On the equivariant K- and KO-homology of some special linear groups

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Abstract

We compute the equivariant KO-homology of the classifying space for proper actions of $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{GL}_3(\mathbb{Z})$. We also compute the Bredon homology and equivariant K-homology of the classifying spaces for proper actions of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ for each prime p. Finally, we prove the unstable Gromov-Lawson-Rosenberg conjecture for $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ when $p \equiv 11 \pmod{12}$.

1 Introduction

There has been considerable interest in the Baum-Connes conjecture; which states that for a group Γ a certain 'assembly map', from the equivariant K-homology of the classifying space for proper actions $\underline{E}\Gamma$ to the topological K-theory of the reduced group C^*-algebra, is an isomorphism. The Baum-Connes conjecture is known to hold for several families of groups, including word-hyperbolic groups, CAT(0)-cubical groups and groups with the Haagerup property.

Conjecture 1.1 (The Baum-Connes Conjecture)**.** *Let Γ be a discrete group, then the following assembly map is an isomorphism*

$$\mu : K_*^\Gamma(\underline{E}\Gamma) \to K_*^{\mathrm{top}}(C_r^*(\Gamma)).$$

There is also a 'real' Baum-Connes conjecture which predicts that an assembly map from the equivariant KO-homology of $\underline{E}\Gamma$ to the topological K of the real group C^*-algebra is an isomorphism. It is known that the two conjectures are equivalent, that is, one of the assembly maps is an isomorphism if and only if the other is [2].

Conjecture 1.2 (The Real Baum-Connes Conjecture)**.** *Let Γ be a discrete group, then the following assembly map is an isomorphism*

$$\mu_{\mathbb{R}} : KO_*^\Gamma(\underline{E}\Gamma) \to K_*^{\mathrm{top}}(C_r^*(\Gamma;\mathbb{R})).$$

In spite of the interest, to date there have been very few computations of K^Γ- and KO^Γ-homology. Indeed, on the K^Γ-homology side of things there are complete calculations for one relator groups [20], NEC groups [18], some Bianchi groups and hyperbolic reflection groups [16, 21, 22], some Coxeter groups [9, 27, 28], and $\mathrm{SL}_3(\mathbb{Z})$ [26]. For KO^Γ-homology the author is aware of two complete computations; the first, due to Davis and Lück, on a family of Euclidean crystallographic groups [7], and the second, due to Mario Fuentes-Rumí, on simply connected graphs of cyclic groups of odd order and of some Coxeter groups [10].

In this paper we compute the topological side of the (real) Baum-Connes conjecture for the S-arithmetic lattices $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$, for p a prime, and for the arithmetic group $\mathrm{SL}_3(\mathbb{Z})$. In particular, we compute the equivariant K-homology of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ and the equivariant KO-homology of $\mathrm{SL}_3(\mathbb{Z})$. We give the relevant background and the connection to Bredon homology in Section 2. The calculation for KO^Γ-homology is of particular interest because it is (to the author's knowledge) the first computation of KO_*^Γ for a property (T) group.

This interest stems from the fact that property (T) is a strong negation of the Haagerup property. Moreover, the (real) Baum-Connes conjecture is still open for $\mathrm{SL}_n(\mathbb{Z})$ when $n \geq 3$. We note that there are counterexamples for the Baum-Connes conjecture for groupoids constructed from $\mathrm{SL}_3(\mathbb{Z})$ and more generally a discrete group with property (T) for which the assembly map is known to be injective [12].

Theorem 1.3. *Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO^\Gamma_{-n}(\underline{E}\Gamma) = \mathbb{Z}^8, \quad \mathbb{Z}_2^8, \quad \mathbb{Z}_2^8, \quad 0, \quad \mathbb{Z}^8, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Applying a Künneth type theorem to the isomorphism $\mathrm{GL}_3(\mathbb{Z}) \cong \mathrm{SL}_3(\mathbb{Z}) \times \mathbb{Z}_2$ on the level of Bredon homology, we obtain the following result for $\mathrm{GL}_3(\mathbb{Z})$.

Corollary 1.4. *Let $\Gamma = \mathrm{GL}_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO^\Gamma_{-n}(\underline{E}\Gamma) = \mathbb{Z}^{16}, \quad \mathbb{Z}_2^{16}, \quad \mathbb{Z}_2^{16}, \quad 0, \quad \mathbb{Z}^{16}, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Moving onto $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ or $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$, for p a prime, we compute the equivariant K-homology groups $K_n^\Gamma(\underline{E}\Gamma)$. There has been considerable interest in determining homological properties of the groups $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{m}])$ and groups related to them [1, 4, 13]. It appears, however, that even with computer based methods the problem of determining the cohomology of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{m}])$ for m a product of 3 primes is out of reach [4]. In Lemma 5.3 we give a short proof of the Baum-Connes conjecture for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ and so we obtain the topological K-theory of the reduced group C^*-algebra of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ as well.

Theorem 1.5. *Let p be a prime and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then $K_n^\Gamma(\underline{E}\Gamma)$ is a free abelian group with rank as given in Table 1. Moreover, since the Baum-Connes and Bost conjectures hold for Γ we have $K_*^\Gamma(\underline{E}\Gamma) \cong K_*^{top}(C_r^*(\Gamma)) \cong K_*^{top}(\ell^1(\Gamma))$.*

	$p=2$	$p=3$	$p \equiv 1 \pmod{12}$	$p \equiv 5 \pmod{12}$	$p \equiv 7 \pmod{12}$	$p \equiv 11 \pmod{12}$
$n=0$	7	6	$4 + \frac{1}{6}(p-7)$	$6 + \frac{1}{6}(p+1)$	$5 + \frac{1}{6}(p-1)$	$7 + \frac{1}{6}(p+7)$
$n=1$	0	0	3	1	2	0

Table 1: $\mathbb{Z}$-rank of the equivariant K-homology of the classifying space for proper actions of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

Corollary 1.6. *Let p be a prime and $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ then $K_n^\Gamma(\underline{E}\Gamma)$ is additively isomorphic to the direct sum of two copies of the equivariant K-homology of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.*

Finally, we will give a proof of the unstable Gromov-Lawson-Rosenberg conjecture for positive scalar curvature for the groups $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ when $p \equiv 11 \pmod{12}$. The statement and background concerning this conjecture is given in Section 6.

Theorem 1.7. *The unstable Gromov-Lawson-Rosenberg conjecture holds for the group $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ when $p \equiv 11 \pmod{12}$.*

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2 Preliminaries

In this section we introduce the relevant background from Bredon homology and its interactions with equivariant K- and KO-homology. We follow the treatment given in Mislin's notes [20].

2.1 Classifying spaces for families

Let Γ be a discrete group. A Γ-CW complex X is a CW-complex equipped with a cellular Γ-action. We say the Γ action is proper if all the cell stabilisers are finite.

Let $\mathcal{F}$ be a family of subgroups of Γ which is closed under conjugation and finite intersections. A *model* for the classifying space $E_{\mathcal{F}}\Gamma$ for the family $\mathcal{F}$ is a Γ-CW complex such that all cell stabilisers are in $\mathcal{F}$ and the fixed point set of every $H \in \mathcal{F}$ is weakly-contractible. This is equivalent the following universal property; for every Γ-CW complex Y there is exactly one Γ-map $Y \to E_{\mathcal{F}}\Gamma$ up to Γ-homotopy.

