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One-loop open-string integrals from differential equations: all-order α'-expansions at n points

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Abstract: We study generating functions of moduli-space integrals at genus one that are expected to form a basis for massless n-point one-loop amplitudes of open superstrings and open bosonic strings. These integrals are shown to satisfy the same type of linear and homogeneous first-order differential equation w.r.t. the modular parameter τ which is known from the A-elliptic Knizhnik–Zamolodchikov–Bernard associator. The expressions for their τ-derivatives take a universal form for the integration cycles in planar and non-planar one-loop open-string amplitudes. These differential equations manifest the uniformly transcendental appearance of iterated integrals over holomorphic Eisenstein series in the low-energy expansion w.r.t. the inverse string tension α'. In fact, we are led to conjectural matrix representations of certain derivations dual to Eisenstein series. Like this, also the α'-expansion of non-planar integrals is manifestly expressible in terms of iterated Eisenstein integrals without referring to twisted elliptic multiple zeta values. The degeneration of the moduli-space integrals at $\tau \to i\infty$ is expressed in terms of their genus-zero analogues – $(n{+}2)$-point Parke–Taylor integrals over disk boundaries. Our results yield a compact formula for α'-expansions of n-point integrals over boundaries of cylinder- or Möbius-strip worldsheets, where any desired order is accessible from elementary operations.

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1 Introduction

Recent studies of scattering amplitudes in string theories have extended our computational reach into several directions and led to a variety of structural insights. Most of these developments were driven by a separation of string amplitudes into kinematic factors and moduli-space integrals. While kinematic factors carry the entire dependence on the external polarizations, the accompanying integrals over moduli spaces of punctured worldsheets depend on dimensionless combinations of the inverse string tension α' and external momenta.

In this work, we describe a new method to integrate over open-string punctures in generating functions of genus-one integrals in one-loop amplitudes of bosonic strings and superstrings. These integrations will be performed in a Laurent expansion w.r.t. α' – for any number of open-string punctures in both planar and non-planar one-loop amplitudes. A brief introduction of the method in a letter format has been given in [1].

In general, α'-expansions of string amplitudes have been identified as rewarding laboratory to encounter the periods of the underlying moduli spaces in a simple context, leading to fruitful crosstalk between string theory, particle phenomenology and number theory. Open- and closed-string tree amplitudes for instance are governed by disk and sphere worldsheets, and the associated periods are multiple zeta values (MZVs) [2–5] and their single-valued[1] analogues [4, 8–13], respectively[2].

One-loop open-string amplitudes in turn were shown [20, 21] to yield elliptic multiple zeta values (eMZVs) defined by Enriquez [22] upon integration over punctures on a cylinder or Möbius-strip worldsheet. Both of these worldsheet topologies are captured by more general integrals over A-cycles of a torus by different specializations of its complex modular parameter τ. The main target of this work is the α'-expansion of such A-cycle integrals, and we will unravel the patterns of eMZVs therein for any number of punctures.

The techniques in this work are driven by first-order differential equations for generating functions of A-cycle integrals w.r.t. the modular parameter τ of the torus. Moreover, the degeneration of the n-point genus-one integrals at the cusp $\tau \to i\infty$ is expressed in terms of $(n+2)$-point genus-zero integrals whose α'-expansions can be imported from [4, 5, 23, 24]. The desired α'-expansions at genus one are generated by solving the differential equations (along with an initial value at $\tau \to i\infty$) via standard Picard iteration. Accordingly, the accompanying eMZVs will appear as iterated integrals over holomorphic Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$ [22, 25, 26], "iterated Eisenstein integrals" for short. In this way, all relations among eMZVs are automatically built in[3], and the α'-expansions of A-cycle integrals are available in a maximally simplified form.

One can view this work as a one-loop generalization of the following structural results on tree-level α'-expansions:

- First, the coefficients of Riemann zeta values ζ_w in the α'-expansion of disk integrals are related to those of products or higher-depth MZVs [4]. These relations are most succinct in the f-alphabet description of (motivic) MZVs [28, 29] and imply [30] that disk integrals are stable under the motivic coaction [28, 29, 31, 32]. Based on the differential equations at genus one, we will identify similar patterns among the coefficients of different eMZVs or iterated Eisenstein integrals in the α'-expansion. Accordingly, the A-cycle integrals under investigation are preserved by the coaction of iterated Eisenstein integrals [33, 34] once we take a suitable quotient by their degeneration at $\tau \to i\infty$.

- Second, α'-expansions of n-point disk integrals can be recursively generated from matrix representations of the Drinfeld associator [5]. The specific representations follow from Knizhnik–Zamolodchikov (KZ) equations of higher-multiplicity disk integrals with an additional puncture and line up with the construction of more general braid ma-

[1] The notion of single-valued MZVs was introduced in [6, 7].

[2] See for instance [14–17] for earlier work on tree-level α'-expansions at $n \leq 7$ points, in particular [15, 16, 18, 19] for synergies with hypergeometric-function representations.

[3] This relies on the linear-independence result of [27] on iterated Eisenstein integrals.

trices in [35]. As a genus-one generalization, the A-cycle integrals under investigation are shown to obey the same type of differential equation in τ as the elliptic Knizhnik–Zamolodchikov–Bernard (KZB) associator [36–38]. In particular, n-point A-cycle integrals induce $(n-1)! \times (n-1)!$ matrix representations of certain derivations dual to Eisenstein series [39] which accompany the iterated Eisenstein integrals in the α'-expansions.

- Third, only $(n-3)!$ choices for n-point disk integrands are inequivalent under integration by parts, i.e. the so-called twisted cohomology at genus zero has dimension $(n-3)!$. This counting follows from the work of Aomoto in the mathematics literature [40] and has been independently conjectured and applied in the context of string tree amplitudes [41–44]. As a tentative one-loop analogue, the A-cycle integrands of this work are proposed to furnish representatives of an $(n-1)!$-dimensional twisted cohomology at genus one[4]. Our $(n-1)!$-family of A-cycle integrals is stable under τ-derivatives and should capture massless n-point one-loop amplitudes of both superstrings and open bosonic strings.

The differential-equation setup outlined above will be used to manifest the following properties of open-string α'-expansions at genus one:

- **Uniform transcendentality:** In conventional basis choices for disk integrals [4, 41, 42], the order in the α'-expansion is correlated with the transcendental weight of the accompanying MZVs, as can for instance be seen from [5]. The same kind of "uniform transcendentality"[5] becomes manifest for the A-cycle integrals in this work when generating their α'-expansions from the differential equations: Among other things, we will exploit that the $(n-1)! \times (n-1)!$ matrix representations of the derivations are linear in α', and that the initial value at $\tau \to i\infty$ is built from uniformly transcendental genus-zero integrals. Our open-string results are complemented by recent investigations of transcendentality properties of one-loop amplitudes of heterotic strings [50] and type-II superstrings [51].

- **No twisted eMZVs in non-planar amplitudes:** The earlier expansion method for non-planar open-string amplitudes [21, 26] introduces twisted eMZVs or cyclotomic analogues of eMZVs in intermediate steps. To the α'-orders considered, however, the end results were found to be expressible in terms of conventional (i.e. untwisted) eMZVs. The dropout of twisted eMZVs was not at all obvious from the techniques of [21, 26], and only some of the cases admitted an explanation by factorization on closed-string poles. The differential equations in this work will expose the absence of twisted eMZVs for any partition of n punctures on the two cylinder boundaries. The appearance of holomorphic Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$ in τ-derivatives turns out to be universal for A-cycle integrals referring to planar and non-planar cylinders, respectively.

[4] We are grateful to Albert Schwarz for discussions on twisted cohomologies at genus one. In collaboration with Dmitry Fuchs he calculated and analyzed the twisted homology of configuration spaces for surfaces of arbitrary genus.

[5] See [45–49] for a discussion of uniform transcendentality in the context of Feynman integrals in dimensional regularization, where the regularization parameter ε takes the role of α'.

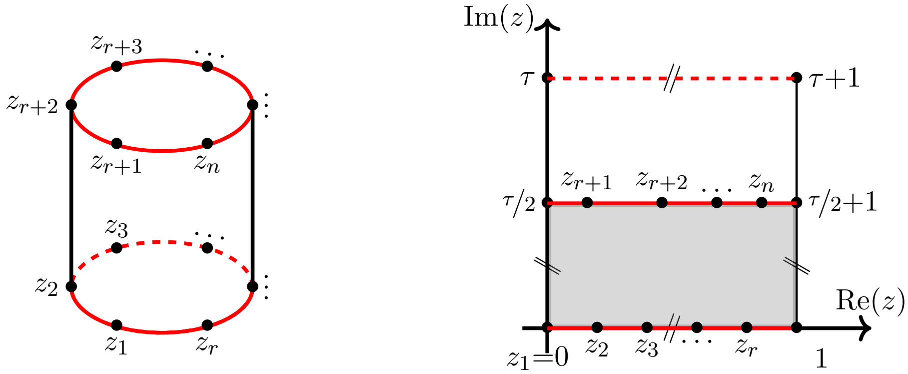


Figure 1: *The cylinder worldsheet for one-loop open-string amplitudes seen in the left panel is parameterized by the gray rectangle in the right panel, where the periodic direction is reflected by the identification of horizontal lines. The cylinder boundaries drawn in red are the integration domain $\mathcal{C}(*)$ of the integrals in (1.1) where a cyclic ordering of the punctures is prescribed within both boundaries. The asterisk is a placeholder for various distributions of the punctures, say $z_1, z_2, \ldots, z_r$ on the lower boundary and $z_{r+1}, \ldots, z_n$ on the upper one for some $0 \leq r \leq n$.*

1.1 Summary of the main results

We will now give a more detailed summary of the main results and display the key equations. First of all, the $(n-1)!$-family of A-cycle integrals to be investigated in this work is given by the generating functions

$$Z^\tau_{\vec{\eta}}(*|1,2,\ldots,n) = \int_{\mathcal{C}(*)} \mathrm{d}z_2\, \mathrm{d}z_3\, \ldots\, \mathrm{d}z_n\, \exp\Big(\sum_{i<j}^n s_{ij}\mathcal{G}(z_{ij},\tau)\Big) \tag{1.1}$$
$$\times\, \Omega(z_{12},\eta_2{+}\eta_3{+}\ldots{+}\eta_n,\tau)\Omega(z_{23},\eta_3{+}\ldots{+}\eta_n,\tau)\ldots\Omega(z_{n-1,n},\eta_n,\tau)$$

and their permutations in the external-state labels $2,3,\ldots,n$. The integration domain $\mathcal{C}(*)$ may refer to any planar or non-planar arrangement of open-string punctures z_j on the two boundaries of the cylinder depicted in figure 1. The two cylinder boundaries are parameterized by the unit interval $z_j \in (0,1)$ and its translate by $\tau/2$, i.e. parallel to the A-cycle of an auxiliary torus. While the cylinder in figure 1 is obtained from a rectangular torus with a purely imaginary modular parameter $\tau \in i\mathbb{R}_+$, our results for A-cycle integrals (1.1) will hold for general tori with $\mathrm{Re}\,\tau \neq 0$. In particular, integrals over the boundary of a Möbius strip can be obtained by specializing the A-cycle integrals in a planar ordering to $\tau \in \frac{1}{2}+i\mathbb{R}_+$ [52].

The one-loop Green function $\mathcal{G}(z,\tau)$ and the doubly-periodic Kronecker–Eisenstein series $\Omega(z,\eta,\tau)$ in (1.1) with first arguments $z_{ij} = z_i{-}z_j$ and formal expansion variables η_j will be introduced below. The Mandelstam variables are taken to be dimensionless,

$$s_{ij} = -2\alpha' k_i \cdot k_j\,, \qquad 1 \leq i < j \leq n\,, \tag{1.2}$$

i.e. the α'-expansions refer to simultaneous Laurent expansion w.r.t. all of (1.2).

Their dependence on $n-1$ bookkeeping variables $\eta_2, \eta_3, \ldots, \eta_n$ promotes the $Z^{\tau}_{\vec{\eta}}$ in (1.1) to generating series of the A-cycle integrals that enter specific one-loop string amplitudes. Open-superstring amplitudes with maximal supersymmetry give rise to component integrals at homogeneity degree -3 in the η_j, and also at degrees $-7, -9, \ldots$ in case of $n \geq 8$ external legs [20, 53]. Similarly, one-loop superstring amplitudes with reduced supersymmetry additionally involve degree-η^{-1} and η^{-5} parts of (1.1) [54, 55], while degree-η^{+1} parts only enter bosonic-string amplitudes or chiral halves of the heterotic string [56]. Higher orders in the η_j are likely to be relevant for one-loop amplitudes involving massive states.

The A-cycle integrals in (1.1) can be viewed as the one-loop generalization of the Parke–Taylor integrals over disk boundaries at genus zero [42]

$$Z^{\rm tree}(a_1, a_2, \ldots, a_n | 1, 2, \ldots, n) = \int\limits_{-\infty < z_{a_1} < z_{a_2} < \ldots < z_{a_n} < \infty} \frac{\mathrm{d}z_1 \, \mathrm{d}z_2 \, \ldots \, \mathrm{d}z_n}{\mathrm{vol} \, \mathrm{SL}_2(\mathbb{R})} \, \frac{\prod_{i<j}^n |z_{ij}|^{-s_{ij}}}{z_{12} z_{23} \ldots z_{n-1,n} z_{n,1}} \, . \quad (1.3)$$

The labelling of both $Z^{\rm tree}(\cdot|\cdot)$ and $Z^{\tau}_{\vec{\eta}}(\cdot|\cdot)$ by two slots "$\cdot$" emphasizes that moduli-space integrals in open-string amplitudes are pairings of integration cycles and differential forms. In the tree-level case (1.3), the cycle is labelled by a permutation $a_1, a_2, \ldots, a_n \in S_n$ of the external legs $1, 2, \ldots, n$ in the first slot which does not need to correlate with the arrangements of the factors z_{ij} in the Parke–Taylor denominator $z_{12} z_{23} \ldots z_{n,1}$. The product of Kronecker–Eisenstein series $\Omega(\ldots)$ in (1.1) takes the role of the Parke–Taylor forms, though the second slot of $Z^{\tau}_{\vec{\eta}}(\cdot|1, 2, \ldots, n)$ is not cyclically identified with $2, \ldots, n, 1$ by the absence of $\Omega(z_{n,1}, \eta, \tau)$ in the integrand.

The permutation symmetric products of $|z_{ij}|^{-s_{ij}}$ and $\exp(s_{ij} \mathcal{G}(z_{ij}, \tau))$ in (1.3) and (1.1) are referred to as the Koba–Nielsen factors at genus zero and one, respectively. They introduce the α'-dependence into the moduli-space integrals and suppress boundary terms when integrating total derivatives w.r.t. the punctures. The resulting integration-by-parts relations define twisted cohomologies at the respective genus [57]. At genus zero, the twisted cohomology entering the second slot of $Z^{\rm tree}(\cdot|\cdot)$ in (1.3) is known to be spanned by $(n-3)!$ Parke–Taylor factors [40].

At genus one, we propose that $(n-1)!$ permutations of $2, 3, \ldots, n$ in the second slot of (1.1) yield valid representatives of the twisted cohomology. A piece of evidence stems from the fact that these $(n-1)!$ permutations are closed under τ-derivatives,

$$2\pi i \partial_\tau Z^{\tau}_{\vec{\eta}}(*|1, a_2, a_3, \ldots, a_n) = \sum_{B \in S_{n-1}} D^{\tau}_{\vec{\eta}}(A|B) Z^{\tau}_{\vec{\eta}}(*|1, b_2, b_3, \ldots, b_n) \, . \quad (1.4)$$

The matrix $D^{\tau}_{\vec{\eta}}(A|B)$ is indexed by permutations $A = a_2, a_3, \ldots, a_n$ and $B = b_2, b_3, \ldots, b_n$ in S_{n-1} that act on the labels $2, 3, \ldots, n$ of both the η_j and s_{ij} in (1.1). As we will see, each entry of $D^{\tau}_{\vec{\eta}}(A|B)$ is linear in s_{ij} which will be used to demonstrate uniform transcendentality of the α'-expansion of $Z^{\tau}_{\vec{\eta}}$. Moreover, $D^{\tau}_{\vec{\eta}}(A|B)$ is composed of second derivatives $\partial_{\eta_i} \partial_{\eta_j}$ and Weierstrass functions evaluated at sums of η_j. Given that the Weierstrass function is the generating series of holomorphic Eisenstein series G_{2m} including $\mathrm{G}_0 = -1$,

$$\wp(\eta,\tau) = -\frac{\mathrm{G}_0}{\eta^2} + \sum_{m=2}^{\infty}(2m-1)\eta^{2m-2}\mathrm{G}_{2m}(\tau)\,, \tag{1.5}$$

the $(n-1)! \times (n-1)!$ matrix $D^{\tau}_{\vec{\eta}}$ in (1.4) can be expanded as follows

$$D^{\tau}_{\vec{\eta}} = \sum_{m=0}^{\infty}(1-2m)\mathrm{G}_{2m}(\tau) r_{\vec{\eta}}(\epsilon_{2m})\,. \tag{1.6}$$

The notation and normalization for the coefficients of the Eisenstein series is motivated by the expectation that $r_{\vec{\eta}}(\epsilon_{2m})$ are $(n-1)! \times (n-1)!$ matrix representations of the derivations $\epsilon_0, \epsilon_2, \epsilon_4, \ldots$ firstly studied by Tsunogai [39]. Each entry of $r_{\vec{\eta}}(\epsilon_{2m>0})$ is of homogeneity degree η^{2m-2}, and $r_{\vec{\eta}}(\epsilon_0)$ contains derivatives $\partial_{\eta_i}\partial_{\eta_j}$ on top of degree-η^{-2} and degree-η^0 terms. The dependence of $r_{\vec{\eta}}(\epsilon_{2m})$ on η_j and ∂_{η_j} should preserve the commutation relations of the derivations [25, 58, 59], that is why they are referred to as a conjectural representation.

Note that differential equations of the form in (1.4) have also been found for Feynman integrals that evaluate to elliptic polylogarithms, see e.g. [34, 48, 60–74] and references therein. In particular, since the differential operator $D^{\tau}_{\vec{\eta}}(A|B)$ is linear in α', (1.4) can be viewed as the string-theory analogue of the ε-form of the differential equations for Feynman integrals in the elliptic case [48]. At genus zero, the KZ equations of higher-multiplicity disk integrals studied in [5] exhibit the same linearity of the right-hand side in α' and thereby furnish the string-theory analogue of the ε-form for Feynman integrals that evaluate to multiple polylogarithms [47].

Once we solve the differential equation (1.4) via Picard iteration, the expansion (1.6) of the differential operator $D^{\tau}_{\vec{\eta}}$ naturally leads to iterated Eisenstein integrals $\gamma(k_1,\ldots,k_r|\tau)$ to be reviewed below. Since the initial values $Z^{i\infty}_{\vec{\eta}}(*|1,B) = \lim_{\tau\to i\infty} Z^{\tau}_{\vec{\eta}}(*|1,B)$ will be assembled from known functions of η_j and genus-zero integrals (1.3), we can extract the α'-expansion of the A-cycle integrals from

$$\begin{aligned} Z^{\tau}_{\vec{\eta}}(*|1,A) = \sum_{r=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}} \prod_{j=1}^{r}(k_j-1)\,\gamma(k_1,k_2,\ldots,k_r|\tau) \\ \times \sum_{B\in S_{n-1}} r_{\vec{\eta}}(\epsilon_{k_r}\epsilon_{k_{r-1}}\ldots\epsilon_{k_2}\epsilon_{k_1})_A{}^B Z^{i\infty}_{\vec{\eta}}(*|1,B)\,. \end{aligned} \tag{1.7}$$

In an expansion w.r.t. η_j and α', each order can be assembled from a finite number of terms in the sum over $\{k_1,k_2,\ldots,k_r\}$ on the right-hand side. Hence, the α'-expansion of A-cycle integrals in open-string amplitudes that occur at specific orders of $Z^{\tau}_{\vec{\eta}}(*|1,A)$ in η_j follows from (1.7) via elementary operations – matrix multiplication and differentiation w.r.t. η_j. Since the eMZVs are represented via iterated Eisenstein integrals $\gamma(\ldots)$, the α'-expansions in (1.7) are available in their minimal form, i.e. all relations among eMZVs are already incorporated. This is analogous to expressing (motivic) MZVs in the f-alphabet [25], so one can think of (1.7) as generalizing the structure of the tree-level α'-expansion unraveled in [4] to genus one.

On top of the structural insights provided by (1.7), it has practical advantages in the explicit evaluation of α'-expansions. In contrast to earlier expansion methods for A-cycle integrals [20, 21], there is no need to rearrange elliptic iterated integrals via so-called "z-removal" techniques[6] when integrating over one puncture after the other. Moreover, all the kinematic poles of the A-cycle integrals such as s_{12}^{-1} are determined by the initial value $Z^{i\infty}_{\vec{\eta}}(*|1,B)$ and do not require any subtraction scheme [26] when integrating over the punctures.

It will turn out to be convenient in this work to consider all the $\frac{n}{2}(n-1)$ Mandelstam invariants in an n-point integral (1.1) as independent. This may appear surprising since the primary application of the α'-expansion of this work are massless string amplitudes with phase-space constraints $\sum_{i=1, i\neq j}^{n} s_{ij}$ $\forall$ $j=1,2,\ldots,n$ and $\sum_{1\leq i<j}^{n-1} s_{ij}=0$. As a practical benefit of an enlarged momentum phase space[7], one can already define non-trivial two- and three-point building blocks which can be recycled to simplify non-planar four and five-point string amplitudes. Moreover, having non-vanishing sums like $s_{12}+s_{13}+s_{23}$ at four points regularizes $\frac{0}{0}$ indeterminates which would otherwise arise from the method of this work.

1.2 Outline

The main body of this paper is organized as follows: After a review of basic material on eMZVs and iterated Eisenstein integrals in section 2, we illustrate the main results and techniques of this work by means of two-point examples in section 3. Section 4 is dedicated to the differential equation (1.4), with rigorous derivations up to five points and an n-point conjecture for the explicit form of the differential operator $D^{\tau}_{\vec{\eta}}$. The associated initial values for planar and non-planar A-cycle integrals at $\tau\to i\infty$ are expressed via linear combinations of disk integrals in section 5 and 6, respectively. In section 7, the results of this work are shown to imply that A-cycle integrals (1.1) exhibit uniformly transcendental α'-expansions and are preserved under the coaction.

The key equations of this work include the n-point proposal for $D^{\tau}_{\vec{\eta}}$ in (4.35) as well as the relations (6.7), (5.29) and (5.12) between their initial values and (n+2)-point disk integrals.

2 Basics of eMZVs and iterated Eisenstein integrals

This section aims to review various properties of elliptic analogues of MZVs and iterated integrals over holomorphic Eisenstein series. Our conventions for MZVs are fixed by ($n_j\in\mathbb{N}$)

$$\zeta_{n_1,n_2,\ldots,n_r}\equiv\sum_{0<k_1<k_2<\ldots<k_r}k_1^{-n_1}k_2^{-n_2}\ldots k_r^{-n_r}\,,\qquad n_r\geq 2\,,\tag{2.1}$$

where r and $n_1+n_2+\ldots+n_r$ are referred to as their depth and transcendental weight, respectively. The customary terminology "transcendental weight" used throughout this work is highly abusive since none of the ζ_n with odd n have been proven to be transcendental so far.

[6]The "z-removal" techniques of [20] have been recently described from the perspective of elliptic symbol calculus [34].

[7]The idea of relaxing momentum conservation goes back to Minahan [75] and was instrumental to describe the structure of one-loop string- and field-theory integrands with half-maximal supersymmetry [55, 76, 77].

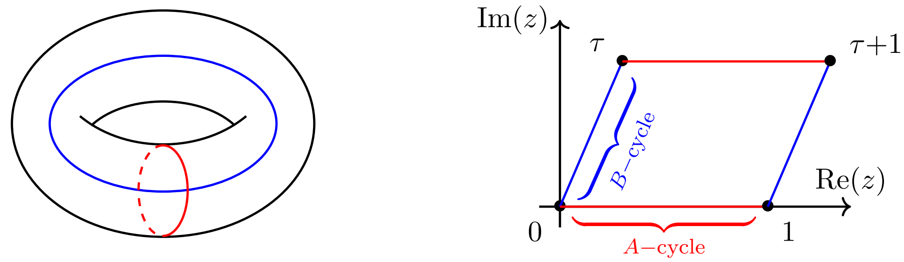


Figure 2: *The torus will be parameterized through a complex coordinate z, where the A-cycle and B-cycle translate into periodicities $z \cong z+1$ and $z \cong z+\tau$, respectively.*

2.1 Kronecker–Eisenstein series

The integrands under investigation in (1.1) are built from a non-holomorphic version of the Kronecker–Eisenstein series [78, 79],

$$\Omega(z,\eta,\tau) \equiv \exp\Big(2\pi i\eta \frac{\operatorname{Im} z}{\operatorname{Im}\tau}\Big)\frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}\,, \tag{2.2}$$

where $\operatorname{Im}\tau > 0$, and the tick of $\theta_1'(0,\tau)$ denotes a derivative w.r.t. the first argument. Our conventions for the modular parameter and the odd Jacobi theta function are $q = e^{2\pi i\tau}$ and

$$\theta_1(z,\tau) \equiv 2q^{1/8}\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-e^{2\pi iz}q^n)(1-e^{-2\pi iz}q^n)\,. \tag{2.3}$$

The double periodicity of the Kronecker–Eisenstein series (2.2)

$$\Omega(z,\eta,\tau) = \Omega(z{+}1,\eta,\tau) = \Omega(z{+}\tau,\eta,\tau) \tag{2.4}$$

qualifies the first argument z to live on a torus $\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$, see figure 2 for the parameterization used in this work.

Upon Laurent expansion in the second argument $\eta \in \mathbb{C}$, the expression for $\Omega(z,\eta,\tau)$ in (2.2) defines an infinite family of doubly-periodic functions $f^{(k)}$ with $k \in \mathbb{N}_0$,

$$\Omega(z,\eta,\tau) = \sum_{k=0}^{\infty}\eta^{k-1}f^{(k)}(z,\tau)\,, \qquad f^{(k)}(z,\tau) = f^{(k)}(z{+}1,\tau) = f^{(k)}(z{+}\tau,\tau)\,, \tag{2.5}$$

starting with $f^{(0)}(z,\tau) = 1$ and $f^{(1)}(z,\tau) = \partial_z \log\theta_1(z,\tau) + 2\pi i\frac{\operatorname{Im} z}{\operatorname{Im}\tau}$. While $f^{(1)}(z,\tau)$ has simple poles on the lattice $z \in \mathbb{Z}+\tau\mathbb{Z}$, the remaining $f^{(k\neq 1)}(z,\tau)$ are regular for any $z \in \mathbb{C}$.

It is worthwhile to highlight two properties of the doubly-periodic Kronecker–Eisenstein series (2.2) that will be used extensively in this work: Its bilinears obey Fay identities [79, 80]

$$\Omega(z_1,\eta_1,\tau)\Omega(z_2,\eta_2,\tau) = \Omega(z_1,\eta_1{+}\eta_2,\tau)\Omega(z_2{-}z_1,\eta_2,\tau) + \Omega(z_2,\eta_1{+}\eta_2,\tau)\Omega(z_1{-}z_2,\eta_1,\tau)\,, \tag{2.6}$$

and its derivatives are related through the mixed heat equation

$$2\pi i\partial_\tau\Omega(u\tau{+}v,\eta,\tau) = \partial_v\partial_\eta\Omega(u\tau{+}v,\eta,\tau)\,, \qquad u,v\in\mathbb{R}\,. \tag{2.7}$$

The τ-derivative in (2.7) is taken at a fixed value of the real coordinates $u,v\in\mathbb{R}$ of the first argument $z=u\tau+v$ in (2.2) which will be shown to be a natural choice in the context of one-loop open-string amplitudes.

2.2 eMZVs and twisted eMZVs

Enriquez defined eMZVs as the iterated integrals of the above Kronecker–Eisenstein coefficients $f^{(k)}(z,\tau)$ over the homology cycles of a torus [22]. We will only consider A-cycle eMZVs due to integration over $z\in(0,1)$ in this work and refer the reader to [22, 81–83] for discussions of B-cycle eMZVs (with $\frac{z}{\tau}\in(0,1)$ as an integration path). Based on the recursively defined elliptic iterated integrals [20, 79]

$$\Gamma\left(\begin{smallmatrix} k_1 & k_2 & \dots & k_r \\ a_1 & a_2 & \dots & a_r\end{smallmatrix};z|\tau\right) = \int_0^z \mathrm{d}t\, f^{(k_1)}(t{-}a_1,\tau)\,\Gamma\left(\begin{smallmatrix} k_2 & \dots & k_r \\ a_2 & \dots & a_r\end{smallmatrix};t|\tau\right)\,, \qquad z\in\mathbb{R},\quad a_j\in\mathbb{C} \tag{2.8}$$

with $\Gamma\left(\begin{smallmatrix}\emptyset\\ \emptyset\end{smallmatrix};z|\tau\right)=1$, Enriquez' A-cycle eMZVs can be obtained by evaluation at $z=1$ and specialization to $a_j=0$,

$$\omega(k_1,k_2,\ldots,k_r|\tau) \equiv \Gamma\left(\begin{smallmatrix} k_r & k_{r-1} & \dots & k_2 & k_1 \\ 0 & 0 & \dots & 0 & 0\end{smallmatrix};1|\tau\right)\,. \tag{2.9}$$

While the elliptic iterated integrals (2.8) are not homotopy invariant by themselves, it is known how to generate homotopy invariant uplifts by adding similar iterated integrals with simpler integration kernels [79]. Following the demands of the integrals over A-cycles in figure 1, we will always take the interval $(0,z)$ with $z\in\mathbb{R}$ as the integration path for (2.8).

The integers r and $k_1{+}k_2{+}\ldots{+}k_r$ characterizing the eMZV (2.9) are referred to as its length and (transcendental) weight. By the simple pole of $f^{(1)}(z,\tau)$ at lattice points $z\in\mathbb{Z}{+}\tau\mathbb{Z}$, both of $\omega(1,\ldots|\tau)$ and $\omega(\ldots,1|\tau)$ generically diverge. We will follow the regularization prescription in section 2.2.1 of [20] which assigns $\lim_{\tau\to i\infty}\omega(0,1|\tau)=\frac{i\pi}{2}$ and preserves the reflection and shuffle identities.

When the shifts a_j of the elliptic iterated integrals (2.8) are taken to be rational points on the torus, evaluation at $z=1$ gives rise to twisted eMZVs [21]

$$\omega\left(\begin{smallmatrix} k_1, & k_2, & \dots, & k_r \\ b_1, & b_2, & \dots, & b_r\end{smallmatrix}\Big|\,\tau\right) \equiv \Gamma\left(\begin{smallmatrix} k_r & k_{r-1} & \dots & k_2 & k_1 \\ b_r & b_{r-1} & \dots & b_2 & b_1\end{smallmatrix};1|\tau\right)\,, \qquad b_j\in\mathbb{Q}+\tau\mathbb{Q} \tag{2.10}$$

of length r and weight $k_1{+}k_2{+}\ldots{+}k_r$. Again, integration over $f^{(k_j=1)}$ may cause divergences if some of the twists b_j are real, see section 2.1 of [21] for a possible regularization. The only twists we encounter in this work are $b_j=0$ and $b_j={}^\tau\!/\!_2$, where the latter only occur in intermediate steps and do not cause any divergence when integrating $f^{(1)}(z{-}{}^\tau\!/\!_2,\tau)$ over the A-cycle $z\in(0,1)$. Note that double periodicity implies that $f^{(k)}(z{-}{}^\tau\!/\!_2,\tau)=f^{(k)}(z{+}{}^\tau\!/\!_2,\tau)$ $\forall\ k\in\mathbb{N}_0$, so one cannot distinguish shifts $a_j=\pm{}^\tau\!/\!_2$ in (2.8) or $b_j=\pm{}^\tau\!/\!_2$ in (2.10).

The functional dependence of twisted eMZVs on τ can be inferred by solving first-order differential equations along with a known initial value at $\tau \to i\infty$ [21]: By the mixed heat equation (2.7), the τ-derivatives of twisted eMZVs of length r are expressible via those of length $(r-1)$ multiplied by holomorphic Eisenstein series of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. At the cusp $\tau \to i\infty$, twisted eMZVs with $\mathrm{Re}\, b_j = 0$ degenerate to $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs. For untwisted eMZVs, the τ-derivative solely introduces Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$ [22, 25]. This leads to an expansion around the cusp of the form

$$\omega(k_1, k_2, \ldots, k_r|\tau) = \sum_{n=0}^{\infty} c_n(k_1, k_2, \ldots, k_r) q^n \,, \tag{2.11}$$

where the Fourier coefficients $c_n(\ldots)$ are $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs [22, 25]. Twisted eMZVs with $b_j \in \{0, \tau/2\}$ allow for a similar expansion,

$$\omega\Big(\begin{smallmatrix} k_1, & k_2, & \ldots & , & k_r \\ b_1, & b_2, & \ldots & , & b_r \end{smallmatrix} \Big| \, \tau\Big) = \sum_{n=0}^{\infty} d_n \Big(\begin{smallmatrix} k_1, & k_2, & \ldots & , & k_r \\ b_1, & b_2, & \ldots & , & b_r \end{smallmatrix}\Big) q^{n/2} \,, \tag{2.12}$$

where the powers of q include half-odd integers, and $d_n(\ldots)$ are again $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs [21].

As a main result of this work, we will derive and solve differential equations for eMZVs at the level of the A-cycle integrals (1.1). Hence, we provide a unified all-order treatment of the α'-dependence introduced by the Koba–Nielsen factor $\sim e^{s_{ij}\mathcal{G}(z_{ij},\tau)}$. Even for the integration cycles of non-planar open-string amplitudes, the τ-derivatives will be shown to solely introduce Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$, manifesting the dropout of twisted eMZVs at all orders in α'. The $\tau \to i\infty$ degenerations of the A-cycle integrals yield series in s_{ij} with $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations MZVs as coefficients. These series in s_{ij} and MZVs will be inferred from disk integrals with known α'-expansion [4, 5, 23, 24] which elegantly resums the degeneration of all the eMZVs in the α'-expansion of the A-cycle integrals (see [22, 25] for a method to determine the individual $\omega(k_1, \ldots, k_r|\tau \to i\infty)$).

2.3 Open-string Green functions

We will now review the connection between elliptic iterated integrals and the open-string Green function $\mathcal{G}(z,\tau)$ that enters the A-cycle integrals (1.1) through the Koba–Nielsen factor

$$\mathrm{KN}^{\tau}_{12\ldots n} \equiv \exp\Big(\sum_{i<j}^{n} s_{ij}\mathcal{G}(z_i - z_j, \tau)\Big) \,. \tag{2.13}$$

According to the parameterization of the cylinder in figure 1, placing the punctures $z_j = u_j\tau + v_j$ on the cylinder boundaries amounts to fixing the first real coordinate to $u_j = 0$ or $u_j = \frac{1}{2}$. The integrations in (1.1) along the boundaries – i.e. along A-cycles of a torus – only concern the second coordinate $v_j \in (0,1)$ with $\mathrm{d}z_j = \mathrm{d}v_j$.

When the Green function connects punctures on the same boundary, the first coordinate of $z_{ij} = z_i - z_j = u_{ij}\tau + v_{ij}$ is set to $u_{ij} = 0$, whereas z_i and z_j on different boundaries give rise to $u_{ij} = \frac{1}{2}$. In both cases, the dependence on $v_{ij} \in (-1, 1)$ can be expressed via elliptic iterated integrals (2.8) [20, 21] (with additional simplifications due to our choice $v_1 = 0$):

- both punctures on the same boundary ("planar Green function")

$$\begin{aligned}\mathcal{G}(v_{1j},\tau) &= \omega(1,0|\tau) - \Gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix};v_j|\tau\right) \\ \mathcal{G}(v_{ij},\tau) &= \omega(1,0|\tau) - \Gamma\left(\begin{smallmatrix}1\\v_j\end{smallmatrix};v_i|\tau\right) - \Gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix};v_j|\tau\right)\end{aligned} \tag{2.14}$$

- punctures on different boundaries ("non-planar Green function")

$$\begin{aligned}\mathcal{G}(v_{1j}+\tau/2,\tau) &= \omega\left(\begin{smallmatrix}1, & 0\\ \tau/2, & 0\end{smallmatrix}\Big|\,\tau\right) - \Gamma\left(\begin{smallmatrix}1\\ \tau/2\end{smallmatrix};v_j\Big|\,\tau\right) + \frac{i\pi\tau}{4} \\ \mathcal{G}(v_{ij}+\tau/2,\tau) &= \omega\left(\begin{smallmatrix}1, & 0\\ \tau/2, & 0\end{smallmatrix}\Big|\,\tau\right) - \Gamma\left(\begin{smallmatrix}1\\ v_j+\tau/2\end{smallmatrix};v_i\Big|\,\tau\right) - \Gamma\left(\begin{smallmatrix}1\\ \tau/2\end{smallmatrix};v_j\Big|\,\tau\right) + \frac{i\pi\tau}{4}\end{aligned} \tag{2.15}$$

When restricted to the on-shell kinematics $\sum_{1\leq i<j}^n s_{ij} = 0$ of massless n-point string amplitudes, the Koba–Nielsen factor (2.13) is invariant under shifts $\mathcal{G}(z_{ij},\tau) \to \mathcal{G}(z_{ij},\tau)+\mathcal{F}(\tau)$ of the Green function, where $\mathcal{F}(\tau)$ does not depend on the punctures. In this work, the A-cycle integrals (1.1) will be studied in the enlarged phase space with unconstrained $\sum_{1\leq i<j}^n s_{ij}$, so we committed to a choice of $\mathcal{F}(\tau)$ in specifying the Green functions in (2.14) and (2.15).

As firstly noticed in [81], the benefit of the additive term $\omega(1,0|\tau)$ in (2.14) is that the planar Green function integrates to zero along the A-cycle $\int_0^1 \mathrm{d}v_j\, \mathcal{G}(v_{1j},\tau) = 0$. Since the planar and non-planar Green functions have to descend from the same Koba–Nielsen factor (2.13), this leaves no further freedom in the non-planar Green function (2.15) which integrates to $\int_0^1 \mathrm{d}v_j\, \mathcal{G}(v_{1j}+\tau/2,\tau) = \frac{i\pi\tau}{4}$. The detailed connection of the above $\mathcal{G}(\ldots,\tau)$ with suitable restrictions of the closed-string Green function $-\log\left|\frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)}\right|^2 + \frac{2\pi}{\mathrm{Im}\,\tau}(\mathrm{Im}\,z)^2$ is explained in section 4.2 of [21]. Note that in section 5 of [81], the term $\frac{i\pi\tau}{4}$ in (2.15) has been absorbed into a redefinition of the Koba–Nielsen factor by products of $q^{s_{ij}/8}$, and we will similarly peel off these products in several examples in sections 3.5 and 6.

The mixed heat equation (2.7) and the recursive definition (2.8) of elliptic iterated integrals can be used to compute the v_j- and τ-derivatives of the Green function. For both choices of $u_{ij} \in \{0, \frac{1}{2}\}$ in $z_{ij} = u_{ij}\tau + v_{ij}$, one can show that[8]

$$\partial_{v_i}\mathcal{G}(z_{ij},\tau) = -f^{(1)}(z_{ij},\tau)\,, \qquad 2\pi i\partial_\tau\mathcal{G}(z_{ij},\tau) = -f^{(2)}(z_{ij},\tau) - 2\zeta_2\,, \tag{2.16}$$

which implies the following differential equations for the Koba–Nielsen factor (2.13):

$$\partial_{v_i}\mathrm{KN}^\tau_{12\ldots n} = -\sum_{\substack{j=1\\ j\neq i}}^n s_{ij} f^{(1)}_{ij}\mathrm{KN}^\tau_{12\ldots n}\,, \qquad 2\pi i\partial_\tau \mathrm{KN}^\tau_{12\ldots n} = -\sum_{1\leq i<j}^n s_{ij}(f^{(2)}_{ij}+2\zeta_2)\mathrm{KN}^\tau_{12\ldots n}\,. \tag{2.17}$$

[8]The following identities have been used in intermediate steps:

$$2\pi i\partial_\tau\Gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix};z|\tau\right) = f^{(2)}(z,\tau) - f^{(2)}(0,\tau)\,, \qquad 2\pi i\partial_\tau\omega(1,0|\tau) = -f^{(2)}(0,\tau) - 2\zeta_2\,, \qquad 2\pi i\partial_\tau\frac{i\pi\tau}{4} = -3\zeta_2$$

$$2\pi i\partial_\tau\Gamma\left(\begin{smallmatrix}1\\ \tau/2\end{smallmatrix};z|\tau\right) = f^{(2)}(z-\tau/2,\tau) - f^{(2)}(\tau/2,\tau)\,, \qquad 2\pi i\partial_\tau\omega\left(\begin{smallmatrix}1, & 0\\ \tau/2, & 0\end{smallmatrix}|\tau\right) = -f^{(2)}(\tau/2,\tau) + \zeta_2$$

Here and in later sections, we employ the shorthands

$$f_{ij}^{(k)} = f^{(k)}(z_{ij},\tau)\,, \tag{2.18}$$

and we emphasize that (2.17) is universally valid for any distribution of the n punctures over the two cylinder boundaries in figure 1, i.e. for any integration cycle $\mathcal{C}(*)$ in the A-cycle integral (1.1).

