6), we have

$$M_o \frac{\partial (p/\rho_o a_o u_o)}{\partial \xi} + \frac{\partial v}{\partial \zeta} = 0, \qquad (6)$$

and the whole system of approximate equations reads

$$\left.\begin{aligned} M_o \frac{\partial (p/\rho_o a_o u_o)}{\partial \xi} + \frac{\partial v}{\partial \zeta} &= 0 \\ \frac{\partial u}{\partial \xi} &= -\frac{1}{2\rho_o u_o}\left(\frac{\partial p}{\partial \xi} + \frac{\partial p}{\partial \eta}\right) \\ \frac{\partial v}{\partial \xi} &= -\frac{1}{\rho_o u_o}\frac{\partial p}{\partial \zeta} \\ \frac{\partial p}{\partial \xi} &= a_o^2 \frac{\partial \rho}{\partial \xi} \end{aligned}\right\} \qquad (7)$$

The general solution to equation (3) is

$$p/\rho_o a_o = f_1^*(\xi - M_o\zeta, \eta) - f_2^*(\xi + M_o\zeta, \eta) \tag{8}$$

where f_1^* and f_2^* are arbitrary functions of two independent variables. From the third equation of (7), we obtain

$$v = f_1^*(\xi - M_o\zeta, \eta) + f_2^*(\xi + M_o\zeta, \eta) + f_6(\zeta, \eta) \tag{9}$$

with another arbitrary function f_6. Putting (8) and (9) into the first equation of (7), we have

$$\frac{\partial f_6}{\partial \zeta} = 0 \tag{10}$$

or

$$f_6 = f_6(\eta) \tag{11}$$

As f_6 is a function of η only, it can be absorbed into f_1^* and f_2^* by letting

$$f_1(\xi - M_o\zeta, \eta) = f_1^*(\xi - M_o\zeta, \eta) + \tfrac{1}{2}f_6(\eta) \tag{12}$$

$$f_2(\xi + M_o\zeta, \eta) = f_2^*(\xi + M_o\zeta, \eta) + \tfrac{1}{2}f_6(\eta), \tag{13}$$

then f_1 and f_2 are arbitrary functions of two independent variables, and we have

$$v = f_1(\xi - M_o\zeta, \eta) + f_2(\xi + M_o\zeta, \eta) \tag{14}$$

$$p/\rho_o a_o = f_1(\xi - M_o\zeta, \eta) - f_2(\xi + M_o\zeta, \eta) \tag{15}$$

From the fourth and the second equations of (7) we obtain

$$a_o\rho/\rho_o = f_1(\xi - M_o\zeta, \eta) - f_2(\xi + M_o\zeta, \eta) + f_3(\zeta, \eta) \tag{16}$$

$$\begin{aligned} -2M_o u = {} & f_1(\xi - M_o\zeta, \eta) - f_2(\xi + M_o\zeta, \eta) + f_4(\zeta, \eta) \\ & + \frac{\partial}{\partial\eta}\int \left[f_1(\xi - M_o\zeta, \eta) - f_2(\xi + M_o\zeta, \eta) \right] d\xi \end{aligned} \tag{17}$$

where f_3 and f_4 are arbitrary functions.

Written in terms of the independent variables x, y and t, the general solution to the approximate perturbation equations (7) can be expressed through four arbitrary functions f_1 to f_4 of two independent variables as follows

$$v = f_1(x + t - 2M_o y,\ x - t) + f_2(x + t + 2M_o y,\ x - t) \tag{18}$$

$$p/\rho_o a_o = f_1(x + t - 2M_o y,\ x - t) - f_2(x + t + 2M_o y,\ x - t) \tag{19}$$

$$a_o \rho/\rho_o = f_1(x + t - 2M_o y,\ x - t) - f_2(x + t + 2M_o y,\ x - t) + f_3(y,\ x - t) \tag{20}$$

$$-2M_o u = f_1(x + t - 2M_o y,\ x - t) - f_2(x + t + 2M_o y,\ x - t) + \left[\frac{\partial}{\partial \eta} \int \left\{ f_1(\xi - M_o \zeta,\ \eta) - f_2(\xi + M_o \zeta,\ \eta) \right\} d\xi \right]_{\substack{\xi = x + t \\ \eta = x - t \\ \zeta = 2y}} + f_4(y,\ x - t) \tag{21}$$

where the functions f_3 and f_4, in which one of the two independent variables 2y is replaced by y, are still arbitrary in form.

The functions $f_1(x + t - 2M_o y,\ x - t)$ and $f_2(x + t + 2M_o y,\ x - t)$ describe the distrubances which propagate along two families of characteristics which are the intersections of two characteristic planes (here the time variable t is considered as one coordinate)

$$x - M_o y = \text{Const.} \tag{22}$$

$$x - t = \text{Const.} \tag{23}$$

and

$$x + M_o y = \text{Const.} \tag{24}$$

$$x - t = \text{Const.} \tag{25}$$

This physical interpretation is easily obtained from the fact that the values of functions f_1 and f_2 should remain constant at different moments of time. Along the first family of characteristics disturbances propagate in the direction towards the bow shock, whereas along the second family of characteristics disturbances propagate in the direction away from the bow shock. These waves — the propagations of disturbances — are seen to be the Mach waves within the approximation of the present theory.

It is easily seen from the general solution given above that the orders of magnitude of the perturbation quantities are as follows

$$\frac{u/u_o}{v/u_o} = 0(\frac{1}{M_o}) \tag{26}$$

$$\frac{p/p_o}{v/u_o} = 0(M_o) \tag{27}$$

$$\frac{\rho/\rho_o}{v/u_o} = 0(M_o) \tag{28}$$

Therefore the orders of magnitude of the perturbation quantities, at least for flows past a wedge-like body, depend mainly upon the Mach number M_o behind the bow shock rather than the free stream Mach number M_∞. Of course, for hypersonic flow past a slender body M_o is of the same order of magnitude as M_∞, and we then arrive at the same conclusion regarding the orders of magnitude of the perturbation quantities as that in the ordinary hypersonic small — disturbance theory.

The boundary conditions are those derived in Chapter I.

In principle, the problem of determining an unsteady flow past a wedge-like body has been reduced to that of finding five unknown functions f_1 to f_5 of two independent variables to satisfy the five boundary conditions and, appropriate initial conditions as well if the motion is an aperiodic one. Mathematically, this is equivalent to solving a system of five first order linear integro — partial differential — functional equations.

For most engineering problems, the only unknown flow parameter required is the pressure distribution p. It is then important to notice that in order to find p, we need find only three unknown functions f_1,f_2 and f_5 to satisfy (through (18) and (19)) the boundary conditions(I.10) and the first and the second of (I.22). We are therefore only required to solve a system of three first order linear partial differential — functional equations. After doing this, the other quantities u and ρ can, if needed, be found from the remaining two boundary conditions.

3 PERIODIC MOTIONS OF A WEDGE

3.1 FORMULATION OF THE PROBLEM

In order to apply the general solution to any practical problem, it would be useful to change the forms of the arbitrary functions f_1 and f_2 as follows*

$$f_1(x + t - 2M_o y, x - t) = f_1(x - M_o y, x - t), \qquad (29)$$

$$f_2(x + t + 2M_o y, x - t) = f_2(x + M_o y, x - t). \qquad (30)$$

In the right hand sides of expressions (29) and (30), as the functions are still arbitrary and to avoid introducing new symbols for these new functions we use the same symbols f_1 and f_2, keeping in mind that they may be different in some way from the original ones.

For a harmonic motion** of a wedge, the wedge surface is given by ***

$$\delta(x, t) = e^{ikt}\Delta(x), \qquad (31)$$

---

* These simpler forms of the arbitrary functions could also be obtained directly if the Gallilean transformation[3] was applied instead of the transformation (I.5). However, by using Gallilean transformation, equations (I.6) would be more complicated.

** Any periodic motion, according to Fourier's theorem, can be expressed as a sum of harmonic motions. As the basic perturbation equations (I.4) and boundary conditions (I.10) and (I.22) are linear, the solution to a periodic motion is therefore the sum of solutions to the harmonic motions. Hence we need only study a harmonic motion of a wedge in detail.

*** It will be understood that throughout this thesis we consider only real parts of all the complex expressions.

where, $\Delta(x)$ is the "amplitude" function and is given in every practical problem, and

$$k = \frac{\omega \bar{\ell}}{u_o}, \tag{32}$$

where ω is the circular frequency. So k is the non-dimensional frequency parameter based on the chord length of the body and the flow speed u_o behind the bow shock in the reference steady flow which is of the same order of magnitude as the free stream velocity U_∞, no matter whether the wedge is thick or slender. For a slender wedge, $u_o \cong U_\infty$, and the parameter k defined by (32) is twice that in American notation and equals to λ, in British notation. For a thick wedge, u_o is less than U_∞, but the parameter k given by (32) retains its physical meaning, as can be seen from its definition.

Because the basic perturbation equations and boundary conditions (I.10) and (I.22) are linear, the motion of the gas as observed from a fixed position should also be harmonic with the same period as the wedge. Therefore the perturbation quantities must be equal to e^{ikt} times some functions of x, y only, and by taking into account the forms of the arbitrary functions f_1 and f_2, we may write

$$\begin{aligned} v &= e^{ik(t-x)}\left[F_1(x - M_o y) + F_2(x + M_o y) \right], \\ \frac{p}{\rho_o a_o} &= e^{ik(t-x)}\left[F_1(x - M_o y) - F_2(x + M_o y) \right], \\ \frac{\rho}{\rho_o} &= e^{ik(t-x)}\left[F_1(x - M_o y) - F_2(x + M_o y) + F_3(y) \right], \\ -2M_o u &= e^{ik(t-x)}\left[F_1(x - M_o y) - F_2(x + M_o y) + 2F_4(y) \right] \\ &\quad + \left\{ \frac{\partial}{\partial \eta}\int \left[f_1(\xi - M_o\zeta, \eta) - f_2(\xi + M_o\zeta, \eta) \right] d\xi \right\}_{\substack{\xi = x+t \\ \eta = x-t \\ \zeta = 2y}}, \end{aligned} \tag{33}$$

where F_3,F_3,F_3 and F_4 are arbitrary functions of one independent variable only. The last expression of (33) can be further simplified as below.

Since

$$f_1(\xi - M_o\zeta, \eta) = e^{-ik\eta}F_1(\frac{\xi + \eta}{2} - M_o y),$$

then

$$\frac{\partial f_1}{\partial \eta} = - ike^{ik\eta}F_1(\frac{\xi + \eta}{2} - M_o y) + \tfrac{1}{2}e^{-ik\eta}F_1',$$

and hence

$$\frac{\partial}{\partial \eta}\int f_1 d\xi = \int \frac{\partial f_1}{\partial \eta} d\xi = - ike^{-ik\eta}\int F_1(\frac{\xi + \eta}{2} - M_o y)d\xi + e^{-ik\eta}\int F_1' d(\frac{\xi}{2})$$

$$= e^{-ik\eta}\left[-2ik\int F_1(x - M_o y)dx + F_1(x - M_o y) \right].$$

Similarly,

$$\frac{\partial}{\partial \eta}\int f_2 d\xi = e^{-ik\eta}\left[-2ik\int F_2(x + M_o y)dx + F_2(x + M_o y) \right].$$

Therefore the last expression of (33) becomes

$$-M_o u = e^{ik(t - x)}\Big[F_1(x - M_o y) - F_2(x + M_o y) + F_4(y)$$
$$-ik\int \{ F_1(x - M_o y) - F_2(x + M_o y) \}dx \Big]. \qquad (34)$$

Now the wedge surface may be accordingly expressed as

$$\delta(x, t) = e^{ik(t - x)}\Delta_1(x), \qquad (35)$$

with

$$\Delta_1(x) = e^{ikx}\Delta(x), \qquad (36)$$

while the shape of the bow shock is described by

$$f_5(x, t) = e^{ik(t - x)}F_5(x), \qquad (37)$$

where the function $F_5(x)$, and the other four functions F_3, F_3, F_3 and F_4, are to be determined by the boundary conditions.

Combining the first equation of (I.22) with the first equation of (33), the second of (I.22) with the second of (33), we obtain

$$F_1(\ (1 - H_a)x\ ) + F_2(\ (1 + H_a)x) = Nu_o e^{-ik(t - x)},$$

$$F_1(\ (1 - H_a)x\ ) - F_2(\ (1 + H_a)x\ ) = Lu_o e^{-ik(t - x)}$$

Solving these two equations to obtain F_1 and F_2 as

$$\begin{aligned} F_1(\ (1 - H_a)x\ ) &= \frac{u_o}{2}(N + L)e^{-ik(t - x)}, \\ F_2(\ (1 + H_a)x\ ) &= \frac{u_o}{2}(N - L)e^{-ik(t - x)} \end{aligned} \tag{38}$$

then

$$\begin{aligned} F_1(x) &= \frac{u_o}{2}\left[\ N(\frac{x}{1 - H_a}, t) + L(\frac{x}{1 - H_a}, t)\ \right]e^{-ik(t - \frac{x}{1 - H_a})} \\ F_2(x) &= \frac{u_o}{2}\left[\ N(\frac{x}{1 + H_a}, t) - L(\frac{x}{1 + H_a}, t)\ \right]e^{-ik(t - \frac{x}{1 + H_a})} \end{aligned} \tag{39}$$

Putting (39) into (I.10) we obtain

$$\begin{aligned} &\left[\ N(\frac{x}{1 - H_a}, t) + L(\frac{x}{1 - H_a}, t)\ \right]e^{-ik(t - \frac{x}{1 - H_a})} \\ &+ \left[\ N(\frac{x}{1 + H_a}, t) - L(\frac{x}{1 + H_a}, t)\ \right]e^{-ik(t - \frac{x}{1 + H_a})} = 2\Delta_1'(x) \end{aligned} \tag{40}$$

The fucctions N and L can be expressed, through the first two equations of (I.22), in terms of $F_5(x)$

$$\begin{aligned} N(x, t) &= \left[\ \bar{N}_2 F_5'(x) + ik\bar{N}_1 F_5(x)\ \right]e^{ik(t - x)}, \\ L(x, t) &= \left[\ \bar{L}_2 F_5'(x) + ik\bar{L}_1 F_5(x)\ \right]e^{ik(t - x)}, \end{aligned} \tag{41}$$

where the non-dimensional constants $\bar{N}_1$, $\bar{N}_2$, $\bar{L}_1$ and $\bar{L}_2$ can be obtained by combining (I.17), (I.19) and the first two equations of (I.22) as follows

$$\begin{aligned} \bar{N}_2 &= \frac{(1 - \frac{\rho_\infty}{\rho_o})\cos^4\varphi}{1 - W}\left[\ 1 + \frac{\rho_o}{\rho_\infty}(M_o^2 - 1)\ \tan^2\varphi - \gamma W(\frac{\rho_o}{\rho_\infty} - 1)\right], \\ \bar{N}_1 &= \frac{(1 - \frac{\rho_\infty}{\rho_o})\sin^2\varphi\ \cos^2\varphi}{1 - W}\left[\ 1 + \frac{\rho_o}{\rho_\infty} - \gamma W(\frac{\rho_o}{\rho_\infty} - 1)\ \right], \\ \bar{L}_2 &= \frac{2H_a(1 - \frac{\rho_\infty}{\rho_o})\cos^4\varphi}{1 - W}\left[\ 1 - \frac{\gamma - 1}{2}\ W(\frac{\rho_o}{\rho_\infty} - 1)\ \right], \\ \bar{L}_1 &= \tan^2\varphi\ \bar{L}_2. \end{aligned} \tag{42}$$

Substituting expressions (41) into equation (40) gives

$$aF_5'\left(\frac{x}{1 - H_a}\right) + bF_5'\left(\frac{x}{1 + H_a}\right) + ik\left[cF_5\left(\frac{x}{1 - H_a}\right) + dF_5\left(\frac{x}{1 + H_a}\right)\right] = 2\Delta_1'(x),$$

where

$$\begin{aligned} a &= \overline{N}_2 + \overline{L}_2, \\ b &= \overline{N}_2 - \overline{L}_2, \\ c &= \overline{N}_1 + \overline{L}_1, \\ d &= \overline{N}_1 - \overline{L}_1, \end{aligned} \qquad (43)$$

Letting

$$m_a = \frac{1 - H_a}{1 + H_a}, \qquad (44)$$

we finally obtain

$$\left[aF_5'(x) + bF_5'(m_a x)\right] + ik\left[cF_5(x) + dF_5(m_a x)\right] = 2\Delta_1'(\,(1 - H_a)x\,), \qquad (45)$$

or

$$\left[\,F_5'(x) - \lambda_a F_5'(m_a x)\,\right] + ik\left[\,\mu F_5(x) + \nu F_5(m_a x)\,\right] = \frac{2}{a}\Delta_1'(\,(1 - H_a)x\,), \qquad (46)$$

where

$$F_5'(m_a x) = \left.\frac{dF_5(\sigma)}{d\sigma}\right|_{\sigma = m_a x} \quad \text{etc., and}$$

$$\lambda_a = -\frac{b}{a}, \qquad \mu = \frac{c}{a}, \qquad \nu = \frac{d}{a}. \qquad (47)$$

Equation (46), a functional-differential equation for $F_5(x)$, is our basic equation for hypersonic and supersonic flows past an oscillating wedge. A special case of equation (46) when the second group of terms on its left hand side disappears was obtained by Chernyi[2] and Chu[16] for steady flows, and by McIntosh[14] for hypersonic flow past a slender wedge. After solving $F_5(x)$ from this equation, we can obtain the two functions N and L from (41), and then $F_1(x)$ and $F_2(x)$ from (39), and finally $p(x, y, t)$ and $v(x, y, t)$ from the first two equations of (33). The final expression for the perturbation pressure $p(x, y, t)$ is

$$\frac{p(x, y, t)}{\frac{1}{2}\rho_o u_o^2} = \frac{e^{ik(t-x)}}{M_o}\left[aF_5'\left(\frac{x - M_o y}{1 - H_a}\right) - bF_5'\left(\frac{x + M_o y}{1 + H_a}\right) + ik\left\{ cF_5\left(\frac{x - M_o y}{1 - H_a}\right) - dF_5\left(\frac{x + M_o y}{1 + H_a}\right) \right\}\right] , \tag{48}$$

and the perturbation pressure at the wedge surface is given by

$$\frac{p(x, o, t)}{\frac{1}{2}\rho_o u_o} = \frac{e^{ik(t-x)}}{M_o}\left[aF_5'\left(\frac{x}{1 - H_a}\right) - bF_5'\left(\frac{x}{1 + H_a}\right) + ik\left\{ cF_5\left(\frac{x}{1 - H_a}\right) - dF_5\left(\frac{x}{1 + H_a}\right) \right\}\right] . \tag{49}$$

3.2 PITCHING MOTION OF A WEDGE

Figure 2 describes a general pitching motion of a wedge about its pivot O_1, ψ being the semi-vertex angle of the wedge, α the angle of attack, ϵe^{ikt} the angular displacement of the wedge, and ϵ the amplitude of this angular displacement. The function $\Delta_1(x)$ can be obtained as follows.

The coordinates x, y of the new position P of a point P_o on the wedge surface ater an angular displacement ϵe^{ikt} have the relation

$$y = O_1P_o\epsilon e^{ikt} \cos \angle xP_oO_1$$

$$\begin{aligned} y &= \epsilon e^{ikt}(x_o - h \cos \psi) \\ &= \epsilon e^{ikt}(x - O_1P_o\epsilon e^{ikt} \sin \angle OP_oO_1 - h \cos \psi) \\ &= \epsilon e^{ikt}(x - h \cos \psi) , \end{aligned}$$

in which the quadratic term of ϵ has been neglected. From this we get

$$\delta(x, t) = e^{ikt}(x - h \cos \psi)$$

and

$$\Delta_1(x) = e^{ikx}(x - h \cos \psi) . \tag{50}$$

For the upper surface, $\theta = \psi - \alpha$ and $\delta(x, t) = e^{ikt}(x - h \cos \psi)$, for the lower surface, $\theta = \psi + \alpha$ and $\delta(x, t) = e^{ikt}(x - h \cos \psi)$.

For determining the flow field between the bow shock and the surface of a wedge with finite length, it is only necessary to prescribe the disturbance over the surface, because in supersonic flow, the downstream disturbance is not felt upstream.

3.2.1. GENERAL FORMULAE FOR THE AERODYNAMIC DERIVATIVES

For any periodic motion of a wedge, once the function $\Delta_1(x)$ is given, the basic equation (45) can be solved by the method of power series.

For pitching motion of a wedge, the function $\Delta_1(x)$ is given by equation (50), and hence we have

$$\Delta_1'(x) = e^{ikx}(1 + ikx - ikh\cos\psi),$$

and

$$\begin{aligned} \Delta_1'((1 - H_a)x) &= e^{ikx}\left[1 + ik(1 - H_a)x - ikh\cos\psi\right] \\ &= \sum_{n=0}^{\infty} (n + 1 - ikh\cos\psi)\frac{\left[ik(1 - H_a)x\right]^n}{n!}. \end{aligned} \tag{51}$$

Let

$$F_5(x) = \sum_{n=0}^{\infty} b_n x^n ,$$

then

$$F_5(m_a x) = \sum_{n=0}^{\infty} b_n m_a^n x^n , \qquad F_5'(x) = \sum_{n=0}^{\infty}(n + 1)b_{n+1}x^n ,$$

$$F_5'(x m_a) = \sum_{n=0}^{\infty}(n + 1)b_{n+1}m_a^n x^n . \tag{52}$$

Putting expressions (51) and (52) into equation (45), we obtain

$$\sum_{n=0}^{\infty}\left[(n + 1)b_{n+1}(a + bm_a^n) + ikb_n(c + dm_a^n)\right]x^n = \sum_{n=0}^{\infty} 2(n + 1 - ikh\cos\psi)\frac{\left[ik(1 - H_a)x\right]^n}{n!} .$$

By equating the coefficients of terms of the same power in x on both sides of the last equation, we have

$$b_1(a + b) + ikb_o(x + b) = 2(1 - ikh\cos\psi),$$

$$2b_2(a + bm_a) + ikb_1(c + dm_a) = 2(2 - ikh\cos\psi)ik(1 - H_a)$$

$$3b(a + bm_a^2) + ikb(c + dm_a^2) = 2(3 - ikh\cos\psi)\frac{\left[ik(1 - H_a)\right]^2}{2},$$

$$\cdots\cdots\cdots\cdots \qquad (53)$$

$$(n + 1)b_{n+1}(a + bm_a^n) + ikb_n(c + dm_a^n) = 2(n + 1 - ikh\cos\psi)\frac{\left[ik(1 - H_a)\right]^n}{n!},$$

and so on.

Therefore the coefficients $b_1, b_2, \ldots b_n, \ldots$ of the power series $F_5(x)$ can be sucessively expressed in terms of b_o, an integral constant to be found by the additional condition that

$$f_5(o, t) = \delta(o, t),$$

which arises because of the attachment of the bow shock to the wedge vertex.

Hence $$b_o = -h\cos\psi \qquad (54)$$

and all of the coefficients b_n can be sucessively determined, and the function $F_5(x)$ found. It is evident that $b_{n+1} = 0(k^n)$ for small k, hence the radius of convergence of the power series for $F_5(x)$ is of order k^{-1}, which is much greater than unity for stability analysis.

We are mainly interested in the pressure distribution over the wedge surface, then by using equalities (52), we obtain from (49):

$$\frac{p(x, o, t)}{\frac{1}{2}\rho_o a_o u_o e^{ik(t - x)}} = aF_5'\left(\frac{x}{1 - H_a}\right) - bF_5'\left(\frac{x}{1 + H_a}\right) + ik\left[cF_5\left(\frac{x}{1 - H_a}\right) - dF_5\left(\frac{x}{1 + H_a}\right)\right]$$

$$= \sum_{n=0}^{\infty}\left[(n + 1)b_{n+1}(a - bm_a^n)\left(\frac{x}{1 - H_a}\right) + ikb_n(c - dm_a^n)\left(\frac{x}{1 - H_a}\right)^n\right]$$

$$= \sum_{n=0}^{\infty} T_n x^n, \qquad (55)$$

where

$$T_n = \frac{1}{(1 - H_a)^n}\left[(n + 1)b_{n+1}(a - bm_a^n) + ikb_n(c - dm_a^n)\right]. \qquad (56)$$

With formula (55) in hand, we are in a position to derive formulae fow aerodynamic derivatives. In doing so, the base flow effects are neglected and we give only formulae for the case of zero incidence.

For zero incidence, $\theta = \psi$, and the perturbation pressure p_ℓ at the lower surface is equal in magnitude and opposite in sign to that at the corresponding point of the upper surface p_u.

The general expression for the pitching moment M about a pivot O_1 (Fig.2) acting on the wedge of unit span is

$$\begin{aligned} M &= \int_0^{1/\cos\psi} (\bar{p}_\ell - \bar{p}_u)\cos\psi\,(z-h)dx + \int_0^{1/\cos\psi} (\bar{p}_\ell - \bar{p}_u)\sin\psi\cdot z\tan\psi\,dx \\ &= \int_0^{1/\cos\theta} (\bar{p}_\ell - \bar{p}_u)\cos\theta\,(x\cos\theta - h)dx + \int_0^{1/\cos\theta} (\bar{p} - \bar{p}_u)x\sin^2\theta\,dx \\ &= \int_0^{1/\cos\theta} (\bar{p}_\ell - \bar{p}_u)(x - h\cos\theta)dx \end{aligned} \tag{57}$$

from which, the non-dimensional pitching moment coefficient c_m can be obtained as

$$c_m = \frac{M}{\frac{1}{2}\rho_\infty U_\infty^2} = \int_0^{1/\cos\theta} (C_{p_\ell} - C_{p_u})(x - h\cos\theta)dx, \tag{58}$$

where

$$C_p = \frac{\bar{p} - p_o}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{\epsilon p}{\frac{1}{2}\rho_\infty U_\infty^2} \tag{59}$$

Now for the case of zero incidence,

$$C_{p_\ell} = -C_{p_u} = \frac{\epsilon e^{ikt}}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2 e^{-ikx}\sum_{n=0}^{\infty} T_n x^n , \tag{60}$$

and hence from (58),

$$c_m = \frac{2\epsilon e^{ikt}}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2 \sum_{n=0}^{\infty} T_n \int_0^{1/\cos\theta} (x^{n+1} - x^n h\cos\theta)e^{-ikx}dx .$$

Upon integration this reduces to

$$\begin{aligned} c_m &= \frac{2\epsilon e^{ikt}}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2 \frac{e^{-ik/\cos\theta}}{ik}\sum_{n=0}^{\infty} T_n(S_{n+1} - h\cos\theta\, S_n) \\ &= \epsilon e^{ikt}(-m_\theta - ikm_{\dot\theta}), \end{aligned} \tag{61}$$

where

$$S_n = -\frac{n!}{(ik)^n} e^{ik/\cos\theta} + \sum_{\ell=0}^{n} \frac{n(n-1)(n-2)\ldots(n-\ell+1)}{(ik)^{\ell}\cos^{n-\ell}\theta}, \tag{62}$$

$$S_n = -\frac{ik}{\cos^{n+1}\theta}\left[\frac{1}{n+1} + \frac{ik}{(n+1)(n+2)\cos\theta} + \frac{(ik)^2}{(n+1)(n+2)(n+3)\cos^2\theta} + \ldots\right]. \tag{63}$$

Expression (61) is the most general formula for the two aerodynamic derivatives $-m_\theta$ and $-m_{\dot\theta}$. The derivative $-m_\theta$, called here the in-phase component of pitching moment, is composed of terms proportional to the steady-state position and higher order even derivatives of the motion. (i.e. the zeroth derivative, the second, the fourth etc.). Thus in $-m_\theta$, the first term $-c_{m_\theta}$, the stiffness derivative in the ordinary sense, is frequency independent and can be calculated from steady flow theory (see Section 4), the other terms are proportional to k^2, k^4, k^6, ect. The derivative $-m_{\dot\theta}$, called here the out-of-phase component of pitching moment, consists of terms proportional to the odd derivatives of the motion. (i.e. the first derivative, the third, etc.). Thus in $-m_{\dot\theta}$, the first term $-c_{m_{\dot\theta}}$, called here the damping-in-pitch derivative*, is frequency independent, whereas the other terms are proportional to k^2, k^4, k^6 etc. The in-phase component $-m_\theta$ and the out-of-phase component $-m_{\dot\theta}$ of pitching moment are proportional to k^2M_3 and kM_4, respectively, in American notation.

In Figs 4 to 7 are plotted curves for both in-phase and out-of phase moment derivatives versus pivot position for various values of free stream Mach numbers and flow deflection angles. Generally both in-phase and

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* Throughout the thesis the damping derivative so defined in smaller by a factor of U_∞/u_o than the damping derivative in the ordinary sense, and this is due to the definition of k.

out-of-phase moment derivatives increase in absolute value with increasing wedge angle θ for a given value of M_∞. However, under certain conditions, negative value of the out-of-phase derivative are obtained, (Fig.12) and this will be discussed in detail in Section 3.2.5.

For small frequency parameters k, of interest for stability analysis, we can obtain some very useful conclusions by analysing the structure of formula (61).

It is evident from equations (53) that T_n is of order k^n. On the other hand, (63) shows that S_n is of order k for any integer (including zero) n. It is therefore concluded that the series in (61) is arranged essentially in the order of ascending powers in k, and hence for unsteady theory of any order (i.e. accurate up to terms proportional to any power in k), closed form formulae for the aerodynamic derivatives can be obtained by truncating the series in (61). Indeed, in order to be accurate up to terms proportional to k^n we need only pick up the first (n + 1) terms in (61) and neglect the terms of higher order than k^n in these first (n + 1) terms. Closed form formulae for the stability derivatives (the stiffness derivative and the damping derivative) will be given later in Section 3.2.3.

3.2.2 WAVE INTERACTION AND REFLECTION

A physical interpretation to the results obtained in Section 3.2.1, is given by rearranging the expression (55) for the pressure distribution over the wedge surface as follows

$$\frac{p(x, o, t)}{\rho_o u_o a_o e^{ik(t-x)}} = \frac{1}{2}\sum_{n=o}^{\infty} \left[(n + 1)b_{n+1}(a - bm_a^n) + ikb_n(c - dm_a^n) \right] \left(\frac{x}{1 - H_a}\right)^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{2}\left[(n + 1)b_{n+1}(a + bm_a^n) + ikb_n(c + dm_a^n) \right]\left(\frac{x}{1 + H_a}\right)^n$$

$$- \sum_{n=0}^{\infty} m_a^n \left[(n + 1)b_{n+1} \cdot b + ikb_n \cdot d \right]\left(\frac{x}{1 - H_a}\right)^n$$

$$= - \sum_{n=o}^{\infty} (n + 1 - ikh \cos\psi) \frac{(ikx)^n}{n!}$$

$$- \sum_{n=o}^{\infty} \Big[(n + 1) b_{n+1} \cdot b + ikb_n \cdot d \Big] \left(\frac{x}{1 + H_a}\right)^n$$

$$\frac{p(x, o, t)}{\rho_o u_o a_o e^{ik(t-x)}} = \Delta_1'(x) + \sum_{n=o}^{\infty} \frac{1}{(1 + H_a)^n} \Big[\frac{b}{a+bm_a^n} \cdot \frac{2(n+1-ikh \cos\psi)(ik(1-H_a)x)^n}{n!}$$

$$- ikb_n \left(\frac{ad - bc}{a + bm_a^n}\right) x^n \Big]$$

$$= \Delta_1'(x) + 2b \sum_{n=o}^{\infty} \frac{(n+1-ikh \cos\psi)}{a + bm_a^n} \frac{(ikm_a x)^n}{n!}$$

$$- ik \sum_{n=o}^{\infty} \frac{ad - bc}{a + bm_a^n} b_n \left(\frac{x}{1 + H_a}\right)^n$$

$$= \Delta_1'(x) - 2\lambda \sum_{n=o}^{\infty} \sum_{\ell=o}^{\infty} (\lambda m_a^n)^{\ell} \frac{(n + 1 - ikh \cos\psi)(ikm_a x)^n}{n!}$$

$$- ik (\lambda_a c + d) \sum_{\ell=o}^{\infty} \lambda^{\ell} \sum_{n=o}^{\infty} b_n \left(\frac{m_a^{\ell} x}{1 + H_a}\right)^n$$

$$= \Delta_1'(x) - 2 \sum_{\ell=1}^{\infty} \lambda_a^{\ell} \sum_{n=o}^{\infty} \frac{(n + 1 - ikh \cos\psi)(ikm_a^{\ell} x)^n}{n!}$$

$$- ik(\lambda_a c + d) \sum_{\ell=o}^{\infty} \lambda^{\ell} F_5\left(\frac{m_a^{\ell} x}{1 + H_a}\right)$$

$$\equiv \Big[\Delta_1'(x) + 2 \sum_{\ell=1}^{\infty} \lambda_a^{\ell} \Delta_1'(m_a^{\ell} x) \Big]$$

$$- ik(\lambda_a c + d) \Big[F_5\left(\frac{x}{1 + H_a}\right) + \sum_{\ell=1}^{\infty} \lambda_a^{\ell} F_5\left(\frac{m_a^{\ell} x}{1 + H_a}\right) \Big] . \qquad (64)$$

Formula (64) for the pressure distribution at the wedge surface is a convenient form for discussing waves and waves reflections.

We conclude from formula (64) that there are two sets of waves which contribute to the pressure distribution at the wedge surface. The first set of waves is due to the disturbance $\Delta_1'(x)$ at the wedge surface, while the second is due to the motion of the bow shock $ikF_5(\frac{x}{1+H_a})$. Both sets of waves have their secondary as well as primary contributions to the pressure distribution at the wedge surface.

From the definition (47) it is obvious that λ_a is independent of the motion of the the wedge, so its physical significance in unsteady flow is exactly the same as that in steady flow which has been discussed in detail by Chernyi[2]. In our case, by putting $k = o$, and combining equations (38) and (41), we obtained

$$\begin{aligned} F_2(\ (1+H_a)x\ ) + \lambda_a F_1(\ (1-H_a)x\ ) &= \tfrac{1}{2}\Big[\ N(1+\lambda_a) - L(1-\lambda_a)\ \Big] \\ &= \tfrac{1}{2}\Big[\ \bar{N}_2(1+\lambda_a) - \bar{L}\ (1-\lambda_a)\ \Big] F_5'(x) \\ &= 0, \end{aligned}$$

therefore

$$\lambda_a = \frac{-F_2(\ (1+H_a)x\ )}{F_1(\ (1-H_a)x\ )}. \tag{65}$$

From (65) and noting the second expression of (33) for p, it follows that in both steady and unsteady flows if we take as a measure of the flow disturbance the corresponding change in pressure, the quantity λ_a will represent the reflection coefficient of a disturbance from the bow shock. This reflection coefficient is the ratio of the amplitude of the disturbance reflected from the bow shock (along the characteristic $x + M_o y = \text{const.}$) to the amplitude of the disturbance incident on the bow shock (along the characteristic $x - M_o y = \text{const.}$)

The first group of terms in the formula (64) represents the first set of waves. It is obvious that with the bow shock present, at time t the pressure acting by the fluid on the wedge surface at a point P with abscissa x (Fig.3) depends not only upon the instantaneous inclination angle (represented by $e^{ik(t - x)}\Delta_1'(x)$) of the surface at P itself, but also upon the instantaneous inclination angle at the same moment of time t of the surface at the points with abscissas $m_a x$, $m_a^2 x$, etc. One can easily show that the ℓth term of the first series in (64) corresponds to a disturbance at the wedge surface which has struck the point P after ℓ multiple reflections from the bow shock surface with a reflection coefficient λ_a (and, of course, also after $(\ell - 1)$ multiple reflections from the wedge surface with a reflection coefficient of unity in accordance with the present linear wave theory).

The second group of terms in the formula (64) represents the second set of waves. It is easy to see that at time t the pressure acting on the wedge surface at the point P (Fig.3) depends also upon the instantaneous distrubance of the bow shock (represented by $ike^{ik(t - x)}F_5(\frac{x}{1 + H_a})$), and upon the instantaneous disturbance at the same moment of time t of the bow shock at the points with abscissas $m_a x/(1 + H_a)$, $m_a^2 x/(1 + H_a)$, etc. In a similar way one can also show that the ℓth term in the second series in (64) corresponds to a disturbance at the bow shock which has struck the point P after ℓ multiple reflections on the bow shock surface with the same reflection coefficient λ_a (and also after ℓ multiple reflections from the wedge surface).

In steady flow, $k = 0$, the second set of waves disappears. For the case of hypersonic flow past a slender wedge, it will be shown in Section 3.2.6 that the quantities μ and ν in the formula (64) are negligible compared with the reflection coefficient λ_a, and hence the second set of waves is negligible. Therefore the second set of waves result from the motion of a thick wedge.

Lighthill[15], Chu[16], Chernyi[2], and McIntosh[14] among others, studied interactions of disturbances with a bow shock and obtined equations which are a special case of equation (46) of the present theory when the second group of terms in the left hand side of this equation is dropped.

They obtained one set of waves (the first set) only. This is so because the cases they considered are equivalent either to a steady flow past a thick wedge (in this case, $k = 0$) or to an unsteady hypersonic flow past a slender wedge (in this case, μ and ν are negligible).

The motion of the bow shock (in our present case) is due to the disturbance at the wedge surface, so it might be thought that the second set of waves has only secondary effects on the pressure distribution at the wedge surface. However, numerical results (Figs 8 and 9) show that it is as important as the first set of waves for hypersonic flow past a thick wedge, and it becomes dominant at the critical situation. It is due to this set of waves that negative damping appears for hypersonic flow flow past a wedge under certain conditions.

It is obvious from equation (64) that for small values of the frequency parameter k, the second set of waves affects mainly the out-of-phase moment derivatives and has little contributuion to the in-phase moment derivatives, and particularly, within the first order unsteady theory it only affects the damping derivatives and not the stiffness derivatives, because the cofficient of the second group of terms in (64) includes a factor ik.

Curves of the reflection coefficient λ_a versus flow deflection angle θ for several values of free stream Mach number M_∞ are plotted in Fig.10. Also plotted there, for comparison, are curves of λ obtained in ChapterIII (or from Ref.2) which are exact within the small perturbation theory and which could be obtained from the first of equations (47), the definition of λ_a, if for the quantity H_a in the parameters a and b is substituded

$(M_o^2 - 1)^{\frac{1}{2}}\tan\varphi$ instead of $M_o \tan\varphi$. It is interesting to notice that the only approximation made in the present theory is to replace the coefficient $(M_o^2 - 1)$ of the fourth term in equation (6) by M_o^2, so putting $H_a = (M_o^2 - 1)\cdot \tan\varphi$ is, to some extent, a correction to the approximation made. But throughout this Chapter, in calculating λ_a, $M_o \tan\varphi$ is used for H_a. The two families of curves are seen from Fig.10 to differ only quantitatively but not qualitatively.

As seen from the figure, the reflection coefficient λ_a can be positive as well as negative, i.e. in some cases the reflection takes place without the disturbance changing sign, while in other cases the sign of the disturbance is changed due to the reflection from the bow shock. Chernyi[2] proved that $|\lambda| < 1$. This is to be expected physically, as it is the ratio of the amplitude of the disturbance reflected from the bow shock to the amplitude of the disturbance incident on the bow shock. With the approximation made in this theory, as proved above, λ_a retains the same physical meaning, therefore it should be expected physically that the same result that $|\lambda_a| < 1$ will still hold. Mathematical proof of this is given in the Appendix A.

Again, from an examination of the curves in Fig.10, it follows that for small values of the free stream Mach number M_∞, the quantity $|\lambda_a|$ is small, but it increases with M_∞ and the flow deflection angle θ, and becomes appreciable for hypersonic flow with sufficiently large flow deflection angles which are smaller than the shock detachment angle θ_{max}. Therefore, the reflected wave effects of both sets of waves which depend mainly upon the value of λ_a become very important for hypersonic flow past a wedge which is not too thin, and can never be neflected. Figs 13 and 14 show how important these reflected wave effects are.

It can be seen that the reflection coefficient λ_a also increases in absolute value for decreasing γ and finally tends to -1 when γ tends to 1, i.e. the disturbances when reflected from the bow shock change their sign but retain their magnitude. (With regard to Newtonian limit, see Section 3.2.5)

3.2.3 STABILITY DERIVATIVES

It has been pointed out in Section 3.2.1 that for unsteady theory of any order, closed form formulae for aerodynamic derivatives can be obtained from the general formula (61) by, trucnating its terms. In this section we derive the unsteady theory of the first order i.e. accurate up to terms proportional to the frequency parameter k. To do so, we need the first two terms in (61) only.

