On the Baum–Connes Conjecture for Groups Acting on CAT(0)-Cubical Spaces

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We give a new proof of the Baum–Connes conjecture with coefficients for any second countable, locally compact topological group that acts properly and cocompactly on a finite-dimensional CAT(0)-cubical space with bounded geometry. The proof uses the Julg–Valette complex of a CAT(0)-cubical space introduced by the 1st three authors and the direct splitting method in Kasparov theory developed by the last author.

1 Introduction

The computation of K-theory groups for crossed product C^*-algebras is a central problem in C^*-algebra theory, the solution of which has deep applications to manifold topology and to the algebra of group rings. The Baum–Connes conjecture proposes a general form of the solution. It asserts that the *Baum–Connes assembly map*

$$K_*^G(\underline{E}G; A) \longrightarrow K_*(C_r^*(G, A)) \tag{1.1}$$

is an isomorphism for any second countable, locally compact group G and any separable coefficient G-C^*-algebra A. Here $\underline{E}G$ is a universal proper G-space and $C_r^*(G, A)$ is the reduced crossed product. See [2] for details and discussion.

Our launching point is the following theorem [9, 10]:

Theorem (Higson, Kasparov).   Let G be a second countable, locally compact group. If G admits an isometric and metrically proper action on a (possibly infinite dimensional) Euclidean space then the Baum–Connes assembly map (1.1) is an isomorphism for G and any separable coefficient algebra A.

Gromov introduced the term *a-T-menable* for the groups in the statement of this theorem. Our purpose here is to reexamine the Baum–Connes isomorphism for a class of a-T-menable groups having a very geometric flavor. We consider groups which admit a proper, cellular action on a *bounded geometry* CAT(0)-cubical space. A CAT(0)-cubical space is a simply-connected topological space that is built by gluing Euclidean cubes along their faces in a manner that satisfies a local combinatorial condition. A motivating example is a finite product of simplicial trees. The bounded geometry condition asserts that there is a uniform bound on the number of edges to which a vertex can belong.

Building our previous work [3], we shall prove the following theorem. While the result itself is a consequence of the Higson–Kasparov theorem, the approach here is direct, geometrical, and better rooted in simple finite-dimensional constructions.

Theorem A.   The Baum–Connes assembly map (1.1) is an isomorphism for a second countable, locally compact topological group G that acts properly and cocompactly on a bounded geometry CAT(0)-cubical space and any separable coefficient algebra A.

We shall refer to the groups that we are considering here as CAT(0)-cubical groups. The essential difference between CAT(0)-cubical and a-T-menable groups, and so also between the Higson–Kasparov theorem and our work, is the difference between a combinatorial and a measured setting. It is very much analogous to the difference between a simplicial tree and an $\mathbb{R}$-tree.

A key feature of CAT(0)-cubical spaces is the existence of *hyperplanes*, and indeed a CAT(0)-cubical space can be essentially reconstructed from the combinatorics of its hyperplanes. Each hyperplane divides the space into precisely two connected components, and a hyperplane *separates* a pair of points if they are in different components. Counting the number of hyperplanes that separate a pair of points defines a distance and, restricting to a G-orbit, we obtain a distance on G, which is intimately

related to the a-T-menability of G through the theory of conditional negative-type functions. Compare [15]. The analogous *measured* notion is that of a space with measured walls (hyperplanes). For this notion, the combinatorics of counting is replaced by a measure on a set of hyperplanes. The existence of a proper action on a space with measured walls characterizes a-T-menability; this was proven for countable discrete groups in [5, 19] and later extended to the locally compact and second countable setting in [4].

The following concrete example is analyzed in [8]. The Baumslag–Solitar group

$$BS(2,3) = \langle\, a, t \mid ta^2t^{-1} = a^3 \,\rangle$$

is a-T-menable but cannot act properly on a CAT(0)-cubical space. Haglund points out that $BS(2,3)$ admits a proper action on the product of a simplicial tree (its Bass–Serre tree) and the hyperbolic plane, which is a space with measured walls, and hence $BS(2,3)$ is a-T-menable. However, since the cyclic subgroup generated by a is distorted, that subgroup must have a fixed point in any action of $BS(2,3)$ on a CAT(0)-cubical space.

A second, less substantial difference between our result and the Higson–Kasparov theorem is related to the bounded geometry hypothesis. Farley proved that Thompson's group F acts properly on a locally finite but *infinite dimensional* CAT(0)-cubical space, and so this group is a-T-menable [7]. Nevertheless, F (apparently) falls outside the scope of our results. Extending our work to the locally finite but infinite dimensional setting may be possible, but would entail significant technical hurdles.

In the balance of this introduction, we shall place this piece in the broader context of our previous work [3, 17] and explain in outline our new proof of Theorem A. Assume given a second countable, locally compact group G acting by automorphisms on a bounded geometry CAT(0)-cubical space. The Kasparov representation ring of G, denoted $R(G) = KK_G(\mathbb{C}, \mathbb{C})$ plays a central role here, and indeed in the Baum–Connes theory in general. Associated to G there are three different Kasparov cycles for $R(G)$. It emerges that they represent the same element, but they have different properties, and it is precisely these differences that allow us to draw conclusions. The three cycles are as follows:

$$
\begin{aligned}
D_{\mathrm{JV}} &\ :\ \text{the Julg–Valette cycle,}\\
D_{\mathrm{PS}} &\ :\ \text{the Pytlik–Szwarc cycle,}\\
D_{\mathrm{dR}} &\ :\ \text{the de Rham cycle.}
\end{aligned}
$$

The 1st cycle was introduced in [3], although it will reappear below in slightly modified form. The 2nd cycle is from [3], and the 3rd cycle is the focus of this paper.

A cycle for $R(G)$ is an odd, self-adjoint operator on a separable, graded Hilbert space H equipped with a representation of G by grading preserving unitary operators. The operator satisfies axioms related to approximate invertibility and approximate G-equivariance. The exact form of the axioms depends on whether one works with bounded or unbounded cycles, and we refer to Section 2 below for precise definitions.

Here then are the differences among the three cycles relevant to this work. The Pytlik–Szwarc cycle is constructed directly from the combinatorics of hyperplanes, and the operator is an *exactly G-equivariant Fredholm operator of index one.* It follows immediately that the Pytlik–Szwarc cycle represents the identity of $R(G)$. The unitary representations appearing in the Julg–Valette cycle are the permutation representations on the set of cubes (of various dimensions) comprising the CAT(0)-cubical space. Assuming G acts properly, these representations are *weakly contained in the regular representation* of G. The operator in the Julg–Valette cycle is *not* exactly G-equivariant (although, it is approximately G-equivariant, as required by the axioms). Finally, assuming the action is both proper and cocompact, the de Rham cycle satisfies *property (γ)* recently introduced in [17]. The de Rham cycle will be defined and analyzed in detail in Sections 4 and 5 below. We shall prove Theorem A by proving the following result:

Theorem B.   Let G be a second countable, locally compact topological group that acts by automorphisms on a bounded geometry CAT(0)-cubical space. The three cycles above all represent the multiplicative unit of the representation ring:

$$[D_{\mathrm{dR}}] = [D_{\mathrm{JV}}] = [D_{\mathrm{PS}}] = 1 \in R(G). \tag{1.2}$$

In particular,

(1)   if G acts properly then it is K-amenable, since $[D_{\mathrm{JV}}] = 1$; and

(2)   if G acts both properly and cocompactly then it satisfies the Baum–Connes conjecture, since $[D_{\mathrm{dR}}] = 1$.

The equalities (1.2) are the heart of this theorem. Assuming these, and the properties of the various cycles outlined above, conclusions (1) and (2) are immediate; the 1st conclusion is essentially the definition of K-amenability [6], and the 2nd conclusion is an application of Nishikawa's direct splitting method [17]. As for the

individual equalities comprising (1.2), the 1st conclusion is the main technical result of this piece, and is the content of Theorem 5.1. The 2nd conclusion is the main technical result of [3] and, as remarked above, the final equality is obvious.

The paper is organized as follows. Sections 2 and 3 contain background material. In particular, in Section 2 we review the direct splitting method from [17] and in Section 3 we recast the definition of the Julg–Valette cycle from [3] in a form convenient for what follows. We introduce the de Rham cycle in Section 4 and prove that it is an unbounded Kasparov cycle for $R(G)$. See Theorem 4.12. Finally, we analyze the de Rham cycle in Section 5. The key results here are the equality of the de Rham and Julg–Valette cycles in $R(G)$ and that the de Rham cycle has Property (γ). See Theorems 5.1 and 5.2, respectively.