In the case where $\mathcal{F} = \mathcal{FIN}(\Gamma)$, the family of all finite subgroups of Γ, we denote $E_{\mathcal{FIN}}(\Gamma)$ by $\underline{\mathrm{E}}\Gamma$. We call such a space, the classifying space for proper actions of Γ. Note that if Γ is torsion-free then $\underline{\mathrm{E}}\Gamma = E\Gamma$.

2.2 Bredon homology

Let Γ be a discrete group and $\mathcal{F}$ be a family of subgroups. We define the *orbit category* $\mathbf{Or}_{\mathcal{F}}(\Gamma)$ to be the catgeory with objects given by left cosets Γ/H for $H \in \mathcal{F}$ and morphisms the Γ-maps $\phi : \Gamma/H \to \Gamma/K$. A morphism in the orbit category is uniquely determined by its image $\varphi(H) = \gamma K$ and $\gamma H \gamma^{-1} \subseteq K$; conversely, each such $\gamma \in \Gamma$ defines a G-map.

A *(left) Bredon module* is a covariant functor $M : \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of Abelian groups. Consider a Γ-CW complex X and a family of subgroups $\mathcal{F}$ containing all cell stabilisers. Let M be a Bredon module and define the *Bredon chain complex with coefficients* in M as follows:

Let $\{c_\alpha\}$ be a set of orbit representatives of the n-cells in X and let Γ_α denote the stabiliser of the cell α. The nth chain group is then

$$C_n := \bigoplus_\alpha M(\Gamma/\Gamma_{c_\alpha}).$$

If $\gamma c'$ is an $(n-1)$-cell in the boundary of c, then $\gamma^{-1}\Gamma_c\gamma \subseteq \Gamma_{c'}$ yielding a Γ-map $\varphi : \Gamma/\Gamma_c \to \Gamma/\Gamma_{c'}$. This gives a induced homomorphism $M(\varphi) : M(\Gamma/\Gamma_c) \to M(\Gamma/\Gamma_{c'})$. Putting this together we obtain a differential $\partial : C_n \to C_{n-1}$. Taking homology of the chain complex (C_*, ∂) gives the *Bredon homology groups* $H_n^{\mathcal{F}}(X; M)$. A right Bredon module and Bredon cohomology is defined analogously with contravariant functors.

2.3 Equivariant K-homology

Let Γ be a discrete group. In the context of the Baum-Connes conjecture we are specifically interested in the case where $X = \underline{\mathrm{E}}\Gamma$, $\mathcal{F} = \mathcal{FIN}(\Gamma)$ and $M = \mathcal{R}_{\mathbb{C}}$ the complex representation ring. We consider

$\mathcal{R}_{\mathbb{C}}(-)$ as a Bredon module in the following way, for $\Gamma/H \in \mathbf{Or}_{\mathcal{F}}(\Gamma)$ set $\mathcal{R}_{\mathbb{C}}(\Gamma/H) = \mathcal{R}_{\mathbb{C}}(H)$, the ring of complex representations of the finite group H. Morphisms are then given by induction of representations.

We note that $\mathcal{R}_{\mathbb{C}}(\Gamma) := H_0^{\mathcal{F}}(\Gamma) = \operatorname{colim}_{\Gamma/H\in\mathbf{Or}(\Gamma)}\mathcal{R}_{\mathbb{C}}(H)$. In the case that Γ has finitely many conjugacy classes of finite subgroups, $\mathcal{R}_{\mathbb{C}}(\Gamma)$ is a finitely generated quotient of $\bigoplus \mathcal{R}_{\mathbb{C}}(H)$, where H runs over conjugacy classes of finite subgroups.

Of course we still have to show how Bredon homology links to $K_*^{\Gamma}(\underline{E}\Gamma)$, the equivariant K-homology of the classifying space for proper actions. The answer is provided by Kasparov's KK-theory and the following equivariant Atiyah-Hirzebruch type spectral sequence.

More specifically for each subgroup $H \leq \Gamma$ we have $K_n^{\Gamma}(\Gamma/H) = K_i(C_r^*(H))$. In the case H is a finite subgroup then $C_r^*(H) = \mathbb{C}H$, $K_0^{\Gamma}(\Gamma/H) = K_0(\mathbb{C}H) = \mathcal{R}_{\mathbb{C}}(H)$, and $K_1^{\Gamma}(\Gamma/H) = K_1(\mathbb{C}H) = 0$. The remaining K^{Γ} groups are given by 2-fold Bott periodicity.

Theorem 2.1. [20, Page 50] *Let Γ be a group and X a proper Γ-CW complex, then there is an Atiyah-Hirzebruch type spectral sequence*

$$E^2_{p,q} := H_p^{\mathcal{FIN}}(X; K_q^{\Gamma}(-)) \Rightarrow K_{p+q}^{\Gamma}(X).$$

2.4 Equivariant KO-homology

Again fixing a discrete group Γ and $\mathcal{F} = \mathcal{FIN}(\Gamma)$, we introduce two more Bredon modules, the real representation ring $\mathcal{R}_{\mathbb{R}}(-)$, and the quaternionic representation ring $\mathcal{R}_{\mathbb{H}}(-)$. These are defined on $\mathbf{Or}_{\mathcal{F}}(\Gamma)$ in exactly the same way as the complex representation ring. We have natural transformations between the functors. Indeed for a finite subgroup $H \leq \Gamma$ we have a diagram (which does not commute):

$$\begin{tikzcd} \mathcal{R}_{\mathbb{R}}(H) \arrow[r, bend left, "\nu"] & \mathcal{R}_{\mathbb{C}}(H) \arrow[l, bend left, "\rho"] \arrow[r, bend left, "\sigma"] & \mathcal{R}_{\mathbb{H}}(H). \arrow[l, bend left, "\eta"] \end{tikzcd}$$

For a real representation ψ, the *complexification* is $\nu(\psi) = \psi \otimes \mathbb{C}$. For a complex representation ϕ, the *symplectification* is $\sigma(\phi) = \phi\otimes\mathbb{H}$. Going the other way, for an n-dimensional quaternionic representation ω, the *complexification* is $\eta(\omega) = \eta$ considered as $2n$ complex representation. Similarly, for an n-dimensional complex representation ϕ, the *realification* is $\rho(\phi) = \phi$ considered as a $2n$-dimensional real representation. Note that any composition of the x-ification natural transformations with the same source and target is necessarily not the identity.

The situation for the equivariant KO-homology, denoted $KO_*^{\Gamma}(-)$, is similar to the equivariant K-homology but more complicated. For a subgroup $H \leq \Gamma$ we set $KO_n^{\Gamma}(\Gamma/H) = KO_n(C_r^*(H;\mathbb{R}))$. In the case that H is a finite subgroup we now have that

$$KO_{-n}^{\Gamma}(\Gamma/H) = KO_{-n}(C_r^*(H;\mathbb{R})) = \begin{cases} \mathcal{R}_{\mathbb{R}}(H) & n=0, \\ \mathcal{R}_{\mathbb{R}}(H)/\rho(\mathcal{R}_{\mathbb{C}}(H) & n=1, \\ \mathcal{R}_{\mathbb{C}}(H)/\eta(\mathcal{R}_{\mathbb{H}}(H)) & n=2, \\ 0 & n=3, \\ \mathcal{R}_{\mathbb{H}}(H) & n=4, \\ \mathcal{R}_{\mathbb{H}}(H)/\sigma(\mathcal{R}_{\mathbb{C}}(H)) & n=5, \\ \mathcal{R}_{\mathbb{C}}(H)/\nu(\mathcal{R}_{\mathbb{R}}(H)) & n=6, \\ 0 & n=7, \end{cases}$$

with the remaining groups given by 8-fold Bott-periodicity. For X a proper Γ-space, the Atiyah-Hirzebruch spectral sequence from before now takes the form

$$E^2_{p,q} := H_p^{\mathcal{FIN}}(X; KO_q^\Gamma(-)) \Rightarrow KO_{p+q}^\Gamma(X).$$

2.5 Spectra and homotopy

The section gives an alternative Γ-equivariant homotopy theoretic viewpoint. Now we consider Γ-equivariant homology theories as functors $\mathbf{E}\colon \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Spectra}$. Technically, to avoid functorial problems one must take composite functors through the categories C^*-**Cat** and **Groupoids**. We do not concern ourselves with this complication and refer the reader to [5] and [8].