From equations (56), (53) and (54), we have

$$T_o = -\frac{2(a-b)(1-ikh\cos\theta)}{a+b} + 2ik\frac{(bc-ad)}{a+b}h\cos\theta ,$$

$$\begin{aligned} T_1 &= \frac{1}{1-H_a}\left[2b_2(a-bm_a) + ikb_1(c-bm_a)\right] \\ &= -4ik\left[\frac{a-bm_a}{a+bm_a} + \frac{bc-ad}{(1+H_a)(a+b)(a+bm_a)}\right] + O(k^2) \end{aligned} \tag{66}$$

while from (63), we get

$$\begin{aligned} S_o &= -\frac{ik}{\cos\theta}\left[1 + \frac{ik}{2\cos\theta} + O(k^2)\right], \\ S_1 &= -\frac{ik}{\cos^2\theta}\left[\tfrac{1}{2} + \frac{ik}{6\cos\theta} + O(k^2)\right], \\ S_2 &= -\frac{ik}{\cos^3\theta}\left[\tfrac{1}{3} + \frac{ik}{12\cos\theta} + O(k^2)\right]. \end{aligned} \tag{67}$$

Putting expressions (67) and (66) into the general formua (61) and neglecting terms of order k^2 or higher, we obtain the closed form formulae for stiffness derivative $-c_{m_\theta}$ and damping derivative $-c_{m_{\dot\theta}}$ as follows

$$-c_{m_\theta} = \frac{4}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2\frac{1}{\cos^2\theta}E_a\left(\tfrac{1}{2} - h\cos^2\theta\right), \tag{68}$$

$$-c_{m_{\dot\theta}} = \frac{4}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2\frac{1}{\cos^3\theta}\left[I_a(h\cos^2\theta)^2 - G_a h\cos^2\theta + \tfrac{1}{3}(2G_a - I_a)\right], \tag{69}$$

where

$$E_a = (1 + \lambda_a)/(1 - \lambda_a), \qquad I_a = E_a - (\lambda_a c + d)/(1 - \lambda_a),$$
$$G_a = \frac{1 + \lambda_a m_a}{1 - \lambda_a m_a} - \frac{\lambda_a \mu + \nu}{(1 + H_a)(1 - \lambda_a)(1 - \lambda_a m_a)} - \frac{1}{2}\frac{\lambda_a c + d}{1 - \lambda_a}. \tag{70}$$

It will be proved in Section 4 that the formula (68) is an exact formula for stiffness derivative in the most general case, despite the approximation made in Section 2. For several combinations of free stream Mach number M_∞ from 1.5 up to 30, and for flow deflection angles θ from zero up to the corresponding detachment angles, the differences between the in-phase derivative $-m_\theta$ and the stiffness derivative $-c_{m_\theta}$, and the out-of-phase derivative $-m_{\dot\theta}$ and the damping derivative $-c_{m_{\dot\theta}}$ are negligible. For k = 0.01 the difference is within 0.01%, whereas for the case k = 0.1 it does not exceed 0.1%, even if k = 1.0, which is most unlikely in any practical situation, its maximum value is about 3%. In table 1, are shown some typical examples.

As seen from Sections 3.2.1 and 3.2.2, the structure of the general formulae (61) and (64) for aerodynamic derivatives is a double series, which may be arranged either (essentially) in the order of ascending powers in the frequency parameter k, or in the order of ascending powers in the reflection coefficient λ_a. In the former case, each term of the series contains all the terms proportional to each power in λ_a, while in the latter case, each term of the series contains all the terms proportional to each power in k. Now it has been shown in Figs.13 to 16 and Table 1 that the reflected wave effects are very important, and the terms proportional to k^2 or higher are negligible. Therefore formulae (68) and (69) are the most useful ones for practical calculations, taking into account the fact that they are in closed form and the condition that $k \ll 1$ is always satisfied for stability analysis. After all, it is important to know that the first order theory includes all the reflected wave effects (of course, up to order of k). This can be confirmed by rewriting (70) as

$$E_a = 1 + 2\sum_{\ell=1}^{\infty} \lambda_a^{\ell} , \qquad I_a = \left[1 + 2\sum_{\ell=1}^{\infty} \lambda_a^{\ell} \right] - (\lambda_a c + d)\left[1 + \sum_{\ell=1}^{\infty} \lambda_a^{\ell} \right], \tag{71}$$

$$G_a = \left[1 + 2\sum_{\ell=1}^{\infty} (\lambda_a m_a)^{\ell} \right] - (\lambda_a c + d)\left[\left\{ \frac{1}{(a+b)(1+H_a)} + \sum_{\ell=1}^{\infty} \frac{1}{a+b}\cdot\frac{(\lambda_a m_a)^{\ell}}{1+H_a} \right\} + \frac{1}{2}\left(1 + \sum_{\ell=1}^{\infty} \lambda_a^{\ell} \right) \right].$$

and noticing that within the unsteady theory of the first order

$$\begin{aligned} \Delta'_1(x) &= 1 + 2ikx - ikh\cos\theta \\ F_5(x) &= \frac{2x}{a+b} - h\cos\theta \end{aligned} \tag{72}$$

Thus from (64) and (71), we conclude that the quantity E_a takes into account all the reflected wave effects of the first set of waves, while I_a and G_a take into account all the reflected wave effects of both the first and the second sets of waves.

3.2.4 SUPERSONIC FLOW, AND HYPERSONIC FLOW PAST THICK WEDGES

For hypersonic flow past a slender wedge, the local Mach number M_o is very large and the present theory is justified by itself. Besides, as shown in Section 2, the present theory can also be applied to cases of hypersonic flow past thick wedges (e.g. $M_\infty = 17$, $\theta = 30^\circ$) or of relatively high supersonic flow past a slender wedge.

For supersonic flow, Figs.15 to 17 give comparisons for the stability derivatives of a wedge between the present theory and the experimental results by Pugh and Woodgate[4], and also the theory of Van Dyke[19] which is a potential theory including the non-linear effect of thickness up to the second order, and also the exact theory in Chapter III. Good agreements are obtained. On the other hand, for the case of hypersonic flow past a thick wedge there exist, to the author's knowledge, no experimental results or other theories for comparison with the present theory. But it is expected that for a given value of M_o the present theory should work for hypersonic flow past thick wedges as well as it does for supersonic flow. This is confirmed by the exact theory in Chapter III which shows that the present approximate theory gives excellent results for $M_o > 2.0$. However, there is still room for improvement when M_o is less than 2.0.

3.2.5 GENERAL CRITERION FOR STABILITY

One of the several important results obtained in this theory is the general criterion for neutral damping.

It is well-known that for supersonic flow, under certain circumstances negative damping is obtained. Neutral damping boundaries can be obtained from theories in Ref.3, 19, 11 and 12.

A general criterion for neutral damping of a wedge in hypersonic or supersonic flows can be obtained from formula (69). Thus we have, by setting $-c_{m_{\dot\theta}}$ equal zero,

$$I_a(h\ \cos^2\theta)^2 - G_a(h\ \cos^2\theta) + \tfrac{1}{3}(2G_a - I_a) = 0, \qquad (73)$$

which is a quadratic equation for $h\ \cos^2\theta$. The discriminant of (73) is

$$\Delta^* = G_a^2 - 4I_a(2G_a - I_a)/3. \qquad (74)$$

If $\Delta^* < o$, the damping derivative is always positive for all values of h, i.e. the wedge is stablized by the gas flow as it oscillates about all pivot positions. If $\Delta^* > 0$, the damping derivative is positive for some values of h and negative for other values of h, i.e. the wedge is stablized by the gas flow when it oscillates about some pivot positions, and destablized for others. Setting Δ^* equal zero we obtain, either

$$2I_a = 3G_a, \qquad (75)$$

or

$$2I_a = G_a. \qquad (76)$$

For a fixed value of M_∞, when θ is increased, equation (75) is first satisfied and hence it is the condition wanted.

If the values of M_∞ and θ are such that condition (75) is satisfied, the damping derivative first becomes zero at

$$h \cos^2\theta = \frac{G_a}{2I_a},$$

from which the critical pivot position h_{cr}, at which the damping derivative first becomes zero, is given by

$$h_{cr} = \frac{1}{3} + \frac{1}{3}\tan^2\theta_{cr}. \qquad (77)$$

Thus the critical pivot position h_{cr} runs from one-third of the chord length for a very thin wedge to about two-thirds of the chord length for hypersonic flow past a very thick wedge. This conclusion agrees very well with the figures given in Ref.3, 4 and 19, and is proved in Chapter III to be exact disregarding the approximation made in this Chapter.

The quantities I_a and G_a defined in (70) are explicit functions of the free stream Mach number M_∞ and the bow shock angle β in the reference steady flow. Therefore equation (75) gives a relation between M_∞ and β, and is solved for β with given M_∞ by the method of iteration. The result is plotted in Fig.18 as a curve of M_∞ versus θ which is a locus of the highest points of all the neutral boundaries (i.e., curves of M_∞ vs h for constant values of θ). It is shown by the exact theory in Chapter III that for a given Mach number M_∞ the critical angle θ_{cr} predicted by the approximate theory is about 2° larger then its exact value.

The curves of Fig. 18 have a number of interesting features. Firstly, from an examination of them it follows that for a given value of M_∞ there is always a critical value θ_{cr} of the flow deflection angle at which the dmaping derivative first falls down to zero for the pivot position at

$h_{cr} = \frac{1}{3} + \frac{1}{3}\tan^2\theta_{cr}$. If $\theta < \theta_{cr}$, the body is stablized by the flow passing it, if $\theta > \theta_{cr}$, the wedge is stronly destablized by the flow passing it ~~is~~ (Figs. 19, 11, 12) for pivot position within a zone which is determind by the difference $\theta - \theta_{cr}$ and the free stream Mach number M_∞. Usually for hypersonic flow when $\theta - \theta_{cr}$ equals 1 degree or so this zone of destablization of pivot position covers nearly the whole chord of the wedge (Fig.19).

For a given flow deflection angle θ, there is always a critical value of free stream Mach number $M_{\infty cr}$, at which the damping derivative first falls to zero for the pivot position at $h_{cr} = \frac{1}{3} + \frac{1}{3}\tan^2\theta$. If $M_\infty > M_{\infty cr}$, the wedge is stablized by the gas passing it, if $M_\infty < M_{\infty cr}$, the wedge is strongly destablized by the gas flow passing it for the pivot position within a zone which is determined by the angle θ and the difference $M_{\infty cr} - M_\infty$. Thus increasing the free stream Mach number increases the stability of the body, other factors being constant, and vice versa.

It is important to notice that the critical value of the flow deflection angle θ_{cr} for a fixed value of M_∞ is a few degrees less than the corresponding detachment angle for the same value of M_∞. Therefore, the wedge is strongly destablized by the flow passing it before shock detachment occurs.

It is also noticed that for hypersonic flow, θ_{cr} is about 41° or so, hence the critical pivot position h_{cr} is about 0.6, near the centre of gravity of a free wedge with uniform mass distributuion. The critical value M_{ocr} of the Mach number behind the shock is about 1.55 or so, thus the flow behind the bow shock is still a purely supersonic flow. In contrast to this, for supersonic flow, θ_{cr} is very small and h_{cr} is about one-third, and for most cases for zero damping to be obtained at a pivot position h = 2/3, the free stream Mach number must be as low as about 1.2[3, 19] and the flow over the wedge surface is therefore not a purely supersonic one.

As seen from Figs.8 and 9, the effect of the second set of waves is to reduce the damping derivative, and near the critical situation it becomes dominant. It is due to this effect that the damping derivative becomes zero and negative.

Decreasing γ increases the values of both θ_{max} and θ_{cr}, but the main picture remains about the same that, for a fixed value of M_∞, the wedge will be stronly destablized by the flow before θ reaches θ_{max}.

In the double limit $M_\infty \longrightarrow \infty$ and $\gamma \longrightarrow 1$, the wedge is always stablized by the flow passing it, and therefore the pressent theory in its limiting case, agrees qualitatively with the Newtonican theory.

3.2.6 HYPERSONIC SLENDER WEDGE THEORY

For hypersonic flow past a slender wedge, the shock angle β, the flow deflection angle θ in the reference steady flow, and their difference φ are all small so that when their quadratic and higher terms are neglected, we obtain,

$$\sin\varphi = \varphi, \qquad \cos\beta = 1, \qquad \text{etc.,}$$

hence,

$$H_a^2 \doteq M_o^2 \tan^2\varphi = \frac{2 + (\gamma - 1)K^2}{2\gamma K^2 - (\gamma - 1)}, \tag{78}$$

where $K = M_\infty\beta$ is the hypersonic similarity parameter based on the shock angle β in the reference steady flow.

Furthermore, we have,

$$\begin{aligned} u_o &= U_\infty, \\ \frac{\rho_\infty}{\rho_o} &= \frac{2 + (\gamma - 1)K^2}{(\gamma + 1)K^2}, \\ M_o &= \frac{(\gamma + 1)K^2}{\beta\left\{\left[2 + (\gamma - 1)K^2\right]\left[2 K^2 - (\quad - 1)\right]\right\}^{\frac{1}{2}}} \end{aligned} \tag{79}$$

and therefore, from (42),

$$\bar{N}_2 = \frac{2(K^2+1)}{(\gamma+1)K^2}, \qquad \bar{L}_2 = \frac{4H_a}{\gamma+1}, \tag{80}$$
$$\bar{N}_1 = 0, \qquad \bar{L}_1 = 0,$$

$$\lambda_a = \frac{K^2(2H_a-1)-1}{K^2(2H_a+1)+1},$$
$$\mu = 0, \tag{81}$$
$$\nu = 0,$$

Therefore the second set of waves due to the motion of the bow shock disappears and we are left with, for the pressure distribution over the wedge surface,

$$\frac{p(x,o,t)}{\rho_o u_o a_o e^{ik(t-x)}} = \Delta_1'(x) + 2\sum_{\ell=1}^{\infty} \lambda_a^{\ell}\, \Delta_1'(m_a^{\ell} x), \tag{82}$$

from which, the in-phase component $-m_\theta$ and the out-of-phase component $-m_{\dot\theta}$ of the pitching moment derivative are obtained as follows

$$-m_\theta = 4\beta\left[\frac{2\gamma K^2-(\gamma-1)}{2+(\gamma-1)K^2}\right]^{\frac{1}{2}}\left[\tfrac{1}{2} - h + 2\sum_{\ell=1}^{\infty}\lambda_a^{\ell}\left\{\frac{(1-h)(m_a^{\ell}-h)\cos k(m_a^{\ell}-1)-h^2}{m_a^{\ell}-1} - \frac{(m_a^{\ell}+1-2h)\sin k(m_a^{\ell}-1)}{k(m_a^{\ell}-1)^2} + \frac{(m_a^{\ell}+1)(1-\cos k(m_a^{\ell}-1))}{k^2(m_a^{\ell}-1)^3}\right\}\right],$$

$$-m_{\dot\theta} = \frac{4\beta}{k}\left[\frac{2\gamma K^2-(\gamma-1)}{2+(\gamma-1)K^2}\right]^{\frac{1}{2}}\left[k(\tfrac{1}{3} - h + h^2) + 2\sum_{\ell=1}^{\infty}\lambda_a^{\ell}\left\{\frac{(1-h)(m_a^{\ell}-h)\sin k(m_a^{\ell}-1)}{(m_a^{\ell}-1)} + \frac{2h+(m_a^{\ell}+1-2h)\cos k(m_a^{\ell}-1)}{k(m_a^{\ell}-1)^2} - \frac{(m_a^{\ell}+1)\sin k(m_a^{\ell}-1)}{k^2(m_a^{\ell}-1)^3}\right\}\right] \tag{83}$$

Formula (82) and the special case of formulae (83) when $h = 0$ are, apart from different symbols used, identical to those obtained in Ref.14. Thus we see, McIntosh's result is a special case of the present general theory when the flow is hypersonic and the wedge is slender. However, in

Ref.14, the formulae for stability derivatives were given only in the form (83) of an infinite series in the order of ascending powers in λ_a, but not in the form equivalent to formula (61), which is arranged essentially in the order of ascending powers in k, and which has been shown to be the most convenient form for practical calculations.

From (68) and (69), we obtain formulae for the stiffness derivative and the damping derivative as follows

$$-c_{m_\theta} = 4\beta\frac{K^2}{K^2+1}(1-2h),$$
$$-c_{m_{\dot\theta}} = 4\beta\left[(\tfrac{2}{3}-h)Q(K,\gamma) - \frac{K^2}{K^2+1}h(1-2h)\right], \qquad (84)$$

where

$$Q(K,\gamma) = \frac{2(\gamma+1)K^6 + 2(2\gamma-1)K^4 + (3-\gamma)K^2 - (\gamma-1)}{(K^2+1)\left[2(2\gamma-1)K^4 + (\gamma+5)K^2 - (\gamma-1)\right]} \qquad (85)$$

Formulae (84) are, apart from different symbols used, identical to those first obtained in Ref.13, and corrected in Ref.22.

In Figs 4 to 7, are plotted results of the hypersonic slender wedge theory compared with that of the general theory. An examination of these curves shows that the hypersonic slender wedge theory under-estimates the effect of thickness, as is to be expected. The range of applicability of the hypersonic slender wedge theory depends on the percentage of error allowed. For the stiffness derivatives at h = 1, its error is 3% for $M_\infty = 17$, $\theta = 9^\circ 9'$, and 24% for $M_\infty = 10$, $\theta = 19^\circ 19'$. Whereas for the damping derivatives at h = 1, its error is 6% for $M_\infty = 17$, $\theta = 9^\circ 9'$, and 28% for $M_\infty = 10$, $\theta = 19^\circ 19'$.

However, the hypersonic slender wedge theory as given by formulae (83) or that in Ref.14 is not convenient for practical calculations, because the reflected wave effects are important for hypersonic flow, and closed form formulae are not possible to obtain. On the other hand, (68) and (69) are the most general formulae in closed form which are more accurate and simple for practical calculations. Thus it is a disadvantage using formulae (83) instead of (68) and (69) even for hypersonic flow past a slender wedge. Formulae (84) are only slightly simpler than their general forms (68) and (69). Therefore, it is sensible to use formulae (68) and (69) for practical calculations as they give more accurate results than those of either formula (83) or (84).

3.3 PLUNGING MOTION OF A WEDGE

For plunging motions of a wedge, we have

$$\frac{\partial\delta}{\partial x} \ll \frac{\partial\delta}{\partial t},$$

which implies that

$$\Delta_1(x) = e^{ikx},$$

and hence

$$\Delta_1'(\,(1 - H_a)x\,) = ik\sum_{n=0}^{\infty}\frac{\left[\,ik(1 - H_a)x\,\right]^n}{n!} \tag{86}$$

Therefore to obtain formulae for aerodynamic derivatives of a wedge in plunging motion, we need only replace the right hand side of equation (45) by (86) instead of (51), and thus we obtain for determining the coefficients b_n in the power series of $F_5(x)$ the following equations

$$\begin{aligned}
b_1(a + b) + ikb_o(c + d) &= 2ik,\\
2b_2(a + bm_a) + ikb_1(c + dm_a) &= 2ik\left[\,ik(1 - H_a)\,\right],\\
3b_3(a + bm_a^2) + ikb_2(c + dm_a^2) &= 2ik\frac{\left[\,ik(1 - H_a)\,\right]^2}{2!},\\
&\cdots\cdots\cdots\cdots \\
(n + 1)b_{n+1}(a + bm_a^n) + ikb_n(c + dm_a^n) &= \frac{2ik\left[\,ik(1 - H_a)\,\right]^n}{n!},
\end{aligned} \tag{87}$$

and so on. Also we have the additional condition

$$b_o = 1, \qquad (88)$$

using the same reasoning as that for a wedge in pitching motion.

The other formulae (55), (56), (61), (63) and (64) are all unchanged in form. Since in both series of $\Delta_1'(x)$ and $F_5(x)$, the first term is proportional to ik, no steady-state part contributes to the in-phase component, and the second set of waves affects the in-phase component more strongly than the out-of-phase component for not too large values of k.

For steady hypersonic and supersonic flows past a wedge-like body, the flow is also considered as a small perturbation to some reference steady flow past a wedge.

This case has been studied in detail in Refs.16 and 2. No approximation is necessary for steady flows, thus the results are exact within the linear perturbation theory. From our scheme, by setting all derivatives with respect to time t in (I.7) equal zero, we obtain, instead of equation(1),

$$(M_o^2 - 1) \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} = 0. \qquad (89)$$

The other perturbation equations may also be simplified.

By solving these perturbation equations to satisfy the boundary conditions (I.10) and (I.22) with all the derivatives $\partial/\partial t$ equal to zero, we can obtain the pressure distribution etc. This has been done by Chu[16] and Chernyi[2]. Their result for pressure distribution over the body surface, written in the present notations, is as follows.

$$\frac{p(x, o)}{\rho_o u_o^2} = \frac{1}{(M_o^2 - 1)^{\frac{1}{2}}} \left[\Delta_1'(x) + 2\sum_{\ell=1}^{\infty} \lambda_a^{\ell} \Delta_1'(m_a^{\ell} x) \right], \qquad (90)$$

where $\Delta_1'(x)$ is now equal to $\Delta'(x)$, the slope of the body surface, m_a and λ_a are calculated, respectively, from formulae (44) and (47) with $H_a = (M_o^2 - 1)^{\frac{1}{2}} \tan\varphi$ instead of $M_o \tan \varphi$.

The stiffness derivative $-c_{m\theta}$, by definition, can be obtained from steady flow theory. For a wedge, it can easily be obtained by completing the integral (58) with $\Delta_1'(x) = 1$ in formula (90). The result is

$$-c_{m\theta} = \frac{4}{(M_o^2 - 1)^{\frac{1}{2}}} \left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2 \frac{1}{\cos^2\theta} E'\left(\tfrac{1}{2} - h \cos^2\theta\right), \qquad (91)$$

where

$$E' = \left[\frac{1+\lambda_a}{1+\lambda_a}\right]_{H_a = (M_o^2 - 1)^{\frac{1}{2}} \tan\varphi} \tag{92}$$

$$= 2(M_o^2 - 1)^{\frac{1}{2}} \tan\varphi \frac{\left[1 - \frac{\gamma - 1}{2} W(\frac{\rho_o}{\rho_\infty} - 1)\right]}{\left[1 + \frac{\rho_o}{\rho_\infty}(M_o^2 - 1)\tan^2\varphi - \gamma W(\frac{\rho_o}{\rho_\infty} - 1)\right]} .$$

Thus the stiffness derivative $-c_{m_\theta}$ of a wedge calculated by the exact steady flow theory, is

$$-c_{m\theta} = \frac{8\tan\varphi}{\cos^2\theta}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2 \frac{\left[1 - \frac{\gamma - 1}{2} W(\frac{\rho_o}{\rho_\infty} - 1)\right](\frac{1}{2} - h\cos^2\theta)}{\left[1 + \frac{\rho_o}{\rho_\infty}(M_o^2 - 1)\tan^2\varphi - \gamma W(\frac{\rho_o}{\rho_\infty} - 1)\right]} . \tag{93}$$

Formula (93) should be recognized to be identical to formula (68) in which the quantity E_a is calculated using $H_a = M_o \tan\varphi$. Summing up, we have derived an exact formula (93) for the stiffness derivative of a wedge which can be applied in the most general cases, i.e. for a thick wedge or a slender wedge in supersonic flow as well as in hypersonic flow, and we have also proved that in the most general cases the present general theory does give an exact formula for the stiffness derivative of a wedge, disregarding the approximation that has been introduced in deriving it. The reason for this is that for the special case of a wedge with small and steady change in its incidence the perturbation quantities are all constants and their second derivatives with regard to x vanish, therefore the approximation that the coefficient $M_o^2 - 1$ of the term $\partial^2 p/\partial x^2$ in equation (I.7) be replaced by M_o^2 does not affect the solution of the problem which is then completely determined by the exact boundary conditions.

CHAPTER III

EXACT THEORY [23, 24]

1 INTRODUCTION

In this Chapter we restrict ourselves to the problem of unsteady hypersonic and supersonic flows past an oscillating wedge. The perturbation equations are solved by expanding the unknown functions as power series of ik, where k is the frequency parameter defined in (II.32). Exact formulae for the stability derivatives of a wedge in inviscid hypersonic and supersonic flows are thus obtained in closed form which can be applied to wedges of any thickness, providing the bow shock is attached to the wedge. Also obtained is an exact criterion for stability. This exact theory will be shown to include both the approximate theory developed in the last Chapter and the theory of Carrier & Van Dyke (for supersonic flow) as special cases. Comparisions with experimental results for supersonic flow are also given.

2 METHOD OF SOLUTION

Because the perturbation equations (I.4) and boundary conditions (I.10) and (I.22) are linear for a harmonic motion of the wedge given by $\delta(x, t) = e^{ikt}\Delta(x)$, the motion of the gas, as observed from a fixed position, must also be harmonic with the same period as the wedge. Therefore the perturbation quantities may be expressed as

$$
\begin{aligned}
u &= u_o e^{ikt} U(x, y) \ , \\
v &= u_o e^{ikt} V(x, y) \ , \\
p &= \rho_o a_o u_o e^{ikt} P(x, y) \ , \\
\rho &= \frac{\rho_o u_o}{a_o} e^{ikt} R(x, y) \ , \\
f_5 &= e^{ikt} Q(x) \ ,
\end{aligned}
\tag{1}
$$

where U, V, P, R, Q are time-independent unknown functions to be determined from the perturbation equations and the boundary conditions.

Putting equation (1) into (I.7) and (I.4), we obtain

$$
\begin{aligned}
(M_o^2 - 1)\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} &= -2ikM_o^2 \frac{\partial P}{\partial x} - (ik)^2 M_o^2 P \ , \\
\frac{\partial P}{\partial y} &= -M_o \frac{\partial V}{\partial x} - ikM_o V \ , \\
\frac{\partial U}{\partial x} &= -\frac{1}{M_o}\frac{\partial P}{\partial x} - ikU \ , \\
ikR + \frac{\partial R}{\partial x} &= ikP + \frac{\partial P}{\partial x} \ , \\
ikP + \frac{\partial P}{\partial x} + \frac{1}{M_o}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) &= 0 \ ,
\end{aligned}
\tag{2}
$$

where the first equation, useful in what follows, is derived from the others.

Similarly, by putting (1) into (I.10) and (I.22) we obtain the following boundary conditions:

$$\text{at the body, } y = 0, \qquad V = \Delta'(x) + ik\Delta(x) \,,$$

$$\text{at the shock, } y = x\tan\varphi, \qquad \begin{aligned} V &= \bar{A}Q'(x) + ik\bar{B}Q(x) \,, \\ P &= \bar{C}Q'(x) + ik\bar{D}Q(x) \,, \\ U &= \bar{E}Q'(x) + ik\bar{F}Q(x) \,, \\ R &= \bar{G}Q'(x) + ik\bar{J}Q(x) \,, \end{aligned} \tag{3}$$

where the constants $\bar{A}$ through $\bar{J}$ are dependent only on the reference steady flow and can be found to be

$$\begin{aligned}
\bar{A} &= \frac{\cot\varphi}{1-W}\Big[B(1+\gamma W)\sin\varphi\cos\varphi + C(W - \gamma W\cos^2\varphi - \sin^2\varphi) \Big] \,, \\
\bar{B} &= \frac{\cot\varphi}{1-W}\Big[A(1+\gamma W)\sin\varphi\cos\varphi - C(\gamma-1)W \Big] \,, \\
\bar{C} &= \frac{H_a\cot\varphi}{1-W}\Big[B(2+\gamma W - W)\sin\varphi\cos\varphi - C(\gamma-1)W\cos^2\varphi \Big] \,, \\
\bar{D} &= \frac{H_a\cot\varphi}{1-W}\Big[A(2+\gamma W - W)\sin\varphi\cos\varphi - C(\gamma-1)W \Big] \,, \\
\bar{E} &= \frac{-1}{1-W}\Big[B(1+\gamma W)\sin\varphi\cos\varphi + C(1-\gamma W)\cos^2\varphi \Big] \,, \\
\bar{F} &= \frac{-1}{1-W}\Big[A(1+\gamma W)\sin\varphi\cos\varphi - C(\gamma-1)W \Big] \,, \\
\bar{G} &= \frac{\cot\varphi}{M_o(1-W)}\Big[B(\gamma+1)W - C(\gamma-1)W\cot\varphi \Big] \,, \\
\bar{J} &= \frac{\cot\varphi}{M_o(1-W)}\Big[A(\gamma+1)W - \frac{C(\gamma-1)W}{\sin\varphi\cos\varphi} \Big] \,,
\end{aligned} \tag{4}$$

For most engineering problems the values of the frequency parameter k defined in (I.32) are small, of order of magnitude 0.01. Now, expanding the five unknown functions and the "amplitude function" in power series in ik,

$$
\begin{aligned}
V &= V^{(o)} + ikV^{(1)} + (ik)^2V^{(2)} + \dots , \\
P &= P^{(o)} + ikP^{(1)} + (ik)^2P^{(2)} + \dots , \\
U &= U^{(o)} + ikU^{(1)} + (ik)^2P^{(2)} + \dots , \\
R &= R^{(o)} + ikR^{(1)} + (ik)^2R^{(2)} + \dots , \\
Q &= Q^{(o)} + ikQ^{(1)} + (ik)^2Q^{(2)} + \dots , \\
\Delta &= \Delta^{(o)} + ik\Delta^{(1)} + (ik)^2\Delta^{(2)} + \dots ,
\end{aligned} \tag{5}
$$

Putting (5) into equations (2) and equating the terms of the same order of ik on both of each equation, we obtain

$$
\begin{aligned}
&(M_o^2 - 1)\frac{\partial^2 P^{(o)}}{\partial x^2} - \frac{\partial^2 P^{(o)}}{\partial y^2} = 0 , \\
&\frac{\partial P^{(o)}}{\partial y} = - M_o \frac{\partial V^{(o)}}{\partial x} , \\
&\frac{\partial U^{(o)}}{\partial x} = - \frac{1}{M_o} \frac{\partial P^{(o)}}{\partial x} , \\
&\frac{\partial R^{(o)}}{\partial x} = \frac{\partial P^{(o)}}{\partial x} , \\
&\frac{\partial P^{(o)}}{\partial x} + \frac{1}{M_o}\left(\frac{\partial U^{(o)}}{\partial x} + \frac{\partial V^{(o)}}{\partial y}\right) = 0 ;
\end{aligned} \tag{6_o}
$$

$$
\begin{aligned}
&(M_o^2 - 1)\frac{\partial^2 P^{(1)}}{\partial x^2} - \frac{\partial^2 P^{(1)}}{\partial y^2} = - 2M_o^2 \frac{\partial P^{(o)}}{\partial x} , \\
&\frac{\partial P^{(1)}}{\partial y} = - M_o \frac{\partial V^{(1)}}{\partial x} - M_o V^{(o)} , \\
&\frac{\partial U^{(1)}}{\partial x} = - \frac{1}{M_o} \frac{\partial P^{(1)}}{\partial x} - U^{(o)} , \\
&R^{(o)} + \frac{\partial R^{(1)}}{\partial x} = P^{(o)} + \frac{\partial P^{(1)}}{\partial x} , \\
&P^{(o)} + \frac{\partial P^{(1)}}{\partial x} + \frac{1}{M_o}\left(\frac{\partial U^{(1)}}{\partial x} + \frac{\partial V^{(1)}}{\partial y}\right) = 0 ;
\end{aligned} \tag{6_1}
$$

and for $n \geq 2$:

$$(M_o^2 - 1)\frac{\partial^2 P^{(n)}}{\partial x^2} - \frac{\partial^2 P^{(n)}}{\partial y^2} = -2M_o^2 \frac{\partial^2 P^{(n-1)}}{\partial x} - M_o^2 P^{(n-2)} ,$$

$$\frac{\partial P^{(n)}}{\partial y} = -M_o \frac{\partial V^{(n)}}{\partial x} - M_o V^{(n-1)} ,$$

$$\frac{\partial U^{(n)}}{\partial x} = -\frac{1}{M_o}\frac{\partial P^{(n)}}{\partial x} - U^{(n-1)} , \qquad (6_n)$$

$$R^{(n-1)} + \frac{\partial R^{(n)}}{\partial x} = P^{(n-1)} + \frac{\partial P^{(n)}}{\partial x} ,$$

$$P^{(n-1)} + \frac{\partial P^{(n)}}{\partial x} + \frac{1}{M_o}\left(\frac{\partial U^{(n)}}{\partial x} + \frac{\partial V^{(n)}}{\partial y}\right) = 0 .$$

Similarly, we obtain for the boundary conditions,

$$y = 0 , \qquad V^{(o)} = \Delta^{(o)\prime}(x) ,$$

$$y = x \tan\varphi , \qquad V^{(o)} = \bar{A}Q^{(o)\prime}(x) ,$$

$$P^{(o)} = \bar{C}Q^{(o)\prime}(x) , \qquad (7_o)$$

$$U^{(o)} = \bar{E}Q^{(o)\prime}(x) ,$$

$$R^{(o)} = \bar{G}Q^{(o)\prime}(x) ,$$

$$y = 0 , \qquad V^{(1)} = \Delta^{(1)\prime}(x) + \Delta^{(o)}(x) ,$$

$$y = x \tan\varphi , \qquad V^{(1)} = \bar{A}Q^{(1)\prime}(x) + \bar{B}Q^{(o)}(x) ,$$

$$P^{(1)} = \bar{C}Q^{(1)\prime}(x) + \bar{D}Q^{(o)}(x) , \qquad (7_1)$$

$$U^{(1)} = \bar{E}Q^{(1)\prime}(x) + \bar{F}Q^{(o)}(x) ,$$

$$R^{(1)} = \bar{G}Q^{(1)\prime}(x) + \bar{J}Q^{(o)}(x) ,$$

and for $n \geq 2$:

$$y = 0 , \qquad V^{(n)} = \Delta^{(n)\prime}(x) + \Delta^{(n-1)}(x) ,$$

$$y = x \tan\varphi , \qquad V^{(n)} = \bar{A}Q^{(n)\prime}(x) + \bar{B}Q^{(n-1)}(x) ,$$

$$P^{(n)} = \bar{C}Q^{(n)\prime}(x) + \bar{D}Q^{(n-1)}(x) , \qquad (7_n)$$

$$U^{(n)} = \bar{E}Q^{(n)\prime}(x) + \bar{F}Q^{(n-1)}(x),$$

$$R^{(n)} = \bar{G}Q^{(n)\prime}(x) + \bar{J}Q^{(n-1)}(x).$$

The problem of solving the equations (2) to satisfy the boundary conditions (3) is now reduced to that of first solving the equations (6_o) to satisfy the boundary conditions (7_o), and then equations (6_1) to satisfy (7_1), and so forth.

The solution of the first problem will give the stiffness derivative, whereas that of the second problem will give the damping derivayive (In principle, the above procedure of solving the equations (2) to satisfy the boundary conditions (3) can be successively carried on to any order in ik, i.e. to give aerodynamic derivatives as accurate as desired. However, it has been shown in Chapter II that terms proportional to k^2 or higher have neligible contribution to the aerodynamic derivatives).

For a wedge with zero incidence , oscillating about a pivot O_1 (Fig.2), the "amplitude function" is given by

$$\Delta(x) = x - h\cos\theta \ . \qquad (8)$$

With (8) given, the solution to the equations (6_o) which satisfies (7_o) is easily found to be

$$V^{(o)} = 1, \quad P^{(o)} = \bar{C}/\bar{A}, \quad U^{(o)} = \bar{E}/\bar{A}, \quad R^{(o)} = \bar{G}/\bar{A}, \qquad (9)$$

which is a special case of the result obtained by Chu[16], and by Chernyi[2]. Also, for later use, we write down the following expression

$$Q^{(o)}(x) = x/\bar{A} - h\cos\theta \ , \qquad (10)$$

in which the value $-h\cos\theta$ of the integral constant is obtained by an additional condition that

$$f_5(o, t) = \delta(o, t)$$

which arises because of the attachment of the bow shock to the wedge.

By making use of the solution (9) for the steady flow, the following equation is derived from (6_1)

$$(M_o^2 - 1)\frac{\partial^2 V^{(1)}}{\partial x^2} - \frac{\partial^2 V^{(1)}}{\partial y^2} = 0 \qquad (11)$$

to which the general solution is

$$V^{(1)} = F_1(x - (M_o^2 - 1)^{\frac{1}{2}}y) + F_2(x + (M_o^2 - 1)^{\frac{1}{2}}y) , \qquad (12)$$

where F_1 and F_2 are two arbitrary functions.

From the second equation of (6_1), is obtained

$$P^{(1)} = \varkappa\left[F_1(x - (M_o^2 - 1)^{\frac{1}{2}}y) - F_2(x + (M_o^2 - 1)^{\frac{1}{2}}y) \right] - M_o y + F_3(x), \qquad (13)$$

in which F_3 is another arbitrary function, and

$$\varkappa = \frac{M_o}{(M_o^2 - 1)^{\frac{1}{2}}} . \qquad (14)$$

By combining the third and the fifth equations of (6_1) to eliminate $U^{(1)}$, and putting (12) and (13) into the resulting equation, we obtain

$$F_3(x) = C_1 x + C_o ,$$

in which C_o is arbitrary constant, and

$$C_1 = \varkappa^2(U^{(o)}/M_o - P^{(o)}) . \qquad (15)$$

To satisfy the boundary conditions at the bow shock, $y = x \tan \varphi$, we have, from equations (12) and (13) and using (7_1),

$$\begin{aligned} F_1(\,(1 - H)x\,) + F_2(\,(1 + H)x\,) &= \bar{A}Q^{(1)}(x) + \bar{B}x/\bar{A} - \bar{B}h \cos \theta , \\ \varkappa\left[F_1(\,(1 - H)x\,) - F_2(\,(1 + H)x\,) \right] &= \bar{C}Q^{(1)\prime}(x) + \bar{D}x/\bar{A} - \bar{D}h \cos \theta \\ &\quad + M_o x \tan \varphi - C_1 x - C_o , \end{aligned} \qquad (16)$$

and hence,

$$2F_1(x) = (\bar{A} + \frac{\bar{C}}{\varkappa})Q^{(1)\prime}(\frac{x}{1-H}) + \left[\frac{\bar{B}}{\bar{A}} + \frac{\bar{D}/\bar{A} + M_o \tan\varphi - C_1}{\varkappa}\right]\frac{x}{1-H} - \frac{C_o}{\varkappa}$$

$$- (\bar{B} + \frac{\bar{D}}{\varkappa})h\cos\theta ,$$

(17)

$$2F_2(x) = (\bar{A} - \frac{\bar{C}}{\varkappa})Q^{(1)\prime}(\frac{x}{1+H}) + \left[\frac{\bar{B}}{\bar{A}} - \frac{\bar{D}/\bar{A} + M_o \tan\varphi - C_1}{\varkappa}\right]\frac{x}{1+H} + \frac{C_o}{\varkappa}$$

$$- (\bar{B} - \frac{\bar{D}}{\varkappa})h\cos\theta ,$$

where

$$H = (M_o^2 - 1)^{\frac{1}{2}} \tan\varphi . \tag{18}$$

To satisfy the boundary condition (7_1) at the wedge surface $y = 0$, we obtain

$$Q^{(1)\prime}(x) - \lambda Q^{(1)\prime}(mx) =$$

$$\frac{2\left[\left\{1 - H^2 - \frac{\bar{B}}{\bar{A}} - H(\frac{\bar{D}}{\bar{A}} + M_o \tan\varphi - C_1)/\varkappa\right\}\frac{x}{1+H} + h\cos\theta(\bar{B} - 1)\right]}{(\bar{A} + \bar{C}/\varkappa)} \tag{19}$$

where

$$m = \frac{1-H}{1+H} , \tag{20}$$

and

$$\lambda = \frac{\bar{C} - \varkappa\bar{A}}{\bar{C} + \varkappa\bar{A}} . \tag{21}$$

By a method similar to that in Chapter II, it is easy to prove that the quantity λ given by (21) represents the reflection coefficient of a disturbance from the bow shock, i.e. λ is equal to the ratio of the amplitude of the disturbance reflected from the bow shock (along the characteristic $x + (M_o^2 - 1)^{\frac{1}{2}}y = \text{const.}$) to the amplitude of the disturbance incident on the bow shock. (along the characteristic $x - (M_o^2 - 1)^{\frac{1}{2}}y = \text{const.}$)

Also this reflection coefficient λ is, at least for periodic unsteady motions, identical to that derived for steady flows in Refs.16 and 2.