Postscript. February 10, 2020. Improvements made to the direct splitting method in [18] make it now unnecessary to assume that the action of G on the cube complex X be cocompact (but the bounded geometry assumption remains). That is, the hypothesis of cocompactness in part (2) of Theorem B above can be dropped. No changes to the constructions or proofs in this paper are required. Instead, in [18] the hypothesis of cocompactness is dropped from the definition of property (γ), including the entirety of condition (2b) in Definition 2.2 below, and the results from [17] that we cite in Section 2 below are shown to hold in the noncocompact case. See [18] for further details.

2   The Direct Splitting Method

In this section, G will be an arbitrary second countable, locally compact topological group that admits a second countable, locally compact and G-compact model $E = \underline{E}G$ of the universal proper G-space (see [2] for more information about $\underline{E}G$).

We begin by recalling the definition of Kasparov cycles for $KK_G(\mathbb{C}, \mathbb{C})$ from [12]. For a Hilbert space H, we denote by $\mathfrak{K}(H)$ the algebra of all compact operators on H.

Definition 2.1.   A Kasparov cycle for $KK_G(\mathbb{C}, \mathbb{C})$ is a pair (H, F) where H is a separable, graded G-Hilbert space and F is an odd, self-adjoint bounded G-continuous operator on H satisfying the following two conditions:

(1) $1 - F^2 \in \mathfrak{K}(H)$; and
(2) $g(F) - F \in \mathfrak{K}(H)$ for any $g \in G$,

where $g(F) = u_g F u_g^*$, with $g \mapsto u_g$ the unitary representation of G on H.

There is a natural notion of homotopy of Kasparov cycles (see [12, Definition 2.3]). The commutative ring $R(G) = KK_G(\mathbb{C}, \mathbb{C})$ is defined as the set of homotopy equivalence classes of Kasparov cycles. We write by $[H, F]$, the element in $R(G)$, which corresponds to a cycle (H, F). Among the Kasparov cycles there is a cycle of the form $[\mathbb{C} \oplus 0, 0]$, which corresponds to the trivial representation of G. This cycle is denoted by 1_G and it is the multiplicative identity element of the ring $R(G)$.

For a locally compact, Hausdorff space X and a C^*-algebra A, we denote by $C_0(X, A)$, the C^*-algebra of continuous A-valued functions on X vanishing at infinity. Recall that a compactly supported, continuous, non-negative function c on a (cocompact) proper G-space X such that the Haar integral

$$\int_{g \in G} g(c)^2 \mathrm{d}\mu_G(g) = 1$$

is called a cut-off function. We also note that for any closed subgroup G' of G, a Kasparov cycle for $KK_G(\mathbb{C}, \mathbb{C})$ can be viewed as a cycle for $KK_{G'}(\mathbb{C}, \mathbb{C})$.

Definition 2.2 (See [17, Definition 2.2]). We say a Kasparov cycle (H, F) for $KK_G(\mathbb{C}, \mathbb{C})$ has Property (γ) if the following conditions are satisfied:

(1) $[H, F] = 1_K$ in $R(K)$ for any compact subgroup K of G;
(2)   there is a non-degenerate, G-equivariant representation of the G-C^*-algebra $C_0(E)$ on H satisfying the following:

(a)   the function $g \mapsto [g(\phi), F]$ belongs to $C_0(G, \mathfrak{K}(H))$ for every $\phi \in C_0(E)$; and

(b)   for some cut-off function c on E

$$\int_{g \in G} g(c) F g(c) \mathrm{d}\mu_G(g) - F \in \mathfrak{K}(H),$$

where the Haar integration appearing here is taken in the strong topology.

The main result of [17] is as follows:

Theorem 2.3 (See [17, Corollary 5.6]). If the element $1_G \in KK_G(\mathbb{C}, \mathbb{C})$ is represented by a Kasparov cycle (H, F) with Property (γ), then the Baum–Connes conjecture with coefficients holds for G.

We end this section by recalling the definition of unbounded cycles for $KK_G(\mathbb{C}, \mathbb{C})$ and sufficient conditions for such a cycle to define a (bounded) cycle with Property (γ).

Definition 2.4 (Compare [1]).   An *unbounded cycle* for $KK_G(\mathbb{C}, \mathbb{C})$ consists of a separable, graded G-Hilbert space H and an odd-graded self-adjoint unbounded operator on H for which

(1)   The operator D has compact resolvent.

(2)   The G-action on H preserves the domain of D, and $D - g(D)$ extends to a bounded operator on H, for every $g \in G$, and defines a strongly continuous, locally bounded, bounded operator-valued function of $g \in G$.

Lemma 2.5 (See [1]).   If (H, D) is an unbounded Kasparov cycle for $KK_G(\mathbb{C}, \mathbb{C})$, and if F is the bounded transform

$$F = D(1 + D^2)^{-\frac{1}{2}},$$

then (H, F) is a bounded Kasparov cycle for $KK_G(\mathbb{C}, \mathbb{C})$.

Theorem 2.6 (See [17, Theorem 6.1]).   Let (H, D) be an unbounded Kasparov cycle for $KK_G(\mathbb{C}, \mathbb{C})$. Let B be a dense G-subalgebra of $C_0(E)$ consisting of compactly supported functions that includes cutoff functions for the G-action. Assume that there is a non-degenerate representation of the G-C^*-algebra $C_0(E)$ on H, which has the following properties:

(1)   Each $b \in B$ preserves the domain of D, the commutator $[D, g(b)]$ extends to a bounded operator on H, and defines a uniformly bounded, bounded operator-valued function of $g \in G$.

(2)   For every $b \in B$ there is a compact subset $K \subseteq E$ such that

$$\mathrm{Supp}\,([D, g(b)]) \subseteq g \cdot K$$

for all $g \in G$.

If F is the bounded transform of D, then the pair (H, F) is a cycle for $KK_G(\mathbb{C}, \mathbb{C})$ that satisfies the conditions (2a) and (2b) of Property (γ) in Definition 2.2.

3   The Julg–Valette Cycle

In this section, we shall briefly review the construction in [3] of the *Julg–Valette complex* for a CAT(0)-cubical space and the associated *Julg–Valette cycle* for $KK_G(\mathbb{C}, \mathbb{C})$. The

construction is based on the ideas that Julg and Valette applied to the case of trees [11], but the extension to general CAT(0)-cubical spaces requires some additional care. One specific issue is the need to equip cubes with *orientations*, and we shall present here an alternative but equivalent approach to the one in [3] that is better suited to our later arguments. Otherwise, our account here is identical to the original.

We first recall some necessary ideas from the theory of CAT(0)-cubical spaces. For more details see [3] and the references given there, especially [16].

For the rest of this paper, X will be a bounded geometry CAT(0)-cubical space, as in [16, Section 2.2]. In particular, X is obtained by identifying cubes, each of which is isometric to the standard Euclidean cube $[0, 1]^q$ of the appropriate dimension, by isometries of their faces. We shall refer to 0-cubes as vertices, 1-cubes as edges, etc. The bounded geometry condition asserts that there is a uniform bound on the number of edges that contain a given vertex, and this implies that X is finite dimensional. Further, we shall assume all group actions on X are by automorphism of its structure as a CAT(0)-cubical space. In particular, a group acting on X permutes the vertices, the edges, and the q-cubes for each fixed q.

The *midplanes* of a q-cube C in X correspond to the intersections of the standard cube with the coordinate hyperplanes $x_j = \frac{1}{2}$ $(j = 1, \ldots, q)$ and there are therefore q midplanes of C. We generate an equivalence relation on the collection of all the midplanes of all the cubes in X by declaring as equivalent any two midplanes whose intersection is itself a midplane. The union of all midplanes in a given class is called a *hyperplane*, and each hyperplane H satisfies the following important properties: (i) H is connected and (ii) the complement $X \setminus H$ has precisely two path components. We shall say that a hyperplane H *separates* two subsets of X if these subsets lie in distinct components of $X \setminus H$. A group acting on X necessarily permutes the hyperplanes. The hyperplanes of X play an important role in what follows.

Example 3.1.   In the case of a tree, the hyperplanes are simply the midpoints of edges. In the case of a product of two CAT(0)-cubical spaces (for instance, a product of two trees), the hyperplanes are the products of one of the spaces with a hyperplane in the other.