2.4 Iterated Eisenstein integrals

By repeatedly integrating the τ-derivative of A-cycle eMZVs, one is lead to iterated integrals of holomorphic Eisenstein series $\mathrm{G}_k(\tau)$ of $\mathrm{SL}_2(\mathbb{Z})$ [22, 25]. We will use conventions where $\mathrm{G}_0(\tau) = -1$ and

$$\mathrm{G}_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} = \begin{cases} 0 & : \ k>0 \text{ odd} \\ 2\zeta_k + \dfrac{2(2\pi i)^k}{(k-1)!}\displaystyle\sum_{m,n=1}^{\infty} m^{k-1}q^{mn} & : \ k>0 \text{ even} \end{cases}\,, \tag{2.19}$$

with Eisenstein summation prescription in case of $k=2$. Iterated Eisenstein integrals in the normalization of [25] are defined by

$$\gamma(k_1,k_2,\ldots,k_r|\tau) = \frac{(-1)^r}{(2\pi i)^{2r}} \int\limits_{0<q_1<q_2<\ldots<q_r<q} \frac{\mathrm{d}q_1}{q_1}\frac{\mathrm{d}q_2}{q_2}\ldots\frac{\mathrm{d}q_r}{q_r}\,\mathrm{G}_{k_1}(\tau_1)\mathrm{G}_{k_2}(\tau_2)\ldots\mathrm{G}_{k_r}(\tau_r)\,. \tag{2.20}$$

Integration over G_0 or the zero mode $2\zeta_k$ in the q-expansions of (2.19) may cause endpoint divergences from the integration region where $q_1\rightarrow 0$ in (2.20). These divergences will be regularized through the tangential-base-point prescription [84] with net effect $\int_0^q \frac{\mathrm{d}q_1}{q_1} = \log q$. This leads to regularized values such as

$$\gamma(0|\tau) = \frac{\tau}{2\pi i}\,, \qquad \gamma(4|\tau) = \frac{i\pi^3\tau}{90} + \frac{4\pi^2}{3}\sum_{m,n=1}^{\infty}\frac{m^2}{n}q^{mn}\,, \tag{2.21}$$

that preserve the shuffle relations of $\gamma(k_1,\ldots,k_r|\tau)$. Alternatively, one can bypass the endpoint divergences by subtracting the zero modes of the non-constant holomorphic Eisenstein series (2.19),

$$\mathrm{G}_k^0(\tau) = \begin{cases} \mathrm{G}_k(\tau) - 2\zeta_k & : \ k>0 \text{ even} \\ 0 & : \ k>0 \text{ odd} \\ -1 & : \ k=0 \end{cases}\,. \tag{2.22}$$

These integration kernels give rise to modified versions of iterated Eisenstein integrals

$$\gamma_0(k_1,k_2,\ldots,k_r|\tau) = \frac{(-1)^r}{(2\pi i)^{2r}} \int\limits_{0<q_1<q_2<\ldots<q_r<q} \frac{\mathrm{d}q_1}{q_1}\frac{\mathrm{d}q_2}{q_2}\ldots\frac{\mathrm{d}q_r}{q_r}\,\mathrm{G}_{k_1}^0(\tau_1)\mathrm{G}_{k_2}^0(\tau_2)\ldots\mathrm{G}_{k_r}^0(\tau_r)\,, \tag{2.23}$$

that converge if $k_1 > 0$ [25]. The q-expansion resulting from (2.19) takes the following compact form in these cases (with $k_j \neq 0$ and $p_j \geq 0$),

$$\gamma_0(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_d, 0^{p_d-1}|\tau) = (-2)^d \Big(\prod_{j=1}^{d} \frac{(2\pi i)^{k_j-2p_j}}{(k_j-1)!} \Big) \tag{2.24}$$
$$\times \sum_{\substack{m_1,m_2,\ldots,m_d=1 \\ n_1,n_2,\ldots,n_d=1}}^{\infty} \frac{m_1^{k_1-1} m_2^{k_2-1} \ldots m_d^{k_d-1} q^{m_1n_1+m_2n_2+\ldots+m_dn_d}}{(m_1n_1)^{p_1}(m_1n_1+m_2n_2)^{p_2} \ldots (m_1n_1+m_2n_2+\ldots+m_dn_d)^{p_d}} \,,$$

and the analogues of (2.21) are $\gamma_0(0|\tau) = \frac{\tau}{2\pi i}$ and $\gamma_0(4|\tau) = \frac{4\pi^2}{3} \sum_{m,n=1}^{\infty} \frac{m^2}{n} q^{mn}$. Both of $\gamma(k_1, k_2, \ldots, k_r|\tau)$ and $\gamma_0(k_1, k_2, \ldots, k_r|\tau)$ are said to have length r, and the number of nonzero entries $k_j \neq 0$ (i.e. the integer d in (2.24)) is referred to as their depth. The transcendental weight of both $\gamma(k_1, k_2, \ldots, k_r|\tau)$ and $\gamma_0(k_1, k_2, \ldots, k_r|\tau)$ is $k_1{+}k_2{+}\ldots{+}k_r{-}r$. This follows from the differential equation of eMZVs as well as the transcendental weights k of $\mathrm{G}_k(\tau), \mathrm{G}^0_k(\tau)$ and $k_1{+}k_2{+}\ldots{+}k_r$ of $\omega(k_1, k_2, \ldots, k_r|\tau)$.

The decomposition of A-cycle eMZVs into iterated Eisenstein integrals usually involves fewer terms when employing $\gamma_0(\ldots)$ instead of $\gamma(\ldots)$, e.g. [25]

$$\begin{aligned} \omega(0, 2k{-}1|\tau) &= \delta_{1,k} \frac{\pi i}{2} + (2k{-}1)\gamma_0(2k|\tau) \\ \omega(0, 0, 2k|\tau) &= -\frac{1}{3}\zeta_{2k} - 2k(2k{+}1)\gamma_0(2k{+}2, 0|\tau) \,, \end{aligned} \tag{2.25}$$

see appendix B of the reference for further examples. Note that eMZVs of length one are constant ($\omega(2k|\tau) = -2\zeta_{2k}$ and $\omega(2k{+}1|\tau) = 0$ for $k \in \mathbb{N}_0$), and eMZVs of length r yield iterated Eisenstein integrals of length $\leq r{-}1$.

2.5 Derivations

Not all the iterated Eisenstein integrals $\gamma(k_1, k_2, \ldots, k_r|\tau)$ are realized in the decomposition of A-cycle eMZVs. The selection rules on whether a given combination of $\gamma(k_1, k_2, \ldots, k_r|\tau)$ descends from eMZVs are encoded in a family of derivations $\epsilon_{2m},\ m \geq 0$ firstly studied by Tsunogai [39]. These derivations appear in the differential equation of the elliptic KZB associator whose A-cycle component $A(x, y, \tau)$ satisfies [36–38]

$$2\pi i \partial_\tau A(x, y, \tau) = \sum_{m=0}^{\infty} (1{-}2m) \mathrm{G}_{2m}(\tau) \epsilon_{2m} A(x, y, \tau) \,. \tag{2.26}$$

The derivations ϵ_{2m} act on two non-commutative variables x, y via ($m > 0$)

$$\begin{aligned} \epsilon_{2m}(x) &= (\mathrm{ad}_x)^{2m}(y) \\ \epsilon_{2m}(y) &= [y, (\mathrm{ad}_x)^{2m-1}(y)] + \sum_{j=1}^{m-1} (-1)^j [(\mathrm{ad}_x)^j(y), (\mathrm{ad}_x)^{2m-1-j}(y)] \,, \end{aligned} \tag{2.27}$$

where $\mathrm{ad}_x(y) \equiv [x,y]$, and ϵ_0 acts via

$$\epsilon_0(x) = y\,, \qquad \epsilon_0(y) = 0\,. \tag{2.28}$$

These definitions imply a variety of relations in the derivation algebra [25, 58, 59], starting with the fact that ϵ_2 is a central element,

$$[\epsilon_{2m}, \epsilon_2] = 0\,, \qquad m \geq 0\ . \tag{2.29}$$

Furthermore, representation theory of SL_2 implies that $[\epsilon_0,[\epsilon_0,[\epsilon_0,\epsilon_4]]] = 0$ and more generally

$$(\mathrm{ad}_{\epsilon_0})^{2m-1}(\epsilon_{2m}) = 0\,, \qquad m > 0\,. \tag{2.30}$$

Moreover, irreducible relations[9] at various depths are related to cusp forms [59], starting with

$$\begin{aligned}
0 &= [\epsilon_{10},\epsilon_4] - 3[\epsilon_8,\epsilon_6]\\
0 &= 2[\epsilon_{14},\epsilon_4] - 7[\epsilon_{12},\epsilon_6] + 11[\epsilon_{10},\epsilon_8]\\
0 &= 80[\epsilon_{12},[\epsilon_4,\epsilon_0]] + 16[\epsilon_4,[\epsilon_{12},\epsilon_0]] - 250[\epsilon_{10},[\epsilon_6,\epsilon_0]]\\
&\quad - 125[\epsilon_6,[\epsilon_{10},\epsilon_0]] + 280[\epsilon_8,[\epsilon_8,\epsilon_0]] - 462[\epsilon_4,[\epsilon_4,\epsilon_8]] - 1725[\epsilon_6,[\epsilon_6,\epsilon_4]]\,,
\end{aligned} \tag{2.31}$$

and further examples can for instance be found in [25, 59] or downloaded from [85]. As detailed in section 4 of [25], the vanishing of $[\epsilon_{10},\epsilon_4]-3[\epsilon_8,\epsilon_6]$ for instance constrains the relative factors of $\gamma(4,10|\tau), \gamma(10,4|\tau), \gamma(6,8|\tau)$ and $\gamma(8,6|\tau)$ in the decomposition of eMZVs.

At each multiplicity $n \geq 2$, the A-cycle integrals (1.1) will induce conjectural $(n{-}1)! \times (n{-}1)!$ matrix representations $r_{\vec{\eta}}(\epsilon_{2m})$ of the above derivations that preserve all their commutator relations including (2.29) to (2.31), see section 4.5. We will later on spell out the $(n{\leq}4)$-examples of $r_{\vec{\eta}}(\epsilon_{2m})$ from the differential equation (1.4) of the A-cycle integrals by isolating the coefficient of $\mathrm{G}_{2m}(\tau)$ in the differential operator, cf. (1.6), and the all-multiplicity generalization is generated by (4.35).

3 Two-point warm-up

The purpose of this section is to illustrate the main results in a two-point context, where the basis of integrands for the $Z^\tau_{\vec{\eta}}(\cdot|\cdot)$ in (1.1) is one-dimensional. Once we fix translation variance by setting $z_1 = 0$, the planar and non-planar arrangement of the two cylinder punctures in figure 1 amount to integration cycles

$$\mathcal{C}(1,2) = \big\{ z_2 \in (0,1) \big\}\,, \qquad \mathcal{C}\big(\tfrac{2}{1}\big) = \big\{ z_2 = \tau/2 + v_2,\ v_2 \in (0,1) \big\}\,, \tag{3.1}$$

cf. section 2.3. The two-point instances of (1.1) are then given by

$$\begin{aligned}
Z^\tau_{\eta_2}(1,2|1,2) &= \int_{\mathcal{C}(1,2)} \mathrm{d}z_2\, \Omega(z_{12},\eta_2,\tau)\, e^{s_{12}\mathcal{G}(z_{12},\tau)} = \int_0^1 \mathrm{d}v_2\, \Omega(v_{12},\eta_2,\tau)\, e^{s_{12}\mathcal{G}(v_{12},\tau)}\\
Z^\tau_{\eta_2}\big(\tfrac{2}{1}|1,2\big) &= \int_{\mathcal{C}(\frac{2}{1})} \mathrm{d}z_2\, \Omega(z_{12},\eta_2,\tau)\, e^{s_{12}\mathcal{G}(z_{12},\tau)} = \int_0^1 \mathrm{d}v_2\, \Omega(v_{12}{+}\tau/2,\eta_2,\tau)\, e^{s_{12}\mathcal{G}(v_{12}+\tau/2,\tau)}\,,
\end{aligned} \tag{3.2}$$

[9] A relation is said to be irreducible if it cannot be written as $\mathrm{ad}_{\epsilon_{n_1}}\mathrm{ad}_{\epsilon_{n_2}}\ldots\mathrm{ad}_{\epsilon_{n_r}}N(\epsilon) = 0$ with $r > 0$ and $N(\epsilon)$ denoting some vanishing expression built from (possibly nested) commutators of ϵ_{2m}.

see (2.2) for the Kronecker–Eisenstein series $\Omega(\dots)$ as well as (2.14) and (2.15) for the planar and non-planar Green functions $\mathcal{G}(\dots)$, respectively. By the expansion methods of [20] and [21], the τ-dependence at any α'-order of $Z^\tau_{\eta_2}(1,2|1,2)$ and $q^{-s_{12}/8}Z^\tau_{\eta_2}\big({2\atop 1}|1,2\big)$ is carried by eMZVs and twisted eMZVs with Fourier expansions (2.11) and (2.12), respectively. As we will see, the half-odd integer powers q^m, $m\in\mathbb{N}-\frac{1}{2}$ in (2.12) drop out.

3.1 The τ-derivative

Given that the derivatives (2.16) of the Green functions w.r.t. v_j and τ take a universal form for the planar and non-planar case, we will evaluate the τ-derivative of both integrals in (3.2) for an unspecified integration cycle $\mathcal{C}(*)$,

$$\begin{aligned}2\pi i\partial_\tau Z^\tau_{\eta_2}(*|1,2) &= 2\pi i\int\limits_{\mathcal{C}(*)}\mathrm{d}z_2\,\Big(\partial_\tau\Omega(z_{12},\eta_2,\tau)\,e^{s_{12}\mathcal{G}(z_{12},\tau)}+\Omega(z_{12},\eta_2,\tau)\partial_\tau\,e^{s_{12}\mathcal{G}(z_{12},\tau)}\Big)\\ &=\int\limits_{\mathcal{C}(*)}\mathrm{d}z_2\,\Big(\partial_z\partial_{\eta_2}\Omega(z_{12},\eta_2,\tau)-s_{12}(f^{(2)}_{12}{+}2\zeta_2)\Omega(z_{12},\eta_2,\tau)\Big)\,e^{s_{12}\mathcal{G}(z_{12},\tau)}\\ &=s_{12}\int\limits_{\mathcal{C}(*)}\mathrm{d}z_2\,\Big(f^{(1)}_{12}\partial_{\eta_2}\Omega(z_{12},\eta_2,\tau)-(f^{(2)}_{12}+2\zeta_2)\Omega(z_{12},\eta_2,\tau)\Big)\,e^{s_{12}\mathcal{G}(z_{12},\tau)}\,.\end{aligned}\tag{3.3}$$

We have used (2.7) and (2.16) in passing to the second line and integrated the z-derivative of $\partial_{\eta_2}\Omega(z_{12},\eta_2,\tau)$ by parts in the last step, see (2.18) for the shorthand $f^{(k)}_{ij}$. Throughout this work, boundary terms w.r.t. the punctures are discarded since the Koba–Nielsen factor always leads to a suppression of the form[10] $\lim_{z_{ij}\to 0}e^{s_{ij}\mathcal{G}(z_{ij},\tau)}=0$ or $\lim_{z_{ij}\to 1}e^{s_{ij}\mathcal{G}(z_{ij},\tau)}=0$.

In order to relate the right-hand side of (3.3) to the original integral $Z^\tau_{\eta_2}(*|1,2)$, we rewrite the combination $f^{(1)}_{12}\partial_{\eta_2}\Omega(z_{12},\eta_2,\tau)-f^{(2)}_{12}\Omega(z_{12},\eta_2,\tau)$ using

$$f^{(1)}_{ij}=-\Omega(z_{ji},\xi,\tau)\big|_{\xi^0}\,,\qquad f^{(2)}_{ij}=\partial_\xi\Omega(z_{ji},\xi,\tau)\big|_{\xi^0}\,.\tag{3.4}$$

The notation $\big|_{\xi^0}$ instructs to pick up the zero$^{\text{th}}$ order in the Laurent expansion w.r.t. an auxiliary variable $\xi\in\mathbb{C}$. Moreover, we will need the key identity [22, 79]

$$(\partial_{\eta_2}+\partial_\xi)\Omega(z_{12},\eta_2,\tau)\Omega(z_{21},\xi,\tau)=\big(\wp(\eta_2,\tau)-\wp(\xi,\tau)\big)\Omega(z_{12},\eta_2-\xi,\tau)\tag{3.5}$$

whose derivation is reviewed in appendix A.1. The Weierstrass functions on the right-hand side generate holomorphic Eisenstein series (2.19) upon Laurent expansion in η,

$$\wp(\eta,\tau)=-\partial^2_\eta\log\theta_1(\eta,\tau)-\mathrm{G}_2(\tau)=-\frac{\mathrm{G}_0}{\eta^2}+\sum_{m=2}^\infty(2m{-}1)\eta^{2m-2}\mathrm{G}_{2m}(\tau)\,.\tag{3.6}$$

[10] In order to see this, one first exploits the local behavior $\mathcal{G}(z,\tau)\to-\log|z|$ of the Green function around the origin to write $\lim_{z_{ij}\to 0}e^{s_{ij}\mathcal{G}(z_{ij},\tau)}=\lim_{z_{ij}\to 0}e^{-s_{ij}\log|z_{ij}|}=\lim_{z_{ij}\to 0}|z_{ij}|^{-s_{ij}}$. Then, the vanishing of $\lim_{z_{ij}\to 0}|z_{ij}|^{-s_{ij}}$ is obvious in case of $\mathrm{Re}(s_{ij})<0$ and otherwise follows from analytic continuation.

Based on (3.4) and (3.5), the last line of (3.3) can be rewritten as

$$\begin{aligned} f_{12}^{(1)}\partial_{\eta_2}\Omega(z_{12},\eta_2,\tau) &- f_{12}^{(2)}\Omega(z_{12},\eta_2,\tau) \\ &= -\Omega(z_{21},\xi,\tau)\partial_{\eta_2}\Omega(z_{12},\eta_2,\tau) - \Omega(z_{12},\eta_2,\tau)\partial_\xi\Omega(z_{21},\xi,\tau)\big|_{\xi^0} \\ &= -(\partial_{\eta_2}+\partial_\xi)\Omega(z_{12},\eta_2,\tau)\Omega(z_{21},\xi,\tau)\big|_{\xi^0} \\ &= \big(\wp(\xi,\tau)-\wp(\eta_2,\tau)\big)\Omega(z_{12},\eta_2-\xi,\tau)\big|_{\xi^0} \\ &= \Big(\frac{1}{2}\partial_{\eta_2}^2 - \wp(\eta_2,\tau)\Big)\Omega(z_{12},\eta_2,\tau)\,. \end{aligned} \tag{3.7}$$

The derivative $\frac{1}{2}\partial_{\eta_2}^2$ in the last line stems from the interplay of the double pole $\wp(\xi,\tau) = \frac{1}{\xi^2} + \mathcal{O}(\xi^2)$ with the Taylor expansion of $\Omega(z_{12},\eta_2-\xi,\tau)$ around $\xi=0$. At fixed order in η_2, one can extract relations such as $f_{12}^{(2)}f_{12}^{(2)} - 2f_{12}^{(3)}f_{12}^{(1)} = 3\mathrm{G}_4 - 2f_{12}^{(4)}$ from (3.7), see (A.6) for further examples and (A.7) for a general formula. More importantly, (3.7) allows to rewrite the τ-derivative of the two-point integrals (3.2) as

$$2\pi i\partial_\tau Z_{\eta_2}^\tau(*|1,2) = s_{12}\int_{\mathcal{C}(*)}\mathrm{d}z_2\,\Big(\frac{1}{2}\partial_{\eta_2}^2 - \wp(\eta_2,\tau) - 2\zeta_2\Big)\Omega(z_{12},\eta_2,\tau)\,e^{s_{12}\mathcal{G}(z_{12},\tau)}\,, \tag{3.8}$$

i.e. in terms of the original integral

$$2\pi i\partial_\tau Z_{\eta_2}^\tau(*|1,2) = s_{12}\Big(\frac{1}{2}\partial_{\eta_2}^2 - \wp(\eta_2,\tau) - 2\zeta_2\Big)Z_{\eta_2}^\tau(*|1,2)\,. \tag{3.9}$$

As we will see, this linear and homogeneous first-order differential equation in τ has a variety of structural implications and can be applied to efficiently perform α'-expansions.

3.2 Derivations and iterated Eisenstein integrals

The upshot (3.9) of the previous section is the two-point instance of our central result (1.4): The A-cycle integrals in (3.2) close under τ-derivatives, and the differential operator in

$$2\pi i\partial_\tau Z_{\eta_2}^\tau(*|1,2) = D_{\eta_2}^\tau Z_{\eta_2}^\tau(*|1,2)\,, \qquad D_{\eta_2}^\tau = s_{12}\Big(\frac{1}{2}\partial_{\eta_2}^2 - \wp(\eta_2,\tau) - 2\zeta_2\Big) \tag{3.10}$$

takes a universal form for the planar and non-planar integration cycles $\mathcal{C}(*)$ in (3.1). Once we expand the Weierstrass function via (3.6), this can be lined up with the differential equation (2.26) of the KZB associator,

$$D_{\eta_2}^\tau = \sum_{m=0}^\infty (1-2m)\mathrm{G}_{2m}(\tau) r_{\eta_2}(\epsilon_{2m})\,, \tag{3.11}$$

and one can read off a scalar representation $r_{\eta_2}(\cdot)$ of the derivations reviewed in section 2.5,

$$\begin{aligned} r_{\eta_2}(\epsilon_0) &= s_{12}\Big(\frac{1}{\eta_2^2} + 2\zeta_2 - \frac{1}{2}\partial_{\eta_2}^2\Big)\,, \qquad r_{\eta_2}(\epsilon_2) = 0 \\ r_{\eta_2}(\epsilon_{2m}) &= s_{12}\eta_2^{2m-2}\,, \qquad m>1\,. \end{aligned} \tag{3.12}$$

The differential equation (3.10) can be solved via standard Picard iteration once an initial value at some reference value τ_0 is available (which is chosen as $\tau_0 \to i\infty$ for convenience),

$$
\begin{aligned}
Z^{\tau}_{\eta_2}(*|1,2) &= \Big(1 + \frac{1}{2\pi i}\int_{i\infty}^{\tau} \mathrm{d}\tau_1\, D^{\tau_1}_{\eta_2} + \frac{1}{(2\pi i)^2}\int_{i\infty}^{\tau} \mathrm{d}\tau_1\, D^{\tau_1}_{\eta_2}\int_{i\infty}^{\tau_1} \mathrm{d}\tau_2\, D^{\tau_2}_{\eta_2} + \ldots\Big) Z^{i\infty}_{\eta_2}(*|1,2) \\
&= \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^{2n}} \int\limits_{0<q_1<q_2<\ldots<q_n<q} \frac{\mathrm{d}q_1}{q_1}\,\frac{\mathrm{d}q_2}{q_2}\ldots\frac{\mathrm{d}q_n}{q_n}\, D^{\tau_n}_{\eta_2}\ldots D^{\tau_2}_{\eta_2} D^{\tau_1}_{\eta_2} Z^{i\infty}_{\eta_2}(*|1,2)\,. \qquad (3.13)
\end{aligned}
$$

By the term $\sim \partial^2_{\eta_2}$ in (3.10), $D^{\tau}_{\eta_2}$ is a differential operator in η_2, so $D^{\tau_i}_{\eta_2}$ and $D^{\tau_j}_{\eta_2}$ do not commute at different $\tau_i \neq \tau_j$ in (3.13). Instead, when all the $D^{\tau_i}_{\eta_2}$ on the right-hand side of (3.13) are expanded in terms of Eisenstein series via (3.11), each term can be identified as an iterated Eisenstein integral (2.20),

$$
Z^{\tau}_{\eta_2}(*|1,2) = \sum_{r=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_r\\ =0,4,6,8,\ldots}} \prod_{j=1}^{r}(k_j-1)\,\gamma(k_1,k_2,\ldots,k_r|\tau) r_{\eta_2}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1}) Z^{i\infty}_{\eta_2}(*|1,2)\,, \quad (3.14)
$$

where $r_{\eta_2}(\epsilon_{k_i}\epsilon_{k_j}) \equiv r_{\eta_2}(\epsilon_{k_i}) r_{\eta_2}(\epsilon_{k_j})$. This is the two-point instance of the general result (1.7), where the derivations appear in a scalar representation (as opposed to $(n-1)! \times (n-1)!$ matrices at n points). Since each $r_{\eta_2}(\epsilon_k)$ in (3.12) is linear in s_{12}, the summation variable r in (3.14) counts the powers of α' introduced by the derivations. Note, however, that $Z^{i\infty}_{\eta_2}(*|1,2)$ may by itself depend on α', see the discussion in the next sections.

Although the representations of ϵ_k are not yet matrix-valued in the two-point case, the term $\sim \partial^2_{\eta_2}$ in $r_{\eta_2}(\epsilon_0)$ prevents it from commuting. Accordingly, it is nontrivial to check that relations (2.30) are preserved by $r_{\eta_2}(\cdot)$, e.g. that $[r_{\eta_2}(\epsilon_0), [r_{\eta_2}(\epsilon_0), [r_{\eta_2}(\epsilon_0), r_{\eta_2}(\epsilon_4)]]]\mathcal{F}(\eta_2) = 0$ for an arbitrary function $\mathcal{F}$ of η_2. However, since all the $r_{\eta_2}(\epsilon_k)$ at positive $k > 0$ are multiplicative, commutators $[r_{\eta_2}(\epsilon_{k_1}), r_{\eta_2}(\epsilon_{k_2})]$ with $k_1, k_2 > 0$ always vanish, and relations such as $[r_{\eta_2}(\epsilon_{10}), r_{\eta_2}(\epsilon_4)] - 3[r_{\eta_2}(\epsilon_8), r_{\eta_2}(\epsilon_6)] = 0$ (see (2.31)) hold trivially. And the vanishing of $r_{\eta_2}(\epsilon_2)$ is of course compatible with ϵ_2 being a central element, cf. (2.29).

As a consequence of $[r_{\eta_2}(\epsilon_{k_1}), r_{\eta_2}(\epsilon_{k_2})] = 0\ \forall\ k_1, k_2 > 0$, iterated Eisenstein integrals of depth ≥ 2 appear in a constrained manner in (3.14). For instance, $\gamma(k_1, k_2|\tau)$ and $\gamma(k_2, k_1|\tau)$ with $k_1, k_2 > 0$ will always have the same coefficient and conspire to the shuffle product $\gamma(k_1, k_2|\tau) + \gamma(k_2, k_1|\tau) = \gamma(k_1|\tau)\cdot\gamma(k_2|\tau)$. This is a genus-one analogue of the dropout of depth-($d \geq 2$) MZVs from four-point disk integrals: The α'-expansion of the latter is still expressible via products of ζ_m whereas ($n \geq 5$)-point disk integrals involve irreducible higher-depth MZVs [4].

On the other hand, since $[r_{\eta_2}(\epsilon_k), r_{\eta_2}(\epsilon_0)] \neq 0\ \forall\ k > 2$, not all the iterated Eisenstein integrals at depth ≥ 2 in (3.14) can be written as shuffles of depth-one instances: Cases like $\gamma(4,4,0,0|\tau), \gamma(4,0,4,0|\tau)$ and $\gamma(4,0,0,4|\tau)$ with entries $k_j = 0$ appear in combinations that cannot be reduced to $\gamma(4,0|\tau)^2$ and $\gamma(4|\tau)\cdot\gamma(4,0,0|\tau)$. Hence, in spite of the vanishing commutators among the $r_{\eta_2}(\epsilon_{k>0})$, the α'-expansion of the two-point A-cycle integrals goes beyond the coefficients of the meta-abelian quotient of the KZB associator [86].

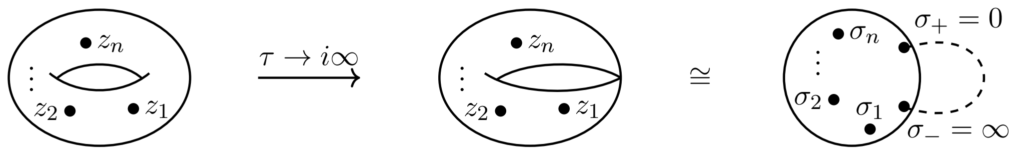


Figure 3: *The degeneration of the torus at $\tau \to i\infty$ pinches the A-cycle and yields the topology of a Riemann sphere. In particular, the pinched A-cycle introduces a pair of identified punctures $\sigma_+ = 0$ and $\sigma_- = \infty$ on the Riemann sphere.*

3.3 The initial value at the cusp: degenerating the integrand

Given that the pattern of iterated Eisenstein integrals in the two-point integrals is fixed by (3.14), the next step is to find the explicit form of their initial value $Z^{i\infty}_{\eta_2}(*|1,2)$, i.e. the degeneration of (3.2) at the cusp $\tau \to i\infty$. In this limit, both the Green function and the Kronecker–Eisenstein series in the integrand are most conveniently described in the variables

$$\sigma_j = e^{2\pi i z_j} \, , \qquad \mathrm{d}z_j = \frac{\mathrm{d}\sigma_j}{2\pi i \, \sigma_j} \, . \tag{3.15}$$

This parameterization of the punctures is tailored to the nodal Riemann sphere which arises from the degeneration of a torus as visualized in figure 3. The functions of σ_j we will encounter can be interpreted as SL_2-fixed expressions on a sphere, where the identified punctures from the pinching of the A-cycle in figure 3 are $\sigma_+ = 0$ and $\sigma_- = \infty$, and the choice $z_1 = 0$ is mapped to $\sigma_1 = 1$ under (3.15). The degeneration of the Kronecker–Eisenstein series can be assembled from the $\tau \to i\infty$ behavior of the functions $f^{(k)}(z,\tau)$ and $f^{(k)}(z-\tau/2,\tau)$ studied in section 3.2.1 of [21]. In the planar and non-planar cases with $v_{ij} \in \mathbb{R}$, the results of the reference imply

$$\lim_{\tau \to i\infty} \Omega(v_{ij},\eta,\tau) = \pi \cot(\pi\eta) + G_{ij} \, , \qquad \lim_{\tau \to i\infty} \Omega(v_{ij}+\tau/2,\eta,\tau) = \frac{\pi}{\sin(\pi\eta)} \, . \tag{3.16}$$

Remarkably, the Kronecker–Eisenstein series $\Omega(v_{ij}+\tau/2,\eta,\tau)$ on the non-planar cycle no longer depends on the punctures at the cusp. In the planar case, the only dependence of $\Omega(v_{ij},\eta,\tau \to i\infty)$ on σ_i and σ_j occurs at the zero$^{\text{th}}$ order in η through the Green function on the nodal Riemann sphere[11],

$$G_{ij} \equiv 2\pi i \frac{\sigma_i + \sigma_j}{2(\sigma_i - \sigma_j)} \, . \tag{3.17}$$

[11]Note the different fonts used for the letter 'G' in the open-string Green function $\mathcal{G}(z,\tau)$, the holomorphic Eisenstein series $\mathrm{G}_k(\tau)$ and the Green function G_{ij} on the nodal sphere in (3.17).

The trigonometric functions of η in (3.16) have the straightforward Laurent expansion

$$\pi\cot(\pi\eta) = \frac{1}{\eta} - 2\zeta_2\eta - 2\zeta_4\eta^3 - 2\zeta_6\eta^5 - \ldots \ = \frac{1}{\eta} - 2\sum_{k=1}^{\infty}\zeta_{2k}\eta^{2k-1} \tag{3.18}$$
$$\frac{\pi}{\sin(\pi\eta)} = \frac{1}{\eta} + \zeta_2\eta + \frac{7}{4}\zeta_4\eta^3 + \frac{31}{16}\zeta_6\eta^5 + \ldots = \frac{1}{\eta} + \sum_{k=1}^{\infty}\frac{2^{2k-1}-1}{2^{2k-2}}\zeta_{2k}\eta^{2k-1}\,.$$

The degeneration of the planar Green function can for instance be determined by inserting the identity $\lim_{\tau\to i\infty}\mathrm{d}z'\,f^{(1)}(z'-z_j,\tau) = \frac{\mathrm{d}\sigma'}{\sigma'-\sigma_j} - \frac{\mathrm{d}\sigma'}{2\sigma'}$ at real values of z' and z_j into (2.14),

$$\lim_{\tau\to i\infty}\mathcal{G}(v_{ij},\tau) = \frac{1}{2}\log(\sigma_i) + \frac{1}{2}\log(\sigma_j) - \log(\sigma_j{-}\sigma_i)\,, \qquad v_i < v_j\,, \tag{3.19}$$

where cases with $v_j < v_i$ give rise to $-\log(\sigma_i{-}\sigma_j)$ in the place of $-\log(\sigma_j{-}\sigma_i)$. For the non-planar Green function (2.15), the reasoning in appendix B yields the regularized value

$$\lim_{\tau\to i\infty}\mathcal{G}(v_{ij}{+}\tau/2,\tau) = 0\,. \tag{3.20}$$

In summary, the results of this subsection pinpoint the following degeneration of the integrands of the $Z^{\tau}_{\eta_2}(*|1,2)$ in (3.2),

$$\lim_{\tau\to i\infty}\Omega(v_{12},\eta_2,\tau)\,e^{s_{12}\mathcal{G}(v_{12},\tau)} = (\pi\cot(\pi\eta_2) + G_{12})\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}} \tag{3.21}$$
$$\lim_{\tau\to i\infty}\Omega(v_{12}{+}\tau/2,\eta_2,\tau)\,e^{s_{12}\mathcal{G}(v_{12}+\tau/2,\tau)} = \frac{\pi}{\sin(\pi\eta_2)}\,. \tag{3.22}$$

In the non-planar case, the integrand at $\tau\to i\infty$ no longer depends on the puncture, so one can immediately perform the integral $\int_{\mathcal{C}\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)}\mathrm{d}z_2 = 1$ and obtain the degeneration

$$Z^{i\infty}_{\eta_2}\left(\begin{smallmatrix}2\\1\end{smallmatrix}|1,2\right) = \frac{\pi}{\sin(\pi\eta_2)}\,. \tag{3.23}$$

Upon insertion into the solution (3.14) to the differential equation in τ, this completes the result for the α'-expansion of the non-planar A-cycle integral,

$$Z^{\tau}_{\eta_2}\left(\begin{smallmatrix}2\\1\end{smallmatrix}|1,2\right) = \sum_{r=0}^{\infty}\sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}}\prod_{j=1}^{r}(k_j{-}1)\,\gamma(k_1,k_2,\ldots,k_r|\tau)r_{\eta_2}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})\frac{\pi}{\sin(\pi\eta_2)}\,. \tag{3.24}$$

Since $\frac{\pi}{\sin(\pi\eta_2)}$ does not depend on α' and $r_{\eta_2}(\epsilon_k)$ are linear in s_{12}, the order of $(\alpha')^r$ in (3.24) exclusively features iterated Eisenstein integrals of length r. As will be detailed in section 3.5, the integral over a specific $f^{(k)}(v{-}\tau/2,\tau)$ rather than $\Omega(v{-}\tau/2,\eta,\tau)$ can be obtained by expanding $\frac{\pi}{\sin(\pi\eta_2)}$ as in (3.18) and isolating the desired power of η_2 after summing over the words in $r_{\eta_2}(\epsilon_k)$. The analogous statements in the planar case will be derived in the next section – the dependence of (3.21) on σ_2 will require extra care.

It is also worth highlighting that (3.24) manifests the absence of twisted eMZVs which are naively expected when the α'-expansion of $\int_0^1\mathrm{d}v\,f^{(k)}(v{-}\tau/2,\tau)e^{s_{12}\mathcal{G}(v-\tau/2,\tau)}$ is performed by the methods of [21]. Hence, half-odd integer powers q^m, $m\in\mathbb{N}{-}\frac{1}{2}$ are guaranteed to drop out at any α'-order of $q^{-s_{12}/8}Z^{\tau}_{\eta_2}\left(\begin{smallmatrix}2\\1\end{smallmatrix}|1,2\right)$, where the factor of $q^{-s_{12}/8}$ eliminates the contribution $\frac{i\pi\tau}{4}$ to the non-planar Green function.

3.4 The initial value at the cusp: deforming the integration contour

In this section, the degeneration $Z^{i\infty}_{\eta_2}(1,2|1,2)$ of the planar two-point A-cycle integral will be reduced to a four-point disk integral. The integrand (3.21) yields expressions of the form

$$I^{\rm tree}(1,2|\mathcal{F}(\sigma_2)) = \int_{\mathcal{C}(1,2)} \frac{\mathrm{d}\sigma_2}{2\pi i \sigma_2}\, \sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}}\mathcal{F}(\sigma_2) \tag{3.25}$$

with $\mathcal{F}(\sigma_2)\to 1$ and $\mathcal{F}(\sigma_2)\to G_{12}$,

$$Z^{i\infty}_{\eta_2}(1,2|1,2) = \pi\cot(\pi\eta_2) I^{\rm tree}(1,2|1) + I^{\rm tree}(1,2|G_{12})\,. \tag{3.26}$$

The integrand of $I^{\rm tree}(1,2|G_{12})$ is in fact a total derivative and integrates to zero,

$$\frac{G_{12}}{2\pi i \sigma_2}\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}} = \frac{1}{s_{12}}\frac{\partial}{\partial\sigma_2}\big(\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}}\big) \quad\Rightarrow\quad I^{\rm tree}(1,2|G_{12}) = 0\,. \tag{3.27}$$

Hence, the leftover work is to simplify the expression (3.25) for $I^{\rm tree}(1,2|1)$. In the $\sigma_2 = e^{2\pi i z_2}$ variable, the integration contour $\mathcal{C}(1,2)$ is the unit circle visualized in the left panel of figure 4 instead of the unit interval $z_2\in(0,1)$. In order to connect with genus-zero techniques, it is convenient to deform the unit circle $\mathcal{C}(1,2)$ to the homotopy-equivalent contour in the right panel of figure 4. The contour deformation must not cross the branch points $\sigma_2=0$ and $\sigma_2=1$ of the multivalued part of the integrand $\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}}$. Accordingly, one arrives at the two straight paths parallel to the unit interval $\sigma_2\in(0,1)$ whose small displacement from the real axis indicates that they are loaded with different branches of $\sigma_2^{s_{12}/2}$. The phases associated with the straight paths above and below the real axis are $-e^{-\frac{i\pi}{2}s_{12}}$ and $e^{+\frac{i\pi}{2}s_{12}}$, respectively, and the minus sign of the former stems from the negative orientation of the path.

The contour deformation in figure 4 has been used in [22] and [21] to evaluate the $\tau\to i\infty$ degeneration of A-cycle eMZVs and twisted eMZVs, respectively. In these references, the small circle around the origin introduces separate contributions to the (twisted) eMZVs at the cusp. In our situation with the additional factor of $\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}}$, the net effect of the small circle is to attribute phases to the straight paths that differ by $\frac{(e^{2\pi i}\sigma_2)^{s_{12}/2}}{(\sigma_2)^{s_{12}/2}} = e^{2\pi i\frac{s_{12}}{2}}$, and we arrive at

$$\begin{aligned}\int_{\mathcal{C}(1,2)} \mathrm{d}\sigma_2\, \sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}} &= \bigg(e^{-\frac{i\pi}{2}s_{12}}\int_1^0 \mathrm{d}\sigma_2 + e^{\frac{i\pi}{2}s_{12}}\int_0^1\mathrm{d}\sigma_2\bigg)\,\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}}\\ &= 2i\sin\Big(\frac{\pi s_{12}}{2}\Big)\int_0^1 \mathrm{d}\sigma_2\,\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}}\,.\end{aligned} \tag{3.28}$$

Contour deformations and s_{ij}-dependent phases of this type are well-known from the Kawai–Lewellen–Tye relations [87] and monodromy relations [88, 89] among string tree amplitudes. More recently, these techniques have been connected [35, 57] with the framework of intersection theory (see e.g. [90]), where the choices of branch for the Koba–Nielsen factor $\sim\sigma_{ij}^{-s_{ij}}$ lead to the notion of twisted cycles. Accordingly, (3.28) can be viewed as a relation between

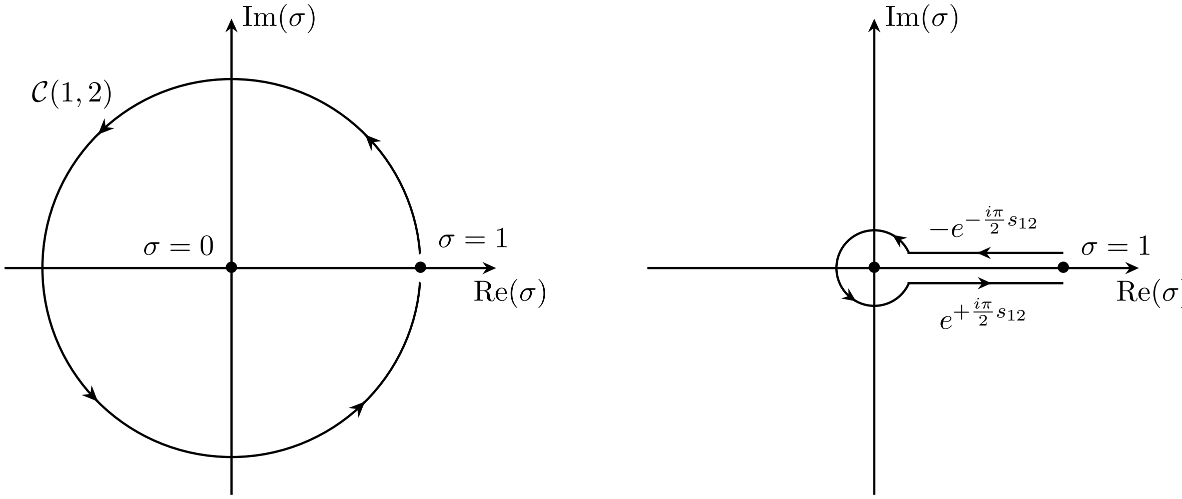


Figure 4: *As depicted in the left panel, the integration contour $\mathcal{C}(1,2)$ in the $\sigma_2 = e^{2\pi i z_2}$ variable is the unit circle $|\sigma_2| = 1$. When replacing $\mathcal{C}(1,2)$ by the homotopy-equivalent combination of paths visualized the right panel, the multivalued part of the integrand $\sigma_2^{s_{12}/2}$ introduces phases $e^{\pm\frac{i\pi}{2}s_{12}}$ depending on the sign of the small imaginary part of σ_2.*

twisted cycles which holds for any rational function of σ_j in the integrand. Upon insertion into (3.25), the desired integral can be evaluated in terms of the well-known Euler Beta function

$$
\begin{aligned}
I^{\text{tree}}(1,2|1) &= \frac{2i}{2\pi i} \sin\Big(\frac{\pi s_{12}}{2}\Big) \int_0^1 \frac{\mathrm{d}\sigma_2}{\sigma_2}\, \sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}} \\
&= \frac{1}{\pi} \sin\Big(\frac{\pi s_{12}}{2}\Big) \frac{\Gamma(\frac{s_{12}}{2})\Gamma(1-s_{12})}{\Gamma(1-\frac{s_{12}}{2})} = \frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}\,,
\end{aligned}
\tag{3.29}
$$

or equivalently, a four-point disk integral (1.3) at special arguments $(-\frac{s_{12}}{2}, s_{12})$ in the place of (s_{12}, s_{23}). In section 5, we will spell out the general dictionary between $(\tau \to i\infty)$-limits of planar n-point A-cycle integrals and $(n{+}2)$-point disk integrals of Parke–Taylor type.