The solution to the functional equation (19) is a linear function of x,

$$Q^{(1)\prime}(x) = \frac{2\left[1 - H^2 - \frac{\bar{B}}{\bar{A}} - \frac{H(\bar{D}/\bar{A} + M_o \tan\varphi - C_1)}{\varkappa}\right]}{(1 + \lambda m)(1 + H)(\bar{A} + \bar{C}/\varkappa)} x + \frac{h\cos\theta(\bar{B} - 1)}{\bar{A}} ,$$

and therefore, the pressure distribution over the wedge surface is given by

$$P^{(1)}(x, o) = (2G - I)x - Ih\cos\theta , \qquad (22)$$

where

$$\begin{aligned} I &= E + \bar{D} - E\bar{B} , \\ E &= \varkappa \frac{1 + \lambda}{1 - \lambda} , \\ G &= \frac{1}{2}\Bigg[I + \varkappa \frac{1 + \lambda m}{1 - \lambda m} + \varkappa^2 \frac{\bar{E}/M_o - \bar{C}}{\bar{A}} + \\ &\quad \frac{(1 - \lambda)\left\{ \frac{\bar{D}}{\bar{A}} + M_o \tan\varphi - \varkappa^2 \frac{\bar{E}/M_o - \bar{C}}{\bar{A}} - \varkappa \frac{(1 + \lambda)\bar{B}}{(1 - \lambda)\bar{A}} \right\}}{(1 - \lambda m)(1 + H)} \Bigg] \end{aligned} \qquad (23)$$

3 THE STABILITY DERIVATIVES USING THE EXACT THEORY

To the unsteady theory of the first order, the pitching moment coefficient c_m consists of the stiffness derivative $-c_{m_\theta}$ and the damping derivative $-c_{m_{\dot\theta}}$,

$$c_m = \epsilon e^{ikt}\left[(-c_{m_\theta}) + ik(-c_{m_{\dot\theta}})\right]. \tag{24}$$

By the same reasoning as in Section II.3.2.1, and using the solutions (9) and (22) we obtain

$$-c_{m_\theta} = \frac{4}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)\frac{1}{\cos^2\theta}E\left(\tfrac{1}{2} - h\cos^2\theta\right), \tag{25}$$

$$-c_{m_{\dot\theta}} = \frac{4}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)\frac{1}{\cos^3\theta}\left[I(h\cos^2\theta)^2 - Gh\cos^2\theta + \tfrac{1}{3}(2G - I)\right]. \tag{26}$$

The quantities E and I in (25) and (26) are seen to be, respectively, equal to the quantities E_a and I_a in (II.70), so the stiffness derivative obtained in this Chapter is the the same as that in ChapterII, as it should be.

On the other hand, the quantity G is, in general, not equal to the quantity G_a in (II.70), so the formula (26) for the damping derivative is not the same as the formula (II.69). However, as shown in Applendix B, if $M_o^2 \gg 1$ so that the terms proportional to M_o^{-2} are negligible, G is reduced to G_a, and therefore the present exact theory includes (to the first order of k) the approximate theory in Chapter II as a special case.

Comparisons between this exact theory and the previous approximate theory for various free stream Mach numbers M_∞ and flow deflection angles θ are plotted in Figs.11, 12, 17. Also plotted for supersonic flows are the results of Van Dyke's theory[19] and the experimental results by Pugh and Woodgate[4].

It can be concluded from these figures that the approximate theory works very well for the Mach number M_o behind the bow shock bigger than 2.0, but not so otherwise.

4 GENERAL CRITERION FOR STABILITY

The stability of an oscillating pitching wedge in supersonic and hypersonic flows may be examined by considering an exact criterion for the conditions at which the damping derivative first becomes zero at some cirtical position of the axis of oscillation. This can be obtained by solving the following equation

$$I(h \cos^2\theta)^2 - Gh \cos^2\theta + \frac{1}{3}(2G - I) = 0 , \qquad (27)$$

and this is done by an iteration method and the result is plotted in Fig.18. Also plotted in the same figure, for comparison, are the results of the approximate theory. It is seen that the approximate theory underestimates the thickness effect by a few degrees for supersonic flow and by about two degrees for hypersonic flow.

Although in general, $G_a \neq G$, the critical pivot position h_{cr}, at which the damping derivative first becomes zero, is the same in both the exact theory and the approximate theory, and is given by

$$h_{cr} = \frac{1}{3} + \frac{1}{3} \tan^2\theta_{cr} , \qquad (28)$$

where θ_{cr} is the critical flow deflection angle, at which the damping derivative first becomes zero.

Neutral damping boundaries for given wedges can also be obtained from equation (27), and the results for a thin wedge and a very thick wedge are plotted in Figs.20 and 21. Plotted in Fig.20, for comparison, is also the theory of Carrier and Van Dyke[11,12] it's seen that both theories give the same result for supersonic flow, as is expected.

CHAPTER IV

EFFECTS OF VISCOSITY [28, 29]

1 INTRODUCTION

In this Chapter we shall study the effect of viscosity on the stability derivatives of an oscillating wedge.

For moderate supersonic flow, the pressure field over the wedge, and hence the stability derivatives of the wedge is not significantly altered by the presence of the viscous boundary layer which is relatively thin. On the other hand, with increasing Mach number the boundary layer becomes relatively thicker and the pressure field may be quite different from the inviscid one. This causes a significant change in the stability derivatives. Orlik-Ruckmann[17] has examined the viscous effect on the stability derivatives of a thin, sharp wedge. Using piston theory, closed form formulae for the stability derivatives were obtained which showed that for a free stream Mach number of 17 and a wedge of 3 degrees semi-vertex angle the viscous effect may alter the stability derivatives by about 50 percent of the inviscid value at the wedge vertex to 200 percent at the end of the wedge. However, in Ref.17 the effect of the reflected waves coming from the bow shock is neglected, and the assumption of a slender wedge is made.

It is the purpose of this Chapter to develop a theory which includes both both the thickness effect and the effect of wave reflection. Comparison of the present theory with Orlik-Ruckmann's theory is given in Section 6.

In this Chapter we consider only a sharp oscillating wedge in viscous hypersonic or supersonic flows at zero incidence. An empirical formula for estimating the effect of the leading edge bluntness has been proposed by East[5].

The boundary layer is assumed to respond instantaneously to the unsteady flow quantities. Physically this assumption is valid if the time required for the boundary layer to adjust itself to the instantaneous incidence of the body is very small compared with any characteristic time in the problem. In stability analysis, particularly for hypersonic flow, the frequency parameter k is usually very small, this means that the period of oscillation, which may be taken as a characteristic time for comparison with the time required for the adjustment of the boundary layer, is very large compared with the time required for a fluid particle to pass through a characteristic length in the problem (say, the chord length of the wedge). The latter is easily shown to be of the same order of magnitude of the adjustment time(Refs 25, 26, 27). Hence the boundary layer adjustment time is very small comparied with the period of oscillation of the body.

Therefore this quasi-steady assumption for the unsteady boundary layer should be expected to te valid for the purpose of stability analyses. More discussion is given in Ref.17.

With viscosity present, the wedge is thickened by the displacement boundary layer. This introduces an effective wedge at zero incidence which is of semi-vertex angle θ_1 equal to the semi-vertex angle θ_o of the original wedge plus the average inclination of the displacement boundary layer to the surface of the original wedge (Fig.22). From the viewpoint of the perturbation theory the viscous flow past the original oscillating wedge is considered as the inviscid flow past the effective wedge which is oscillating at the same frequency as the original wedge and which is deforming according to the changing displacement thickness of the boundary layer on the original wedge. This implies an assumption that waves reflect from the boundary layer as from a solid surface. The problem of finding an unsteady viscous flow field is thus transformed to a suitably formulated problem of finding an unsteady inviscid flow field for which the methods developed in Chapters II and III can be applied.

For calculating the stability derivatives the exact theory in Chapter III may be used. However it is only in the cases for which the Mach number behind the bow shock is large that the viscous effect will be significant, and in these cases the approximate theory gives excellent results as compared with the exact theory. Therefore, in this Chapter when calculating the viscous effect we use the approximate theory but follow the same framework as that in the exact theory, so it is easy to extend, if required, the present method to give exact results.

2 INVISCID FLOW

Let the system of coordinates $\bar{x}O\bar{y}$ be such that $O\bar{x}$ coincide with the surface of the effective wedge in its average position, and O the vertex of the wedge (Fig.22). Let $\bar{t}$ be the time variable, $\bar{u}(\bar{x}, \bar{y}, \bar{t})$, $\bar{v}(\bar{x}, \bar{y}, \bar{t})$ the velocity components in $\bar{x}$, $\bar{y}$ directions, respectively, $\bar{p}(\bar{x}, \bar{y}, \bar{t})$ the pressure, and $\bar{\rho}(\bar{x}, \bar{y}, \bar{t})$ the density, of the fluid particles. Subscripts o and 1 refer, respectively, to the quantities of the steady flows past the original wedge and the effective wedge, thus u_1, v_1, p_1, ρ_1, a_1, M_1 and β_1 are, respectively, the velocity components in the $\bar{x}$, $\bar{y}$ directions, the pressure, the density, the speed of sound, the Mach number and the shock angle in the steady flow past the effective wedge, etc. Obvisously $v_1 = 0$. For simplicity, let $\varphi_1 = \beta_1 - \theta_1$.

For small perturbation from the pure wedge flow past the effective wedge, we have, say,

$$\bar{p} = p_1 + \epsilon p + \ldots, \qquad \text{etc.} \tag{1}$$

The non-dimensionalized variables x, y, and t are introduced to relate to the dimensional variables $\bar{x}$, $\bar{y}$, $\bar{t}$ by

$$x = \frac{\bar{x}}{\bar{\ell}}, \qquad y = \frac{\bar{y}}{\bar{\ell}}, \qquad t = \frac{u_1 \bar{t}}{\bar{\ell}}. \tag{2}$$

To study the gas motion resulting from a harmonic motion of the wedge with a given circular frequency ω we have, by a method similar to that in the previous Chapter, the following equations respectively for the wedge surface and the bow shock,

$$y = \epsilon\delta(x, t) = \epsilon e^{ik_1 t}\Delta(x), \tag{3}$$

and

$$y = x \tan\varphi_1 + \epsilon e^{ik_1 t}Q(x), \tag{4}$$

where

$$k_1 = \omega\bar{\ell}/u_1 , \tag{5}$$

is the frequency parameter based on the wedge length $\bar{\ell}$ and the speed of gas behind the bow shock in the steady flow past the effective wedge. We may also express, say

$$p = \rho_1 a_1 u_1 e^{ik_1 t} P(x, y), \quad \text{etc.} \tag{6}$$

After making the same assumption as in Chapter II that the coefficient $M_1^2 - 1$ in (I.7) be replaced by M_1^2, we obtain the following equations for the time-independent perturbation quantities

$$\begin{aligned}
M_1^2 \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} &= -2ik_1 M_1^2 \frac{\partial P}{\partial x} - (ik_1)^2 M_1^2 P, \\
\frac{\partial P}{\partial y} &= -M_1 \frac{\partial V}{\partial x} - ik_1 M_1 V, \\
\frac{\partial U}{\partial x} &= -\frac{1}{M_1}\frac{\partial P}{\partial x} - ik_1 U, \\
ik_1 R + \frac{\partial R}{\partial x} &= ik_1 P + \frac{\partial P}{\partial x}, \\
ik_1 P + \frac{\partial P}{\partial x} + \frac{1}{M_1}\frac{\partial V}{\partial y} &= 0.
\end{aligned} \tag{7}$$

Similarly, the boundary conditions are

$$\begin{aligned}
y = 0, \quad & V = \Delta'(x) + ik_1\Delta(x), \\
y = x \tan\varphi_1, \quad & V = \bar{A}_1 Q'(x) + ik_1 \bar{B}_1 Q(x), \\
& P = \bar{C}_1 Q'(x) + ik_1 \bar{D}_1 Q(x), \\
& U = \bar{E}_1 Q'(x) + ik_1 \bar{F}_1 Q(x), \\
& R = \bar{G}_1 Q'(x) + ik_1 \bar{J}_1 Q(x),
\end{aligned} \tag{8}$$

where the constants $\bar{A}_1$ through $\bar{J}_1$ are given by (III.4) with the subscript o being replaced by the subscript 1.

Further, the five unknown functions may be expanded as a power series in ik_1,

$$
\begin{aligned}
V &= V^{(o)} + ik_1 V^{(1)} + \dots, \\
P &= P^{(o)} + ik_1 P^{(1)} + \dots, \\
U &= U^{(o)} + ik_1 U^{(1)} + \dots, \\
R &= R^{(o)} + ik_1 R^{(1)} + \dots, \\
Q &= Q^{(o)} + ik_1 Q^{(1)} + \dots,
\end{aligned} \tag{9}
$$

also,

$$\Delta = \Delta^{(o)} + ik_1 \Delta^{(1)} + \dots, \tag{10}$$

Substituting (9) into equations (7) and equating the terms of the same power in ik_1 on both sides of each equation, we obtain the zeroth order and the first order perturbation equations as follows

$$
\begin{aligned}
& M_1^2 \frac{\partial^2 P^{(o)}}{\partial x^2} - \frac{\partial^2 P^{(o)}}{\partial y^2} = 0, \\
& \frac{\partial P^{(o)}}{\partial y} = - M_1 \frac{\partial V^{(o)}}{\partial x}, \\
& \frac{\partial U^{(o)}}{\partial x} = - \frac{1}{M_1} \frac{\partial P^{(o)}}{\partial x}, \\
& \frac{\partial R^{(o)}}{\partial x} = \frac{\partial P^{(o)}}{\partial x}, \\
& \frac{\partial P^{(o)}}{\partial x} = \frac{1}{M_1} \frac{\partial V^{(o)}}{\partial y} = 0,
\end{aligned} \tag{11$_o$}
$$

and,

$$M_1^2 \frac{\partial^2 P^{(1)}}{\partial x^2} - \frac{\partial^2 P^{(1)}}{\partial y^2} = - 2M_1^2 \frac{\partial P^{(o)}}{\partial x},$$

$$\frac{\partial P^{(1)}}{\partial y} = - M_1 \frac{\partial V^{(1)}}{\partial x} - M_1 V^{(o)},$$

$$\frac{\partial U^{(1)}}{\partial x} = -\frac{1}{M_1}\frac{\partial P^{(1)}}{\partial x} - U^{(o)}, \qquad (11_1)$$

$$R^{(o)} + \frac{\partial R^{(1)}}{\partial x} = P^{(o)} + \frac{\partial P^{(1)}}{\partial x},$$

$$\frac{\partial P^{(1)}}{\partial x} + \frac{1}{M_1}\frac{\partial V^{(1)}}{\partial y} + P^{(o)} = 0$$

Similarly, we obtain the following zeroth and the first order boundary conditions

$$\begin{aligned}
y &= 0, & V^{(o)} &= \Delta^{(o)\prime}(x), \\
y &= x\tan\varphi_1, & V^{(o)} &= \bar{A}_1 Q^{(o)\prime}(x), \\
& & P^{(o)} &= \bar{C}_1 Q^{(o)\prime}(x), \qquad (12_o)\\
& & U^{(o)} &= \bar{E}_1 Q^{(o)\prime}(x), \\
& & R^{(o)} &= \bar{G}_1 Q^{(0)}(x),
\end{aligned}$$

and,

$$\begin{aligned}
y &= 0, & V^{(1)} &= \Delta^{(1)\prime}(x) + \Delta^{(o)}(x), \\
y &= x\tan\varphi_1, & V^{(1)} &= \bar{A}_1 Q^{(1)\prime}(x) + \bar{B}_1 Q^{(o)}(x), \\
& & P^{(1)} &= \bar{C}_1 Q^{(1)\prime}(x) + \bar{D}_1 Q^{(o)}(x), \qquad (12_1)\\
& & U^{(1)} &= \bar{E}_1 Q^{(1)\prime}(x) + \bar{E}_1 Q^{(o)}(x), \\
& & R^{(1)} &= \bar{G}_1 Q^{(1)\prime}(x) + \bar{J}_1 Q^{(o)}(x).
\end{aligned}$$

Equations (11) and boundary conditions (12) will be used in Sections 3 and 4.

To describe the displacement boundary layer, another system of coordinates $x'O'y$ is introduced for which Ox' is along the original wedge surface at its average position (Fig.22). Denote by $\delta^*(x', t)$ the displacement thickness of the viscous flow past the original wedge at time t, and by $\delta_o^*(x')$ that in the steady flow past the original wedge. (of course, both of them are always measured from the original wedge surface). Then it is easy to see that the average inclination of the displacement boundary layer to the original wedge surface is equal to $\delta_o^*(1)$ for a very thin wedge and approximately so for thicker wedges. We therefore define the semi-vertex angle θ_1 of the effective wedge as

$$\theta_1 = \theta_o + \delta_o^*(1). \tag{13}$$

In order to obtain the displacement thickness at any time, it is necessary to know the unsteady inviscid flow past the original oscillating wedge. This has been done in Chapters II and III, the results are quoted as follows. Let $\bar{u}'$, $\bar{v}'$ represent the velocity components behind the bow shock in x', y' directions, respectively, and subscipt orig refersto the quantities in the unsteady inviscid flow past the original wedge, then

$$\begin{aligned}
\bar{u}'_{orig} &= u'_o(1 + \epsilon e^{ik_1 t} U_{inv}), \\
\bar{v}'_{orig} &= u'_o \epsilon e^{ik_1 t} V_{inv}, \\
\bar{p}_{orig} &= p_o(1 + \gamma M_o \epsilon e^{ik_1 t} P_{inv}), \\
\bar{\rho}_{orig} &= \rho_o(1 + M_o \epsilon e^{ik_1 t} R_{inv}),
\end{aligned} \tag{14}$$

where u'_o is the velocity component of the gas behind the bow shock in the x' direction in the steady flow past the original wedge, and

$$\begin{aligned}
U_{inv} &= U_{inv}^{(o)} + ik_1 U_{inv}^{(1)} + \dots, \\
V_{inv} &= V_{inv}^{(o)} + ik_1 V_{inv}^{(1)} + \dots, \\
P_{inv} &= P_{inv}^{(o)} + ik_1 P_{inv}^{(1)} + \dots, \\
R_{inv} &= R_{inv}^{(o)} + ik_1 R_{inv}^{(1)} + \dots,
\end{aligned} \tag{15}$$

in which,

$$
\begin{aligned}
P^{(o)}_{inv}(x,\, o) &= \bar{C}_o/\bar{A}_o \equiv E_o, \\
R^{(o)}_{inv}(x,\, o) &= \bar{G}_o/\bar{A}_o \equiv N_o, \\
P^{(1)}_{inv}(x,\, 0) &= (2G_o - E_o)x + \left[E_o(\bar{B}_o - 1) - \bar{D}_o \right] h \cos\theta_o, \\
R^{(1)}_{inv}(x,\, o) &= (2G_o - N_o)x + \left[N_o(\bar{B}_o - 1) - \bar{J}_o \right] h \cos\theta_o,
\end{aligned}
\tag{16}
$$

where h is the non-dimensionalized pivot position measured from the apex of the wedge, the constants $\bar{A}_o$ through $\bar{G}_o$, $\bar{J}_o$ are equal to the corresponding quantities given by (III.4), thus $\bar{A}_o = \bar{A}$, etc. The constant G_o is

$$
G_o = \frac{1 + \lambda_o m_o}{1 - \lambda_o m_o} - \frac{\lambda_o u_o + \nu_o}{(1 - \lambda_o)(1 - \lambda_o m_o)(1 + H_o)}, \tag{17}
$$

with

$$
m_o = \frac{1 - H_o}{1 + H_o}, \qquad \lambda_o = \frac{\bar{C}_o - \bar{A}_o}{\bar{C}_o + \bar{A}_o}, \qquad H_o = M_o \tan\varphi_o,
$$

$$
\mu_o = \left[\bar{B}_o - \bar{A}_o + \bar{C}_o \tan^2\varphi_o \right] / (\bar{A}_o + \bar{C}_o) \tag{18}
$$

$$
\nu_o = \left[\bar{B}_o - \bar{A}_o - \bar{C}_o \tan^2\varphi_o \right] / (\bar{A}_o + \bar{C}_o)
$$

In (15), the other quantities such as $U^{(1)}_{inv}$, being of no use in this Chapter, are not quoted here.

3 WEAK INTERACTION

As the boundary layer responds instantaneously to the flow quantities in the unsteady flow past the original wedge, for the case of weak interaction, in the $x'y'$ system of coordinates the displacement thickness $\delta^*(x', t)$ at time t can be written by[1]

$$\frac{\partial \delta^*(x', t)}{\partial x'} = d_{orig} \frac{\bar{\chi}_{orig}}{M_{orig}} , \qquad (19)$$

where the hypersonic interaction parameter $\bar{\chi}$ is as follows

$$\bar{\chi} = \frac{(C)^{\frac{1}{2}} M^3}{(R_{ex'})^{\frac{1}{2}}} , \qquad (20)$$

in which Rex' is the Reynolds number based on the non-dimensionalized length x' from the wedge vertex, C is the constant of proportionality in the linear viscosity-temperature law and is determined by matching the viscosity relation, usually evaluated at the surface temperature, with the exact value of the viscosity μ for which the power law

$$\mu \propto T^{\Omega} \qquad (21)$$

is taken and the values of the index Ω are taken as 0.76 for air and 0.647 for heluim throughout this Chapter. Thus

$$C_{orig} = \left(\frac{\mu_b}{\mu_{orig}}\right)\left(\frac{T_{orig}}{T_b}\right) = \left(\frac{T_{orig}}{T_b}\right)^{1 - \Omega} \qquad (22)$$

where T is temperature of the gas and the subscript b refers to the body surface. The quantity d_{orig} in (19) is

$$d_{orig} = \frac{A(P_r)}{M^2_{orig}} \frac{T_b}{T_{orig}} + (\gamma - 1)B(P_r) , \qquad (23)$$

The two constants $A(P_r)$ and $B(P_r)$ are dependent on the Prandtl number P_r and given below[1]:

P_r	A	B
1.0	0.865	0.166
0.725	0.968	0.145

For an insulated body,

$$T_b/T_{orig} = 1 + (\gamma - 1)(P_r)^{\frac{1}{2}} M_{orig}^2/2, \tag{24}$$

and hence

$$d_{orig} = \frac{A(P_r)}{M_{orig}^2} + (\gamma - 1)B(P_r) + A(P_r)(P_r)^{\frac{1}{2}}\frac{\gamma - 1}{2}, \tag{25}$$

whereas for a very cold body,

$$T_b/T_{orig} = 1, \tag{26}$$

and hence

$$d_{orig} = \frac{A(P_r)}{M_{orig}^2} + (\gamma - 1)B(P_r). \tag{27}$$

Using equations (14), and denoting $\epsilon e^{ik_1 t}$ by $\epsilon(t)$, the quantities M_{orig}, $\bar{X}_{orig}$ and d_{orig} can be expressed as

$$M_{orig}^2 = \frac{\bar{u}'^2_{orig} + \bar{v}'^2_{orig}}{\gamma \bar{P}_{orig}/\bar{\rho}_{orig}} = \frac{\rho_o\left[1 + \epsilon(t)M_o R_{inv}\right]u_o^2\left[1 + 2\epsilon(t)U_{inv}\right]}{p_o\left[1 + \epsilon(t)\gamma M_o P_{inv}\right]}$$

$$= M_o^2\left[1 + \epsilon(t)(M_o R_{inv} - \gamma M_o P_{inv} + 2U_{inv})\right],$$

$$(R_{ex'_{orig}})^{\frac{1}{2}} = (x')^{\frac{1}{2}}(R_{e1,o})^{\frac{1}{2}}\left[1 + \epsilon(t)(M_o R_{inv} - \frac{\gamma}{2}M_o P_{inv} + \frac{1}{2}U_{inv})\right],$$

$$d_{orig} = d_o\left[1 - \epsilon(t)\frac{A(P_r)}{M_o^2 d_o}(M_o R_{inv} - \gamma M_o P_{inv} + 2U_{inv})\right], \tag{28}$$

where $R_{e1,o}$ is the Reynolds number at $x' = 1$, using the quantities in the steady inviscid flow past the original wedge, and d_o is given by

$$d_o = \frac{A(P_r)}{M_o^2} + (\gamma - 1)B(P_r) + \frac{\gamma - 1}{2}A(P_r)(P_r)^{\frac{1}{2}} \text{ for an insulated body}$$

$$d_o \doteq \frac{A(P_r)}{M_o^2} + (\gamma - 1)B(P_r) \quad \text{for a very cold body} \tag{29}$$

Putting (28) into (19) we obtain

$$\frac{\partial\delta^*(x', t)}{\partial x'} = \frac{Y_w}{(x')^{\frac{1}{2}}}\Big[1 + \epsilon(t)\Big\{ \gamma M_o(\frac{A(P_r)}{M_o^2 d_o} - \frac{1}{2})P_{inv} - \frac{M_o A(P_r)}{M_o^2 d_o}R_{inv} + (\frac{3}{2} - \frac{2A(P_r)}{M_o^2 d_o})U_{inv} \Big\}\Big], \tag{30}$$

where

$$Y_w = \frac{d_o M_o^2}{(R_{e1,o})^{\frac{1}{2}}} = \frac{d_o \bar{X}_{o,1}}{M_o} \tag{31}$$

It has been shown in Chapter II that for large value of M_o^2 the ratio of U_{inv} to $M_o P_{inv}$ is of order of M_o^{-2} and thus the term U_{inv} in (30) may be neglected within the approximate theory.

Making use of the inviscid solution (14) to (16) and integrating (30) to give the displacement thickness

$$\delta^*(x', t) = 2Y_w(x')^{\frac{1}{2}} + \epsilon(t)\Big[2Y_w Z_w x' + ik_1 Y_w M_o\Big\{ \gamma(\frac{A(P_r)}{M_o^2 d_o} - \frac{1}{2}) \int \frac{P_{inv}^{(1)}(x', o)}{(x')^{\frac{1}{2}}}dx' - \frac{A(P_r)}{M_o^2 d_o} \int \frac{R_{inv}^{(1)}(x', o)}{(x')^{\frac{1}{2}}}dx'\Big\}\Big], \tag{32}$$

where

$$Z_w = M_o\Big[\gamma(\frac{A(P_r)}{M_o^2 d_o} - \frac{1}{2})E_o - \frac{A(p_r)}{M_o^2 d_o}N_o \Big], \tag{33}$$

is a constant depending only on the steady flow over the original wedge. As seen from (32) we have

$$\delta_o^*(1) = 2Y_w, \tag{34}$$

and hence for weak interaction the semi-vertex angle θ_1 of the effect wedge is given by (see Fig.22)

$$\theta_1 = \theta_o + 2Y_w. \tag{35}$$

In transforming from the $x'y'$ system of coordinates to the xy system, we notice that the two systems of coordinates differ only by an angle of rotation $\delta_o^*(1)$, which is neverthless very small compared with unity, thus we shall use the equalities

$$\sin \delta_o^*(1) = \delta_o^*(1), \qquad \cos \delta_o^*(1) = 1,$$

throughout this Chapter, and hence we can easily obtain the disturbance function $y = \delta(x, t)$ over the effective wedge surface in the xy system of coordinates as below

$$\delta(x, t) = \delta_o^*(1)\left((x)^{\frac{1}{2}} - x \right) + \epsilon(t)(x - h \cos \theta_o) + \epsilon(t)\delta_o^*(1)\Big[Z_w x^{\frac{1}{2}} + ik_1 \frac{M_o}{2}\Big\{ \gamma\Big(\frac{A(P_r)}{M_o^2 d_o} - \frac{1}{2}\Big) \int \frac{P_{inv}^{(1)}(x, o)}{(x)^{\frac{1}{2}}}dx - \frac{A(P_r)}{M_o^2 d_o} \int \frac{R_{inv}^{(1)}(x, o)}{(x)^{\frac{1}{2}}}dx \Big\}\Big] . \tag{36}$$

It is clear now that this disturbance consists of three parts.

The first part of the disturbance can be seen to be samll compared with the effective wedge over most of its surface, except near the apex. Because, for weak interaction $\delta_o^*(1)$ is small compared with θ_o, and even smaller after multiplying by a small factor $((x)^{\frac{1}{2}} - x)$ and compared with θ_1; for strong interaction $\delta_o^*(1)$ may be comparable with θ_o, but it is then multiplied by a even smaller factor $(x^{\frac{3}{4}} - x)$ (see Section 4), so the first term is still small compared with θ_1. Therefore the small perturbation theory can be applied to this first part of disturbance. As the perturbation equantions are linear, the perturbation pressure and hence the stability derivatives contributed by these three parts of the disturbance can be calculated separately. Now the perturbation pressure caused by the first part of the disturbance is independent of time and therefore gives no contribution to the moment coefficient and hence the stability derivatives as the effective wedge is at zero incidence to the free stream. Therefore the first term on the right hand side of (36) can be dropped out for the purpose of stability analysis.

In the second term on the right hand side of (36) $\cos\theta_o$ could be well replaced by $\cos\theta_1$ (this would cause an error less than o.5%) as $\delta_1^*(1)$ is usually very small.

Therefore the contribation to the stability derivatives due to this term is that of a wedge with semi-vertex angle θ_1 in the inviscid flow and can be found from the previous chapter.

We need therefore calculate the perturbation pressure due to the last term on the right hand side of equation (36) only. To do this, we write all the unknown quantities with a subscript v, thus the zeroth order solution for the perturbation quantities $P_v^{(o)}$, $V_v^{(o)}$, is obtained, from equations (11_o), as

$$
\begin{aligned}
V_v^{(o)} &= F_1^{(o)}(x - M_1 y) + F_2^{(o)}(x + M_1 y) , \\
P_v^{(o)} &= F_1^{(o)}(x - M_1 y) - F_2^{(o)}(x + M_1 y) + F_3^{(o)}(x) ,
\end{aligned}
\qquad (37)
$$

where $F_1^{(o)}$, $F_2^{(o)}$ and $F_3^{(o)}$ are arbitrary functions. Substituting the above equations into the fifth equation of (11_o) is obtained

$$
\begin{aligned}
dF_3^{(o)}/dx &= o, \\
\therefore\ F_3^{(o)}(x) &= \text{const.}
\end{aligned}
\qquad (38)
$$

To satisfy the boundary conditions (12_o) at $y = x\tan\varphi_1$, we have

$$F_1^{(o)}(\ (1 - H_1)x\ ) + F_2^{(o)}(\ (1 + H_1)x\ ) \;=\; \bar{A}_1 Q_v^{(o)\prime}(x)\ ,$$

$$F_1^{(o)}(\ (1 - H_1)x\ ) - F_2^{(o)}(\ (1 + H_1)x\ ) \;=\; \bar{C}_1 Q_v^{(o)\prime}(x) - F_3^{(o)},$$

and hence

$$\begin{aligned} 2F_1^{(o)}(x) &= (\bar{A}_1 + \bar{C}_1) Q_v^{(o)\prime}\left(\frac{x}{1 - H_1}\right) - F_3^{(o)}\ , \\ 2F_2^{(o)}(x) &= (\bar{A}_1 - \bar{C}_1) Q_v^{(o)\prime}\left(\frac{x}{1 + H_1}\right) + F_3^{(o)}\ , \end{aligned} \tag{39}$$

where

$$H_1 \;=\; M_1 \tan \varphi_1 . \tag{40}$$

To satisfy the boundary condition (12_o) at $y = 0$, we obtain

$$Q_v^{(o)\prime}(x) - \lambda_1 Q_v^{(o)\prime}(m_1 x) \;=\; \frac{2Y_w Z_w}{(\bar{A}_1 + \bar{C}_1)(1 - H_1)^{\frac{1}{2}}} \cdot \frac{1}{(x)^{\frac{1}{2}}}\ , \tag{41}$$

the solution to which is

$$Q_v^{(o)}(x) \;=\; \frac{4Y_w Z_w}{(\ (1 - \lambda_1/(m_1)^{\frac{1}{2}}\ )(\bar{A}_1 + \bar{C}_1)(1 - H_1)^{\frac{1}{2}}} (x)^{\frac{1}{2}} + \text{const.} \tag{42}$$

where

$$\lambda_1 \;=\; (\bar{C}_1 - \bar{A}_1)/(\bar{C}_1 + \bar{A}_1), \qquad m_1 \;=\; (1 - H_1)/(1 + H_1).$$

The integral constant in (42) is equal to zero, as

$$f_{5v}(o,\ t) \;=\; \delta(o,\ t).$$

Finally we find the zeroth order solution as

$$\begin{aligned} V_v^{(o)}(x,\ y) &= \frac{Y_w Z_w}{1 - \lambda_1/(m_1)^{\frac{1}{2}}} \left[\frac{1}{(x - M_1 y)^{\frac{1}{2}}} - \frac{\lambda_1}{(m_1)^{\frac{1}{2}}} \frac{1}{(x + M_1 y)^{\frac{1}{2}}} \right], \\ P_v^{(o)}(x,\ y) &= \frac{Y_w Z_w}{1 - \lambda_1/(m_1)^{\frac{1}{2}}} \left[\frac{1}{(x - M_1 y)^{\frac{1}{2}}} + \frac{\lambda_1}{(m_1)^{\frac{1}{2}}} \frac{1}{(x + M_1 y)^{\frac{1}{2}}} \right]. \end{aligned} \tag{43}$$

and, particularly, the pressure at the surface

$$P_v^{(o)}(x, o) = \frac{Y_w Z_w(1 + \lambda_1/(m_1)^{\frac{1}{2}})}{(1 - \lambda_1/(m_1)^{\frac{1}{2}})} \frac{1}{(x)^{\frac{1}{2}}} . \qquad (44)$$

To find the first order solution we should solve equation (11_1) to satisfy the boundary conditions (12_1), in which the first condition becomes

$$y = 0: \quad V_v^{(1)} = 2Y_w Z_w x^{\frac{1}{2}} + \frac{Y_w M_o}{(x)^{\frac{1}{2}}}\Big[\gamma(\frac{A(P_r)}{M_o^2 d_o} - \frac{1}{2}) P_{inv}^{(1)}(x, o) - \frac{A(P_r)}{M_o^2 d_o} R_{inv}^{(1)}(x, o) \Big] .$$

Using the zeroth order solution (43) and a method similar to that for the zeroth order solution, the perturbation pressure $P_v^{(1)}(x, o)$ at the surface is found to be (the problem could be easier solved by making a transformation $\xi = x - M_1 y$, $\eta = x + M_1 y$)

$$P_v^{(1)}(x, o) = Y_w\Big[W_q x^{\frac{1}{2}} - \frac{W_r h \cos \theta_1}{(x)^{\frac{1}{2}}} \Big] , \qquad (45)$$

where,

$$W_q = \frac{1}{2}\Big[\frac{q_w}{(1 - H_1)} \frac{1 + \lambda_1(m_1)^{\frac{1}{2}}}{1 - \lambda_1(m_1)^{\frac{1}{2}}} + Z_w(\frac{\overline{K}_w - \overline{T}_w}{1 - \lambda_1/(m_1)^{\frac{1}{2}}} - \frac{1 + \lambda_1/(m_1)^{\frac{1}{2}}}{1 - \lambda_1/(m_1)^{\frac{1}{2}}}) \Big]$$

$$W_r = r_w(1 - H_1)^{\frac{1}{2}} \frac{1 + \lambda_1/(m_1)^{\frac{1}{2}}}{1 - \lambda_1/(m_1)^{\frac{1}{2}}} ,$$

and

$$\overline{K}_w = \frac{4(\overline{B}_1 + \overline{D}_1)}{(1 - H_1)(\overline{A}_1 + \overline{C}_1)} + \frac{1}{2}(\frac{1}{m_1} - \lambda_1) + \frac{1 + \lambda_1}{1 - H_1} - \frac{1}{2}(1 + \frac{\lambda_1}{m_1}) ,$$

$$\overline{T}_w = (m_1)^{\frac{1}{2}}\Big[\frac{4(\overline{B}_1 - \overline{D}_1)}{(1 - H_1)(\overline{A}_1 + \overline{C}_1)} + \frac{1}{2}(\frac{1}{m_1} - \lambda_1) - \frac{1 + \lambda_1}{1 - H_1} + \frac{1}{2}(1 + \frac{\lambda_1}{m_1}) \Big]$$

$$q_w = Z_w(5 - \frac{\overline{K}_w + \overline{T}_w}{1 - \lambda_1/(m_1)^{\frac{1}{2}}}) + 2M_o\Big[\gamma(\frac{A(P_r)}{M_o^2 d_o} - \frac{1}{2})(2G_o - E_o) - \frac{A(P_r)}{M_o^2 d_o}(2G_o - N_o) \Big] \qquad (46)$$

$$r_w = M_o\Big[\frac{A(P_r)}{M_o^2 d_o}\Big\{ N_o(\overline{B}_o - 1) - \overline{J}_o\Big\} - \gamma(\frac{A(P_r)}{M_o^2 d_o} - \frac{1}{2})\Big\{ E_o(\overline{B}_o - 1) - \overline{D}_o\Big\}\Big]$$

For weak interaction, the tolal moment coefficient c_m of the wedge in viscous flow is equal to that of the effective wedge $(c_m)_{eff}$. plus that due to the deformation of this effective wedge $(c_m)_w$,

$$c_m = (c_m)_{eff} + (c_m)_w . \qquad (47)$$

The moment coefficient $(c_m)_{eff}$, as stated before, can be found from previous chapters, whereas the moment coefficient $(c_m)_w$ can be obtained by using the solution (44) and (45) and performing the integration in the general formula for the moment coefficient about a pivot position h as

$$(c_m)_w = \frac{4\epsilon(t)}{M_1}\left(\frac{\rho_0}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2 \int_0^{1/\cos\theta_1}\left[P_v^{(0)}(x, 0) + ik_1 P_v^{(1)}(x, 0) \right](x - h\cos\theta_1)dx$$

$$\equiv \epsilon(t)\left[(-c_{m_\theta})_w + ik_1(-c_{m_{\dot\theta}})_w \right], \qquad (48)$$

where, as before, U_∞ is the speed of gas in the free stream. And therefore the stiffness derivatifes $(-c_{m_\theta})_w$ and the damping derivative $(-c_{m_{\dot\theta}})_w$ due to the deformation of the effective wedge in the case of weakinteraction are as follows

$$(-c_{m_\theta})_w = \frac{4}{M_1}\left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2 \frac{2Y_w Z_w(1 + \lambda_1/(m_1)^{\frac{1}{2}})}{\cos^{\frac{3}{2}}\theta_1(1 - \lambda_1/(m_1)^{\frac{1}{2}})}\left(\tfrac{1}{3} - h\cos^2\theta_1\right) ,$$

$$(-c_{m_{\dot\theta}})_w = \frac{4}{M_1}\left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2 \frac{2Y_w}{\cos^{\frac{5}{2}}\theta_1}\left[W_r(h\cos^2\theta_1)^2 - \tfrac{1}{3}(W_r + W_q)h\cos^2\theta_1 + \tfrac{1}{5}W_q \right] . \qquad (49)$$

4 STRONG INTERACTION

The case of strong inteaction has been discussed in Ref.1 for a flat plate at zero angle of attack. With θ_o small the results in Ref.1 need only be slightly changed by replacing quantities in the free stream by the corresponding ones in the unsteady inviscid flow past the original wedge. Thus the displacement thickness $\delta^*(x', t)$ at time t, expressed in the $x'y'$ system of coordinates, is given by

$$\delta^*(x', t) = \sigma x \frac{X_{orig}^{-\frac{1}{2}}}{M_{orig}} , \qquad (50)$$

with the constant σ tabulated below.

Values of σ, ($P_r = 1.0$)

	very cold body	insulated body
$\gamma = 1.4$	0.397	0.738
$\gamma = 1.667$	0.457	0.858

Perturbing equation (50) for the small quantity ε gives

$$\delta^*(x', t) = 2Y_s x'^{\frac{3}{4}}\left[1 - \tfrac{1}{4}\epsilon(t) M_o R_{inv} \right] , \qquad (51)$$

in which

$$Y_s = \frac{\sigma M_o^{\frac{1}{2}}}{2(R_{eo,1})^{\frac{1}{4}}} = \frac{\sigma}{2}\frac{X_{o,1}^{-\frac{1}{2}}}{M_o} . \qquad (52)$$

From (51), we obtain, in the case of case of strong interaction,

$$\delta_o^*(1) = 2Y_s , \qquad (53)$$

and hence,

$$\theta_1 = \theta_o + 2Y_s . \qquad (54)$$

Following the same line as that in Section 3, the disturbance function over the surface of the effective wedge, written in the xy system, is expressed as

$$\delta(x, t) = \delta_o^*(1)(x^{\frac{3}{4}} - x) + \varepsilon(t)(x - h\cos\theta_o)$$

$$+ \varepsilon(t)\delta_o^*(1)\left[Z_s x^{\frac{3}{4}} + ik_1\left(- \frac{M_o}{4} R^{(1)}_{inv}(x, o)x^{\frac{3}{4}}\right)\right] \qquad (55)$$

where

$$Z_s = - \frac{M_o N_o}{4} \qquad (56)$$

By a method similar to that in Section 3, we obtain the following formulae for the stiffness derivative $(-c_{m_\theta})$s and the damping derivative $(-c_{m_{\dot\theta}})$s caused by the deformation of the effective wedge in the case of strong interaction.

$$(-c_{m_\theta})_s = \frac{4}{M_1}\left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2 \frac{2Y_s X_s(1 + \lambda_1/(m_1)^{\frac{1}{4}})}{\cos^{\frac{7}{4}}\theta_1(1 - \lambda_1/(m_1)^{\frac{1}{4}})}\left(\frac{3}{7} - h\cos^2\theta_1\right) , \qquad (57)$$

$$(-c_{m_{\dot\theta}})_s = \frac{4}{M_1}\left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2 \frac{2Y_s}{\cos^{\frac{11}{4}}\theta_1}\left[\frac{2}{3}S_r(h\cos^2\theta_1)^2 - \frac{2}{7}(S_r + S_q)h\cos^2\theta_1 + \frac{2}{11}S_q \right] ,$$

where

$$S_q = \frac{1}{2}\left[\frac{q_s}{(1 - H_1)^{\frac{3}{4}}} \frac{1 + \lambda_1/(m)^{\frac{3}{4}}}{1 - \lambda_1/(m_1)^{\frac{3}{4}}} + Z_s\left(\frac{\bar K_s - \bar T_s}{1 - \lambda_1/(m_1)^{\frac{1}{4}}} - \frac{3}{2}\,\frac{1 + \lambda_1/(m_1)^{\frac{1}{4}}}{1 - \lambda_1/(m_1)^{\frac{1}{4}}}\right) \right] ,$$

$$S_r = r_s(1 - H_1)^{\frac{1}{4}} \frac{1 + \lambda_1/(m_1)^{\frac{1}{4}}}{1 - \lambda_1/(m_1)^{\frac{1}{4}}} ,$$

and,

$$\bar K_s = \frac{4(\bar B_1 + \bar D_1)}{(1 - H_1)(\bar A_1 + \bar C_1)} + \frac{3}{4}\left(\frac{1}{m_1} - \lambda_1\right) + \frac{3}{2}\,\frac{1 + \lambda_1}{1 - H_1} - \frac{3}{4}\left(1 + \frac{\lambda_1}{m_1}\right), \qquad (58)$$

$$\bar T_s = m_1^{\frac{3}{4}}\left[\frac{4(\bar B_1 - \bar D_1)}{(1 - H_1)(\bar A_1 + \bar C_1)} + \frac{3}{4}\left(\frac{1}{m_1} - \lambda_1\right) - \frac{3}{2}\,\frac{1 + \lambda_1}{1 - H_1} + \frac{3}{4}\left(1 + \frac{\lambda_1}{m_1}\right) \right] ,$$

$$q_s = Z_s\left(\frac{11}{2} - \frac{\bar K_s + \bar T_s}{1 - \lambda_1/(m_1)^{\frac{1}{4}}}\right) - \frac{7}{4}M_o(2G_o - N_o) ,$$

$$r_s = \frac{3}{8}M_o\left[N_o(\bar B_o - 1) - \bar J_o \right] .$$

5 MATCHING OF THE STRONG AND WEAK INTERACTIONS

In the case for which a strong pressure interaction exists over the whole wedge surface the formulae (57) for the stability derivatives caused by the deformation of the effective wedge should be used together with that for the effective wedge oscillating in inviscid flow to give complete formulae for the stability derivatives of the original wedge in viscous flow; whereas in the case for which a weak pressure interaction exists over the whole wedge surface the formulae (49) should used instead of (57).