We shall handle the issue of orientations that we mentioned above by using differential forms. By a *smooth form* on a cube C we shall always mean a complex-valued C^∞-differential form defined on all of C, including its boundary. So on the cube

$[0, 1]^q$ the smooth p-forms can be written

$$\alpha = \sum_I f_I \mathrm{d}x_I,$$

where $I = \{ i_1 < \cdots < i_p \}$ is a multi-index, and f_I is a smooth, complex-valued function on the cube $[0, 1]^q$. The interior of each cube is a Riemannian manifold of volume one, and so each space $\Omega^p(C)$ of smooth p-forms carries a natural inner product.

Definition 3.2. If C is an individual cube in X, then we shall denote by $A(C)$ the one-dimensional Hilbert space of top-degree differential forms on C with constant coefficients.

Definition 3.3. For $q = 0, \ldots, \dim(X)$ we shall denote by $A^q(X)$ the algebraic direct sum of the spaces $A(C)$ over all q-cubes in X. We shall denote by $A^q_{L^2}(X)$ the Hilbert space orthogonal direct sum of the spaces $A(C)$, and by $A^*_{L^2}(X)$ the direct sum over q of the spaces $A^q_{L^2}(X)$.

We now define a differential

$$d_{\mathrm{JV}} \colon A^q(X) \longrightarrow A^{q+1}(X) \tag{3.1}$$

using the following notion of adjacency.

Definition 3.4. Let C be a cube in X. A hyperplane H is *adjacent to C* if it is disjoint from C, but intersects a cube that includes C as a codimension-one face.

Example 3.5. In the case of a tree, each hyperplane is adjacent to precisely two vertices, namely the endpoints of the edge in which the hyperplane is included.

At this point we need to choose a *base vertex P* in X, which will remain fixed throughout the rest of the paper.

Definition 3.6. Let H be a hyperplane and let D be a cube in X that intersects H (necessarily in a midplane of D). Define a coordinate function

$$x_H \colon D \longrightarrow \mathbb{R}$$

as follows:

    (1) x_H is an affine function on D.

    (2) x_H assumes the value 1 on the codimension-one face of D that is disjoint from H and separated by H from the base vertex, while it assumes the value 0 on the opposite codimension-one face, disjoint from H but not separated by H from the base vertex.

Then define $\alpha_H = dx_H$. This is a constant-coefficient one-form on D.

Definition 3.7.   A *weight function* for X is a positive function on the set of hyperplanes.

Definition 3.8.   A weight function w is *proper* if the set $\{H : w(H) \leq M\}$ is finite for any $M > 0$.

Definition 3.9.   Let G be a second countable and locally compact group acting on the CAT(0)-cubical space X by automorphisms. A weight function w is G-adapted if

$$\sup_H \left| w(H) - w(gH) \right| < \infty$$

for every $g \in G$ and if there is an open subgroup of G such that $w(gH) = w(H)$ for every H and every element g of the subgroup.

Remark 3.10.   Of course, if G is discrete the final condition is automatic. In general it would be sufficient to require that $g \mapsto w(gH)$ is continuous for every H, but the stronger condition we have chosen is more convenient.

    In what follows, our preferred weight function will be

$$w(H) = \text{distance}(P, H), \tag{3.2}$$

which is both G-adapted and proper, if the complex X has bounded geometry (see Section 1). However, the construction that follows can be carried out for a general weight function and it will be helpful for us not to fix a weight function until the final section of the paper.

Definition 3.11.   If $\beta \in A(C)$, then we define

$$d_{\mathrm{JV}} \colon \beta \longmapsto \sum_{H \in \mathrm{SAH}(C)} w(H) \cdot \alpha_H \wedge \beta, \tag{3.3}$$

where

(1)   The sum is over the set $\mathrm{SAH}(C)$ of all hyperplanes that are adjacent to C and separate C from the base vertex.

(2)   On the right-hand side of the formula, β is pulled back from C to the cube D_H that is intersected by H and that contains C as a codimension-one face (via the orthogonal projection from D_H to its face C). Thus, the summands in the formula above lie in different direct summands of $A^{q+1}(X) = \oplus_{\dim(D)=q+1} A(D)$.

It is shown in [3] that d_{JV} is indeed a differential, and that the cohomology of the *Julg–Valette complex*

$$A^0(X) \xrightarrow{\ d_{\mathrm{JV}}\ } A^1(X) \xrightarrow{\ d_{\mathrm{JV}}\ } \cdots \qquad \xrightarrow{\ d_{\mathrm{JV}}\ } A^{\mathrm{N}}(X)\,, \quad \mathrm{N} = \dim(X) \tag{3.4}$$

vanishes in positive degrees and is one-dimensional in degree zero.

Example 3.12.   If X is a tree, then for each 0-cube except the base vertex there is a unique hyperplane that is adjacent to the cube and separates it from the base vertex. If we set $w(H) \equiv 1$ then the single differential in the Julg–Valette complex identifies with operator studied in [11].

Example 3.13.   If X is a product of two CAT(0)-cubical spaces (for instance, a product of trees), then the Julg–Valette complex for X identifies with the tensor product of the Julg–Valette complexes for the factors. Note that in this case a pair of weight functions, one on each factor, determines a weight function on the product; see Example 3.1.

The Julg–Valette differential in Definition 3.11 has a formal adjoint

$$\delta_{\mathrm{JV}} \colon A^{q+1}(X) \longrightarrow A^q(X)$$

with respect to the inner products of Definition 3.3, given by the formula

$$\delta_{\mathrm{JV}}\colon \beta \longmapsto \sum_{H \in \mathrm{Mid}(C)} w(H) \cdot \iota_H(\beta), \tag{3.5}$$

for $\beta \in A(C)$, where:

(1) The sum is over the set $\mathrm{Mid}(C)$ of hyperplanes that cut through C (i.e., they include a midplane from C).

(2) On the right-hand side, $\iota_H(\beta)$ is contraction of β by the coordinate vector field dual to the coordinate x_H, followed by restriction to the face of C where $x_H = 1$.

See [3, Prop. 3.19].

Definition 3.14.   The *Julg–Valette operator* is the unbounded operator

$$D_{\mathrm{JV}} = d_{\mathrm{JV}} + \delta_{\mathrm{JV}}$$

on the Hilbert space

$$A^*_{L^2}(X) = \bigoplus_q A^q_{L^2}(X),$$

with domain $A^*(X) = \bigoplus_q A^q(X)$.

If $\beta \in A(C)$, then it is not difficult to compute that

$$D^2_{\mathrm{JV}}\beta = \sum_{H \in \mathrm{SAH}(C)} w(H)^2\beta + \sum_{H \in \mathrm{Mid}(C)} w(H)^2\beta. \tag{3.6}$$

See [3, Prop. 3.25]. The following result is an easy consequence:

Lemma 3.15 ([3, Lemmas 9.4 and 9.5]).   If w is a proper weight function, then D_{JV} is an essentially self-adjoint Hilbert space operator with compact resolvent.

The Julg–Valette operator is also nearly equivariant in the sense appropriate to equivariant *KK*-theory:

Lemma 3.16 ([3, Lemma 9.6]).   If a group G acts by automorphisms of X, and if the weight function is G-adapted, then for every $g \in G$ the operator $D_{\mathrm{JV}} - g(D_{\mathrm{JV}})$ extends to

a bounded operator on H, and the formula

$$g \longmapsto D_{\mathrm{JV}} - g(D_{\mathrm{JV}})$$

defines a strongly continuous, locally bounded, bounded operator-valued function on G.

Remark 3.17.   The continuity and local boundedness of the function in the lemma are automatic, since in fact the function is locally constant: if g lies in a sufficiently small open subgroup of G, then $g(D_{\mathrm{JV}}) = D_{\mathrm{JV}}$.

Therefore, given any proper and G-adpated weight function, we obtain an unbounded Kasparov cycle for $KK_G(\mathbb{C}, \mathbb{C})$ in the sense of Definition 2.4. It is actually independent of the choice of weight function thanks to the homotopy invariance of KK-theory, and we shall use the notation

$$[D_{\mathrm{JV}}] \in KK_G(\mathbb{C}, \mathbb{C}).$$

The following is the main technical result from the paper [3].

