Modified Tseng's Extragradient Methods for Solving Pseudo-Monotone Variational Inequalities

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Abstract

We propose two modified Tseng's extragradient methods (also known as Forward-Backward-Forward methods) for solving non-Lipschitzian and pseudo-monotone variational inequalities in real Hilbert spaces. Under mild and standard conditions, we obtain the weak and strong convergence of the proposed methods. Numerical examples for illustrating the behavior of the proposed methods are also presented.

Keywords: Forward-Backward-Forward method, Extragradient method, Mann type method, Variational inequality, Pseudomonotone operator.

1 Introduction

In this paper, we are interested in the classical variational inequality (VI) [11, 12], which consists in finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1}$$

where C is a nonempty closed convex subset in a real Hilbert space H, and $A : H \to H$ is a single-valued mapping. We denote (1) by $VI(C,A)$ and its the solution set by Ω, which is assumed to be non-empty.

Variational inequalities are fundamental in a broad range of mathematical and applied sciences; the theoretical and algorithmic foundations as well as the applications of variational inequalities have been extensively studied in the literature and continue to attract intensive research [10, 18, 19]. For the current state of the art in finite dimensional setting, see for instance [10] and the extensive list of references therein.

Solution methods solving the variational inequality (1) have been developed extensively in the literature [4, 5, 6, 10, 16, 17, 18, 19, 23, 24, 28, 30, 31, 33, 34]. The simplest method is the classical projection algorithm, which generates an iterative sequence via

$$x_{n+1} = P_C(x_n - \lambda A x_n), \qquad \forall n \geq 0, \tag{2}$$

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where P_C denotes the metric projection from H onto C. This method is an extension of the projected gradient method for solving optimization problems. It is known that the convergence of this projection method only holds under quite restrictive assumptions that A is L-Lipschitz continuous and α-strongly (pseudo)-monotone and the setpsize λ is chosen satisfying $\lambda \in \left(0, \frac{2\alpha}{L^2}\right)$ [10, 16].

Korpelevich [20] (and also independently Antipin [1]) proposed a double projection method in Euclidean space, known as the extragradient method for solving VIs when A is monotone and L-Lipschitz continuous

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \end{cases} \qquad \forall n \geq 0, \tag{3}$$

where $\lambda \in \left(0, \frac{1}{L}\right)$. It is well known that the extragradient method can be applied to solve pseudo-monotone, Lipschitz continuous VIs in finite dimensional spaces [10, 28]. The weak convergence of this method in infinite dimensional Hilbert spaces was studied in [7] under an additional assumption that A is sequentially *weak-strong* continuous, i.e., A maps a weakly convergent sequence to a strongly convergent sequence. This assumption is rather strong and is not satisfied even for a simple example when A is the identity operator. In [33], the author has weakened this assumption to sequentially *weak-weak* continuity of A.

The extragradient method and its variants require (at least) two projections per iteration. Censor, Gibali and Reich [4, 5, 6] proposed the following scheme, called subgradient extragradient method

$$\begin{cases} y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda F(y_n)) \end{cases} \qquad \forall n \geq 0,$$

where

$$T_n = \{w \in H, \langle x_n - \lambda F(x_n) - y_n, w - y_n \rangle \leq 0\}.$$

Since the projection onto the half-space T_n can be explicitly calculated [2], the subgradient extragradient requires only one projection per iteration. This method converges for pseudo-monotone VIs in finite dimensional Euclidean spaces [6] and monotone VIs in infinite dimensional Hilbert spaces [4, 5].

An alternative method of the extragradient method is the following remarkable scheme studied by Tseng [32], which also requires only one projection per iteration

$$\begin{cases} y_n = P_C(x_n - \lambda F(x_n)), \\ x_{n+1} = y_n + \lambda (F(x_n) - F(y_n)) \end{cases} \qquad \forall n \geq 0.$$

The weak convergence of Tseng's extragradient method (also known as the Forward-Backward-Forward method) for solving monotone Lipschitz continuous VIs was established in [32], and is recently studied in [3] for solving pseudo-monotone Lipschitz continuous VIs under sequentially *weak-weak* continuity of A.

The aim of this paper has two folds. We first incorporate the Tseng's extragradient method studied in [3, 32] with a suitable linesearch to remove the dependence on the Lipschitz continuity modulus of A when choosing stepsize λ. We also weaken the Lipschitz continuity of A to the uniform continuity. This is crucial when the operator is not Lipschitz continuous and/or the Lipschitz modulus is difficult to estimate in advance. Doing so, we obtain the weak convergence of the iterative sequence. As we are working in infinite dimensional Hilbert spaces, the strong convergence is essential. Therefore, in the second part of the paper, we combine the linesearch method with a Mann-type iteration step to obtain the strong convergence of the iterative sequence.

The rest of the paper is organized as follows. We first recall some basic definitions and results in Section 2. The weak convergence method and its convergence analysis are presented in Section 3. Section 4 contains the analysis of the strong convergence method. In Section 5 we present some elementary numerical experiments which demonstrate the performances of the proposed methods. Finally, we give some conclusion remarks in the last section.