Instead we will take for granted that there is a composite functor

$$\mathbf{KO}\colon \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Spectra}$$

which satisfies $\pi_n \mathbf{KO}(\Gamma/H) = KO_n(C_r^*(H;\mathbb{R}))$. When $\mathcal{F} = \mathcal{FIN}$ this perspective gives a homotopy theoretic construction of the (real) Baum-Connes assembly map. Indeed, we have maps

$$B\Gamma_+ \wedge \mathbf{KO} \simeq \underset{\mathbf{Or}_{\mathcal{TRV}}(\Gamma)}{\operatorname{hocolim}} \mathbf{KO} \to \underset{\mathbf{Or}_{\mathcal{F}}(\Gamma)}{\operatorname{hocolim}} \mathbf{KO} \to \underset{\mathbf{Or}_{\mathcal{ALL}}(\Gamma)}{\operatorname{hocolim}} \mathbf{KO} \simeq \mathbf{KO}(C_r^*(\Gamma;\mathbb{R})).$$

The assembly map $\mu_{\mathbb{R}}$ is then π_n applied to the composite.

3 Equivariant KO-homology of $\mathrm{SL}_3(\mathbb{Z})$

3.1 A classifying space for proper actions

A model for $X = \underline{\mathrm{E}}\mathrm{SL}_3(\mathbb{Z})$ can be constructed as a $\mathrm{SL}_3(\mathbb{Z})$-equivariant deformation retract of the symmetric space $\mathrm{SL}_3(\mathbb{R})/O(3)$. This construction has been detailed several times in the literature ([31, Theorem 2], [11, Theorem 2.4] or [26, Theorem 13]), so rather than detailing it again here, we simply extract the relevant cell complex and cell stabilisers. Specifically, we follow the notation of Sánchez-García [26] and collect the information in Table 2.

3.2 Proof of Theorem 1.3

The calculation of the equivariant KO-groups will follow from the following proposition and an analysis of the representation theory of the finite subgroups of $\mathrm{SL}_3(\mathbb{Z})$. We remark that one could prove a dozen subtle variations on the theme of the following proposition. However, rather than do this we introduce a slogan: "Computations with coefficients in $KO_n^\Gamma(-)$ can be greatly simplified by looking for chain maps to the Bredon chain complex with coefficients in $\mathcal{R}_{\mathbb{C}}(-)$."

Proposition 3.1. *Let Γ be a discrete group, $\mathcal{F} = \mathcal{FIN}(\Gamma)$ and $X = \underline{\mathrm{E}}\Gamma$. Assume that for every cell stabiliser the real, complex and quarternionic character tables are equal, then the Atiyah-Hirzebruch spectral sequence converging to $KO_*^\Gamma(\underline{\mathrm{E}}\Gamma)$ has E^2-page isomorphic to*

$$E^2_{p,q} = H_p^{\mathcal{FIN}}(\underline{\mathrm{E}}\Gamma); K_0^\Gamma(-)) \otimes KO_q(*)$$

where for $n = 0, \dots, 7$ we have

$$KO_{-n}(*) = \mathbb{Z}, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad 0, \quad \mathbb{Z}, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Dimension	Cell	Boundary	Stabiliser
3	T_1	$-t_1+t_2-t_3+t_4-t_5$	$\{1\}$
2	t_1	$e_1-e_2-e_4$	$\mathbb{Z}_2$
	t_2	$e_4-e_5+e_6$	$\{1\}$
	t_3	$e_6-e_7+e_8$	$\mathbb{Z}_2^2$
	t_4	$e_1-e_3+e_5+e_8$	$\mathbb{Z}_2$
	t_5	$e_2-e_3+e_6-e_6+e_7$	$\mathbb{Z}_2$
1	e_1	v_1-v_2	$\mathbb{Z}_2^2$
	e_2	v_3-v_1	D_3
	e_3	v_5-v_1	D_3
	e_4	v_3-v_2	$\mathbb{Z}_2$
	e_5	v_4-v_2	$\mathbb{Z}_2$
	e_6	v_4-v_3	$\mathbb{Z}_2^2$
	e_7	v_5-v_3	D_4
	e_8	v_5-v_4	D_4
0	v_1	-	$\mathrm{Sym}(4)$
	v_2		D_6
	v_3		$\mathrm{Sym}(4)$
	v_4		D_4
	v_5		$\mathrm{Sym}(4)$

Table 2: Cell structure and stabilisers of a model for $\underline{\mathrm{E}}\mathrm{SL}_3(\mathbb{Z})$.

Proof. First note that when $n=3$ or 7 the result is immediate since $KO_n^\Gamma(-)=0$. Now, since the three character tables are equal the complexification from $\mathcal{R}_\mathbb{R}(-)$ to $\mathcal{R}_\mathbb{C}(-)$ and the symplectification from $\mathcal{R}_\mathbb{C}(-)$ to $\mathcal{R}_\mathbb{H}(-)$ are isomorphisms. In particular, $KO_5^\Gamma(-)=\mathcal{R}_\mathbb{H}(-)/\sigma(\mathcal{R}_\mathbb{C}(-))=0$ and $KO_6^\Gamma(-)=\mathcal{R}_\mathbb{C}(-)/\nu(\mathcal{R}_\mathbb{R}(-))=0$. Moreover, it follows they induce isomorphisms on the Bredon chain complexes $C_*(X;\mathcal{R}_\mathbb{R})\cong C_*(X;\mathcal{R}_\mathbb{C})\cong C_*(X;\mathcal{R}_\mathbb{H})$. In particular, $KO_0^\Gamma(-)\cong KO_4^\Gamma(-)\cong K_0^\Gamma(-)$. Finally, the complexification from $\mathcal{R}_\mathbb{H}(-)$ to $\mathcal{R}_\mathbb{C}(-)$ and the realification from $\mathcal{R}_\mathbb{C}$ to $\mathcal{R}_\mathbb{R}$ correspond to multiplication by 2. Thus, when we take cokernels and pass to the Bredon chain complex for the bredon modules $KO_1^\Gamma(-)$ and $KO_2^\Gamma(-)$, we obtain an isomorphism to the modulo 2 reduction of Bredon chain complex for $K_0^\Gamma(-)$. ☐

Proof of Theorem 1.3. Let $\Gamma=\mathrm{SL}_3(\mathbb{Z})$, $\mathcal{F}=\mathcal{FIN}(\Gamma)$ and $X=\underline{\mathrm{E}}\mathrm{SL}_3(\mathbb{Z})$. We can now complete the calculation for the equivariant KO-homology groups. First, we recap calculation of the Bredon chain complex with complex representation ring coefficients due to Sánchez-García. We have a chain complex

$$0\longrightarrow\mathbb{Z}\xrightarrow{\partial_3}\mathbb{Z}^{11}\xrightarrow{\partial_2}\mathbb{Z}^{28}\xrightarrow{\partial_1}\mathbb{Z}^{26}\longrightarrow 0$$

where

$$\partial_3\sim\begin{bmatrix}1 & \mathbf{0}_{1\times 10}\end{bmatrix},\quad \partial_2\sim\begin{bmatrix}I_{10} & \mathbf{0}_{10\times 18}\\ \mathbf{0}_{10\times 1} & \mathbf{0}_{1\times 18}\end{bmatrix},\quad\text{and}\quad \partial_1\sim\begin{bmatrix}I_{18} & \mathbf{0}_{18\times 8}\\ \mathbf{0}_{10\times 18} & \mathbf{0}_{10\times 8}\end{bmatrix}.$$

Therefore, the homology groups of the chain complex are isomorphic to $\mathbb{Z}^8$ in dimension 0 and to 0 in every other dimension.