Theorem 3.18 ([3, Theorem 9.14]).   Assume that X is a bounded geometry (and finite dimensional) CAT(0)-cubical space, and that a locally compact, second countable group G acts on X by automorphisms. Then $[D_{\mathrm{JV}}] = 1_G \in KK_G(\mathbb{C}, \mathbb{C})$.

Remark 3.19.   Actually, Theorem 9.14 in [3] proves that $[D_{\mathrm{JV}}]$ is equal to the Pytlik–Szwarc element $[D_{\mathrm{PS}}]$ discussed in Section 1 of this paper. But as we noted there, the Pytlik–Szwarc element is in turn equal to the identity element 1_G for elementary reasons.

4   A de Rham-Type Complex

In this section we shall use differential forms and the de Rham operator to construct a new unbounded Kasparov cycle D_{dR} for the Kasparov group $KK_G(\mathbb{C}, \mathbb{C})$. In the next section we shall show that $[D_{\mathrm{dR}}] = [D_{\mathrm{JV}}]$ and that, in the case of a proper and cocompact action, the bounded transform of D_{dR} has Property (γ).

We begin by reviewing a few simple facts about the de Rham differential on a single cube C in X.

Definition 4.1.   For $p \geq 0$, we shall denote by $\Omega^p(C)$ the space of smooth differential p-forms on C, by $\Omega^p_{L^2}(C)$ the Hilbert space of L^2-differential p-forms on C, by $\Omega^p_0(C) \subseteq \Omega^p(C)$ the space of those smooth p-forms on C that pull back to zero on each open face of C, and by $\Omega^p_{00}(C)$ the space of smooth forms that are compactly supported in the interior of C.

We shall consider the de Rham differential d as an unbounded Hilbert space operator from $\Omega^p_{L^2}(C)$ to $\Omega^{p+1}_{L^2}(C)$ with domain $\Omega^p_0(C)$. If we denote by $d^\diamond$ the formal adjoint of d in the sense of partial differential equations, so that

$$\langle d\sigma, \tau \rangle = \langle \sigma, d^\diamond \tau \rangle \tag{4.1}$$

for all $\sigma \in \Omega^p_{00}(C)$ and all $\tau \in \Omega^{p+1}(C)$, then in fact the relation (4.1) holds for all $\sigma \in \Omega^p_0(C)$ and all $\tau \in \Omega^{p+1}(C)$. Indeed, if we choose an orientation on C and introduce the associated Hodge $\star$-operator, then for all $\sigma \in \Omega^p(C)$ and all $\tau \in \Omega^{p+1}(C)$,

$$\langle d\sigma, \tau \rangle - \langle \sigma, d^\diamond \tau \rangle = \int_{\partial C} \overline{\sigma} \wedge \star \tau.$$

Henceforth, we shall consider $d^\diamond$ as an unbounded Hilbert space operator with domain $\Omega^{p+1}(C)$. Form the direct sums

$$\Omega^*_0(C) = \bigoplus_p \Omega^p_0(C) \quad \text{and} \quad \Omega^*_{L^2}(C) = \bigoplus_p \Omega^p_{L^2}(C),$$

and then form the unbounded operator

$$D = d + d^\diamond : \Omega^*_{L^2}(C) \longrightarrow \Omega^*_{L^2}(C) \tag{4.2}$$

whose domain is the intersection of the domains of d and $d^\diamond$, which is $\Omega^*_0(C)$. This is a symmetric operator.

Lemma 4.2.   The operator D in (4.2) is essentially self-adjoint, and the self-adjoint closure of D has compact resolvent. The kernel of the closure is precisely the one-dimensional space $A(C)$ of top-degree constant coefficient forms.

Proof.   Let q be the dimension of C and let $H_1, \ldots, H_q$ be the hyperplanes that intersect C. Let $k_1, \ldots, k_q$ be nonnegative integers, and form the span of all differential forms of

the type

$$\sigma_1 \wedge \cdots \wedge \sigma_q,$$

where each σ_j is one of $\sin(\pi k_j x_{H_j})$ or $\cos(\pi k_j x_{H_j})d_{X_j}$. The span is a finite-dimensional subspace of $\Omega_0^*(C)$; it is invariant under D; the square of D on the span is the scalar multiple

$$\pi^2 k_1^2 + \cdots + \pi^2 k_q^2$$

of the identity operator; and the family of all such spans (as the list of integers $k_1, \ldots, k_q$ varies) is an orthogonal decomposition of $\Omega_{L^2}^*(C)$. The lemma follows from these facts. ∎

Let us now fix, for the rest of this section, a proper and G-adapted weight function. For each cube C in X define an affine function $w_C \colon C \to \mathbb{R}$ by the formula

$$w_C = \sum_{H \in \mathrm{Mid}(C)} w(H) x_H , \tag{4.3}$$

and form the Witten-type perturbation

$$\mathrm{d}_w = e^{-w_C} \mathrm{d} e^{w_C} \colon \Omega_0^*(C) \longrightarrow \Omega^*(C).$$

It follows from the definition that

$$\mathrm{d}_w \colon \beta \longmapsto \mathrm{d}\beta + \sum_{H \in \mathrm{Mid}(C)} w(H)\mathrm{d}x_H \wedge \beta, \tag{4.4}$$

so that d_w is in fact a bounded perturbation of d. Let $d_w^\diamond$ be its formal adjoint, considered as an unbounded operator with domain $\Omega^*(C)$, and then form the Witten-type de Rham operator

$$D_w = \mathrm{d}_w + \mathrm{d}_w^\diamond \colon \Omega_{L^2}^*(C) \longrightarrow \Omega_{L^2}^*(C) \tag{4.5}$$

with the same domain as D, namely $\Omega_0^*(C)$. Being a bounded perturbation of an essentially self-adjoint operator with compact resolvent, the operator D_w is itself essentially self-adjoint with compact resolvent; see [13, Ch. 4 Theorem 3.1.11] and [13, Ch. 5, Theorem 4.4.4].

We are ready now to proceed from a single cube to the cubical space X.

Definition 4.3.   Denote by $\Omega^*_{L^2}(X)$ the Hilbert space direct sum

$$\Omega^*_{L^2}(X) = \bigoplus_C \Omega^*_{L^2}(C)$$

over all the cubes in X of all dimensions, and denote by $\Omega^*_0(X) \subseteq \Omega^*_{L^2}(X)$ the algebraic direct sum of the spaces $\Omega^*_0(C)$.

Our general aim is to assemble the operators D_w in (4.5) into one essentially self-adjoint operator with compact resolvent on the full space $\Omega^*_{L^2}(X)$. However, this requires some care, as the following computation shows.

Let C be a one-dimensional cube and let H be the hyperplane that intersects C. For each integer $k > 0$ the span in $\Omega^*_0(C)$ of the forms

$$\sin(\pi k x_H) \quad \text{and} \quad d_w \sin(\pi k x_H) = \pi k \cos(\pi k x_H) d_{x_H} + w(H) \sin(\pi k x_H) d_{x_H}$$

is invariant under the action of D_w, and furthermore $D_w^2 = \pi^2 k^2 + w(H)^2$ on the span. The spans for distinct k are orthogonal to one another, and their mutual orthogonal complement in $\Omega^*_{L^2}(C)$ is spanned by

$$e^{w(H)x_H} dx_H \in \Omega^*_0(C),$$

which lies in the kernel of D_w, and indeed spans it. It follows that

$$\text{Spectrum}(D_w^2) = \{0\} \sqcup \{\pi^2 k^2 + w(H)^2 \ : \ k = 1, 2 \ldots\}$$

(we mean here the spectrum of the square of the self-adjoint extension of D_w).

We find that in the case of a tree, for example, the operator D_w on each cube has a one-dimensional kernel, and in order to construct a single Witten-type de Rham operator on the tree we need to "cancel" these kernels against one another so as to obtain an essentially self-adjoint, compact resolvent operator on $\Omega^*_{L^2}(X)$. The next two definitions are dedicated to solving this problem (for CAT(0)-cubical spaces in all dimensions).

Definition 4.4.   If D is any cube in X, and if H is any hyperplane that intersects D, then let

$$y_H = x_H - 1 : D \longrightarrow \mathbb{R},$$

where x_H is as in Definition 3.6. Observe that $dy_H = \alpha_H = dx_H$. Define a linear operator

$$e_w \colon \Omega^p(X) \longrightarrow \Omega^{p+1}(X)$$

by the formula

$$e_w \colon \beta \longmapsto \sum_{H \in \mathrm{SAH}(C)} w(H) \cdot e^{w(H)y_H} \cdot \alpha_H \wedge \beta$$

for $\beta \in \Omega^*(C)$, where

(1)   As in Definition 3.11, $\mathrm{SAH}(C)$ is the set of all hyperplanes that are adjacent to C and separate C from the base vertex.