2 Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y, z \in H$ and for all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle; \tag{4}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2; \tag{5}$$

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Definition 1. *Let $T : H \to H$ be an operator.*

1. *The operator T is called L-`Lipschitz continuous` with $L > 0$ if*

$$\|Tx - Ty\| \leq L\|x-y\| \quad \forall x, y \in H; \tag{6}$$

   if $L = 1$ then the operator T is called `nonexpansive` and if $L \in (0,1)$, T is called `contraction`.

2. *The operator T is called `monotone` if*

$$\langle Tx - Ty, x-y\rangle \geq 0 \quad \forall x, y \in H; \tag{7}$$

3. *The operator T is called `pseudo-monotone` if*

$$\langle Tx, y-x\rangle \geq 0 \Longrightarrow \langle Ty, y-x\rangle \geq 0 \quad \forall x, y \in H; \tag{8}$$

4. *The operator T is called `sequentially weakly continuous` if for each sequence $\{x_n\}$ we have: $\{x_n\}$ converges weakly to x implies Tx_n converges weakly to Tx.*

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\| \ \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and the following properties hold.

Lemma 1. ([13]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x \Longleftrightarrow \langle x - z, z - y \rangle \geq 0 \ \ \forall y \in C$.*

Lemma 2. ([13]) *Let C be a closed and convex subset in a real Hilbert space H, $x \in H$. Then*

i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \ \forall y \in C$;
ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2 \ \forall y \in C$;
iii) $\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2 \ \forall y \in C$.

The following Lemmas are useful for the convergence of our proposed methods.

Lemma 3. ([9]) *For $x \in H$ and $\alpha \geq \beta > 0$ the following inequalities hold.*

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \leq \frac{\|x - P_C(x - \beta Ax)\|}{\beta},$$

$$\|x - P_C(x - \beta Ax)\| \leq \|x - P_C(x - \alpha Ax)\|.$$

Lemma 4. ([14, 15]) *Let H_1 and H_2 be two real Hilbert spaces. Suppose $A : H_1 \to H_2$ is uniformly continuous on bounded subsets of H_1 and M is a bounded subset of H_1. Then $A(M)$ (the image of M under A) is bounded.*

Lemma 5. [8, Lemma 2.1] *Consider the problem $VI(C,A)$ with C being a nonempty, closed, convex subset of a real Hilbert space H and $A : C \to H$ being pseudo-monotone and continuous. Then, x^* is a solution of $VI(C,A)$ if and only if*

$$\langle Ax, x - x^* \rangle \geq 0 \ \ \forall x \in C.$$

Lemma 6. ([26]) *Let C be a nonempty set of H and $\{x_n\}$ be a squence in H such that the following two conditions hold:*
i) for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;
ii) every sequential weak cluster point of $\{x_n\}$ is in C.
Then $\{x_n\}$ converges weakly to a point in C.

The next technical lemma is very useful and used by many authors, for example Liu [21] and Xu [36]. Furthermore, a variant of Lemma 7 has already been used by Reich in [27].

Lemma 7. *Let $\{a_n\}$ be sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence such that
a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
b) $\limsup_{n\to\infty} b_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Weak Convergence Method

Through this section, we make the following conditions on $VI(C,A)$:

Condition 1. *The feasible set C of $VI(C,A)$ is a nonempty, closed, and convex subset of the real Hilbert space H.*

Condition 2. *The operator $A : H \to H$ is a pseudo-monotone, sequentially weakly continuous on C, and uniformly continuous on bounded subsets of H.*

Condition 3. *The solution set of $VI(C,A)$ is nonempty, that is $\Omega \neq \emptyset$.*

We are now in the position to present our first method.

Algorithm 1.

Initialization: *Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_1 \in C$ be arbitrary*

Iterative Steps: *Given the current iterate x_n, calculate x_{n+1} as follows:*

Step 1. *Compute*

$$y_n = P_C(x_n - \lambda_n Ax_n)$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|. \tag{9}$$

If $x_n = y_n$ or $Ay_n = 0$ then Stop, y_n is a solution of VI(C,A). Otherwise
Step 2. *Compute*

$$x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n).$$

*Set $n := n+1$ and go to **Step 1**.*

We start the convergence analysis by proving that (9) terminates after finite steps.

Lemma 8. *Assume that Conditions 1–3 hold. Then the Armijo-line search rule (9) is well defined. In addition, we have $\lambda_n \leq \gamma$.*

Proof. If $x_n \in \Omega$ then $x_n = P_C(x_n - \gamma Ax_n)$, therefore (9) holds with $m = 0$. If $x_n \notin \Omega$ and assume the contrary that for all m we have

$$\gamma l^m \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| > \mu \|x_n - P_C(x_n - \gamma l^m Ax_n)\|. \tag{10}$$

This implies that

$$\|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| > \mu \frac{\|x_n - P_C(x_n - \gamma l^m Ax_n)\|}{\gamma l^m}. \tag{11}$$