By (3.27) and (3.29), the planar two-point A-cycle integral at the cusp takes the form

$$
Z^{i\infty}_{\eta_2}(1,2|1,2) = \pi \cot(\pi\eta_2) \frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}\,. \tag{3.30}
$$

When inserted into the general solution (3.14) of the differential equation for $Z^{\tau}_{\eta_2}(*|1,2)$, one can assemble the α'-expansion of the planar A-cycle integral from

$$
\begin{aligned}
Z^{\tau}_{\eta_2}(1,2|1,2) = \frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2} \sum_{r=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_r\\ =0,4,6,8,\ldots}} \prod_{j=1}^{r} (k_j{-}1)\, \gamma(k_1,k_2,\ldots,k_r|\tau) \\
\times\, r_{\eta_2}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1}) \pi \cot(\pi\eta_2)\,,
\end{aligned}
\tag{3.31}
$$

along with the straightforward expansion of the Γ-functions in terms of Riemann zeta values

$$\begin{aligned}\frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2} &= \exp\left(\sum_{k=2}^{\infty}\frac{\zeta_k}{k}(1-2^{1-k})s_{12}^k\right)\\ &= 1+\frac{1}{4}s_{12}^2\zeta_2+\frac{1}{4}s_{12}^3\zeta_3+\frac{19}{160}s_{12}^4\zeta_2^2+\frac{1}{16}s_{12}^5\zeta_2\zeta_3+\frac{3}{16}s_{12}^5\zeta_5\\ &\quad +\frac{55}{896}s_{12}^6\zeta_2^3+\frac{1}{32}s_{12}^6\zeta_3^2+\mathcal{O}(\alpha'^7)\,. \end{aligned} \tag{3.32}$$

The non-planar α'-expansion in (3.24) does not exhibit any analogue of the α'-dependent factor (3.32), and in fact, no MZVs at all. By the planar result (3.31), the coefficient of $(\alpha')^r$ in $Z^\tau_{\eta_2}(1,2|1,2)$ comprises products $\gamma(k_1,k_2,\ldots,k_\ell|\tau)\zeta_{w_1}\zeta_{w_2}\ldots\zeta_{w_k}$ such that the length ℓ and the weights w_j conspire to $r=\ell+w_1+w_2+\ldots+w_k$.

3.5 Extracting component integrals

One-loop string amplitudes can be reduced to A-cycle integrals over products of $f_{ij}^{(m_k)}$ at fixed overall weight $\sum_k m_k$ [20, 53–56] and do not involve the full η-dependent $\Omega(z_{ij},\eta,\tau)$ in the integrand. In order to apply the results on the generating functions $Z^\tau_{\vec\eta}$ in (1.1) to the integrals in string amplitudes, one has to extract particular orders in η_j. In the two-point case, this amounts to studying the following component integrals (with the usual placeholder $*$ for planar or non-planar cycles)

$$Z^\tau_{(m)}(*) \equiv Z^\tau_{\eta_2}(*|1,2)\,\big|_{\eta_2^{m-1}} = \int_{\mathcal{C}(*)} \mathrm{d}z_2\, f^{(m)}(z_{12},\tau)\, e^{s_{12}\mathcal{G}(z_{12},\tau)}\,. \tag{3.33}$$

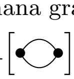
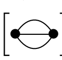
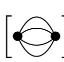
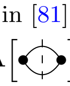
As will be explained below, each order in α' and η_2 only receives a finite number of contributions from the general formula (3.14), and we will spell out detailed cutoffs on the relevant infinite sums. Like this, one can efficiently obtain two-point α'-expansions to high orders from the differential-equation method of this work.

The α'-expansion of $Z^\tau_{(0)}(1,2)$ and $Z^\tau_{(0)}\big(\begin{smallmatrix}2\\1\end{smallmatrix}\big)$ with a plain Koba–Nielsen factor in the integrand has already been studied in [81], and the coefficients where dubbed planar and non-planar A-cycle graph functions[12] A_w and $A_{\underline{w}}$, respectively,

$$Z^\tau_{(0)}(1,2) = \sum_{w=0}^{\infty}\frac{s_{12}^w}{w!}A_w(\tau)\,,\qquad Z^\tau_{(0)}\big(\begin{smallmatrix}2\\1\end{smallmatrix}\big) = q^{s_{12}/8}\sum_{w=0}^{\infty}\frac{s_{12}^w}{w!}A_{\underline{w}}(\tau)\,. \tag{3.34}$$

In the expansion of $Z^\tau_{(0)}\big(\begin{smallmatrix}2\\1\end{smallmatrix}\big)$, the factor of $q^{s_{12}/8}$ has been separated to ensure that both A_w and $A_{\underline{w}}$ have a Fourier expansion in q. Accordingly, A-cycle graph functions take a more compact

[12] Following earlier work on closed-string α'-expansions [91, 92], the property $\int_1^0 \mathrm{d}z\,\mathcal{G}(z,\tau)=0$ of the open-string Green function has been exploited to organize the α'-expansion of $Z^\tau_{(0)}(1,2)$ and generalizations via one-particle irreducible graphs Γ [81]. The notation A_w and $A_{\underline{w}}$ for A-cycle graph functions in (3.34) refers to planar banana graphs with w edges, and the translation to the notation $\mathbf{A}[\Gamma]$ in [81] is exemplified as follows:

$A_2 = \mathbf{A}\Big[$  $\Big]$ $\quad A_3 = \mathbf{A}\Big[$  $\Big]$ $\quad A_4 = \mathbf{A}\Big[$  $\Big]$ $\quad A_{\underline{2}} = \mathbf{A}\Big[$  $\Big]$ $\quad A_{\underline{3}} = \mathbf{A}\Big[$  $\Big]$

form[13] when expressed in terms of the convergent iterated Eisenstein integrals $\gamma_0(k_1,k_2,\ldots|\tau)$ with $k_1{>}0$, see (2.23) for their definition and (2.24) for their q-expansion. In the planar case, we have $A_0(\tau)=1,\ A_1(\tau)=0$ and [81]

$$\begin{aligned}
A_2(\tau) &= \frac{\zeta_2}{2} - 6\gamma_0(4,0|\tau) \\
A_3(\tau) &= \frac{3\zeta_3}{2} + 144\zeta_2\gamma_0(4,0,0|\tau) - 60\gamma_0(6,0,0|\tau) \\
A_4(\tau) &= \frac{57\zeta_4}{8} - 18\zeta_2\gamma_0(4,0|\tau) - 3456\zeta_4\gamma_0(4,0,0,0|\tau) + 216\gamma_0(4,0,4,0|\tau) - 432\gamma_0(4,4,0,0|\tau) \\
&\quad + 5760\zeta_2\gamma_0(6,0,0,0|\tau) - 3024\gamma_0(8,0,0,0|\tau)\,.
\end{aligned} \tag{3.35}$$

The simplest non-planar examples are $A_{\underline{0}}(\tau)=1,\ A_{\underline{1}}(\tau)=0$ and [81]

$$\begin{aligned}
A_{\underline{2}}(\tau) &= -6\gamma_0(4,0|\tau) \\
A_{\underline{3}}(\tau) &= -72\zeta_2\gamma_0(4,0,0|\tau) - 60\gamma_0(6,0,0|\tau) \\
A_{\underline{4}}(\tau) &= -3456\zeta_4\gamma_0(4,0,0,0|\tau) + 216\gamma_0(4,0,4,0|\tau) - 432\gamma_0(4,4,0,0|\tau) \\
&\quad - 2880\zeta_2\gamma_0(6,0,0,0|\tau) - 3024\gamma_0(8,0,0,0|\tau)\,.
\end{aligned} \tag{3.36}$$

In order to efficiently generate A-cycle graph functions from (3.14), we reiterate that

- $Z^\tau_{(0)}(*)$ are the coefficients of η_2^{-1} in $Z^\tau_{\eta_2}(*|1,2)$, and $A_w, A_{\underline{w}}$ occur at their α'^w-order
- the initial values $Z^{i\infty}_{\eta_2}(*|1,2)$ depend on η_2 via straightforward trigonometric series (3.18)
- the derivations $r_{\eta_2}(\epsilon_k)$ in (3.12) are linear in α' and change the order in η_2 by $\geq k{-}2$

The last point implies that the order-$(\alpha')^w$ contribution $r_{\eta_2}(\epsilon_{k_w}\ldots\epsilon_{k_1})$ to (3.14) cannot lower the order in η_2 by more than $2w$ units. Hence, the A-cycle graph functions $A_w, A_{\underline{w}}$ along with α'^w are only sensitive to the orders $\eta_2^{\leq 2w-1}$ in the expansion of $\pi\cot(\pi\eta_2)$ in $Z^{i\infty}_{\eta_2}(*|1,2)$.

Similarly, only a small number of length-w words $\gamma(k_1,\ldots,k_w|\tau)r_{\eta_2}(\epsilon_{k_w}\ldots\epsilon_{k_1})$ in (3.14) contributes to a given A_w or $A_{\underline{w}}$: Since the initial values $Z^{i\infty}_{\eta_2}(*|1,2)$ have no poles of higher order than $\frac{1}{\eta_2}$, the combined effect of the $r_{\eta_2}(\epsilon_k)$ must not raise the power of η_2 in order to contribute to $Z^\tau_{(0)}(*)$. This leads to the bound $k_1+k_2+\ldots+k_w\leq 2w$ for the $\gamma(k_1,\ldots,k_w|\tau)$ that contribute to A_w and $A_{\underline{w}}$, in lines with the examples in (3.35) and (3.36).

So far, we did not take the MZVs in the planar α'-expansion (3.31) into account. Since the series (3.32) in ζ_m can be factored out from $Z^\tau_{\eta_2}(1,2|1,2)$, some of the terms in $A_w, A_{\underline{w}}$ are products of ζ_m multiplying A-cycle graph functions of lower weight.

[13]In case of A_2 and A_3, rewriting $\gamma(\ldots|\tau)$ in terms of $\gamma_0(\ldots|\tau)$ will absorb the contributions $\sim\zeta_4\gamma(0,0|\tau)$ and $\sim\zeta_6\gamma(0,0,0|\tau)$ from

$$A_2(\tau)=\frac{\zeta_2}{2}-12\zeta_4\gamma(0,0|\tau)-6\gamma(4,0|\tau)\,,\qquad A_3(\tau)=\frac{3\zeta_3}{2}+384\zeta_6\gamma(0,0,0|\tau)+144\zeta_2\gamma(4,0,0|\tau)-60\gamma(6,0,0|\tau)\,.$$

This kind of counting can be straightforwardly generalized to higher-order component integrals $Z^\tau_{(m>0)}(*)$ in (3.33), resulting in sharp bounds on the possible contributions to specific α'-orders. While integrals over odd functions $f^{(2k-1)}$ vanish identically,

$$Z^\tau_{(2k-1)}(1,2) = Z^\tau_{(2k-1)}\big(\begin{smallmatrix}2\\1\end{smallmatrix}\big) = 0\,, \qquad k\geq 1\,, \tag{3.37}$$

the simplest examples beyond A-cycle graph functions read

$$\begin{aligned}
Z^\tau_{(2)}(1,2) &= -2\zeta_2 + 3s_{12}\gamma_0(4|\tau) + s_{12}^2\Big(-\frac{5\zeta_4}{4} - 18\zeta_2\gamma_0(4,0|\tau) + 10\gamma_0(6,0|\tau)\Big)\\
&\quad + s_{12}^3\Big(-\frac{\zeta_2\zeta_3}{2} + \frac{3\zeta_2}{4}\gamma_0(4|\tau) + 24\zeta_4\gamma_0(4,0,0|\tau) - 9\gamma_0(4,0,4|\tau) + 18\gamma_0(4,4,0|\tau)\\
&\qquad - 220\zeta_2\gamma_0(6,0,0|\tau) + 126\gamma_0(8,0,0|\tau)\Big) + \mathcal{O}(\alpha'^4) \qquad\qquad (3.38)\\
Z^\tau_{(4)}(1,2) &= -2\zeta_4 + s_{12}\Big(-6\zeta_2\gamma_0(4|\tau) + 5\gamma_0(6|\tau)\Big)\\
&\quad + s_{12}^2\Big(-\frac{7\zeta_6}{8} - 42\zeta_4\gamma_0(4,0|\tau) + 9\gamma_0(4,4|\tau) - 100\zeta_2\gamma_0(6,0|\tau) + 63\gamma_0(8,0|\tau)\Big) + \mathcal{O}(\alpha'^3)\\
Z^\tau_{(6)}(1,2) &= -2\zeta_6 + s_{12}\Big(-6\zeta_4\gamma_0(4|\tau) - 10\zeta_2\gamma_0(6|\tau) + 7\gamma_0(8|\tau)\Big) + \mathcal{O}(\alpha'^2)
\end{aligned}$$

as well as

$$\begin{aligned}
q^{-\frac{s_{12}}{8}} Z^\tau_{(2)}\big(\begin{smallmatrix}2\\1\end{smallmatrix}\big) &= \zeta_2 + 3s_{12}\gamma_0(4|\tau) + s_{12}^2\Big(9\zeta_2\gamma_0(4,0|\tau) + 10\gamma_0(6,0|\tau)\Big)\\
&\quad + s_{12}^3\Big(114\zeta_4\gamma_0(4,0,0|\tau) - 9\gamma_0(4,0,4|\tau) + 18\gamma_0(4,4,0|\tau)\\
&\qquad + 110\zeta_2\gamma_0(6,0,0|\tau) + 126\gamma_0(8,0,0|\tau)\Big) + \mathcal{O}(\alpha'^4) \qquad\qquad (3.39)\\
q^{-\frac{s_{12}}{8}} Z^\tau_{(4)}\big(\begin{smallmatrix}2\\1\end{smallmatrix}\big) &= \frac{7\zeta_4}{4} + s_{12}\Big(3\zeta_2\gamma_0(4|\tau) + 5\gamma_0(6|\tau)\Big)\\
&\quad + s_{12}^2\Big(\frac{147\zeta_4}{4}\gamma_0(4,0|\tau) + 9\gamma_0(4,4|\tau) + 50\zeta_2\gamma_0(6,0|\tau) + 63\gamma_0(8,0|\tau)\Big) + \mathcal{O}(\alpha'^3)\\
q^{-\frac{s_{12}}{8}} Z^\tau_{(6)}\big(\begin{smallmatrix}2\\1\end{smallmatrix}\big) &= \frac{31\zeta_6}{16} + s_{12}\Big(\frac{21\zeta_4}{4}\gamma_0(4|\tau) + 5\zeta_2\gamma_0(6|\tau) + 7\gamma_0(8|\tau)\Big) + \mathcal{O}(\alpha'^2)\,.
\end{aligned}$$

There is no bottleneck in generating much high orders in α' from (3.14), even without optimizing the performance of a computer implementation.

3.6 On B-cycle graph functions and modular graph functions

The motivation for defining A-cycle graph functions in [81] was to find an open-string analogue of the modular graph functions in closed-string α'-expansions [91, 92]. Indeed, the B-cycle graph functions obtained from modular $\tau \to -\frac{1}{\tau}$ transformation of their A-cycle analogues were taken as a starting point to propose an elliptic single-valued map connecting genus-one integrals in open- and closed-string one-loop amplitudes [81].

The asymptotics of various graph functions at the cusp $\tau \to i\infty$ has recently attracted a lot of attention. In the case of B-cycle and modular graph functions, the behavior at the cusp is governed by Laurent polynomials in $(\pi\tau)$ and $(\pi \operatorname{Im} \tau)$, respectively, whose coefficients are $\mathbb{Q}$-linear combinations of MZVs[14]. For the planar two-point B-cycle graph functions, i.e. the modular transformations of A_w, the Laurent polynomials at all $w \in \mathbb{N}$ have been expressed in terms of Riemann zeta values [94]. Similarly, the Laurent polynomials of all two-point modular graph functions were determined in terms of odd (i.e. single-valued) Riemann zeta values [94, 95].

While the recent results on B-cycle graph functions at the cusp settle the behavior of $A_w(\tau)$ at $\tau \to 0$, their behavior at $\tau \to i\infty$ has not yet been spelled out for all weights. By isolating the η^{-1} order of (3.31) and exploiting that the iterated Eisenstein integrals do not contribute at the cusp, we infer the generating function

$$\sum_{w=0}^{\infty} \frac{s_{12}^w}{w!} A_w(i\infty) = \frac{\Gamma(1-s_{12})}{\left[\Gamma(1-\frac{s_{12}}{2})\right]^2}\,, \tag{3.40}$$

that of course reproduces the results $A_w(i\infty) = \frac{\zeta_2}{2}$, $A_3(i\infty) = \frac{3\zeta_3}{2}$ and $A_4(i\infty) = \frac{57\zeta_4}{8}$ in (3.35). Similarly, the non-planar α'-expansion (3.24) implies the vanishing of all non-trivial non-planar A-cycle graph functions $A_{\underline{w}}$ at the cusp

$$A_{\underline{w}}(i\infty) = 0\,, \qquad w \geq 1\,. \tag{3.41}$$

In later sections, similar generating series will be given for the asymptotics of higher-point A-cycle graph functions. We hope that the techniques of this work can be helpful to determine the much richer patterns of MZVs in the Laurent polynomials of B-cycle and modular graph functions beyond two points. Also, it would be interesting to study the generalization of B-cycle and modular graph functions to additional factors of $f_{ij}^{(m)}$ in the integrand, i.e. the B-cycle and closed-string analogues of the higher $Z^\tau_{(m>0)}(*)$ functions in (3.33).

4 Differential equations at n points

In the previous section, we have illustrated the key ideas of this work by a two-point example. The purpose of the remaining sections is to propose the generalization of all the steps to the n-point A-cycle integrals in (1.1). In this section, we explain the derivation of the homogeneous and linear first-order differential equation (1.4) at n points and describe the explicit form of the $(n-1)! \times (n-1)!$ matrix-valued differential operator $D^\tau_{\vec{\eta}}$. While the expressions up to and including $n = 5$ are based on rigorous calculations, the form of $D^\tau_{\vec{\eta}}$ at $n \geq 6$ is conjectural.

The subsequent differential equations hold for any planar or non-planar integration cycle $\mathcal{C}(*)$ in (1.1) that is realized in one-loop open-string amplitudes, cf. figure 1. We will always set $z_1 = 0$ and use the notation

$$\mathcal{C}(1,2,3,\ldots,n) \equiv \{z_2, z_3, \ldots, z_n \in \mathbb{R},\ 0 = z_1 < z_2 < z_3 < \ldots < z_n < 1\} \tag{4.1}$$

[14] For B-cycle graph functions, the appearance of $\mathbb{Q}$-linear rather than $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs has been proven in [81] and, with an improved bound on the degree of the Laurent polynomial, in [83]. For modular graph functions, the coefficients in the Laurent polynomial are proven to be $\mathbb{Q}$-linear combinations of cyclotomic MZVs and conjectured to be single-valued MZVs [82, 93].

in the planar case as well as

$$\mathcal{C}\big(\begin{smallmatrix} r+1,\ldots,n \\ 1,2,3,\ldots,r \end{smallmatrix}\big) \equiv \mathcal{C}(1,2,\ldots,r) \times \Big\{ z_j = \tau/2 + v_j \ \forall \ j = r{+}1,\ldots,n, \qquad (4.2)$$
$$\bigvee_{\substack{\rho \ \text{cyclic}(r+1, \\ r+2,\ldots,n)}} 0 < v_{\rho(r+1)} < v_{\rho(r+2)} < \ldots < v_{\rho(n)} < 1 \Big\}$$

for non-planar integration cycles, in lines with the two-point cases in (3.1). The second line of (4.2) ensures that, for fixed $z_1 = 0$, all the cyclic arrangements of punctures $z_{r+1},\ldots,z_n$ on the upper boundary in figure 1 are taken into account. The Koba–Nielsen factors will henceforth be denoted by the shorthand $\mathrm{KN}^\tau_{12\ldots n}$ defined in (2.13).

4.1 Three points

The three-point instances of the A-cycle integrals (1.1) are given by

$$Z^\tau_{\eta_2,\eta_3}(*|1,2,3) = \int_{\mathcal{C}(*)} \mathrm{d}z_2\,\mathrm{d}z_3\,\Omega(z_{12},\eta_2{+}\eta_3,\tau)\Omega(z_{23},\eta_3,\tau)\,\mathrm{KN}^\tau_{123} \qquad (4.3)$$

and allow for two inequivalent planar integration cycles $\mathcal{C}(1,2,3), \mathcal{C}(1,3,2)$ as well as three inequivalent non-planar ones $\mathcal{C}\big(\begin{smallmatrix} 3 \\ 1,2 \end{smallmatrix}\big), \mathcal{C}\big(\begin{smallmatrix} 2 \\ 1,3 \end{smallmatrix}\big), \mathcal{C}\big(\begin{smallmatrix} 2,3 \\ 1 \end{smallmatrix}\big)$. The τ-derivative of (4.3) can be evaluated by iteration of the steps in the two-point computation (3.3),

$$\begin{aligned} 2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3) = \int_{\mathcal{C}(*)} \mathrm{d}z_2\,\mathrm{d}z_3 \Big(&- 2\zeta_2 s_{123}\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau) \\ &+ s_{12}\big[f^{(1)}_{12}\partial_{\beta_1}\Omega(z_{12},\beta_1,\tau) - f^{(2)}_{12}\Omega(z_{12},\beta_1,\tau)\big]\Omega(z_{23},\beta_2,\tau) \\ &+ s_{23}\big[f^{(1)}_{23}\partial_{\beta_2}\Omega(z_{23},\beta_2,\tau) - f^{(2)}_{23}\Omega(z_{23},\beta_2,\tau)\big]\Omega(z_{12},\beta_1,\tau) \\ &+ s_{13}\big[f^{(1)}_{13}\Omega(z_{23},\beta_2,\tau)\partial_{\beta_1}\Omega(z_{12},\beta_1,\tau) + f^{(1)}_{13}\Omega(z_{12},\beta_1,\tau)\partial_{\beta_2}\Omega(z_{23},\beta_2,\tau) \\ &\quad - f^{(2)}_{13}\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\big]\Big)\,\mathrm{KN}^\tau_{123}\,, \end{aligned} \qquad (4.4)$$

where we have again used the mixed heat equation (2.7), integration by parts to eliminate $\partial_z\Omega(z_{ij},\ldots)$ as well as the Koba–Nielsen derivatives (2.17). The second arguments of the $\Omega(\ldots)$ are denoted by $\beta_1 = \eta_2{+}\eta_3$ and $\beta_2 = \eta_3$ in intermediate steps to make the scope of the ∂_{β_j} more transparent. In the next step, we express $f^{(1)}_{ij}$ and $f^{(2)}_{ij}$ in terms of the ξ^0-order of another Kronecker–Eisenstein series $\Omega(z_{ji},\xi,\tau)$, see (3.4), and obtain

$$\begin{aligned} 2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3) = \int_{\mathcal{C}(*)} \mathrm{d}z_2\,\mathrm{d}z_3 \Big(&- 2\zeta_2 s_{123}\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau) \\ &- s_{12}\Omega(z_{23},\beta_2,\tau)(\partial_{\beta_1}{+}\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{21},\xi,\tau) \\ &- s_{23}\Omega(z_{12},\beta_1,\tau)(\partial_{\beta_2}{+}\partial_\xi)\Omega(z_{23},\beta_2,\tau)\Omega(z_{32},\xi,\tau) \\ &- s_{13}(\partial_{\beta_1}{+}\partial_{\beta_2}{+}\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\Omega(z_{31},\xi,\tau)\Big)\,\mathrm{KN}^\tau_{123}\Big|_{\xi^0}\,. \end{aligned} \qquad (4.5)$$

The main goal in the simplification of the τ-derivative is to have at most two factors of $\Omega(\ldots)$ in each term – this is a necessary condition for writing the right-hand side in terms of integrals

$Z^\tau_{\eta_2,\eta_3}(\ldots)$. In the second and third line of (4.5), this can be readily achieved by means of (3.5) whereas the last line requires the more general identity

$$\begin{aligned}
&(\partial_{\beta_1}+\partial_{\beta_2}+\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\Omega(z_{31},\xi,\tau) \\
&\quad= -\wp(\beta_1,\tau)\Omega(z_{23},\beta_2{-}\beta_1,\tau)\Omega(z_{31},\xi{-}\beta_1,\tau) \\
&\qquad - \wp(\beta_2,\tau)\Omega(z_{12},\beta_1{-}\beta_2,\tau)\Omega(z_{31},\xi{-}\beta_2,\tau) \\
&\qquad - \wp(\xi,\tau)\Omega(z_{12},\beta_1{-}\xi,\tau)\Omega(z_{23},\beta_2{-}\xi,\tau)
\end{aligned} \tag{4.6}$$

to be derived in appendix A.2. After inserting (3.5) and (4.6) into (4.5) and picking up the zero$^{\text{th}}$ order in ξ, one arrives at

$$\begin{aligned}
2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3) = \int_{\mathcal{C}(*)} &\mathrm{d}z_2\,\mathrm{d}z_3\,\Big(-2\zeta_2 s_{123}\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau) \\
&+ s_{12}\Omega(z_{23},\beta_2,\tau)\Big(\frac{1}{2}\partial^2_{\beta_1}-\wp(\beta_1,\tau)\Big)\Omega(z_{12},\beta_1,\tau) \\
&+ s_{23}\Omega(z_{12},\beta_1,\tau)\Big(\frac{1}{2}\partial^2_{\beta_2}-\wp(\beta_2,\tau)\Big)\Omega(z_{23},\beta_2,\tau) \\
&+ s_{13}\Big[\big(\wp(\beta_1,\tau)-\wp(\beta_2,\tau)\big)\Omega(z_{13},\beta_1,\tau)\Omega(z_{32},\beta_1{-}\beta_2,\tau) \\
&\quad + \Big(\frac{1}{2}(\partial_{\beta_1}{+}\partial_{\beta_2})^2-\wp(\beta_2,\tau)\Big)\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\Big]\Big)\,\mathrm{KN}^\tau_{123}\,,
\end{aligned} \tag{4.7}$$

where the Fay identity (2.6) has been used to eliminate one of the three arrangements of the first arguments $\Omega(z_{ij},\ldots)\Omega(z_{jk},\ldots)$:

$$\Omega(z_{12},\beta_1{-}\beta_2,\tau)\Omega(z_{13},\beta_2,\tau) = \Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)+\Omega(z_{13},\beta_1,\tau)\Omega(z_{32},\beta_1{-}\beta_2,\tau)\,. \tag{4.8}$$

At this point, one can see the benefit of rewriting $\beta_1=\eta_2{+}\eta_3$ and $\beta_2=\eta_3$ in terms of the η_j variables: The two contributions $\Omega(z_{12},\eta_2{+}\eta_3,\tau)\Omega(z_{23},\eta_3,\tau)$ and $\Omega(z_{13},\eta_2{+}\eta_3,\tau)\Omega(z_{32},\eta_2,\tau)$ to the integrand of (4.7) are permutations of each other w.r.t. $(z_2,\eta_2)\leftrightarrow(z_3,\eta_3)$. Hence, the entire right-hand side of (4.7) can be rewritten in terms of the original integral and one permutation under $2\leftrightarrow 3$,

$$\begin{aligned}
2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3) &= \Big(s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2}-\wp(\eta_2{+}\eta_3,\tau)\big]+s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}{-}\partial_{\eta_3})^2-\wp(\eta_3,\tau)\big] \\
&\qquad + s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3}-\wp(\eta_3,\tau)\big]-2\zeta_2 s_{123}\Big)Z^\tau_{\eta_2,\eta_3}(*|1,2,3) \\
&\quad + s_{13}\big[\wp(\eta_2{+}\eta_3,\tau)-\wp(\eta_3,\tau)\big]Z^\tau_{\eta_2,\eta_3}(*|1,3,2) \\
&= \sum_{B\in S_2} D^\tau_{\eta_2,\eta_3}(2,3|b_2,b_3)Z^\tau_{\eta_2,\eta_3}(*|1,b_2,b_3)\,.
\end{aligned} \tag{4.9}$$

In the last step, $2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3)$ has been lined up with the general form (1.4) of the differential equation. The words $B=b_2b_3\ldots b_n$ in a summation $B\in S_{n-1}$ are always understood to be permutations of $2,3,\ldots,n$. The first row of the 2×2 differential operator $D^\tau_{\eta_2,\eta_3}$

can be immediately read off from (4.9),

$$\begin{aligned}D^\tau_{\eta_2,\eta_3}(2,3|2,3) &= s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2{+}\eta_3,\tau)\big] + s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}{-}\partial_{\eta_3})^2 - \wp(\eta_3,\tau)\big]\\ &\qquad + s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3} - \wp(\eta_3,\tau)\big] - 2\zeta_2 s_{123}\\ D^\tau_{\eta_2,\eta_3}(2,3|3,2) &= s_{13}\big[\wp(\eta_2{+}\eta_3,\tau) - \wp(\eta_3,\tau)\big]\,,\end{aligned} \tag{4.10}$$

whereas the second row is obtained by relabeling $s_{12}\leftrightarrow s_{13}$ and $\eta_2\leftrightarrow\eta_3$,

$$\begin{aligned}D^\tau_{\eta_2,\eta_3}(3,2|3,2) &= s_{13}\big[\tfrac{1}{2}\partial^2_{\eta_3} - \wp(\eta_2{+}\eta_3,\tau)\big] + s_{23}\big[\tfrac{1}{2}(\partial_{\eta_2}{-}\partial_{\eta_3})^2 - \wp(\eta_2,\tau)\big]\\ &\qquad + s_{12}\big[\tfrac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau)\big] - 2\zeta_2 s_{123}\\ D^\tau_{\eta_2,\eta_3}(3,2|2,3) &= s_{12}\big[\wp(\eta_2{+}\eta_3,\tau) - \wp(\eta_2,\tau)\big]\,.\end{aligned} \tag{4.11}$$

The expansion (3.6) of the Weierstrass function casts the differential operator into the form

$$D^\tau_{\eta_2,\eta_3} = \sum_{m=0}^{\infty}(1{-}2m)\mathrm{G}_{2m}(\tau)r_{\eta_2,\eta_3}(\epsilon_{2m})\,, \tag{4.12}$$

and yields the following (2×2)-matrix representation of the derivations

$$\begin{aligned}r_{\eta_2,\eta_3}(\epsilon_k) &= \delta_{k,0}\Big(2\zeta_2 s_{123} - \frac{1}{2}s_{12}\partial^2_{\eta_2} - \frac{1}{2}s_{13}\partial^2_{\eta_3} - \frac{1}{2}s_{23}(\partial_{\eta_2}-\partial_{\eta_3})^2\Big)1_{2\times2}\\ &\quad + \eta_{23}^{k-2}\begin{pmatrix} s_{12} & -s_{13}\\ -s_{12} & s_{13}\end{pmatrix} + \eta_2^{k-2}\begin{pmatrix} 0 & 0\\ s_{12} & s_{12}{+}s_{23}\end{pmatrix} + \eta_3^{k-2}\begin{pmatrix} s_{13}{+}s_{23} & s_{13}\\ 0 & 0\end{pmatrix}\,,\qquad k\neq 2\end{aligned} \tag{4.13}$$

with $r_{\eta_2,\eta_3}(\epsilon_2)=0$ and shorthand $\eta_{23}\equiv\eta_2{+}\eta_3$. All the matrix entries are linear in s_{ij} and therefore in α'. As a three-point instance of (1.7), the solution of (4.9) via Picard iteration yields the following α'-expansion

$$\begin{aligned}Z^\tau_{\eta_2,\eta_3}(*|1,2,3) &= \sum_{r=0}^{\infty}\sum_{\substack{k_1,k_2,\ldots,k_r\\ =0,4,6,8,\ldots}}\prod_{j=1}^{r}(k_j{-}1)\,\gamma(k_1,k_2,\ldots,k_r|\tau)\\ &\qquad\times\sum_{B\in S_2} r_{\eta_2,\eta_3}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})_{23}{}^{B}\,Z^{i\infty}_{\eta_2,\eta_3}(*|1,B)\,.\end{aligned} \tag{4.14}$$

In contrast to the scalar representations (3.12) at two points, the $r_{\eta_2,\eta_3}(\epsilon_{k\geq4})$ no longer commute. By $[r_{\eta_2,\eta_3}(\epsilon_4),r_{\eta_2,\eta_3}(\epsilon_6)]\neq 0$, for instance, $\gamma(4,6|\tau)$ and $\gamma(6,4|\tau)$ enter (4.14) with different coefficients and introduce an irreducible iterated Eisenstein integral at depth two which is not expressible via $\gamma(4|\tau)$ and $\gamma(6|\tau)$.

The initial values $Z^{i\infty}_{\eta_2,\eta_3}(*|1,b_2,b_3)$ will be later on assembled from $(n{\leq}5)$-point disk integrals, see section 5.4 for the planar cycles $\mathcal{C}(*)$ and section 6.2 for the non-planar ones.

4.2 Four points

Starting from the four-point example of the A-cycle integrals (1.1), we will use the vector-notation $\vec\eta$ to refer to the η_j in the subscript. I.e. we have $\vec\eta=\eta_2,\eta_3,\eta_4$ in the definition

$$Z^\tau_{\vec\eta}(*|1,2,3,4) = \int_{\mathcal{C}(*)}\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4\,\Omega(z_{12},\eta_2{+}\eta_3{+}\eta_4,\tau)\Omega(z_{23},\eta_3{+}\eta_4,\tau)\Omega(z_{34},\eta_4,\tau)\,\mathrm{KN}^\tau_{1234} \tag{4.15}$$

that applies to 3, 8 and 6 cyclically inequivalent integration cycles of the type $\mathcal{C}\big({}^{k,l}_{i,j}\big), \mathcal{C}\big({}^{\;\;l}_{i,j,k}\big)$ and $\mathcal{C}(i,j,k,l)$, respectively. The τ-derivative of (4.15) can be computed by repeating the steps (4.4), (4.5) in the three-point case[15], and it is convenient to use the variables $\beta_1 = \eta_2{+}\eta_3{+}\eta_4$, $\beta_2 = \eta_3{+}\eta_4$ and $\beta_3 = \eta_4$ in intermediate steps:

$$\begin{aligned}
2\pi i \partial_\tau Z^\tau_{\vec\eta}(*|1,2,3,4) = \int_{\mathcal{C}(*)} &\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4\,\mathrm{KN}^\tau_{1234}\Big(-2\zeta_2 s_{1234}\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)\\
&- s_{12}\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)(\partial_{\beta_1}{+}\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{21},\xi,\tau)\\
&- s_{23}\Omega(z_{12},\beta_1,\tau)\Omega(z_{34},\beta_3,\tau)(\partial_{\beta_2}{+}\partial_\xi)\Omega(z_{23},\beta_2,\tau)\Omega(z_{32},\xi,\tau)\\
&- s_{34}\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)(\partial_{\beta_3}{+}\partial_\xi)\Omega(z_{34},\beta_3,\tau)\Omega(z_{43},\xi,\tau)\\
&- s_{13}\Omega(z_{34},\beta_3,\tau)(\partial_{\beta_1}{+}\partial_{\beta_2}{+}\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\Omega(z_{31},\xi,\tau)\\
&- s_{24}\Omega(z_{12},\beta_1,\tau)(\partial_{\beta_2}{+}\partial_{\beta_3}{+}\partial_\xi)\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)\Omega(z_{42},\xi,\tau)\\
&- s_{14}(\partial_{\beta_1}{+}\partial_{\beta_2}{+}\partial_{\beta_3}{+}\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)\Omega(z_{41},\xi,\tau)\Big)\Big|_{\xi^0}\,.
\end{aligned} \tag{4.16}$$

The cycles of Kronecker–Eisenstein series can be resolved using the earlier lemmata (3.5) and (4.6) as well as the following identity explained in appendix A.3

$$\begin{aligned}
&(\partial_{\beta_1}+\partial_{\beta_2}+\partial_{\beta_3}+\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)\Omega(z_{41},\xi,\tau)\\
&\quad= -\wp(\beta_1,\tau)\Omega(z_{23},\beta_2{-}\beta_1,\tau)\Omega(z_{34},\beta_3{-}\beta_1,\tau)\Omega(z_{41},\xi{-}\beta_1,\tau)\\
&\qquad - \wp(\beta_2,\tau)\Omega(z_{12},\beta_1{-}\beta_2,\tau)\Omega(z_{34},\beta_3{-}\beta_2,\tau)\Omega(z_{41},\xi{-}\beta_2,\tau)\\
&\qquad - \wp(\beta_3,\tau)\Omega(z_{12},\beta_1{-}\beta_3,\tau)\Omega(z_{23},\beta_2{-}\beta_3,\tau)\Omega(z_{41},\xi{-}\beta_3,\tau)\\
&\qquad - \wp(\xi,\tau)\Omega(z_{12},\beta_1{-}\xi,\tau)\Omega(z_{23},\beta_2{-}\xi,\tau)\Omega(z_{34},\beta_3{-}\xi,\tau)\,.
\end{aligned} \tag{4.17}$$

After extracting the ξ^0-order in (4.16), one can enforce by a sequence of Fay identities (4.8) that the first arguments of the Kronecker–Eisenstein series match one of the six permutations of $\Omega(z_{12},\ldots)\Omega(z_{23},\ldots)\Omega(z_{34},\ldots)$ in $2,3,4$. This is analogous to repeated partial-fraction manipulations which can be used to expand expressions of the form $\frac{1}{z_{ij}z_{jk}z_{kl}}$ and $\frac{1}{z_{ij}z_{ik}z_{il}}$ (with i,j,k,l pairwise distinct) in a six-dimensional basis. When the β_j are rewritten in terms of the η_j variables, we arrive at a linear combination of various $Z^\tau_{\vec\eta}(*|1,i,j,k)$ in (4.15):

$$\begin{aligned}
2\pi i\partial_\tau Z^\tau_{\vec\eta}(*|1,2,3,4) = \Big(&\frac{1}{2}\sum_{j=2}^4 s_{1j}\partial^2_{\eta_j} + \frac{1}{2}\sum_{2\leq i<j}^4 s_{ij}(\partial_{\eta_i}{-}\partial_{\eta_j})^2 - s_{12}\wp(\eta_2{+}\eta_3{+}\eta_4,\tau)\\
&- (s_{13}{+}s_{23})\wp(\eta_3{+}\eta_4,\tau) - (s_{14}{+}s_{24}{+}s_{34})\wp(\eta_4,\tau) - 2\zeta_2 s_{1234}\Big) Z^\tau_{\vec\eta}(*|1,2,3,4)
\end{aligned}$$

[15] Note that two integrations by parts w.r.t. z_3, z_4 are required to remove the z-derivative from $2\pi i\partial_\tau\Omega(z_{23},\beta_2,\tau) = \partial_{z_2}\partial_{\beta_2}\Omega(z_{23},\beta_2,\tau)$. More specifically, the Koba–Nielsen derivative (2.17) implies that

$$\begin{aligned}
\partial_{z_2}\partial_{\beta_2}\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)\mathrm{KN}^\tau_{1234} &= \partial_{\beta_2}\Omega(z_{23},\beta_2,\tau)\mathrm{KN}^\tau_{1234}(\partial_{z_3}{+}s_{13}f^{(1)}_{13}{+}s_{23}f^{(1)}_{23}{-}s_{34}f^{(1)}_{34})\Omega(z_{34},\beta_3,\tau)\\
&= (s_{13}f^{(1)}_{13}{+}s_{23}f^{(1)}_{23}{+}s_{14}f^{(1)}_{14}{+}s_{24}f^{(1)}_{24})\partial_{\beta_2}\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)\mathrm{KN}^\tau_{1234}
\end{aligned}$$

after discarding total derivatives $(\partial_{z_3}{+}\partial_{z_4})\big[\partial_{\beta_2}\Omega(z_{23},\beta_2,\tau)\Omega(z_{34},\beta_3,\tau)\mathrm{KN}^\tau_{1234}\big]$.