(2)   For a hyperplane $H \in \mathrm{SAH}(C)$, if D_H is the cube that is intersected by H, and that includes C as a codimension-one face, then we view $\alpha_H \wedge \beta$ as a differential form on D_H by pulling back β along the orthogonal projection from D_H to C.

Definition 4.5.   We shall denote by D_{dR} the following symmetric operator on $\Omega^*_{L^2}(X)$ with domain $\Omega^*_0(X)$:

$$D_{\mathrm{dR}} = d_w + e_w + d_w^\diamond + e_w^\diamond.$$

By d_w we mean here the direct sum of all the Witten-type operators (4.4) over all the cubes of X, and the same for the formal adjoint $d_w^\diamond$.

We shall prove the following result:

Theorem 4.6.   If the weight function w is proper, then the operator D_{dR} is essentially self-adjoint and has compact resolvent.

For the purposes of the proof, and for other purposes as well, it will be convenient to decompose $\Omega^*_{L^2}(X)$ into a direct sum of "blocks" in such a way that the operator D_{dR} is block-diagonal.

Definition 4.7.   Let Q be a vertex of X. We shall say that a cube C *leads Q to the base vertex P* if the following conditions hold:

(1)   The cube C includes the vertex Q.

(2)   All the hyperplanes that intersect C also separate Q from P. That is, $\mathrm{Mid}(C) \subseteq$ $\mathrm{SAH}(Q)$.

We shall denote by

$$\Omega^*_{L^2}(X)_Q \subseteq \Omega^*_{L^2}(X)$$

the direct sum of the spaces $\Omega^*_{L^2}(C)$ over those cubes C that lead Q to P. We define $\Omega^*_0(X)_Q$ and $\Omega^*(X)_Q$ similarly.

Remark 4.8.   For any vertex Q, the collection $\mathrm{SAH}(Q)$ coincides with collection $\mathrm{Mid}(D)$ for a distinguished cube D in X, namely the 1st cube in the normal cube path from Q to P in the sense of [16, Sec. 3]. The cubes in X that lead Q to P are the faces of D that include the vertex Q.

We have the following Hilbert space direct sum decomposition:

$$\Omega^*_{L^2}(X) = \bigoplus_Q \Omega^*_{L^2}(X)_Q. \tag{4.6}$$

This is because each cube C in X has a unique vertex Q that is separated from the base vertex by all the hyperplanes intersecting C. Inspecting the formulas, we see immediately that:

Lemma 4.9.   The operator D_{dR} is block-diagonal with respect to the Hilbert space decomposition (4.6).

To begin the proof of Theorem 4.6, let us examine the case of a vertex Q for which $\mathrm{SAH}(Q)$ consists of a single hyperplane H, so that

$$\Omega^*_0(X)_Q = \Omega^*_0(Q) \oplus \Omega^*_0(E), \tag{4.7}$$

where the edge E leads Q toward P. A little more generally, let Q be any vertex and let E be any edge leading Q to P, and consider the operator

$$\begin{bmatrix} 0 & e^\diamond \\ e & d_w + d_w^\diamond \end{bmatrix} : \Omega^*_{L^2}(Q) \oplus \Omega^*_{L^2}(E) \longrightarrow \Omega^*_{L^2}(Q) \oplus \Omega^*_{L^2}(E) \tag{4.8}$$

with domain $\Omega_0^*(Q) \oplus \Omega_0^*(E)$, where e acts as

$$\Omega_0^0(Q) \ni 1 \longmapsto w(H)e^{w(H)y_H}\,\mathrm{d}x_H \in \Omega_0^1(E).$$

In the special case described by (4.7), this is the restriction of the operator D_{dR} to $\Omega_{L^2}^*(X)_Q$. The images of symmetric operators

$$\begin{bmatrix} 0 & e^\diamond \\ e & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & d_w + d_w^\diamond \end{bmatrix}$$

are orthogonal to one another, and indeed the image of the first is precisely the kernel of the second. This kernel is spanned by

$$1 \in \Omega_0^0(Q) \quad \text{and} \quad e^{w(H)x_H}\,\mathrm{d}x_H \in \Omega_0^1(E),$$

and the 1st operator not only leaves this space invariant, but its square there is

$$w(H)^2 \cdot \|e^{w(H)y_H}\|_{L^2(0,1)}^2 = \tfrac{1}{2}w(H)(1 - e^{-2w(H)}).$$

As a result there is a Hilbert space orthonormal basis consisting of eigenvectors in the domain of (4.8), and the squares of the eigenvalues are the scalars in the list

$$\left\{ \tfrac{1}{2}w(H)(1 - e^{-2w(H)}) \right\} \sqcup \left\{ \pi^2 k^2 + w(H)^2 \ : \ k = 1, 2 \dots \right\}. \tag{4.9}$$

Proof of Theorem 4.6.   Keeping in mind Lemma 4.9, we shall determine the structure of D_{dR} on each block $\Omega_{L^2}^*(X)_Q$.

Let us start with the block for the base vertex P. It is the de Rham complex of P alone, and hence is one-dimensional. The operator D_{dR} is identically zero on this block.

Suppose next that $Q \neq P$. Let $H_1, \dots, H_q$ be the hyperplanes adjacent to Q and separating Q from P. Denote by E_j the unique edge leading Q to P that is cut by H_j. There is then a unitary isomorphism of graded Hilbert spaces

$$\widehat{\bigotimes_j}\left(\Omega_{L^2}^*(Q) \oplus \Omega_{L^2}^*(E_j)\right) \xrightarrow{\ \cong\ } \Omega_{L^2}^*(X)_Q \tag{4.10}$$

given by exterior multiplication of forms, as follows. If C is a cube that leads Q to P, then define a map

$$\Omega_{L^2}^*(Q) \oplus \Omega_{L^2}^*(E_j) \longrightarrow \Omega_{L^2}^*(C)$$

by pulling back forms from E_j to C along the orthogonal projection, when E_j is included in C, and by pulling back from the point Q when E_j is not included in C. The morphism (4.10) is defined by pulling back forms in this way, then forming the exterior products of the pullbacks.

The isomorphism (4.10) restricts to an inclusion

$$\widehat{\bigotimes_{j}}\left(\Omega_0^*(Q) \oplus \Omega_0^*(E_j)\right) \xrightarrow{\cong} \Omega_0^*(X)_Q \tag{4.11}$$

Denote by

$$D_j \colon \Omega_0^*(Q) \oplus \Omega_0^*(E_j) \longrightarrow \Omega^*(Q) \oplus \Omega^*(E_j)$$

the operator in (4.8). The operator D_{dR} on $\Omega^*(X)_Q$ identifies, via (4.10), with the sum

$$\sum_j I\widehat{\otimes}\cdots\widehat{\otimes}I\widehat{\otimes}D_j\widehat{\otimes}I\widehat{\otimes}\cdots\widehat{\otimes}I,$$

from which we see that there is an orthonormal basis for $\Omega_{L^2}^*(X)_Q$ consisting of eigenvectors for D_{dR} in $\Omega_0^*(X)_Q$, and indeed in the image of (4.11), for which in addition the squares of eigenvalues are sums of one term from each of the lists

$$\left\{ \tfrac{1}{2}w(H_j)(1 - e^{-2w(H_j)}) \right\} \sqcup \left\{ \pi^2 k^2 + w(H_j)^2 \,:\, k = 1, 2\dots \right\},$$

for $j = 1, \dots q$. This proves the theorem.   ∎

The proof above also yields the following estimate, which we shall use in the next section.

Lemma 4.10.   If $\beta \in \Omega_0^*(X)_Q$, and if β lies in the kernel of $d_w + d_w^\diamond$, then

$$\|D_{\mathrm{dR}}\beta\|^2 = \sum_{H\in\mathrm{SAH}(Q)} \tfrac{1}{2}w(H)(1 - e^{-2w(H)})\|\beta\|^2.$$

If $\beta \in \Omega_0^*(X)_Q$, and if β is orthogonal to the kernel of $d_w + d_w^\diamond$, then $\|D_{\mathrm{dR}}\beta\|^2 \geq \pi^2\|\beta\|^2$.