Now, the cell stabiliser subgroups of $\mathrm{SL}_3(\mathbb{Z})$ acting on X are isomorphic to $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, D_3, D_4, $\mathrm{Sym}(4)$ and D_6. Each of which satisfies the conditions of the proposition above. Applying this to the previous calculation we obtain a single non-trivial column when $p = 0$ in the Atiyah-Hirzebruch spectral sequence and so it collapses trivially. □

Proof of Corollary 1.4. The result for $\mathrm{GL}_3(\mathbb{Z})$ follows from the observation that the direct product of $\mathbb{Z}_2$ with any of the cell stabiliser subgroups still satisfies the conditions of the Proposition 3.1 and so we have isomorphisms

$$KO_n^{\mathrm{GL}_3(\mathbb{Z})}(\underline{\mathrm{E}}\mathrm{GL}_3(\mathbb{Z})) \cong KO_n^{\mathrm{SL}_3(\mathbb{Z})}(\underline{\mathrm{E}}\mathrm{SL}_3(\mathbb{Z})) \otimes KO_n^{\mathbb{Z}_2}(*).$$

□

4 Equivariant K-homology of Fuchsian Groups

In this section we compute the equivariant K-homology of every finitely generated Fuchsian group, that is a finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Note that Theorem 4.1(a) was computed in [18] along with a more general result for cocompact NEC groups. Moreover, their integral cohomology was determined by the author in [13]. Finally, we remark that the equivariant K-homology for orientable (and some non-orientable) NEC groups can be easily determined via an appropriate graph of groups construction. However, we leave this as an exercise for the interested reader.

The reason for this apparent detour is that we will later split the groups $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ as amalgamated free products of certain Fuchsian subgroups. Thus, we can use a Mayer-Vietoris type argument to compute their K-homology. This is made easier by the fact that every finitely generated Fuchsian group is described by piece of combinatorial data called a *signature*. Indeed, a Fuchsian group of *signature* $[g, s; m_1, \ldots, m_r]$ has presentation

$$\left\langle a_{1,b}, \ldots, a_g, b_g, c_1, \ldots, c_s, d_1 \ldots, d_r \mid \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} d_j \prod_{k=1}^{s} c_k = d_1^{m_1} = \cdots = d_r^{m_r} = 1 \right\rangle$$

and acts on the hyperbolic plane $\mathbb{R}\mathbf{H}^2$ with a $4g + 2s + 2r$ sided fundamental polygon. The tessellation of the polygon under the group action has $1 + s + r$ orbits of vertices, s of which are on the boundary $\partial\mathbb{R}\mathbf{H}^2$, $2g + s + r$ orbits of edges and 1 orbit of faces. All edge and face stabilisers are trivial. All vertex stabilisers are trivial except for r orbits of vertices, each of which is stabilised by some $\mathbb{Z}_{m_j}$. Note that if $s = 0$ we say Γ is cocompact.

Theorem 4.1. *Let Γ be a Fuchsian group of signature $[g, s; m_1, \ldots, m_r]$, then*

(a) if $s = 0$,

$$K_n^{\Gamma}(\underline{\mathrm{E}}\Gamma) = \begin{cases} \mathbb{Z}^{2+r+\sum_{j=1}^{r} m_j} & n \text{ even}, \\ \mathbb{Z}^{2g} & n \text{ odd}. \end{cases}$$

(b) if $s > 0$,

$$K_n^{\Gamma}(\underline{\mathrm{E}}\Gamma) = \begin{cases} \mathbb{Z}^{1+r+\sum_{j=1}^{r} m_j} & n \text{ even}, \\ \mathbb{Z}^{2g+s-1} & n \text{ odd}. \end{cases}$$

Proof of (a). Let Γ be a Fuchsian group of signature $[g, s; m_1, \dots, m_r]$ with $s = 0$. The hyperbolic plane with the induced cell structure of the Γ action is a model for $\underline{E}G$ (see for instance [19]). Recall that the cell structure has $r+1$ orbits of vertices, $2g+r$ orbits of edges and exactly 1 orbit of 2-cells. One vertex v_0 is stabilised by the trivial group and for $j = 1, \dots, r$ the vertex v_j is stabilised y $\mathbb{Z}_{m_j}$. Thus, we have a Bredon chain complex

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r} R_{\mathbb{C}}(\mathbb{Z}_{m_j}) \right) \xleftarrow{\partial_1} \mathbb{Z}^{2g+r} \xleftarrow{\partial_2} \mathbb{Z} \longleftarrow 0,$$

substituting in $R_{\mathbb{C}}(\mathbb{Z}_{m_j}) = \mathbb{Z}^{m_j}$ we obtain

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}^{m_j} \right) \xleftarrow{\partial_1} \mathbb{Z}^{2g+r} \xleftarrow{\partial_2} \mathbb{Z} \longleftarrow 0.$$

We fix the following basis for each chain group: In degree 0 we have generators $x_{j,l}$, for $j = 1, \dots, r$ and $l = 1, \dots, m_j$, and the generator z. In degree 1 we have $a_1, b_1, \dots, a_g, b_g$ and $y_1, \dots, y_r$, and in degree 2, the generator w. A good stare reveals that $\partial_2(w) = 0$, $\partial_1(a_i) = \partial_1(b_i) = 0$, and $\partial_1(y_j) = \sum_{l=1}^{m_j} x_{j,l} - z$. Thus,

$$H_n^{\mathcal{FIN}}(\underline{E}\Gamma; R_{\mathbb{C}}) = \begin{cases} \mathbb{Z}^{1+\sum_{j=1}^{r}(m_j - 1)} & \text{if } n = 0; \\ \mathbb{Z}^{2g} & \text{if } n = 1; \\ \mathbb{Z} & \text{if } n = 2; \\ 0 & \text{otherwise.} \end{cases}$$

From here we apply the equivariant AHSS, since the homology in concentrated in degrees less than or equal to 2 there are no non-trivial differentials. Moreover, since every term is free abelian, the extension problems are resolved trivially. It follows that $K_0^{\Gamma}(\underline{E}\Gamma) = H_0^{\mathcal{FIN}}(\underline{E}\Gamma; R_{\mathbb{C}}) \oplus H_2^{\mathcal{FIN}}(\underline{E}\Gamma; R_{\mathbb{C}})$ and $K_1^{\Gamma}(\underline{E}\Gamma) = H_1^{\mathcal{FIN}}(\underline{E}\Gamma; R_{\mathbb{C}})$. $\square$

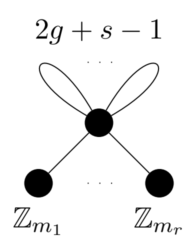

Figure 1: A graph of groups for a non-cocompact Fuchsian group.