However there are cases for which a strong interaction exists in a region near the apex of the wedge and a weak interaction exists in a region near the trailing edge of the wedge. In these cases, there must be an intermediate region between the two for which neither the strong interaction theory nor the weak interaction theory can be applied. The author knows no theory which can be applied for this intermediate region. The two regions are thus assumed to extend and match at some intermediate point. Physically, the real flow field over the wedge surface has a continuous pressure distribution, continuous displacement thickness, etc. Unfortunately, it is impossible to choose a matching point such that all these conditions are satisfied. Therefore it is an artificial method to choose a matching point, and the flow field is distorted to some extent by any choice.

Orlik-Ruckemann[17] matched the two regions at the point (denoted here by x_p) where the pressures calculated by the two asymptotic theories of the strong interaction and the weak interaction are equal. It is preferred to match the two regions at the point x_m where the displacement thicknesses in the steady flow case calculated by the two theories are equal. Thus the formula for x_m is

$$x_m = \left(\frac{2d_o}{\sigma}\right)^4 \bar{\chi}_{o,1}^{2} \qquad (59)$$

One advantage by using x_m rather than x_p as the matching point is that for slender wedges at the point x_m the slope of the displacement boundary layer calculated by the weak interaction theory is less than θ_o, and that calculated by the strong interaction theory is larger than θ_o, and this justifies the use of two asymptotic theories for the two regions so constructed. (However, for thick wedges, for which the whole viscous effect is then small, the strong interaction region so constructed is overextended by a samll percentage of the length of the wedge, e.g. for $M_\infty = 17$, $\theta_o = 10^o$, it is overextended by 2.4% of the wedge length). Whereas at the point x_p the two slopes are generally either both larger than θ_o or both less than θ_o, and this makes it incorrect to use the asymptoticformulae of the weak or the strong interaction theories for their regions so constructed.

After determining the matching point position x_m, it is straight forward to obtain the following formulae for the stiffness derivative $(-c_{m_\theta})_m$ and the damping derivative $(-c_{m_{\dot\theta}})_m$ due to the deformation of the effective wedge in cases for which both weak and strong interaction exist over the wedge surface.

$$\begin{aligned}(-c_{m_\theta})_m &= \frac{4}{M_1}\left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2\Bigg[\frac{2Y_sZ_s(1+\lambda_1/(m_1)^{\frac{1}{4}})}{\cos^{7/4}\theta_1(1-\lambda_1/(m_1)^{\frac{1}{4}})}\left(\frac{3}{7}X_m^{\frac{7}{4}} - h\cos^2\theta_1 X_m^{\frac{3}{4}}\right) \\ &\quad + \frac{2Y_wZ_w(1+\lambda_1/(m_1)^{\frac{1}{4}})}{\cos^{3/2}\theta_1(1-\lambda_1/(m_1)^{\frac{1}{4}})^{\frac{1}{2}}}\left\{\tfrac{1}{3}(1-X_m^{\frac{3}{2}}) - h\cos^2\theta_1(1-X_m^{\frac{1}{2}})\right\}\Bigg],\end{aligned} \qquad (60)$$

$$\begin{aligned}(-c_{m_{\dot\theta}})_m &= \frac{4}{M_1}\left(\frac{\rho_1}{\rho_\infty}\right)\left(\frac{u_1}{U_\infty}\right)^2\Bigg[\frac{2Y_s}{\cos^{11/4}\theta_1}\left\{\tfrac{2}{3}S_r(h\cos^2\theta)^2X_m^{\frac{3}{4}} - \tfrac{2}{7}(S_r+S_q)h\cos^2\theta X_m^{\frac{7}{4}}\right. \\ &\quad \left. + \tfrac{2}{11}S_qX_m^{\frac{11}{4}}\right\} + \frac{2Y_w}{\cos^{5/2}\theta_1}\left\{W_r(h\cos^2\theta_1)^2(1-X_m^{\frac{1}{2}}) - \tfrac{1}{3}(W_r+W_q)h\cos^2\theta(1-X_m^{\frac{5}{2}})\right. \\ &\quad \left. + \tfrac{1}{5}W_q(1-X_m^{5/2})\right\}\Bigg]\end{aligned}$$

where

$$X_m = x_m \cos \theta_1 . \qquad (61)$$

Obviously, formuale (60) can be used only for the values of X_m lying between zero and 1.0, and include both formulae (49) and (57) as special cases when X_m is equal to zero and 1.0, respectively.

6 NUMERICAL EXAMPLES AND DISCUSSIONS

Numerical examples are given in Figs.23 to 30. All calculations were made for an insulated body, and for the Reynolds number $R_{e1,o} = 10^6$, and for a gas with $P_r = 0.725$, $\Omega = 0.76$ and $\gamma = 1.4$. except in Figs.29,30 for which $\gamma = 1.667$ and $\Omega = 0.647$.

It is seen that for the free stream Mach number up to 17, the viscous effect is important for a thin wedge, but negligible for a relatively thicker wedge, say the semi-vertex angle $\theta_o = 15^o$.

The viscosity of the gas is seen to play two roles: first, it thickens the wedge by a semi-vertex angle equal to the average inclination of the displacement boundary layer; secondly, it makes this effective wedge deformable. For a very thin wedge the second role is dominant; whereas for a relatively thicker wedge, both effect have the same order of magnitude, and with increasing wedge thickness, the first role becomes dominant.

For the stiffness derivatives, these two roles produce opposite effects thus,the first role only would overestimate the magnitude of the viscous effect, while the second role cancels part of this effect. For the damping derivatives the first role is to stablize the wedge, whereas the second role is to destablize the wedge for forward pivot positions and stablize it for rearward pivot positions. The combined effect is roughly the same as that of the second role.

Although the overall effect of viscouity is to decrease the damping derivatives for forward pivot positions, for $R_{e1,o} = 10^6$, no negative values of the damping derivatives are obtained due to the viscous effect. With decreasing Reynolds number, the damping derivative may become negative for a slender wedge

The viscous effect depends, strongly on the flow Mach number M_o behind the bow shock which represents the combined effect of the free stream Mach Mach number M_∞ and the semi-vertex angle θ_o. The viscous effect depends also upon the Reynolds number. Thus decreasing the value of the Reynolds number by a factor of one half increases roughly the viscous effect by about 40% for cases of weak interaction, and by about 20% for cases of strong interaction. The viscous effect tends to zero as the Reynolds number tends to infinity. The value of C'_{orig} is larger for a very cold body than for an insulated body, but the value of d_{orig} is much smaller for a very cold body than for an insulated body. However, $\bar{X}$ is smaller for a very cold body than for an insulated body and therefore the viscous effect for a very cold body is smaller than that for an insulated body. All these conclusions can be easily seen from the general formulae (49), (57) and (60).

For a very high Mach number, as seen from Figs.27 and 28 the effect of reflected waves coming from the bow shock is important even for very thin wedges, and can no longer be neglected for hypersonic flow. This is so, because the wedge is now thickened by the viscous layer. The effect of wave reflection consists of two parts: the inviscid part and the viscous part. Numerical results show that for $M_\infty = 17$ and θ_o up to 10° the inviscid wave reflection has negligible contribution. (about 2% of the overall effect of wave reflection)

In Figs.29 and 30 are plotted comparisons between the results from the present theory and the Orlik-Ruckemann's theory[17] for $M_\infty = 17$, $\theta_o = 3^\circ$, $\gamma = 5/3$, and $\Omega = 0.647$. Very good agreements are obtained except for the stiffness derivatives for pivot positions near the nose of the wedge.

A step-by-step comarison between the two theories is difficult because the theoretical models used are different. In Ref.17, the piston theory is used, and the viscous effect is divided into a static viscous pressure interaction and a dynamic pressure interaction, which is related to the effective change of the flow deflection angle and the normal velocity, respectively, of the wedge surface. Whereas the present theory is developed using a small perturbation method and the viscous effect is shown to thicken the wedge to an effective one and to make this effective wedge deformable. There are also several minor points which are different in the two theories, e.g. the Reynolds numbers used, the difinitions of the frequency parameter, the positions of the matching point of the weak and the strong interaction regions, and some empirical coefficients used in Ref.17,etc. The slightly different results shown in these figures might be attributed to these mionor points. However some more discussions are useful.

Within the small perturbation theory, which is a basic assumption common to both theories, the present theory is exact with regard to the thickness effect and the reflected wave effect. But in Orlik-Ruckemann's theory the effect of viscous wave reflection is neglected, though that of the inviscid part is included which is nevertheless negligible as shown by the present theory. Also in Ref.17 the thickness effect is not fully included as it is based on McIntoch's theory(see Chapter II). For the case plotted in these figures the wedge is thin, but it is now thickened by the displacement boundary layer to more than double its genmetrical thickness. As seen from Fig.28 the effect of the wave reflection is to decrease the damping derivative, but that of the thickness is to increase it, they tend to cancel each other and as a result, Orlik-Ruckemann's theory agrees very well with the present theory. (Fig.30) Also from Figs 27, the effect of the wave reflection is to decrease in absolute value the stiffness derivative, while the effect of the thickness is to increase it in absolute value. These two effects also tend to cancel to some extent (Fig.29). It is therefore concluded that Orlik-Ruchemann's theory is a special case in the present theory when the wedge is very thin.

In both theories there is a singularity at the wedge apex. Strictly speaking, neither of these two theories can be applied to any region near this singular point. Also the present theory by assuming the attachment of the bow shock to the wedge apex, can only be applied to cases for which the wedge is sharp and the bluntness at the wedge apex due to viscous effect is very small compared with the wedge length. For a preliminary estimate of these effects of bluntness, Ref. 5 should be referred to.

CHAPTER V

STABILITY OF NONWEILER WINGS [43,44]

1 INTRODUCTION

In recent years, several methods[30-34] have been suggested for designing wing shapes of hypersonic lifting vehicles to support known inviscid two-dimensional and conical flow fields. Among these, the caret wing proposed by Nonweiler[30] is the simplest one which, at design condition, generates a uniform two-dimensional flow on its lower surface and a plane shock attached to the swept leading edges. The upper surface is designed to be either parallel or at some negative incidence to the free stream and hence will, particularly at hypersonic or high supersonic speeds, contribute a very small amount to the pressure on the body which will therefore be neglected for the purpose of stability analyses. Since at the design conditions the upper and lower surface flows on caret wings are independent there is no need to define the upper surface shapes and in this thesis the caret wing will be simply represented by its lower surface.

Theoretical work has been reported by Collingbourne and Peckam[35] and Cooke[36], among others, investigating the effects of several variables on the aerodynamic efficiency of caret wings. Whereas experimental work has been reported concerning the overall force[37], pressure distribution and flow visualisation[38,39], and heat transfer rates[40].

As ther is no restriction with a caret wing on its aspect ratio in achieving the attached-shock condition, Peckham[41] pointed out that an aspect ratio could be chosen which gave adequate lift and longitudinal as well as lateral stability at low speeds. Apart from this comment, however, no theoretical or experimental work exists concerning the longitudinal stability of caret wings (Bagley[42] has given some preliminary theoretical estimates of the lateral forces and moment on yawed caret wings).

It is the purpose of this Chapter to develop an exact theory for the stability of pitching caret wings at design conditions in hypersonic flow. By extending the two-dimentional perturbation theory in Chapter III for oscillating wedges, exact formulae for the stability derivatives of caret wings at design conditions in hypersonic flow are obtained which are valid for any incidence of the lower ridge of the caret wings.

It will be shown that the stability derivatives of a caret wing at its design condition are independent of its aspect ratio but dependent only on the flight Mach number M_∞ and the incidence θ of its lower ridge. For a given value of M_∞ the damping derivatives are shown to become negative for some positions of the pitching axes several degrees before θ is increased* to the shock detachment angle. A general criterion for the stability of caret wings is obtained and numberical results included.

It is also shwon in this Chapter that for a small departure of the caret wing from its design condition, the flow field below the wing remains, within the linearized theory, two-dimensional for a steady disturbance, and the cross flow is caused by the unsteadiness of the disturbance. This cross flow is seen to greatly increase the damping derivative for forward pivot positions and to decrease the damping derivatives for rearward pivot positions.

---

*It should be noted that in this Chapter, whenever we compare cases of different values of θ for a given value of M_∞, we are comparing different caret wings at their design conditions and not the same caret wing at different incidences.

2 FORMULATION OF THE PROBLEM

2.1 GEOMETRY

Fig.31 shows a caret wing at design condition of delta planform and of length $\bar{\ell}$ and span $2\bar{s}$ whose lower ridge and the plane of leading edges are at incidences θ and β to the free stream direction, respectively, measured in the central plane of the wing. The base of the wing is chosen to be perpendicular to the free stream direction in all cases, but there is no difficulty in applying the present theory to other cases for which the base of the wing is not perpendicular to the free stream direction.

Let the right-hand system of co-ordinates $O\bar{x}\bar{y}\bar{z}$ be such that O is at the apex of the wing, $O\bar{x}$ along its lower ridge, $O\bar{y}$ lies in the central plane of the wing and is positive downwards, when the wing is at its design condition. The length $\bar{\ell}$ of the wing is again taken as a unit length to non-dimensionalize all the length quantities, e.g.

$$s = \frac{\bar{s}}{\bar{\ell}}, \quad x = \frac{\bar{x}}{\bar{\ell}}, \quad \text{etc.} \tag{1}$$

For any oscillation of the wing about an axis parallel to Oz, the flow field must be symmetric with respect to the central plane Oxy, we need therefore only consider the flow field for which $z \leq 0$. When the caret wing is flying steadily at its design condition, the equation of the plane of leading edges is

$$y - x\tan\varphi = 0, \tag{2}$$

where

$$\varphi = \beta - \theta \tag{3}$$

whereas the equation of the wing surface is

$$y + z\cot\Gamma = 0 \tag{4}$$

where Γ is the angle between the central plane and the wing surface, and can be determined from the following equation

$$\tan\Gamma = s\cos\beta/\sin\varphi \tag{5}$$

Once any set of three independent parameters is given, all the other geometrical quantities can be determined. These independent parameters may be, say, θ, φ and Γ. For practical calculations it is convenient to use the free stream Mach number M_∞, incidence β of the plane of leading edges and the aspect ratio s, since then all the other quantities can be expressed explicitly in terms of these three parameters.

2.2 THE PERTURBATION EQUATIONS

To obtain the stability derivatives of a pitching caret wing, it is necessary to know the flow field on its lower surface This is done by solving the equations of motion by the perturbation method developed in Chapter III.

Let $\bar{t}$ be the time variable, $\bar{u}(\bar{x},\bar{y},\bar{z},\bar{t})$, $\bar{v}(\bar{x},\bar{y},\bar{z},\bar{t})$ and $\bar{w}(\bar{x},\bar{y},\bar{z},\bar{t})$ the velocity components in the $\bar{x},\bar{y},\bar{z}$, directions, respectively. Denote by $\bar{p}(\bar{x},\bar{y},\bar{z},\bar{t})$ and $\bar{\rho}(\bar{x},\bar{y},\bar{z},\bar{t})$, the pressure and density of the fluid, respectively. Then the basic equations for the motion of the fluid between the shock and the lower surface of the caret wing are

$$\frac{\partial\bar{u}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}} + \bar{w}\frac{\partial\bar{u}}{\partial\bar{z}} = -\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{x}},$$

$$\frac{\partial\bar{v}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{v}}{\partial\bar{y}} + \bar{w}\frac{\partial\bar{v}}{\partial\bar{z}} = -\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{y}},$$

$$\frac{\partial\bar{w}}{\partial\bar{t}} + \bar{u}\frac{\partial\bar{w}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{w}}{\partial\bar{y}} + \bar{w}\frac{\partial\bar{w}}{\partial\bar{z}} = -\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial\bar{z}}, \qquad (6)$$

$$\frac{\partial(\bar{p}/\bar{\rho}^\gamma)}{\partial\bar{t}} + \bar{u}\frac{\partial(\bar{p}/\bar{\rho}^\gamma)}{\partial\bar{x}} + \bar{v}\frac{\partial(\bar{p}/\bar{\rho}^\gamma)}{\partial\bar{y}} + \bar{w}\frac{\partial(\bar{p}/\bar{\rho}^\gamma)}{\partial\bar{z}} = 0,$$

$$\frac{\partial\bar{\rho}}{\partial\bar{t}} + \frac{\partial(\bar{\rho}\bar{u})}{\partial\bar{x}} + \frac{\partial(\bar{\rho}\bar{v})}{\partial\bar{y}} + \frac{\partial(\bar{\rho}\bar{w})}{\partial\bar{z}} = 0.$$

Denote by u_o, v_o, w_o, p_o and ρ_o, respectively, the velocity components, the pressure, and the density of the gas in the steady flow when the wing is at its design condition. Obviously, $v_o = w_o = 0$, under the system of co-ordinates chosen. For a small oscillation of the body the flow quantities may be expressed as

$$
\begin{aligned}
\bar{u} &= u_o + \epsilon u + \dots , \\
\bar{v} &= \epsilon v + \dots , \\
\bar{w} &= \epsilon w + \dots , \\
\bar{p} &= p_o + \epsilon p + \dots , \\
\bar{\rho} &= \rho_o + \epsilon \rho + \dots .
\end{aligned}
\tag{7}
$$

where ϵ is a small quantity which characterises the deviation of the flow field from that for the design condition and for which the maximum angular displacement of the oscillation of the wing will be taken in this Chapter.

Putting (7) into (6) and upon non-dimensionalising the independent variables, we obtain

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} &= -\frac{1}{\rho_o u_o} \frac{\partial p}{\partial u} , \\
\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} &= -\frac{1}{\rho_o u_o} \frac{\partial p}{\partial y} , \\
\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} &= -\frac{1}{\rho_o u_o} \frac{\partial p}{\partial z} , \\
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} &= a_o^2 \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\right) , \\
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} + \frac{\rho_o a_o^2}{u_o} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) &= 0.
\end{aligned}
\tag{8}
$$

in which a_o is the speed of sound in the steady flow at design condition, and the non-dimensionalized time variable t is defined by

$$
t = \bar{t} \, u_o / \ell \tag{9}
$$

Making a transformation

$$\begin{aligned} \xi &= x + t, \\ \eta &= x - t, \\ \zeta &= 2y \\ \chi &= 2z \end{aligned} \qquad (10)$$

we obtain from **Eqs.(8)**

$$\frac{\partial u}{\partial \xi} = -\frac{1}{2\rho_o u_o}\left(\frac{\partial p}{\partial \xi} + \frac{\partial p}{\partial \eta}\right),$$

$$\frac{\partial v}{\partial \xi} = -\frac{1}{\rho_o u_o}\frac{\partial p}{\partial \zeta},$$

$$\frac{\partial w}{\partial \xi} = -\frac{1}{\rho_o u_o}\frac{\partial p}{\partial \chi}, \qquad (11)$$

$$\frac{\partial p}{\partial \xi} = a_o^2 \frac{\partial \rho}{\partial \xi},$$

$$\frac{\partial \rho}{\partial \xi} + \frac{a_o^2 \rho_o}{u_o}\left[\frac{1}{2}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial v}{\partial \zeta} + \frac{\partial w}{\partial \chi}\right] = 0.$$

$$\therefore \quad \frac{u_o^2}{a_o^2}\frac{\partial^2 p}{\partial \xi^2} = -\rho_o u_o\left[\frac{1}{2}\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}\right) + \frac{\partial^2 v}{\partial \xi \partial \zeta} + \frac{\partial^2 w}{\partial \xi \partial \chi}\right]$$

$$= \frac{1}{4}\left[\left(\frac{\partial^2 p}{\partial \xi^2} + \frac{\partial^2 p}{\partial \xi \partial \eta}\right) + \left(\frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{\partial^2 p}{\partial \eta^2}\right)\right] + \frac{\partial^2 p}{\partial \zeta^2} + \frac{\partial^2 p}{\partial \chi^2},$$

$$\therefore \quad M_o^2 \frac{\partial^2 p}{\partial \xi^2} - \frac{\partial^2 p}{\partial \zeta^2} - \frac{\partial^2 p}{\partial \chi^2} = \frac{1}{4}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 p,$$

where M_o is the Mach number behind the shock in the steady flow at the design condition. Hence equations (8) become

$$(M_o^2 - 1)\frac{\partial^2 p}{\partial x^2} + 2M_o^2 \frac{\partial^2 p}{\partial x \partial t} + M_o^2 \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 p}{\partial z^2} = 0 ,$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = - \frac{1}{\rho_o u_o}\frac{\partial p}{\partial x} ,$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = - \frac{1}{\rho_o u_o}\frac{\partial p}{\partial y} , \qquad (12)$$

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = - \frac{1}{\rho_o u_o}\frac{\partial p}{\partial z} ,$$

$$a_o^2\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \right) = \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} ,$$

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} + \frac{\rho_o a_o^2}{u_o}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 .$$

In the perturbation equations (12), there are only five independent equations but the first one is useful for what follows.

The flow field can now be found by successively solving the first equation of (12) for p, the second for u, the third for v, the fourth for w, and the fifth for ρ.

For a periodic motion of the body, the resulting flow field, as observed from a fixed point, must also be periodic with the same period as the body, because the perturbation equations (12) and the boundary conditions (see Section 2.3) are linear. As shown in Chapter II, we need only study the flow field resulting from a harmonic motion of the body with a given circular frequency ω. Hence the flow quantities may be expressed as

$$\begin{aligned} u &= u_o\, e^{ikt}\, U(x, y, z) , \\ v &= u_o\, e^{ikt}\, V(x, y, z) , \\ w &= u_o\, e^{ikt}\, W(x, y, z) , \qquad (13) \\ p &= \rho_o a_o u_o\, e^{ikt}\, P(x, y, z) , \\ \rho &= \frac{\rho_o u_o}{a_o}\, e^{ikt}\, R(x, y, z) , \end{aligned}$$

in which, again, the non-dimensional parameter k is given by

$$k = \omega \bar{\ell}/u_o. \tag{14}$$

Thus k is seen to be the frequency parameter based on the length $\bar{\ell}$ of the wing and the flow speed u_o of the gas in the steady flow below the surface at the design condition. k is generally very small, particularly at hypersonic and high supersonic speeds.

Putting (13) into (12), we obtain

$$\begin{aligned}
&(M_o^2 - 1) \frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} = - ik2M_o^2 \frac{\partial P}{\partial x} - (ik)^2 M_o^2 P, \\
&\frac{\partial U}{\partial x} = - \frac{1}{M_o} \frac{\partial P}{\partial x} - ikU, \\
&\frac{\partial V}{\partial x} = - \frac{1}{M_o} \frac{\partial P}{\partial y} - ikV, \\
&\frac{\partial W}{\partial x} = - \frac{1}{M_o} \frac{\partial P}{\partial z} - ikW \\
&\frac{\partial R}{\partial x} = \frac{\partial P}{\partial x} - ik(R - P), \\
&\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = -M_o \frac{\partial P}{\partial x} - ikM_o P.
\end{aligned} \tag{15}$$

Expanding the time-independent unknown quantities as a power series in ik

$$\begin{aligned}
U &= U^{(o)} + ik\, U^{(1)} + (ik)^2 U^{(2)} + \dots, \\
V &= V^{(o)} + ik\, V^{(1)} + (ik)^2 V^{(2)} + \dots, \\
W &= W^{(o)} + ik\, W^{(1)} + (ik)^2 W^{(2)} + \dots, \\
P &= P^{(o)} + ik\, P^{(1)} + (ik)^2 P^{(2)} + \dots, \\
R &= R^{(o)} + ik\, R^{(1)} + (ik)^2 R^{(2)} + \dots,
\end{aligned} \tag{16}$$

and putting these expressions into (14) we obtain a sequence of systems of partial differential equations by equating terms of the same power in ik on both sides of each equation in (15). However, for stability analysis, only the systems of the zeroth order and the first order equations are required. These are:

$$
\begin{aligned}
&(M_o{}^2 - 1)\frac{\partial^2 P^{(o)}}{\partial x^2} - \frac{\partial^2 P^{(o)}}{\partial y^2} - \frac{\partial^2 P^{(o)}}{\partial z^2} = 0, \\
&\frac{\partial U^{(o)}}{\partial x} = -\frac{1}{M_o}\frac{\partial P^{(o)}}{\partial x}, \\
&\frac{\partial V^{(o)}}{\partial x} = -\frac{1}{M_o}\frac{\partial P^{(o)}}{\partial y}, \\
&\frac{\partial W^{(o)}}{\partial x} = -\frac{1}{M_o}\frac{\partial P^{(o)}}{\partial z}, \\
&\frac{\partial R^{(o)}}{\partial x} = \frac{\partial P^{(o)}}{\partial x}, \\
&\frac{\partial P^{(o)}}{\partial x} + \frac{\kappa^2}{M_o}\left(\frac{\partial V^{(o)}}{\partial y} + \frac{\partial W^{(o)}}{\partial z}\right) = 0,
\end{aligned}
\qquad (17_o)
$$

and,

$$
\begin{aligned}
&(M_o{}^2 - 1)\frac{\partial^2 P^{(1)}}{\partial x^2} - \frac{\partial^2 P^{(1)}}{\partial y^2} - \frac{\partial^2 P^{(1)}}{\partial z^2} = -2M_o{}^2\frac{\partial P^{(o)}}{\partial x}, \\
&\frac{\partial U^{(1)}}{\partial x} = -\frac{1}{M_o}\frac{\partial P^{(1)}}{\partial x} - U^{(o)}, \\
&\frac{\partial V^{(1)}}{\partial x} = -\frac{1}{M_o}\frac{\partial P^{(1)}}{\partial y} - V^{(o)}, \\
&\frac{\partial W^{(1)}}{\partial x} = -\frac{1}{M_o}\frac{\partial P^{(1)}}{\partial z} - W^{(o)}, \\
&\frac{\partial R^{(1)}}{\partial x} = \frac{\partial P^{(1)}}{\partial x} + (P^{(o)} - R^{(o)}), \\
&\frac{\partial P^{(1)}}{\partial x} + \frac{\kappa^2}{M_o}\left(\frac{\partial V^{(1)}}{\partial y} + \frac{\partial W^{(1)}}{\partial z}\right) = 0,
\end{aligned}
\qquad (17_1)
$$

where

$$\kappa = M_o/(M_o^{\,2} - 1)^{\frac{1}{2}}, \tag{18}$$

and

$$e_o = \kappa^2(U^{(o)}/M_o - P^{(o)}). \tag{19}$$

2.3 BOUNDARY CONDITIONS

1). At $z = 0$, $W = 0$, (20)

which arises from the symmetry of the flow field with respect to the xy plane.

2). At the wing surface

For a pitching caret wing, the wing surface at time t is given by

$$f_b(x, y, z, t) \equiv \epsilon(t)\,(x - x_o) - y - z \cot\Gamma = 0, \tag{21}$$

with $\epsilon(t) \equiv \epsilon\, e^{ikt}$ and ϵ is the maximum amplitude of the oscillation of the wing. The axis of pitching is parallel to z axis and intersecting the xy plane at a point $P_o(x_o, y_o)$.

The condition to be satisfied at the wing surface is that the relative normal velocity is equal to zero, i.e.

$$u_o \frac{\partial f_b}{\partial t} + \bar{u}\frac{\partial f_b}{\partial x} + \bar{v}\frac{\partial f_b}{\partial y} + \bar{w}\frac{\partial f_b}{\partial z} = 0$$

Thus, the linearized boundary condition is,

$$\text{at} \quad y + z\cot\Gamma = 0, \qquad V + W\cot\Gamma = 1 + ik(x - x_o). \tag{22}$$

3). Across the shock wave

Let the equation of the bow shock at time t be

$$G(x, y, z, t) \equiv \epsilon(t)\,Q(x, z) + x\tan\varphi - y = 0, \tag{23}$$

with $Q(x, z)$ unknown, to be determined as part of the solution. Then the conditions to be satisfied at the bow shock are

$$\text{Continuity:} \qquad \left[\rho\frac{\partial G}{\partial t} + \vec{v}\cdot\nabla G\right]_{(\infty)}^{(s)} = 0, \tag{24}$$

momentum: $$\left[\vec{v} \cdot \vec{t}_1 \right]_{(\infty)}^{(s)} = 0, \tag{25}$$

$$\left[\vec{v} \cdot \vec{t}_2 \right]_{(\infty)}^{(s)} = 0, \tag{26}$$

$$\left[\rho\left(\frac{\partial G}{\partial t} + \vec{v} \cdot \nabla G\right)^2 + (\nabla G)^2 p \right]_{(\infty)}^{(s)} = 0, \tag{27}$$

Energy: $$\left[\tfrac{1}{2}\left(\frac{\partial G}{\partial t} + \vec{v} \cdot \nabla G\right)^2 + (\nabla G)^2 h \right]_{(\infty)}^{(s)} = 0, \tag{28}$$

where h is the specific enthalpy of the gas, $\vec{t}_1$ and $\vec{t}_2$ are two different vectors on the surface $G = 0$ for which $\vec{t}_1$ is taken to be parallel to the xy plane.

Now the components of the vectors in (24) to (28) can be found to be

	x-component	y-component	z-component
$\vec{v}$	$u_o(1 + \epsilon(t)U)$	$u_o\epsilon(t)V$	$u_o\epsilon(t)W$,
$\vec{v}_\infty$	$U_\infty \cos\theta$	$-U_\infty \sin\theta$	0 ,
∇G	$\tan\varphi + \epsilon(t)Q_x$	-1	$\epsilon(t)Q_z$,
$\vec{t}_1$	1	$\tan\varphi + \epsilon(t)Q_x$	0 ,
$\vec{t}_2$	$-\epsilon(t)Q_z \tan\varphi$	$\epsilon(t)Q_z$	$\sec^2\varphi + \epsilon(t)(Q_x+Q_z)\tan\varphi$,

where U_∞ is the flow speed in the free stream. Although ∇G is different from that for the two-dimensional flow past a wedge with semi-vertex angle θ, within the linear perturbation theory, $(\nabla G)^2$, $\vec{v} \cdot \nabla G$, and $\vec{v} \cdot \vec{t}_1$ remain the same in form as those for the two-dimensional flow except that in the present case z appears in Q as a parameter. Because of this, the boundary conditions across the bow shock which are derived from (24), (25), (27), and (28) should remain the same as those for the two-dimensional flow except that for the present case z is included in the function Q as a parameter. These boundary conditions can be written down from Chapters I and III as follows:

at $y = x \tan \varphi$,

$$\begin{aligned}
V &= \bar{A}\, Q_x(x, z) + ik\, \bar{B}\, Q(x, z), \\
P &= \bar{C}\, Q_x(x, z) + ik\, \bar{D}\, Q(x, z), \\
U &= \bar{E}\, Q_x(x, z) + ik\, \bar{F}\, Q(x, z), \\
R &= \bar{G}\, Q_x(x, z) + ik\, \bar{J}\, Q(x, z),
\end{aligned} \tag{29}$$

where the constants $\bar{A}$ through $\bar{J}$ are defined in (III.4). Whereas from (26), the fifth boundary condition across the bow shock is

at $y = x \tan \varphi$,

$$W = \bar{K}\, Q_z(x, z) \tag{30}$$

where

$$\bar{K} = -\sin \varphi \cos \varphi \left(\frac{\rho_o}{\rho_\infty} - 1 \right) = -C \tag{31}$$

As in all the boundary conditions (20), (22), (29) and (30), the ordinate y_o does not appear, for pivot axes parallel to z-axis with the same value of x_o but different values of y_o, the pressure distributions should be the same. Further, the perturbation force vector $\vec{p}$ is

$$\vec{p} = \varepsilon(t)\, \rho_o a_o u_o P(\, \varepsilon(t)\sin\Gamma,\ -\sin\Gamma,\ \cos\Gamma\,).$$

The z-component of $\vec{p}$ has no contribution to the moment about an axis parallel to z direction, and the x-component of $\vec{p}$ is of smaller order than the y-component and should be neglected. Hence in the formula for the oment of perturbation force the parameter y_o disappears. It is therefore concluded that the stability derivatives of a caret wing for pivot axes parallel to z-direction with the same value of x_o should remain the same for different values of y_o. This conclusion is very important for experimental performance, as the axis of pi hing normal to the ridge line may be fixed wherever is convenient.

Letting

$$Q(x, z) = Q^{(o)}(x, z) + ik\, Q^{(1)}(x, z) + (ik)^2 Q^{(2)}(x, z) + \ldots \quad (32)$$

we obtain the following boundary conditiins for the zeroth and the first order solutions.

$$\begin{aligned}
z = 0, \quad & W^{(o)} = 0, \\
y + z\cot\Gamma = 0, \quad & V^{(o)} + W^{(o)}\cot\Gamma = 1, \\
y - x\tan\varphi = 0, \quad & V^{(o)} = \bar{A}\, Q_x^{(o)}(x, z), \\
& P^{(o)} = \bar{C}\, Q_x^{(o)}(x, z), \\
& U^{(o)} = \bar{E}\, Q_x^{(o)}(x, z), \\
& R^{(o)} = \bar{G}\, Q_x^{(o)}(x, z), \\
& W^{(o)} = \bar{K}\, Q_z^{(o)}(x, z).
\end{aligned} \quad (33_o)$$

$$\begin{aligned}
\text{and,}\quad z = 0 \quad & W^{(1)} = 0, \\
y + z\cot\Gamma = 0 \quad & V^{(1)} + W^{(1)}\cot\Gamma = x - x_o, \\
y - x\tan\varphi = 0 \quad & V^{(1)} = \bar{A}\, Q_x^{(1)}(x, z) + \bar{B}\, Q^{(o)}(x, z), \\
& P^{(1)} = \bar{C}\, Q_x^{(1)}(x, z) + \bar{D}\, Q^{(o)}(x, z),
\end{aligned} \quad (33_1)$$

$$U^{(1)} = \bar{E}\, Q_x^{(1)}(x, z) + \bar{F}\, Q^{(o)}(x, z),$$
$$R^{(1)} = \bar{G}\, Q_x^{(1)}(x, z) + \bar{J}\, Q^{(o)}(x, z),$$
$$W^{(1)} = \bar{K}\, Q_z^{(1)}(x, z).$$

In order to find the aerodynamic derivatives of a caret wing the equations (15) should be solved to satisfy the boundary conditions (20), (22), (29) and (30). In this thesis we restrict ourselves to finding only the stiffness and the damping derivatives. For this purpose equations (17_o) and (17_1) should be successively solved to satisfy the boundary conditions (33_o) and (33_1), respectively.

3 THE STIFFNESS DERIVATIVES

In order to find the stiffness derivative of a pitching caret wing, we need only solve equations (17_o) to satisfy the boundary conditions (36_o). This can be easily found to be a constant flow:

$$V^{(o)} = 1, \quad P^{(o)} = \frac{\bar{C}}{\bar{A}} \equiv E, \quad U^{(o)} = \frac{\bar{E}}{\bar{A}}, \quad R^{(o)} = \frac{\bar{G}}{\bar{A}}, \quad W^{(o)} = 0, \tag{34}$$

also,

$$Q_x^{(o)} = \frac{1}{\bar{A}}, \quad Q_z^{(o)} = 0,$$

hence

$$Q^{(o)}(x, z) = \frac{x}{\bar{A}} - x_o, \tag{35}$$

in which the constant of integration is determined from an additional condition that the bow shock is always attached to the apex of the caret wing.

Since $W^{(o)} = 0$, and all the zeroth order flow quantities are independent of z, it is concluded that the steady flow field below the lower surface of the wing due to a small steady departure of the wing from its design condition is (within the linearized theory) still a two-dimensional flow. The three-dimensional effect can only arise from the unsteadiness of the disturbance.

As seen from equation (35), the shock wave resulting from an in-phase (steady) disturbance remains a plane attached to the apex of the caret wing and parallel to the z-direction. Since $\bar{A}$ is generally smaller than unity, the shock will be, according to (35), detached from the leading edges of the wing for an increase of incidence of its lower ridge, and vice versa.

Having obtained the expression for $P^{(o)}$, the stiffness derivative can be easily obtained by performing the integration for the moment of perturbation pressure $\epsilon(t)\gamma M_o p_o P^{(o)}$ about the pivot axis over the wing surface. In doing so the pressure acting on the upper surface is assumed to be equal to zero and thus has no contribution.

The moment of pressure is obviously twice that contributed from that half of the wing for which $z \leq 0$, because of the symmetry of the flow field with respect to the xy plane.

$$\therefore \quad M = 2\int_{S\omega}\int \epsilon(t)\gamma M_o p_o\left[P^{(o)} + ik\ P^{(1)}\right] \sin\Gamma\ (\bar{x} - \bar{x}_o)\,ds$$

$$\doteq 2\int_D\int \epsilon(t)\gamma M_o p_o\left[P^{(o)} + ik\ P^{(1)}\right] \tan\Gamma\ (\bar{x} - \bar{x}_o)\,d\bar{x}d\bar{y}, \qquad (36)$$

where $S\omega$ is the surface of half a wing and D its projection on the xy plane.

Now the pitching moment coefficient c_m is

$$c_m \equiv \frac{M}{\frac{1}{2}\rho_\infty U_\infty^{\ 2}\tau} \equiv \epsilon(t)\left[-c_{m_\theta} + ik\ (-c_{m_{\dot\theta}})\right], \qquad (37)$$

where τ is some characteristic quantity proportional to the volume of the caret wing. For simplicity, we take

$$\tau = \bar{\ell}\cdot\bar{S} \equiv \bar{\ell}^2\bar{s}, \qquad (38)$$

where $\bar{S}$ is the plan area of the caret wing. Then the stiffness derivative $-c_{m_\theta}$ of a caret wing at its design condition is

$$-c_{m_\theta} = \frac{4}{\rho_\infty U_\infty^{\ 2} s}\ \rho_o a_o u_o\ \tan\Gamma\ P^{(o)}\ .$$

$$\cdot\left[\int_0^{1/\cos\theta} dx\int_0^{x\tan\varphi}(x-x_o)\,dy + \int_{1/\cos\theta}^{\cos\theta/\cos\beta} dx\int_{(x-\frac{1}{\cos\theta})\cot\theta}^{x\tan\varphi}(x-x_o)\,dy\right],$$

$$\therefore \quad -c_{m_\theta} = \frac{4}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2 \frac{E}{2}\left[\tfrac{1}{3}(g + j) - x_o\right], \qquad (39)$$

in which

$$g = 1/\cos\theta, \qquad j = \cos\varphi/\cos\beta \qquad (40)$$

are quantities with an obvious geometric significance.

One conclusion to be drawn from formula (39) is that the stiffness derivative of a catet wing at its design condition is independent of its aspect ratio but dependent only on the flight Mach number and the incidence of its lower ridge.