Lemma 4.11 (Compare [3, Lemma 9.6].).   If the weight function w is G-adapted, then for any g in G, the difference $g(D_{\mathrm{dR}}) - D_{\mathrm{dR}}$ is a bounded operator. Moreover, $\|g(D_{\mathrm{dR}}) - D_{\mathrm{dR}}\|$

is bounded by a constant times

$$\sup \left\{ \left| gw(H) - w(H) \right| : H \text{ is a hyperplane of } X \right\} + \max \left\{ w(H) : H \text{ separates } P \text{ and } gP \right\}.$$

The constant depends only on the dimension of X.

Proof. For the purposes of this proof it will be convenient to make explicit note of the dependence of D_{dR} on the choice of basepoint, and on the choice of weight function. So we shall write

$$D_{\mathrm{dR}} = D_{w,P},$$

so that $g(D_{\mathrm{dR}}) = D_{gw,gP}$, where $gw(H) = w(g^{-1}H)$. We shall bound the two differences

$$D_{gw,gP} - D_{w,gP} \quad \text{and} \quad D_{w,gP} - D_{w,P}$$

separately to obtain the estimate in the statement of the lemma.

For the first, let us write

$$D_{gw,gP} - D_{w,gP} = (d_{gw,gP} - d_{w,gP}) + (e_{gw,gP} - e_{w,gP}) + (d_{gw,gP}^{\diamond} - d_{w,gP}^{\diamond}) + (e_{w,P}^{\diamond} - e_{gw,gP}^{\diamond}).$$

Since w is G-adapted, there is some $M \geq 0$ for which

$$\left| gw(H) - w(H) \right| \leq M \qquad \forall H.$$

It follows from the formula for d_w that

$$\| d_{gw,gP} - d_{w,gP} \| \leq \dim(X) \cdot M,$$

and of course from this we also get $\| d_{gw,gP}^{\diamond} - d_{w,gP}^{\diamond} \| \leq \dim(X) \cdot M$. Next, if $x, w \geq 0$, and if $M \geq m \geq 0$, then

$$\left| we^{-wx} - (w+m)e^{-(w+m)x} \right| \leq 2M, \tag{4.12}$$

since for instance one can write

$$\left| we^{-wx} - (w+m)e^{-(w+m)x} \right| \leq m \left| we^{-wx} \frac{(1 - e^{-mx})}{m} \right| + m$$

$$\leq m \left| e^{-wx} wx \right| + m,$$

from which (4.12) follows easily. From (4.12) and the formula for $e_{w,P}$ in Definition 4.4 we obtain

$$\|e_{gw,gP} - e_{w,gP}\| \le 2\dim(X)M,$$

and hence also $\|e^{\diamond}_{gw,gP} - e^{\diamond}_{w,gP}\| \le 2\dim(X)M$. Putting everything together, we obtain

$$\|D_{gw,gP} - D_{w,gP}\| \le 6\dim(X)M.$$

Next, let us turn to the difference $D_{w,gP} - D_{w,P}$. We shall analyze it part by part, more or less as we did above. To begin,

$$d_{w,gP}\beta - d_{w,P}\beta = \sum_{H \in \mathrm{Mid}(C)} w(H)\big(\alpha_{H,gP} - \alpha_{H,P}\big) \wedge \beta.$$

The difference $\alpha_{H,gP} - \alpha_{H,P}$ is zero unless the hyperplane H separates P and gP, in which case the difference is $2\alpha_{H,gP}$. So $\|d_{w,gP} - d_{w,P}\|$ is bounded by a constant times the largest value of $w(H)$ among the hyperplanes separating P and gP, as required, and in addition the same is true of $\|d^{\diamond}_{w,gP} - d^{\diamond}_{w,P}\|$.

The remaining terms in $D_{w,gP} - D_{w,P}$ are treated in a similar way. Since

$$e_{w,gP}\beta - e_{w,P}\beta = \sum w(H)\left(e^{w(H)\gamma_{H,P}}\alpha_{H,P}\wedge\beta \; - \; e^{w(H)\gamma_{H,gP}}\alpha_{H,gP}\wedge\beta\right),$$

and since again the summands are zero except when H separates P and gP, we can again bound the sum by a constant times the largest value of $w(H)$ among the hyperplanes separating P and gP, and of course we can do the same for $e^{\diamond}_{w,gP} - e^{\diamond}_{w,P}$, too.   ∎

Now let us equip the Hilbert space $\Omega^{*}_{L^2}(X)$ with the $\mathbb{Z}/2$-grading it acquires from the differential form degree. The operator D_{dR} is odd with respect to this grading. Combined, Theorem 4.6 and Lemma 4.11 give the following result.

Theorem 4.12.   The operator D_{dR} on $\Omega^{*}_{L^2}(X)$, constructed from the weight function (3.2), defines an unbounded cycle for the group $KK_G(\mathbb{C}, \mathbb{C})$.

Definition 4.13.   The *de Rham cycle* for the Kasparov group $KK_G(\mathbb{C}, \mathbb{C})$ is $(\Omega^{*}_{L^2}(X), D_{\mathrm{dR}})$, as in Theorem 4.12. We shall denote by $[D_{\mathrm{dR}}] \in KK_G(\mathbb{C}, \mathbb{C})$ the associated *KK*-class.

5 Proofs of the Main Results

In this section we shall prove the following two theorems:

Theorem 5.1.   If G acts on X by automorphisms, the de Rham and Julg–Valette classes are equal to one another in $KK_G(\mathbb{C}, \mathbb{C})$.

Theorem 5.2.   If G acts properly and cocompactly on X by automorphisms, the de Rham cycle has Property (γ).

We shall prove Theorem 5.1 by constructing an explicit homotopy between the Julg-Valette and de Rham cycles, or in other words we shall construct an unbounded cycle for $KK_G(\mathbb{C}, C[0, 1])$ that restricts to the Julg–Valette and de Rham cycles at $t=0$ and $t=1$, respectively.

The underlying Hilbert $C[0, 1]$-module for the homotopy will be

$$\mathcal{H}(X) = \big\{ \beta : [0, 1] \to \Omega^*_{L^2}(X) \,|\, \beta \text{ is norm-continuous and } \beta_0 \in A^*_{L^2}(X) \big\}. \tag{5.1}$$

Here $A^*_{L^2}(X)$ is the Hilbert space for the Julg–Valette cycle from Definition 3.3, which we note is a Hilbert subspace of $\Omega^*_{L^2}(X)$.

Let w be a proper and G-adapted weight function. The operator $\mathcal{D}$ for our cycle is defined by

$$\mathcal{D}\beta : s \longmapsto \begin{cases} s^{-1}D_{\mathrm{dR},s}\beta_s & s \in (0, 1] \\ D_{\mathrm{JV}}\beta_0 & s = 0, \end{cases} \tag{5.2}$$

where

(1) $D_{\mathrm{dR},s}$ is the de Rham operator from Definition 4.5 associated to the weight function $H \mapsto s \cdot w(H)$.

(2) D_{JV} is the Julg–Valette operator from Definition 3.14 associated to the weight function w itself.

Of course we need to specify the domain of $\mathcal{D}$. But before we do that, let us carry out some preliminary computations that should help make Theorem 5.1 plausible, and help justify our choice for the operator $\mathcal{D}$.

Definition 5.3.   Let C be a q-cube in X, and let $H_1, \ldots, H_q$ be the hyperplanes that cut C (i.e., the hyperplanes in $\mathrm{Mid}(C)$). We shall denote by

$$A_s(C) \subseteq \Omega_0^q(C)$$

the one-dimensional subspace that is spanned by the form $e^{swc}\, dx_{H_1} \cdots dx_{H_q}$.

Lemma 5.4.   The space $A_s(C)$ is the kernel of the operator

$$d_{sw} + d_{sw}^\diamond : \Omega_0^*(C) \longrightarrow \Omega_0^*(C),$$

and of its self-adjoint extension.

Proof.   This follows from the computations after Definition 4.3.   ∎

Definition 5.5.   We shall denote by $A_s^q(X) \subseteq \Omega_0^q(X)$ the algebraic direct sum

$$A_s^q(X) = \bigoplus_{\dim(C)=q} A_s(C),$$

by $A_s^*(X) \subseteq \Omega_0^*(X)$ the direct sum of all $A_s^q(X)$, and by $A_{L^2,s}^*(X) \subseteq \Omega_{L^2}^*(X)$ the Hilbert space completion of $A_s^*(X)$.