Proof of (b). Let Γ be a Fuchsian group of signature $[g, s; m_1, \dots, m_r]$ with $s > 0$. In this case we can rearrange the presentation of Γ such that we have a splitting of Γ as an amalgamated free product $\Gamma \cong \mathbb{Z}^{s-1} * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_r}$. Now, Γ splits as a finite graph of finite groups (Figure 1) and it is easy to see that the Bass-Serre tree of Γ is a model for $\underline{E}\Gamma$.

We first compute the Bredon homology $H_*^{\mathcal{FIN}}(\underline{E}\Gamma; R_{\mathbb{C}})$ with coefficients in the representation ring, from here we can apply the equivariant Atiyah-Hirzebruch spectral sequence. For $\underline{E}\Gamma$ we have a Bredon chain complex

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r} R_{\mathbb{C}}(\mathbb{Z}_{m_j}) \right) \xleftarrow{\partial} \mathbb{Z}^{2g+s-1} \longleftarrow 0,$$

substituting in $R_{\mathbb{C}}(\mathbb{Z}_{m_j}) = \mathbb{Z}^{m_j}$ we obtain

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}^{m_j} \right) \xleftarrow{\ \partial\ } \mathbb{Z}^{2g+s-1} \longleftarrow 0.$$

Let the first non-zero term have generating set $\langle x_{j,l}, z \mid j = 1, \dots, r,\ l = 1, \dots, m_j \rangle$ and the second term $\langle a_1, b_1 \dots, a_g, b_g, c_1, \dots, c_{s-1}, d_1, \dots, d_r \rangle$. It is easy to see the differential ∂ is given by $\partial(a_i) = \partial(b_i) = \partial(c_k) = 0$ and $\partial(d_j) = \sum_{l=1}^{m_j} x_{j,l} - z$. It follows that $H_0^{\mathcal{FIN}}(\underline{E}\Gamma; R_{\mathbb{C}}) = \mathbb{Z}^{1+\sum_{j=1}^{r}(m_j-1)}$, $H_1^{\mathcal{FIN}}(\underline{E}\Gamma; R_{\mathbb{C}}) = \mathbb{Z}^{2g+s-1}$ and 0 otherwise. Since the Bredon homology is concentrated in degrees 0 and 1 it follows the equivariant AHSS collapses trivially. In particular, we have $K_n^{\Gamma}(\underline{E}\Gamma) = H_n^{\mathcal{F}}(\underline{E}\Gamma; R_{\mathbb{C}})$ for $n = 0, 1$. □

5 Computations for $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$

5.1 Preliminaries

In an abuse of notation, throughout this section we will denote the image $\{\pm A\}$ of matrix $A \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ in $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ by the matrix A. Recall that for p a prime we have $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$, where $\Gamma_0(p)$ is the level p Hecke principle congruence subgroup (see for instance Serre's book "Trees" [29]). The amalgamation is specified by two embeddings of the congruence subgroup $\Gamma_0(p)$ into $\mathrm{PSL}_2(\mathbb{Z})$. The first is given by

$$\Gamma_0(p) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z})\colon c \equiv 0 \pmod{p} \right\}$$

and the second via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & pb \\ p^{-1}c & d \end{bmatrix}.$$

In light of this we will collect some facts about each of the groups in the amalgamation. We begin by recording (Table 3) the Fuchsian signatures and the associated Bredon homology for each of the groups $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z})$.

p	Signature of $\Gamma_0(p)$	$H_0^{\mathcal{F}}(\underline{E}\Gamma_0(p); R_{\mathbb{C}})$	$H_1^{\mathcal{F}}(\underline{E}\Gamma_0(p); R_{\mathbb{C}})$
2	$[0, 2; 2]$	$\mathbb{Z}^2$	$\mathbb{Z}$
3	$[0, 2; 3]$	$\mathbb{Z}^3$	$\mathbb{Z}$
$p \equiv 1 \pmod{12}$	$[0, \frac{1}{6}(p-7)+1; 2, 2, 3, 3]$	$\mathbb{Z}^7$	$\mathbb{Z}^{\frac{1}{6}(p-7)}$
$p \equiv 5 \pmod{12}$	$[0, \frac{1}{6}(p+1)+1; 2, 2]$	$\mathbb{Z}^3$	$\mathbb{Z}^{\frac{1}{6}(p+1)}$
$p \equiv 7 \pmod{12}$	$[0, \frac{1}{6}(p-1)+1; 3, 3]$	$\mathbb{Z}^5$	$\mathbb{Z}^{\frac{1}{6}(p-1)}$
$p \equiv 11 \pmod{12}$	$[0, \frac{1}{6}(p+7)+1;]$	$\mathbb{Z}$	$\mathbb{Z}^{\frac{1}{6}(p+7)}$
$\mathrm{PSL}_2(\mathbb{Z})$	$[0, 1; 2, 3]$	$\mathbb{Z}^4$	0

Table 3: Fuchsian signatures and Bredon homology groups of $\mathrm{P}\Gamma_0(p)$.

Lemma 5.1. *The signatures and Bredon homology groups listed in Table 3 are the signatures and Bredon homology groups of $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z})$.*

Proof. The Bredon homology follows from the computation of the signatures and Theorem 4.1. Note that the fact the signature of $\mathrm{PSL}_2(\mathbb{Z})$ is $[0, 1; 2, 3]$ is well known. To determine the signatures for the other groups, we first note that it follows from [13] that two finitely generated Fuchsian groups are isomorphic if and only if their (co)homology groups are isomorphic. As $\Gamma_0(p)$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$, both the torsion of $\Gamma_0(p)$ and the torsion in the cohomology of $\Gamma_0(p)$ contains elements of order 2 or 3. Moreover, in their signature $g = 0$.

Next, we consult the computations of the homology of principal congruence subgroups $\widetilde{\Gamma_0}(p)$ of $\mathrm{SL}_2(\mathbb{Z})$ in [1]. Now, consider the Lyndon-Hochschild-Serre spectral sequence for the group extension $\mathbb{Z}_2 \rightarrowtail \widetilde{\Gamma_0}(p) \twoheadrightarrow \Gamma_0(p)$ which takes the form

$$E_2^{r,s} = H^r(\Gamma_0(p); H^s(\mathbb{Z}_2;\mathbb{Z})) \Rightarrow H_{r+s}(\widetilde{\Gamma_0}(p);\mathbb{Z}).$$

We immediately deduce that $s = b_1(\widetilde{\Gamma_0}(p)) + 1$ and the m_j can be deduced from comparing the E_2-page, the results in [13] and the results in [1]. Indeed, each m_j can only equal 2 or 3 and corresponds to a $\mathbb{Z}_2$ or $\mathbb{Z}_3$ summand in the cohomology of $\Gamma_0(p)$. $\square$

We shall also record the conjugacy classes of finite order elements of $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. Note that the only conjugacy classes of finite subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ are one class of groups isomorphic to $\mathbb{Z}_2$ and one to $\mathbb{Z}_3$ since $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. The conjugacy classes of finite subgroups of $\Gamma_0(p)$ can be read off of the signature, there is exactly one of order m_j for each $j = 1, \ldots, r$.