4 THE DAMPING DERIVATIVES

In order to find the damping derivative of a caret wing, the first order equations (17_1) must be solved to satisfy the boundary conditions (33_1). This seems quite complicated, but a detailed examination of it shows that the solution to (17_1) is represented by linear functions of x, y, z. Thus putting

$$
\begin{aligned}
P^{(1)} &= A_1 x + B_1 y + C_1 z + D_1 x_o, \\
V^{(1)} &= A_2 x + B_2 y + C_2 z + D_2 x_o, \\
W^{(1)} &= A_3 x + B_3 y + C_3 z + D_3 x_o, \\
U^{(1)} &= A_4 x + B_4 y + C_4 z + D_4 x_o, \\
R^{(1)} &= A_5 x + B_5 y + C_5 z + D_5 x_o,
\end{aligned}
\tag{41}
$$

into (17_1) and (33_1), the constants A_1 through D_5 can be determined uniquely. For our interest, we write down the following results only

$$
\begin{aligned}
P^{(1)} &= A_1 x - 2M_o y + D_1 x_o, \\
V^{(1)} &= x + B_2 y - x_o, \\
W^{(1)} &= C_3 z, \\
Q^{(1)} &= \frac{A_1 - 2\kappa H - \bar{D}/\bar{A}}{2\bar{C}} x^2 + \frac{D_1 + 1}{\bar{C}} x + \frac{C_3}{2\bar{K}} z^2 + \text{Const.},
\end{aligned}
\tag{42}
$$

in which

$$
\begin{aligned}
A_1 &= e_o - \frac{2\kappa^2}{M_o} C_3 , \\
D_1 &= E(\bar{B} - 1) - \bar{D}, \\
C_3 = B_2 &= \frac{M_o}{2\kappa^2}\left[\frac{e_o + D_1/\bar{A} + E(\frac{1}{\bar{A}} - 1) - 2\kappa H}{1 + EH/2\kappa^2} \right],
\end{aligned}
\tag{43}
$$

The solution for $U^{(1)}$ and $R^{(1)}$ may, if desired, also be written down, here we just point out two interesting points that $C_4 = C_5 = 0$, and all the constants are independent of Γ.

The constant of integration in (42) may be found to be equal to zero by using the additional condition that the shock wave is always attached to the apex of the caret wing.

The whole flow field does not depend on the aspect ratio s (or Γ), though the latter does appear in the boundary conditions (33_1).

All the flow quantities except $W^{(1)}$ do not vary with z, therefore the flow field will be two-dimensional if and only if an additional condition that $C_3 = 0$ is satisfied. Generally this is not the case, and the cross flow does affect the pressure distribution. Therefore the flow field caused by an unsteady disturbance is essentially a three-dimensional one. Further, C_3 is always negative as shown by a series of numerical calculations, thus when the wing is oscillating to increase its incidence, the fluid will be flowing towards the central plane and vice versa. Also as seen from (43) the three-dimensional effect increases the pressure by an amount $-2\kappa^2 C_3 x/M_o$ which is proportional to x, and hence greatly stabilizes the body for forward pivot positions and destabilizes it for rearward pi ot positions. A numerical example is shown in Fig. 33.

The shape of the bow shock resulting from an out-of-phase disturbance is seen to be a surface which intersects at parabolas with planes either parallel to the central plane or perpendicular to the lower ridge of the caret wing. It is concave for an increase of incidence and convex for a decrease of incidence.

Having obtained the solution (42) for the pressure distribution $p^{(1)}$, the damping derivative $-c_{m_{\dot\theta}}$ of a caret wing at its design condition may be obtained from (37) combined with (36) as follows

$$-c_{m_{\dot\theta}} = \frac{4}{M_o}\left(\frac{\rho_o}{\rho_\infty}\right)\left(\frac{u_o}{U_\infty}\right)^2 \frac{\cos\beta}{\sin\varphi}\left[\, G_2 x_o^{\,2} + G_3 x_o + G_4 \,\right], \qquad (44)$$

where

$$G_2 = -\frac{D_1 \sin\varphi}{2\cos\theta\cos\beta},$$

$$G_3' = \frac{(D_1 - A_1)(j^2 - g^2)}{6\sin\theta} + \frac{M_o \sin^2\varphi}{3\cos^2\beta\cos\theta},$$

$$G_4 = \frac{A_1(j^3 - g^3)}{12\sin\theta} + M_o\left[\frac{\cot^2\theta\,(j^4 - g^4)}{4} - \frac{j^4\tan^2\varphi}{4} - \frac{3\cos\theta}{2\sin^2\theta}(j^3 - g^3) + \frac{(j^2 - g^2)}{2\sin^2\theta}\right] \tag{45}$$

It is interesting to notice that like the stiffness derivative, the damping derivative of a caret wing at its design condition is also independent of its aspect ratio and dependent only on the flight Mach number and the incidence of its lower ridge.

In using formula (44), care should be taken that due to the definition of the frequency parameter k, in order to obtain the value of the damping derivative in the usual sense, the value calculated from (44) should be multiplied by a factor U_∞/u_o. However, the author feels that the definition for $-c_{m_{\dot\theta}}$ given by (37) is the most natural one.

5 GENERAL CRITERION FOR STABILITY

The stability of a caret wing at its design condition may be investigated through a general criterion for the critical condition for which the damping derivative first becomes zero. Since the damping derivative $-c_{m_{\dot\theta}}$ is a quadratic function of x_o, the condition for which it first becomes zero is obviously that

$$G_3^2 - 4\,G_4 G_2 = 0 \qquad (46)$$

and the corresponding critical pivot position $x_{o_{cr}}$ is given by

$$x_{o_{cr}} = -\,G_3/2G_4 \qquad (47)$$

Equation (46) gives an explicit relation between the flight Mach number M_∞ and the bow shock angle β at the design condition. Unfortunately this relation is too complicated to solve and express any quantity explicitly in terms of the other. However the method of iteration may be used and the result is plotted as a curve of M_∞ vs θ (Fig.34). In this figure, for a given value of M_∞, different θ correspond to different caret wings at design condition and not to the same wing at different incidences.

As seen from Fig 34 for a fixed value of M_∞ there exists a maximum angle of attack θ_{cr} for which the damping derivative of the corresponding caret wing at its design condition first becomes zero for the pivot position at $x_{o_{cr}}$ given by (47). At hypersonic speeds this critical angle θ_{cr} is only a few degrees below the shock detachment angle (i.e. the flow deflection angle in two-dimensional flow case at which the bow shock begins to be detached from the body). On the other hand for a fixed value of incidence, there exists a minimum value of Mach number M_∞ for which the damping derivative of the corresponding caret wing at its design condition first becomes zero for pivot position at $x_{o_{cr}}$ given by (47).

Neutral damping boundaries for given caret wings may also be obtained from equation (46).

6 ILLUSTRATIVE EXAMPLES

Typical results for the stability derivatives of a caret wing versus the position of the pivot axis are plotted in Figs 32 to 35. In Fig. 35 as well as in Fig. 34, different θ correspond to different caret wings at design condition and not to the same wing at different incidences. In all these figures a gas with $\gamma = 1.4$ is used. For a given M_∞ and varying θ, the stiffness derivative of the corresponding caret wings at design condition increases with increasing θ, but after θ reaches some value depending on the flight Mach number, it rapidly decreases to very large negative value.

The three-dimensional effect is seen to greatly increase the damping derivative for forward pivot positions and to decrease it for rearward pivot positions.

7 DISCUSSIONS

The formulae for the stability derivatives of an oscillating caret wing obtained in Sections 4 and 5 are exact with regard to the incidence so far as that part of the moment of pressure contributed from the upper surface of the wing can be neglected. This may be the case for hypersonic and high supersonic speeds. For moderate or low supersonic speeds, these formulae should be modified to take into account the contribution to the moment of pressure from the upper surface. However, the present work is concerned primarily with the operation of the caret wing at the design conditiions, i.e. hypersonic or high supersonic speeds, and the contribution of the upper surface has been neglected.

As for the viscous effect, assuming, as it is likely to be, that for a caret wing it is of the same order of magnitude as that for a two-dimensional wedge-shaped wing, then it is negligible for practical interests. As shown in Ch.IV, , if the Reynolds number based on the wing length is of the order 10^6, which is much lower than that in practical flight condition[40], the viscous effect is very small for the flow deflection angle θ greater than, say, 10° and at hypersonic speed. For a caret wing, any incidence θ of the lower ridge less than, say, 10° would be out of practical interests, for the caret wing should have a finite volume and its upper ridge should be at most at zero incidence. Therefore, the formulae (39) and (44) for the stability derivatives obtained from inviscid flow theory are adequate for practical use, and the viscous effect may be neglected.

CHAPTER VI

GENERAL DISCUSSIONS AND CONCLUSIONS

1 DISCUSSIONS

Perturbation method is one of the most useful methods in Mathematical Physics, and particularly in Fluid Mechanics[45]. If the perturbation from a known flow field is small, one can linearize the perturbation equations and thus reduces a non-linear problem to a linear one. That the perturbation from a pure wedge flow is small is the main assumption in this thesis and is valid for the purpose of analising stability and flutter. For cases for which the known flow field is simple an analytical study of the linearized perturbation equations may be possible. Such is the case in the thesis. In general, numerical solution to the perturbation equations is necessary, and little physical insight can be obtained. It is not always possible to get physical insight even in analytical cases, for instance, a complete analytical study has been made in Chapter V of the stability of a pitching Nonweiler wing, but it is still difficult to know how the unsteady waves, which are included in the formulae for the stability derivatives, are reflected from the shock waves.

The inviscid perturbation method in Chapters II and III is a regular one, whereas the viscous perturbation method in Chapter IV is singular with a singularity at the wedge apex which is inherented in any boundary layer theory.

In Chapter II, the perturbation equations are approximate but the boundary conditions are exact. This inconsistence might be easily removed by neglecting all terms proportional to M_0^{-2} in the boundary conditions. However, consistency of an approximate theory does not necessarily give the most accurate result. It is believed that by exactly satisfying the boundary conditions, as is done in the thesis one may expect a more accurate result, as one starts with exact rather than approximate data. Also, keeping terms proportional to M_o^{-2} in the boundary conditions may, to some extent, keep in more characteristics of the supersonic flow. These expectations are confirmed partly by comparisons with experimental data for supersonic flow, and partly by comparisons with the exact theory in Chapter III which shows that in this way the approximate theory gives excellent results for M_o as low as 2.0.

The method used in Chapter III for solving the perturbation equations is a powerful one for stability analysis for which the frequency parameter k is very small. This method of solution gives exact formulae for the stability derivatives, i.e. the stiffness derivative and the damping derivative. Should terms proportional to k^2, k^3, k^4, ... in the aerodynamic derivatives be required, the higher order equations ($III.6_n$) should be successively solved. However, Chapter II shows that for pitching motions of a wedge these higher order terms in k have negligible contribution to the aerodynamic derivatives even if k is not small compared with unity.

As for flutter, for which k may be comparable with unity, this method should cease to be used unless the convergence of the power series (5) in Chapter III is proved.

On the other hand, the solution given in Chapter II is exact with regard to the frequency parameter k, as long as k is not too large compared with unity (otherwise the velocity disturbance at the body surface is, for a given value of ϵ, not small compared with the local velocity of the gas). Hence it can be applied for the case of flutter as well as for stability analysis, and also for cases of aperiodic motions. It should be noticed that this solution is approximate with regard to the thickness and the free stream Mach number (more precisely, to the combined effect of the two, as representedby M_o). However, the method can be improved to give more accurate solutions. Thus it is proposed here that the flow quantities u, v, p, ρ and f_5 be expanded in power series of M_o^{-2}, say,

$$p = p^{(o)} + M_o^{-2} p^{(1)} + M_o^{-4} p^{(2)} + \dots \qquad (1)$$

Putting these expressions into the exact perturbation equations(I.4) and (I.7) and equating like terms of M_o^{-2}, we obtain a sequence of systems of equations to be solved successively, for example, we have, for p,

$$\frac{\partial^2 p^{(0)}}{\partial \xi^2} - \frac{\partial^2 p^{(0)}}{\partial \zeta'^2} = 0 \qquad (2_o)$$

$$\frac{\partial^2 p^{(n)}}{\partial \xi^2} - \frac{\partial^2 p^{(n)}}{\partial \zeta'^2} = \frac{1}{4}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right)^2 p^{(n-1)} \qquad (2_n)$$

$$(n = 1,2,3, \dots)$$

where $\zeta' = M_o \zeta$. The boundary conditions should also be treated in the same manner. Now $p^{(o)}$ has been obtained in Chapter II and one may easily find $p^{(1)}$ by the same method of solution as that for $p^{(o)}$, and so forth. In this way, one may obtain a solution as accurate as required, assuming that all the $p^{(n)}$'s are of the same order of magnitude. For cases where the bow shock is attached to the wedge, we have $M_o > 1$ and these series must converge. As $p^{(o)}$ by itself gives excellent results for M_o as low as 2.0, it is believed that only terms up to $p^{(2)}$ are required to give a good solution which can be applied for M_0 as low as about 1.2.

Success has been made of extending the two-dimensional perturbation method to a three-dimensional flow past a special type of bodies — the Nonweiler wing, and an analytical solution obtained. However, to apply the perturbation method to flow past a three-dimensional body of general shape, even a circular cone, analytical treatment is not possible and numerical methods must be used instead.

As mentioned in the Introduction of Chapter I, only a few experimental results for unsteady hypersonic flow have been reported, it is therefore suggested that the following experiments may be performed at the Department of Aeronautics and Astronautics of the University of Southampton:

a) Measurements of the stability derivatives of relatively thick and very thick wedges in hypersonic and supersonic flows in order to check formulae (III. 25) and (III. 26) and the stability criterion for a wedge.

b) Measurements of the stability derivatives of very thin wedges in hypersonic flow to check formulae (IV. 60). Also to obtain information of bluntness effect of a slender wedge, and of the phase lag between the unsteady boundary layer and the motion of the wedge. As the bluntness effect and the phase lag are neglected in the thesis, such experiments could give a range of applicability of the present theory.

c) Measurements of the stability derivatives of Nonweiler wings in hypersonic flow at their design conditions to check the theoretical results obtained in Chapter V. It would be useful also to perform these experiments for lower Mach number to check the assumption made in the thesis that the perturbation pressure on the upper surface of the wing is negligible compared with that on the lower surface.

2 CONCLUDING REMARKS

1 Exact (linearized) perturbation equations and boundary conditions for inviscid hypersonic and supersonic flows past a wedge-like body, for which the bow shock is attached to the body, are derived, and the problem of finding the flow field reduced to that of solving a wave equation containing only one unknown function. This system of equations can be applied to both unsteady and steady flows over both slender and thick, rigid or flexible bodies performing either periodic or aperiodic motions.

2 These equations include those obtained by Chu[16] and by Chernyi[2] as a special case when the flow is steady.

3 Two methods of solution have been developed to solve the perturbation equations. The first is applicable to periodic motions only in which all the unknown functions are expanded into power series in the frequency parameter k. Thus it can be applied to stability analyses for which k is very small.

4 The second method is applicable for any motion — periodic or aperiodic — in which all the unknown functions are expanded into power series in M_0^{-2}. It can be applied to flutter as well as stability analysis, for there is no restriction on the frequency parameter.

5 The application of the first method to a pitching wedge in inviscid hypersonic and supersonic flows gives exact formulae for the stability derivatives in the most general cases. In these formulae the effects of wave reflection and of thickness are fully included.

6 This exact theory includes the theory of Carrier & Van Dyke[11,12] as a special case when the flow is supersonic.

7 It is shown that whereas the stiffness derivative of a wedge increases in magnitude with its thickness, the damping derivative first increases with the thickness and then falls to very large negative values several degrees before the semi-vertex angle reaches the shock detachment angle.

8 Increasing flight Mach number tends to increase the stability of the wedge, however, it is shown that even at hypersonic speeds,under certain circumstances the body may be destablized by the flow passing it. A general criterion for stability is also obtained.

9 Solution to the perturbation equations by the second method has been carried out to the first term in the power series in M_o^{-2}. The application of the solution to a pitching wedge in inviscid hypersonic and supersonic flows yields approximate formulae for the aerodynamic derivatives in two forms of power series in the frequency parameter k and in the reflection coefficient λ, the value of which in the unsteady flow is shown to be the same as that in the corresponding steady flow. While the damping derivative so obtained is approximate, the stiffness derivative is proved to be exact.

10 In addition to the set of waves due to the disturbance at the body surface which has been discussed previously[15,16,2,14], a new set of waves due to the motion of the bow shock is discovered which exists for relatively thick wedges only. This set of waves is found to be a factor which tends to strongly destablize the motion of the body. It is shown to be important for thick wedges

and become dominant near the critical situation. Effects of the reflected waves of both sets are shown to be important for relatively thick wedges.

11 For pitching motions of a wedge, it is shown that the terms proportional to k^2 or higher orders of k in the aerodynamic derivatives are negligible.

12 It is shown that for stability analysis the exact theory developed in Chapter III includes the approximate theory developed in Chapter II as a special case which, in turn, includes both McIntosh's theory[14] and Appleton's theory[13] (see also Ref. 22) as special cases when the flow is hypersonic and the wedge is slender. As to the ranges of applicability of the theories of McIntosh and of Appleton, it depends on the error allowed. However, both the exact and the approximate theories give more accurate and simpler formulae for the stability derivatives than either McIntosh's theory or Appleton's theory does.

13 Viscous effect is included by modifying the body shape to account for the displacement boundary layer. Thus the concept of an effective wedge is introduced which is the original wedge thickened by a semi-vertex angle equal to the average inclination of the displacement boundary layer, and the viscous flow field past the original oscillating wedge reduced to an inviscid flow field past the effective wedge which is oscillating and deforming according to the actual growth of the boundary layer, which is assumed to be in-phase with the motion of the body.

14 Closed form formulae for the stability derivatives of a sharp pitching wedge in viscous hypersonic flow are obtained which

fully includes the effects of wave reflection and of thickness. The viscosity of gas is shown to play two roles: first it thickens the wedge to an effective one, secondly it makes this effective wedge deformable.

15 The effect of viscosity is shown to destablize the body for forward pivot positions and to stablize it for rearward pivot positions. It is important for thin wedges in hypersonic flow, but decreases with increasing thickness. Comparisons with inviscid flow are given.

16 The viscous perturbation theory appears to include Orlik-Ruckemann's viscous piston theory[17] as a special case when the wedge is very thin.

17 Extension of the two-dimensional perturbation theory to a three-dimensional one is made and the results applied to the study of the stability of a pitching caret wing at its design condition in hypersonic or high supersonic flows. Exact formulae for the stability derivatives and a general criterion for stability are obtained. The stability derivatives of a caret wing at design condition are shown to be independent of its aspect ratio but dependent only on the flight Mach number and the incidence of its lower ridge.

18 It is shown that there is no three-dimensional effect on the stiffness derivative of a caret wing, but this effect is dominant for the damping derivative and greatly increases it for forward pivot positions and decreases it for rearward pivot positions.

* * * * *

CONCLUSIONS

A general perturbation theory of unsteady hypersonic and supersonic flow past a wedge-like body has been developed which can be applied to inviscid and viscous flows over both slender and thick, rigid or flexible bodies performing either periodic or aperiodic motions, providing the bow shock is attached to the body. The theory is applied to study the stability of a pitching wedge, and gives results which include as special cases most of the up-to-date theories for oscillating wedges : the theory of Carrier & Van Dyke ~~theory of Carrier~~ for supersonic flow, McIntosh theory for hypersonic flow past a slender wedge and Orlik-Ruckemann's theory for viscous hypersonic flow past a slender wedge. A general criterion for stability of a wedge is also given. An exact theory for the stability of a pitching Nonweiler wing at design condition in hypersonic flow is also obtained using the same perturbation method.

APPENDIX A

PROOF OF THE INEQUALITY $|\lambda_a| < 1$

Let

$$N_2' = 1 + \frac{\rho_o}{\rho_\infty}(M_o^2 - 1)\tan^2\varphi - \gamma W\left(\frac{\rho_o}{\rho_\infty} - 1\right),$$

$$L_2' = 2H_a\left[1 - \frac{\gamma - 1}{2}W\left(\frac{\rho_o}{\rho_\infty} - 1\right)\right], \qquad (A_1)$$

then, it is clear that

$$\lambda_a = \frac{L_2' - N_2'}{L_2' + N_2'} \qquad (A_2)$$

The density ratio ρ_∞/ρ_o is given by

$$\frac{\rho_o}{\rho_\infty} \equiv \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)M_\infty^2 \sin^2\beta}. \qquad (A_3)$$

This relation at the bow shock is symmetric with respect to the subscripts o and ∞, hence we have

$$\frac{\rho_o}{\rho_\infty} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)W}. \qquad (A_4)$$

From (A_3) and (A_4) it follows that

$$1 < \frac{\rho_o}{\rho_\infty} < \frac{\gamma + 1}{\gamma - 1}, \qquad \frac{\gamma - 1}{2\gamma} < W < 1, \qquad \frac{\gamma + 1}{2\gamma} < W\frac{\rho_o}{\rho_\infty} < 1 . \qquad (A_5)$$

Replacing every term on the right hand sides of expressions (A_1) by its minima, we obtain

$$N_2' = 1 + \frac{\rho_o}{\rho_\infty}(M_o^2 - 1)\tan^2\varphi + \gamma W - \gamma\left(\frac{\gamma - 1}{\gamma + 1}W + \frac{2}{\gamma + 1}\right)$$

$$= \frac{\rho_o}{\rho_\infty}(M_o^2 - 1)\tan^2\varphi + \frac{2\gamma}{\gamma + 1}W - \frac{\gamma - 1}{\gamma + 1} > \frac{\rho_o}{\rho_\infty}(M_o^2 - 1)\tan^2\varphi > o ,$$

$$L_2' > 2H_a\left[1 - \frac{\gamma - 1}{2} + \frac{\gamma - 1}{2}\,\frac{\gamma - 1}{2\gamma}\right] > o .$$

Therefore,

$$\lambda_a = 1 - \frac{2N_2'}{L_2' + N_2'} < 1 ,$$

and

$$\lambda_a = -1 + \frac{2L_2'}{L_2' + N_2'} > -1 .$$

It follows that

$$|\lambda_a| < 1 .$$

APPENDIX B

THE REDUCTION OF G TO G_a

We want to prove in this Appendix that the quantity G in Chapter III reduces to the quantity G_a in Chapter II when the terms proportion to M_o^{-2} are neglected.

Firstly, the two quantities are,

$$G_a = \frac{1 + \lambda_a m_a}{1 - \lambda_a m_a} - \frac{\lambda_a \mu + \nu}{(1 - \lambda_a)(1 - \lambda m_a)(1 + H_a)} - \frac{1}{2} \frac{\lambda_a c + d}{1 - \lambda_a}, \qquad (B_1)$$

$$2G = I + \kappa \frac{1 + \lambda m}{1 - \lambda m} + \kappa^2 \frac{\overline{E}/M_o - \overline{C}}{\overline{A}} +$$

$$\frac{(1 - \lambda)\left\{\frac{\overline{D}}{\overline{A}} + M_o \tan\varphi - \kappa^2 \frac{\overline{E}/M_o - \overline{C}}{\overline{A}} - \frac{(1 + \lambda)}{(1 - \lambda)} \frac{\overline{B}}{\overline{A}}\right\}}{(1 - \lambda m)(1 + \lambda H)}, \qquad (B_2)$$

where, the quantities c, d, λ_a, μ, ν, m_a, and H_a are defined in Chapter II.

We can establish the following relations immediately by their definitions:

$$\overline{A} = \overline{N}_2, \qquad \overline{B} = \overline{N}_1 + \overline{N}_2, \qquad \overline{C} = \overline{L}_2,$$

$$\overline{D} = \overline{L}_2 \sec^2\varphi, \qquad \overline{E} = -\overline{N}_1 \cot\varphi, \qquad (B_3)$$

in which, the quantities $\overline{N}_1$, $\overline{N}_2$ and $\overline{L}_2$ are defined in Chapter II. Also, we have, generally

$$I = I_a = E_a - \frac{\lambda_a c + d}{1 - \lambda_a}, \qquad (B_4)$$

and

$$E = \frac{\overline{C}}{\overline{A}} = E_a,$$

where I_a and E_a are defined in Chapter II

When all the terms proportional to M_o^{-2} are neglected, we obtain

$$\lambda_a = \lambda, \quad H_a = H, \quad m_a = m,$$
$$\varkappa = 1, \quad \bar{E}/\bar{A} = O(M_o^{-2}) = 0. \tag{B_5}$$

Therefore, making use of (B_4) and (B_5), we have

$$2G = \frac{1 + \lambda_a m_a}{1 - \lambda_a m_a} + \frac{(1 - \lambda_a)(\bar{D}/\bar{A} + H_a + \bar{C}/\bar{A} - \frac{1 + \lambda_a}{1 - \lambda_a}\frac{\bar{B}}{\bar{A}})}{(1 - \lambda_a m_a)(1 + H_a)} - \frac{\lambda_a c + d}{1 - \lambda_a}$$

$$= \frac{1 + \lambda_a m_a}{1 - \lambda_a m_a} - \frac{\lambda_a(\bar{B} + \bar{D} + \bar{C} + H_a\bar{A}) + (\bar{B} - \bar{D} - \bar{C} - H_a\bar{A})}{\bar{A}(1 - \lambda_a m_a)(1 + H_a)} - \frac{\lambda_a c + d}{1 - \lambda_a}$$

but

$$\bar{A} + \bar{C} = \bar{N}_2 + \bar{L}_2 = a$$

$$\bar{B} + \bar{D} + \bar{C} + H_a\bar{A} = \bar{N}_1 + \bar{N}_2 + \bar{L}_1(1 + \cot^2\varphi) + \bar{L}_2 + H_a\bar{N}_2$$

$$= \bar{N}_1 + \bar{L}_1 + \bar{N}_2(1 + H_a) + 2\bar{L}_2$$

$$= c + a(1 + H_a) + \bar{L}_2(1 - H_a),$$

and similarly,

$$\bar{B} - \bar{D} - \bar{C} - H_a\bar{A} = d + b(1 - H_a) - \bar{L}_2(1 + H_a),$$

where a, b, c and d are defined in Chapter II.

By making use of all these relations, we finally obtain

$$2G = \frac{1 + \lambda_a m_a}{1 - \lambda_a m_a} - 2\frac{\lambda_a\mu + \nu}{(1 - \lambda_a m_a)(1 - \lambda_a)(1 + H_a)} - \frac{\lambda_a c + d}{1 - \lambda_a}$$

$$+ \frac{-\lambda_a(\, a(1 + H_a) + \bar{L}_2(1 - H_a)\,) - (\, b(1 - H_a) - \bar{L}_2(1 + H_a)\,)}{\bar{N}_2(1 - \lambda_a m_a)(1 + H_a)},$$

the last term on the right hand side of the above equation can be simplified as follows,

$$\text{the last term} = \frac{1}{1 - \lambda_a m_a} \frac{b(a + \bar{L}_2 m_a) - a(bm_a - \bar{L}_2)}{a\bar{N}_2}$$

$$= \frac{b(1 - m_a) + \bar{L}_2(1 - \lambda_a m_a)}{\bar{N}_2(1 - \lambda_a m_a)}$$

$$= \frac{1 - m_a + (1 - \lambda_a) m_a \bar{L}_2 / \bar{N}_2}{1 - \lambda_a m_a}$$

$$= \frac{1 + \lambda_a m_a}{1 - \lambda_a m_a} \; ,$$

and therefore,

$$G = G_a \; . \qquad\qquad \text{Q.E.D.}$$

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TABLE I

COMPARISON OF THE AERODYNAMIC DERIVATIVES BETWEEN THE GENERAL THEORY AND THE FIRST ORDER THEORY $\gamma = 1.4$

	$M_\infty = 17,\ \theta = 3^\circ 25'$				$M_\infty = 10,\ \theta = 11^\circ 28'$				$M_\infty = 17,\ \theta = 28^\circ$			
h	$-c_{m_\theta}$	$-m_\theta$ $k = 0.01$	$-c_{m_{\dot\theta}}$	$-m_{\dot\theta}$ $k = 0.01$	$-c_{m_\theta}$	$-m_\theta$ $k = 0.1$	$-c_{m_{\dot\theta}}$	$-m_{\dot\theta}$ $k = 0.1$	$-c_{m_\theta}$	$-m_\theta$ $k = 1.0$	$-c_{m_{\dot\theta}}$	$-m_{\dot\theta}$ $k = 1.0$
0.0	0.31837	0.31839	0.23018	0.23020	0.99865	0.99860	0.78340	0.78342	2.64418	2.54887	2.25348	2.28262
0.1	0.25492	0.25494	0.17040	0.17046	0.80682	0.80683	0.59275	0.59322	2.23188	2.16480	1.80664	1.83272
0.2	0.19147	0.19149	0.12325	0.12334	0.61499	0.61503	0.43910	0.43979	1.81957	1.77591	1.42753	1.45028
0.3	0.12802	0.12804	0.08873	0.08882	0.43215	0.43322	0.32243	0.32313	1.40728	1.38218	1.11614	1.13530
0.4	0.06457	0.06459	0.06685	0.06692	0.23132	0.23139	0.24274	0.24325	0.99497	0.98363	0.87247	0.83779
0.5	0.00113	0.00114	0.05760	0.05761	0.03949	0.03955	0.20005	0.20014	0.58267	0.58024	0.69653	0.70774
0.6	-0.06232	-0.06230	0.06097	0.06091	-0.15234	-0.15231	0.19433	0.19381	0.17037	0.17202	0.58831	0.59516
0.7	-0.12577	-0.12575	0.07699	0.07681	-0.34418	-0.34419	0.22561	0.22456	-0.24193	-0.24103	0.54781	0.55005
0.8	-0.18922	-0.18920	0.10564	0.10532	-0.53602	-0.53609	0.29387	0.29147	-0.65427	-0.65891	0.57504	0.57240
0.9	-0.25267	-0.25264	0.14692	0.14643	-0.72786	-0.72800	0.39912	0.39546	-1.06554	-1.08161	0.66999	0.66221
1.0	-0.31611	-0.31609	0.20083	0.20014	-0.91969	-0.91994	0.54136	0.53621	-1.47884	-1.50916	0.83266	0.81950

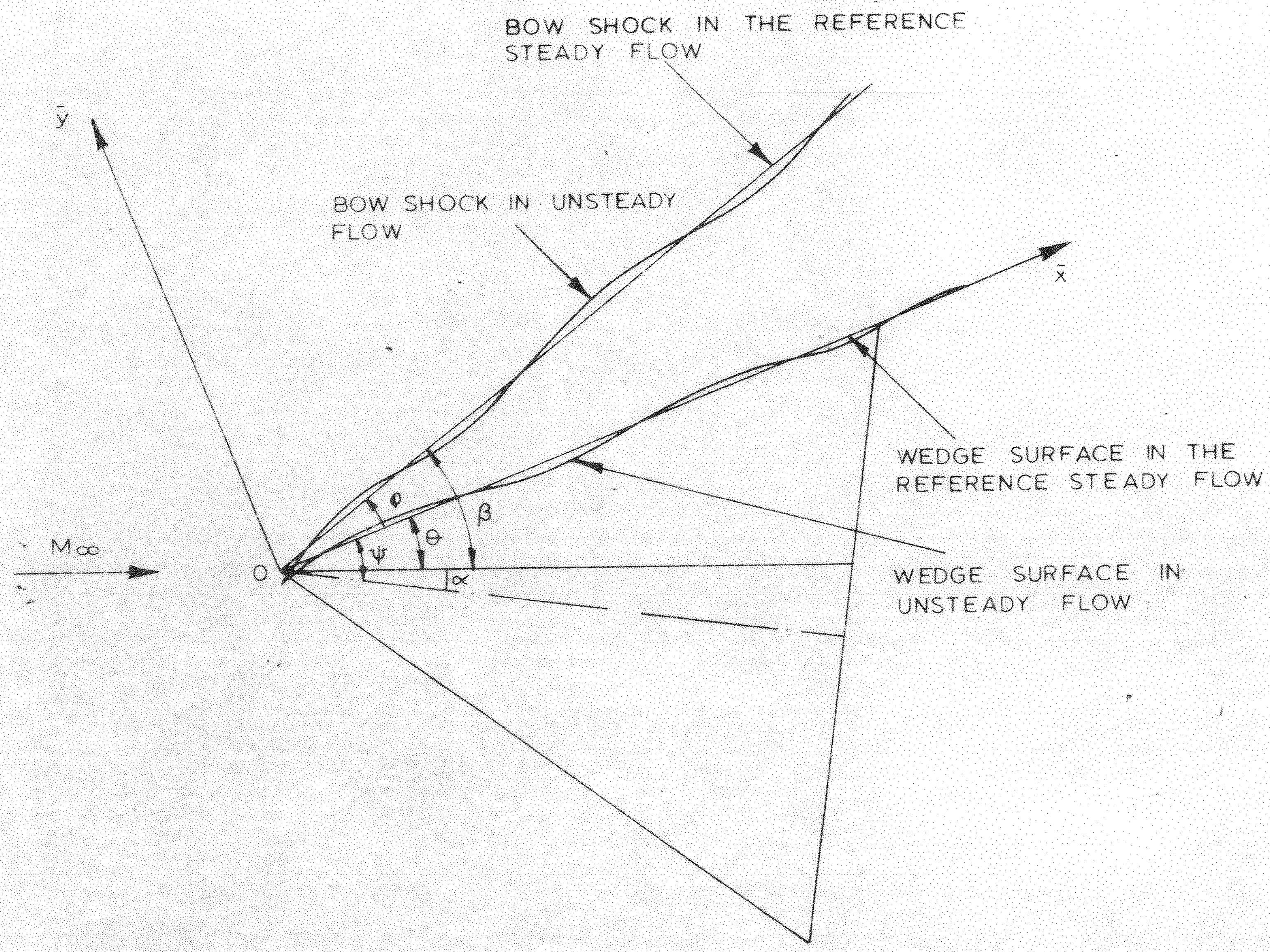

Fig 1. System of co-ordinates

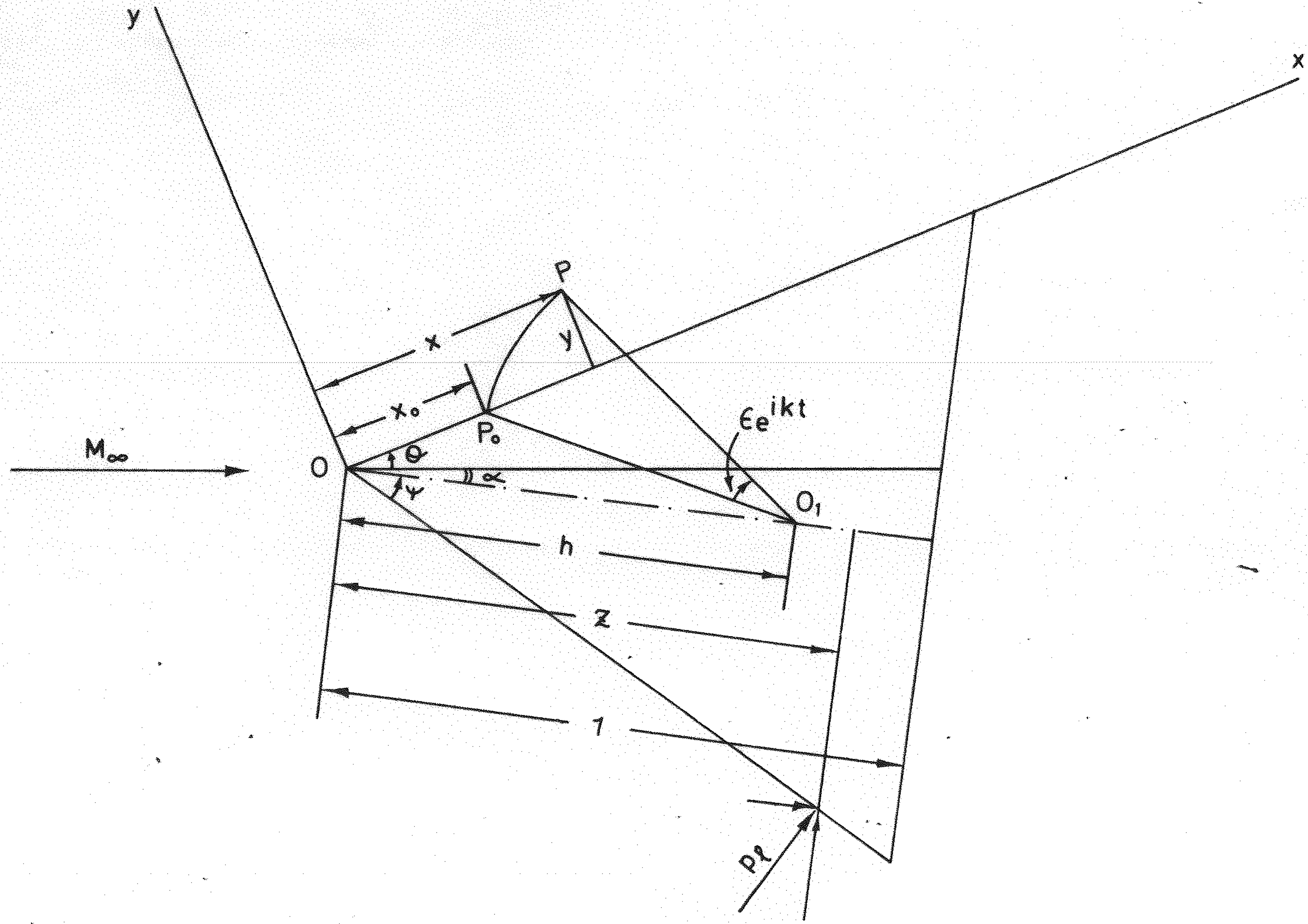

Fig. 2 Pitching motion of a wedge.