Lemma 5.6.   For every $s \in (0,1]$ the operators

$$e_{sw}, e_{sw}^\diamond : \Omega^*(X) \longrightarrow \Omega^*(X)$$

map $A_s^*(X)$ into itself.

Proof.   Let $s \in (0,1]$. The operator e_{sw} has the form

$$e_{sw} : \beta \longmapsto \sum_{H \in \mathrm{SAH}(C)} sw(H)\, e^{sw(H)y_H}\, \alpha_H \wedge \beta \tag{5.3}$$

for $\beta \in \Omega^*(C)$; see Definition 4.4. Recall that in this formula $\alpha_H \wedge \beta$ is to be viewed as a form on the cube D_H that is cut by H and contains C as a codimension-one face; the form β is pulled back to D_H along the face projection $D_H \to C$. If C is cut by the hyperplanes $H_1, \ldots, H_q$, then of course the cube D_H is cut by the hyperplanes

$H, H_1, \ldots, H_q$. In addition, if w_C is viewed as a function on D_H by pulling back along the face projection $D_H \to C$, then

$$w_{D_H} = w_C + w(H) x_H.$$

It follows that

$$sw_{D_H} = sw_C + sw(H) x_H = sw_C + sw(H) y_H + sw(H),$$

and hence we find that

$$e_{sw} : e^{sw_C} \, \mathrm{d}x_{H_1} \cdots \mathrm{d}x_{H_q} \longmapsto \sum_{H \in \mathrm{SAH}(C)} (sw(H) e^{-sw(H)}) e^{sw_{D_H}} \, \mathrm{d}x_H \mathrm{d}x_{H_1} \cdots \mathrm{d}x_{H_q}.$$

This proves the lemma for e_{sw}. A similar computation shows that if D is a $(q+1)$-cube cut by hyperplanes $H_0, \ldots, H_q$, and if C_j denotes the q-dimensional face of D that is separated from the basepoint by H_j, then

$$e_{sw}^\diamond : e^{sw_D} \, \mathrm{d}_{H_0} \cdots \mathrm{d}x_{H_q} \longmapsto \sum_{j=0}^{q} (-1)^j \| e^{sw(H_j)x} \|_{L^2(0,1)}^2 \cdot sw(H_j) e^{-sw(H_j)} \cdot e^{sw_{C_j}} \, \mathrm{d}x_{H_0} \cdots \widehat{\mathrm{d}x_{H_j}} \cdots \mathrm{d}x_{H_q}.$$

This proves the lemma for $e_{sw}^\diamond$.   ■

Now $A_s^*(X)$ has an orthogonal complement in the inner product space $\Omega_0^*(X)$, since each one-dimensional space $A_s(C)$ is certainly complemented in $\Omega_0^*(C)$, and it follows from Lemmas 5.4 and 5.6 that the operator $s^{-1} D_{\mathrm{dR},s}$ preserves the direct sum decomposition of $\Omega_0^*(X)$ into $A_s^*(X)$ and its orthogonal complement in $\Omega_0^*(X)$. On the former, we have, in view of Lemma 5.4,

$$s^{-1} D_{\mathrm{dR},s} = s^{-1} e_{sw} + s^{-1} e_{sw}^\diamond : A_s^*(X) \longrightarrow A_s^*(X).$$

Moreover, it follows from (5.3) that

$$s^{-1} e_{sw} : \beta \longmapsto \sum_{H \in \mathrm{SAH}(C)} w(H) \, e^{sw(H) y_H} \, \alpha_H \wedge \beta \tag{5.4}$$

for $\beta \in A_s(C)$. But when $s=0$ the space $A_s^*(X)$ is the space $A^*(X)$ from Definition 3.3 that we used in our construction of the Julg–Valette cycle, and it is at least informally clear from (5.4) that as $s \to 0$ the operator $s^{-1} e_{sw}$ "converges" to the Julg–Valette differential.

It follows that, in the same informal sense, $s^{-1}D_{\mathrm{dR},s}$ converges to the Julg–Valette operator D_{JV}.

So roughly speaking, to prove Theorem 5.1 we need to:

(1)   Make the above idea of convergence precise.

(2)   Show that the summand of operator $s^{-1}D_{\mathrm{dR},s}$ that acts on the orthogonal complement $A_s^*(X)^\perp \subseteq \Omega_0^*(X)$ contributes nothing in *KK*-theory.

This is what we shall do rigorously below. An additional complication is that the direct sum decomposition above is not *G*-equivariant, but this too will be properly taken into account in our proof of the theorem.

Let us now return to the issue of the domain for the operator $\mathcal{D}$ in (5.2). First we define

$$\mathcal{H}_0(C) = \left\{ \left. \beta\colon [0,1] \to \Omega_0^*(C) \ \right| \ \begin{array}{l} \beta \text{ is continuous for the } C^\infty\text{-topology on } \Omega_0^*(C), \\ \text{and } \beta_s \in A_s(C) \text{ for all sufficiently small } s \end{array} \right\},$$

and then we form the algebraic direct sum

$$\mathcal{H}_0(X) = \bigoplus_C \mathcal{H}_0(C),$$

which is a dense $C[0,1]$-submodule of $\mathcal{H}(X)$. This will be our domain. The additional smoothness and continuity hypotheses ensure that $\mathcal{D}$ maps $\mathcal{H}_0(X)$ into $\mathcal{H}(X)$.

Theorem 5.7.   The operator $\mathcal{D}$, with the above domain $\mathcal{H}_0(X)$, is a regular and essentially self-adjoint operator on the Hilbert module $\mathcal{H}(X)$.

To prove the theorem we shall use the following simple technical result, which is proved by explicitly constructing the resolvent operators for the operator in question. For further information about regularity for Hilbert module operators, which is needed to guarantee a reasonable spectral theory for these operators, see [14].

Lemma 5.8.   Let $\{H_s\}_{s\in[0,1]}$ be a continuous field of Hilbert spaces over [0, 1] with $H_0=0$, and let $\mathcal{E}$ be a dense submodule of its Hilbert $C[0,1]$-module of continuous sections. Let $\{D_s\}_{s\in[0,1]}$ be a family of symmetric operators on the fibers of the field. Assume that

(1)   If $e \in \mathcal{E}$, then $e_s \in \mathrm{dom}(D_s)$ for every $s \in [0,1]$.

(2)   If $e \in \mathcal{E}$, then $s \mapsto D_s e_s$ is a continuous section of $\{H_s\}_{s\in[0,1]}$.

(3)   For every $\delta > 0$ the family $\{D_s\}_{s\in[\delta,1]}$ defines an essentially self-adjoint and regular operator on the Hilbert $C[\delta,1]$-module of continuous sections of the restricted field $\{H_s\}_{s\in[\delta,1]}$, with domain the module of all restrictions of sections in $\mathcal{E}$ to $[\delta,1]$.

Then $\{D_s\}_{s\in[0,1]}$ defines an essentially self-adjoint and regular operator on the Hilbert $C[0,1]$-module of all continuous sections of $\{H_s\}_{s\in[0,1]}$, with domain $\mathcal{E}$.

Proof of Theorem 5.7.   If Q is a vertex in X, then define

$$\mathcal{H}(X)_Q = \big\{ \beta \in \mathcal{H}(X) \ \big| \ \beta_s \in \Omega_{L^2}^*(X)_Q \quad \forall s \in [0,1] \big\},$$

and also define

$$\mathcal{H}_0(X)_Q = \mathcal{H}_0(X) \cap \mathcal{H}(X)_Q.$$

Then

$$\mathcal{H}(X) = \bigoplus_Q \mathcal{H}(X)_Q \quad \text{and} \quad \mathcal{H}_0(X) = \bigoplus_Q \mathcal{H}_0(X)_Q,$$

where the former is a direct sum decomposition of Hilbert modules, and the latter is an algebraic direct sum according to which the operator $\mathcal{D}$ decomposes as a direct sum, $\mathcal{D} = \oplus_Q \mathcal{D}_Q$. To show that $\mathcal{D}$ is a regular operator, we only need to show the same for each $\mathcal{D}_Q$.