$$
\begin{aligned}
&+ s_{13}\big[\wp(\eta_2{+}\eta_3{+}\eta_4,\tau) - \wp(\eta_3{+}\eta_4,\tau)\big]\big(Z^\tau_{\vec\eta}(*|1,3,2,4) + Z^\tau_{\vec\eta}(*|1,3,4,2)\big)\\
&+ s_{24}\big[\wp(\eta_3{+}\eta_4,\tau) - \wp(\eta_4,\tau)\big]Z^\tau_{\vec\eta}(*|1,2,4,3)\\
&+ s_{14}\big[\wp(\eta_3{+}\eta_4,\tau) - \wp(\eta_2{+}\eta_3{+}\eta_4,\tau)\big]Z^\tau_{\vec\eta}(*|1,4,3,2)\\
&+ s_{14}\big[\wp(\eta_3{+}\eta_4,\tau) - \wp(\eta_4,\tau)\big]\big(Z^\tau_{\vec\eta}(*|1,2,4,3) + Z^\tau_{\vec\eta}(*|1,4,2,3)\big)\\
= &\sum_{B\in S_3} D^\tau_{\vec\eta}(2,3,4|b_2,b_3,b_4) Z^\tau_{\vec\eta}(*|1,b_2,b_3,b_4)\,.
\end{aligned}
\tag{4.18}
$$

The last line defines the first row of a 6×6 differential operator $D^\tau_{\vec\eta}$,

$$
\begin{aligned}
D^\tau_{\vec\eta}(2,3,4|2,3,4) &= \frac{1}{2}\sum_{j=2}^4 s_{1j}\partial^2_{\eta_j} + \frac{1}{2}\sum_{2\le i<j}^4 s_{ij}(\partial_{\eta_i}{-}\partial_{\eta_j})^2 - s_{12}\wp(\eta_2{+}\eta_3{+}\eta_4,\tau)\\
&\quad - (s_{13}{+}s_{23})\wp(\eta_3{+}\eta_4,\tau) - (s_{14}{+}s_{24}{+}s_{34})\wp(\eta_4,\tau) - 2\zeta_2 s_{1234}\\
D^\tau_{\vec\eta}(2,3,4|2,4,3) &= (s_{14}{+}s_{24})\big[\wp(\eta_3{+}\eta_4,\tau) - \wp(\eta_4,\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|3,2,4) &= s_{13}\big[\wp(\eta_2{+}\eta_3{+}\eta_4,\tau) - \wp(\eta_3{+}\eta_4,\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|3,4,2) &= s_{13}\big[\wp(\eta_2{+}\eta_3{+}\eta_4,\tau) - \wp(\eta_3{+}\eta_4,\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|4,2,3) &= s_{14}\big[\wp(\eta_3{+}\eta_4,\tau) - \wp(\eta_4,\tau)\big]\\
D^\tau_{\vec\eta}(2,3,4|4,3,2) &= s_{14}\big[\wp(\eta_3{+}\eta_4,\tau) - \wp(\eta_2{+}\eta_3{+}\eta_4,\tau)\big]\,,
\end{aligned}
\tag{4.19}
$$

and the remaining rows are obtained from relabelings of both s_{ij} and η_j w.r.t. 2,3,4. The 6×6 representation of the derivations resulting from the expansion (1.6) in terms of Eisenstein series can be conveniently organized as

$$
\begin{aligned}
r_{\vec\eta}(\epsilon_k) &= \delta_{k,0}\Big(2\zeta_2 s_{1234} - \frac{1}{2}\sum_{j=2}^4 s_{1j}\partial^2_{\eta_j} - \frac{1}{2}\sum_{2\le i<j}^4 s_{ij}(\partial_{\eta_i}{-}\partial_{\eta_j})^2\Big)1_{6\times 6} + \eta_{234}^{k-2} r_{\vec\eta}(e_{234})\\
&+ \eta_{23}^{k-2} r_{\vec\eta}(e_{23}) + \eta_{24}^{k-2} r_{\vec\eta}(e_{24}) + \eta_{34}^{k-2} r_{\vec\eta}(e_{34}) + \eta_{2}^{k-2} r_{\vec\eta}(e_{2}) + \eta_{3}^{k-2} r_{\vec\eta}(e_{3}) + \eta_{4}^{k-2} r_{\vec\eta}(e_{4})\,, \quad k\neq 2
\end{aligned}
\tag{4.20}
$$

with $r_{\vec\eta}(\epsilon_2)=0$, and we used the shorthands $\eta_{ij}\equiv\eta_i{+}\eta_j$ and $\eta_{234}\equiv\eta_2{+}\eta_3{+}\eta_4$. The $r_{\vec\eta}(e_{\ldots})$ refer to 6×6 matrices whose entries are linear in the s_{ij}. Representative examples for their explicit form are

$$
r_{\vec\eta}(e_{234}) = \begin{pmatrix}
s_{12} & 0 & -s_{13} & -s_{13} & 0 & s_{14}\\
0 & s_{12} & 0 & s_{13} & -s_{14} & -s_{14}\\
-s_{12} & -s_{12} & s_{13} & 0 & s_{14} & 0\\
0 & s_{12} & 0 & s_{13} & -s_{14} & -s_{14}\\
-s_{12} & -s_{12} & s_{13} & 0 & s_{14} & 0\\
s_{12} & 0 & -s_{13} & -s_{13} & 0 & s_{14}
\end{pmatrix}
$$

$$r_{\vec\eta}(e_{34})=\begin{pmatrix} s_{13}{+}s_{23} & -s_{14}{-}s_{24} & s_{13} & s_{13} & -s_{14} & -s_{14}\\ -s_{13}{-}s_{23} & s_{14}{+}s_{24} & -s_{13} & -s_{13} & s_{14} & s_{14}\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix} \tag{4.21}$$

$$r_{\vec\eta}(e_{4})=\begin{pmatrix} s_{14}{+}s_{24}{+}s_{34} & s_{14}{+}s_{24} & 0 & 0 & s_{14} & 0\\ 0&0&0&0&0&0\\ 0&0& s_{14}{+}s_{24}{+}s_{34} & s_{14}{+}s_{34} & 0 & s_{14}\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

and the expressions for $r_{\vec\eta}(e_{23})$, $r_{\vec\eta}(e_{24})$, $r_{\vec\eta}(e_{2})$ and $r_{\vec\eta}(e_{3})$ obtained from relabelings are given in appendix C.1. The solution to (4.18) via Picard iteration

$$\begin{aligned} Z^\tau_{\vec\eta}(*|1,2,3,4) &= \sum_{r=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_r\\ =0,4,6,8,\ldots}} \prod_{j=1}^{r}(k_j{-}1)\,\gamma(k_1,k_2,\ldots,k_r|\tau) \\ &\quad\times \sum_{B\in S_3} r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})_{234}{}^{B}\, Z^{i\infty}_{\vec\eta}(*|1,B) \end{aligned} \tag{4.22}$$

will be completed by the discussion of the initial values $Z^{i\infty}_{\vec\eta}(*|1,B)$ in sections 5.5 and 6.3.

4.3 Five points

The five-point A-cycle integrals (1.1) with $\vec\eta=\eta_2,\eta_3,\eta_4,\eta_5$ are given by

$$\begin{aligned} Z^\tau_{\vec\eta}(*|1,2,3,4,5) &= \int_{\mathcal{C}(*)} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4\,\mathrm{d}z_5\,\Omega(z_{12},\eta_2{+}\eta_3{+}\eta_4{+}\eta_5,\tau)\Omega(z_{23},\eta_3{+}\eta_4{+}\eta_5,\tau) \\ &\quad\times\Omega(z_{34},\eta_4{+}\eta_5,\tau)\Omega(z_{45},\eta_5,\tau)\,\mathrm{KN}^\tau_{12345} \end{aligned} \tag{4.23}$$

and may be applied to 20, 30 and 24 cyclically inequivalent integration cycles $\mathcal{C}\big(\begin{smallmatrix} l,m\\ i,j,k\end{smallmatrix}\big)$, $\mathcal{C}\big(\begin{smallmatrix} m\\ i,j,k,l\end{smallmatrix}\big)$ and $\mathcal{C}(i,j,k,l,m)$, respectively. Their τ-derivatives can be computed by iterating the steps of the earlier sections and inserting the five-point version of the lemma (A.12) to resolve the five-cycle of Kronecker–Eisenstein series in the last line of

$$\begin{aligned} 2\pi i\partial_\tau Z^\tau_{\vec\eta}(*|1,2,3,4,5) &= \int_{\mathcal{C}(*)} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4\,\mathrm{d}z_5\,\mathrm{KN}^\tau_{12345} \\ \times\Big(&-2\zeta_2 s_{12345}\Omega(z_{12},\beta_1)\Omega(z_{23},\beta_2)\Omega(z_{34},\beta_3)\Omega(z_{45},\beta_4) \\ &- s_{12}\Omega(z_{23},\beta_2)\Omega(z_{34},\beta_3)\Omega(z_{45},\beta_4)(\partial_{\beta_1}{+}\partial_\xi)\Omega(z_{12},\beta_1)\Omega(z_{21},\xi) \\ &- s_{23}\Omega(z_{12},\beta_1)\Omega(z_{34},\beta_3)\Omega(z_{45},\beta_4)(\partial_{\beta_2}{+}\partial_\xi)\Omega(z_{23},\beta_2)\Omega(z_{32},\xi) \\ &- s_{34}\Omega(z_{12},\beta_1)\Omega(z_{23},\beta_2)\Omega(z_{45},\beta_4)(\partial_{\beta_3}{+}\partial_\xi)\Omega(z_{34},\beta_3)\Omega(z_{43},\xi) \end{aligned} \tag{4.24}$$

$$
\begin{aligned}
&- s_{45}\Omega(z_{12},\beta_1)\Omega(z_{23},\beta_2)\Omega(z_{34},\beta_3)(\partial_{\beta_4}+\partial_\xi)\Omega(z_{45},\beta_4)\Omega(z_{54},\xi) \\
&- s_{13}\Omega(z_{34},\beta_3)\Omega(z_{45},\beta_4)(\partial_{\beta_1}+\partial_{\beta_2}+\partial_\xi)\Omega(z_{12},\beta_1)\Omega(z_{23},\beta_2)\Omega(z_{31},\xi) \\
&- s_{24}\Omega(z_{12},\beta_1)\Omega(z_{45},\beta_4)(\partial_{\beta_2}+\partial_{\beta_3}+\partial_\xi)\Omega(z_{23},\beta_2)\Omega(z_{34},\beta_3)\Omega(z_{42},\xi) \\
&- s_{35}\Omega(z_{12},\beta_1)\Omega(z_{23},\beta_2)(\partial_{\beta_3}+\partial_{\beta_4}+\partial_\xi)\Omega(z_{34},\beta_3)\Omega(z_{45},\beta_4)\Omega(z_{53},\xi) \\
&- s_{14}\Omega(z_{45},\beta_4)(\partial_{\beta_1}+\partial_{\beta_2}+\partial_{\beta_3}+\partial_\xi)\Omega(z_{12},\beta_1)\Omega(z_{23},\beta_2)\Omega(z_{34},\beta_3)\Omega(z_{41},\xi) \\
&- s_{25}\Omega(z_{12},\beta_1)(\partial_{\beta_2}+\partial_{\beta_3}+\partial_{\beta_4}+\partial_\xi)\Omega(z_{23},\beta_2)\Omega(z_{34},\beta_3)\Omega(z_{45},\beta_4)\Omega(z_{52},\xi) \\
&- s_{15}(\partial_{\beta_1}+\partial_{\beta_2}+\partial_{\beta_3}+\partial_{\beta_4}+\partial_\xi)\Omega(z_{12},\beta_1)\Omega(z_{23},\beta_2)\Omega(z_{34},\beta_3)\Omega(z_{45},\beta_4)\Omega(z_{51},\xi)\Big)\Big|_{\xi^0}\,,
\end{aligned}
$$

where $\beta_i = \sum_{j=i+1}^5 \eta_j$ for $i = 1,2,3,4$. A sequence of Fay relations (4.8) allows to cast the products of $\Omega(z_{ij},\ldots)$ due to (A.12) into a 24-dimensional basis. In view of the n-point generalization, we present the 24×24 differential operator $D^\tau_{\vec\eta}$

$$
2\pi i\partial_\tau Z^\tau_{\vec\eta}(*|1,2,3,4,5) = \sum_{B\in S_4} D^\tau_{\vec\eta}(2,3,4,5|b_2,b_3,b_4,b_5) Z^\tau_{\vec\eta}(*|1,b_2,b_3,b_4,b_5) \tag{4.25}
$$

from a slightly different angle and express the τ-derivative in terms of an overcomplete set of $n!$ integrals with leg 1 in an arbitrary position of the second entry:

$$
\begin{aligned}
2\pi i\partial_\tau Z^\tau_{\vec\eta}(*|1,2,3,4,5) &= \Big(\frac{1}{2}\sum_{j=2}^5 s_{1j}\partial^2_{\eta_j} + \frac{1}{2}\sum_{2\le i<j}^5 s_{ij}(\partial_{\eta_i}-\partial_{\eta_j})^2 - 2\zeta_2 s_{12345}\Big) Z^\tau_{\vec\eta}(*|1,2,3,4,5) \\
&- \wp(\eta_{2345},\tau)\Big(s_{12}Z^\tau_{\vec\eta}(*|1,2,3,4,5) - s_{13}\big(Z^\tau_{\vec\eta}(*|1,3,2,4,5)+Z^\tau_{\vec\eta}(*|1,3,4,2,5)+Z^\tau_{\vec\eta}(*|1,3,4,5,2)\big) \\
&\quad + s_{14}\big(Z^\tau_{\vec\eta}(*|1,4,5,3,2)+Z^\tau_{\vec\eta}(*|1,4,3,5,2)+Z^\tau_{\vec\eta}(*|1,4,3,2,5)\big) - s_{15}Z^\tau_{\vec\eta}(*|1,5,4,3,2)\Big) \\
&- \wp(\eta_{345},\tau)\Big(s_{23}Z^\tau_{\vec\eta}(*|1,2,3,4,5) - s_{24}\big(Z^\tau_{\vec\eta}(*|1,2,4,3,5)+Z^\tau_{\vec\eta}(*|1,2,4,5,3)\big) + s_{25}Z^\tau_{\vec\eta}(*|1,2,5,4,3) \\
&\quad - s_{13}Z^\tau_{\vec\eta}(*|2,1,3,4,5) + s_{14}\big(Z^\tau_{\vec\eta}(*|2,1,4,3,5)+Z^\tau_{\vec\eta}(*|2,1,4,5,3)\big) - s_{15}Z^\tau_{\vec\eta}(*|2,1,5,4,3)\Big) \\
&- \wp(\eta_{45},\tau)\Big(s_{34}Z^\tau_{\vec\eta}(*|1,2,3,4,5) - s_{24}\big(Z^\tau_{\vec\eta}(*|1,3,2,4,5)+Z^\tau_{\vec\eta}(*|3,1,2,4,5)\big) + s_{14}Z^\tau_{\vec\eta}(*|3,2,1,4,5) \\
&\quad - s_{35}Z^\tau_{\vec\eta}(*|1,2,3,5,4) + s_{25}\big(Z^\tau_{\vec\eta}(*|1,3,2,5,4)+Z^\tau_{\vec\eta}(*|3,1,2,5,4)\big) - s_{15}Z^\tau_{\vec\eta}(*|3,2,1,5,4)\Big) \\
&- \wp(\eta_5,\tau)\Big(s_{45}Z^\tau_{\vec\eta}(*|1,2,3,4,5) - s_{35}\big(Z^\tau_{\vec\eta}(*|1,2,4,3,5)+Z^\tau_{\vec\eta}(*|1,4,2,3,5)+Z^\tau_{\vec\eta}(*|4,1,2,3,5)\big) \\
&\quad + s_{25}\big(Z^\tau_{\vec\eta}(*|4,3,1,2,5)+Z^\tau_{\vec\eta}(*|4,1,3,2,5)+Z^\tau_{\vec\eta}(*|1,4,3,2,5)\big) - s_{15}Z^\tau_{\vec\eta}(*|4,3,2,1,5)\Big)\,.
\end{aligned} \tag{4.26}
$$

In an n-point context it will be useful to define

$$
\eta_1 \equiv -(\eta_2+\eta_3+\ldots+\eta_n)\,, \tag{4.27}
$$

since a repeated application of Fay identities can be used to show that $Z^\tau_{\vec\eta}(*|P\sqcup Q) = 0$ for $P,Q\neq\emptyset$. Therefore, the $n!$ integrands of $Z^\tau_{\vec\eta}(*|a_1,a_2,\ldots,a_n)$, $A\in S_n$, can be reduced to an $(n{-}1)!$-element basis of $Z^\tau_{\vec\eta}(*|1,c_2,\ldots,c_n)$, $C\in S_{n-1}$. More precisely [96],

$$
Z^\tau_{\vec\eta}(*|A,1,B) = (-1)^{|A|} Z^\tau_{\vec\eta}\big(*|1,(A^t\sqcup B)\big)\,, \tag{4.28}
$$

where $|A|$ is the number of letters in the word $A = a_1 a_2 \ldots a_{|A|}$, and $A^t = a_{|A|} \ldots a_2 a_1$ is obtained by reversing the order of the letters. The shuffle symbol yields a formal sum over all words that preserve the order of A^t and B, and any object labelled by words is understood to obey a linearity property such as $Z^\tau_{\vec{\eta}}(*|1, C{+}D) = Z^\tau_{\vec{\eta}}(*|1, C) + Z^\tau_{\vec{\eta}}(*|1, D)$.

Remarkably, (4.28) is the same shuffle symmetry obeyed by the tree-level functions $\frac{1}{z_{12}z_{23}\ldots z_{k-1,k}}$. This can be seen from the fact that Fay identities among $\Omega(z_{ij}, \ldots)\Omega(z_{jk}, \ldots)$ mirror the partial-fraction manipulations among $\frac{1}{z_{ij}z_{jk}}$ when disregarding the second argument. The latter in turn are fixed by requiring both sides of the Fay identity to have the same behavior under antiholomorphic derivatives $\partial_{\bar{z}}\Omega(z, \eta, \tau) = -\frac{\pi\eta}{\operatorname{Im}\tau}\Omega(z, \eta, \tau)$.

By virtue of the identity (4.28), the right-hand side of (4.26) can be uniquely expanded in the 24-element basis of $Z^\tau_{\vec{\eta}}\big(*|1, i, j, k, l\big)$. The result is noted in appendix C.2, and one can read off the first row of the 24×24 differential operator $D^\tau_{\vec{\eta}}$ in (4.25). Similarly, the expansion (1.6) in terms of Eisenstein series straightforwardly yields the 24×24 matrix representation of the derivations which enters the solution of (4.25) via Picard iteration,

$$
\begin{aligned}
Z^\tau_{\vec{\eta}}(*|1,2,3,4,5) = \sum_{r=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}} \prod_{j=1}^{r} (k_j{-}1)\, \gamma(k_1, k_2, \ldots, k_r|\tau) \\
\times \sum_{B\in S_4} r_{\vec{\eta}}(\epsilon_{k_r} \ldots \epsilon_{k_2}\epsilon_{k_1})_{2345}{}^{B} Z^{i\infty}_{\vec{\eta}}(*|1, B)\,.
\end{aligned}
\tag{4.29}
$$

The initial values $Z^{i\infty}_{\vec{\eta}}(*|1, B)$ are determined by the discussion in later sections.

4.4 n points

There is no obstacle to extending the rigorous computations of the previous sections to the τ-derivatives of higher-point A-cycle integrals (1.1). The mixed heat equation and the Koba–Nielsen derivatives lead to the intermediate step

$$
\begin{aligned}
2\pi i \partial_\tau Z^\tau_{\vec{\eta}}(*|1,2,\ldots,n) = \int_{\mathcal{C}(*)} \mathrm{d}z_2\, \mathrm{d}z_3\, \ldots\, \mathrm{d}z_n\, \mathrm{KN}^\tau_{12\ldots n}\Big(-2\zeta_2 s_{12\ldots n} \prod_{k=1}^{n-1} \Omega(z_{k,k+1}, \beta_k) \\
- \sum_{1\leq i<j}^{n} s_{ij}(\partial_{\beta_i}{+}\partial_{\beta_{i+1}}{+}\ldots{+}\partial_{\beta_{j-1}}{+}\partial_\xi)\Omega(z_{ji}, \xi) \prod_{k=1}^{n-1}\Omega(z_{k,k+1}, \beta_k)\Big)\Big|_{\xi^0}\,,
\end{aligned}
\tag{4.30}
$$

where $\beta_i = \sum_{j=i+1}^{n} \eta_j$ for $i = 1, 2, \ldots, n{-}1$. And the leftover work in extracting the form of the differential operator $D^\tau_{\vec{\eta}}$ in (1.4) is to resolve the cycles of $\Omega(z_{ij}, \ldots)$ through the lemma (A.12) and to reduce the result to the $(n{-}1)!$ basis of $Z^\tau_{\vec{\eta}}(*|1, B)$, $B \in S_{n-1}$ via Fay identities.

Instead of performing these tedious but straightforward calculations on a case-by-case basis at $n{\geq}6$ points, we take inspiration from certain patterns in the $(n{\leq}5)$-point results to propose a closed formula for $2\pi i\partial_\tau Z^\tau_{\vec{\eta}}(*|1, 2, \ldots, n)$. These patterns are based on a combinatorial operation dubbed the "S-map" [97, 98]. For any object $X(A)$ labelled by words

$A = a_1 a_2 \ldots a_{|A|}$, the S-map is defined by

$$X(S[A,B]) = \sum_{i=1}^{|A|} \sum_{j=1}^{|B|} (-1)^{i-j+|A|-1} s_{a_i b_j} \times X\big((a_1 a_2 \ldots a_{i-1} ⧢ a_{|A|} a_{|A|-1} \ldots a_{i+1}), a_i, b_j, (b_{j-1} \ldots b_2 b_1 ⧢ b_{j+1} \ldots b_{|B|})\big)\,, \tag{4.31}$$

for instance

$$X(S[1,2]) = s_{12} X(1,2)\,, \qquad X(S[12,3]) = s_{23} X(1,2,3) - s_{13} X(2,1,3)\,. \tag{4.32}$$

The S-map is antisymmetric $X(S[A,B]) = -X(S[B,A])$ if X obeys the shuffle identity $X(A,1,B) = (-1)^{|A|} X(1,(A^t ⧢ B))$ as in (4.28).

In the previous section, we have chosen to present the τ-derivative at five points in the form (4.26) because it can be compactly written in terms of the above S-map (4.31):

$$\begin{aligned} 2\pi i \partial_\tau Z^\tau_{\vec\eta}(*|1,2,3,4,5) &= \Big(\frac{1}{2} \sum_{j=2}^{5} s_{1j} \partial^2_{\eta_j} + \frac{1}{2} \sum_{2\leq i<j}^{5} s_{ij} (\partial_{\eta_i} - \partial_{\eta_j})^2 - 2\zeta_2 s_{12345} \Big) Z^\tau_{\vec\eta}(*|1,2,3,4,5) \\ &\quad - \wp(\eta_2+\eta_3+\eta_4+\eta_5,\tau) Z^\tau_{\vec\eta}(*|S[1,2345]) - \wp(\eta_3+\eta_4+\eta_5,\tau) Z^\tau_{\vec\eta}(*|S[12,345]) \\ &\quad - \wp(\eta_4+\eta_5,\tau) Z^\tau_{\vec\eta}(*|S[123,45]) - \wp(\eta_5,\tau) Z^\tau_{\vec\eta}(*|S[1234,5])\,. \end{aligned} \tag{4.33}$$

The structure of the right-hand side harmonizes with a rewriting of the ($n\leq 4$)-point results (3.9), (4.9) and (4.18) as

$$\begin{aligned} 2\pi i \partial_\tau Z^\tau_{\eta_2}(*|1,2) &= \Big(\frac{1}{2} s_{12} \partial^2_{\eta_2} - 2\zeta_2 s_{12} \Big) Z^\tau_{\eta_2}(*|1,2) - \wp(\eta_2,\tau) Z^\tau_{\eta_2}(*|S[1,2]) \\ 2\pi i \partial_\tau Z^\tau_{\vec\eta}(*|1,2,3) &= \Big(\frac{1}{2} \sum_{j=2}^{3} s_{1j} \partial^2_{\eta_j} + \frac{1}{2} s_{23} (\partial_{\eta_2} - \partial_{\eta_3})^2 - 2\zeta_2 s_{123} \Big) Z^\tau_{\vec\eta}(*|1,2,3) \\ &\quad - \wp(\eta_2+\eta_3,\tau) Z^\tau_{\vec\eta}(*|S[1,23]) - \wp(\eta_3,\tau) Z^\tau_{\vec\eta}(*|S[12,3]) \\ 2\pi i \partial_\tau Z^\tau_{\vec\eta}(*|1,2,3,4) &= \Big(\frac{1}{2} \sum_{j=2}^{4} s_{1j} \partial^2_{\eta_j} + \frac{1}{2} \sum_{2\leq i<j}^{4} s_{ij} (\partial_{\eta_i} - \partial_{\eta_j})^2 - 2\zeta_2 s_{1234} \Big) Z^\tau_{\vec\eta}(*|1,2,3,4) \\ &\quad - \wp(\eta_2+\eta_3+\eta_4,\tau) Z^\tau_{\vec\eta}(*|S[1,234]) - \wp(\eta_3+\eta_4,\tau) Z^\tau_{\vec\eta}(*|S[12,34]) \\ &\quad - \wp(\eta_4,\tau) Z^\tau_{\vec\eta}(*|S[123,4])\,, \end{aligned} \tag{4.34}$$

where the equivalence to earlier results can be checked by means of (4.28). By inspecting the pattern in (4.34) and (4.33), it is natural to propose the following generalization to n points

$$\begin{aligned} 2\pi i \partial_\tau Z^\tau_{\vec\eta}(*|1,2,\ldots,n) &= \Big(\frac{1}{2} \sum_{j=2}^{n} s_{1j} \partial^2_{\eta_j} + \frac{1}{2} \sum_{2\leq i<j}^{n} s_{ij} (\partial_{\eta_i} - \partial_{\eta_j})^2 - 2\zeta_2 s_{123\ldots n} \Big) Z^\tau_{\vec\eta}(*|1,2,\ldots,n) \\ &\quad - \sum_{j=2}^{n} \wp(\eta_j+\eta_{j+1}+\ldots+\eta_n,\tau) Z^\tau_{\vec\eta}(*|S[12\ldots j{-}1\,,\, j(j{+}1)\ldots n])\,, \end{aligned} \tag{4.35}$$

which is conjectural at $n\geq 6$. There is no bottleneck in checking (4.35) at fixed multiplicities $n=6,7,\ldots$ by starting from the intermediate step (4.30) and proceeding as outlined above. By carrying out the S-map (4.31) and applying Fay identities (4.28) to the result, (4.35) yields a well-defined proposal for the differential operator $D^\tau_{\vec{\eta}}$ at n points. Accordingly, its expansion (1.6) in Eisenstein series leads to concrete $(n-1)!\times(n-1)!$ representations of the derivations $r_{\vec{\eta}}(\epsilon_k)$. Like this, the appearance of iterated Eisenstein integrals in the α'-expansion of $Z^\tau_{\vec{\eta}}(*|1,A)$ can be made fully explicit from (1.7), and it remains to fix the initial values $Z^{i\infty}_{\vec{\eta}}(*|1,B)$ that introduce MZVs.

The $r_{\vec{\eta}}(\epsilon_k)$ resulting from (4.35) are linear in α' since each term in the first line and the expansion of the S-map $Z^\tau_{\vec{\eta}}(*|S[12\ldots j{-}1\,,\, j(j{+}1)\ldots n])$ features one factor of s_{ij}, see (4.31). This will be important in section 7.1, where the structure of the α'-expansion (1.7) is argued to imply uniform transcendentality of the $Z^\tau_{\vec{\eta}}(*|1,A)$.

Finally, the absence of the twisted Eisenstein series $f^{(k)}(\tau/2,\tau)$ on the right-hand side of (4.35) implies that the α'-expansion (1.7) of non-planar A-cycle integrals cannot involve any twisted eMZVs (which would naively arise from the expansion method in [21]). As a hallmark of twisted eMZVs, their τ-derivatives involve Eisenstein series of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ [21], i.e. $f^{(k)}(\tau/2,\tau)$ in case of the twists of the non-planar Green function. Still, the absence of twisted eMZVs by itself does not guarantee that the iterated Eisenstein integrals in non-planar α'-expansions (1.7) conspire to eMZVs, see the discussion in the next subsection.

4.5 Representations of the derivation algebra

We shall now clarify to which extent the above matrices $r_{\vec{\eta}}(\epsilon_k)$ are known to preserve the commutation relations of the derivations, i.e. to which extent they qualify to be called a "representation".

The reasoning will rely on the fact that the planar instances of the A-cycle integrals (1.1) can be α'-expanded in terms of eMZVs whose coefficients are $\mathbb{Q}$-linear combinations of MZVs. This can be seen by applying the methods of [20, 26] to each component integral over $f^{(m_1)}_{12}f^{(m_2)}_{23}\ldots f^{(m_{n-1})}_{n-1,n}$, $m_j\in\mathbb{N}_0$ that arises in the η_j-expansion: When integrating one puncture after the other, Fay identities can be used to have only one instance of the integration variable among the subscripts of the $f^{(m_k)}_{ij}$ in each step. Regardless on the accompanying monomials in planar Green functions (2.14), the integral over each puncture can therefore be performed within the space of elliptic iterated integrals (2.8)[16]. The integration over the last puncture sets the argument of the elliptic iterated integrals $\Gamma(\ldots;z)$ to $z=1$ and yields eMZVs by (2.9). At higher multiplicity or order in α', this method may be slowed down by a nesting of kinematic poles and the multitude of relations among eMZVs. Still, it clarifies that the iterated Eisenstein integrals in the α'-expansion (1.7) of planar n-point A-cycle integrals

[16] See [20] for details on the "z-removal" in intermediate steps and [26] for examples on the handling of kinematic poles.

must conspire to eMZVs, with admixtures of MZVs through "z-removal" identities[17].

We can now demonstrate that, when acting on the initial value $Z^{i\infty}_{\vec\eta}$ of planar A-cycle integrals, any failure of the $r_{\vec\eta}(\epsilon_k)$ to satisfy the commutation relations of the derivations would lead to a contradiction: As detailed in [25], each relation in the derivation algebra selects a linear combination of iterated Eisenstein integrals that cannot be realized as an eMZV, see section 2.5. If one of these relations was violated by the $r_{\vec\eta}(\epsilon_k)$ acting on $Z^{i\infty}_{\vec\eta}$, then (1.7) would introduce such a non-eMZV combination of iterated Eisenstein integrals (say an isolated $\gamma(6,8|\tau)$) into the α'-expansion of some component integral over $f^{(m_1)}_{12} f^{(m_2)}_{23} \ldots f^{(m_{n-1})}_{n-1,n}$, cf. section 4.3 of [25]. Since all the τ-dependence in the α'-expansion must stem from eMZVs by the previous paragraph, we conclude that setting $\epsilon_k \to r_{\vec\eta}(\epsilon_k)$ in vanishing commutators must annihilate the planar initial values $Z^{i\infty}_{\vec\eta}$. I.e. when restricted to act on the subspace of functions of η_j set by the initial values, the $r_{\vec\eta}(\epsilon_k)$ furnish a matrix representation of the derivation algebra.

One might wonder whether this annihilation is a peculiarity of the η_j-dependence of $Z^{i\infty}_{\vec\eta}$, say the factor of $\cot(\pi\eta_2)$ in the planar two-point case (3.30). By the commutativity $r_{\eta_2}([\epsilon_{k_1},\epsilon_{k_2}]) = 0 \ \forall \ k_1,k_2>0$ at two points, most of the nontrivial checks of $r_{\vec\eta}(\epsilon_k)$ preserving the derivation algebra have been performed at $n{\geq}3$ points, based on the matrix representations in (4.13) and (4.20). For instance, we have verified that arbitrary functions of η_j are annihilated when setting $\epsilon_k \to r_{\vec\eta}(\epsilon_k)$ in vanishing combinations of commutators of the form

- $[\epsilon_{k_1},\epsilon_{k_2}]$ at $k_1{+}k_2 \leq 30$ and $n=3,4$,
- $[\epsilon_{\ell_1},[\epsilon_{\ell_2},\epsilon_{\ell_3}]]$ at $\ell_1{+}\ell_2{+}\ell_3 \leq 30$ and $n=3,4$,
- $[\epsilon_{p_1},[\epsilon_{p_2},[\epsilon_{p_3},\epsilon_{p_4}]]]$ at $p_1{+}p_2{+}p_3{+}p_4 \leq 26,\ n=3$ and $p_1{+}p_2{+}p_3{+}p_4 \leq 18,\ n=4$,

where some of the four-point checks rely on numerical methods.

For non-planar A-cycle integrals, the integration methods of [21, 26] guarantee that their α'-expansions are expressible in terms of twisted eMZVs. The result (1.7) of our differential-equation method implies these twisted eMZVs to conspire to iterated Eisenstein integrals of $\mathrm{SL}_2(\mathbb{Z})$, and one could wonder if they are necessarily eMZVs on these grounds. However, at the time of writing, we cannot rule out the following loophole beyond the range of our explicit checks: Setting $\epsilon_k \to r_{\vec\eta}(\epsilon_k)$ in some vanishing combination of ϵ_k-commutators might fail to annihilate the initial value $Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}\big|\,\cdot\,\big)$ of a *non-planar* A-cycle integral with $P,Q\neq\emptyset$. This is related to the open question whether all iterated Eisenstein integrals expressible in terms of twisted eMZVs are necessarily eMZVs.

[17] The appearance of MZVs in "z-removal" identities is exemplified by [20]

$$\Gamma\left(\begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix}; z|\tau\right) = 2\Gamma\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}; z|\tau\right) + \Gamma\left(\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}; z|\tau\right) - 2\Gamma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}; z|\tau\right) + \zeta_2\,.$$

5 Planar genus-one integrals at the cusp

Starting from this section, the proposal (4.35) for the τ-derivatives of n-point A-cycle integrals will be supplemented by an initial value at the cusp. We will now determine the degenerations $Z^{i\infty}_{\vec{\eta}}(P|\cdot)$ in case of planar integration cycles $\mathcal{C}(P)$ and express the results in terms of $(n{+}2)$-point Parke–Taylor integrals (1.3) over disk boundaries:

$$Z^{i\infty}_{\vec{\eta}}(1,A|1,P) = \frac{1}{(2\pi i)^{n-1}} \sum_{B,Q\in S_{n-1}} \mathcal{H}_{\alpha'}(A|B)\mathcal{K}_{\vec{\eta}}(P|Q) Z^{\rm tree}(+,B,1,-|+,Q,-,1)\,. \tag{5.1}$$

The explicit form of the $(n{-}1)! \times (n{-}1)!$ matrices $\mathcal{H}_{\alpha'}(A|B)$ and $\mathcal{K}_{\vec{\eta}}(P|Q)$ will be given below. The normalization by powers of $2\pi i$ is due to the change of variables $\prod_{j=2}^n \mathrm{d}z_j = \frac{1}{(2\pi i)^{n-1}} \prod_{j=2}^n \frac{\mathrm{d}\sigma_j}{\sigma_j}$, cf. (3.15). The extra legs $+,-$ in the $(n{+}2)$-point disk integrals $Z^{\rm tree}$ are associated with the identified punctures $\sigma_+ = 0$ and $\sigma_- = \infty$ due to the pinching of the A-cycle at $\tau \to i\infty$, see figure 3. The Mandelstam invariants associated with the extra legs are determined by the degeneration of the planar Koba–Nielsen factor that follows from (3.19)

$$\mathrm{KN}^{i\infty}_{12\ldots n}\big|_{\mathcal{C}(1,a_2,a_3,\ldots,a_n)} = \prod_{j=2}^n \sigma_j^{\frac{1}{2}\sum_{i\neq j}^n s_{ij}} \prod_{j=2}^n (1{-}\sigma_j)^{-s_{1j}} \prod_{2\leq i<j}^n (\sigma_{a_j}{-}\sigma_{a_i})^{-s_{a_i a_j}}\,, \tag{5.2}$$

i.e. one can read off[18]

$$s_{j+} = s_{j-} = -\frac{1}{2}\sum_{1\leq i\neq j}^n s_{ij}\,, \qquad s_{+,-} = \sum_{1\leq i<j}^n s_{ij} \tag{5.3}$$

by identifying (5.2) as an SL$_2$-fixed version of $\sigma_{+,-}^{-s_{+,-}} \prod_{j=1}^n \sigma_{j+}^{-s_{j+}}\sigma_{j-}^{-s_{j-}} \prod_{1\leq i<j}^n \sigma_{a_j a_i}^{-s_{a_i a_j}}$.

The first matrix $\mathcal{H}_{\alpha'}(A|B)$ in (5.1) translates the twisted cycles defined by $\mathcal{C}(1,A)$ and (5.2) into disk orderings in an SL$_2$-frame[19] with $(\sigma_+,\sigma_1,\sigma_-) = (0,1,\infty)$,

$$\mathcal{D}_B \equiv \mathcal{D}(+,b_2,b_3,\ldots,b_n,1,-) = \{\sigma_{b_2},\sigma_{b_3},\ldots,\sigma_{b_n}\in\mathbb{R},\ 0{<}\sigma_{b_2}{<}\sigma_{b_3}{<}\ldots{<}\sigma_{b_k}{<}1\}\,. \tag{5.4}$$

The relation (3.28) among the twisted cycles at two points amounts to $\mathcal{H}_{\alpha'}(2|2) = 2i\sin(\frac{\pi}{2}s_{12})$, and the entries of $\mathcal{H}_{\alpha'}(\cdot|\cdot)$ at higher multiplicity will be determined in sections 5.1 and 5.2 in terms of s_{ij}-dependent phases.

The second matrix $\mathcal{K}_{\vec{\eta}}(P|Q)$ in (5.1) tracks the transformation between the basis integrands at genus zero & genus one and carries the dependence on η_j through the $\pi\cot(\pi\eta)$

[18] We leave it as an open problem whether the identification of Mandelstam variables $\{s_{ij},\ 1\leq i<j\leq n\}$ with dot products of momenta can be extended to the formal variables (5.3) via suitable choices of $k_\pm$. We will use $s_{j\pm}$ and $s_{+,-}$ as auxiliary variables that are uniquely determined by SL$_2$-invariance and whose virtue for the subsequent calculations does not rely on a physical interpretation.

[19] Note that (5.4) takes the following form in a generic SL$_2$-frame:

$$\mathcal{D}(a_1,a_2,\ldots,a_k) = \{\sigma_1,\sigma_2,\ldots,\sigma_k\in\mathbb{R},\ -\infty<\sigma_{a_1}<\sigma_{a_2}<\ldots<\sigma_{a_k}<\infty\}\,.$$

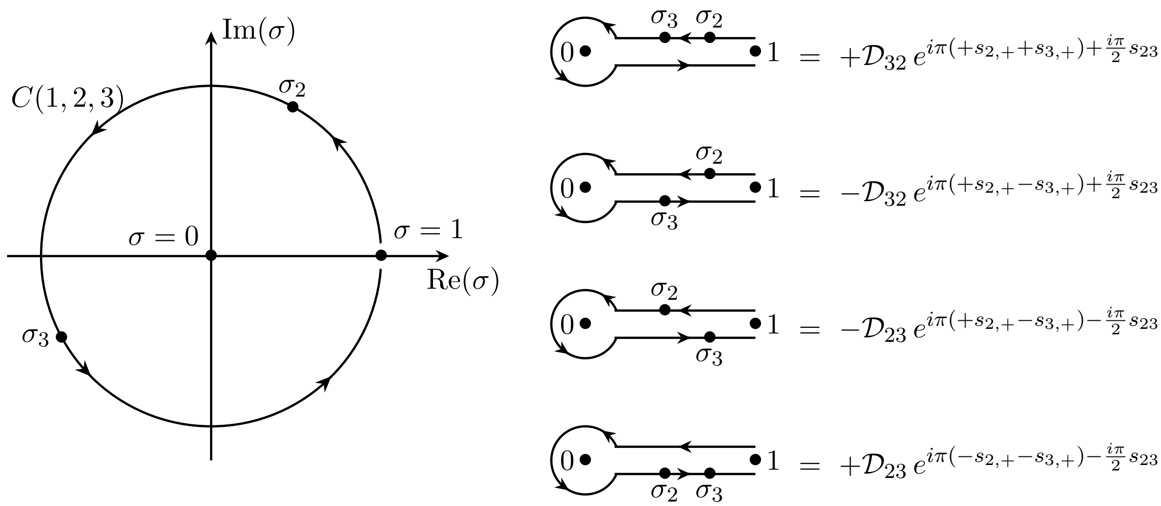


Figure 5: *In the $\sigma_j = e^{2\pi i z_j}$ variables, the integration contour $\mathcal{C}(1,2,3)$ is the unit circle in the left panel, where the phases of σ_2 and σ_3 are ordered according to $z_2 < z_3$. Similar to figure 4, we replace $\mathcal{C}(1,2,3)$ by the homotopy-equivalent combination of paths visualized in the right panel. For each of the four inequivalent relative positions of σ_2 and σ_3, the Koba–Nielsen factor (5.2) introduces a different phase that depends on s_{2+}, s_{3+} and s_{23}.*

function in the degeneration (3.16) of the Kronecker–Eisenstein series. At two points, the fact that $\frac{1}{\sigma_2}$ descends from the Parke–Taylor factor $(\sigma_{+2}\sigma_{2-}\sigma_{-1}\sigma_{1+})^{-1}$ in an SL_2-frame with $(\sigma_+, \sigma_1, \sigma_-) = (0, 1, \infty)$ yields $\mathcal{K}_{\eta_2}(2|2) = \pi \cot(\pi\eta_2)$. The extension to higher multiplicity will be the topic of section 5.3.

5.1 Recovering twisted cycles on the disk boundary up to five points

The strategy of section 3.4 to deform the integration contour from the unit circle $|\sigma_j| = 1$ to the interval $0 < \sigma_j < 1$ carries over to higher multiplicity. The only additional feature at $n \geq 3$ points concerns the relative ordering of $\sigma_2, \sigma_3, \ldots, \sigma_n$ on the unit circle with relative phases according to $0<z_2<z_3<\ldots<z_n<1$ in case of $\mathcal{C}(1,2,\ldots,n)$.