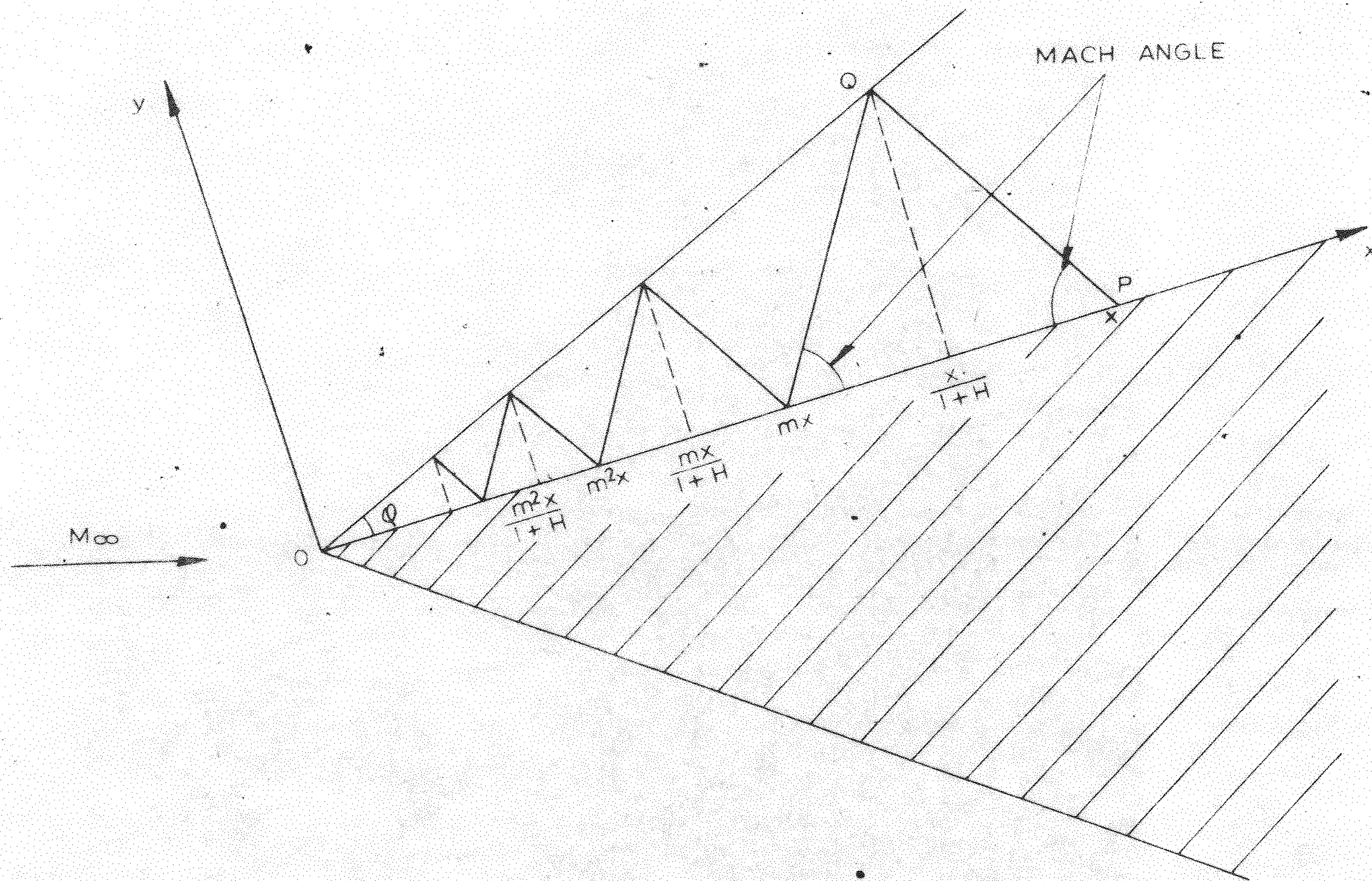

Fig. 3 Wave reflections

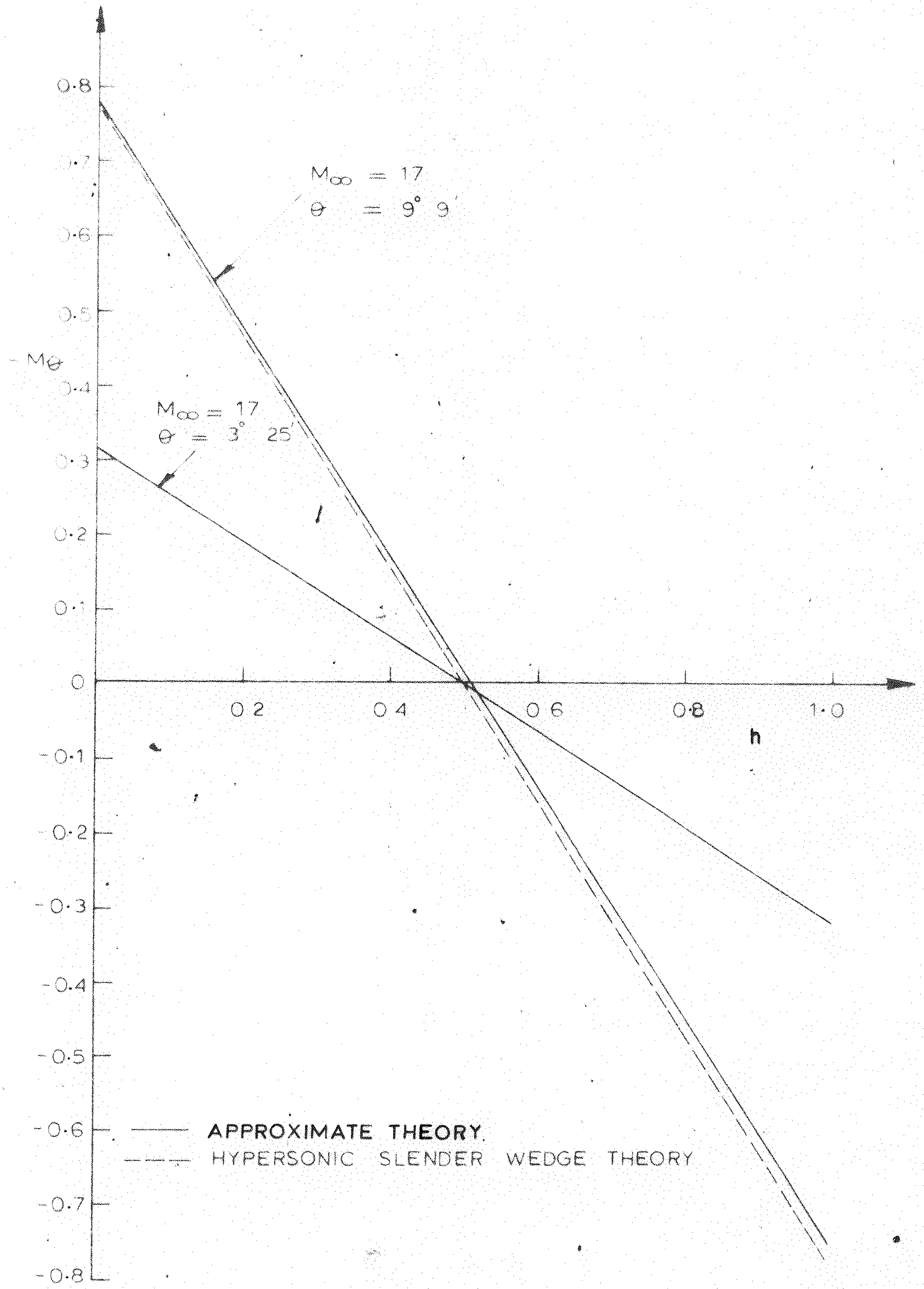

Fig 4 Comparison of the in-phase moment derivation $-m_\theta$
$k = 0{\cdot}05$

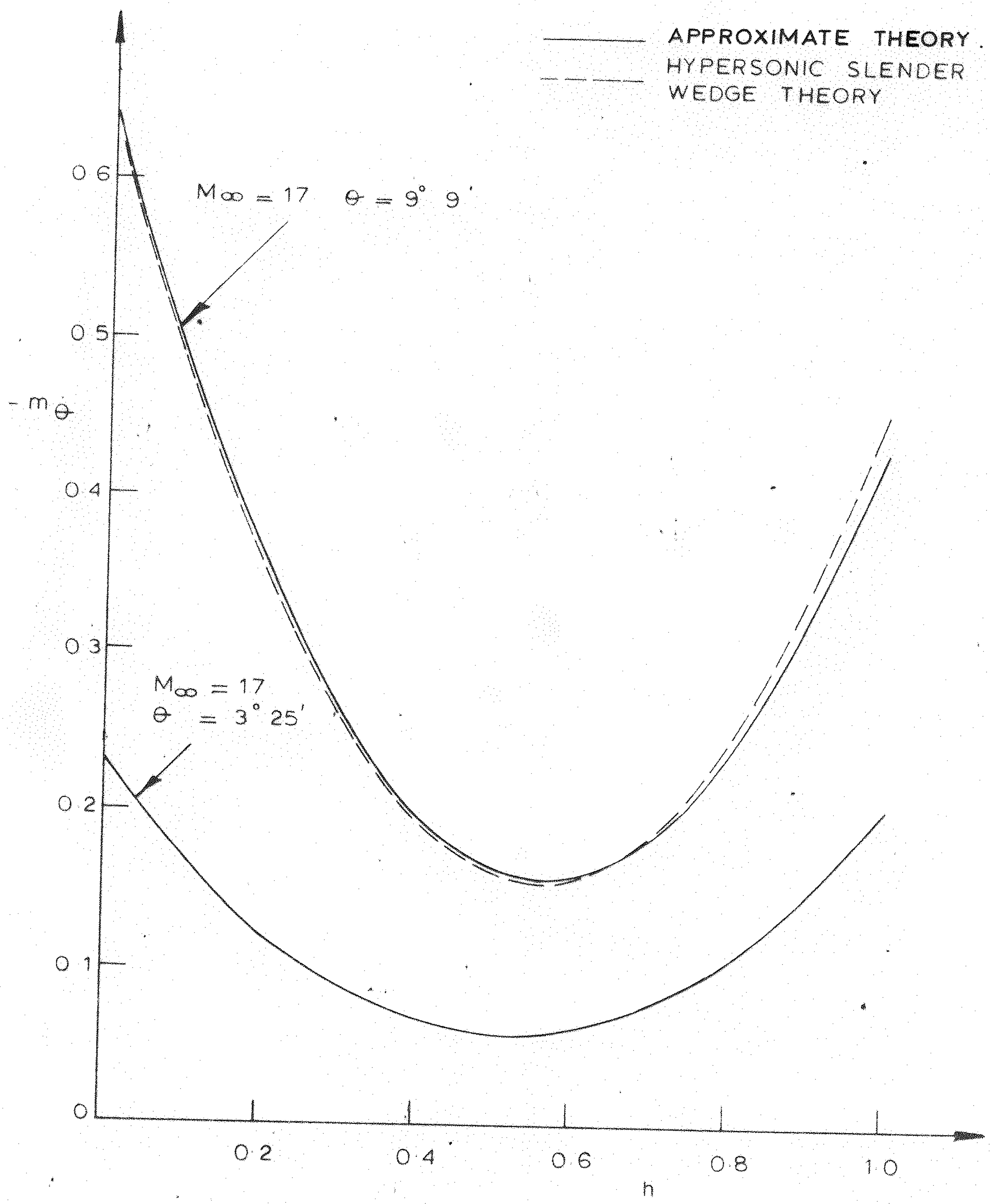

Fig. 5. Comparison of the out-of-phase moment derivative $-m_{\dot{\theta}}$ $k = 0{\cdot}05$

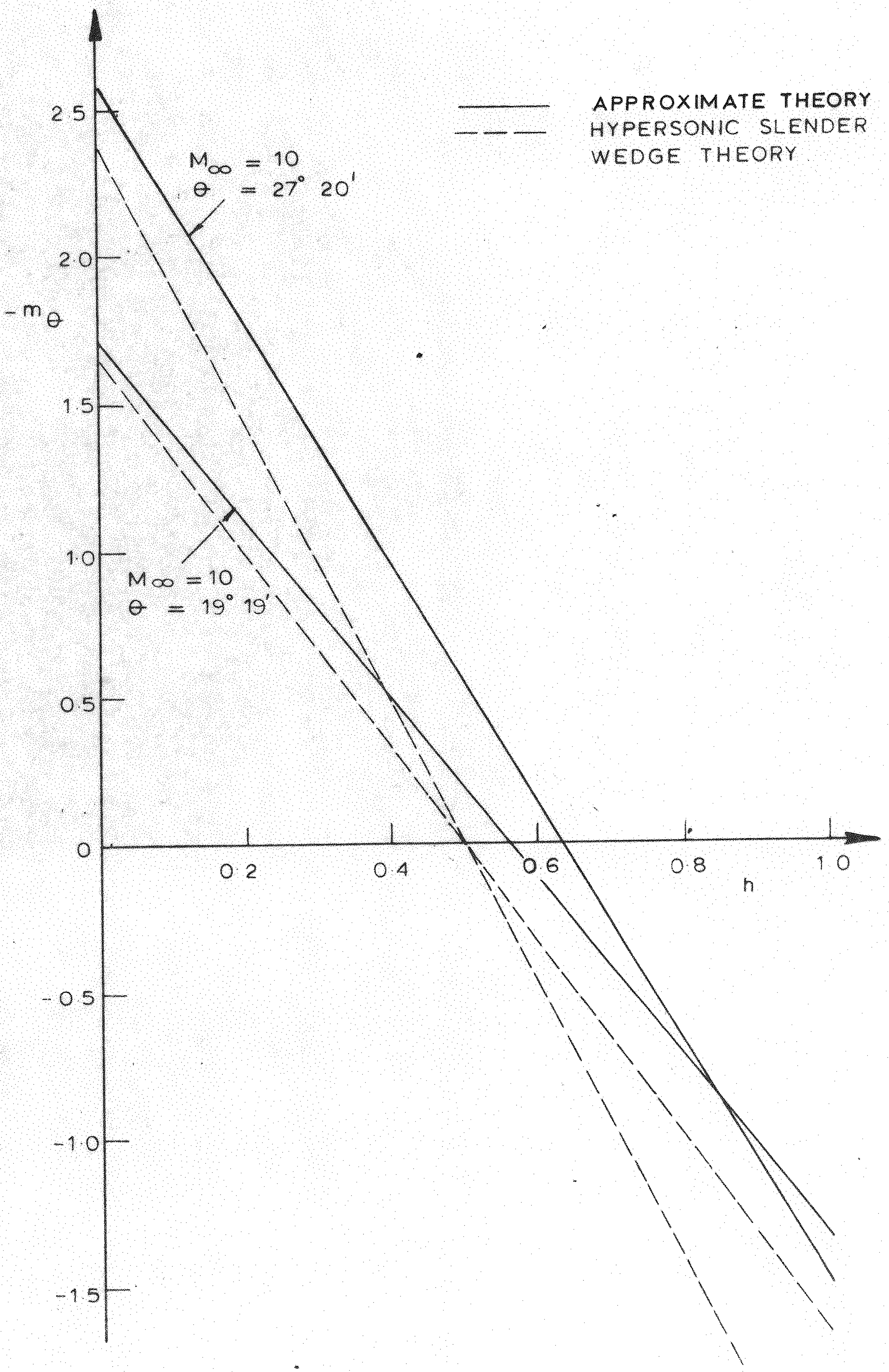

Fig 6 Comparison of the in-phase moment derivative $-m_\theta$ $k = 0.05$

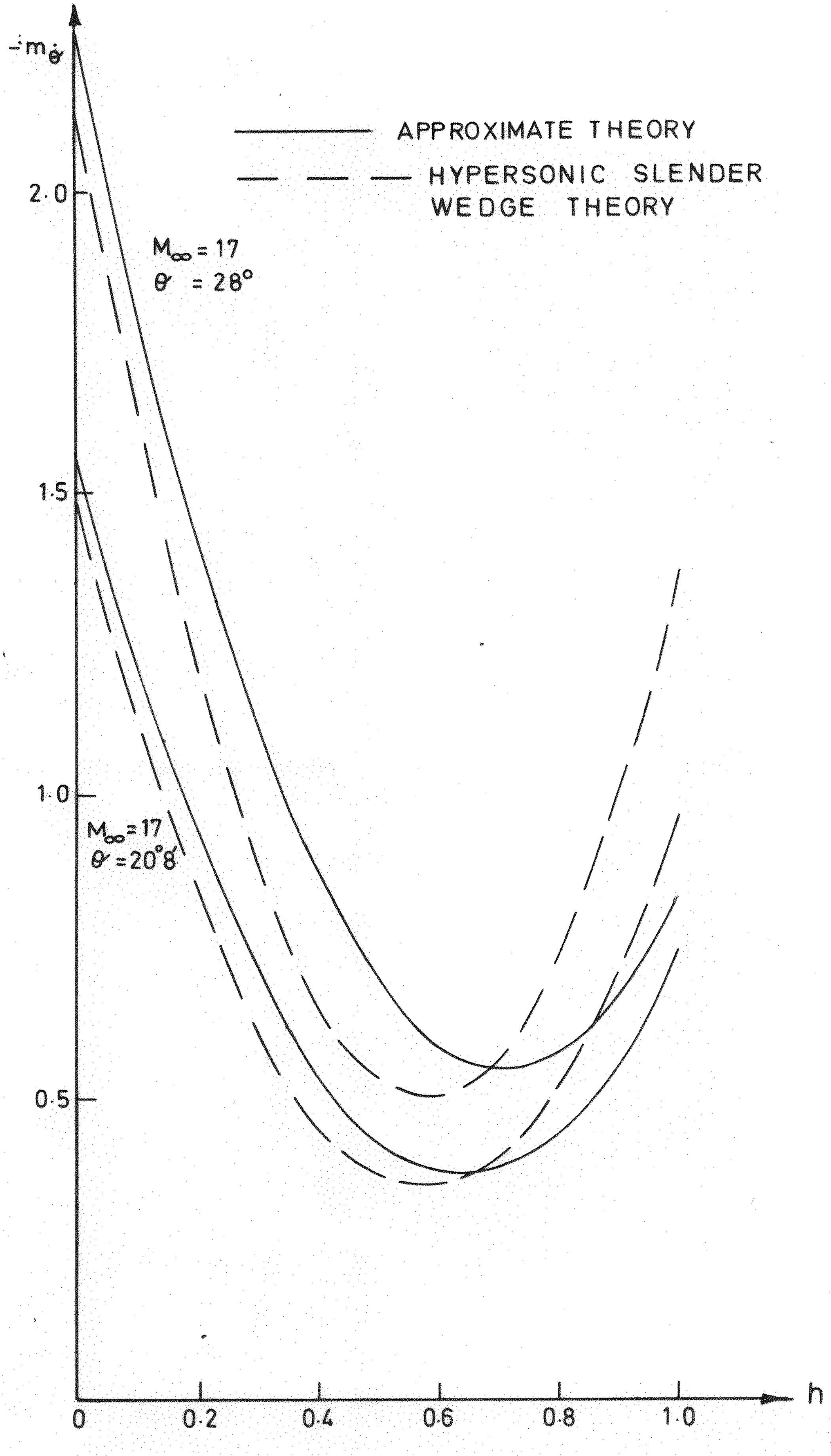

Fig. 7 Comparison of the out-of-phase moment derivative $-m_{\dot\theta}$ k = 0.5.

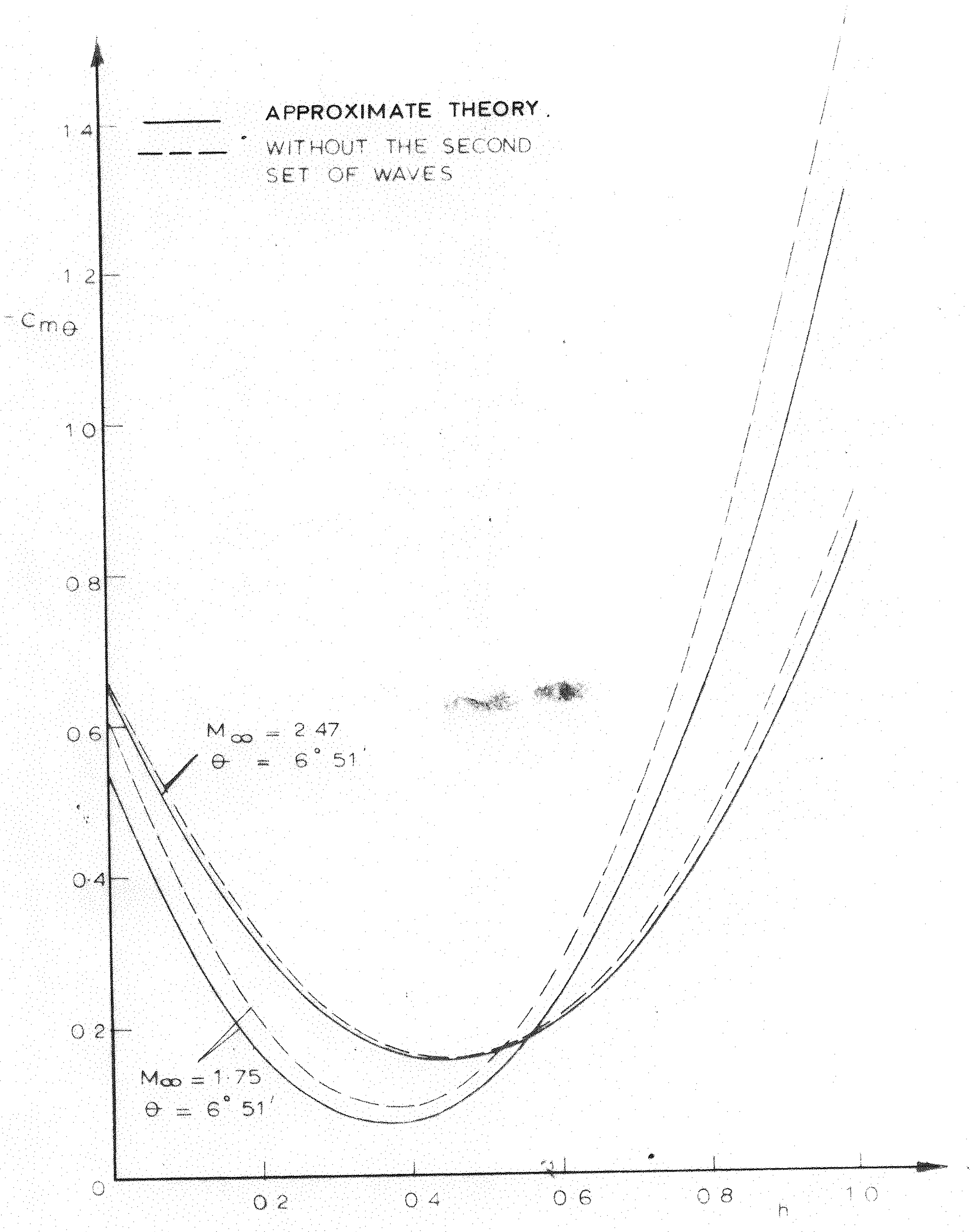

Fig 8. Effects of the second set of waves

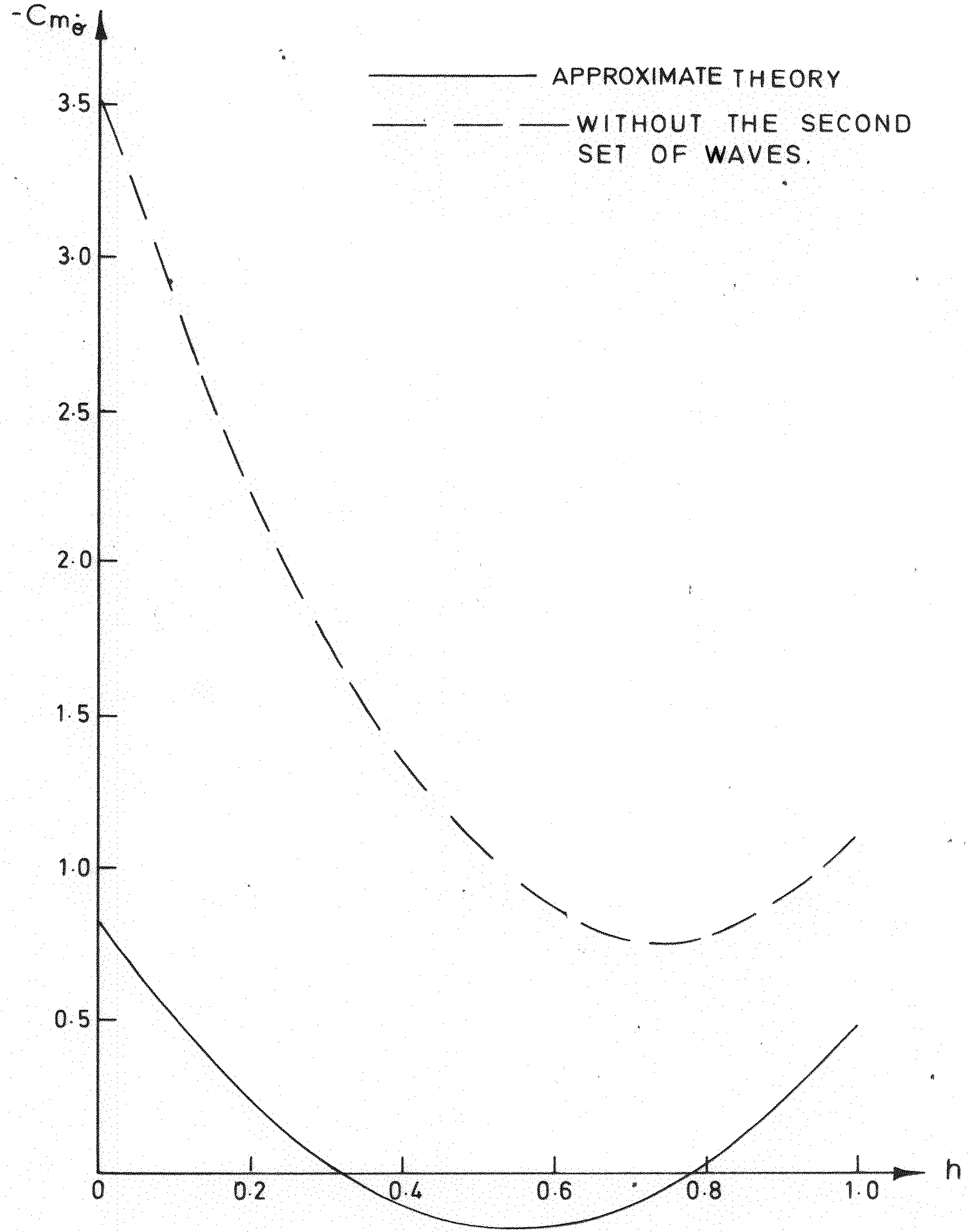

Fig. 9 Effect of the second set of waves on the damping derivative. $M_\infty = 10$ $\theta = 41°46'$

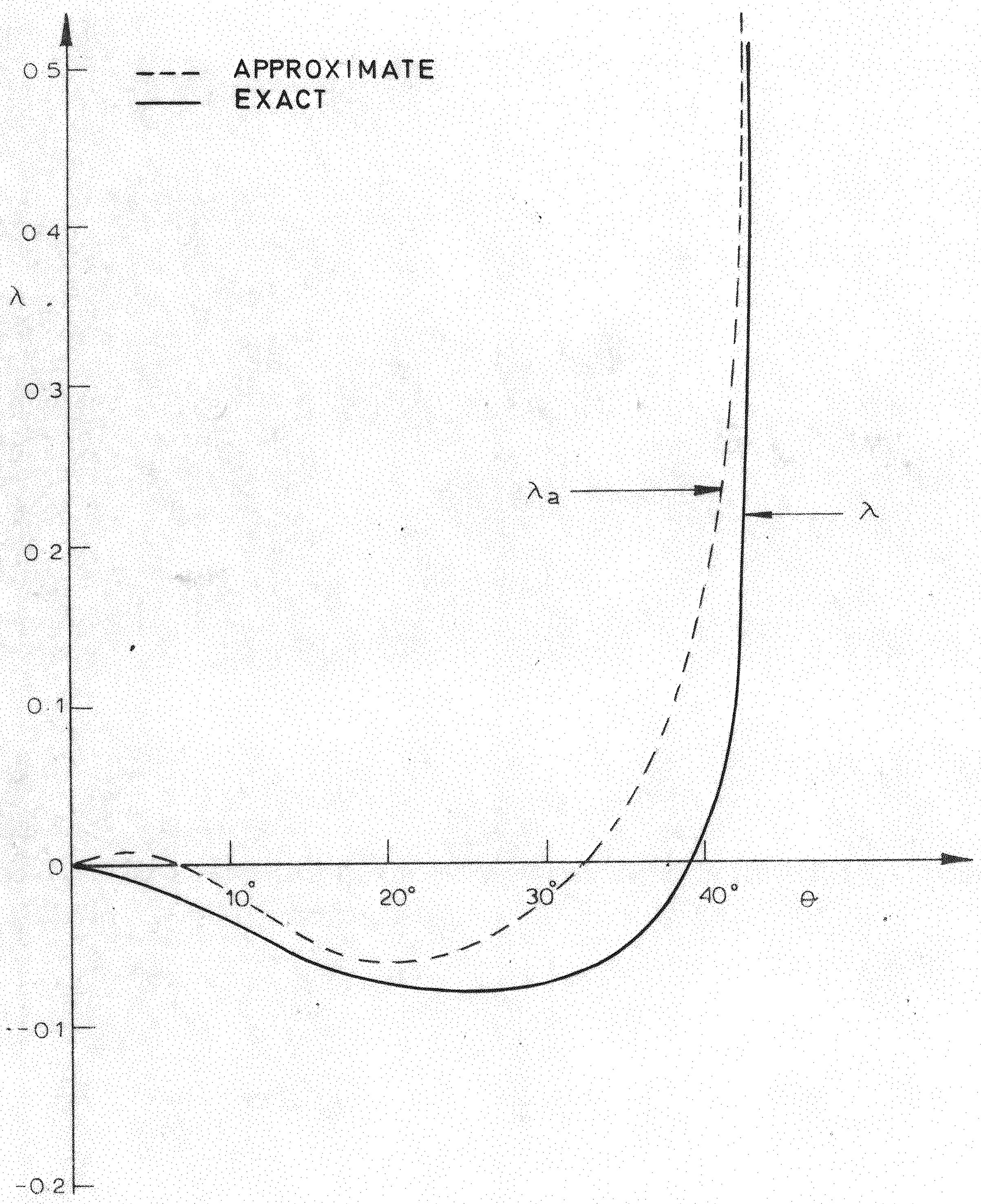

Fig. 10 Reflection coefficient $M_\infty = 7$ $\gamma = 1.4$

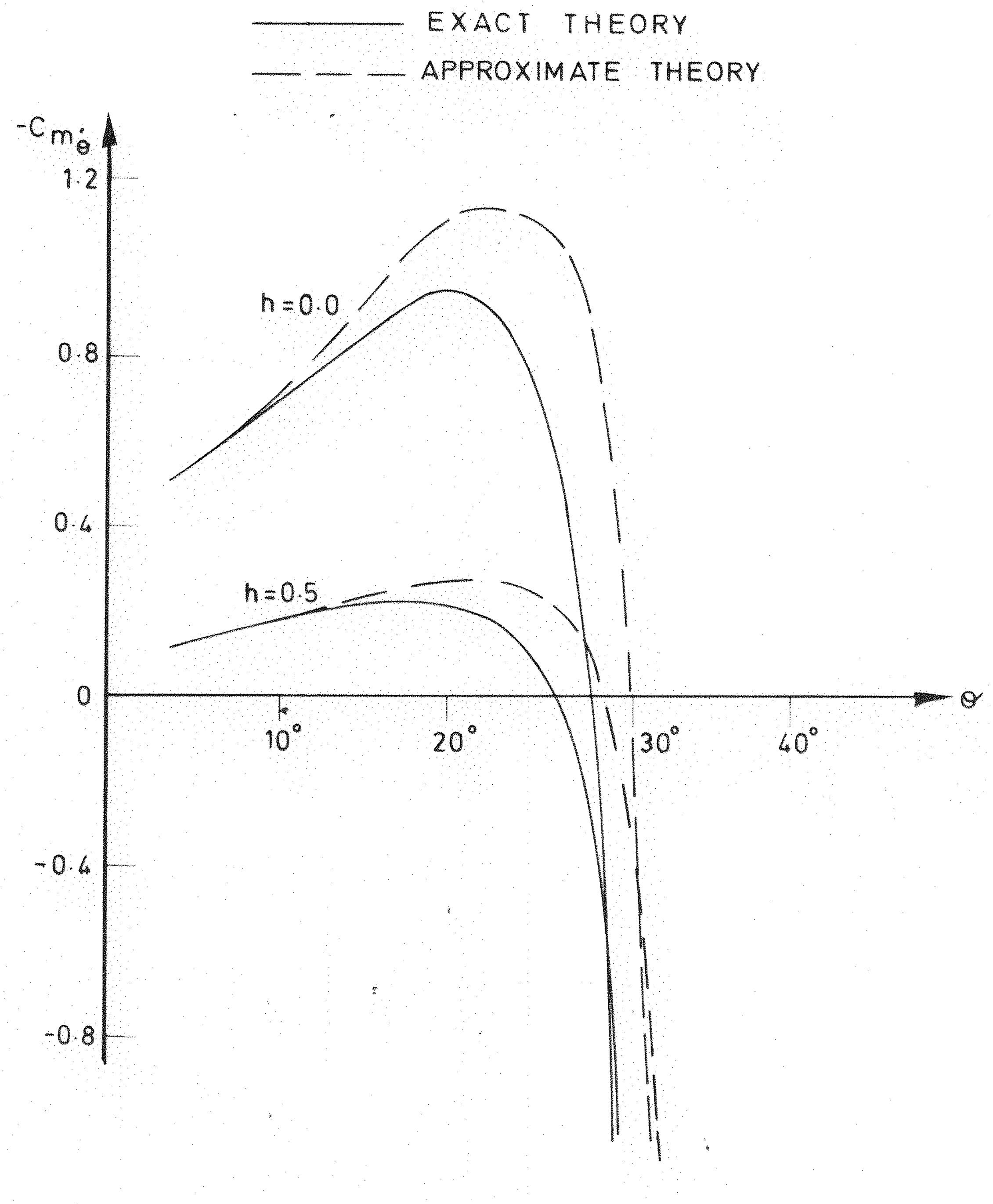

Fig. 11 Comparison of $-C_{m_{\dot{\theta}}}$
$M_{\infty} = 3$, $\gamma = 1.4$

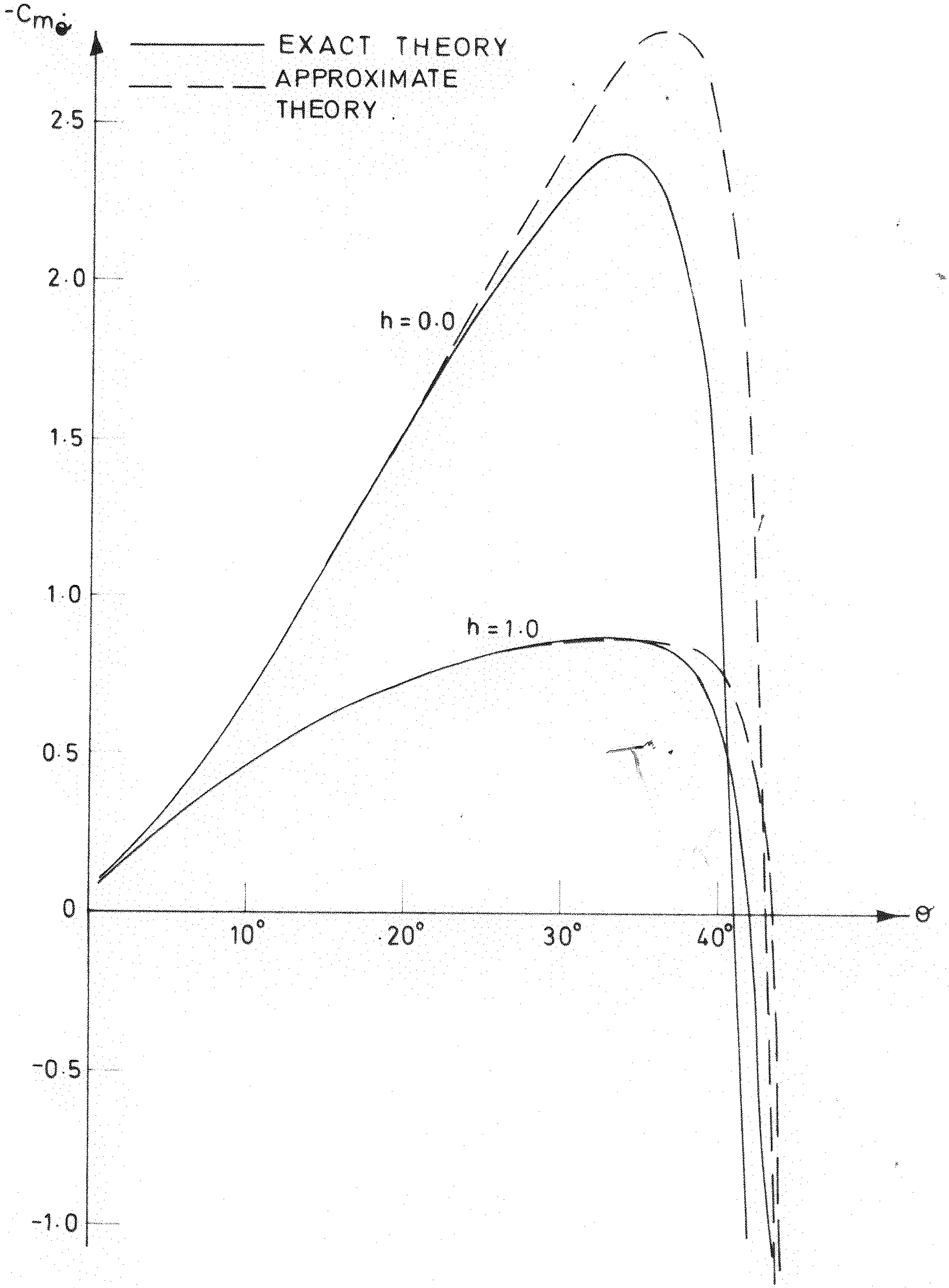

Fig. 12. Comparison of $-C_{m\dot{\theta}}$
$M_\infty = 17$, $\gamma = 1.4$

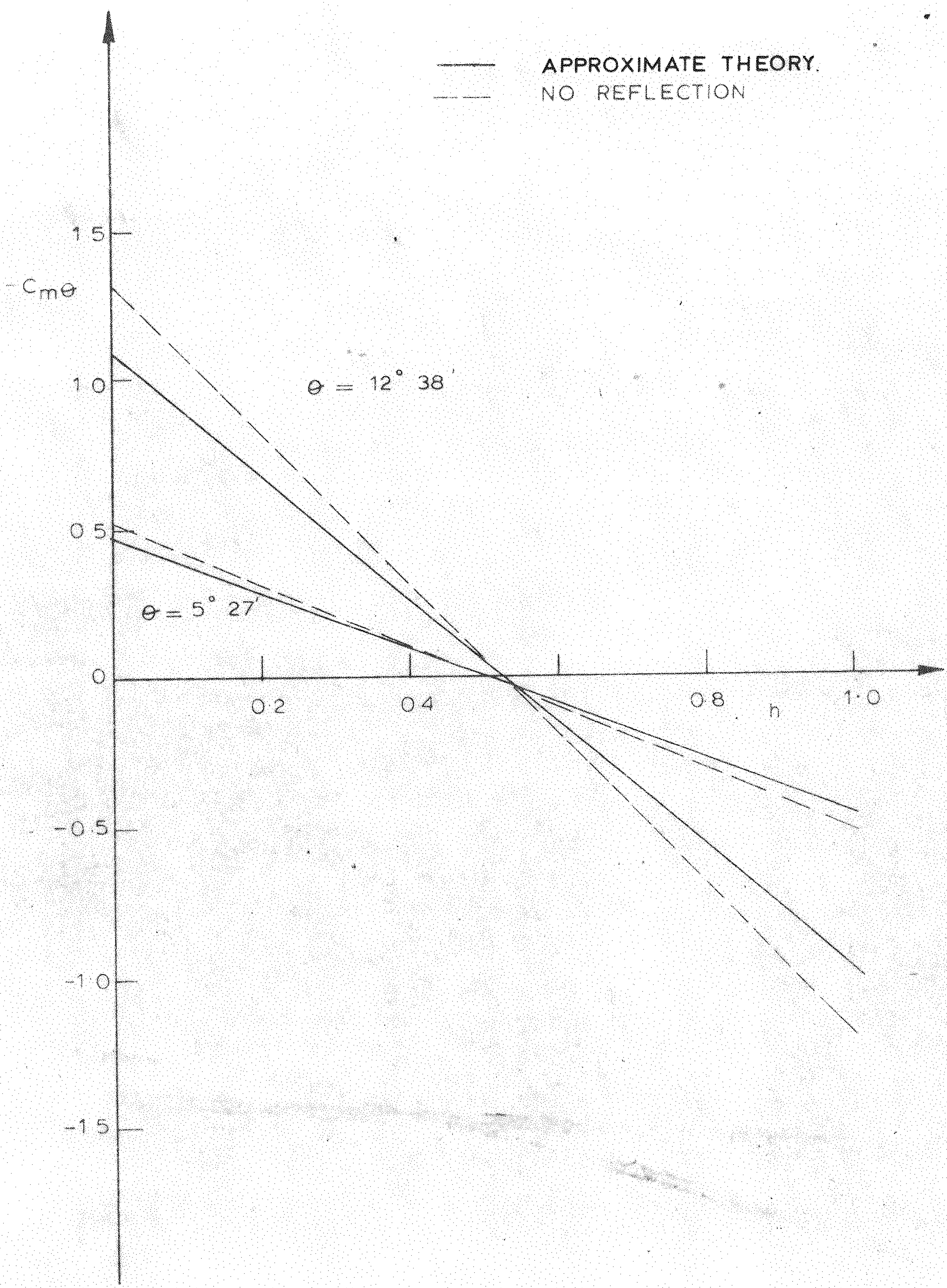

Fig 13. Effects of the reflected waves $M_\infty = 17$.