Lemma 5.2. *The number of conjugacy classes of finite order elements in* $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ *are those given in Table 4.*

	$p=2$	$p=3$	$p \equiv 1 \pmod{12}$	$p \equiv 5 \pmod{12}$	$p \equiv 7 \pmod{12}$	$p \equiv 11 \pmod{12}$
Identity	1	1	1	1	1	1
Order 2	1	2	1	1	2	2
Order 3	4	2	2	4	2	4
Total	6	5	4	6	5	7

Table 4: Number of conjugacy classes of finite order elements of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

Proof. The result follows from the following observation: If there is a conjugacy of elements of order 2 (resp. 3) in $\Gamma_0(p)$, then each of class of elements of order 2 (resp. 3) in $\mathrm{PSL}_2(\mathbb{Z})$ fuses in $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. To see this, consider an element in the first copy of $\mathrm{PSL}_2(\mathbb{Z})$, conjugate it to an element in $\Gamma_0(p)$, and then conjugate it to an element in the other copy of $\mathrm{PSL}_2(\mathbb{Z})$. $\square$

Lemma 5.3. *Both* $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ *and* $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ *satisfy the Baum-Connes Conjecture.*

Proof. Since $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$, the Bass-Serre tree of the amalgamation is a locally-finite 1-dimensional contractible $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$-CW complex. Moreover, each of the stabilisers Γ_c have $\mathrm{cd}_{\mathbb{Q}}(\Gamma_c) = 1$, being a graph of finite groups. Now, we apply [20, Corollary 5.14] to see the stabilisers satisfy Baum-Connes and [20, Theorem 5.13] to see that $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ does. The proof is identical for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. $\square$

5.2 Computations

There is a long exact Mayer-Vietoris sequence for computing the Bredon homology of an amalgamated free product.

Theorem 5.4. [20, Corollary 3.32] *Let $\Gamma = H *_L K$ and let M be a Bredon module. There is a long exact Mayer-Vietoris sequence:*

$$\begin{array}{ccc} \cdots \longrightarrow H_n^{\mathcal{FIN}}(\underline{E}L \times_\Gamma \Gamma; M) & \longrightarrow & H_n^{\mathcal{FIN}}(\underline{E}H \times_\Gamma \Gamma; M) \oplus H_n^{\mathcal{FIN}}(\underline{E}K \times_\Gamma \Gamma; M) \\ & & \downarrow \\ \cdots \longleftarrow H_{n-1}^{\mathcal{FIN}}(\underline{E}L \times_\Gamma \Gamma; M) & \longleftarrow & H_n^{\mathcal{FIN}}(\underline{E}\Gamma; M) \end{array}$$

We are now ready to compute the K-theory of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

Proof of Theorem 1.5. There are 6 cases to consider. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $R_\mathbb{C} = R_\mathbb{C}(\Gamma)$. In each case we have the following long exact Mayer-Vietoris sequence

$$0 \longrightarrow H_2^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) \longrightarrow H_1^{\mathcal{FIN}}(\underline{E}\Gamma \times_\Gamma \Gamma_0(p); R_\mathbb{C}) \longrightarrow \left(H_1^{\mathcal{FIN}}(\underline{E}\Gamma \times_\Gamma \mathrm{PSL}_2(\mathbb{Z}); R_\mathbb{C})\right)^2$$

$$\longrightarrow H_1^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) \longrightarrow H_0^{\mathcal{FIN}}(\underline{E}\Gamma \times_\Gamma \Gamma_0(p); R_\mathbb{C}) \longrightarrow \left(H_0^{\mathcal{FIN}}(\underline{E}\Gamma \times_\Gamma \mathrm{PSL}_2(\mathbb{Z}); R_\mathbb{C})\right)^2$$

$$\longrightarrow H_0^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) \longrightarrow 0.$$

For $\Lambda \le \Gamma$ we have $H_n^{\mathcal{FIN}}(\underline{E}\Gamma \times_\Gamma \Lambda; R_\mathbb{C}(\Gamma)) \cong H_n^{\mathcal{FIN}}(\Lambda; R_\mathbb{C}(\Lambda))$. We also computed the Bredon homology groups of $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0(p)$ in Table 3. Thus, we can separate the above sequence into two sequences. Indeed, $H_1^{\mathcal{FIN}}(\mathrm{PSL}_2(\mathbb{Z}); R_\mathbb{C}(\mathrm{PSL}_2(\mathbb{Z}))) = 0$, so it follows that $H_2^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) \cong H_1^{\mathcal{FIN}}(\Gamma_0(p); R_\mathbb{C}(\Gamma_0(p)))$. The other sequence is then given by the remaining terms.

We will treat the case $p = 2$, the other cases are identical. We have $H_2^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) = \mathbb{Z}$ and an exact sequence

$$0 \longrightarrow H_1^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^8 \longrightarrow H_0^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) \longrightarrow 0.$$

We now compute the colimit $H_0^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) = \mathrm{colim}_{\Gamma/H \in \mathbf{Or}(\Gamma)} R_\mathbb{C}(H)$. Since we have a complete description of the conjugacy classes of finite subgroups of Γ and the only inclusions are given by $\{1\} \hookrightarrow \mathbb{Z}_2$ and $\{1\} \hookrightarrow \mathbb{Z}_3$, it follows that $H_0^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) = \mathbb{Z}^6$. Moreover, for the sequence to be exact, if follows the map $\mathbb{Z}^2 \to \mathbb{Z}^8$ must be an isomorphism onto the kernel of the first map. In particular $H_1^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) = 0$.

The equivariant K-homology then follows from applying the equivariant Atiyah-Hirzebruch spectral sequence, where we find $K_0^\Gamma(\underline{E}\Gamma) = H_0^{\mathcal{FIN}}(\Gamma; R_\mathbb{C}) \oplus H_2^{\mathcal{FIN}}(\Gamma; R_\mathbb{C})$ and $K_0^\Gamma(\underline{E}\Gamma) = H_1^{\mathcal{FIN}}(\Gamma; R_\mathbb{C})$. We record the Bredon homology groups for the remaining cases in Table 5, the reader can easily verify these. Note that they are always torsion-free and so are completely determined by their $\mathbb{Z}$-rank. $\square$

To extend the calculations to $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ we use the following proposition and then apply the equivariant Atiyah-Hirzebruch spectral sequence.

	$p=2$	$p=3$	$p\equiv 1 \pmod{12}$	$p\equiv 5 \pmod{12}$	$p\equiv 7 \pmod{12}$	$p\equiv 11 \pmod{12}$
$n=0$	6	5	4	6	5	7
$n=1$	0	0	3	1	2	0
$n=2$	1	1	$\frac{1}{6}(p-7)$	$\frac{1}{6}(p+1)$	$\frac{1}{6}(p-1)$	$\frac{1}{6}(p+7)$

Table 5: $\mathbb{Z}$-rank of the Bredon homology of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

Proposition 5.5. *Let $A \rightarrowtail \Gamma \twoheadrightarrow Q$ be a short exact sequence of groups, where A is a central finite abelian group of order n. If each finite subgroup F of Γ is abelian, then*

$$H_*^{\mathcal{FIN}}(\Gamma; R_{\mathbb{C}}) = \bigoplus_{i=1}^{n} H_*^{\mathcal{FIN}}(Q; R_{\mathbb{C}}).$$

Proof. Consider $\underline{E}Q$ as a Γ-space via the quotient map, clearly $\underline{E}Q$ is a model for $\underline{E}\Gamma$. Let $\overline{F}$ be the image of a cell-stabiliser F in Q, then $F \cong \overline{F} \times A$ or $F \cong \overline{F} \cdot A$, where $\cdot$ denotes a non-split central extension such that F is abelian. Passing to $C_*^{\Gamma}(\underline{E}\Gamma; R_{\mathbb{C}})$, the Bredon chain complex with coefficients in the representation ring, we see that each chain group is isomorphic to $|A|$ copies of the corresponding chain group in $C_*^{Q}(\underline{E}Q; R_{\mathbb{C}})$.