This key ideas become clear from the twisted three-point cycle defined by $\mathcal{C}(1,2,3)$ and $\mathrm{KN}^{i\infty}_{123} = \sigma_2^{-s_{2+}}\sigma_3^{-s_{3+}}(\sigma_3{-}\sigma_2)^{-s_{23}}$ with $s_{2+} = -\frac{1}{2}(s_{12}+s_{23})$ and $s_{3+} = -\frac{1}{2}(s_{13}+s_{23})$, see (5.2) and (5.3). When deforming $\mathcal{C}(1,2,3)$ to the homotopy-equivalent contour in the right panel of figure 5, there are four different scenarios for the phases introduced by $\mathrm{KN}^{i\infty}_{123}$ depending on the relative positions of σ_2 and σ_3. The rules for determining these phases can be easily written down in an n-point context:

- As in the two-point case, $\sigma_j^{-s_{j+}}$ with $j = 2, 3, \ldots, n$ introduce phases $e^{+i\pi s_{j+}}$ and $e^{-i\pi s_{j+}}$ when σ_j is slightly above and below the real axis, respectively. This realizes the desired phase difference $e^{-2\pi i s_{j+}}$ when transporting σ_+ on a circle around the origin.

- Starting from three points, factors of $(\sigma_j-\sigma_i)^{-s_{ij}}$ with $2 \leq i < j \leq n$ introduce phases $e^{+\frac{i\pi}{2}s_{ij}}$ and $e^{-\frac{i\pi}{2}s_{ij}}$ when $\mathrm{Re}(\sigma_j) < \mathrm{Re}(\sigma_i)$ and $\mathrm{Re}(\sigma_i) < \mathrm{Re}(\sigma_j)$, respectively. This realizes the desired phase difference $e^{i\pi s_{ij}}$ when transporting σ_j on a semicircle around σ_i.

By adding up the four contributions in figure 5 and adjoining a factor of (-1) for the negative path orientation of each puncture above the real line, we find the following relation among twisted cycles,

$$\begin{aligned}\mathcal{C}(1,2,3) &= 2i\sin(\pi s_{3+})e^{i\pi(s_{2+}+\frac{1}{2}s_{23})}\mathcal{D}_{32} - 2i\sin(\pi s_{2+})e^{i\pi(-s_{3+}-\frac{1}{2}s_{23})}\mathcal{D}_{23} \\ &= 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)e^{\frac{i\pi}{2}s_{13}}\mathcal{D}_{23} - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23})\Big)e^{-\frac{i\pi}{2}s_{12}}\mathcal{D}_{32}\,,\end{aligned} \tag{5.5}$$

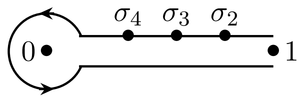
where $\mathcal{D}_{23}$ refers to $0<\sigma_2<\sigma_3<1$ according to the general notation (5.4). For the sake of brevity, we employ the abusive notation in (5.5) and later equations to keep the Koba–Nielsen factor (5.2) defining the twisted cycles implicit.

The rules for assigning phases can be straightforwardly applied to $\mathcal{C}(1,2,3,4)$, where the iteration of the usual path deformation yields eight different scenarios for the phases, depending on the relative positions of $\sigma_2,\sigma_3,\sigma_4$ in figure 6,

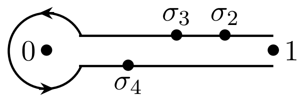
$$\begin{aligned}\mathcal{C}(1,2,3,4) &= -2i\sin(\pi s_{2+})\,e^{i\pi(-s_{3+}-s_{4+})}\,e^{\frac{i\pi}{2}(-s_{23}-s_{24}-s_{34})}\mathcal{D}_{234} \\ &\quad + 2i\sin(\pi s_{3+})\,e^{i\pi(+s_{2+}-s_{4+})}\,e^{\frac{i\pi}{2}(+s_{23}-s_{24}-s_{34})}\mathcal{D}_{324} \\ &\quad + 2i\sin(\pi s_{3+})\,e^{i\pi(+s_{2+}-s_{4+})}\,e^{\frac{i\pi}{2}(+s_{23}+s_{24}-s_{34})}\mathcal{D}_{342} \\ &\quad - 2i\sin(\pi s_{4+})\,e^{i\pi(+s_{2+}+s_{3+})}\,e^{\frac{i\pi}{2}(+s_{23}+s_{24}+s_{34})}\mathcal{D}_{432} \\ &= 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24})\Big)\,e^{\frac{i\pi}{2}s_{134}}\mathcal{D}_{234} \\ &\quad - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\,e^{\frac{i\pi}{2}(-s_{12}+s_{14}-s_{24})}\mathcal{D}_{324} \\ &\quad - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\,e^{\frac{i\pi}{2}(-s_{12}+s_{14}+s_{24})}\mathcal{D}_{342} \\ &\quad + 2i\sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34})\Big)\,e^{-\frac{i\pi}{2}s_{123}}\mathcal{D}_{432}\,.\end{aligned} \tag{5.6}$$

In the five-point generalization of the above bookkeeping, one has to take sixteen different scenarios into account for the phases (depending on the relative positions of $\sigma_2,\sigma_3,\sigma_4,\sigma_5$), and we simply quote the end result of adding these contributions:

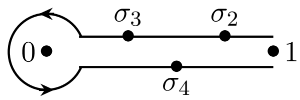
$$\begin{aligned}\mathcal{C}(1,2,3,4,5) &= 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24}+s_{25})\Big)\,e^{+\frac{i\pi}{2}s_{1345}}\mathcal{D}_{2345} \\ &\quad - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34}+s_{35})\Big)\,e^{\frac{i\pi}{2}(-s_{12}+s_{14}+s_{15}-s_{24}-s_{25}+s_{45})}\mathcal{D}_{3245} \\ &\quad - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34}+s_{35})\Big)\,e^{\frac{i\pi}{2}(-s_{12}+s_{14}+s_{15}+s_{24}-s_{25}+s_{45})}\mathcal{D}_{3425} \\ &\quad - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34}+s_{35})\Big)\,e^{\frac{i\pi}{2}(-s_{12}+s_{14}+s_{15}+s_{24}+s_{25}+s_{45})}\mathcal{D}_{3452} \\ &\quad + 2i\sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34}+s_{45})\Big)\,e^{\frac{i\pi}{2}(-s_{12}-s_{13}+s_{15}-s_{23}-s_{25}-s_{35})}\mathcal{D}_{4325}\end{aligned}$$



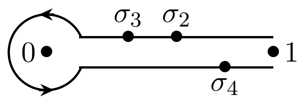
$$= -\mathcal{D}_{432}\, e^{i\pi(+s_{2+}+s_{3+}+s_{4+})}\, e^{\frac{i\pi}{2}(+s_{23}+s_{24}+s_{34})}$$



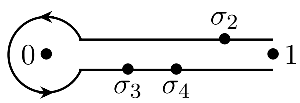
$$= +\mathcal{D}_{432}\, e^{i\pi(+s_{2+}+s_{3+}-s_{4+})}\, e^{\frac{i\pi}{2}(+s_{23}+s_{24}+s_{34})}$$



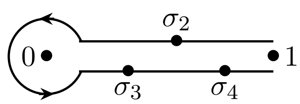
$$= +\mathcal{D}_{342}\, e^{i\pi(+s_{2+}+s_{3+}-s_{4+})}\, e^{\frac{i\pi}{2}(+s_{23}+s_{24}-s_{34})}$$



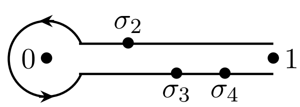
$$= +\mathcal{D}_{324}\, e^{i\pi(+s_{2+}+s_{3+}-s_{4+})}\, e^{\frac{i\pi}{2}(+s_{23}-s_{24}-s_{34})}$$



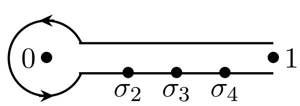
$$= -\mathcal{D}_{342}\, e^{i\pi(+s_{2+}-s_{3+}-s_{4+})}\, e^{\frac{i\pi}{2}(+s_{23}+s_{24}-s_{34})}$$



$$= -\mathcal{D}_{324}\, e^{i\pi(+s_{2+}-s_{3+}-s_{4+})}\, e^{\frac{i\pi}{2}(+s_{23}-s_{24}-s_{34})}$$



$$= -\mathcal{D}_{234}\, e^{i\pi(+s_{2+}-s_{3+}-s_{4+})}\, e^{\frac{i\pi}{2}(-s_{23}-s_{24}-s_{34})}$$



$$= +\mathcal{D}_{234}\, e^{i\pi(-s_{2+}-s_{3+}-s_{4+})}\, e^{\frac{i\pi}{2}(-s_{23}-s_{24}-s_{34})}$$

Figure 6: *Genus-zero contributions $\mathcal{D}_{ijk}$ to the degenerate genus-one integral over the four-point cycle $\mathcal{C}_{1234} = \{0 < z_2 < z_3 < z_4 < 1\}$.*

$$\begin{aligned}
&+ 2i \sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34}+s_{45})\Big)\, e^{\frac{i\pi}{2}(-s_{12}-s_{13}+s_{15}-s_{23}+s_{25}-s_{35})}\mathcal{D}_{4352} \\
&+ 2i \sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34}+s_{45})\Big)\, e^{\frac{i\pi}{2}(-s_{12}-s_{13}+s_{15}-s_{23}+s_{25}+s_{35})}\mathcal{D}_{4532} \\
&- 2i \sin\Big(\frac{\pi}{2}(s_{15}+s_{25}+s_{35}+s_{45})\Big)\, e^{-\frac{i\pi}{2}s_{1234}}\mathcal{D}_{5432}\,.
\end{aligned} \tag{5.7}$$

5.2 Recovering twisted cycles on the disk boundary at n points

In order to extend the above relations between twisted genus-one and genus-zero integration cycles to higher multiplicity, we start by introducing some notation: It is convenient to absorb the exponentials of (5.5), (5.6) and (5.7) into

$$\begin{aligned}
\widehat{\mathcal{D}}\Big(2,\, {}^{\varepsilon_3}_{3}\Big) &\equiv e^{\frac{i\pi}{2}\varepsilon_3 s_{13}}\mathcal{D}_{23} \\
\widehat{\mathcal{D}}\Big(2,\, {}^{\varepsilon_3}_{3}\, {}^{\varepsilon_4}_{4}\Big) &\equiv e^{\frac{i\pi}{2}(\varepsilon_3(s_{13}+s_{34})+\varepsilon_4 s_{14})}\mathcal{D}_{234} \\
\widehat{\mathcal{D}}\Big(2,\, {}^{\varepsilon_3}_{3}\, {}^{\varepsilon_4}_{4}\, {}^{\varepsilon_5}_{5}\Big) &\equiv e^{\frac{i\pi}{2}(\varepsilon_3(s_{13}+s_{34}+s_{35})+\varepsilon_4(s_{14}+s_{45})+\varepsilon_5 s_{15})}\mathcal{D}_{2345}\,,
\end{aligned} \tag{5.8}$$

where $\varepsilon_j \in \{1,-1\}$ or in short $\varepsilon_j = \pm$. These definitions allow to compactly rewrite

$$\begin{aligned}
\mathcal{C}(1,2,3) &= 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)\widehat{\mathcal{D}}\Big(2,\, {}^{+}_{3}\Big) - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23})\Big)\widehat{\mathcal{D}}\Big(3,\, {}^{-}_{2}\Big) \\
\mathcal{C}(1,2,3,4) &= 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24})\Big)\,\widehat{\mathcal{D}}\Big(2,\, {}^{+}_{3}\, {}^{+}_{4}\Big) \\
&\quad - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\,\widehat{\mathcal{D}}\Big(3,\, \Big({}^{-}_{2} \shuffle {}^{+}_{4}\Big)\Big) \\
&\quad + 2i\sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34})\Big)\,\widehat{\mathcal{D}}\Big(4,\, {}^{-}_{3}\, {}^{-}_{2}\Big) \\
\mathcal{C}(1,2,3,4,5) &= 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24}+s_{25})\Big)\,\widehat{\mathcal{D}}\Big(2,\, {}^{+}_{3}\, {}^{+}_{4}\, {}^{+}_{5}\Big) \\
&\quad - 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34}+s_{35})\Big)\,\widehat{\mathcal{D}}\Big(3,\, \Big({}^{-}_{2} \shuffle {}^{+}_{4}\, {}^{+}_{5}\Big)\Big) \\
&\quad + 2i\sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34}+s_{45})\Big)\,\widehat{\mathcal{D}}\Big(4,\, \Big({}^{-}_{3}\, {}^{-}_{2} \shuffle {}^{+}_{5}\Big)\Big) \\
&\quad - 2i\sin\Big(\frac{\pi}{2}(s_{15}+s_{25}+s_{35}+s_{45})\Big)\,\widehat{\mathcal{D}}\Big(5,\, {}^{-}_{4}\, {}^{-}_{3}\, {}^{-}_{2}\Big)\,.
\end{aligned} \tag{5.9}$$

The shuffle symbol is understood to act on the combined letters ${}^{\varepsilon_j}_{j}$, e.g.

$$\widehat{\mathcal{D}}\Big(3,\, \Big({}^{-}_{2} \shuffle {}^{+}_{4}\Big)\Big) = \widehat{\mathcal{D}}\Big(3,\, {}^{-}_{2}\, {}^{+}_{4}\Big) + \widehat{\mathcal{D}}\Big(3,\, {}^{+}_{4}\, {}^{-}_{2}\Big)\,, \tag{5.10}$$

and one can formally align the two-point case into the same pattern with $\widehat{\mathcal{D}}(2) = \mathcal{D}_2$ and $\mathcal{C}(1,2) = 2i\sin\big(\frac{\pi}{2}s_{12}\big)\widehat{\mathcal{D}}(2)$. Based on the obvious n-point generalization of (5.8),

$$\begin{aligned}
\widehat{\mathcal{D}}\Big(2,\, {}^{\varepsilon_3}_{3}\, {}^{\varepsilon_4}_{4}\, {}^{\dots}_{\dots}\, {}^{\varepsilon_n}_{n}\Big) &\equiv e^{\frac{i\pi}{2}(\varepsilon_3(s_{13}+s_{34}+\ldots+s_{3n})+\varepsilon_4(s_{14}+s_{45}+\ldots+s_{4n})+\ldots+\varepsilon_{n-1}(s_{1,n-1}+s_{n-1,n})+\varepsilon_n s_{1,n})}\mathcal{D}_{23\ldots n} \\
&= \prod_{j=3}^{n} e^{\frac{i\pi}{2}\varepsilon_j\left(s_{1j}+\sum_{m=j+1}^{n} s_{jm}\right)}\mathcal{D}(+,2,3,\ldots,n,1,-)\,,
\end{aligned} \tag{5.11}$$

the patterns in (5.9) can be extended to the following all-multiplicity proposal (which is conjectural at $n \geq 6$ points):

$$\mathcal{C}(1,2,3,\ldots,n) = 2i\sum_{j=2}^{n}(-1)^j\sin\Big(\frac{\pi}{2}\sum_{\substack{i=1\\ i\neq j}}^{n} s_{ij}\Big)\widehat{\mathcal{D}}\Big(j,\, \Big({}^{-}_{j-1}\, {}^{-}_{j-2}\, {}^{\dots}_{\dots}\, {}^{-}_{3}\, {}^{-}_{2} \shuffle {}^{+}_{j+1}\, {}^{+}_{j+2}\, {}^{\dots}_{\dots}\, {}^{+}_{n}\Big)\Big)\,. \tag{5.12}$$

These relations among twisted cycles determine the first row of the $(n-1)! \times (n-1)!$ matrices $\mathcal{H}_{\alpha'}(A|B)$ in (5.1) at all multiplicities by matching (5.11) and (5.12) with

$$\mathcal{C}(1,2,3,\ldots,n) = \sum_{B\in S_{n-1}} \mathcal{H}_{\alpha'}(2,3,\ldots,n|b_2,b_3,\ldots,b_n)\mathcal{D}(+,b_2,b_3,\ldots,b_n,1,-)\,. \tag{5.13}$$

The remaining rows of $\mathcal{H}_{\alpha'}(A|B)$ can be obtained by relabelings as for instance seen in the three-point case, cf. (5.5)

$$\begin{aligned}
&\mathcal{H}_{\alpha'}(2,3|2,3) = 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)e^{\frac{i\pi}{2}s_{13}}\,, \quad \mathcal{H}_{\alpha'}(2,3|3,2) = -2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23})\Big)e^{-\frac{i\pi}{2}s_{12}} \\
&\mathcal{H}_{\alpha'}(3,2|3,2) = 2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23})\Big)e^{\frac{i\pi}{2}s_{12}}\,, \quad \mathcal{H}_{\alpha'}(3,2|2,3) = -2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)e^{-\frac{i\pi}{2}s_{13}}\,.
\end{aligned} \tag{5.14}$$

Four-point examples of the entries $\mathcal{H}_{\alpha'}(A|B)$ can be found in appendix D.1.

Symmetrized cycles: The definition of planar A-cycle graph functions at $n \geq 3$ points [81] is based on symmetrized combinations of the integration cycles $\mathcal{C}(1,a_2,a_3,\ldots,a_n)$,

$$\mathcal{C}^{(n)}_{\rm symm} \equiv \sum_{A\in S_{n-1}} \mathcal{C}(1,a_2,a_3,\ldots,a_n) = \{0<z_j<1,\ j=2,3,\ldots,n\}\,. \tag{5.15}$$

Instead of adding up permutations of $Z^\tau_{\vec\eta}(1,a_2,\ldots,a_n|*)$ and separately applying (5.12) to each twisted cycle, it is rewarding to simplify (5.15) via standard trigonometric identities[20]. For instance, by adding up 2, 6 and 24 permutations of (5.5), (5.6) and (5.7), respectively, one arrives at

$$\begin{aligned}
\mathcal{C}^{(3)}_{\rm symm} &= -4\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{13}\Big)\mathcal{D}(+,2,3,1,-) + (2\leftrightarrow 3) \\
\mathcal{C}^{(4)}_{\rm symm} &= -8i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24})\Big)\sin\Big(\frac{\pi}{2}(s_{13}+s_{34})\Big) \\
&\qquad \times\sin\Big(\frac{\pi}{2}s_{14}\Big)\,\mathcal{D}(+,2,3,4,1,-) + \mathrm{perm}(2,3,4) \\
\mathcal{C}^{(5)}_{\rm symm} &= 16\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24}+s_{25})\Big)\sin\Big(\frac{\pi}{2}(s_{13}+s_{34}+s_{35})\Big) \\
&\qquad \times\sin\Big(\frac{\pi}{2}(s_{14}+s_{45})\Big)\sin\Big(\frac{\pi}{2}s_{15}\Big)\,\mathcal{D}(+,2,3,4,5,1,-) + \mathrm{perm}(2,3,4,5)\,,
\end{aligned} \tag{5.16}$$

where the permutations of $2,3,\ldots,n$ affect the labels of both s_{ij} and $\mathcal{D}(+,a_2,a_3,\ldots,a_n,1,-)$. The representations of $\mathcal{C}^{(n)}_{\rm symm}$ in (5.16) manifest that $n-1$ powers of πs_{ij} can be factored out from the α'-expansion of each $\sin(\pi x) = \pi x(1-\zeta_2 x^2+\frac{3}{4}\zeta_4 x^4+\ldots)$. Extrapolating the patterns in (5.16) leads us to propose the general formula

$$\mathcal{C}^{(n)}_{\rm symm} = (2i)^{n-1}\prod_{j=2}^{n}\sin\Big(\frac{\pi}{2}\Big(s_{1j}+\sum_{m=j+1}^{n}s_{j,m}\Big)\Big)\,\mathcal{D}(+,2,3,\ldots,n,1,-)+\mathrm{perm}(2,3,\ldots,n) \tag{5.17}$$

[20] More specifically, one can straightforwardly derive (5.16) by (possibly repeated) use of

$$e^{\pm ix} = \cos(x)\pm i\sin(x)\,, \quad \cos(a\pm b) = \cos(a)\cos(b)\mp\sin(a)\sin(b)\,, \quad \sin(a\pm b) = \sin(a)\cos(b)\pm\sin(b)\cos(a)\,.$$

which reproduces $\mathcal{C}^{(2)}_{\rm symm} = \mathcal{C}(1,2) = 2i\sin(\frac{\pi}{2}s_{12})\mathcal{D}(+,2,1,-)$ and is conjectural at $n \geq 6$. Note that the product of sine functions in (5.17) matches the diagonal entry $S_{\alpha'/2}(n,\ldots,3,2|\, n,\ldots,3,2)_1$ of the string-theory KLT kernel at $\alpha' \to \alpha'/2$ which also plays a role in the simplification of symmetrized disk integrals $\sum_{A\in S_{n-1}} Z^{\rm tree}(1,A|*)$ [99].

5.3 Recovering Parke–Taylor disk integrands

In order to complete the dictionary between the initial values $Z^{i\infty}_{\vec{\eta}}$ of the A-cycle integrals and disk integrals $Z^{\rm tree}$, it remains to recover SL_2-fixed Parke–Taylor factors from the degeneration of the Kronecker–Eisenstein integrands. Based on the expression (3.16) for the planar $\Omega(v,\eta,i\infty)$ at $v\in\mathbb{R}$, we will determine the η_j-dependent entries of the $(n{-}1)!\times(n{-}1)!$ matrix $\mathcal{K}_{\vec{\eta}}(P|Q)$ in (5.1).

As a first step, we generalize the definition (3.25) to planar n-point cycles $A\in S_n$

$$I^{\rm tree}(A|\mathcal{F}(\sigma_j)) \equiv \int_{\mathcal{C}(A)} \frac{\mathrm{d}\sigma_2\,\mathrm{d}\sigma_3\,\ldots\,\mathrm{d}\sigma_n}{(2\pi i)^{n-1}\,\sigma_2\,\sigma_3\,\ldots\,\sigma_n}\,\mathrm{KN}^{i\infty}_{12\ldots n}\,\mathcal{F}(\sigma_j)\,, \tag{5.18}$$

where $\mathcal{F}(\sigma_j)$ may denote an arbitrary rational function of $\sigma_2,\ldots,\sigma_n$, and the degenerate Koba–Nielsen factor $\mathrm{KN}^{i\infty}_{12\ldots n}$ can be found in (5.2). In any instance of (5.18) that arises from the degeneration $Z^{i\infty}_{\vec{\eta}}$ of an n-point A-cycle integral (1.1), the rational function $\mathcal{F}(\sigma_j)$ is a product of up to $n{-}1$ factors of $G_{ij} = i\pi\frac{\sigma_i+\sigma_j}{\sigma_i-\sigma_j}$, the Green function on the nodal sphere. The three- and four-point generalizations of the two-point result (3.26) are given by

$$\begin{aligned} Z^{i\infty}_{\vec{\eta}}(1,a_2,a_3|1,2,3) &= \pi^2\cot(\pi\eta_{23})\cot(\pi\eta_3)I^{\rm tree}(1,a_2,a_3|1) + I^{\rm tree}(1,a_2,a_3|G_{12}G_{23}) \\ &\quad + \pi\cot(\pi\eta_{23})I^{\rm tree}(1,a_2,a_3|G_{23}) + \pi\cot(\pi\eta_3)I^{\rm tree}(1,a_2,a_3|G_{12}) \end{aligned} \tag{5.19}$$

as well as

$$\begin{aligned} &Z^{i\infty}_{\vec{\eta}}(1,a_2,a_3,a_4|1,2,3,4) = \pi^3\cot(\pi\eta_{234})\cot(\pi\eta_{34})\cot(\pi\eta_4)I^{\rm tree}(1,a_2,a_3,a_4|1) \\ &\quad + \pi\cot(\pi\eta_{234})I^{\rm tree}(1,a_2,a_3,a_4|G_{23}G_{34}) + \pi\cot(\pi\eta_{34})I^{\rm tree}(1,a_2,a_3,a_4|G_{12}G_{34}) \\ &\quad + \pi\cot(\pi\eta_4)I^{\rm tree}(1,a_2,a_3,a_4|G_{12}G_{23}) + \pi^2\cot(\pi\eta_{234})\cot(\pi\eta_{34})I^{\rm tree}(1,a_2,a_3,a_4|G_{34}) \\ &\quad + \pi^2\cot(\pi\eta_{234})\cot(\pi\eta_4)I^{\rm tree}(1,a_2,a_3,a_4|G_{23}) + \pi^2\cot(\pi\eta_{34})\cot(\pi\eta_4)I^{\rm tree}(1,a_2,a_3,a_4|G_{12}) \\ &\quad + I^{\rm tree}(1,a_2,a_3,a_4|G_{12}G_{23}G_{34})\,, \end{aligned} \tag{5.20}$$

where the expansion of the cotangent can be found in (3.18), and we use the shorthand $\eta_{ij\ldots l} = \eta_i{+}\eta_j{+}\ldots{+}\eta_l$ throughout this section.

The degeneration of the n-point integrand $\prod_{j=2}^n \Omega(z_{j-1,j},\eta_{j,j+1\ldots n},\tau)$ of $Z^{i\infty}_{\vec{\eta}}(*|1,2,\ldots,n)$ can be described by summing over all the 2^{n-1} possibilities to distribute the $n{-}1$ factors into two distinguishable and disjoint sets P and Q. For each label j in set P and Q, the factor of $\Omega(z_{j-1,j},\eta_{j,j+1\ldots n},i\infty)$ is mapped to its contribution $\pi\cot(\pi\eta_{j,j+1\ldots n})$ and $G_{j-1,j}$, respectively:

$$Z^{i\infty}_{\vec{\eta}}(1,A|1,2,3,\ldots,n) = \sum_{\substack{\{2,3,\ldots,n\}=P\cup Q\\ P\cap Q=\emptyset}} \Big(\prod_{i\in P}\pi\cot(\pi\eta_{i,i+1\ldots n-1,n})\Big) I^{\rm tree}\big(1,A|\prod_{j\in Q} G_{j-1,j}\big)\,. \tag{5.21}$$

Since the integrands in (5.18) only depend on the punctures via $(\sigma_2\sigma_3\ldots\sigma_n)^{-1}$ and products of G_{ij}, one can apply the algorithmic method of [77] to recover combinations of SL_2-fixed Parke–Taylor factors in $n{+}2$ punctures,

$$\mathrm{PT}(a_1,a_2,a_3,\ldots,a_{n+1},a_{n+2}) \equiv \frac{1}{\sigma_{a_1a_2}\sigma_{a_2a_3}\sigma_{a_3a_4}\ldots\sigma_{a_{n+1}a_{n+2}}\sigma_{a_{n+2}a_1}}\,. \tag{5.22}$$

Genus-one integrands $I^{\mathrm{tree}}(*|1)$ without any insertion of G_{ij} descend from[21]

$$\begin{aligned}\prod_{j=2}^{n}\frac{1}{\sigma_j} &= (-1)^n \lim_{\substack{\sigma_-\to\infty\\ \sigma_+\to 0}} |\sigma_-|^2 \sum_{A\in S_{n-1}} \mathrm{PT}(+,a_2,a_3,\ldots,a_n,-,1)\\ &= (-1)^{n-1} \lim_{\substack{\sigma_-\to\infty\\ \sigma_+\to 0}} |\sigma_-|^2 \sum_{B\in S_n} \mathrm{PT}(+,b_1,b_2,\ldots,b_n,-)\,,\end{aligned} \tag{5.23}$$

where Kleiss–Kuijf relations [100] have been used to trade $(n{-}1)!$ permutations A of $2,3,\ldots,n$ for $n!$ permutations B of $1,2,\ldots,n$. The insertion of $|\sigma_-|^2$ stems from the Jacobian $|\sigma_{1,+}\sigma_{1,-}\sigma_{+,-}|$ of the $\mathrm{SL}_2(\mathbb{R})$ frame $(\sigma_+,\sigma_1,\sigma_-)=(0,1,\infty)$.

The structure of the $n!$-term Parke–Taylor expansion in (5.23) is preserved when adjoining factors of G_{ij}: As one can anticipate from the example

$$\begin{aligned}\frac{G_{12}}{\sigma_2\sigma_3} = i\pi \lim_{\substack{\sigma_-\to\infty\\ \sigma_+\to 0}} |\sigma_-|^2\Big(&-\mathrm{PT}(+,1,2,3,-)-\mathrm{PT}(+,1,3,2,-)-\mathrm{PT}(+,3,1,2,-)\\ &+\mathrm{PT}(+,2,1,3,-)+\mathrm{PT}(+,2,3,1,-)+\mathrm{PT}(+,3,2,1,-)\Big)\,,\end{aligned} \tag{5.24}$$

the net effect of the factor G_{ij} is to attribute relative minus signs to individual Parke–Taylor factors, depending on the relative position of legs i and j in its cyclic ordering. These signs can be compactly encoded in the shorthand

$$\mathrm{sgn}^B_{ij} = \begin{cases} +1 : \ i \text{ is right of } j \text{ in } B=b_1,b_2,\ldots,b_n \\ -1 : \ i \text{ is left of } j \text{ in } B=b_1,b_2,\ldots,b_n \end{cases}\,, \tag{5.25}$$

which enters the following all-multiplicity formula for $k\leq n{-}1$ factors of $G_{ij}\to i\pi\,\mathrm{sgn}^B_{ij}$ [77]:

$$\begin{aligned}\Big(\prod_{j=2}^{n}\frac{1}{\sigma_j}\Big)G_{p_1q_1}G_{p_2q_2}\ldots G_{p_kq_k} &= (-1)^{n-1}(i\pi)^k \lim_{\substack{\sigma_-\to\infty\\ \sigma_+\to 0}} |\sigma_-|^2\\ &\times \sum_{B\in S_n} \mathrm{PT}(+,b_1,b_2,\ldots,b_n,-)\mathrm{sgn}^B_{p_1q_1}\mathrm{sgn}^B_{p_2q_2}\ldots\mathrm{sgn}^B_{p_kq_k}\,.\end{aligned} \tag{5.26}$$

This result cannot be applied to products $G_{p_1q_1}G_{p_2q_2}\ldots G_{p_kq_k}$ whose labels form a cycle, e.g. G_{12}^2 and $G_{12}G_{23}G_{31}$ do not admit a Parke–Taylor decomposition via (5.26). Cycles of this

[21] This can be seen from the following corollary of repeated partial-fraction identities:

$$\frac{1}{\sigma_2\sigma_3\ldots\sigma_n} = \frac{(-1)^n}{\sigma_2\sigma_{23}\sigma_{34}\ldots\sigma_{n-2,n-1}\sigma_{n-1,n}} + \mathrm{perm}(2,3,\ldots,n)\,.$$

type do not appear in the integrands of $I^{\rm tree}(\cdot|\cdot)$ in (5.21) – the arrangement of G_{ij} inherits the chain structure from the integrands $\Omega(z_{12},\ldots)\Omega(z_{23},\ldots)\ldots\Omega(z_{n-1,n},\ldots)$ of $Z^\tau_{\vec\eta}(\cdot|\cdot)$.

The main results of this subsection follow from applying the integrand manipulations (5.23) and (5.26) to the general definition of $I^{\rm tree}(\cdot|\cdot)$ in (5.18). In absence of G_{ij}, each term in the Parke–Taylor decomposition

$$\begin{aligned} I^{\rm tree}(1,A|1) &= (-1)^{n-1}\int_{\mathcal{C}(A)} \frac{{\rm d}\sigma_+\,{\rm d}\sigma_-\,{\rm d}\sigma_1\,\ldots\,{\rm d}\sigma_n}{(2\pi i)^{n-1}\,{\rm vol}\,{\rm SL}_2(\mathbb{R})}\,{\rm KN}^{i\infty}_{12\ldots n}\sum_{B\in S_n}{\rm PT}(+,B,-)\\ &= \frac{1}{(-2\pi i)^{n-1}}\sum_{P\in S_{n-1}}\mathcal{H}_{\alpha'}(A|P)\sum_{B\in S_n} Z^{\rm tree}(+,P,1,-|+,B,-) \end{aligned} \tag{5.27}$$

reproduces the definition (1.3) of disk integrals, with s_{j+}, s_{j-} and $s_{+,-}$ given by (5.3). The matrix $\mathcal{H}_{\alpha'}(A|P)$ arises from the translation of planar genus-one cycles $\mathcal{C}(1,A)$ into disk orderings via (5.13). The same reasoning applies to products of G_{ij} without subcycles,

$$\begin{aligned} I^{\rm tree}(1,A|G_{p_1q_1}G_{p_2q_2}\ldots G_{p_kq_k}) &= \frac{(i\pi)^k}{(-2\pi i)^{n-1}}\sum_{P\in S_{n-1}}\mathcal{H}_{\alpha'}(A|P)\\ &\quad\times\sum_{B\in S_n}{\rm sgn}^B_{p_1q_1}{\rm sgn}^B_{p_2q_2}\ldots{\rm sgn}^B_{p_kq_k}Z^{\rm tree}(+,P,1,-|+,B,-)\,. \end{aligned} \tag{5.28}$$

Upon insertion into (5.21), this implies that any initial value $Z^{i\infty}_{\vec\eta}(\cdot|\cdot)$ is expressible in terms of $(n{+}2)$-point Parke–Taylor integrals. Note that each factor of G_{ij} introduces a minus sign into the expressions (5.28) when trading the original integration cycle $A=a_2a_3\ldots a_n$ for its reversal $A^t=a_n\ldots a_3a_2$. Hence, integrands of (5.18) with a fixed number k of G_{ij} factors integrate to zero on $\mathcal{C}(1,A)-(-1)^k\mathcal{C}(1,A^t)$, and one may replace $\mathcal{H}_{\alpha'}(A|P)$ by the parity-weighted combination $\frac{1}{2}\big[\mathcal{H}_{\alpha'}(A|P)+(-1)^k\mathcal{H}_{\alpha'}(A^t|P)\big]$ in (5.28),

$$\begin{aligned} I^{\rm tree}(1,A|G_{p_1q_1}G_{p_2q_2}\ldots G_{p_kq_k}) &= (-1)^{n-1}\frac{(i\pi)^{k-n+1}}{2^n}\sum_{P\in S_{n-1}}\big[\mathcal{H}_{\alpha'}(A|P)+(-1)^k\mathcal{H}_{\alpha'}(A^t|P)\big]\\ &\quad\times\sum_{B\in S_n}{\rm sgn}^B_{p_1q_1}{\rm sgn}^B_{p_2q_2}\ldots{\rm sgn}^B_{p_kq_k}Z^{\rm tree}(+,P,1,-|+,B,-)\,. \end{aligned} \tag{5.29}$$

As one can see from the examples in appendix D.2, the coefficients in the α'-expansion of $\mathcal{H}_{\alpha'}(A|P)+(-1)^k\mathcal{H}_{\alpha'}(A^t|P)$ are real and imaginary if $n{-}k$ is odd and even, respectively. Hence, by the prefactor $(i\pi)^{k-n+1}$ in its first line, (5.29) always yields real linear combinations of disk integrals $Z^{\rm tree}$.

The combination of (5.21) and (5.29) determines the transformation matrix $\mathcal{K}_{\vec\eta}(P|Q)$ between the basis integrands in (5.1) once the $n!$ Parke–Taylor factors of $Z^{\rm tree}(*|+,B,-)$ are reduced to an $(n{-}1)!$-element basis of $Z^{\rm tree}(*|+,Q,-,1)$ via BCJ relations [42, 101]. One can also attain this BCJ basis by removing any appearance of $i,j=1$ from G_{ij} via Fay relations

$$G_{12}G_{23}+G_{23}G_{31}+G_{31}G_{12}=\pi^2 \tag{5.30}$$

combined with integration by parts [77]

$$\partial_{v_i} \mathrm{KN}^{i\infty}_{12\ldots n}\big|_{\mathcal{C}(A)} = -\sum_{\substack{j=1\\ j\neq i}}^{n} s_{ij} G_{ij} \mathrm{KN}^{i\infty}_{12\ldots n}\,. \tag{5.31}$$

The latter follows from the limit $\tau\to i\infty$ of (2.16) on a planar integration cycle.

The techniques of this subsection will now be illustrated by various examples. The two-point instance of (5.27) with $\mathcal{H}_{\alpha'}(2|2) = 2i\sin(\frac{\pi}{2}s_{12})$ reproduces the earlier result (3.29) via

$$\begin{aligned} I^{\rm tree}(1,2|1) &= -\frac{1}{\pi}\sin\Big(\frac{\pi}{2}s_{12}\Big)\big[Z^{\rm tree}(+,2,1,-|+,1,2,-) + Z^{\rm tree}(+,2,1,-|+,2,1,-)\big] \\ &= \frac{1}{\pi}\sin\Big(\frac{\pi}{2}s_{12}\Big) Z^{\rm tree}(+,2,1,-|+,2,-,1) = \frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}\,. \end{aligned} \tag{5.32}$$

Several three- and four-point examples can be found in the next subsections.

5.4 Three points

As a first application of the general results above, we will now relate the constituents of $Z^{i\infty}_{\eta_2,\eta_3}(1,2,3|1,2,3)$ in (5.19) to five-point disk integrals. In absence of G_{ij} in the integrand at $\tau\to i\infty$, one can symmetrize the cycle by $I^{\rm tree}(1,2,3|1) = I^{\rm tree}(1,3,2|1)$, and (5.27) implies

$$\begin{aligned} I^{\rm tree}(1,2,3|1) &= \frac{1}{2\pi^2}\Big[\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{13}\Big)\sum_{B\in S_3} Z^{\rm tree}(+,2,3,1,-|+,B,-) \\ &\qquad + \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{12}\Big)\sum_{B\in S_3} Z^{\rm tree}(+,3,2,1,-|+,B,-)\Big] \\ &= -\frac{1}{2\pi^2}\Big[\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{13}\Big)Z^{\rm tree}(+,2,3,1,-|+,(2\sqcup\!\sqcup 3),-,1) \\ &\qquad + \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{12}\Big)Z^{\rm tree}(+,3,2,1,-|+,(2\sqcup\!\sqcup 3),-,1)\Big]\,, \end{aligned} \tag{5.33}$$

see (D.2) for the trigonometric functions in $\frac{1}{2}\big[\mathcal{H}_{\alpha'}(2,3|P)+\mathcal{H}_{\alpha'}(3,2|P)\big]$. The disk integrals have been rewritten in a BCJ basis in passing to the last two lines. Integrals over a single G_{ij} in turn are asymmetric in the cycle, $I^{\rm tree}(1,2,3|G_{23}) = -I^{\rm tree}(1,3,2|G_{23})$, i.e.

$$\begin{aligned} I^{\rm tree}(1,2,3|G_{23}) &= \frac{1}{2\pi} \\ &\times\Big[\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23})\Big)\cos\Big(\frac{\pi}{2}s_{13}\Big)\Big(Z^{\rm tree}(+,2,3,1,-|+,2,3,-,1) - Z^{\rm tree}(+,2,3,1,-|+,3,2,-,1)\Big) \\ &\quad -\sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23})\Big)\cos\Big(\frac{\pi}{2}s_{12}\Big)\Big(Z^{\rm tree}(+,3,2,1,-|+,2,3,-,1) - Z^{\rm tree}(+,3,2,1,-|+,3,2,-,1)\Big)\Big] \end{aligned} \tag{5.34}$$

after reduction to a BCJ basis. The remaining configurations of G_{ij} in (5.19) follow from the Fay identity (5.30) and integration by parts (5.31) at the level of the G_{ij},

$$\begin{aligned} s_{12} I^{\rm tree}(1,a_2,a_3|G_{12}) &= s_{23} I^{\rm tree}(1,a_2,a_3|G_{23}) \\ I^{\rm tree}(1,a_2,a_3|G_{12}G_{23}) &= \frac{\pi^2 s_{13}}{s_{123}} I^{\rm tree}(1,a_2,a_3|1)\,, \end{aligned} \tag{5.35}$$

so (5.19) can be decomposed as follows in terms of (5.33) and (5.34):

$$
\begin{aligned}
Z^{i\infty}_{\eta_2,\eta_3}(1,a_2,a_3|1,2,3) &= \Big(\pi^2\cot(\pi\eta_{23})\cot(\pi\eta_3)+\frac{\pi^2 s_{13}}{s_{123}}\Big) I^{\rm tree}(1,a_2,a_3|1) \\
&\quad + \Big(\pi\cot(\pi\eta_{23})+\frac{s_{23}}{s_{12}}\pi\cot(\pi\eta_3)\Big) I^{\rm tree}(1,a_2,a_3|G_{23})\,.
\end{aligned}
\tag{5.36}
$$

Since the $I^{\rm tree}(\cdot|\cdot)$ have been expanded in the 2×2 basis of $Z^{\rm tree}(+,P,1,-|+,Q,-,1)$, one can read off the entries of the transformation matrix $\mathcal{K}_{\eta_2,\eta_3}(\cdot|\cdot)$ for the integrands in (5.1):

$$
\begin{aligned}
\mathcal{K}_{\eta_2,\eta_3}(2,3|2,3) &= -\Big(\pi^2\cot(\pi\eta_{23})\cot(\pi\eta_3)+\frac{\pi^2 s_{13}}{s_{123}}\Big) + i\pi\Big(\pi\cot(\pi\eta_{23})+\frac{s_{23}}{s_{12}}\pi\cot(\pi\eta_3)\Big) \\
\mathcal{K}_{\eta_2,\eta_3}(2,3|3,2) &= -\Big(\pi^2\cot(\pi\eta_{23})\cot(\pi\eta_3)+\frac{\pi^2 s_{13}}{s_{123}}\Big) - i\pi\Big(\pi\cot(\pi\eta_{23})+\frac{s_{23}}{s_{12}}\pi\cot(\pi\eta_3)\Big)\,.
\end{aligned}
\tag{5.37}
$$

α'-expansions: One can now insert the α'-expansion of the disk integrals (see for instance [23, 24]) into (5.33) and (5.34). By rewriting $s_{j+}=s_{j-}=-\frac{1}{2}(s_{1j}+s_{23})$ and $\pi^2=6\zeta_2$, we arrive at the following permutation symmetric series in s_{ij},

$$
\begin{aligned}
I^{\rm tree}(1,2,3|1) &= \frac{1}{2}+\frac{1}{8}\zeta_2(s_{12}^2+s_{13}^2+s_{23}^2)+\frac{1}{8}\zeta_3(s_{12}^3+s_{13}^3+s_{23}^3+s_{12}s_{13}s_{23}) \\
&\quad + \frac{1}{320}\zeta_2^2\Big[19s_{12}^4+10s_{12}^2s_{13}^2+12s_{12}^2s_{13}s_{23}+{\rm cyc}(1,2,3)\Big] \\
&\quad + \frac{1}{32}\zeta_5\Big[3s_{12}^5+2s_{12}^3s_{13}s_{23}+3s_{12}^2s_{13}^2s_{23}+{\rm cyc}(1,2,3)\Big] \\
&\quad + \frac{1}{32}\zeta_2\zeta_3(s_{12}^2+s_{13}^2+s_{23}^2)(s_{12}^3+s_{13}^3+s_{23}^3+s_{12}s_{13}s_{23})+\mathcal{O}(\alpha'^6)
\end{aligned}
\tag{5.38}
$$

as well as

$$
\begin{aligned}
s_{23}I^{\rm tree}(1,2,3|G_{23}) &= 1+\frac{1}{4}\zeta_2(s_{12}+s_{13}+s_{23})^2 \\
&\quad + \frac{1}{4}\zeta_3\Big\{\Big[s_{12}^3+3s_{12}^2s_{13}+3s_{12}s_{13}^2+{\rm cyc}(1,2,3)\Big]+7s_{12}s_{13}s_{23}\Big\} \\
&\quad + \frac{1}{160}\zeta_2^2(s_{12}+s_{13}+s_{23})\Big\{\Big[19s_{12}^3+57s_{12}^2s_{13}+57s_{12}s_{13}^2+{\rm cyc}(1,2,3)\Big]+126s_{12}s_{13}s_{23}\Big\} \\
&\quad + \frac{1}{16}\zeta_5\Big[3s_{12}^5+15s_{12}^4s_{13}+15s_{12}s_{13}^4+30s_{12}^3s_{13}^2+30s_{12}^2s_{13}^3+62s_{12}^3s_{13}s_{23} \\
&\qquad\qquad + 93s_{12}^2s_{13}^2s_{23}+{\rm cyc}(1,2,3)\Big] \\
&\quad + \frac{1}{16}\zeta_2\zeta_3(s_{12}+s_{13}+s_{23})^2\Big\{\Big[s_{12}^3+3s_{12}^2s_{13}+3s_{12}s_{13}^2+{\rm cyc}(1,2,3)\Big]+7s_{12}s_{13}s_{23}\Big\}+\mathcal{O}(\alpha'^6)\,.
\end{aligned}
\tag{5.39}
$$

At higher orders in α', the irreducible depth-($d{\geq}2$) MZVs $\zeta_{3,5},\zeta_{3,7}$ and $\zeta_{3,5,3}$ of the five-point $Z^{\rm tree}(\cdot|\cdot)$ drop out from both (5.38) and (5.39). This is a peculiarity of the arguments $s_{j+}=s_{j-}=-\frac{1}{2}(s_{1j}+s_{23})$ which cause the $I^{\rm tree}(\cdot|\cdot)$ to be functions of three variables instead of the five Mandelstam invariants of a five-point disk integral.