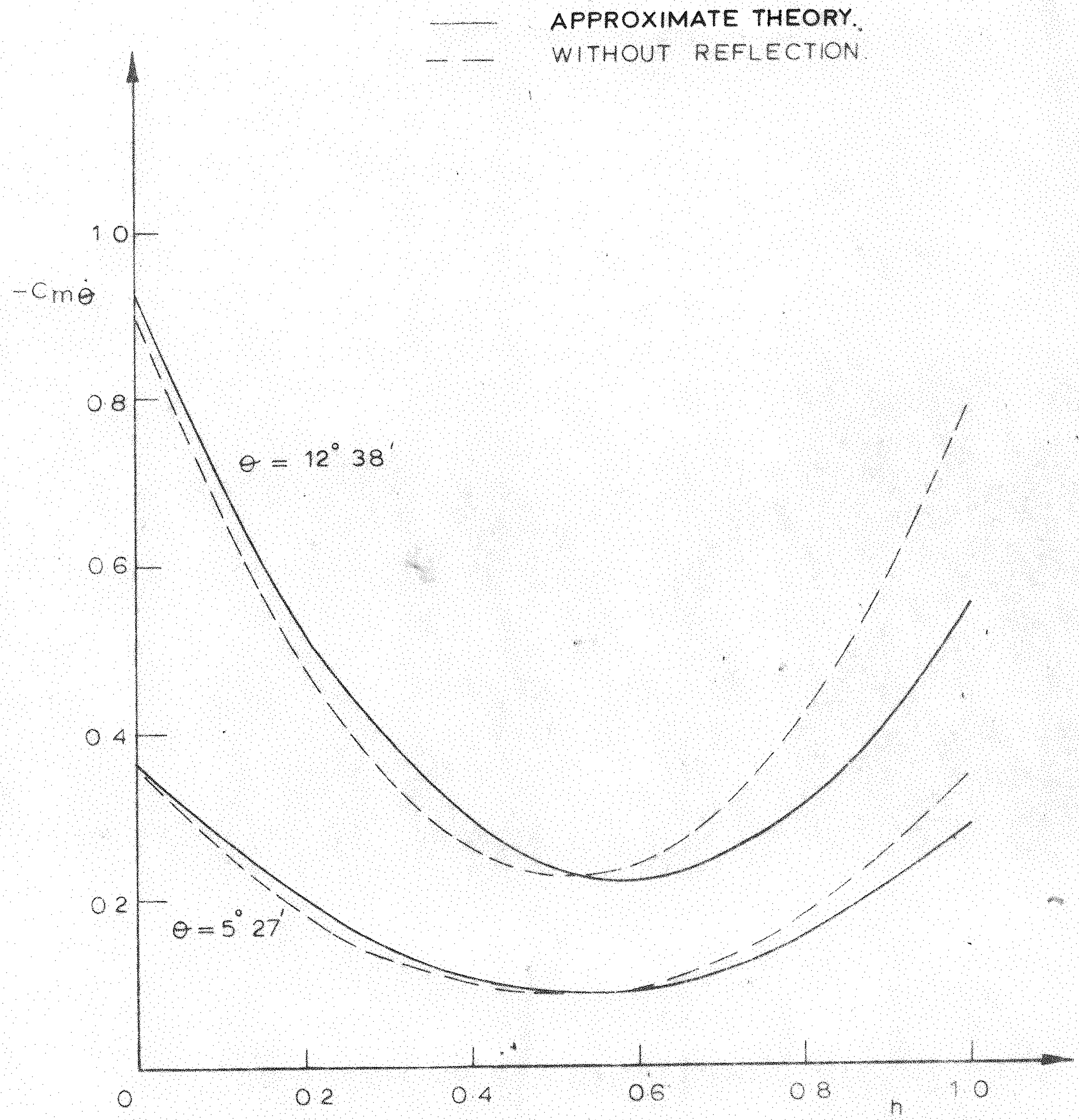

Fig 14. Effects of the reflected waves $M_\infty = 17$

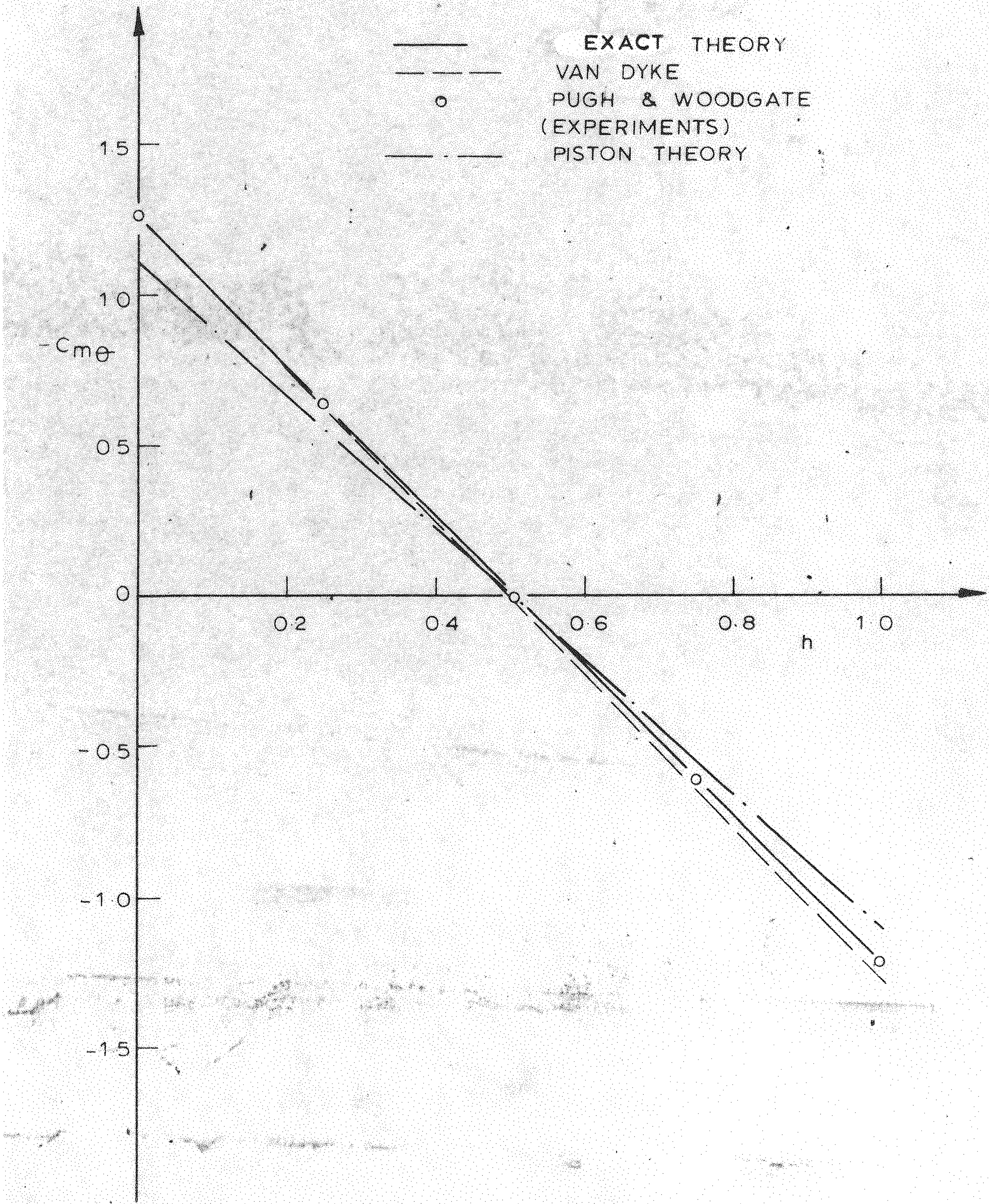

Fig. 15. Comparison of the stiffness derivative $-c_{m\theta}$
$M_\infty = 2{\cdot}47$, $\theta = 6^\circ 51'$

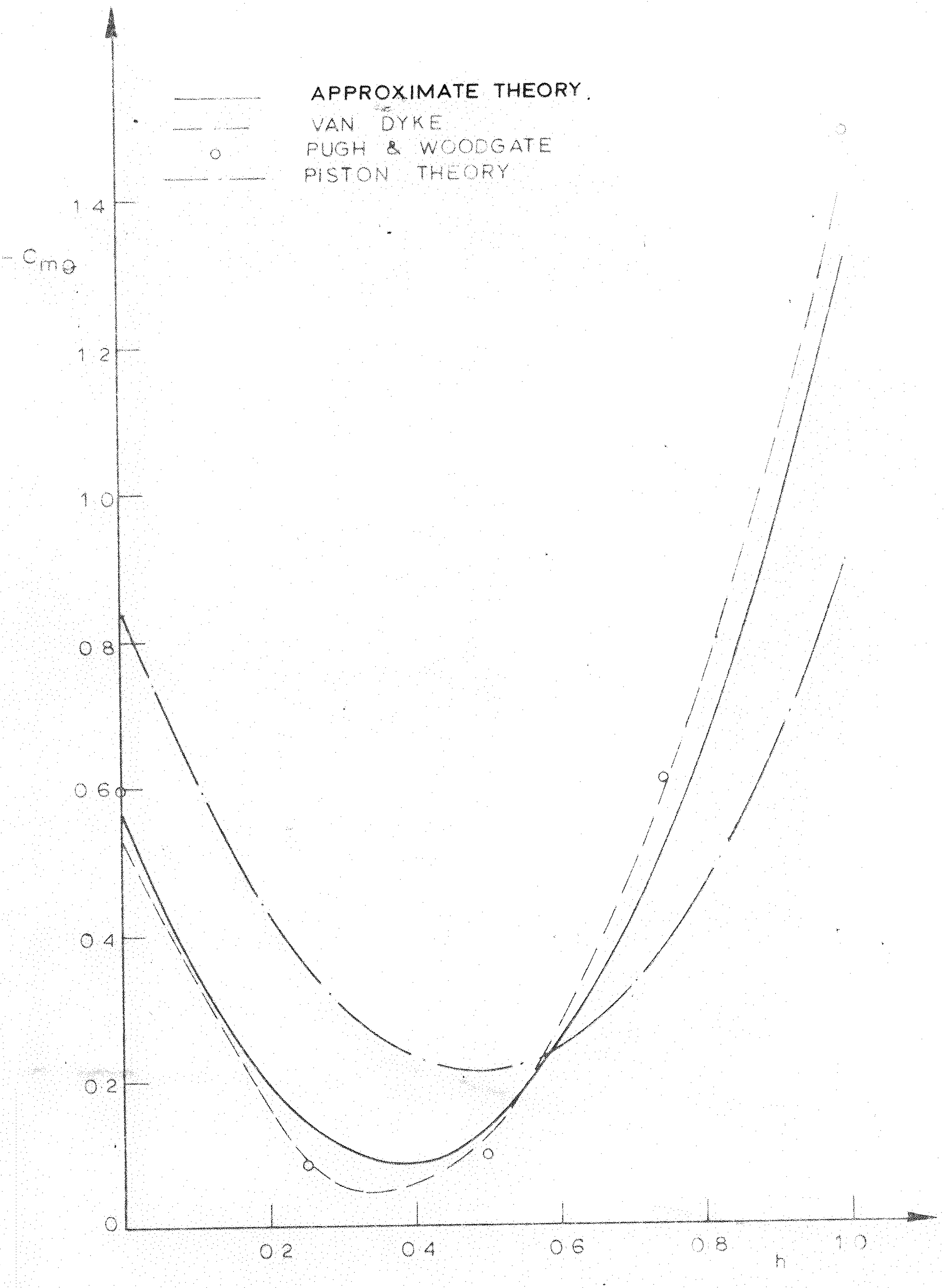

Fig. 16 Comparison of the damping derivative $-c_{m\dot{\theta}}$
$M_\infty = 1.75 \quad \theta = 4^\circ 35'$

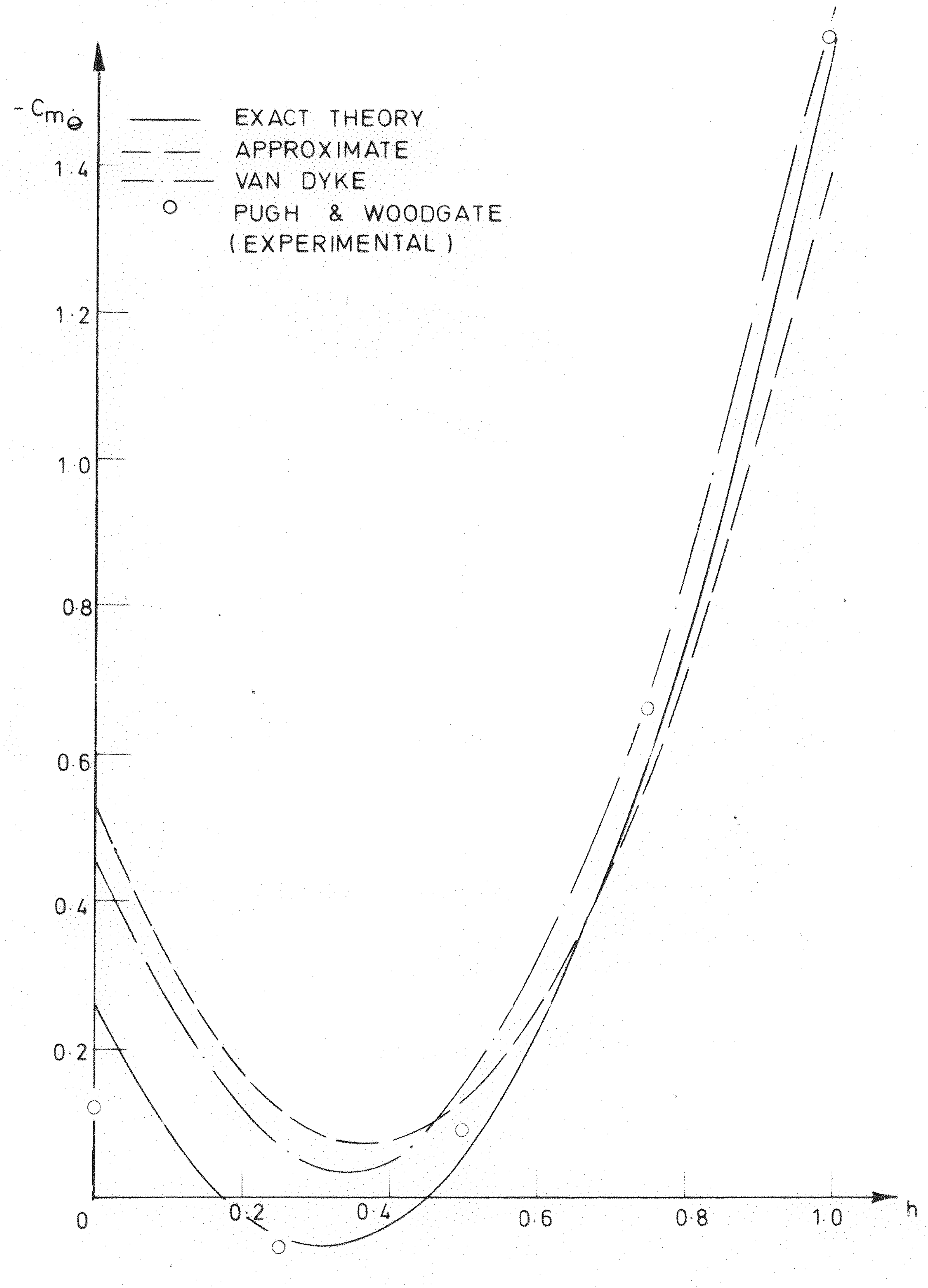

Fig. 17. . Comparison of $-C_{m_{\dot\theta}}$
$M_\infty = 1{\cdot}75$, $\theta = 6^\circ\ 51'$, $(M_o = 1{\cdot}51)$, $\gamma = 1{\cdot}4$

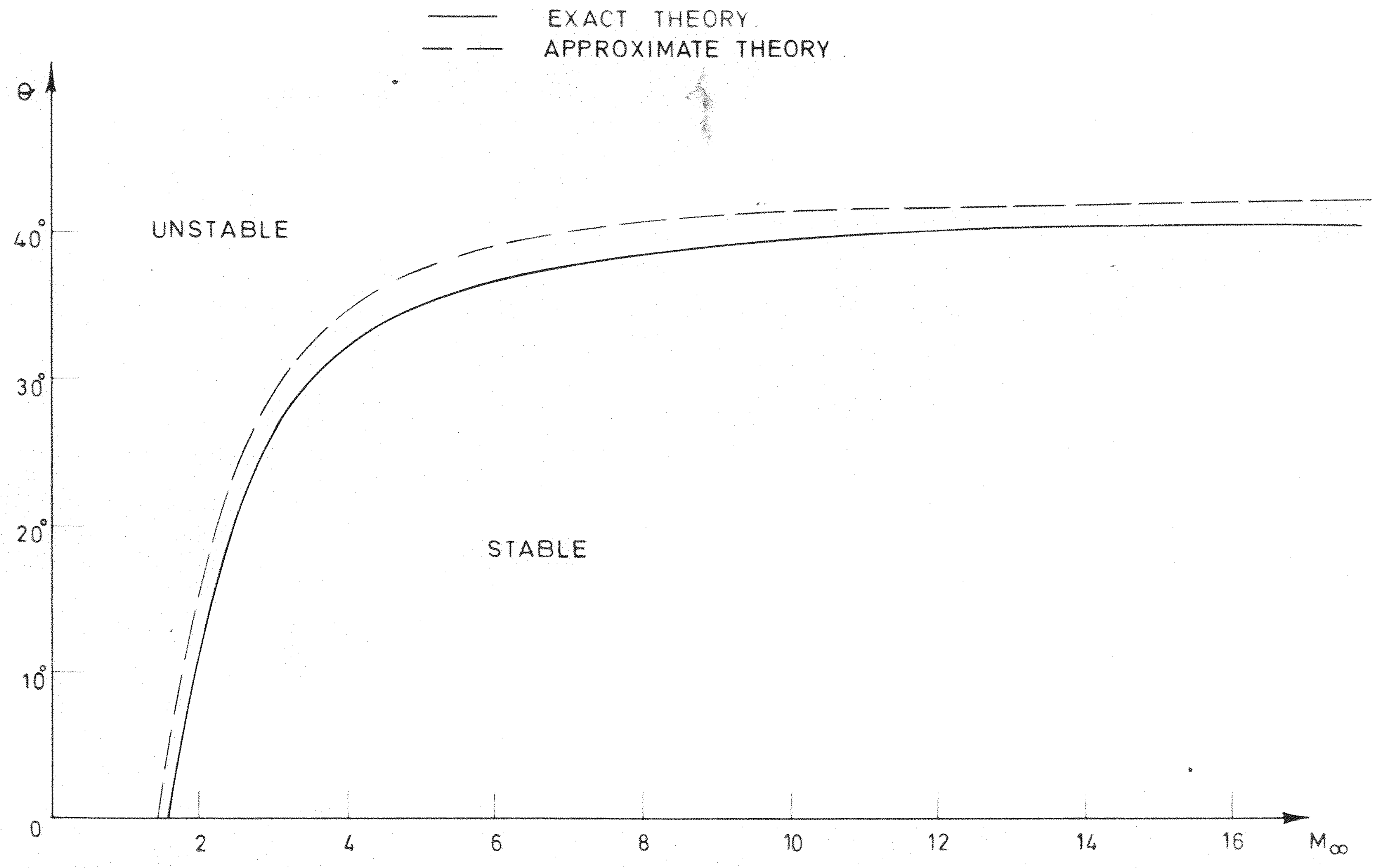

Fig 18. General criterion for the stability of a wedge $\gamma = 1{\cdot}4$, $h = 1/3 + \tan^2\theta / 3$.

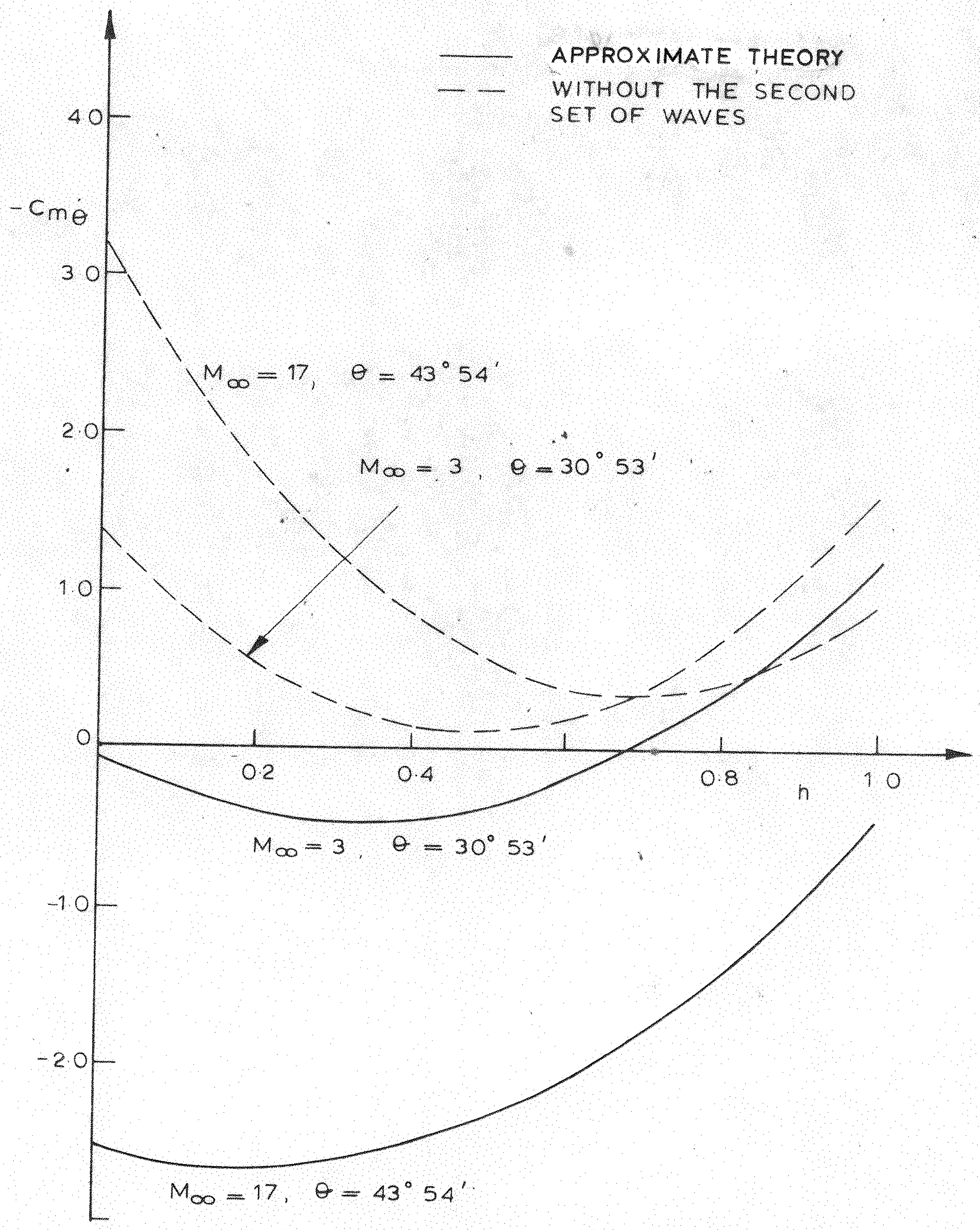

Fig.19. Effect of the second set of waves on the damping derivative.

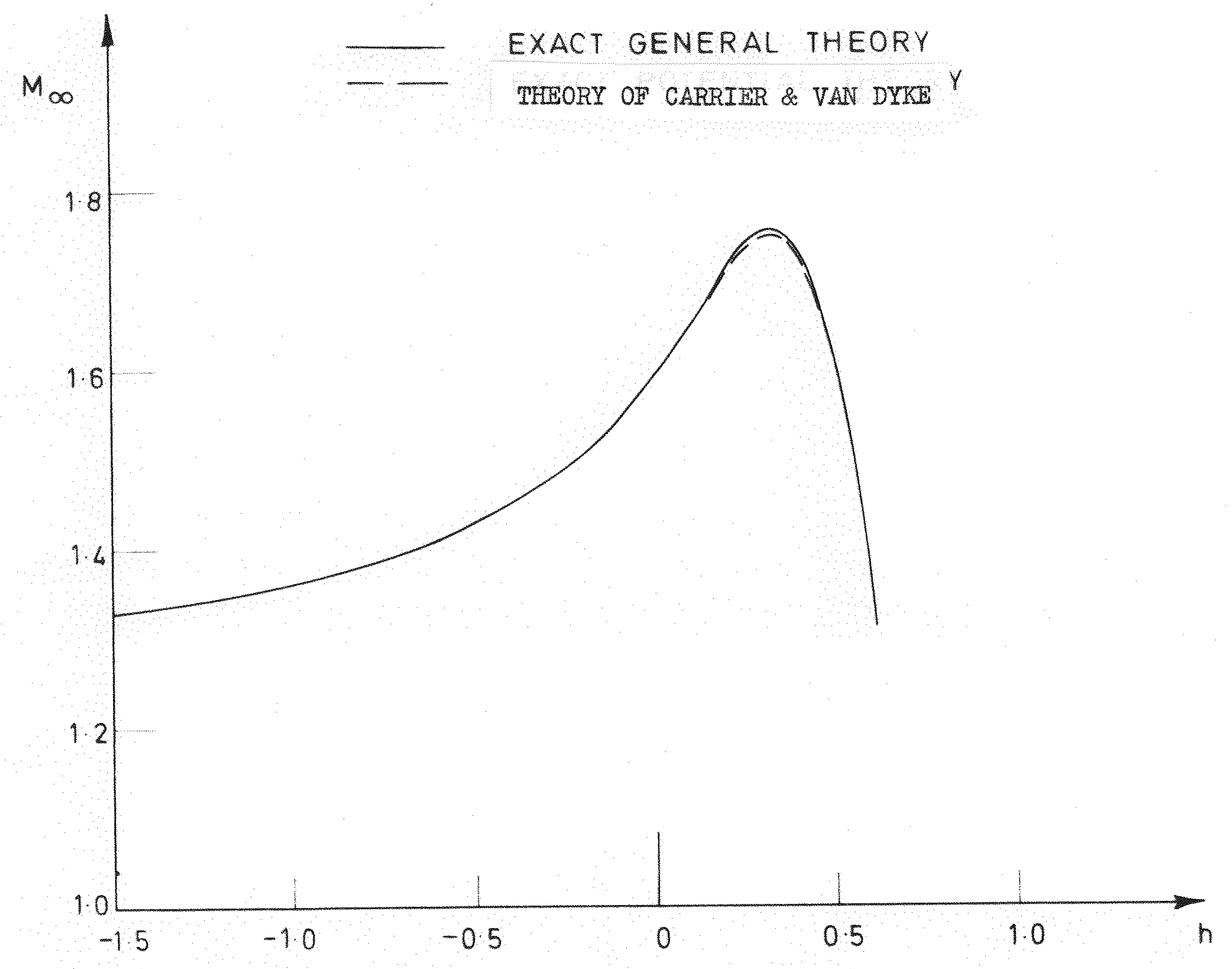

Fig.20. Neutral damping boundary for a wedge of 5° semi-vertex angle, $\gamma = 1{\cdot}4$.

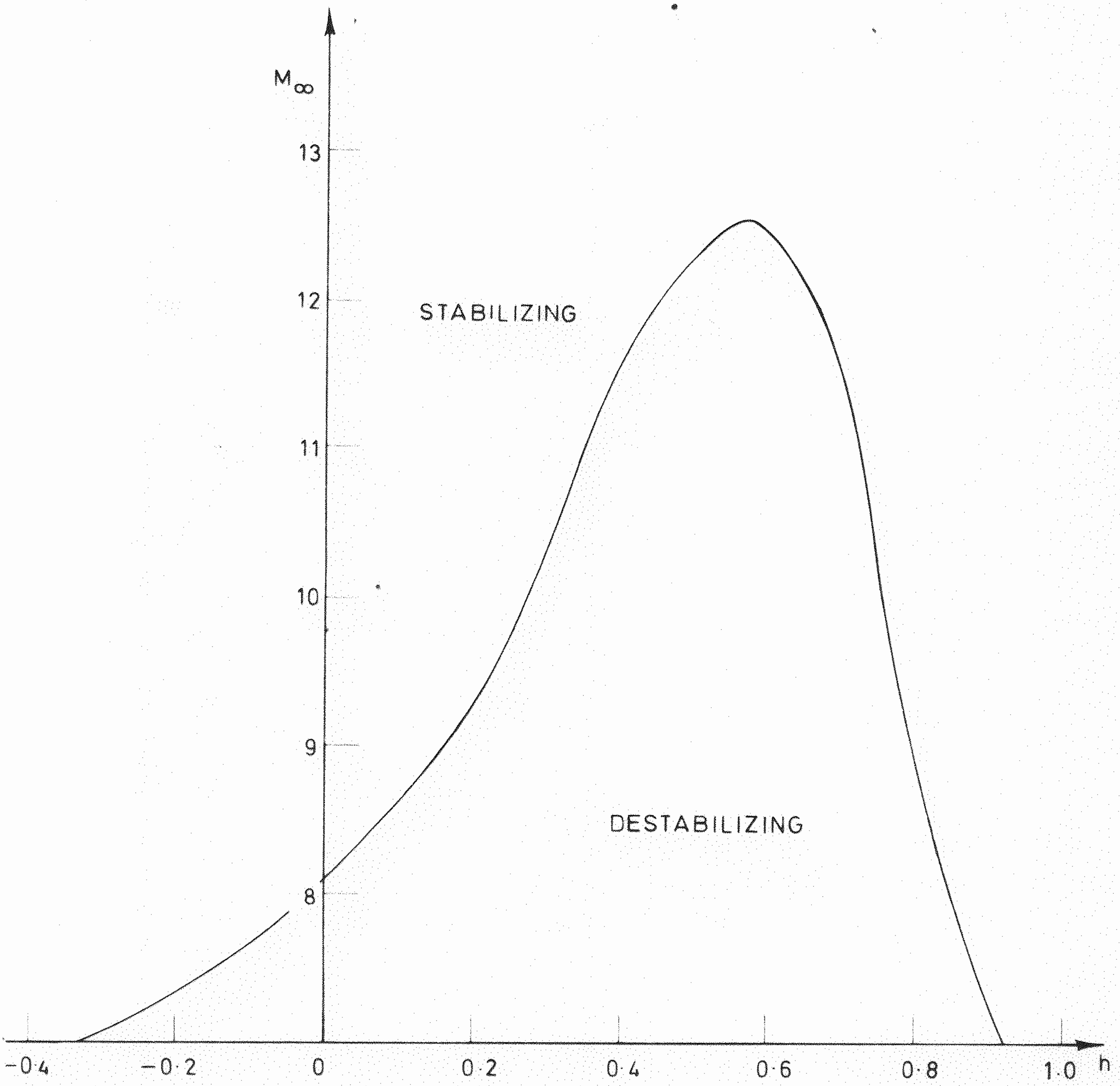

Fig. 21 . Neutral damping boundary for a wedge of 40° semivertex angle. $\gamma = 1{\cdot}4$.

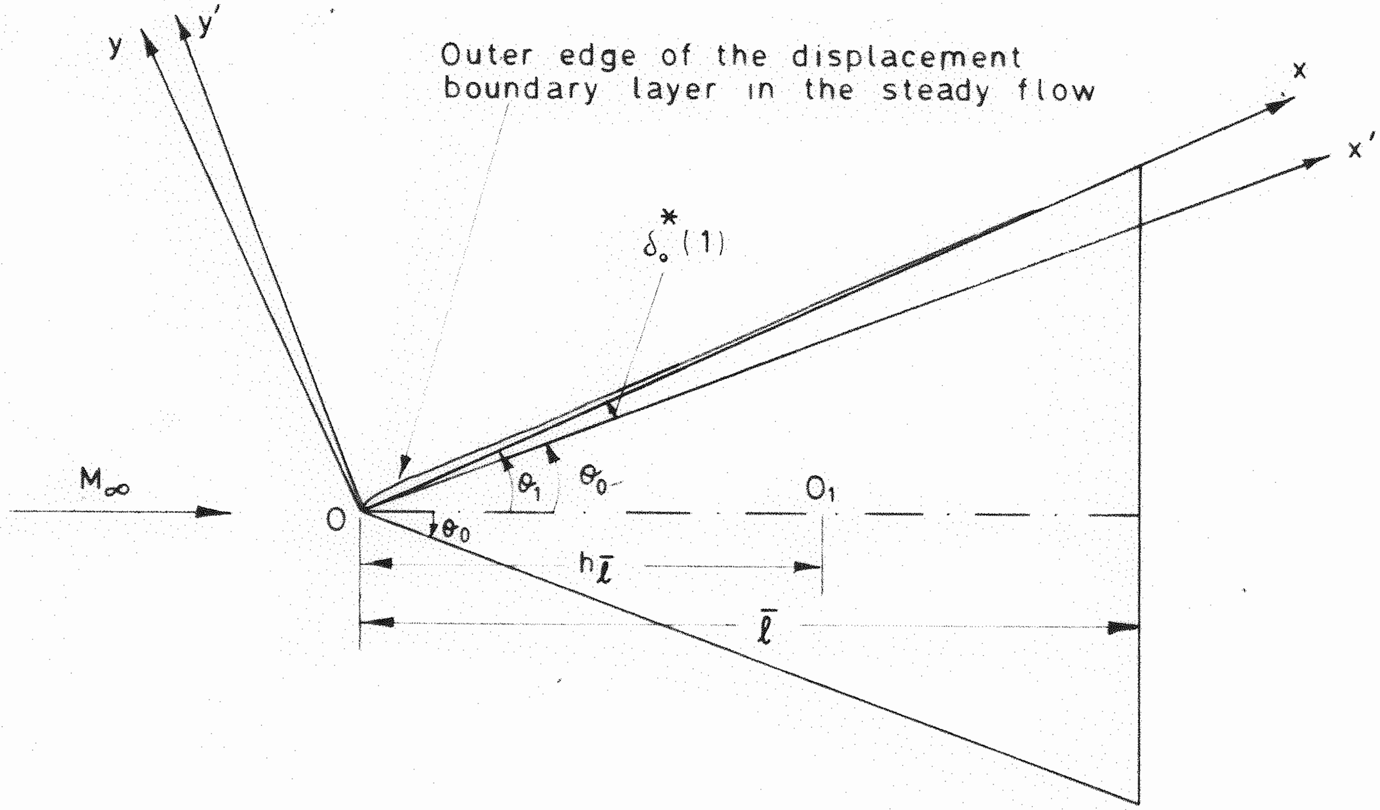

Fig.22 Effective wedge related to the original wedge.

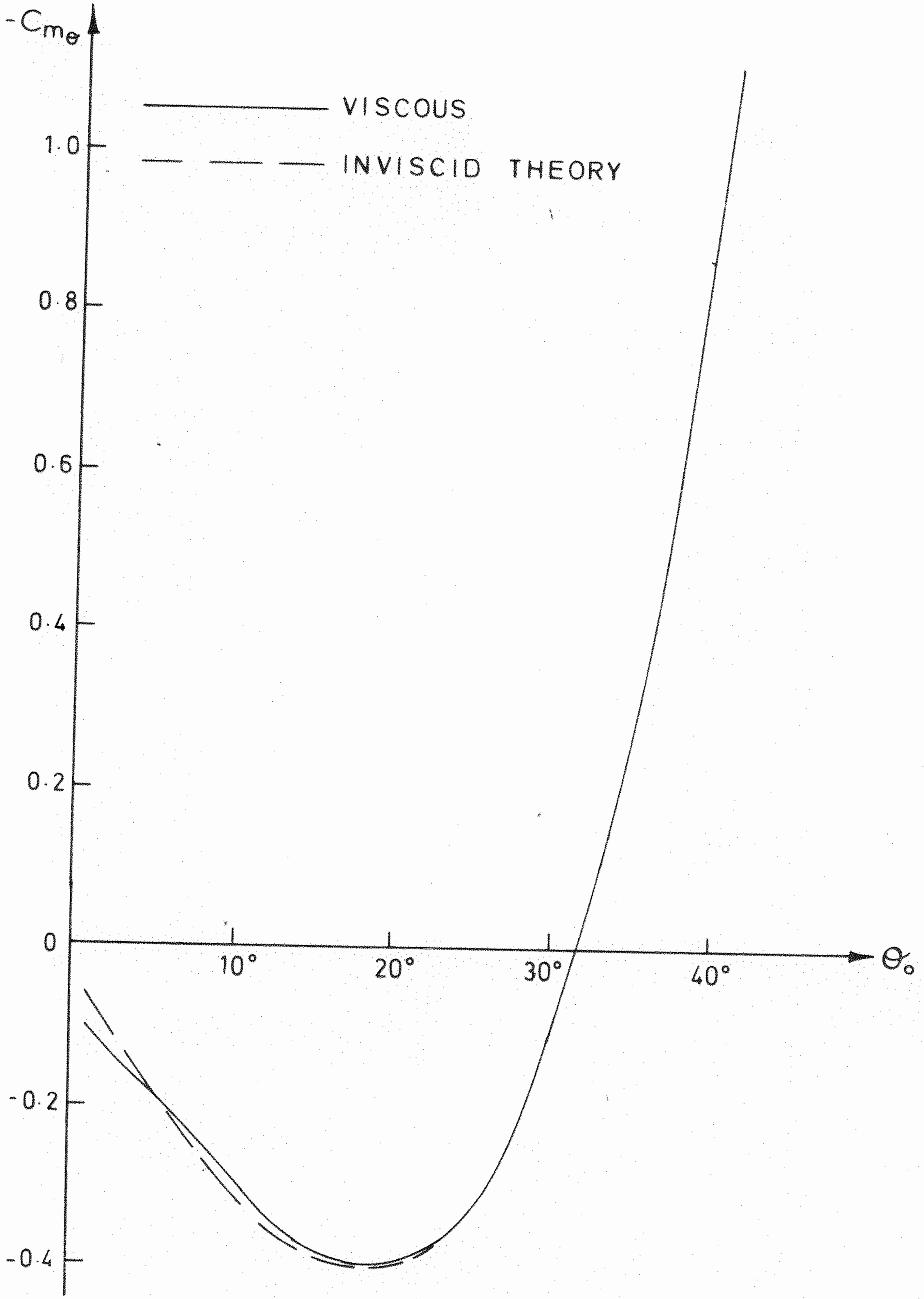

Fig. 23 Stiffness derivatives of sharp wedges. $M_\infty = 17$, $\gamma = 1{\cdot}4$, $h = 0{\cdot}7$

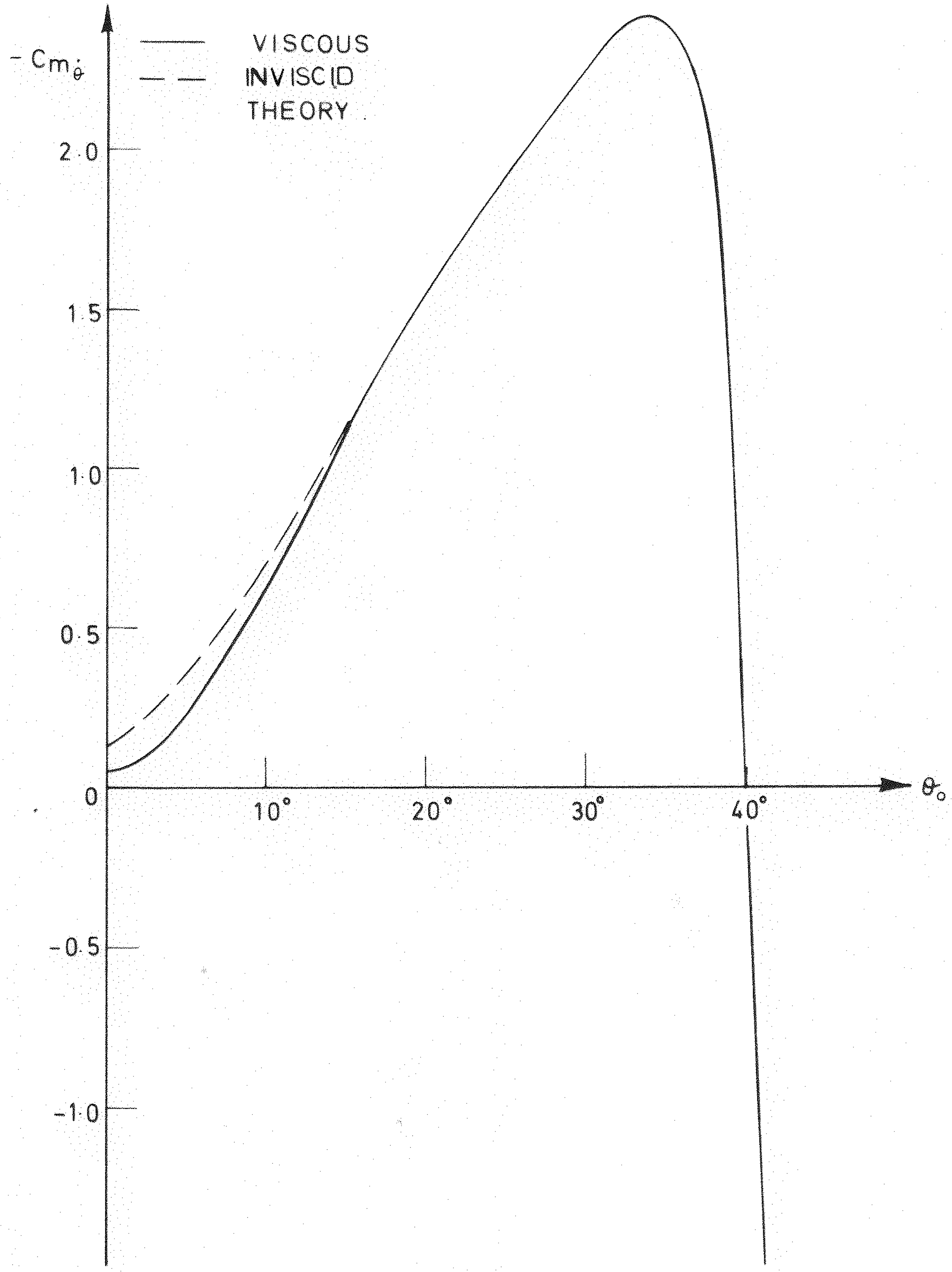

Fig.24 Damping derivatives of sharp wedges. $M_\infty = 17$, $\gamma = 1{\cdot}4$, $h = 0$.