If each stabiliser splits as $F \cong \overline{F} \times A$, then result is immediate by the Künneth formula. If not then the problem reduces to a computation of induced representations of finite abelian groups. Indeed, the only situation where problems may occur is if A and $\overline{F}$ both contain a $\mathbb{Z}_n$ summand, such that F contains a $\mathbb{Z}_{n^2}$ summand. Assume H is the stabiliser of a cell one dimension higher, such that $H \leq F$, so $A \leq H$. There are two cases, either the differential in $C_*^{Q}(\underline{E}Q; R_{\mathbb{C}})$ restricted to $R_{\mathbb{C}}(\overline{H})$ is 0, and so the differential in $C_*^{\Gamma}(\underline{E}\Gamma; R_{\mathbb{C}})$ is also 0. Or, the differential restricts to an inclusion and so the differential in $C_*^{\Gamma}(\underline{E}\Gamma; R_{\mathbb{C}})$ is an inclusion as well. $\square$

6 The unstable Gromov-Lawson-Rosenberg conjecture

Given a smooth closed n-manifold M a classical question is to ask whether M admits a Riemannian metric of positive scalar curvature. In a vast generalisation of the Atiyah-Singer index theorem, Rosenberg [24] exhibits a class in $KO_n(C_r^*(\pi_1(M), \mathbb{R}))$ which is an obstruction to M admitting a metric of positive scalar curvature.

More precisely, let M be a closed spin n-manifold and $f\colon M \to B\Gamma$ be a continuous map for some discrete group Γ. Let $\alpha\colon \Omega_n^{\mathrm{Spin}}(B\Gamma) \to KO_n(C_r^*(\Gamma; \mathbb{R}))$ be the index of the Dirac operator. If M admits a metric of positive scalar curvature, then $\alpha[M, f] = 0 \in KO_n(C_r^*(\Gamma; \mathbb{R}))$

Conjecture 6.1 (The (unstable) GLR conjecture)**.** *Let M be a closed spin n-manifold and $\Gamma = \pi_1(M)$. If $f\colon M \to B\Gamma$ is a continuous map which induces the identity on the fundamental groups, then M admits positive scalar curvature if and on if $\alpha[M, f] = 0 \in KO_n(C_r^*(\Gamma; \mathbb{R}))$.*

The conjecture has been verified in the case of some finite groups [32, 25, 15, 23], when the group has periodic cohomology, torsion-free groups for which the dimension of $B\Gamma$ is less than 9 [14], and cocompact Fuchsian groups [8]. However, there are counterexamples, the first, isomorphic to a semi-direct product $\mathbb{Z}^4 \rtimes \mathbb{Z}_3$, is due to Schick [30], however, other counterexamples have since been constructed [14].

6.1 Proof of Theorem 1.7

We will now prove the conjecture for $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. Our proof is structurally similar to the proof by Davis-Pearson [8] so we will summarise their method and highlight any differences.

Let ko be the connective cover of KO with covering map p and let D be the ko-orientation of spin bordism. The map α (from above) is obtained by the following composition

$$\Omega_n^{\mathrm{Spin}}(B\Gamma) \xrightarrow{D} ko_n(B\Gamma) \xrightarrow{p} KO_n(B\Gamma) \xrightarrow{\mu_{\mathbb{R}}} KO_n(C_r^*(\Gamma;\mathbb{R}))$$

We note that $ko_n(*) = 0$ for $n < 0$ and that p is an isomorphism for $n \geq 0$ on the one point space.

Proposition 6.2. *Let $p \equiv 11 \pmod{12}$ be prime. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $X = \underline{E}\Gamma/\Gamma$ and $\mathcal{F} = \mathcal{FIN}$. Let Λ be a set of conjugacy classes of finite subgroups of Γ. There is a commutative diagram with exact rows*

$$\begin{array}{ccccccc}
\widetilde{KO}_{n+1}(X) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n(BH) & \longrightarrow & \widetilde{KO}_n(B\Gamma) & \longrightarrow & \widetilde{KO}_n(X) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \mu_{\mathbb{R}}} & & \downarrow{\scriptstyle \mu_{\mathbb{R}}} & & \downarrow{\scriptstyle id} \\
\widetilde{KO}_{n+1}(X) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n(C_r^*(H;\mathbb{R}))) & \longrightarrow & \widetilde{KO}_n(C_r^*(\Gamma;\mathbb{R}))) & \longrightarrow & \widetilde{KO}_n(X).
\end{array}$$

Proof. First, we observe that $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfies the following two conditions:

(M) Every finite subgroup is contained in a unique maximal finite subgroup.

(NM) If M is a maximal finite subgroup, then the normaliser $N_\Gamma(M)$ of M is equal to M.

These are both easily seen from the classification of finite subgroups and the amalgamated product decomposition. Note that the failure of (NM) for the other primes is the main obstruction to generalising this proof. Now, by either [6, Corollary 3.13] or the proof of [6, Theorem 4.1] for any constant functor $\mathbf{E}_c \colon \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Spectra}$ by $\Gamma/H \mapsto \mathbf{E}$ there are long exact sequences

$$\cdots \longrightarrow \bigoplus_{(H)\in\Lambda} H_n(BH;\mathbf{E}) \longrightarrow \left(\bigoplus_{(H)\in\Lambda} \pi_n(\mathbf{E})\right) \oplus H_n(B\Gamma;\mathbf{E}) \longrightarrow H_n(X;\mathbf{E}) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \bigoplus_{(H)\in\Lambda} \widetilde{H}_n(BH;\mathbf{E}) \longrightarrow \widetilde{H}_n(B\Gamma;\mathbf{E}) \longrightarrow \widetilde{H}_n(X;\mathbf{E}) \longrightarrow \cdots$$

The result then follows a diagram chase exactly as in [8, Proposition 4], taking $\mathbf{E} = \mathbf{KO}$ and the isomorphism

$$\pi_n\left(\operatorname*{hocolim}_{\mathbf{Or}_{\mathcal{F}}(\Gamma)}(\mathbf{E}_c)\right) \cong H_n(X;\mathbf{E}).$$

$\square$

Proof of Theorem 1.7. Let $p \equiv 11 \pmod{12}$ be prime. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $X = \underline{E}\Gamma/\Gamma$. Let $\mathbf{ko}$ be the spectrum of the connective cover of $\mathbf{KO}$. Via the cover we obtain a natural transformation $p : \mathbf{ko}_c \to \mathbf{KO}_c$ of constant $\mathbf{Or}_{\mathcal{F}}(\Gamma)$-$\mathbf{Spectra}$. From the previous proposition we obtain a commutative diagram

$$\begin{tikzcd}
\widetilde{ko}_{n+1}(X) \arrow[r] \arrow[d, "p"] & \bigoplus_{(H)\in\Lambda} \widetilde{ko}_n(BH) \arrow[r] \arrow[d, "\mu_{\mathbb{R}}\circ p"] & \widetilde{ko}_n(B\Gamma) \arrow[r] \arrow[d, "\mu_{\mathbb{R}}\circ p"] & \widetilde{ko}_n(X) \arrow[d, "p"] \\
\widetilde{KO}_{n+1}(X) \arrow[r] & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n(C_r^*(H;\mathbb{R}))) \arrow[r] & \widetilde{KO}_n(C_r^*(\Gamma;\mathbb{R}))) \arrow[r] & \widetilde{KO}_n(X).
\end{tikzcd} \tag{$\dagger$}$$