Component integrals: By inserting (5.38) and (5.39) into the initial value (5.36), one can access any order in the α'-expansion of the three-point A-cycle integrals via (4.14). In particular, inspection of specific orders in $\eta_2+\eta_3$ and η_3 yields explicit results for the component integrals

$$\begin{aligned} Z^{\tau}_{(m_1,m_2)}(A|1,2,3) &\equiv Z^{\tau}_{\eta_2,\eta_3}(A|1,2,3)\,\big|_{(\eta_2+\eta_3)^{m_1-1}\eta_3^{m_2-1}} \\ &= \int_{\mathcal{C}(A)} \mathrm{d}z_2\, \mathrm{d}z_3\, f^{(m_1)}(z_{12},\tau) f^{(m_2)}(z_{23},\tau)\, \mathrm{KN}^{\tau}_{123}\,. \end{aligned} \tag{5.40}$$

Since all the $r_{\eta_2,\eta_3}(\epsilon_k)$ in (4.13) are even under $(\eta_2,\eta_3) \to -(\eta_2,\eta_3)$, one can take a convenient shortcut in extracting component integrals from the generating series: Any $Z^{\tau}_{(m_1,m_2)}(A|1,2,3)$ with even parity $m_1{+}m_2 \in 2\mathbb{N}_0$ (odd parity $m_1{+}m_2 \in 2\mathbb{N}_0{+}1$) is determined by the even (odd) part of the initial value $Z^{i\infty}_{\eta_2,\eta_3}$ in (5.36) w.r.t. $\eta_j \to -\eta_j$. Hence, $I^{\text{tree}}(A|G_{ij})$ with odd coefficients $\sim \cot(\pi\eta_{23})$ or $\cot(\pi\eta_j)$ do not contribute to $Z^{\tau}_{(m_1,m_2)}(A|1,2,3)\big|_{m_1+m_2 \text{ even}}$, and one can similarly disregard $I^{\text{tree}}(A|1)$ in (5.36) when computing $Z^{\tau}_{(m_1,m_2)}(A|1,2,3)\big|_{m_1+m_2 \text{ odd}}$.

Similar to the two-point case (3.34), the simplest integrand $(m_1,m_2)=(0,0)$ yields a generating series of planar A-cycle graph functions involving up to three vertices[22],

$$\begin{aligned} Z^{\tau}_{(0,0)}&(1,2,3|1,2,3) + Z^{\tau}_{(0,0)}(1,3,2|1,2,3) = 1 + \frac{1}{2}(s_{12}^2{+}s_{13}^2{+}s_{23}^2)A_2(\tau) \\ &+ s_{12}s_{13}s_{23}A_{111}(\tau) + \frac{1}{3!}(s_{12}^3{+}s_{13}^3{+}s_{23}^3)A_3(\tau) + \frac{1}{4}(s_{12}^2s_{13}^2{+}s_{12}^2s_{23}^2{+}s_{13}^2s_{23}^2)A_2(\tau)^2 \\ &+ \frac{1}{2}s_{12}s_{13}s_{23}(s_{12}{+}s_{13}{+}s_{23})A_{211}(\tau) + \frac{1}{4!}(s_{12}^4{+}s_{13}^4{+}s_{23}^4)A_4(\tau) + \mathcal{O}(\alpha'^5)\,, \end{aligned} \tag{5.41}$$

see (3.35) for the planar two-vertex graph functions $A_w(\tau)$. The α'-expansion (4.14) has been checked to reproduce all the planar three-vertex A-cycle graph functions

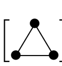
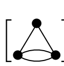
$$A_{ijk}(\tau) = i!j!k!\big[Z^{\tau}_{(0,0)}(1,2,3|1,2,3) + Z^{\tau}_{(0,0)}(1,3,2|1,2,3)\big]\,\big|_{s_{12}^i s_{23}^j s_{13}^k} = 0\,, \qquad i,j,k \geq 1 \tag{5.42}$$

of weight $i{+}j{+}k \leq 6$ known from [81], e.g.

$$\begin{aligned} A_{111}(\tau) &= \frac{\zeta_3}{4} + 36\zeta_2\gamma_0(4,0,0|\tau) - 60\gamma_0(6,0,0|\tau) \\ A_{211}(\tau) &= \frac{3\zeta_4}{8} - 144\zeta_4\gamma_0(4,0,0,0|\tau) - 36\gamma_0(4,4,0,0|\tau) \\ &\quad + 1680\zeta_2\gamma_0(6,0,0,0|\tau) - 756\gamma_0(8,0,0,0|\tau)\,. \end{aligned} \tag{5.43}$$

Moreover, the differential-equation method of this work directly yields the fully simplified iterated-Eisenstein-integral representation of $A_{ijk}(\tau)$ which is particularly helpful at higher

[22] In the graphical notation of [81], the three-vertex graph functions in (5.41) are represented as follows:

$A_{111} = \mathbf{A}\big[$  $\big]$ $\qquad$ $A_{211} = \mathbf{A}\big[$  $\big]$

weights $i+j+k$ beyond the current reach of the eMZV datamine [85]. Moreover, the degeneration of planar three-vertex A-cycle graph functions at the cusp follows from

$$\begin{aligned} Z^{i\infty}_{(0,0)}(1,2,3|1,2,3) + Z^{i\infty}_{(0,0)}(1,3,2|1,2,3) &= 2I^{\rm tree}(1,2,3|1) \\ &= -\frac{1}{\pi^2}\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{13}\Big) Z^{\rm tree}(+,2,3,1,-|+,(2\sqcup\!\sqcup 3),-,1) + (2\leftrightarrow 3) \end{aligned} \tag{5.44}$$

upon insertion into (5.42).

The expansion of integrals (5.40) over non-constant $f^{(m_1)}_{12}f^{(m_2)}_{23}$ with $m_j \neq 0$ has not yet been discussed in the literature, and the leading orders of some representative examples read

$$\begin{aligned} Z^{\tau}_{(2,0)}(1,2,3|1,2,3) &= -\zeta_2 + \frac{3}{2}s_{12}\gamma_0(4|\tau) - \frac{1}{4}(s_{12}^2+s_{13}^2+s_{23}^2)\zeta_2^2 \\ &\quad + 3(s_{13}^2+s_{23}^2-2s_{13}s_{23}-3s_{12}^2)\zeta_2\gamma_0(4,0|\tau) + 5(s_{12}^2+2s_{13}s_{23})\gamma_0(6,0|\tau) + \mathcal{O}(\alpha'^3) \\ Z^{\tau}_{(3,3)}(1,2,3|1,2,3) &= 21s_{13}\zeta_4\gamma_0(4|\tau) + 5s_{13}\zeta_2\gamma_0(6|\tau) - \frac{7}{2}s_{13}\gamma_0(8|\tau) \\ &\quad + 27s_{13}(5s_{12}-8s_{13}+5s_{23})\zeta_6\gamma_0(4,0|\tau) + 15s_{13}(17s_{12}+10s_{13}+17s_{23})\zeta_4\gamma_0(6,0|\tau) \\ &\quad + 9s_{13}(3s_{12}+s_{13}+3s_{23})\zeta_2\gamma_0(4,4|\tau) + 21s_{13}(5s_{12}+7s_{13}+5s_{23})\zeta_2\gamma_0(8,0|\tau) - \frac{15}{2}s_{13}^2\gamma_0(4,6|\tau) \\ &\quad - \frac{15}{2}s_{13}(3s_{12}+s_{13}+3s_{23})\gamma_0(6,4|\tau) - \frac{9}{2}s_{13}(19s_{12}+20s_{13}+19s_{23})\gamma_0(10,0|\tau) + \mathcal{O}(\alpha'^3)\,. \end{aligned} \tag{5.45}$$

Integrals over $f^{(1)}_{ij}f^{(m_2)}_{jk}$ with even m_2 introduce kinematic poles that stem from the initial value (5.36) when computing the α'-expansion via (4.14), e.g.

$$\begin{aligned} Z^{\tau}_{(1,0)}(1,2,3|1,2,3) &= \frac{1}{s_{12}}\Big\{1 + (s_{12}+s_{13}+s_{23})^2\Big(\frac{\zeta_2}{4} - 3\gamma_0(4,0|\tau)\Big) \\ &\quad + (s_{12}+s_{13}+s_{23})^3\Big(\frac{\zeta_3}{4} - 10\gamma_0(6,0,0|\tau) + 24\zeta_2\gamma_0(4,0,0|\tau)\Big) \\ &\quad + s_{12}s_{13}s_{23}\Big(\frac{\zeta_3}{4} - 90\gamma_0(6,0,0|\tau)\Big) + \mathcal{O}(\alpha'^4)\Big\}\,. \end{aligned} \tag{5.46}$$

Integrals over $f^{(1)}_{ij}f^{(m_2)}_{jk}$ with odd $m_2 \geq 3$ in turn are regular as $s_{ij}\to 0$, and the $\alpha'^{\leq 3}$-orders of the example $Z^{\tau}_{(1,3)}(1,2,3|1,2,3)$ can be found in (E.1). Finally, integration over $f^{(1)}_{ij}f^{(1)}_{jk}$ introduces the pole structure $\sim \frac{1}{s_{123}}$ as one can anticipate from the behavior (5.35) of the $\tau\to i\infty$ degeneration.

Note that (5.45) and (E.1) exemplify some of the simplest situations to encounter irreducible iterated Einstein integrals of depth two with different entries: Neither $\gamma_0(6,4|\tau)$ at the α'^2-order of $Z^{\tau}_{(3,3)}(1,2,3|1,2,3)$ nor $\gamma_0(4,4,0|\tau)$ or $\gamma_0(4,6,0|\tau)$ at the α'^3-order of $Z^{\tau}_{(1,3)}(1,2,3|1,2,3)$ are expressible as shuffle products of depth-one representatives.

5.5 Four points

All the constituents of $Z^{i\infty}_{\vec{\eta}}(1,A|1,2,3,4)$ in (5.20) can be expressed in terms of six-point disk integrals via (5.29), for instance

$$
\begin{aligned}
I^{\rm tree}(1,2,3,4|1) = -\frac{1}{4\pi^3}\sum_{B\in S_3}\Big[&\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23}{+}s_{24})\Big)\cos\Big(\frac{\pi}{2}s_{134}\Big) Z^{\rm tree}(+,2,3,4,1,-|+,B,-,1)\\
&- \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23}{+}s_{34})\Big)\cos\Big(\frac{\pi}{2}(-s_{12}{+}s_{14}{-}s_{24})\Big) Z^{\rm tree}(+,3,2,4,1,-|+,B,-,1)\\
&- \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23}{+}s_{34})\Big)\cos\Big(\frac{\pi}{2}(-s_{12}{+}s_{14}{+}s_{24})\Big) Z^{\rm tree}(+,3,4,2,1,-|+,B,-,1)\\
&+ \sin\Big(\frac{\pi}{2}(s_{14}{+}s_{24}{+}s_{34})\Big)\cos\Big(\frac{\pi}{2}s_{123}\Big) Z^{\rm tree}(+,4,3,2,1,-|+,B,-,1)\Big]
\end{aligned} \tag{5.47}
$$

as well as

$$
\begin{aligned}
I^{\rm tree}(1,2,3,4|G_{23}) = \frac{1}{4\pi^2}&\sum_{B\in S_3} {\rm sgn}^B_{23}\\
\times\Big[&\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23}{+}s_{24})\Big)\sin\Big(\frac{\pi}{2}s_{134}\Big) Z^{\rm tree}(+,2,3,4,1,-|+,B,-,1)\\
&- \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23}{+}s_{34})\Big)\sin\Big(\frac{\pi}{2}(-s_{12}{+}s_{14}{-}s_{24})\Big) Z^{\rm tree}(+,3,2,4,1,-|+,B,-,1)\\
&- \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23}{+}s_{34})\Big)\sin\Big(\frac{\pi}{2}(-s_{12}{+}s_{14}{+}s_{24})\Big) Z^{\rm tree}(+,3,4,2,1,-|+,B,-,1)\\
&- \sin\Big(\frac{\pi}{2}(s_{14}{+}s_{24}{+}s_{34})\Big)\sin\Big(\frac{\pi}{2}s_{123}\Big) Z^{\rm tree}(+,4,3,2,1,-|+,B,-,1)\Big]
\end{aligned} \tag{5.48}
$$

and

$$
\begin{aligned}
I^{\rm tree}(1,2,3,4|G_{12}G_{34}) = -\frac{1}{4\pi}&\sum_{B\in S_4} {\rm sgn}^B_{12}{\rm sgn}^B_{34}\\
\times\Big[&\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23}{+}s_{24})\Big)\cos\Big(\frac{\pi}{2}s_{134}\Big) Z^{\rm tree}(+,2,3,4,1,-|+,B,-)\\
&- \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23}{+}s_{34})\Big)\cos\Big(\frac{\pi}{2}(-s_{12}{+}s_{14}{-}s_{24})\Big) Z^{\rm tree}(+,3,2,4,1,-|+,B,-)\\
&- \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23}{+}s_{34})\Big)\cos\Big(\frac{\pi}{2}(-s_{12}{+}s_{14}{+}s_{24})\Big) Z^{\rm tree}(+,3,4,2,1,-|+,B,-)\\
&+ \sin\Big(\frac{\pi}{2}(s_{14}{+}s_{24}{+}s_{34})\Big)\cos\Big(\frac{\pi}{2}s_{123}\Big) Z^{\rm tree}(+,4,3,2,1,-|+,B,-)\Big]\,.
\end{aligned} \tag{5.49}
$$

When the integrand of $I^{\rm tree}(\cdot|\cdot)$ in (5.20) is $G_{12}G_{23}G_{34}$ or one of $G_{12}G_{23}$, $G_{23}G_{34}$ with overlapping labels, repeated use of the Fay identity (5.30) and integration by parts (5.31) can be used to reduce these cases to (permutations of) (5.47), (5.48), and (5.49), for instance

$$
I^{\rm tree}(A|G_{12}G_{23}) = \frac{\pi^2 s_{13}}{s_{123}} I^{\rm tree}(A|1) + \frac{s_{34}}{s_{123}} I^{\rm tree}(A|G_{12}G_{34}) - \frac{s_{14}}{s_{123}} I^{\rm tree}(A|G_{14}G_{23}) \tag{5.50}
$$

as well as

$$
\begin{aligned}
s_{1234} I^{\rm tree}(A|G_{12}G_{13}G_{14}) &= -\pi^2\big[s_{34}I^{\rm tree}(A|G_{12}) + s_{24}I^{\rm tree}(A|G_{13}) + s_{23}I^{\rm tree}(A|G_{14})\big] \qquad (5.51)\\
s_{1234} I^{\rm tree}(A|G_{12}G_{23}G_{34}) &= \pi^2\big[s_{124}I^{\rm tree}(A|G_{12}) + s_{134}I^{\rm tree}(A|G_{34}) + (s_{14}{-}s_{23})I^{\rm tree}(A|G_{23})\big]\,.
\end{aligned}
$$

α'-expansions: Given the α'-expansion of six-point disk integrals (see for instance [23, 24]), we arrive at expressions such as

$$
\begin{aligned}
I^{\rm tree}(1,2,3,4|1) &= \frac{1}{6} + \frac{\zeta_3}{4\pi^2}(s_{12}{-}2s_{13}{+}s_{14}{+}s_{23}{-}2s_{24}{+}s_{34}) \\
&+ \zeta_2\Big(\Big[\frac{13s_{12}^2}{240} - \frac{s_{12}s_{14}}{120} + \frac{s_{12}(s_{13}{+}s_{24})}{240} + {\rm cyc}(1,2,3,4)\Big] \\
&\quad + \frac{s_{13}s_{24}}{60} - \frac{(s_{12}s_{34}{+}s_{14}s_{23})}{120} + \frac{(s_{13}^2{+}s_{24}^2)}{60}\Big) \\
&+ \frac{\zeta_3}{96}\Big(\Big[5s_{12}^3{+}3s_{12}^2s_{14}{+}3s_{14}^2s_{23}{+}3s_{14}^2s_{34}{-}3s_{12}s_{13}^2{-}3s_{13}^2s_{14}{+}4s_{12}s_{13}s_{23} \\
&\quad +6s_{12}s_{14}s_{34}{-}6s_{12}s_{13}s_{24} + {\rm cyc}(1,2,3,4)\Big] + 2s_{13}^3{+}2s_{24}^3{-}6s_{13}^2s_{24}{-}6s_{13}s_{24}^2\Big) \\
&+ \frac{\zeta_5}{16\pi^2}\Big(\Big[2s_{12}^3{-}2s_{12}^2s_{13}{-}2s_{13}s_{23}^2{+}7s_{12}s_{13}^2{+}7s_{13}^2s_{23}{-}5s_{12}^2s_{23}{-}5s_{12}s_{23}^2{-}5s_{12}^2s_{34} \\
&\quad -11s_{12}s_{23}s_{34}{+}11s_{12}s_{13}s_{24} + {\rm cyc}(1,2,3,4)\Big] - 4s_{13}^3{-}4s_{24}^3{+}10s_{13}^2s_{24}{+}10s_{13}s_{24}^2\Big) + \mathcal{O}(\alpha'^4)\,.
\end{aligned}
\tag{5.52}
$$

Similar expansions for $I^{\rm tree}(1,2,3,4|*)$ with integrands G_{12}, G_{13}, $G_{12}G_{34}$, $G_{13}G_{24}$ and $G_{12}G_{13}$ are displayed in appendix E.2.

The inverse powers of π in the combinations $\frac{\zeta_3}{\pi^2}$ and $\frac{\zeta_5}{\pi^2}$ at the α'- and α'^3-orders of (5.52) are a generic feature of $I^{\rm tree}(\cdot|\cdot)$ at $n{\geq}4$ points: The MZVs in the α'-expansion of n-point $I^{\rm tree}(\cdot|\cdot)$ may be accompanied by up to $n{-}2$ inverse powers of π (up to $n{-}3$ inverse powers if n is odd). This can be seen from the following properties of their decomposition (5.29) into disk integrals:

- the overall prefactors $(i\pi)^{k-n+1}$, where k is the number of G_{ij} in the integrand
- the sine functions in (5.12) render the α'-expansion of any $\mathcal{H}_{\alpha'}(A|P)$ proportional to π
- in case of odd multiplicity $n \in 2\mathbb{N}{+}1$ and $k = 0$ powers of $i\pi$ due to G_{ij}, the combination $\frac{1}{2}(\mathcal{H}_{\alpha'}(A|P){+}\mathcal{H}_{\alpha'}(A^t|P))$ allows to factorize π^2, leading to $3{-}n$ powers of π in (5.29)
- the α'-expansion of generic $Z^{\rm tree}(\cdot|\cdot)$ involves $\mathbb{Q}$-linear combinations of MZVs

In the limit of on-shell kinematics $(s_{13}, s_{14}, s_{24}, s_{34}) \rightarrow (-s_{12}{-}s_{23}, s_{23}, -s_{12}{-}s_{23}, s_{12})$ of four massless particles, the α'-expansion of $I^{\rm tree}(1,2,3,4|1)$ governs the $\tau \rightarrow i\infty$ contribution to the one-loop four-point amplitude of the open superstring [102]. The on-shell limit of the six Mandelstam invariants s_{ij} in (5.52) is compatible with the all-order result [103]

$$
I^{\rm tree}(1,2,3,4|1) \rightarrow -\frac{1}{2\pi^2 s_{12}s_{23}}\alpha'\frac{\partial}{\partial\alpha'}\frac{\Gamma(1-s_{12})\Gamma(1-s_{23})}{\Gamma(1-s_{12}-s_{23})}\,. \tag{5.53}
$$

Still, it is beneficial to retain the dependence of various $I^{\rm tree}(1,2,3,4|B)$ on six s_{ij} since this will turn out to yield important building blocks for non-planar six-point A-cycle integrals.

Component integrals: The α'-expansion (4.22) of the generating function $Z^\tau_{\eta_2,\eta_3,\eta_4}$ allows to extract component integrals

$$\begin{aligned} Z^\tau_{(m_1,m_2,m_3)}(A|1,2,3,4) &\equiv Z^\tau_{\eta_2,\eta_3,\eta_4}(A|1,2,3,4)\,\big|_{(\eta_2+\eta_3+\eta_4)^{m_1-1}(\eta_3+\eta_4)^{m_2-1}\eta_4^{m_3-1}} \\ &= \int_{\mathcal{C}(A)} \mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4\, f^{(m_1)}(z_{12},\tau) f^{(m_2)}(z_{23},\tau) f^{(m_3)}(z_{34},\tau)\,\mathrm{KN}^\tau_{1234}\,, \end{aligned} \tag{5.54}$$

see (5.40) for the analogous definition at three points. Again, since the derivations in the four-point representation (4.20) are even under $\eta_j \to -\eta_j$, component integrals with even and odd values of $m_1{+}m_2{+}m_3$ decouple in the following sense: $Z^\tau_{(m_1,m_2,m_3)}$ with $m_1{+}m_2{+}m_3 \in 2\mathbb{N}_0$ ($m_1{+}m_2{+}m_3 \in 2\mathbb{N}_0{+}1$) only receive contributions from the $I^{\rm tree}$ in the initial value (5.20) with an even (odd) number of G_{ij} in the integrand, respectively. Note that even and odd powers of G_{ij} do not mix under Fay relations (5.30) and integration by parts (5.31).

The α'-expansion of the simplest component integral (5.54) starts with

$$\begin{aligned} Z^\tau_{(0,0,0)}(1,2,3,4|1,2,3,4) &= I^{\rm tree}(1,2,3,4|1) + 6(s_{12}{-}2s_{13}{+}s_{14}{+}s_{23}{-}2s_{24}{+}s_{34})\gamma_0(4,0,0|\tau) \\ &\quad + 60\Big[s_{12}^2{-}s_{13}^2{+}s_{13}(s_{12}{+}s_{23}){-}2s_{12}s_{23}{+}s_{13}s_{24}{-}s_{12}s_{34} + \mathrm{cyc}(1,2,3,4)\Big]\gamma_0(6,0,0,0|\tau) \\ &\quad - \frac{1}{2}\Big(\sum_{1\leq i<j}^4 s_{ij}^2\Big)\gamma_0(4,0|\tau) + \mathcal{O}(\alpha'^3)\,, \end{aligned} \tag{5.55}$$

in agreement with [20].

The symmetrization of (5.52) w.r.t. planar integration cycles no longer involves inverse powers of π since the permutation sum $\sum_{A\in S_3}\mathcal{H}_{\alpha'}(A|P)$ yields products of three sine functions by (5.16) and therefore an overall factor of π^3,

$$\begin{aligned} \sum_{A\in S_3} I^{\rm tree}(1,A|1) &= \frac{1}{\pi^3}\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23}{+}s_{24})\Big)\sin\Big(\frac{\pi}{2}(s_{13}{+}s_{34})\Big)\sin\Big(\frac{\pi}{2}s_{14}\Big) \\ &\quad \times \sum_{B\in S_3} Z^{\rm tree}(+,2,3,4,1,-|+,B,-,1) + \mathrm{perm}(2,3,4)\,. \end{aligned} \tag{5.56}$$

As will be detailed in appendix F, symmetrized combinations $\sum_{A\in S_{n-1}} I^{\rm tree}(1,A|*)$ at any multiplicity (possibly involving G_{ij} in the integrand) yield α'-expansions with $\mathbb{Q}$-linear as opposed to $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs in their coefficients.

At the leading orders in α', (5.56) amounts to

$$\begin{aligned} \sum_{A\in S_3} I^{\rm tree}(1,A|1) &= 1 + \frac{\zeta_2}{4}\sum_{1\leq i<j}^4 s_{ij}^2 + \frac{\zeta_3}{4}\sum_{1\leq i<j}^4 s_{ij}^3 \\ &\quad + \frac{\zeta_3}{4}(s_{12}s_{13}s_{23}{+}s_{12}s_{14}s_{24}{+}s_{13}s_{14}s_{34}{+}s_{23}s_{24}s_{34}) + \mathcal{O}(\alpha'^4)\,. \end{aligned} \tag{5.57}$$

This is in fact the $\tau \to i\infty$ degeneration of the simplest A-cycle graph functions which are generated by the component integral $\sum_{A\in S_3} Z^\tau_{(0,0,0)}(1,A|1,2,3,4)$. We have checked the differential-equation method to reproduce the simplest example of four-point A-cycle graph functions [81]

$$A_{ijkl}(\tau) = i!j!k!l! \sum_{A\in S_3} Z^{\tau}_{(0,0,0)}(1,A|1,2,3,4)\,\big|_{s_{12}^i s_{23}^j s_{34}^k s_{41}^l}\,, \tag{5.58}$$

namely

$$A_{1111}(\tau) = \frac{1}{8}\zeta_4 + 960\zeta_2\gamma_0(6,0,0,0|\tau) - 840\gamma_0(8,0,0,0|\tau)\,. \tag{5.59}$$

It will be shown in appendix F that the coefficients in the q-expansion of A-cycle graph functions at any multiplicity are $\mathbb{Q}$-linear as opposed to $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs.

Four-point one-loop amplitudes with half- or quarter-maximal spacetime supersymmetry involve moduli-space integrals over $f^{(2)}_{ij}$ and $f^{(1)}_{ij}f^{(1)}_{pq}$ [54, 55]. Hence, their α'-expansions will require the on-shell limit of $Z^{\tau}_{(2,0,0)}(1,2,3,4|1,b_2,b_3,4)$ and $Z^{\tau}_{(1,0,1)}(1,2,3,4|1,b_2,b_3,4)$ with $(b_2,b_3)\in S_2$, which address the cyclically inequivalent integrands $f^{(2)}_{12}, f^{(2)}_{13}, f^{(1)}_{12}f^{(1)}_{34}$ and $f^{(1)}_{13}f^{(1)}_{24}$. Integrals over $f^{(1)}_{12}f^{(1)}_{23}$ with overlapping labels again follow from Fay relations and integration by parts.

6 Non-planar genus-one integrals at the cusp

We shall now extend the above discussion of initial values $Z^{i\infty}_{\vec\eta}(A|*)$ associated with planar integration cycles $\mathcal{C}(A)$ to non-planar ones $\mathcal{C}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}\big)$. The α'-dependence of the desired initial values $Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|*\big)$ will be shown to reside in products $I^{\rm tree}(P|*)I^{\rm tree}(Q|*)$ of lower-multiplicity integrals (5.18) that boil down to products of disk integrals by virtue of (5.29).

One of the main reasons for this simplified α'-dependence is the factorization of the degenerate Koba–Nielsen on a non-planar cycle into

$$\mathrm{KN}^{i\infty}_{12\ldots n}\,\big|_{\mathcal{C}\left(\begin{smallmatrix}Q\\P\end{smallmatrix}\right)} = \big(\mathrm{KN}^{i\infty}_{P}\,\big|_{\mathcal{C}(P)}\big)\times\big(\mathrm{KN}^{i\infty}_{Q}\,\big|_{\mathcal{C}(Q)}\big)\,, \tag{6.1}$$

see (5.2) for the degeneration of the planar Koba–Nielsen factors on the right-hand side. This factorization follows from (3.20) and can be intuitively understood from figure 1: The cylinder worldsheet becomes infinitely long as $\tau\to i\infty$, so its boundaries decouple in this limit and give rise to separate Koba–Nielsen factors.

6.1 General result

The combinations and coefficients of $I^{\rm tree}(P|*)I^{\rm tree}(Q|*)$ in the initial values $Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|\,*\,\big)$ can be conveniently described by means of the following notation: For a given integrand $\Omega(z_{12},\eta_{23\ldots n},\tau)\Omega(z_{23},\eta_{3\ldots n},\tau)\ldots\Omega(z_{n-1,n},\eta_n,\tau)$ associated with $*\to 1,2,\ldots,n$, we define the following collection of integrated punctures z_j such that both z_{j-1} and z_j belong to the same boundary:

$$X^{12\ldots n}_P = \{j\in\{2,3,\ldots,n\},\ j\in P\ \wedge\ j{-}1\in P\}\,. \tag{6.2}$$

By the parameterization in (4.2), each $z_{j-1,j}$ with $j \in X_P^{12\dots n}$ or $j \in X_Q^{12\dots n}$ is real on $\mathcal{C}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|\ast\big)$. Similarly, the complement in the sense of $X_P^{12\dots n}\cup X_Q^{12\dots n}\cup Y_{P,Q}^{12\dots n}=\{2,3,\dots,n\}$ is given by

$$Y_{P,Q}^{12\dots n}=\{j\in\{2,3,\dots,n\},\ (j\in P\wedge j{-}1\in Q)\ \vee\ (j\in Q\wedge j{-}1\in P)\}\,, \tag{6.3}$$

such that $\operatorname{Im} z_{j-1,j}=\pm\operatorname{Im}(\tau/2)\ \forall\ j\in Y_{P,Q}^{12\dots n}$ on $\mathcal{C}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|\ast\big)$. Given that the degeneration $\lim_{\tau\to i\infty}\Omega(v_{ij}\pm\tau/2,\eta,\tau)=\frac{\pi}{\sin(\pi\eta)}$ does not depend on $v_{ij}\in\mathbb{R}$, the factorization (6.1) of the non-planar Koba–Nielsen factor propagates to

$$\begin{aligned}Z_{\vec\eta}^{i\infty}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|1,2,\dots,n\big)=\prod_{i\in Y_{P,Q}^{12\dots n}}\frac{\pi}{\sin(\pi\eta_{i,i+1\dots n-1,n})}\bigg(\int_{\mathcal{C}(P)}\mathrm{KN}_P^{i\infty}\prod_{j\in X_P^{12\dots n}}\Omega(z_{j-1,j},\eta_{j,j+1\dots n},i\infty)\bigg)\\ \times\bigg(\int_{\mathcal{C}(Q)}\mathrm{KN}_Q^{i\infty}\prod_{k\in X_Q^{12\dots n}}\Omega(z_{k-1,k},\eta_{k,k+1\dots n},i\infty)\bigg)\,\mathrm{d}z_2\,\mathrm{d}z_3\,\dots\,\mathrm{d}z_n\,.\end{aligned} \tag{6.4}$$

Once we insert the degeneration (3.16) of the remaining factors $\Omega(\dots)$, the two decoupled integrals in the first and second line both follow the degeneration (5.21) in the planar case,

$$\begin{aligned}Z_{\vec\eta}^{i\infty}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|1,2,\dots,n\big)=&\prod_{i\in Y_{P,Q}^{12\dots n}}\frac{\pi}{\sin(\pi\eta_{i,i+1\dots n-1,n})}\\ &\times\sum_{\substack{X_P^{12\dots n}=A\cup B\\ A\cap B=\emptyset}}\Big(\prod_{j\in A}\pi\cot(\pi\eta_{j,j+1\dots n})\Big)I^{\mathrm{tree}}(P|\prod_{\bar j\in B}G_{\bar j-1,\bar j})\\ &\times\sum_{\substack{X_Q^{12\dots n}=C\cup D\\ C\cap D=\emptyset}}\Big(\prod_{k\in C}\pi\cot(\pi\eta_{k,k+1\dots n})\Big)I^{\mathrm{tree}}(Q|\prod_{\bar k\in D}G_{\bar k-1,\bar k})\,.\end{aligned} \tag{6.5}$$

For P,Q with one or two elements (say $P=i$ or $P=i,j$), one can readily simplify (6.5) via,

$$I^{\mathrm{tree}}(i|1)=1\,,\qquad I^{\mathrm{tree}}(i,j|G_{ij})=0\,,\qquad I^{\mathrm{tree}}(i,j|1)=\frac{\Gamma(1-s_{ij})}{\big[\Gamma(1-\frac{s_{ij}}{2})\big]^2}\,, \tag{6.6}$$

cf. section 3.4. On these grounds, the non-planar two-point result (3.23) is recovered from (6.5) with $P=1,\ Q=2$ as well as $X_1^{12}=X_2^{12}=\emptyset$ and $Y_{1,2}^{12}=\{2\}$. For permutations of the integrand in (6.5), say $\Omega(z_{e_{j-1}e_j},\eta_{e_je_{j+1}\dots e_n},\tau)$ with $E=e_1e_2\dots e_n$, the sets in (6.2) and (6.3) have to be adjusted to $Y_{P,Q}^{12\dots n}\to Y_{P,Q}^{E}$ and similarly for $X_P^{\cdot},X_Q^{\cdot}$,

$$\begin{aligned}Z_{\vec\eta}^{i\infty}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|E\big)=&\prod_{i\in Y_{P,Q}^{E}}\frac{\pi}{\sin(\pi\eta_{e_i,e_{i+1}\dots e_n})}\\ &\times\sum_{\substack{X_P^{E}=A\cup B\\ A\cap B=\emptyset}}\Big(\prod_{j\in A}\pi\cot(\pi\eta_{e_j,e_{j+1}\dots e_n})\Big)I^{\mathrm{tree}}(P|\prod_{\bar j\in B}G_{e_{\bar j-1},e_{\bar j}})\\ &\times\sum_{\substack{X_Q^{E}=C\cup D\\ C\cap D=\emptyset}}\Big(\prod_{k\in C}\pi\cot(\pi\eta_{e_k,e_{k+1}\dots e_n})\Big)I^{\mathrm{tree}}(Q|\prod_{\bar k\in D}G_{e_{\bar k-1},e_{\bar k}})\,.\end{aligned} \tag{6.7}$$

In this way, the labels of $G_{e_{\bar{j}-1},e_{\bar{j}}}$ and $G_{e_{\bar{k}-1},e_{\bar{k}}}$ are contained in the first entry P of $I^{\rm tree}(P|*)$ and Q of $I^{\rm tree}(Q|*)$, respectively. Given the general dictionary (5.29) between $I^{\rm tree}(P|\cdot)$ and $(|P|{+}2)$-point disk integrals, (6.7) reduces the initial values for non-planar cycles $\mathcal{C}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}|*\big)$ to products of disk integrals. As an additional simplification in comparison to the planar case, these disk integrals have multiplicities $\leq \max(|P|,|Q|){+}2$ instead of $n{+}2 = |P|{+}|Q|{+}2$.

6.2 Three points

At three points, the general prescription (6.7) for non-planar initial values yields the following two inequivalent cases

$$\begin{aligned} Z^{i\infty}_{\eta_2,\eta_3}\big(\begin{smallmatrix} 3 \\ 1,2 \end{smallmatrix}|1,2,3\big) &= \frac{\pi^2\cot(\pi\eta_{23})}{\sin(\pi\eta_3)} I^{\rm tree}(1,2|1) = \frac{\pi^2\cot(\pi\eta_{23})}{\sin(\pi\eta_3)} \frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2} \\ Z^{i\infty}_{\eta_2,\eta_3}\big(\begin{smallmatrix} 3 \\ 1,2 \end{smallmatrix}|1,3,2\big) &= \frac{\pi^2}{\sin(\pi\eta_{23})\sin(\pi\eta_2)} I^{\rm tree}(1,2|1) = \frac{\pi^2}{\sin(\pi\eta_{23})\sin(\pi\eta_2)} \frac{\Gamma(1-s_{12})}{\big[\Gamma(1-\frac{s_{12}}{2})\big]^2}\,. \end{aligned} \tag{6.8}$$

We have exploited the vanishing of $I^{\rm tree}(1,2|G_{12})$, and the α'-expansion of the gamma functions in terms of Riemann zeta values $\zeta_{m\geq 2}$ can be found in (3.32). The α'-expansion of non-planar A-cycle integrals at three points is completely determined by (4.14) and (6.8). As before, any order in the α'-expansion of the component integrals

$$\begin{aligned} \widehat{Z}^{\tau}_{(m_1,m_2)}\big(\begin{smallmatrix} k \\ i,j \end{smallmatrix}|1,2,3\big) &\equiv q^{-\frac{1}{8}(s_{ik}+s_{jk})} Z^{\tau}_{\eta_2,\eta_3}\big(\begin{smallmatrix} k \\ i,j \end{smallmatrix}|1,2,3\big)\,\big|_{(\eta_2+\eta_3)^{m_1-1}\eta_3^{m_2-1}} \\ &= q^{-\frac{1}{8}(s_{ik}+s_{jk})} \int_{\mathcal{C}\big(\begin{smallmatrix} k \\ i,j \end{smallmatrix}\big)} \mathrm{d}z_2\,\mathrm{d}z_3\, f^{(m_1)}(z_{12},\tau) f^{(m_2)}(z_{23},\tau)\,\mathrm{KN}^{\tau}_{123} \end{aligned} \tag{6.9}$$

can be assembled from a finite number of elementary operations. By the extra factor $q^{-\frac{1}{8}(s_{ik}+s_{jk})}$ in comparison to the planar component integrals (5.40), any α'-order of $\widehat{Z}^{\tau}_{(m_1,m_2)}\big(\begin{smallmatrix} k \\ i,j \end{smallmatrix}|1,2,3\big)$ admits a Fourier expansion w.r.t. q.