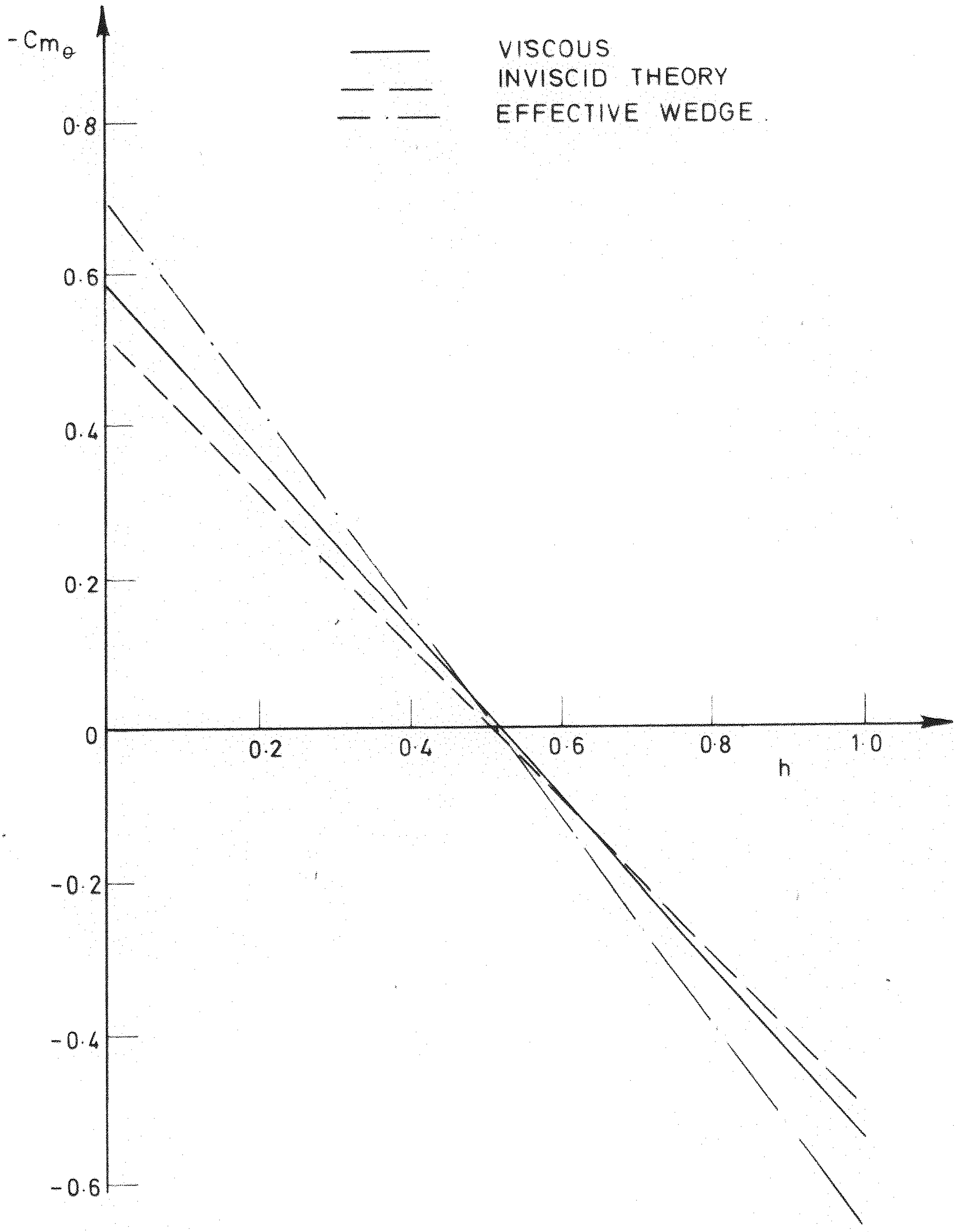

Fig.25 Stiffness derivatives of a sharp wedge. $M_\infty = 17$, $\gamma = 1{\cdot}4$, $\theta_0 = 6°$

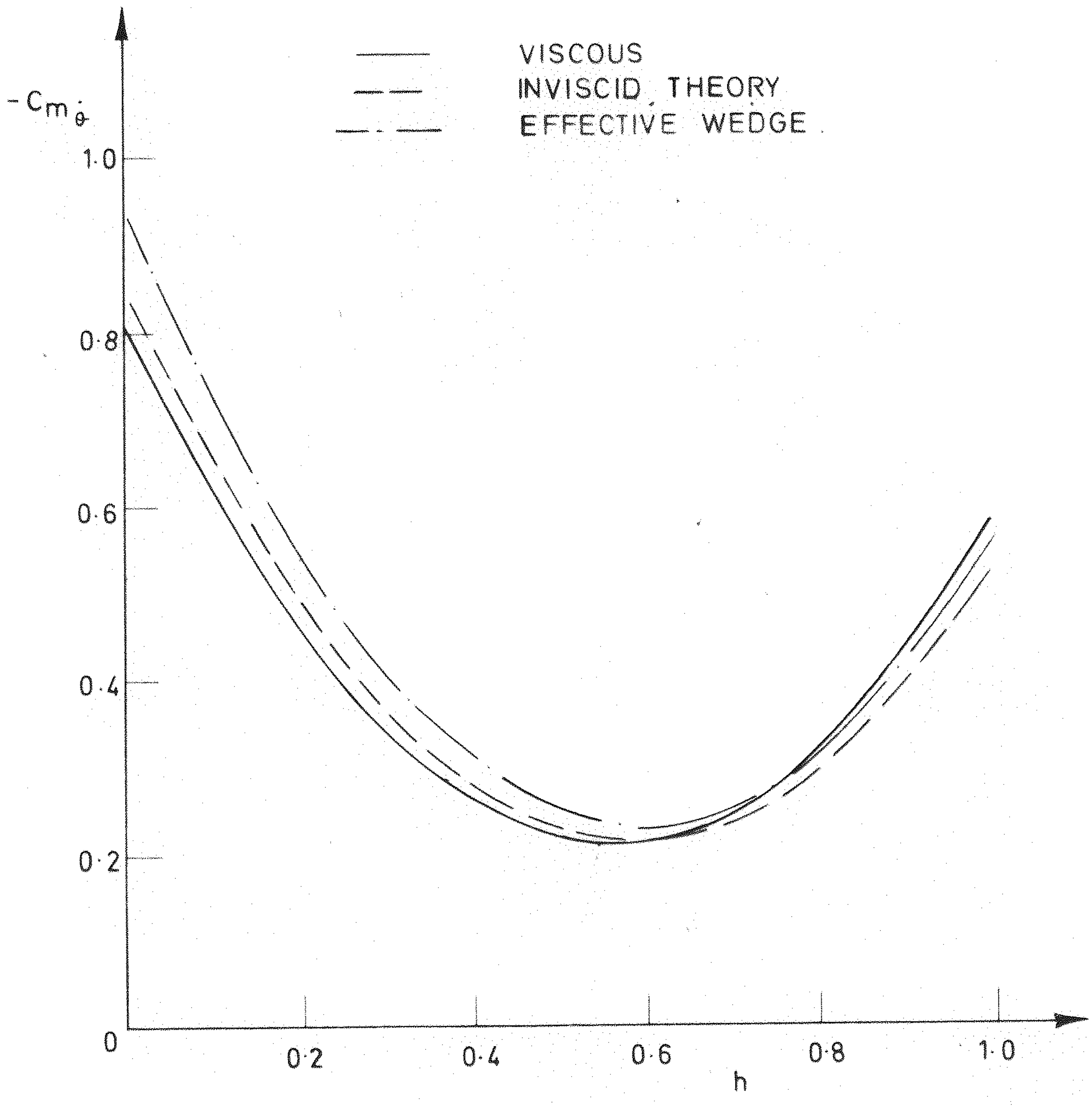

Fig 26 Damping derivatives of a sharp wedge.
$M_\infty = 17$, $\gamma = 1{\cdot}4$, $\theta_0 = 12°$

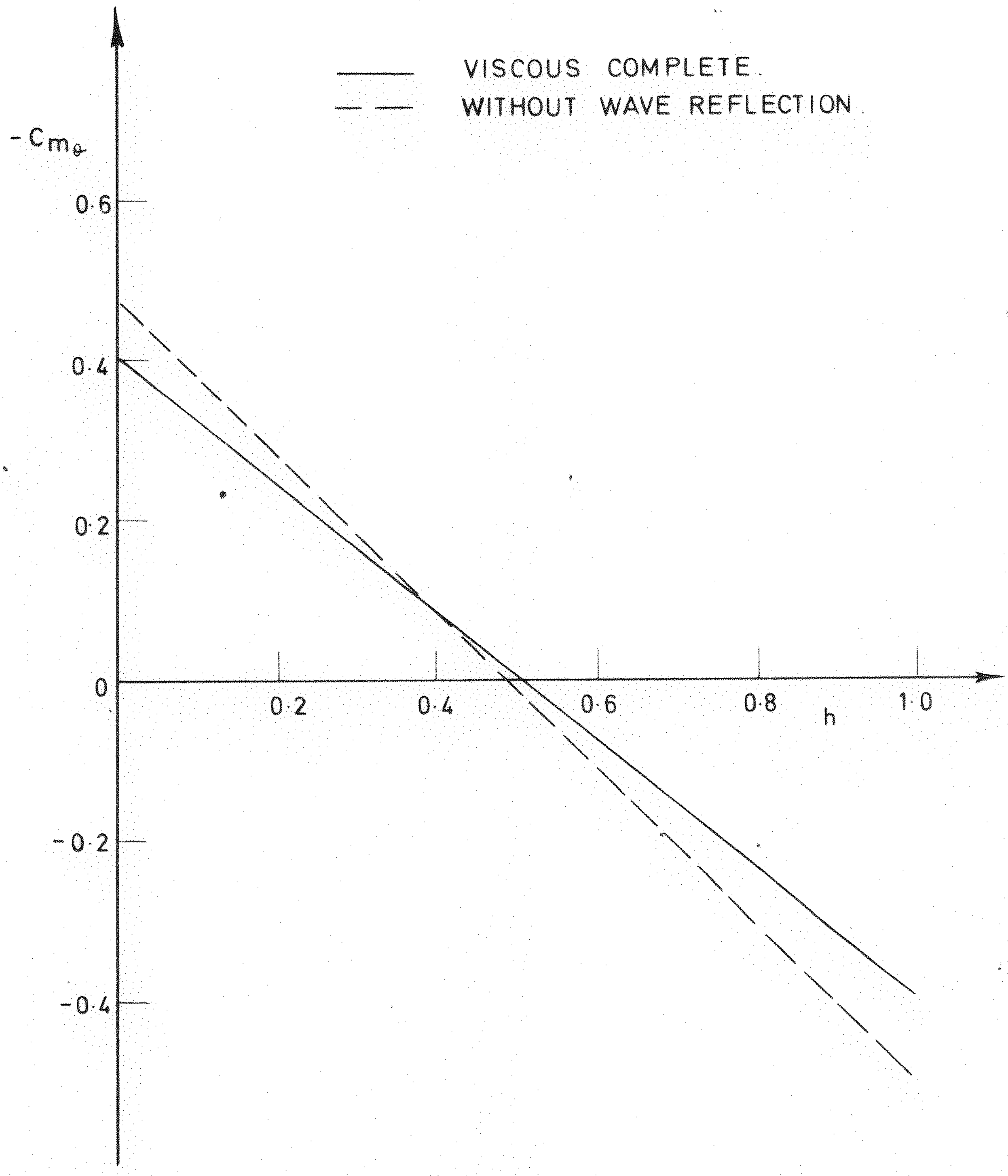

Fig. 27 Effects of wave reflection on the stiffness derivatives. $M_\infty = 17$, $\gamma = 1.4$ $\theta_0 = 3°$

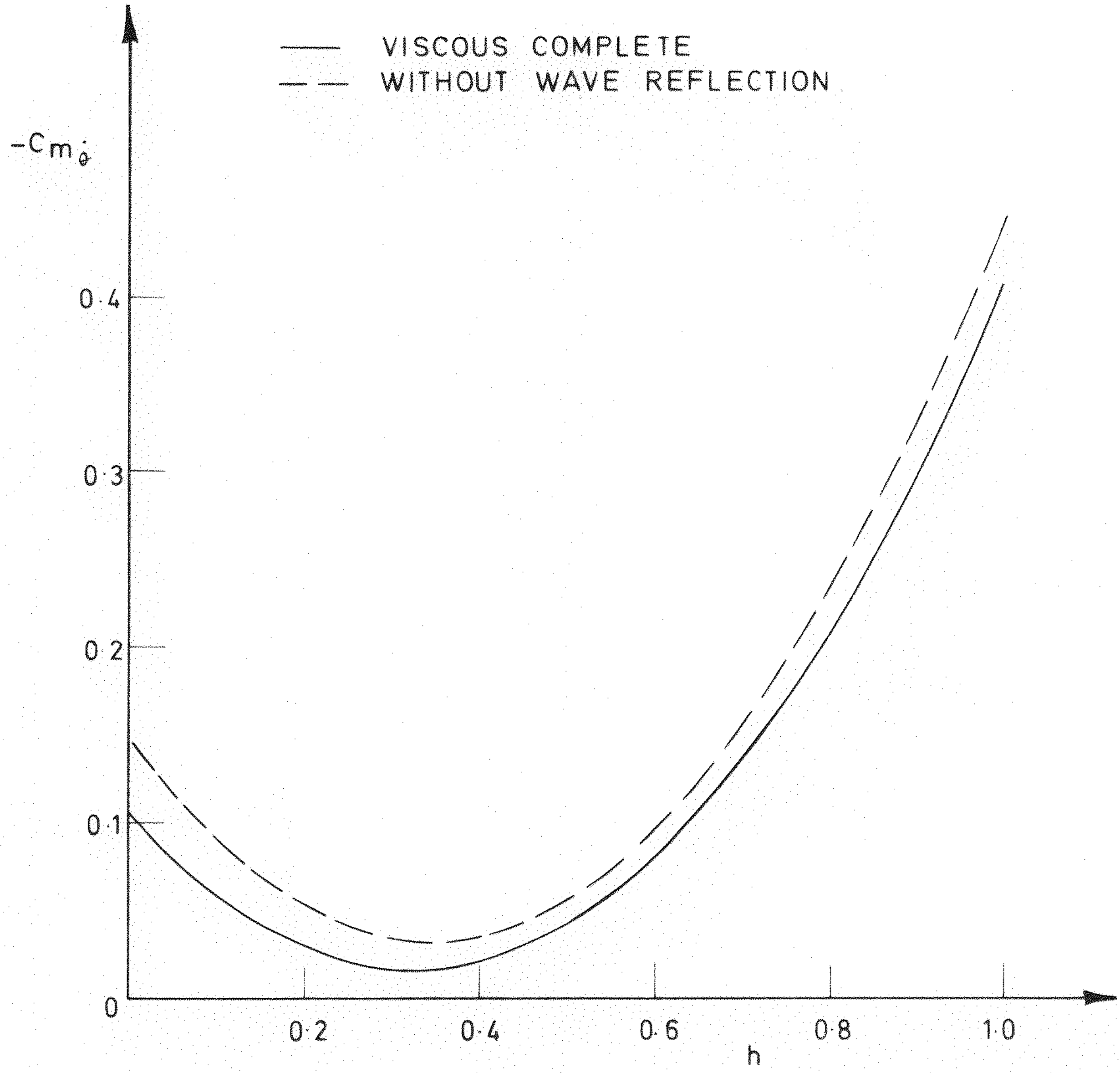

Fig 28 Effects of wave reflection on the damping derivatives. $M_\infty = 17$, $\gamma = 1{\cdot}4$, $\theta_0 = 3°$

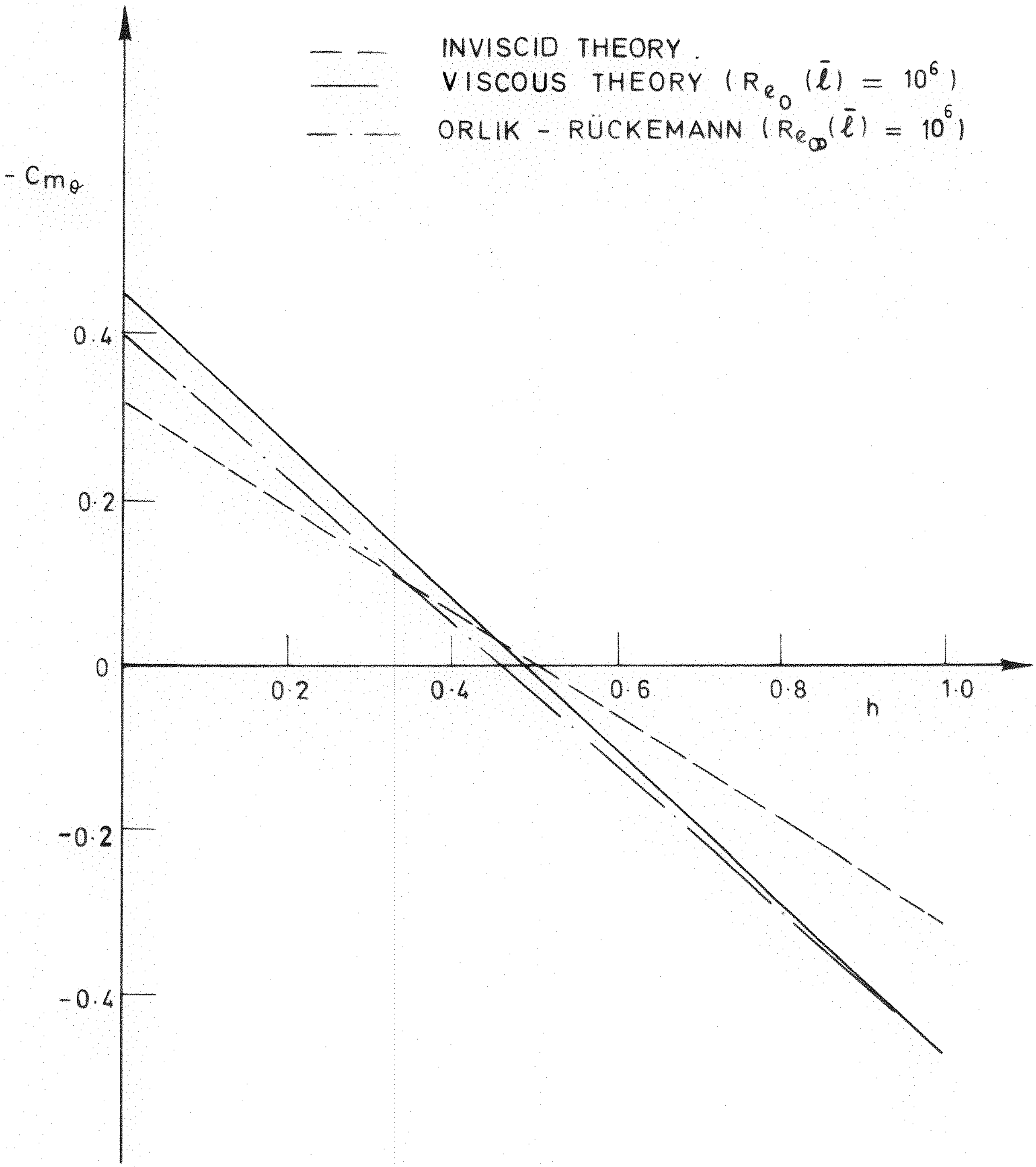

Fig 29 Comparison of the stiffness derivatives of a sharp insulated wedge. $M_\infty = 17$, $\gamma = \frac{5}{3}$, $\theta_0 = 3°$.

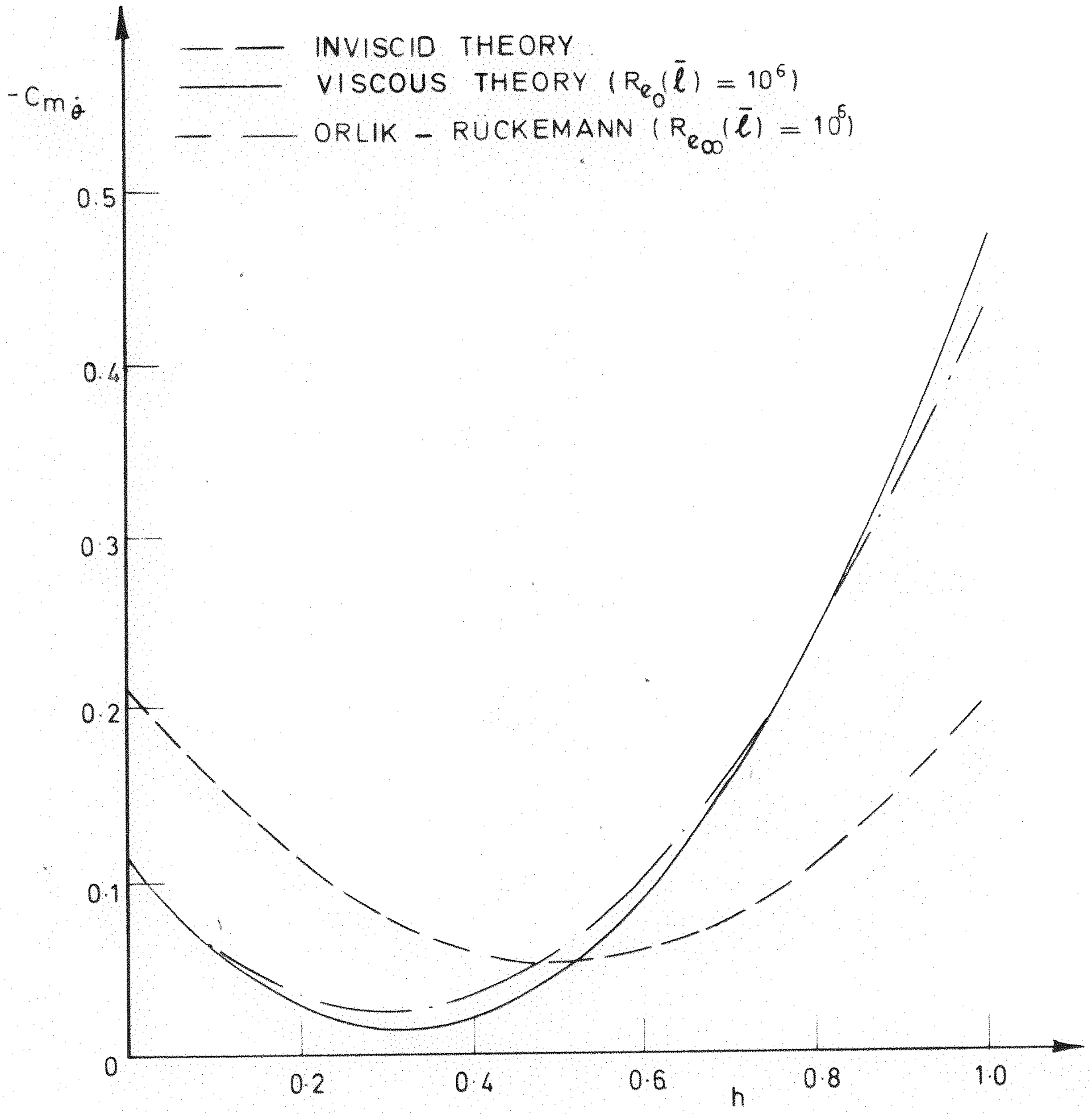

Fig 30 Comparison of the damping derivatives of a sharp insulated wedge. $M_\infty = 17$, $\gamma = \frac{5}{3}$, $\theta_0 = 3°$

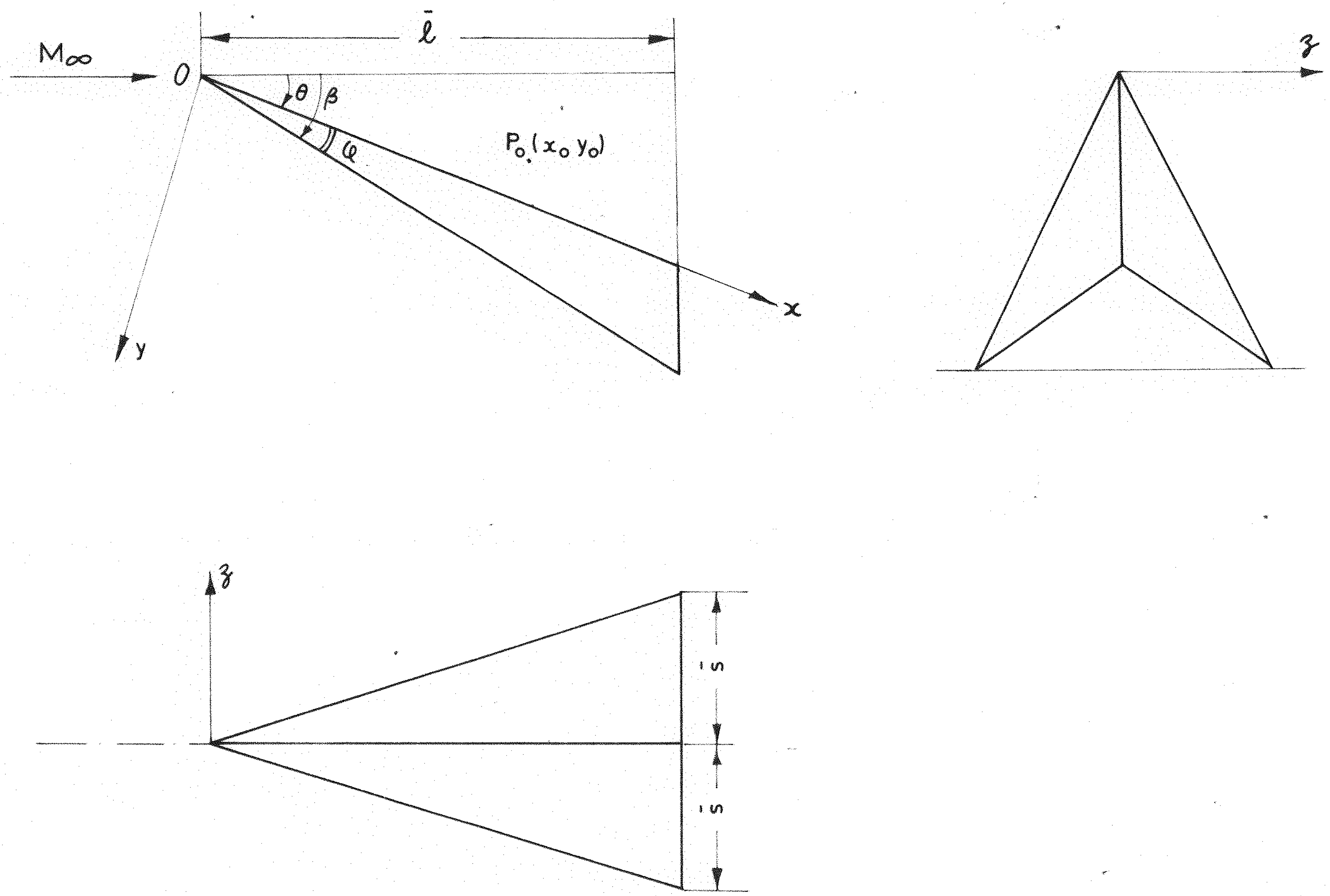

Fig. 31 Caret wing showing notation.

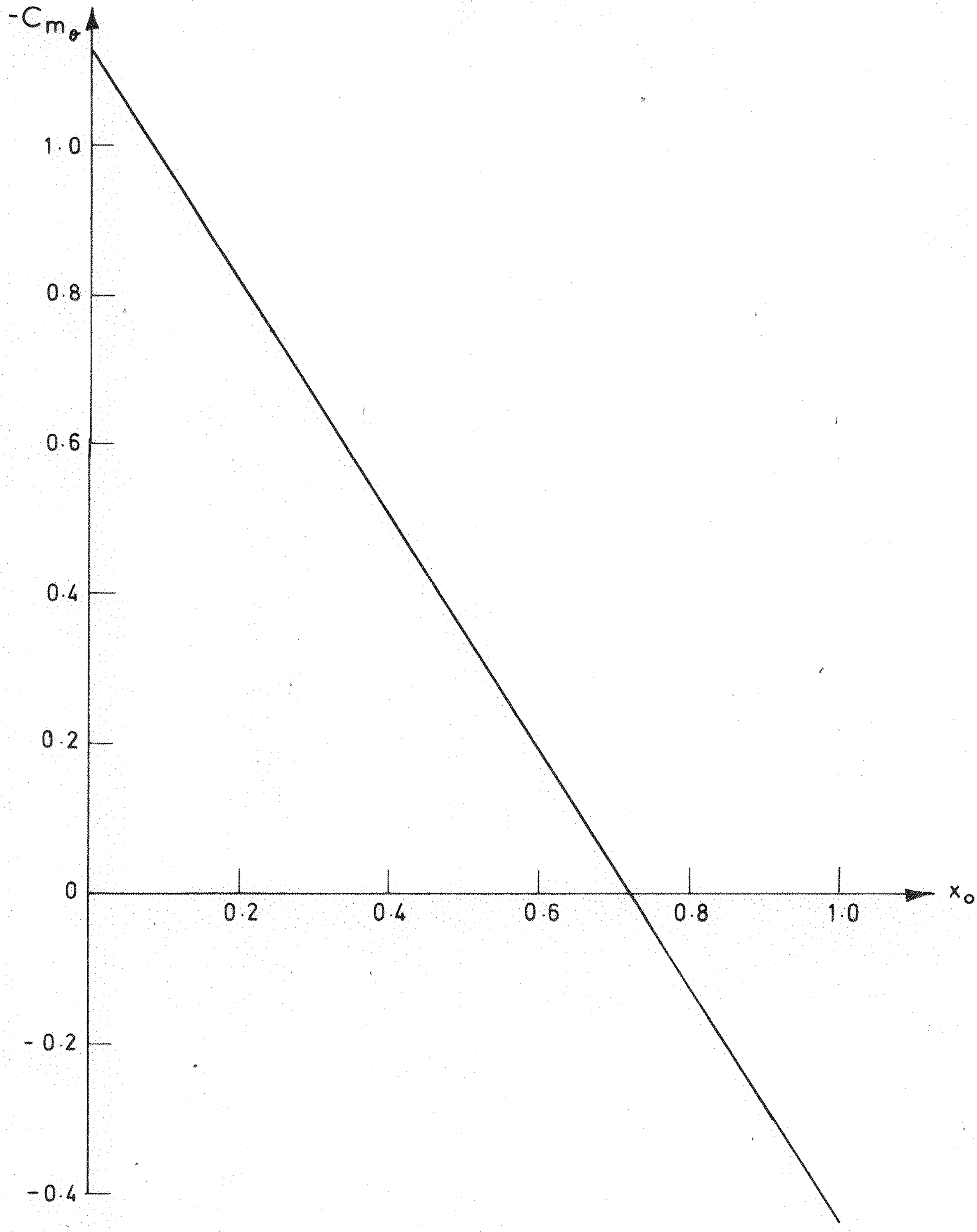

Fig 32 Stiffness derivative of a caret wing at its design condition. $M_\infty = 17$, $\theta = 20°$

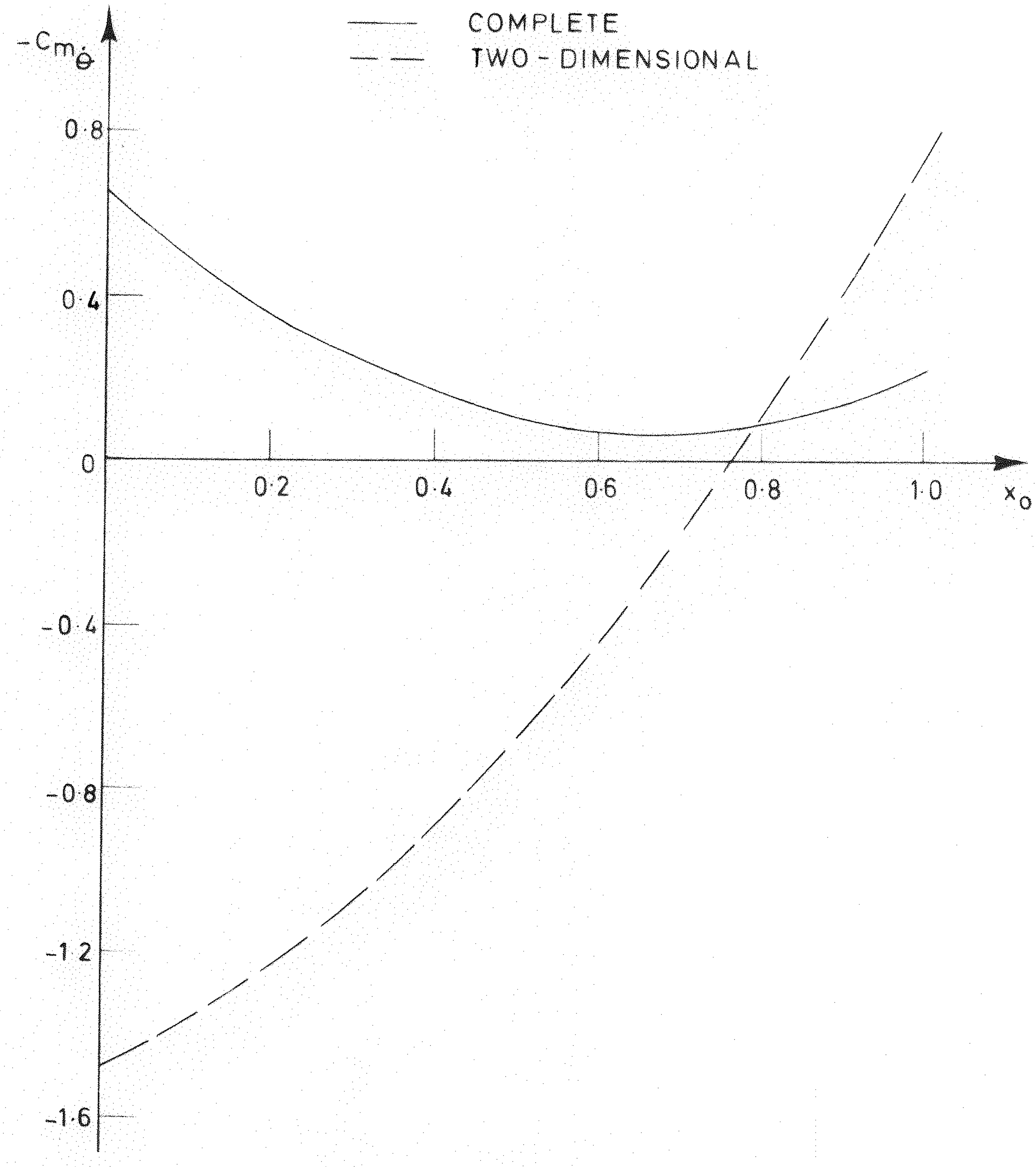

Fig. 33 Damping derivative of a caret wing at design condition. $M_\infty = 5$ $\theta = 15°38'$.

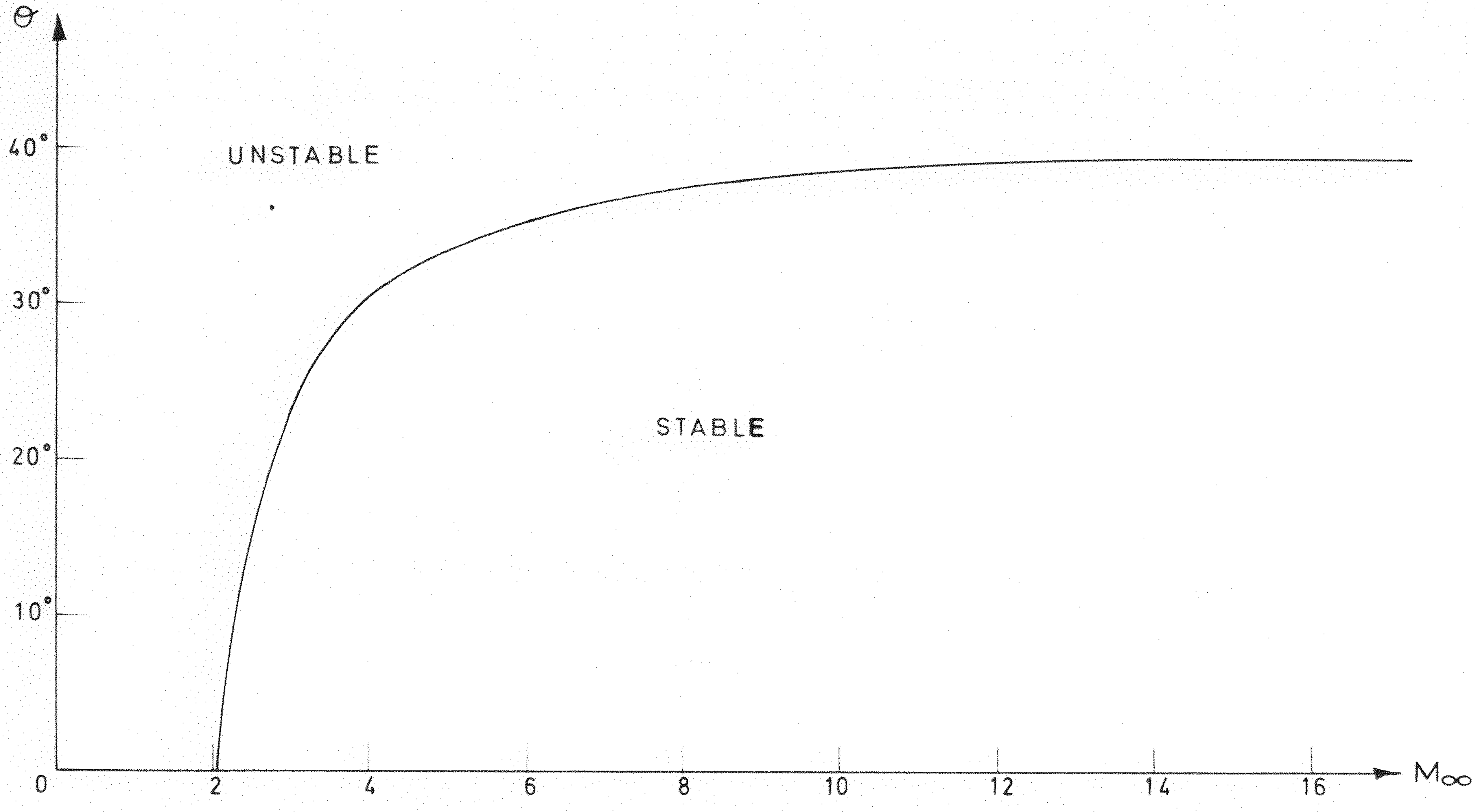

Fig. 34 General criterion for the stability of caret wings at design condition.

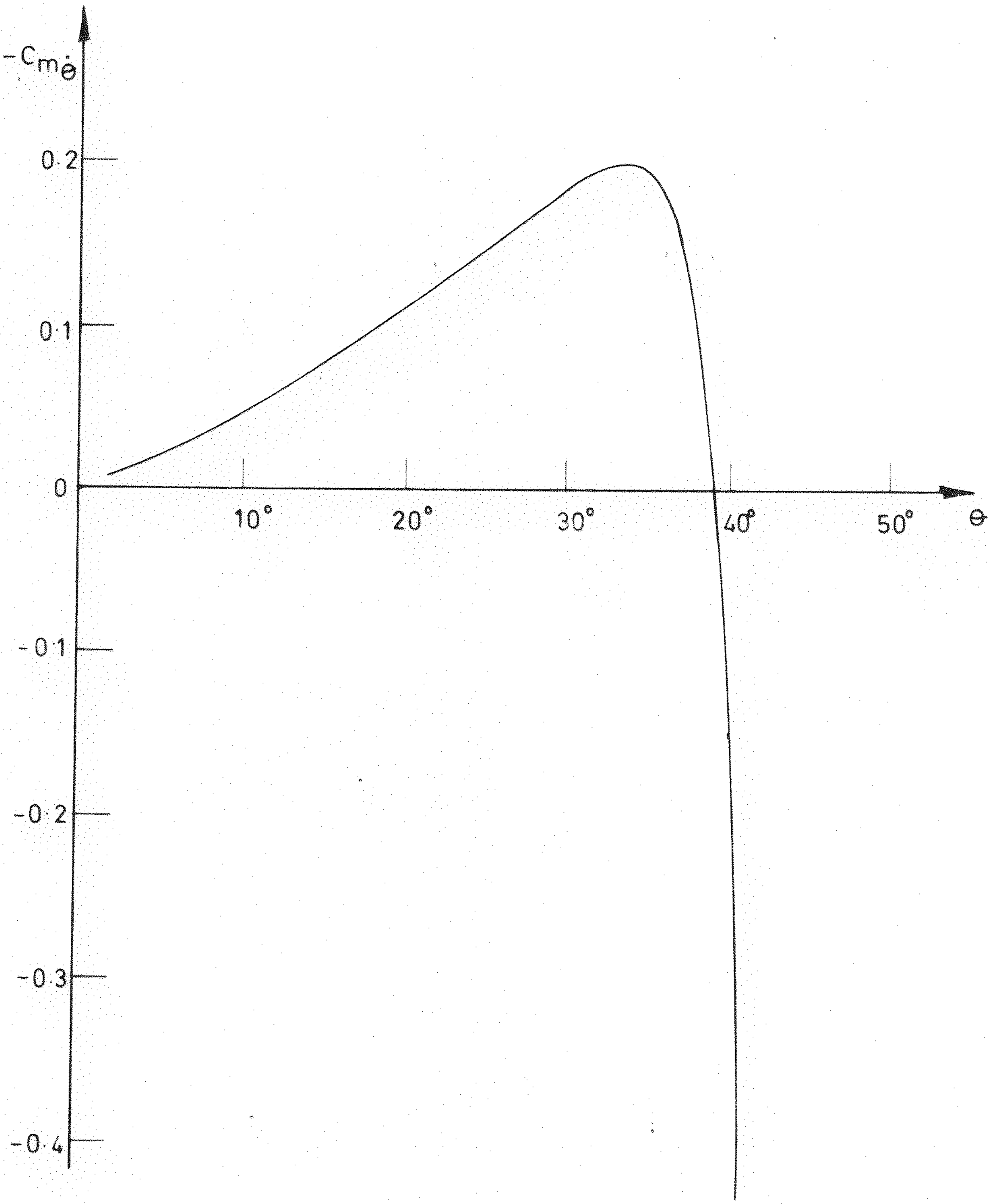

Fig 35 Damping derivative of caret wings at design condition. $M_\infty = 17$, $x_0 = 0{\cdot}7$