We claim that $\Sigma X = \Sigma(\underline{E}\Gamma/\Gamma) \simeq \bigvee_{b_1(\Gamma_0(p))} S^3$, where $b_1(\Gamma_0(p))$ is the first Betti number of $\Gamma_0(p)$. Indeed, we have

$$\begin{aligned}
X &\simeq \operatorname*{hocolim}_{\mathbf{Top}} \left((\underline{E}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma) \leftarrow (\underline{E}\Gamma_0(p) \times_{\Gamma_0(p)} \Gamma) \to (\underline{E}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma)\right)/\Gamma, \\
&\simeq \operatorname*{hocolim}_{\mathbf{Top}} \left((\underline{E}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma)/\Gamma \leftarrow (\underline{E}\Gamma_0(p) \times_{\Gamma_0(p)} \Gamma)/\Gamma \to (\underline{E}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma)/\Gamma\right).
\end{aligned}$$

Since $\underline{E}\mathrm{PSL}_2(\mathbb{Z})/\mathrm{PSL}_2(\mathbb{Z})$ is an interval and $\underline{E}\Gamma_0(p)/\Gamma_0(p)$ is a finite graph, we have

$$X \simeq \operatorname*{hocolim}_{\mathbf{Top}} \left(I \leftarrow \bigvee_{b_1(\Gamma_0(p))} S^1 \to I \right),$$

but I is contractible, so up to homotopy this becomes a suspension of a wedge of circles. In particular,

$$X \simeq \bigvee_{b_1(\Gamma_0(p))} S^2.$$

It follows that $\widetilde{ko}(X) \cong ko_{n-2}(*)^{b_1(\Gamma_0(p))}$ and $\widetilde{KO}(X) \cong KO_{n-2}(*)^{b_1(\Gamma_0(p))}$, therefore, the natural transformation p is an isomorphism for $n \geq 2$. Now, suppose that $n \geq 5$ so we are in the setting of the GLR conjecture. Consider an element

$$\beta \in K := \mathrm{Ker}\left(\mu_{\mathbb{R}} \circ p : ko_n(B\Gamma) \to KO_n(C_r^*(\Gamma;\mathbb{R}))\right)$$

and note $K \cong \mathrm{Ker}(\mu_{\mathbb{R}} \circ p : \widetilde{ko}_n(B\Gamma) \to \widetilde{KO}_n(C_r^*(\Gamma;\mathbb{R}))$. Combining the diagram ($\dagger$) with the isomorphism $p \colon \widetilde{ko}_n(X) \to \widetilde{KO}_n(X)$, we can deduce that there exists

$$\gamma \in \mathrm{Ker}\left(\bigoplus_{(H)\in\Lambda} ko_n(BH) \to \bigoplus_{(H)\in\Lambda} KO_n(BH) \right)$$

which maps to β.

For a group L let $ko_n^+(BL)$ be the subgroup of $ko_n(BL)$ given by $D[M,f]$ where M is a positively curved spin manifold and f is a continuous map. In [3] the authors prove for any finite cyclic group H that $ko_n^+(BH) = \mathrm{Ker}(A \circ p : ko_n(BH) \to KO_n(C_r^*(H;\mathbb{R}))$. Thus, we have $\gamma \in ko_n^+(BH)$ and $\beta \in ko_n^+(B\Gamma)$. Now, in [33] it is proven that if $D[M,f] \in ko_n^+(BG)$, then M admits a metric of positive scalar curvature. In particular, we are done. $\square$

An alternative direct proof of the calculations of the K-groups of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ when $p \equiv 11 \pmod{12}$ is given as follows. Note that this bypasses the computation of the Bredon homology but does not give us a means to compute either invariant for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$.

Alternative proof of Theorem 1.5. Let Λ by a set of representatives of finite subgroups of Γ. By [6, Theorem 4.1(a)] we have a short exact sequence

$$0 \to \bigoplus_{(H)\in\Lambda} \widetilde{K}_n(C_r^*(H)) \to K_n(C_r^*(\Gamma)) \to K_n(\underline{\mathrm{E}}\Gamma/\Gamma) \to 0.$$

The only nontrivial part now is computing $K_n(\underline{\mathrm{E}}\Gamma/\Gamma)$, but we have already shown that $\underline{\mathrm{E}}\Gamma/\Gamma$ is homotopy equivalent to a wedge of $\bigvee_{b_1(\Gamma_0(p))} S^2$, i.e. a wedge of 2-spheres. Thus, we can simply apply the homological Atiyah-Hirzebruch spectral sequence (which collapses trivially) to obtain that $K_0(\underline{\mathrm{E}}\Gamma/\Gamma) = \mathbb{Z}^{b_1(\Gamma_0(p)+1)}$ and $K_1(\underline{\mathrm{E}}\Gamma/\Gamma) = 0$. □

6.2 Some other groups

Let Γ be a discrete group. In light of the proof of Theorem 1.7 we introduce the following condition on the homotopy type of the suspension of $\underline{\mathrm{E}}\Gamma/\Gamma$.

(WOS)$_n$ The space $\Sigma(\underline{\mathrm{E}}\Gamma/\Gamma)$ has the homotopy type of a wedge of spheres of dimension less than n.

We also introduce the following notation:

(BC) Γ satisfies the Baum-Connes conjecture.

(GLR) Γ satisfies the unstable GLR conjecture.

(GLR)$_n$ Γ the unstable GLR conjecture for all manifolds of dimension greater than or equal to n.

Corollary 6.3. *Let Γ be a discrete group satisfying (BC), (M), (NM) and (WOS)$_n$, then Γ satisfies (GLR)$_n$. In particular, if $n \leq 5$ then Γ satisfies (GLR).*

Let $X = \underline{\mathrm{E}}\Gamma/\Gamma$. One source of examples of groups satisfying (WOC)$_{n+2}$, for some n, is provided by groups Γ such that $H_*(X;\mathbb{Z})$ is torsion-free and concentrated in at most 2 consecutive non-zero dimensions (dimensions n and $n+1$). In this case we have a cofibration sequence:

$$\vee_k S^n \to \vee_l S^n \to X \to \vee_k S^{n+1} \to \vee_l S^{n+1} \to \Sigma X \to \vee_k S^{n+2} \to \cdots$$

Or if $H_*(X;\mathbb{Z})$ is concentrated in one non-zero degree:

$$\vee_k S^n \to * \to X \to \vee_k S^{n+1} \to * \to \Sigma X \to \vee_k S^{n+2} \to \cdots$$

Now, applying H_* to either sequence, from the Hurewicz Theorem we obtain a homotopy equivalence $\Sigma X \simeq (\vee_k S^{n+2}) \vee (\vee_l S^{n+1})$ or $\Sigma X \simeq \vee_k S^{n+2}$.

In particular, arbitrary free products of the following groups satisfy (GLR)$_n$: One relator groups where the exponent sum of each letter in the relation is equal to 0, 1 or -1, cyclic groups, Fuchsian groups, $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$, and $\mathbb{Z}^k$ for $k < n$.

Another source of examples is given by Leary's Kan-Thurston type theorem for $\underline{\mathrm{E}}\Gamma/\Gamma$ [17].

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