Define a self-adjoint projection operator

$$\mathcal{P}_Q \colon \mathcal{H}(X)_Q \longrightarrow \mathcal{H}(X)_Q$$
$$\big(\mathcal{P}\beta\big)_s = P_s \beta_s,$$

where P_s is the orthogonal projection from $\Omega_{L^2}^*(X)_Q$ onto the finite-dimensional subspace $A^*(X)_Q = A^*(X) \cap \Omega_{L^2}^*(X)_Q$. The operator $\mathcal{P}$ maps the domain $\mathcal{H}_0(X)_Q$ into itself, and thanks to Lemmas 5.4 and 5.6 it commutes with $\mathcal{D}$.

Let us denote by

$$\mathcal{A}(X)_Q \subseteq \mathcal{H}(X)_Q.$$

the range of the operator $\mathcal{P}_Q$. It is an orthocomplemented Hilbert submodule, and indeed a finitely generated and projective module in its own right. The restriction of $\mathcal{D}$ to $\mathcal{A}(X)_Q$

is the bounded and norm-continuous family of self-adjoint operators

$$
\begin{cases}
s^{-1}e_{sw} + s^{-1}e_{sw}^{\diamond} & s > 0 \\
d_{\mathrm{JV}} + d_{\mathrm{JV}}^{\diamond} & s = 0,
\end{cases}
$$

and it is a regular and self-adjoint Hilbert module operator.

As for the restriction of $\mathcal{D}$ to the orthogonal complement $\mathcal{A}(X)_Q^{\perp} \subseteq \mathcal{H}(X)_Q$, whose domain is the orthogonal complement of the submodule $\mathcal{A}(X)_Q$ in $\mathcal{H}_0(X)_Q$, it follows immediately from the preceeding lemma that this too is a regular operator. ∎

In order to construct a *KK*-cycle we need a bit more than regularity: we also need an operator with compact resolvent. For this, we shall use the following equally easy but slightly stronger version of Lemma 5.8:

Lemma 5.9.   Under the hypotheses of Lemma 5.8, assume in addition that

(1)   For every $\delta > 0$ the family $\{D_s\}_{s \in [\delta,1]}$ defines an essentially self-adjoint and regular operator with compact resolvent.

(2)   For every $K > 0$ there exists $\delta > 0$ so that if $s \in [0,\delta]$, then D_s is bounded below by K (i.e., $\|D_s e_s\| \geq K\|e_s\|$ for every $e_s \in \mathrm{dom}(D_s)$).

Then $\{D_s\}_{s \in [0,1]}$ defines an essentially self-adjoint and regular operator with compact resolvent on the Hilbert $C[0,1]$-module of all continuous sections of $\{H_s\}_{s \in [0,1]}$, with domain $\mathcal{E}$.

With this, we are finally ready to prove Theorem 5.1.

Proof of Theorem 5.1.   We need to show that the operator $\mathcal{D}$ has compact resolvent and is almost-equivariant, so that it determines a class in the equivariant group $KK_G(\mathbb{C}, C[0,1])$.

We already noted that each Hilbert $C[0,1]$-module $\mathcal{A}(X)_Q$ is actually a finitely generated and projective $C[0,1]$-module. Therefore, the regular self-adjoint operator $\mathcal{D}$ on $\mathcal{A}(X)_Q$ automatically has compact resolvent, since any adjointable operator on a finitely generated and projective module is automatically compact. Moreover, it follows from Lemma 4.10 that for $s \in (0,1]$ the operator $s^{-1}D_{\mathrm{dR},s}$ on $A_s^*(X)_Q$ has square bounded

below by

$$s^{-2} \sum_{H \in \mathrm{SAH}(Q)} s^2 w(H)^2 \frac{(1 - e^{-2sw(H)})}{2sw(H)} \geq \sum_{H \in \mathrm{SAH}(Q)} w(H)^2 \frac{(1 - e^{-2w(H)})}{2w(H)}$$

$$= \sum_{H \in \mathrm{SAH}(Q)} \tfrac{1}{2} w(H)(1 - e^{-2w(H)}).$$

So the resolvents $(\mathcal{D} \pm i)^{-1}$ on the summands $\mathcal{A}(X)_Q$ converge in norm to zero as $Q \to \infty$, and therefore the direct sum $\mathcal{D}$ has compact resolvent on $\mathcal{A}(X)$.

Lemma 4.10 also shows that on each $A_s^*(X)_Q^\perp$, and therefore on $A_s^*(X)^\perp$ as a whole, the operator $s^{-1} D_{\mathrm{dR},s}$ is bounded below by $s^{-1}\pi$. So it follows from Lemma 5.9 that $\mathcal{D}$ has compact resolvent on $\mathcal{H}(X)$.

From the proof of Lemma 4.11, we see that the operator family

$$\big\{ g(D_{\mathrm{dR},s}) - D_{\mathrm{dR},s} \big\}_{s \in (0,1]}$$

extends to a bounded operator on $\mathcal{H}(X)$, for every $g \in G$, and defines a strongly continuous, locally bounded, bounded operator-valued function of $g \in G$, as required. ∎

Let us turn now to Theorem 5.2. We just showed that $[D_{\mathrm{dR}}] = [D_{\mathrm{JV}}]$, and Theorem 3.18 asserts that $[D_{\mathrm{JV}}] = 1_G$. Therefore, $[D_{\mathrm{dR}}] = 1_G$, and so D_{dR} satisfies the condition (1) in Definition 2.2. It remains to show that D_{dR} satisfies the condition (2) in Definition 2.2.

Proof of Theorem 5.2.   We shall apply Theorem 2.6. For the space E we shall take the cubical complex X, of course. This is a cocompact model for the universal proper G-space. We shall use the natural non-degenerate representation of the G-C^*-algebra $C_0(X)$ on $\Omega_{L^2}^*(X)$ by pointwise multiplication on forms.

For the algebra B we shall take the dense G-subalgebra of $C_0(X)$ consisting of compactly supported complex-valued functions that are smooth on each cube. It contains a cut-off function of X. For each $b \in B$, we have

$$\begin{aligned}
\big[D_{\mathrm{dR}}, g(b)\big] &= \big[d_w + e_w + d_w^\diamond + e_w^\diamond, g(b)\big] \\
&= \big[d_w + d_w^\diamond, g(b)\big] + \big[e_w + e_w^\diamond, g(b)\big] \\
&= g(c(b)) + \big[e_w + e_w^\diamond, g(b)\big],
\end{aligned}$$

where $c(b)$ denotes Clifford multiplication by the gradient of b in each cube, which is a bounded operator, uniformly bounded in g, and supported in the g-translate of the support of b. The operators $e_w, e_w^\diamond$ and $g(b)$ respect the decomposition (4.6) of the Hilbert space $\Omega^*_{L^2}(X)$. Moreover, b is non-zero on only finitely many blocks $\Omega^*_{L^2}(X)_Q$, and on these blocks e_w and $e_w^\diamond$ are bounded. From this, we see that the commutator $[D, g(b)]$ extends to a bounded operator on $\mathcal{H}$ whose support is contained in gK where K is the union of cubes which intersect the support of b.

In order to apply Theorem 2.6, it remains to show that the commutator $[e_w, g(b)]$ is uniformly bounded in g. For this, choose C so that

$$\left| b(x) - b(y) \right| \leq C \cdot \text{distance}(x, y) \qquad \forall x, y \in X.$$

From the definition of e_w, we see that for all $\beta \in \Omega^*(C)$,

$$\begin{aligned}
\| [e_w, g(b)] \beta \|^2 &\leq \sum_{H \in \text{SAH}(C)} C^2 \cdot \left\| x w(H) e^{-w(H)x} \right\|^2_{L^2(0,1)} \cdot \| \beta \|^2 \\
&\leq \dim(X) \cdot C^2 \cdot \max \{ (x e^{-x})^2 : x \geq 0 \} \cdot \| \beta \|^2 \\
&\leq \dim(X) \cdot C^2 \cdot \| \beta \|^2.
\end{aligned}$$

So $[e_w, g(b)]$ is indeed uniformly bounded in $g \in G$, as required. ∎

To conclude the paper, let us repeat the overall argument as it was presented in Section 1. Suppose G is a second countable, locally compact group acting on a bounded geometry $CAT(0)$-cubical space X by automorphisms. Theorems 5.1 and 5.2, combined with the Theorem 3.18, imply that the identity element of Kasparov's representation ring $R(G)$ is represented by a cycle with Property (γ). It follows therefore from Theorem 2.3 that the Baum–Connes conjecture with coefficients holds for G.

Funding

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