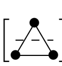
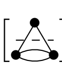
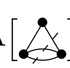
The component integral $\widehat{Z}^{\tau}_{(0,0)}\big(\begin{smallmatrix} 3 \\ 1,2 \end{smallmatrix}|1,2,3\big)$ generates non-planar A-cycle graph functions[23] with two and three vertices [81]

$$\begin{aligned} \widehat{Z}^{\tau}_{(0,0)}\big(\begin{smallmatrix} 3 \\ 1,2 \end{smallmatrix}|1,2,3\big) &= 1 + \frac{1}{2}\big[s_{12}^2 A_2(\tau) + (s_{13}^2{+}s_{23}^2) A_{\underline{2}}(\tau)\big] + \frac{1}{3!}\big[s_{12}^3 A_3(\tau) + (s_{13}^3{+}s_{23}^3) A_{\underline{3}}(\tau)\big] \\ &\quad + s_{12}s_{13}s_{23} A_{1\underline{11}}(\tau) + \frac{1}{2} s_{12}s_{13}s_{23}(s_{13}{+}s_{23}) A_{1\underline{21}}(\tau) + \frac{1}{2} s_{12}^2 s_{13}s_{23} A_{2\underline{11}}(\tau) \\ &\quad + \frac{1}{4}\big[s_{12}^2(s_{13}^2{+}s_{23}^2) A_2(\tau) A_{\underline{2}}(\tau) + s_{13}^2 s_{23}^2 A_{\underline{2}}(\tau)^2\big] + \frac{1}{4!}\big[s_{12}^4 A_4(\tau) + (s_{13}^4{+}s_{23}^4) A_{\underline{4}}(\tau)\big] + \mathcal{O}(\alpha'^5)\,, \end{aligned} \tag{6.10}$$

and the methods of this work have been used to reproduced all examples of

$$A_{i\underline{jk}}(\tau) = i!j!k!\widehat{Z}^{\tau}_{(0,0)}\big(\begin{smallmatrix} 3 \\ 1,2 \end{smallmatrix}|1,2,3\big)\,\big|_{s_{12}^i s_{23}^j s_{13}^k} = 0\,, \qquad i,j,k \geq 1 \tag{6.11}$$

[23] In the graphical notation of [81], the three-vertex graph functions in (6.10) are represented as follows:

$A_{1\underline{11}} = \mathbf{A}\Big[$  $\Big]$ $\qquad A_{2\underline{11}} = \mathbf{A}\Big[$  $\Big]$ $\qquad A_{1\underline{21}} = \mathbf{A}\Big[$  $\Big]$

computed in [81], namely

$$\begin{aligned}
A_{1\underline{11}}(\tau) &= -60\gamma_0(6,0,0|\tau) \\
A_{2\underline{11}}(\tau) &= -144\zeta_4\gamma_0(4,0,0,0|\tau) - 36\gamma_0(4,4,0,0|\tau) + 960\zeta_2\gamma_0(6,0,0,0|\tau) - 756\gamma_0(8,0,0,0|\tau) \\
A_{1\underline{21}}(\tau) &= -144\zeta_4\gamma_0(4,0,0,0|\tau) - 36\gamma_0(4,4,0,0|\tau) - 480\zeta_2\gamma_0(6,0,0,0|\tau) - 756\gamma_0(8,0,0,0|\tau)\,.
\end{aligned} \tag{6.12}$$

Given that the initial value (6.8) is independent on s_{13} and s_{23}, the generating function (6.11) implies that any non-planar three-vertex graph function vanishes at the cusp,

$$A_{i\underline{jk}}(i\infty) = 0\,, \qquad i,j,k \geq 1\,. \tag{6.13}$$

Both the initial values in (6.8) and the three-point derivations (4.13) are even functions under $(\eta_2,\eta_3) \to -(\eta_2,\eta_3)$. Hence, any component integral (6.9) with an integrand of odd weight vanishes at any value of τ,

$$\widehat{Z}^\tau_{(m_1,m_2)}\big(\begin{smallmatrix}3\\1,2\end{smallmatrix}|1,a_2,a_3\big)\Big|_{m_1+m_2 \text{ odd}} = 0\,. \tag{6.14}$$

In particular, there are no kinematic poles in the non-planar component integrals $\widehat{Z}^\tau_{(m_1,m_2)}(\cdot|\cdot)$, as one can also see from the absence of $I^{\rm tree}(*|G_{ij})$ in (6.8). The simplest example of a component integral beyond (6.10) is given by,

$$\begin{aligned}
\widehat{Z}^\tau_{(2,0)}\big(\begin{smallmatrix}3\\1,2\end{smallmatrix}|1,2,3\big) &= -2\zeta_2 + 3s_{12}\gamma_0(4|\tau) - \frac{s_{12}}{2}\zeta_2^2 \\
&\quad + 6(s_{13}^2+s_{23}^2+4s_{13}s_{23}-3s_{12}^2)\zeta_2\gamma_0(4,0|\tau) + 10(s_{12}^2+2s_{13}s_{23})\gamma_0(6,0|\tau) + \mathcal{O}(\alpha'^3)\,,
\end{aligned} \tag{6.15}$$

see (5.45) for its planar analogue $Z^\tau_{(2,0)}(1,2,3|1,2,3)$.

6.3 Four points

At four points, the general prescription (6.7) for initial values yields

$$\begin{aligned}
Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}3,4\\1,2\end{smallmatrix}|1,2,3,4\big) &= \frac{\pi^3\cot(\pi\eta_{234})\cot(\pi\eta_4)}{\sin(\pi\eta_{34})} I^{\rm tree}(1,2|1)I^{\rm tree}(3,4|1) \\
Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}3,4\\1,2\end{smallmatrix}|1,3,2,4\big) &= \frac{\pi^3}{\sin(\pi\eta_{234})\sin(\pi\eta_{24})\sin(\pi\eta_4)} I^{\rm tree}(1,2|1)I^{\rm tree}(3,4|1) \\
Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}3,4\\1,2\end{smallmatrix}|1,3,4,2\big) &= \frac{\pi^3\cot(\pi\eta_{24})}{\sin(\pi\eta_{234})\sin(\pi\eta_2)} I^{\rm tree}(1,2|1)I^{\rm tree}(3,4|1)
\end{aligned} \tag{6.16}$$

in case of a "2+2" cycle $\mathcal{C}\big(\begin{smallmatrix}k,l\\i,j\end{smallmatrix}\big)$ as well as

$$\begin{aligned}
Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}2,3,4\\1\end{smallmatrix}|1,2,3,4\big) &= \frac{\pi}{\sin(\pi\eta_{234})}\Big\{\pi^2\cot(\pi\eta_{34})\cot(\pi\eta_4)I^{\rm tree}(2,3,4|1) \\
&\quad + \pi\cot(\pi\eta_{34})I^{\rm tree}(2,3,4|G_{34}) + \pi\cot(\pi\eta_4)I^{\rm tree}(2,3,4|G_{23}) + I^{\rm tree}(2,3,4|G_{23}G_{34})\Big\}
\end{aligned} \tag{6.17}$$

in case of a "3+1" cycle $\mathcal{C}\big(\begin{smallmatrix} l \\ i,j,k \end{smallmatrix}\big)$. The latter can be further simplified by importing the integration-by-parts relations (5.35) from the planar three-point case,

$$\begin{aligned} Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix} 2,3,4 \\ 1 \end{smallmatrix}|1,2,3,4\big) = \frac{\pi}{\sin(\pi\eta_{234})}\Big\{\Big(\pi^2\cot(\pi\eta_{34})\cot(\pi\eta_4) + \frac{\pi^2 s_{24}}{s_{234}}\Big) I^{\rm tree}(2,3,4|1) \\ + \Big(\pi\cot(\pi\eta_{34}) + \frac{s_{34}}{s_{23}}\pi\cot(\pi\eta_4)\Big) I^{\rm tree}(2,3,4|G_{34})\Big\}\,. \end{aligned} \tag{6.18}$$

The remaining permutations of the integrands follow from (6.16) upon relabeling in $3 \leftrightarrow 4$ and from (6.18) upon relabeling $2,3,4$ on the respective right-hand sides. Non-planar component integrals are defined by

$$\begin{aligned} \widehat Z^\tau_{(m_1,m_2,m_3)}\big(\begin{smallmatrix} k,l \\ i,j \end{smallmatrix}|1,2,3,4\big) &\equiv q^{-\frac{1}{8}(s_{ik}+s_{jk}+s_{il}+s_{jl})} Z^\tau_{\eta_2,\eta_3,\eta_4}\big(\begin{smallmatrix} k,l \\ i,j \end{smallmatrix}|1,2,3,4\big)\,\big|_{\eta_{234}^{m_1-1}\eta_{34}^{m_2-1}\eta_4^{m_3-1}} \\ \widehat Z^\tau_{(m_1,m_2,m_3)}\big(\begin{smallmatrix} l \\ i,j,k \end{smallmatrix}|1,2,3,4\big) &\equiv q^{-\frac{1}{8}(s_{il}+s_{jl}+s_{kl})} Z^\tau_{\eta_2,\eta_3,\eta_4}\big(\begin{smallmatrix} l \\ i,j,k \end{smallmatrix}|1,2,3,4\big)\,\big|_{\eta_{234}^{m_1-1}\eta_{34}^{m_2-1}\eta_4^{m_3-1}}\,, \end{aligned} \tag{6.19}$$

where the extra factors $q^{-\frac{1}{8}s_{ij}}$ in comparison to the planar component integrals (5.54) ensure that any α'-order admits a Fourier expansion w.r.t. q.

The on-shell limits $(s_{13},s_{14},s_{24},s_{34}) \to (-s_{12}-s_{23}, s_{23}, -s_{12}-s_{23}, s_{12})$ of the component integrals $\widehat Z^\tau_{(0,0,0)}$ enter the four-point one-loop amplitude of the open superstring [102], see the discussion around (5.53) for its planar sector. As we will see, the contributions to the non-planar amplitude from the cusp can be related to the above initial values:

- Any component integral on the "2+2 cycle" $\mathcal{C}\big(\begin{smallmatrix} 3,4 \\ 1,2 \end{smallmatrix}\big)$ is proportional to the universal factor of

$$I^{\rm tree}(1,2|1) I^{\rm tree}(3,4|1) = \frac{\Gamma(1-s_{12})\Gamma(1-s_{34})}{\big(\Gamma(1-\frac{s_{12}}{2})\big)^2\big(\Gamma(1-\frac{s_{34}}{2})\big)^2} \tag{6.20}$$

  seen in each line of (6.16). In the on-shell limit $s_{34} \to s_{12}$, this reproduces the $\tau \to i\infty$ behavior of the non-planar "2+2 cycle"-amplitude determined in [104]:

$$I^{\rm tree}(1,2|1) I^{\rm tree}(3,4|1) \to \frac{\big(\Gamma(1-s_{12})\big)^2}{\big(\Gamma(1-\frac{s_{12}}{2})\big)^4} = \frac{2^{-s_{12}}}{\pi}\left(\frac{\Gamma(\frac12-\frac{s_{12}}{2})}{\Gamma(1-\frac{s_{12}}{2})}\right)^2 \tag{6.21}$$

- On a "3+1 cycle", component integrals $\widehat Z^\tau_{(m_1,m_2,m_3)}\big(\begin{smallmatrix} 2,3,4 \\ 1 \end{smallmatrix}|\ast\big)$ at even $m_1{+}m_2{+}m_3 \in 2\mathbb{Z}$ are determined by the odd part of (6.18) w.r.t. $\eta_j \to -\eta_j$ which can be isolated by discarding $I^{\rm tree}(2,3,4|G_{34})$,

$$Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix} 2,3,4 \\ 1 \end{smallmatrix}|1,2,3,4\big)\,\big|^{\textrm{odd in}}_{\eta_j\to-\eta_j} = \frac{\pi^3 I^{\rm tree}(2,3,4|1)}{\sin(\pi\eta_{234})}\Big\{\cot(\pi\eta_{34})\cot(\pi\eta_4) + \frac{s_{24}}{s_{234}}\Big\}\,. \tag{6.22}$$

  Hence, the non-planar "3+1 cycle" amplitude determined by $\widehat Z^\tau_{(0,0,0)}\big(\begin{smallmatrix} 2,3,4 \\ 1 \end{smallmatrix}|1,2,3,4\big)$ is proportional to the on-shell limit of $I^{\rm tree}(2,3,4|1)$. In order to match the known $\tau \to i\infty$ degeneration of the "3+1 cycle" amplitude [103], the on-shell limit $s_{234} \to 0$ should yield

$$I^{\rm tree}(2,3,4|1) \to -\frac{1}{\pi^2}\Big(\frac{\Gamma(-s_{23})\Gamma(-s_{34})}{\Gamma(1{-}s_{23}{-}s_{34})} + \frac{\Gamma(-s_{24})\Gamma(-s_{34})}{\Gamma(1{-}s_{24}{-}s_{34})} + \frac{\Gamma(-s_{23})\Gamma(-s_{24})}{\Gamma(1{-}s_{23}{-}s_{24})}\Big)\Big|_{s_{234}=0}\,. \tag{6.23}$$

This is consistent with the α'-orders in (5.38), and it would be interesting to find a general proof based on the relation (5.33) with five-point disk integrals.

At generic values of τ, the α'-expansions of $\widehat{Z}^{\tau}_{(0,0,0)}\big({}^{2,3,4}_{\ 1} |*\big)$ and $\widehat{Z}^{\tau}_{(0,0,0)}\big({}^{3,4}_{1,2} |*\big)$ generate planar and non-planar A-cycle graph functions [21, 81]. As a consistency check for the differential-equation method of this work, (4.22) with initial values (6.16) and (6.18) has been verified to reproduce all A-cycle graph functions at the orders of $\alpha'^{\leq 4}$ including the simplest instances

$$\begin{aligned} A_{11\underline{11}}(\tau) &= 240\zeta_2\gamma_0(6,0,0,0|\tau) - 840\gamma_0(8,0,0,0|\tau) = A_{1\underline{11}1}(\tau) \\ A_{\underline{1111}}(\tau) &= -480\zeta_2\gamma_0(6,0,0,0|\tau) - 840\gamma_0(8,0,0,0|\tau) \end{aligned} \tag{6.24}$$

of non-planar four-vertex graph functions

$$\begin{aligned} A_{ij\underline{kl}}(\tau) &= 2i!j!k!l!\widehat{Z}^{\tau}_{(0,0,0)}\big({}^{2,3,4}_{\ 1} |1,2,3,4\big)\big|_{s_{23}^i s_{34}^j s_{41}^k s_{12}^l} \\ A_{i\underline{jk}l}(\tau) &= i!j!k!l!\widehat{Z}^{\tau}_{(0,0,0)}\big({}^{3,4}_{1,2} |1,2,3,4\big)\big|_{s_{12}^i s_{23}^j s_{34}^k s_{41}^l} \\ A_{\underline{ijkl}}(\tau) &= i!j!k!l!\widehat{Z}^{\tau}_{(0,0,0)}\big({}^{3,4}_{1,2} |1,2,3,4\big)\big|_{s_{13}^i s_{23}^j s_{24}^k s_{41}^l}\,, \end{aligned} \tag{6.25}$$

see (5.58) and (5.59) for their planar counterparts. One can easily check via (2.24) that the non-planar ($n{\leq}4$)-point A-cycle graph functions in (3.36), (6.12) and (6.25) do not exhibit any negative powers of π in their q-expansion, in lines with the n-point discussion in appendix F.

6.4 Higher points

Non-planar initial values $Z^{i\infty}_{\vec\eta}\big({}^{Q}_{P} |1,2,\ldots,n\big)$ at $n{\geq}5$ points lead to a rapidly growing number of inequivalent cases. Instead of spelling out an exhaustive list of their reduction to $I^{\rm tree}(\cdot|\cdot)$, we content ourselves to representative examples of the general prescription (6.7). At five points, cycles of "3+2" and "4+1" type give rise to initial values such as

$$\begin{aligned} Z^{i\infty}_{\vec\eta}\big({}^{2,4}_{1,3,5} |1,2,3,4,5\big) &= \frac{\pi^4 I^{\rm tree}(1,3,5|1) I^{\rm tree}(2,4|1)}{\sin(\pi\eta_{2345})\sin(\pi\eta_{345})\sin(\pi\eta_{45})\sin(\pi\eta_5)} \\ Z^{i\infty}_{\vec\eta}\big({}^{3}_{1,2,4,5} |1,2,3,4,5\big) &= \frac{\pi^2}{\sin(\pi\eta_{345})\sin(\pi\eta_{45})}\Big\{\pi^2\cot(\pi\eta_{2345})\cot(\pi\eta_5) I^{\rm tree}(1,2,4,5|1) \\ &\quad + \pi\cot(\pi\eta_{2345}) I^{\rm tree}(1,2,4,5|G_{45}) + \pi\cot(\pi\eta_5) I^{\rm tree}(1,2,4,5|G_{12}) + I^{\rm tree}(1,2,4,5|G_{12}G_{45})\Big\}\,, \end{aligned} \tag{6.26}$$

and representative six-point analogues read

$$\begin{aligned} Z^{i\infty}_{\vec\eta}\big({}^{2,5,6}_{1,3,4} |1,2,3,4,5,6\big) &= \frac{\pi^3}{\sin(\pi\eta_{23456})\sin(\pi\eta_{3456})\sin(\pi\eta_{56})} \\ &\quad\times\Big\{\pi\cot(\pi\eta_{456}) I^{\rm tree}(1,3,4|1) + I^{\rm tree}(1,3,4|G_{34})\Big\} \\ &\quad\times\Big\{\pi\cot(\pi\eta_{6}) I^{\rm tree}(2,5,6|1) + I^{\rm tree}(2,5,6|G_{56})\Big\} \\ Z^{i\infty}_{\vec\eta}\big({}^{3,5}_{1,2,4,6} |1,2,3,4,5,6\big) &= \frac{\pi^4\, I^{\rm tree}(3,5|1)}{\sin(\pi\eta_{3456})\sin(\pi\eta_{456})\sin(\pi\eta_{56})\sin(\pi\eta_6)} \\ &\quad\times\Big\{\pi\cot(\pi\eta_{23456}) I^{\rm tree}(1,2,4,6|1) + I^{\rm tree}(1,2,4,6|G_{12})\Big\}\,. \end{aligned} \tag{6.27}$$

Note that even the on-shell limit of $Z^{i\infty}_{\vec{\eta}}\big(\begin{smallmatrix} 3,5 \\ 1,2,4,6 \end{smallmatrix}|*\big)$ (with nine independent Mandelstam invariants instead of fifteen s_{ij}) requires the full expressions for $I^{\rm tree}(1,2,4,6|*)$ in terms of the six Mandelstam invariants s_{ij} with $i,j \in \{1,2,4,6\}$. This is a major motivation for performing the computations in this work without assuming any relations among the s_{ij}. We reiterate that non-planar $Z^{i\infty}_{\vec{\eta}}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}|\cdot\big)$ characterized by $|P|$- and $|Q|$-point cycles can be reduced to disk integrals $Z^{\rm tree}(\cdot|\cdot)$ at multiplicities $\leq \max(|P|,|Q|){+}2$ via (5.29).

Higher-point analogues of the component integrals (6.9) and (6.19) can be generated from

$$\widehat{Z}^{\tau}_{\vec{\eta}}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}|1,2,\ldots,n\big) \equiv \Big(\prod_{i\in P}\prod_{j\in Q} q^{-\frac{1}{8}s_{ij}}\Big) Z^{\tau}_{\vec{\eta}}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}|1,2,\ldots,n\big)\,, \tag{6.28}$$

where the extra factors of $q^{-\frac{1}{8}s_{ij}}$ ensure that each order in α' admits a Fourier expansion w.r.t. q. However, these extra factors modify the differential equation, i.e. the τ-derivative of $\widehat{Z}^{\tau}_{\vec{\eta}}(\cdot|\cdot)$ can no longer be described by a universal differential operator $D^{\tau}_{\vec{\eta}}$ as in (1.4): The cycle-dependent redefinition in (6.28) yields

$$2\pi i \partial_\tau \widehat{Z}^{\tau}_{\vec{\eta}}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}|1,A\big) = \sum_{B\in S_{n-1}} \widehat{D}^{\tau}_{\vec{\eta},P,Q}(A|B)\widehat{Z}^{\tau}_{\vec{\eta}}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}|1,B\big) \tag{6.29}$$

instead of (1.4), with P- and Q-dependent modifications in the diagonal entries,

$$\widehat{D}^{\tau}_{\vec{\eta},P,Q}(A|B) = D^{\tau}_{\vec{\eta}}(A|B) + 3\delta_{A,B}\zeta_2 \sum_{i\in P}\sum_{j\in Q} s_{ij}\,. \tag{6.30}$$

The two-point instance of (6.28) obeys $2\pi i\partial_\tau \widehat{Z}^{\tau}_{\eta_2}\big(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}|1,2\big) = s_{12}\big(\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2,\tau) + \zeta_2\big)\widehat{Z}^{\tau}_{\eta_2}\big(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}|1,2\big)$ with $+\zeta_2$ in place of $-2\zeta_2$ in (3.9).

7 Formal properties

In this section, we take advantage of the new representations and structures of n-point A-cycle integrals to derive some of their formal properties.

7.1 Uniform transcendentality

As one of the central results of this work, we can prove on the basis of the differential-equation method that the α'-expansion of the integrals

$$\int_{\mathcal{C}(*)} \mathrm{d}z_2\,\mathrm{d}z_3\,\ldots\,\mathrm{d}z_n\, f^{(m_1)}_{12} f^{(m_2)}_{23} f^{(m_3)}_{34} \ldots f^{(m_{n-1})}_{n-1,n}\, \mathrm{KN}^{\tau}_{12\ldots n}\,, \qquad m_1,m_2,\ldots,m_{n-1} \geq 0 \tag{7.1}$$

is uniformly transcendental with weight $k{+}m_1{+}m_2{+}\ldots{+}m_{n-1}$ at the order of ${\alpha'}^k$ – for any planar or non-planar integration cycle $\mathcal{C}(P)$ or $\mathcal{C}\big(\begin{smallmatrix} Q \\ P \end{smallmatrix}\big)$. We will derive the equivalent claim

$$\begin{gathered}\text{The } {\alpha'}^k\text{-order of the } A\text{-cycle integrals } Z^{\tau}_{\vec{\eta}}(\cdot|\cdot) \text{ in (1.1)} \\ \text{is uniformly transcendental with weight } k{+}n{-}1.\end{gathered} \tag{7.2}$$

where the η_j are assigned transcendental weight -1. With this choice, the expansion (3.18) of each trigonometric factor $\pi\cot(\pi\eta)$ or $\frac{\pi}{\sin(\pi\eta)}$ carries transcendental weight $+1$. In the subsequent derivation of (7.2), it is convenient to introduce the shorthand

$$\Psi^\tau_{\vec\eta} \equiv \sum_{r=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}} \prod_{j=1}^{r}(k_j-1)\,\gamma(k_1,k_2,\ldots,k_r|\tau) r_{\vec\eta}(\epsilon_{k_r}\epsilon_{k_{r-1}}\ldots\epsilon_{k_2}\epsilon_{k_1}) \tag{7.3}$$

for the path-ordered exponential of $D^\tau_{\vec\eta}$ in the α'-expansion (1.7) of $Z^\tau_{\vec\eta}(\cdot|\cdot)$, i.e. we will determine the transcendentality properties of both ingredients on the right-hand side of

$$Z^\tau_{\vec\eta}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|1,A\big) = \sum_{B\in S_{n-1}} \Psi^\tau_{\vec\eta}(A|B) Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|1,B\big)\,. \tag{7.4}$$

- The expressions (5.21) and (6.7) for the initial values $Z^{i\infty}_{\vec\eta}(A|B)$ and $Z^{i\infty}_{\vec\eta}\big(\begin{smallmatrix}Q\\P\end{smallmatrix}|B\big)$ are linear or quadratic in $I^{\rm tree}(\cdot|\cdot)$. The simplest contributions due to $I^{\rm tree}(\cdot|1)$ with constant integrand involve $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs of overall weight k at the order of ${\alpha'}^k$. This follows from (5.27) and the genus-zero result that the combinations $(2\pi i)^{1-n}Z^{\rm tree}(\cdot|\cdot)$ with $(n{+}2)$-point disk integrals (1.3) are uniformly transcendental with weight k at the order of ${\alpha'}^k$ (and the same is true for the trigonometric entries of $\mathcal{H}_{\alpha'}(A|P)$). Any appearance of $I^{\rm tree}(\cdot|1)$ is accompanied by $n{-}1$ trigonometric factors $\pi\cot(\pi\eta)$ or $\frac{\pi}{\sin(\pi\eta)}$ of weight one each, resulting in overall weight $k{+}n{-}1$ at the order of ${\alpha'}^k$ in the respective contribution to n-point $Z^{i\infty}_{\vec\eta}$.

  The same transcendentality counting holds in presence of $0\leq \ell \leq n{-}1$ factors of G_{ij} in the integrands of $I^{\rm tree}(\cdot|\cdot)$: Each insertion of $G_{ij}\to i\pi\,{\rm sgn}^B_{ij}$ in the expansion (5.29) of $I^{\rm tree}(\cdot|\cdot)$ increases the weight by one, but at the same time suppresses one of the trigonometric weight-one prefactors, cf. (3.16). This leads to the conclusion that

  $$\begin{gathered}\text{Initial values } Z^{i\infty}_{\vec\eta}(\cdot|\cdot) \text{ for planar or non-planar } n\text{-point } A\text{-cycle integrals}\\ \text{are uniformly transcendental with weight } k{+}n{-}1 \text{ at the order of } {\alpha'}^k.\end{gathered} \tag{7.5}$$

- For the τ-dependent factor of $\Psi^\tau_{\vec\eta}(A|B)$ in (7.4), the transcendentality properties are determined by the combinations of $\gamma(k_1,k_2,\ldots,k_r|\tau)r_{\vec\eta}(\epsilon_{k_r}\epsilon_{k_{r-1}}\ldots\epsilon_{k_2}\epsilon_{k_1})$ in (7.3). The transcendental weight of the above iterated Eisenstein integral is $k_1{+}k_2{+}\ldots{+}k_r{-}r$, see section 2.4, and the accompanying derivations are of homogeneity degree $r_{\vec\eta}(\epsilon_{k_j})\sim s_{pq}\eta^{k_j-2}$ if $k_j>0$. The exceptional term $2\zeta_2 s_{12\ldots n}$ in $r_{\vec\eta}(\epsilon_0)$ follows the same transcendentality counting as its remaining terms $\sim s_{ij}\eta^{-2}$ or $s_{ij}\partial^2_\eta$.

  Hence, each product $\gamma(k_1,k_2,\ldots,k_r|\tau)r_{\vec\eta}(\epsilon_{k_r}\epsilon_{k_{r-1}}\ldots\epsilon_{k_2}\epsilon_{k_1})$ contributes transcendental weight $k_1{+}k_2{+}\ldots{+}k_r{-}r$ and $2r{-}k_1{-}k_2{-}\ldots{-}k_r$ from the iterated Eisenstein integrals and the derivations, respectively. The combined transcendental weight r matches the order of α' due to the linearity of each $r_{\vec\eta}(\epsilon_{k_j\geq 0})$ in s_{ij}. As a result,

  $$\begin{gathered}\text{The path-ordered exponentials } \Psi^\tau_{\vec\eta} \text{ in (7.3) are uni-}\\ \text{formly transcendental with weight } k \text{ at the order of } {\alpha'}^k.\end{gathered} \tag{7.6}$$

Uniform transcendentality (7.2) of the A-cycle integrals follows from combining the observations (7.5) and (7.6) on the two ingredients in their α'-expansion (7.4). Therefore, by isolating the coefficient of $\prod_{j=1}^{n-1} \eta_{j+1\ldots n}^{m_j-1}$ in $Z^\tau_{\vec{\eta}}(*|1,2,\ldots,n)$, the α'-expansion of the integral (7.1) over $f_{12}^{(m_1)} f_{23}^{(m_2)} \ldots f_{n-1,n}^{(m_{n-1})}$ is shown to be uniformly transcendental with weight $k{+}m_1{+}m_2{+}\ldots{+}m_{n-1}$ at the order of ${\alpha'}^k$.

7.2 Coaction

As will be shown in this section, A-cycle integrals and in particular their τ-dependent building blocks $\Psi^\tau_{\vec{\eta}}$ in (7.4) are preserved under the coaction. This generalizes the behavior of N-point disk integrals (1.3): The result of applying the motivic coaction Δ [28, 29, 31, 32] order by order to the MZVs in the α'-expansion can be resumed to [4, 30]

$$\Delta Z^{\rm tree}(A|B) = \sum_{C,D\in S_{N-3}} S_0(C|D) Z^{\rm tree}(A|1,C,N,N{-}1) \otimes Z^{\rm tree}_{\mathfrak{dr}}(1,D,N{-}1,N|B)\,, \qquad (7.7)$$

where $S_0(C|D)$ is a local representation[24] of the $(N{-}3)!\times(N{-}3)!$ KLT matrix in field theory [87, 105, 106] and $A,B\in S_N$. Generalizations of the coaction formula to genus-zero integrals with additional fixed punctures have been recently studied in a physics context of Feynman integrals [107–110] and a mathematics context of Lauricella hypergeometric functions [111].

In view of the unsettled transcendentality properties of MZVs, only their motivic versions allow for a well-defined coaction. Motivic MZVs $\zeta^{\mathfrak{m}}_{n_1,n_2,\ldots,n_r}$ are objects in algebraic geometry whose elaborate definition can for instance be found in [7, 29, 31, 32]. All the $\mathbb{Q}$-relations known from MZVs $\zeta_{n_1,n_2,\ldots,n_r}$ (see e.g. [112, 113]) by definition hold for motivic MZVs, and (7.7) is understood to apply to the motivic version of disk integrals, where all the MZVs in their α'-expansion are replaced by the respective $\zeta^{\mathfrak{m}}_{n_1,n_2,\ldots,n_r}$.

The subscript $_{\mathfrak{dr}}$ in the second factor on the right-hand side of (7.7) instructs to replace each MZV in the α'-expansion by the associated de Rham period $\zeta^{\mathfrak{dr}}_{n_1,n_2,\ldots,n_r}$. Loosely speaking, the net effect of passing to de Rham periods is to mod out by ζ_2 or $\zeta_2^{\mathfrak{m}}$, see e.g. [33] for the mathematical background.

The genus-one analogue of (7.7) is based on the following coaction of iterated Eisenstein integrals [33, 34]:

$$\Delta\gamma(k_1,k_2,\ldots,k_r|\tau) = \sum_{j=0}^{r} \gamma(k_1,k_2,\ldots,k_j|\tau) \otimes \mathfrak{S}[\gamma(k_{j+1},\ldots,k_r|\tau)]\,. \qquad (7.8)$$

In the same way as the second entry of the motivic coaction of MZVs involves de Rham periods, the iterated Eisenstein integrals in the second entry of (7.8) are replaced by abstract symbols

$$\mathfrak{S}[\gamma(k_1,k_2,\ldots,k_r|\tau)] = \Big[-\frac{\mathrm{d}\tau}{2\pi i}\mathrm{G}_{k_1}(\tau) \Big| -\frac{\mathrm{d}\tau}{2\pi i}\mathrm{G}_{k_2}(\tau) \Big| \ldots \Big| -\frac{\mathrm{d}\tau}{2\pi i}\mathrm{G}_{k_r}(\tau) \Big]\,, \qquad (7.9)$$

[24] At four and five points, for instance, the KLT matrix is a scalar $S_0(2|2)=-s_{12}$ and a 2×2 matrix with entries such as $S_0(23|23)=s_{12}(s_{13}{+}s_{23})$ and $S_0(23|32)=s_{12}s_{13}$, respectively.

which leads to examples like $\Delta(\tau)=\tau\otimes 1+1\otimes[\mathrm{d}\tau]$ or $\Delta\gamma(k|\tau)=\gamma(k|\tau)\otimes 1+1\otimes\big[-\frac{\mathrm{d}\tau\,\mathrm{G}_k}{2\pi i}\big]$. When applying (7.8) to each term in the α'-expansion of (7.3), we obtain

$$\begin{aligned}\Delta\Psi^\tau_{\vec\eta}(A|B)&=\sum_{r=0}^\infty\sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}}\prod_{j=1}^r(k_j-1)\,r_{\vec\eta}(\epsilon_{k_r}\epsilon_{k_{r-1}}\ldots\epsilon_{k_2}\epsilon_{k_1})_A{}^B\\&\qquad\times\sum_{j=0}^r\gamma(k_1,k_2,\ldots,k_j|\tau)\otimes\mathfrak{S}[\gamma(k_{j+1},\ldots,k_r|\tau)]\\&=\sum_{r=0}^\infty\sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}}\sum_{C\in S_{n-1}}r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})_A{}^C\sum_{s=0}^\infty\sum_{\substack{\ell_1,\ell_2,\ldots,\ell_s\\=0,4,6,8,\ldots}}r_{\vec\eta}(\epsilon_{\ell_s}\ldots\epsilon_{\ell_2}\epsilon_{\ell_1})_C{}^B\\&\qquad\times\prod_{j=1}^s(\ell_j-1)\gamma(\ell_1,\ell_2,\ldots,\ell_s|\tau)\otimes\prod_{j=1}^r(k_j-1)\mathfrak{S}[\gamma(k_1,k_2,\ldots,k_r|\tau)]\,.\end{aligned}\tag{7.10}$$

Since both entries of the tensor product can be recombined to the series-representation of $\Psi^\tau_{\vec\eta}$, one may compactly rewrite (7.10) as

$$\Delta\Psi^\tau_{\vec\eta}(A|B)=\sum_{C\in S_{n-1}}\Psi^\tau_{\vec\eta}(C|B)\otimes\mathfrak{S}\big[\overleftarrow{\Psi}^\tau_{\vec\eta}(A|C)\big]\,.\tag{7.11}$$

The notation $\overleftarrow{\Psi}^\tau_{\vec\eta}$ indicates that the derivations in the second entry (in particular the ∂_{η_j} in the entries of $r_{\vec\eta}(\epsilon_0)$) act from the left on the derivations in the first entry. This reflects the relative positions of the $r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_1})_A{}^C$ and $r_{\vec\eta}(\epsilon_{\ell_s}\ldots\epsilon_{\ell_1})_C{}^B$ in (7.10). The order of the derivations within $\overleftarrow{\Psi}^\tau_{\vec\eta}$ is unchanged in comparison to $\Psi^\tau_{\vec\eta}$.

The coaction of the full A-cycle integrals (7.4) requires an extension of (7.11) to also incorporate the MZVs of the initial values $Z^{i\infty}_{\vec\eta}(\cdot|\cdot)$. In the planar case, their decomposition (5.1) into disk integrals and the genus-zero coaction (7.7) imply a result of the schematic form

$$\Delta Z^{i\infty}_{\vec\eta}(1,A|1,P)=\sum_{B,C\in S_{n-1}}T_{\vec\eta}(B|C)Z^{i\infty}_{\vec\eta}(1,A|1,B)\otimes Z^{i\infty}_{\vec\eta,\mathfrak{dr}}(1,C|1,P)\,.\tag{7.12}$$

The $(n-1)!\times(n-1)!$ matrix $T_{\vec\eta}(B|C)$ may be reconstructed from the KLT matrix S_0 as well as the objects $\mathcal{H}_{\alpha'},\mathcal{K}_{\vec\eta}$ in (5.1). It depends on η_j,s_{ij} and does not affect the second entry through its coaction $\Delta T_{\vec\eta}(B|C)=T_{\vec\eta}(B|C)\otimes 1$ since it comprises no MZVs other than powers of π. On these grounds, one can combine (7.11) and (7.12) to find

$$\begin{aligned}\Delta Z^\tau_{\vec\eta}(1,A|1,B)&=\sum_{C\in S_{n-1}}\Delta\Psi^\tau_{\vec\eta}(B|C)\cdot\Delta Z^{i\infty}_{\vec\eta}(1,A|1,C)\\&=\sum_{C,D,E,F\in S_{n-1}}\Psi^\tau_{\vec\eta}(D|C)T_{\vec\eta}(E|F)Z^{i\infty}_{\vec\eta}(1,A|1,E)\otimes\mathfrak{S}\big[\overleftarrow{\Psi}^\tau_{\vec\eta}(B|D)\big]Z^{i\infty}_{\vec\eta,\mathfrak{dr}}(1,F|1,C)\\&=\sum_{C,D,E,F,G\in S_{n-1}}\Psi^\tau_{\vec\eta}(D|C)T_{\vec\eta}(E|F)\big[\Psi^\tau_{\vec\eta}(E|G)^{-1}Z^\tau_{\vec\eta}(1,A|1,G)\big]\\&\qquad\qquad\otimes\mathfrak{S}\big[\overleftarrow{\Psi}^\tau_{\vec\eta}(B|D)\big]Z^{i\infty}_{\vec\eta,\mathfrak{dr}}(1,F|1,C)\,,\end{aligned}\tag{7.13}$$

where the inverse $(\Psi^\tau_{\vec{\eta}})^{-1}$ of the path-ordered exponential features an expansion with alternating signs $(-1)^r$ and a reversed order of composing the derivations in comparison to $\Psi^\tau_{\vec{\eta}}$:

$$\Psi^\tau_{\vec{\eta}}(A|B)^{-1} = \sum_{r=0}^{\infty}(-1)^r \sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}} \prod_{j=1}^{r}(k_j-1)\,\gamma(k_1,k_2,\ldots,k_r|\tau) r_{\vec{\eta}}(\epsilon_{k_1}\epsilon_{k_2}\ldots\epsilon_{k_r})_A{}^B\,. \tag{7.14}$$

The derivations in the object $\mathfrak{S}\big[\overleftarrow{\Psi}{}^\tau_{\vec{\eta}}(B|D)\big]$ in the second entry of (7.13) are understood to act from the left on all the functions of η_j in the first entry as well as $Z^{i\infty}_{\vec{\eta},\mathfrak{dr}}(1,F|1,C)$ in the second entry. Similarly, the derivations in $\Psi^\tau_{\vec{\eta}}(D|C)$ act on all the η_j in $T_{\vec{\eta}}(E|F)$, $Z^{i\infty}_{\vec{\eta},\mathfrak{dr}}(1,F|1,C)$ and $Z^{i\infty}_{\vec{\eta}}(1,A|1,E)$ (or equivalently $\sum_{G\in S_{n-1}}\Psi^\tau_{\vec{\eta}}(E|G)^{-1}Z^\tau_{\vec{\eta}}(1,A|1,G)$). However, as indicated by $[\ldots]$ in the third line of (7.13), the scope of the derivations in $\Psi^\tau_{\vec{\eta}}(E|G)^{-1}$ is limited to the factor $Z^\tau_{\vec{\eta}}(1,A|1,G)$ enclosed by the bracket.

As a bottom line of (7.13), the coaction of the planar A-cycle integral $Z^\tau_{\vec{\eta}}(1,A|1,B)$ is expressible via linear operations acting on an $(n-1)!$ basis of A-cycle integrals $Z^\tau_{\vec{\eta}}(1,A|1,G)$, $G\in S_{n-1}$ with the same integration cycle $\mathcal{C}(1,A)$. Said linear operations include infinite series in $\gamma(k_1,\ldots,k_r|\tau)$ and η_j-derivatives that enter through the factor of $\Psi^\tau_{\vec{\eta}}$ and its inverse.

The same kind of conclusion holds for the coaction of non-planar A-cycle integrals $Z^\tau_{\vec{\eta}}(\genfrac{}{}{0pt}{}{Q}{P}|\cdot)$: Given that the initial values $Z^{i\infty}_{\vec{\eta}}(\genfrac{}{}{0pt}{}{Q}{P}|\cdot)$ are expressible via products of disk integrals, see (6.7) and (5.29), their coaction is still determined by (7.7). With a bit of additional bookkeeping effort to track the information of P and Q, one can adjust the details of (7.12) and (7.13) to the non-planar setting.

8 Conclusions

In this work, we have studied iterated integrals over A-cycles on a torus that generate the contributions of a cylinder or Möbius-strip surface to n-point one-loop open-string amplitudes. These A-cycle integrals were shown to satisfy linear and homogeneous first-order differential equations w.r.t. the modular parameter τ. Moreover, their degeneration as $\tau\to i\infty$ is reduced to explicitly known combinations of genus-zero integrals from open-string tree amplitudes.

The solution to this initial-value problem via standard Picard iteration exposes the structure of the α'-expansion of the A-cycle integrals, see (1.7): Any order in α' is expressible via iterated Eisenstein integrals and MZVs whose composition follows from elementary operations – matrix multiplications and differentiation in auxiliary variables η_j. Then, by isolating specific orders in the Laurent expansion w.r.t. $\eta_2,\eta_3,\ldots,\eta_n$, one recovers the α'-expansions relevant to one-loop open-string amplitudes.

The form of the differential equations is universal to any integration cycle in planar- and non-planar one-loop amplitudes and shared by the A-cycle component of the elliptic KZB associator [36–38]. The differential equations of n-point A-cycle integrals induce conjectural $(n-1)!\times(n-1)!$ matrix representations of Tsunogai's derivations [39], reflecting the fact that the iterated Eisenstein integrals in the α'-expansion correspond to eMZVs [20, 22, 25]. In particular, the appearance of twisted eMZVs in the α'-expansion of non-planar cylinder integrals [21] is completely bypassed, corroborating and generalizing the four-point observations of the reference.

The homogeneity of the differential equations is tied to considering a collection of $(n-1)!$ Kronecker–Eisenstein-type integrands. The latter are believed to span the twisted cohomology at genus one, i.e. to generalize the basis of $(n-3)!$ Parke–Taylor factors for the integration-by-parts inequivalent genus-zero integrands [40]. Accordingly, the A-cycle integrals $Z^{\tau}_{\vec{\eta}}(\cdot|\cdot)$ are proposed to be the genus-one generalization of the disk- or Z-integrals that are universal to open-string tree amplitudes [41–44]. The α'-expansion of Z-integrals at genus zero can be interpreted in terms of effective-field-theory amplitudes of (bi-)colored scalars [99, 114] and allows for Berends–Giele recursions [24]. Hence, it would be interesting to investigate a genus-one echo of these features at the level of A-cycle integrals $Z^{\tau}_{\vec{\eta}}(\cdot|\cdot)$.

The differential equations of A-cycle integrals presented in this work suggest a variety of follow-up investigations:

- The techniques of this work call for an extension to closed strings. As will be demonstrated elsewhere [115], similar methods can be used to derive Cauchy-Riemann and Laplace equations for modular graph forms (see e.g. [92, 116–120]) at the level of generating series. Closed-string differential equations of this type should harbor valuable input on relations between open- and closed-string one loop amplitudes through an elliptic single-valued map [50, 81, 94]. These research directions are hoped to clarify the role the non-holomorphic modular forms and single-valued iterated Eisenstein integrals of Brown [121, 122] in closed-string α'-expansions.

- The solution (1.7) to the differential equations of A-cycle integrals only depends on the integration cycle through the initial value at $\tau \to i\infty$. Hence, relations between A-cycle integrals for different planar and non-planar orderings should be encoded in the genus-zero information at the cusp. It would be interesting connect the monodromy relations among open-string tree-level amplitudes [88, 89] with their loop-level extensions [104, 123] along these lines.

- In the final expressions for one-loop open-string amplitudes, A-cycle integrals are integrated over the modular parameter τ, at least at the level of individual orders in their expansion w.r.t. η_j. The differential equations in this work should be instrumental for performing the desired τ-integrals, for instance by identifying various contributions as total derivatives in τ. Also, it would be interesting to relate techniques for τ-integrations in open-string amplitudes with recent progress on the closed-string side [51, 91, 116, 124].

- The strategy of this work to infer α'-expansions from the solution of an initial-value problem should be universally applicable at various genera. Similar linear and homogeneous differential equations in the complex-structure moduli are expected to yield recursions for moduli-space integrals in string amplitudes w.r.t. the loop order. This is analogous to the recursions for disk integrals w.r.t. the number of legs that descend from KZ equations in a puncture on a genus-zero surface [5].

At genus two, for instance, a promising intermediate goal is to find differential equations for moduli-space integrals w.r.t. off-diagonal entry of the 2×2 period matrix. The separating degeneration of the surface could then provide an initial value, where the dependence on the diagonal entries of the period matrix enter via products of genus-one integrals. We hope that these directions contribute to understanding the structure & explicit examples of α'-expansions beyond one loop and to advancing the study of modular graph functions at higher genus [125, 126].

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A Resolving cycles of Kronecker–Eisenstein series

This appendix is dedicated to a series of lemmata to simplify cyclic products of Kronecker–Eisenstein series. Many steps in the subsequent calculations are based on the meromorphic Kronecker–Eisenstein series

$$F(z,\eta,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}\,, \tag{A.1}$$

which does not share the double-periodicity of its non-holomorphic counterpart (2.2). Its expansion coefficient at the order of η^0 is denoted by $g^{(1)}(z,\tau)$ and satisfies

$$g^{(1)}(z,\tau) = \partial_z \log\theta_1(z,\tau)\,, \qquad \partial_z g^{(1)}(z,\tau) = -\wp(z,\tau) - \mathrm{G}_2(\tau)\,. \tag{A.2}$$

A.1 Two points

We start by proving the two-point identity[25] (3.5) which plays a key role in the derivation of the differential equation (3.9) of two-point A-cycle integrals. The starting point is the

[25] This identity is equivalent to the identities (3.3) from [79] and (20) from [22].

following version of the Fay identity (2.6):

$$F(z_{12},\eta_2,\tau)F(z_{21}{+}\varepsilon,\xi,\tau) = F(\varepsilon,\eta_2,\tau)F(z_{21}{+}\varepsilon,\xi{-}\eta_2,\tau) + F(\varepsilon,\xi,\tau)F(z_{12},\eta_2{-}\xi,\tau)\,. \quad \text{(A.3)}$$

Then, we evaluate the limit $\varepsilon \to 0$ based on $F(\varepsilon,\eta,\tau) = \frac{1}{\varepsilon} + g^{(1)}(\eta,\tau) + \mathcal{O}(\varepsilon)$, where $\frac{1}{\varepsilon}F(z_{21}{+}\varepsilon,\xi{-}\eta_2,\tau)$ introduces the z-derivative $\partial_z F(z_{21},\xi{-}\eta_2)$ at the zero$^{\rm th}$ order in ε:

$$F(z_{12},\eta_2,\tau)F(z_{21},\xi,\tau) = F(z_{12},\eta_2{-}\xi,\tau)\big(g^{(1)}(\xi,\tau) - g^{(1)}(\eta_2,\tau)\big) + \partial_z F(z_{21},\xi{-}\eta_2)\,. \quad \text{(A.4)}$$

The next step is to apply the differential operator $(\partial_{\eta_2} + \partial_\xi)$ to both sides, which does not affect functions of $\eta_2{-}\xi$ and only needs to be applied to the $g^{(1)}$:

$$(\partial_{\eta_2} + \partial_\xi)F(z_{12},\eta_2,\tau)F(z_{21},\xi,\tau) = \big(\wp(\eta_2,\tau) - \wp(\xi,\tau)\big)F(z_{12},\eta_2{-}\xi,\tau)\,. \quad \text{(A.5)}$$

Note the cancellation of G_2 from $(\partial_{\eta_2}{+}\partial_\xi)(g^{(1)}(\xi,\tau){-}g^{(1)}(\eta_2,\tau)) = \wp(\eta_2,\tau){-}\wp(\xi,\tau)$, cf. (A.2). Finally, we arrive at (3.5) by multiplying both sides of (A.5) by $\exp(2\pi i(\eta_2{-}\xi)\frac{\operatorname{Im} z_{12}}{\operatorname{Im}\tau})$. This exponential commutes with $(\partial_{\eta_2}{+}\partial_\xi)$ and promotes all instances of the meromorphic Kronecker–Eisenstein series (A.1) to the doubly-periodic one (2.2).

When applied to (3.7), corollaries of (A.5) include $f^{(2)}_{12}f^{(2)}_{12} - 2f^{(3)}_{12}f^{(1)}_{12} = 3\mathrm{G}_4 - 2f^{(4)}_{12}$ as well as

$$\begin{aligned}
f^{(3)}_{12}f^{(2)}_{12} - 3f^{(4)}_{12}f^{(1)}_{12} &= 3\mathrm{G}_4 f^{(1)}_{12} - 5f^{(5)}_{12}\\
f^{(4)}_{12}f^{(2)}_{12} - 4f^{(5)}_{12}f^{(1)}_{12} &= 3\mathrm{G}_4 f^{(2)}_{12} + 5\mathrm{G}_6 - 9f^{(6)}_{12}\\
f^{(5)}_{12}f^{(2)}_{12} - 5f^{(6)}_{12}f^{(1)}_{12} &= 3\mathrm{G}_4 f^{(3)}_{12} + 5\mathrm{G}_6 f^{(1)}_{12} - 14f^{(7)}_{12}\\
f^{(6)}_{12}f^{(2)}_{12} - 6f^{(7)}_{12}f^{(1)}_{12} &= 3\mathrm{G}_4 f^{(4)}_{12} + 5\mathrm{G}_6 f^{(2)}_{12} + 7\mathrm{G}_8 - 20f^{(8)}_{12}
\end{aligned} \quad \text{(A.6)}$$

and more generally ($n \in \mathbb{N}_0$)

$$f^{(n)}_{12}f^{(2)}_{12} - nf^{(n+1)}_{12}f^{(1)}_{12} = \sum_{k=4}^{n+2}(k{-}1)\mathrm{G}_k f^{(n+2-k)}_{12} - \frac{1}{2}(n{-}1)(n{+}2)f^{(n+2)}_{12}\,. \quad \text{(A.7)}$$

A.2 Three points

The identity (A.4) to resolve a two-cycle of Kronecker–Eisenstein series can be generalized to longer cycles of the form $F(z_{12},\beta_1,\tau)F(z_{23},\beta_2,\tau)\ldots F(z_{n-1,n},\beta_{n-1},\tau)F(z_{n,1},\xi,\tau)$. Three-cycles can be reduced to (A.4) by applying the Fay identity (2.6) in a first step,

$$\begin{aligned}
&F(z_{12},\beta_1,\tau)F(z_{23},\beta_2,\tau)F(z_{31},\xi,\tau) \quad \text{(A.8)}\\
&\quad = \big[F(z_{13},\beta_1,\tau)F(z_{23},\beta_2{-}\beta_1,\tau) + F(z_{13},\beta_2,\tau)F(z_{12},\beta_1{-}\beta_2,\tau)\big]F(z_{31},\xi,\tau)\\
&\quad = F(z_{23},\beta_2{-}\beta_1,\tau)\big[F(z_{13},\beta_1{-}\xi,\tau)\big(g^{(1)}(\xi,\tau) - g^{(1)}(\beta_1,\tau)\big) + \partial_z F(z_{31},\xi{-}\beta_1)\big]\\
&\qquad + F(z_{12},\beta_1{-}\beta_2,\tau)\big[F(z_{13},\beta_2{-}\xi,\tau)\big(g^{(1)}(\xi,\tau) - g^{(1)}(\beta_2,\tau)\big) + \partial_z F(z_{31},\xi{-}\beta_2)\big]\,.
\end{aligned}$$

When acting with $(\partial_{\beta_1}+\partial_{\beta_2}+\partial_\xi)$, the contributions from $\partial_z F(\ldots)$ again vanish:

$$
\begin{aligned}
&(\partial_{\beta_1}+\partial_{\beta_2}+\partial_\xi)F(z_{12},\beta_1,\tau)F(z_{23},\beta_2,\tau)F(z_{31},\xi,\tau) \\
&\quad = F(z_{13},\beta_1{-}\xi,\tau)F(z_{23},\beta_2{-}\beta_1,\tau)\big(\wp(\beta_1,\tau)-\wp(\xi,\tau)\big) \\
&\qquad + F(z_{12},\beta_1{-}\beta_2,\tau)F(z_{13},\beta_2{-}\xi,\tau)\big(\wp(\beta_2,\tau)-\wp(\xi,\tau)\big) \\
&\quad = -\wp(\beta_1,\tau)F(z_{23},\beta_2{-}\beta_1,\tau)F(z_{31},\xi{-}\beta_1,\tau) \\
&\qquad - \wp(\beta_2,\tau)F(z_{12},\beta_1{-}\beta_2,\tau)F(z_{31},\xi{-}\beta_2,\tau) \\
&\qquad - \wp(\xi,\tau)F(z_{12},\beta_1{-}\xi,\tau)F(z_{23},\beta_2{-}\xi,\tau)\,.
\end{aligned} \tag{A.9}
$$

The Fay identity has been used in the last step to manifest the cyclic symmetry of the result under simultaneous exchange of z_j and β_1,β_2,ξ. Multiplication of both sides by $\exp(\frac{2\pi i}{\mathrm{Im}\,\tau}(\beta_1 z_{12}+\beta_2 z_{23}+\xi z_{31}))$ which is inert under the combination $\partial_{\beta_1}{+}\partial_{\beta_2}{+}\partial_\xi$ replaces the $F(\ldots)$ in (A.9) by $\Omega(\ldots)$ and implies the lemma (4.6).

A.3 Higher points

The methods of the previous subsections can be straightforwardly uplifted to simplify the four-cycle analogue of (A.9),

$$
\begin{aligned}
&(\partial_{\beta_1}+\partial_{\beta_2}+\partial_{\beta_3}+\partial_\xi)F(z_{12},\beta_1,\tau)F(z_{23},\beta_2,\tau)F(z_{34},\beta_3,\tau)F(z_{41},\xi,\tau) \\
&\quad = -\wp(\beta_1,\tau)F(z_{23},\beta_2{-}\beta_1,\tau)F(z_{34},\beta_3{-}\beta_1,\tau)F(z_{41},\xi{-}\beta_1,\tau) \\
&\qquad - \wp(\beta_2,\tau)F(z_{12},\beta_1{-}\beta_2,\tau)F(z_{34},\beta_3{-}\beta_2,\tau)F(z_{41},\xi{-}\beta_2,\tau) \\
&\qquad - \wp(\beta_3,\tau)F(z_{12},\beta_1{-}\beta_3,\tau)F(z_{23},\beta_2{-}\beta_3,\tau)F(z_{41},\xi{-}\beta_3,\tau) \\
&\qquad - \wp(\xi,\tau)F(z_{12},\beta_1{-}\xi,\tau)F(z_{23},\beta_2{-}\xi,\tau)F(z_{34},\beta_3{-}\xi,\tau)\,.
\end{aligned} \tag{A.10}
$$

Upon multiplication by $\exp(\frac{2\pi i}{\mathrm{Im}\,\tau}(\beta_1 z_{12}+\beta_2 z_{23}+\beta_3 z_{34}+\xi z_{41}))$, one arrives at the analogous statement (4.17) for a cycle of $\Omega(\ldots)$. The same logic can be repeated at higher multiplicity and is expected to yield the cyclic result

$$
\begin{aligned}
&(\partial_{\beta_1}{+}\partial_{\beta_2}{+}\ldots{+}\partial_{\beta_{n-1}}{+}\partial_\xi)F(z_{12},\beta_1,\tau)F(z_{23},\beta_2,\tau)\ldots F(z_{n-1,n},\beta_{n-1},\tau)F(z_{n,1},\xi,\tau) \\
&\quad = -\sum_{j=1}^{n-1}\wp(\beta_j,\tau)F(z_{n,1},\xi{-}\beta_j,\tau)\prod_{\substack{i=1\\ i\neq j}}^{n-1}F(z_{i,i+1},\beta_i{-}\beta_j,\tau) - \wp(\xi,\tau)\prod_{i=1}^{n-1}F(z_{i,i+1},\beta_i{-}\xi,\tau)\,,
\end{aligned} \tag{A.11}
$$

which is tested up to and including $n=5$ and conjectural at $n\geq 6$. Note that both sides have the same poles in $\xi,\beta_1,\beta_2,\ldots,\beta_{n-1}$ and monodromies as $z_j\to z_j{+}\tau$. It would be interesting to prove (A.11) by induction.

As before, multiplication by $\exp(\frac{2\pi i}{\mathrm{Im}\,\tau}(\sum_{j=1}^{n-1}\beta_j z_{j,j+1}+\xi z_{n,1}))$ promotes all the meromorphic Kronecker–Eisenstein series in (A.11) to the doubly-periodic ones,

$$
\begin{aligned}
&(\partial_{\beta_1}{+}\partial_{\beta_2}{+}\ldots{+}\partial_{\beta_{n-1}}{+}\partial_\xi)\Omega(z_{12},\beta_1,\tau)\Omega(z_{23},\beta_2,\tau)\ldots\Omega(z_{n-1,n},\beta_{n-1},\tau)\Omega(z_{n,1},\xi,\tau) \\
&\quad = -\sum_{j=1}^{n-1}\wp(\beta_j,\tau)\Omega(z_{n,1},\xi{-}\beta_j,\tau)\prod_{\substack{i=1\\ i\neq j}}^{n-1}\Omega(z_{i,i+1},\beta_i{-}\beta_j,\tau) - \wp(\xi,\tau)\prod_{i=1}^{n-1}\Omega(z_{i,i+1},\beta_i{-}\xi,\tau)\,.
\end{aligned} \tag{A.12}
$$

B The non-planar Green function at the cusp

The purpose of this appendix is to justify the regularized value (3.20), i.e. the vanishing of the non-planar Green function $\mathcal{G}(v_{ij}+\tau/2,\tau)$ at the cusp. In order to do so, we first establish a result that does not require any regularization:

$$\lim_{\tau\to i\infty}\left[\mathcal{G}(v_{ij}+\tau/2,\tau)-\frac{i\pi\tau}{4}\right]=0\,. \tag{B.1}$$

The vanishing of this limit follows from two observations on the difference $\mathcal{G}(v_{ij}+\tau/2,\tau)-\frac{i\pi\tau}{4}$:

- Its derivatives (2.16) w.r.t. v_i and τ vanish at the cusp since $\lim_{\tau\to i\infty} f^{(1)}(z-\tau/2,\tau)=0$ and $\lim_{\tau\to i\infty} f^{(2)}(z-\tau/2,\tau)=\zeta_2$. Hence, the right-hand side of (B.1) must be a constant.
- The representation (2.15) of $\mathcal{G}(v_{ij}+\tau/2,\tau)$ in terms of elliptic iterated integrals implies that $\int_0^1 \mathrm{d}v_j\left[\mathcal{G}(v_{ij}+\tau/2,\tau)-\frac{i\pi\tau}{4}\right]=0$, so the constant in the previous step must be zero.

The degeneration of the non-planar Green function itself (rather than $\mathcal{G}(v_{ij}+\tau/2,\tau)-\frac{i\pi\tau}{4}$) can be obtained from the regularized value $\lim_{\tau\to i\infty}\tau=0$. This choice lines up with the net effect $\log q=\int_0^q\frac{\mathrm{d}q_1}{q_1}=2\pi i\lim_{\tau'\to i\infty}(\tau-\tau')$ of the tangential-base-point regularization [84] quoted below (2.20). This assigns a vanishing degeneration to both terms in (B.1), and we arrive at the desired result (3.20).

C More on the τ-derivatives of A-cycle integrals

C.1 The 6×6 representation of the derivations at four points

In this appendix, we complete the expression (4.20) for the 6×6 representation of the derivations due to the four-point A-cycle integrals in (4.15). While the matrices $r_{\vec\eta}(e_{234}), r_{\vec\eta}(e_{34})$ and $r_{\vec\eta}(e_4)$ can be found in (4.21), we still have to supply the expressions for $r_{\vec\eta}(e_{23}), r_{\vec\eta}(e_{24}), r_{\vec\eta}(e_2)$ and $r_{\vec\eta}(e_3)$: As one can reconstruct from permutations of (4.19) in $2,3,4$, they are given by

$$\begin{aligned}
r_{\vec\eta}(e_{23})&=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\s_{12}&s_{12}&-s_{13}&-s_{13}&s_{12}+s_{24}&-s_{13}-s_{34}\\-s_{12}&-s_{12}&s_{13}&s_{13}&-s_{12}-s_{24}&s_{13}+s_{34}\end{pmatrix}\\
r_{\vec\eta}(e_{24})&=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\s_{12}&s_{12}&s_{12}+s_{23}&-s_{14}-s_{34}&-s_{14}&-s_{14}\\-s_{12}&-s_{12}&-s_{12}-s_{23}&s_{14}+s_{34}&s_{14}&s_{14}\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}
\end{aligned}\tag{C.1}$$

as well as

$$r_{\vec{\eta}}(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_{12} & 0 & s_{12}{+}s_{23} & s_{12}{+}s_{23}{+}s_{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{12} & 0 & 0 & s_{12}{+}s_{24} & s_{12}{+}s_{23}{+}s_{24} \end{pmatrix}$$

$$r_{\vec{\eta}}(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ s_{13}{+}s_{23} & s_{13}{+}s_{23}{+}s_{34} & s_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{13} & s_{13}{+}s_{23}{+}s_{34} & s_{13}{+}s_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.2}$$

C.2 The τ-derivative at five points in a 24-element basis

The purpose of this appendix is to give the minimal form of (4.26), where all the integrals $Z^\tau_{\vec{\eta}}(*|i,j,k,l,m)$ on the right-hand side are in the 24-element basis with $i=1$:

$$\begin{aligned}
2\pi i\partial_\tau Z^\tau_{\vec{\eta}}(*|1,2,3,4,5) = \Big(&\frac{1}{2}\sum_{j=2}^{5} s_{1j}\partial^2_{\eta_j} + \frac{1}{2}\sum_{2\leq i<j}^{5} s_{ij}(\partial_{\eta_i}{-}\partial_{\eta_j})^2 \\
&- s_{12}\wp(\eta_{2345},\tau) - (s_{13}{+}s_{23})\wp(\eta_{345},\tau) - (s_{14}{+}s_{24}{+}s_{34})\wp(\eta_{45},\tau) \\
&- (s_{15}{+}s_{25}{+}s_{35}{+}s_{45})\wp(\eta_5,\tau) - 2\zeta_2 s_{12345}\Big) Z^\tau_{\vec{\eta}}(*|1,2,3,4,5) \\
+ s_{13}&\big[\wp(\eta_{2345},\tau) - \wp(\eta_{345},\tau)\big]\big(Z^\tau_{\vec{\eta}}(*|1,3,2,4,5) + Z^\tau_{\vec{\eta}}(*|1,3,4,2,5) + Z^\tau_{\vec{\eta}}(*|1,3,4,5,2)\big) \\
+ s_{24}&\big[\wp(\eta_{345},\tau) - \wp(\eta_{45},\tau)\big]\big(Z^\tau_{\vec{\eta}}(*|1,2,4,3,5) + Z^\tau_{\vec{\eta}}(*|1,2,4,5,3)\big) \\
+ s_{35}&\big[\wp(\eta_{45},\tau) - \wp(\eta_5,\tau)\big] Z^\tau_{\vec{\eta}}(*|1,2,3,5,4) \\
+ s_{14}&\big[\wp(\eta_{345},\tau) - \wp(\eta_{2345},\tau)\big]\big(Z^\tau_{\vec{\eta}}(*|1,4,3,2,5) + Z^\tau_{\vec{\eta}}(*|1,4,3,5,2) + Z^\tau_{\vec{\eta}}(*|1,4,5,3,2)\big) \\
+ s_{14}&\big[\wp(\eta_{345},\tau) - \wp(\eta_{45},\tau)\big]\big(Z^\tau_{\vec{\eta}}(*|1,2,4,3,5) + Z^\tau_{\vec{\eta}}(*|1,2,4,5,3) \\
&+ Z^\tau_{\vec{\eta}}(*|1,4,2,3,5) + Z^\tau_{\vec{\eta}}(*|1,4,2,5,3) + Z^\tau_{\vec{\eta}}(*|1,4,5,2,3)\big) \\
+ s_{25}&\big[\wp(\eta_{45},\tau) - \wp(\eta_{345},\tau)\big] Z^\tau_{\vec{\eta}}(*|1,2,5,4,3) \\
+ s_{25}&\big[\wp(\eta_{45},\tau) - \wp(\eta_5,\tau)\big]\big(Z^\tau_{\vec{\eta}}(*|1,2,3,5,4) + Z^\tau_{\vec{\eta}}(*|1,2,5,3,4)\big) \\
+ s_{15}&\big[\wp(\eta_{2345},\tau) - \wp(\eta_{345},\tau)\big] Z^\tau_{\vec{\eta}}(*|1,5,4,3,2) \\
+ s_{15}&\big[\wp(\eta_{45},\tau) - \wp(\eta_{345},\tau)\big]\big(Z^\tau_{\vec{\eta}}(*|1,2,5,4,3) + Z^\tau_{\vec{\eta}}(*|1,5,2,4,3) + Z^\tau_{\vec{\eta}}(*|1,5,4,2,3)\big) \\
+ s_{15}&\big[\wp(\eta_{45},\tau) - \wp(\eta_5,\tau)\big]\big(Z^\tau_{\vec{\eta}}(*|1,2,3,5,4) + Z^\tau_{\vec{\eta}}(*|1,2,5,3,4) + Z^\tau_{\vec{\eta}}(*|1,5,2,3,4)\big).
\end{aligned} \tag{C.3}$$

D Transformation matrices between twisted cycles

In this appendix, we give more details on the transformation matrices $\mathcal{H}_{\alpha'}(A|B)$ between planar genus-one cycles and disk orderings $\mathcal{D}_B$ in (5.4). We follow the slightly abusive notation in the main text and relate twisted cycles without explicitly referring to the degenerate Koba–Nielsen factor (5.2).

D.1 Four-point example

Given the definition of $\mathcal{H}_{\alpha'}(\cdot|\cdot)$ via $\mathcal{C}(1,A) = \sum_{B\in S_{n-1}} \mathcal{H}_{\alpha'}(A|B)\mathcal{D}_B$, the four-point relation (5.6) allows to read off

$$\begin{aligned}
\mathcal{H}_{\alpha'}(2,3,4|2,3,4) &= 2i\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24})\Big)\, e^{\frac{i\pi}{2}s_{134}}\\
\mathcal{H}_{\alpha'}(2,3,4|3,2,4) &= -2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\, e^{\frac{i\pi}{2}(-s_{12}+s_{14}-s_{24})}\\
\mathcal{H}_{\alpha'}(2,3,4|3,4,2) &= -2i\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\, e^{\frac{i\pi}{2}(-s_{12}+s_{14}+s_{24})}\\
\mathcal{H}_{\alpha'}(2,3,4|4,3,2) &= 2i\sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34})\Big)\, e^{-\frac{i\pi}{2}s_{123}}\\
\mathcal{H}_{\alpha'}(2,3,4|2,4,3) &= \mathcal{H}_{\alpha'}(2,3,4|4,2,3) = 0\,.
\end{aligned} \tag{D.1}$$

The remaining rows of $\mathcal{H}_{\alpha'}(A|B)$ are obtained by relabeling $2,3,4$ in s_{ij}, A and B.

D.2 Weighted combinations

Depending on the number of G_{ij} insertions in the integrands of (5.29), one encounters combinations $\mathcal{H}_{\alpha'}(A|B) \pm \mathcal{H}_{\alpha'}(A^t|B)$ with the reversed genus-one cycle $\mathcal{C}(1,A^t)$. The weighted three-point cycles are related by

$$\begin{aligned}
\frac{1}{2}\big[\mathcal{C}(1,2,3)+\mathcal{C}(1,3,2)\big] &= -2\Big[\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{13}\Big)\mathcal{D}_{23} + (2\leftrightarrow 3)\Big]\\
\frac{1}{2}\big[\mathcal{C}(1,2,3)-\mathcal{C}(1,3,2)\big] &= 2i\Big[\sin\Big(\frac{\pi}{2}(s_{12}+s_{23})\Big)\cos\Big(\frac{\pi}{2}s_{13}\Big)\mathcal{D}_{23} - (2\leftrightarrow 3)\Big]\,,
\end{aligned} \tag{D.2}$$

and the analogous four-point identity is

$$\begin{aligned}
\frac{1}{2}\big[\mathcal{C}(1,2,3,4)+\mathcal{C}(1,4,3,2)\big] = 2i\Big[&\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24})\Big)\cos\Big(\frac{\pi}{2}s_{134}\Big)\mathcal{D}_{234}\\
&-\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\cos\Big(\frac{\pi}{2}(-s_{12}+s_{14}-s_{24})\Big)\mathcal{D}_{324}\\
&-\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\cos\Big(\frac{\pi}{2}(-s_{12}+s_{14}+s_{24})\Big)\mathcal{D}_{342}\\
&+\sin\Big(\frac{\pi}{2}(s_{14}+s_{24}+s_{34})\Big)\cos\Big(\frac{\pi}{2}s_{123}\Big)\mathcal{D}_{432}\Big]\\
\frac{1}{2}\big[\mathcal{C}(1,2,3,4)-\mathcal{C}(1,4,3,2)\big] = -2\Big[&\sin\Big(\frac{\pi}{2}(s_{12}+s_{23}+s_{24})\Big)\sin\Big(\frac{\pi}{2}s_{134}\Big)\mathcal{D}_{234}\\
&-\sin\Big(\frac{\pi}{2}(s_{13}+s_{23}+s_{34})\Big)\sin\Big(\frac{\pi}{2}(-s_{12}+s_{14}-s_{24})\Big)\mathcal{D}_{324}
\end{aligned} \tag{D.3}$$

$$
\begin{aligned}
&- \sin\Big(\frac{\pi}{2}(s_{13}{+}s_{23}{+}s_{34})\Big)\sin\Big(\frac{\pi}{2}(-s_{12}{+}s_{14}{+}s_{24})\Big)\mathcal{D}_{342} \\
&- \sin\Big(\frac{\pi}{2}(s_{14}{+}s_{24}{+}s_{34})\Big)\sin\Big(\frac{\pi}{2}s_{123}\Big)\mathcal{D}_{432}\Big] \,.
\end{aligned}
$$

The corresponding entries of $\mathcal{H}_{\alpha'}(A|B)$ can be read off by matching these expressions with $\mathcal{C}(1,A) = \sum_{B\in S_{n-1}} \mathcal{H}_{\alpha'}(A|B)\mathcal{D}_B$, e.g.

$$
\begin{aligned}
\frac{1}{2}\big[\mathcal{H}_{\alpha'}(2,3|2,3) + \mathcal{H}_{\alpha'}(3,2|2,3)\big] &= -2\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23})\Big)\sin\Big(\frac{\pi}{2}s_{13}\Big) \\
\frac{1}{2}\big[\mathcal{H}_{\alpha'}(2,3|2,3) - \mathcal{H}_{\alpha'}(3,2|2,3)\big] &= 2i\sin\Big(\frac{\pi}{2}(s_{12}{+}s_{23})\Big)\cos\Big(\frac{\pi}{2}s_{13}\Big)\,. \qquad \text{(D.4)}
\end{aligned}
$$

E Examples of α'-expansions

E.1 Three points: integrating $f^{(1)}_{12} f^{(3)}_{23}$

Among the three-point component integrals (5.40) over $f^{(m_1)}_{12} f^{(m_2)}_{23}$, examples of their α'-expansions at $(m_1,m_2) = (0,0),(2,0),(3,3)$ and $(1,0)$ have been given in (5.41), (5.45) and (5.46). The following α'-expansion at $(m_1,m_2) = (1,3)$ contains further examples of irreducible iterated Eisenstein integrals at depth two such as $\gamma_0(4,4,0|\tau)$ and $\gamma_0(4,6,0|\tau)$:

$$
\begin{aligned}
Z^\tau_{(1,3)}(1,2,3|1,2,3) &= s_{13}\Big(6\zeta_2\gamma_0(4|\tau) - \frac{5}{2}\gamma_0(6|\tau)\Big) \\
&+ s_{13}(s_{12}{+}s_{13})\Big(- 36\zeta_4\gamma_0(4,0|\tau) - \frac{9}{2}\gamma_0(4,4|\tau) + 60\zeta_2\gamma_0(6,0|\tau) - \frac{63}{2}\gamma_0(8,0|\tau)\Big) \\
&+ s_{13}\Big\{\frac{15}{4}(s_{12}^2{+}s_{13}^2{+}s_{23}^2)\zeta_4\gamma_0(4|\tau) - \frac{5}{8}(s_{12}^2{+}s_{13}^2{+}s_{23}^2)\zeta_2\gamma_0(6|\tau) \\
&\quad + 9(36s_{12}^2{-}42s_{12}s_{13}{+}36s_{13}^2{-}31s_{12}s_{23}{-}31s_{13}s_{23}{-}34s_{23}^2)\zeta_6\gamma_0(4,0,0|\tau) \\
&\quad - 30(57s_{12}s_{13}{+}28s_{12}s_{23}{+}28s_{13}s_{23}{-}5s_{23}^2)\zeta_4\gamma_0(6,0,0|\tau) \\
&\quad + 42(36s_{12}^2{+}18s_{12}s_{13}{+}36s_{13}^2{+}29s_{12}s_{23}{+}29s_{13}s_{23}{-}34s_{23}^2)\zeta_2\gamma_0(8,0,0|\tau) \\
&\quad - 9(90s_{12}^2{+}21s_{12}s_{13}{+}90s_{13}^2{+}59s_{12}s_{23}{+}59s_{13}s_{23}{-}85s_{23}^2)\gamma_0(10,0,0|\tau) \qquad \text{(E.1)}\\
&\quad + 18(s_{12}^2{+}s_{13}^2{+}s_{12}s_{23}{+}s_{13}s_{23}{-}s_{23}^2)\zeta_2\gamma_0(4,0,4|\tau) \\
&\quad + 18(6s_{12}^2{+}2s_{12}s_{13}{+}6s_{13}^2{+}3s_{12}s_{23}{+}3s_{13}s_{23}{-}8s_{23}^2)\zeta_2\gamma_0(4,4,0|\tau) \\
&\quad - \frac{15}{2}(9s_{12}^2{-}2s_{12}s_{13}{+}9s_{13}^2{+}4s_{12}s_{23}{+}4s_{13}s_{23}{-}15s_{23}^2)\gamma_0(6,4,0|\tau) \\
&\quad - 15(s_{12}^2{+}s_{13}^2{+}2s_{12}s_{23}{+}2s_{13}s_{23})\gamma_0(6,0,4|\tau) + \frac{15}{2}(s_{12}^2{+}s_{13}^2{+}s_{23}^2)\gamma_0(4,0,6|\tau) \\
&\quad - \frac{15}{2}(9s_{12}^2{+}4s_{12}s_{13}{+}9s_{13}^2{+}2s_{12}s_{23}{+}2s_{13}s_{23}{-}5s_{23}^2)\gamma_0(4,6,0|\tau)\Big\} + \mathcal{O}(\alpha'^4)
\end{aligned}
$$

E.2 Four points: MZVs for the initial values

This appendix contains some further samples of the α'-expansions of $I^{\rm tree}(1,2,3,4|*)$ with one or two powers of G_{ij} in the integrand (see (5.52) for $I^{\rm tree}(1,2,3,4|1)$ at the orders of $\alpha'^{\leq 3}$).

For a given planar integration cycle, there are two cyclically inequivalent representatives to consider for one and two powers of G_{ij}:

$$I^{\rm tree}(1,2,3,4|G_{12}) = \frac{1}{2s_{12}} + \frac{\zeta_2}{8}\Big(\frac{s_{123}^2+(s_{14}{+}s_{24})^2+s_{34}^2}{s_{12}} + 2(s_{24}{-}s_{23})\Big) \tag{E.2}$$
$$+ \frac{\zeta_3}{8}\Big(\frac{s_{123}^3+(s_{14}{+}s_{24})^3+s_{34}^3+s_{123}(s_{14}{+}s_{24})s_{34}}{s_{12}} - 3s_{12}s_{23} - s_{13}s_{23} + s_{14}s_{23}$$
$$- 3s_{23}^2 + 3s_{12}s_{24} - s_{13}s_{24} + 5s_{14}s_{24} + 3s_{24}^2 + s_{23}s_{34} - s_{24}s_{34}\Big) + \mathcal{O}(\alpha'^3)$$

$$I^{\rm tree}(1,2,3,4|G_{13}) = \frac{\zeta_2}{4}(s_{14}-s_{12}-s_{23}+s_{34}) \tag{E.3}$$
$$+ \frac{\zeta_3}{8}\Big(-3s_{12}^2 - 3s_{12}s_{13} + 3s_{13}s_{14} + 3s_{14}^2 - 4s_{12}s_{23} - 3s_{13}s_{23} - 3s_{23}^2 + s_{12}s_{24}$$
$$- s_{14}s_{24} + s_{23}s_{24} + 3s_{13}s_{34} + 4s_{14}s_{34} - s_{24}s_{34} + 3s_{34}^2\Big) + \mathcal{O}(\alpha'^3)$$

$$I^{\rm tree}(1,2,3,4|G_{12}G_{34}) = \frac{1}{s_{12}s_{34}} + \frac{\zeta_2}{4}\frac{s_{1234}^2}{s_{12}s_{34}} \tag{E.4}$$
$$+ \frac{\zeta_3}{4}\Big(\frac{s_{1234}^3}{s_{12}s_{34}} + \frac{(s_{13}{+}s_{14})(s_{23}{+}s_{24})}{s_{34}} + \frac{(s_{13}{+}s_{23})(s_{14}{+}s_{24})}{s_{12}} + s_{14} + s_{23}\Big) + \mathcal{O}(\alpha'^2)$$

$$I^{\rm tree}(1,2,3,4|G_{13}G_{24}) = \frac{\zeta_3}{4}(s_{12}-s_{14}-s_{23}+s_{34}) + \mathcal{O}(\alpha'^2)\,. \tag{E.5}$$

Integrands $G_{ij}G_{jk}$ with overlapping labels i,j,k can be reduced to non-overlapping cases via integration by parts (5.50). They exhibit three-particle poles s_{ijk}^{-1} as for instance seen in

$$I^{\rm tree}(1,2,3,4|G_{12}G_{13}) = \frac{1}{s_{12}s_{123}} - \frac{\zeta_2 s_{23}}{s_{123}} + \frac{\zeta_2}{4}\frac{(s_{123}{+}s_{14}{+}s_{24}{+}s_{34})^2}{s_{12}s_{123}} + \frac{\zeta_3 s_{13}s_{23}}{s_{123}}$$
$$+ \frac{\zeta_3}{4}\frac{(s_{123}{+}s_{14}{+}s_{24}{+}s_{34})^2}{s_{12}s_{123}} + \frac{\zeta_3(s_{14}{+}s_{24})s_{34}}{4s_{12}} + \frac{\zeta_3}{4}(s_{24}{-}s_{23}) + \mathcal{O}(\alpha'^2)\,. \tag{E.6}$$

F Fourier expansion of A-cycle graph functions

In this appendix, we demonstrate on the basis of the conjecture (5.17) that the coefficients in the q-expansion of planar and non-planar A-cycle graph functions are $\mathbb{Q}$-linear as opposed to $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs. By their definition in [81], A-cycle graph functions are symmetrized integrals $\prod_{j=2}^n \int_0^1 \mathrm{d}v_j$ over products of the planar Green functions $\mathcal{G}(v_{ij},\tau)$ in (2.14) and non-planar Green functions $\mathcal{G}(v_{ij}{+}\tau/2,\tau) - \frac{i\pi\tau}{4}$ without the additive $\frac{i\pi\tau}{4}$ in (2.15). This definition implies that they are expressible in terms of $\mathbb{Q}$-linear combinations of A-cycle eMZVs and their twisted counterparts. However, the q-expansion of generic (twisted) eMZVs may involve inverse powers of π, so it remains to show that A-cycle graph functions only realize those combinations of (twisted) eMZVs, where negative powers of π drop out. The subsequent arguments are tailored to ℓ punctures on the A-cycle and $n{-}\ell$ punctures on its displacement by $\tau/2$.

In this work, A-cycle graph functions are generated from the η_j^{1-n}-order of symmetrized A-cycle integrals $\sum_{P\in S_{\ell-1}}\sum_{Q\in S_{n-\ell-1}} Z^\tau_{\vec\eta}\big({}^{n,Q}_{1,P}\,|*\big)$, which does not depend on the choice of the integrand $*$. We will proceed in three steps and establish the absence of negative powers of π first for the symmetrizations $\sum_{A\in S_{n-1}} I^{\rm tree}(1,A|\cdot)$, then for the initial values $Z^{i\infty}_{\vec\eta}$ of the above generating series and finally for the entire q-dependence of A-cycle graph functions.

- The absence of π^{-1} in the symmetrization of $I^{\rm tree}(1,A|\cdot)$ can be understood from (5.29): Its prefactor $(i\pi)^{1-n}$ is compensated by the symmetrization in $\sum_{A\in S_{n-1}} \mathcal{H}_{\alpha'}(A|P)$ that yields $n{-}1$ sine factors of the schematic form $\sin(\frac{\pi}{2}\sum_j s_{ij})$ and therefore $n{-}1$ overall powers of π in its α'-expansion. The appearance of the sine functions follows from the relation (5.17) between twisted cycles which is conjectural at $n\geq 6$.

- The initial values $\sum_{P\in S_{\ell-1}}\sum_{Q\in S_{n-\ell-1}} Z^{i\infty}_{\vec\eta}(\begin{smallmatrix} n,Q \\ 1,P\end{smallmatrix}|*)$ relevant for A-cycle graph functions boil down to products of symmetrized $I^{\rm tree}(1,A|\cdot)$ by (6.7). These products do not involve any inverse powers of π by the previous step, and their coefficients are built from $\pi\cot(\pi\eta)$ and $\frac{\pi}{\sin(\pi\eta)}$ (with η referring to generic sums of $\eta_2,\eta_3,\ldots,\eta_n$).

  Whenever the integrands of the $I^{\rm tree}(1,A|\cdot)$ in (6.7) involve k powers of G_{ij} in total, then the number of trigonometric factors is $n{-}1{-}k$, and they have an expansion of the form $\eta^{k+1-n}\sum_{p=0}^\infty c_p(\pi\eta)^p$ with $c_p\in\mathbb{Q}$, see (3.18). Since each G_{ij} in the integrand of $I^{\rm tree}(1,A|\cdot)$ introduces an extra power of $i\pi$ by (5.29), the counting of relative powers of π and η in the entire initial value is governed by the $k=0$ case: The α'-expansion of $\sum_{P\in S_{\ell-1}}\sum_{Q\in S_{n-\ell-1}} Z^{i\infty}_{\vec\eta}(\begin{smallmatrix} n,Q \\ 1,P\end{smallmatrix}|*)$ has the schematic form of $\eta^{1-n}\sum_{p=0}^\infty \hat c_p(\pi\eta)^p$, $\hat c_p\in\mathbb{Q}$, multiplying series in α' with $\mathbb{Q}$-linear combinations of MZVs from the symmetrized $I^{\rm tree}$.

- A-cycle graph functions arise from the η^{1-n}-order of the full τ-dependent $Z^\tau_{\vec\eta}(\cdot|\cdot)$ upon symmetrization. It remains to check that the operator $\Psi^\tau_{\vec\eta}$ defined in (7.3) does not introduce any inverse powers of π through the q series of $\gamma(k_1,k_2,\ldots,k_r|\tau)r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})$. The two types of Eisenstein series $\gamma(\ldots)$ and $\gamma_0(\ldots)$ are equivalent w.r.t. powers of π since all coefficients of q^0 and $q^{n\geq 1}$ in G_k are rational multiples of π^k, cf. (2.19). Hence, the counting of π in $\Psi^\tau_{\vec\eta}$ is captured by replacing $\gamma(\ldots)\to\gamma_0(\ldots)$ and inserting the q-expansion (2.24) which determines each Fourier coefficient in $\gamma_0(k_1,k_2,\ldots,k_r|\tau)$ to be a rational multiple of $\pi^{-2r+k_1+k_2+\ldots+k_r}$.

  The derivations $r_{\vec\eta}(\epsilon_k)$ are of homogeneity degree $k{-}2$ in η_j, with the exception of the term $2\zeta_2 s_{123\ldots n}$ in $r_{\vec\eta}(\epsilon_0)$ which can be viewed as introducing an extra power $(\pi\eta)^2$. Hence, each term $\gamma_0(k_1,k_2,\ldots,k_r|\tau)r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})$ in $\Psi^\tau_{\vec\eta}$ has a homogeneity degree $\sim(\pi\eta)^{-2r+k_1+k_2+\ldots+k_r}$ up to tentative extra factors of $(\pi\eta)^2$ due to the exceptional term in $r_{\vec\eta}(\epsilon_0)$.

  Contributions with negative powers of π (say $(\pi\eta)^{-2}$ due to $\gamma_0(4,0,0|\tau)r_{\vec\eta}(\epsilon_0\epsilon_0\epsilon_4)$) can only affect the η^{1-n}-order relevant to A-cycle graph functions if the initial value $Z^{i\infty}_{\vec\eta}$ contributes with higher orders in η as compared to the leading term $\sim\eta^{1-n}$. As argued above, the η-dependence of $Z^{i\infty}_{\vec\eta}$ has the schematic form $\eta^{1-n}\sum_{p=0}^\infty \hat c_p(\pi\eta)^p$ with $\hat c_p\in\mathbb{Q}$. Hence, any negative power $(\pi\eta)^{-p}$ due to $\Psi^\tau_{\vec\eta}$ must be compensated by a positive power $(\pi\eta)^p$ from the initial value in order to contribute to the order η^{1-n} relevant to A-cycle graph functions.

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