We consider two possibilities of x_n. First, if $x_n \in C$, then since P_C ans A are continuous, we have $\lim_{m\to\infty} \|x_n - P_C(x_n - \gamma l^m Ax_n)\| = 0$. From the uniform continuity of the operator A on bounded subsets of H it implies that

$$\lim_{m\to\infty} \|Ax_n - AP_C(x_n - \gamma l^m Ax_n)\| = 0. \tag{12}$$

Combining (11) and (12) we get

$$\lim_{m\to\infty} \frac{\|x_n - P_C(x_n - \gamma l^m A x_n)\|}{\gamma l^m} = 0. \tag{13}$$

Setting $z_m = P_C(x_n - \gamma l^m A x_n)$ we have

$$\langle z_m - x_n + \gamma l^m A x_n, x - z_m\rangle \geq 0 \ \ \forall x \in C.$$

This implies that

$$\langle \frac{z_m - x_n}{\gamma l^m}, x - z_m\rangle + \langle A x_n, x - z_m\rangle \geq 0 \ \ \forall x \in C. \tag{14}$$

Taking the limit $m \to \infty$ in (14) and using (13) we obtain

$$\langle A x_n, x - x_n\rangle \geq 0 \ \ \forall x \in C,$$

which implies that $x_n \in \Omega$. This is a contradiction. Now, if $x_n \notin C$, then we have

$$\lim_{m\to\infty} \|x_n - P_C(x_n - \gamma l^m A x_n)\| = \|x_n - P_C x_n\| > 0. \tag{15}$$

and

$$\lim_{m\to\infty} \gamma l^m \|A x_n - A P_C(x_n - \gamma l^m A x_n)\| = 0. \tag{16}$$

Combining (10), (15) and (16) we get a contradiction. □

Remark 1. *1. In the proof of Lemma 8 we do not need the pseudo-monotonicity of A.*

2. Note that if $x_n = y_n$ then x_n is a solution of $VI(C,A)$. Indeed, we have $0 < \lambda_n \leq \gamma$, which together with Lemma 3 we get

$$0 = \frac{\|x_n - y_n\|}{\lambda_n} = \frac{\|x_n - P_C(x_n - \lambda_n A x_n)\|}{\lambda_n} \geq \frac{\|x_n - P_C(x_n - \gamma A x_n)\|}{\gamma}.$$

This implies that x_n is a solution of $VI(C,A)$.

The following Lemma states that the sequence $\{x_n\}$ is Fejér monotone with respect to the solution set Ω.

Lemma 9. *Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then for every $p \in \Omega$ it holds*

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1-\mu^2)\|x_n - y_n\|^2. \tag{17}$$

Proof. We have

$$\begin{aligned}
\|x_{n+1} - p\|^2 =& \|y_n - \lambda_n(A y_n - A x_n) - p\|^2 \\
=& \|y_n - p\|^2 + \lambda_n^2 \|A y_n - A x_n\|^2 - 2\lambda_n \langle y_n - p, A y_n - A x_n\rangle \\
=& \|x_n - p\|^2 + \|x_n - y_n\|^2 + 2\langle y_n - x_n, x_n - p\rangle \\
& + \lambda_n^2 \|A y_n - A x_n\|^2 - 2\lambda_n \langle y_n - p, A y_n - A x_n\rangle \\
=& \|x_n - p\|^2 + \|x_n - y_n\|^2 - 2\langle y_n - x_n, y_n - x_n\rangle + 2\langle y_n - x_n, y_n - p\rangle \\
& + \lambda_n^2 \|A y_n - A x_n\|^2 - 2\lambda_n \langle y_n - p, A y_n - A x_n\rangle \\
=& \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\langle y_n - x_n, y_n - p\rangle \\
& + \lambda_n^2 \|A y_n - A x_n\|^2 - 2\lambda_n \langle y_n - p, A y_n - A x_n\rangle.
\end{aligned} \tag{18}$$

Since $y_n = P_C(x_n - \lambda_n A x_n)$ we obtain

$$\langle y_n - x_n + \lambda_n A x_n, y_n - p \rangle \leq 0,$$

or equivalently

$$\langle y_n - x_n, y_n - p \rangle \leq -\lambda_n \langle A x_n, y_n - p \rangle. \tag{19}$$

From (18) and (19), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq & \|x_n - p\|^2 - \|x_n - y_n\|^2 - 2\lambda_n \langle A x_n, y_n - p \rangle + \lambda_n^2 \|A y_n - A x_n\|^2 \\
& - 2\lambda_n \langle y_n - p, A y_n - A x_n \rangle \\
= & \|x_n - p\|^2 - \|x_n - y_n\|^2 + \lambda_n^2 \|A y_n - A x_n\|^2 - 2\lambda_n \langle y_n - p, A y_n \rangle \\
\leq & \|x_n - p\|^2 - \|x_n - y_n\|^2 + \mu^2 \|x_n - y_n\|^2 - 2\lambda_n \langle y_n - p, A y_n \rangle \\
\leq & \|x_n - p\|^2 - (1 - \mu^2) \|x_n - y_n\|^2 - 2\lambda_n \langle y_n - p, A y_n \rangle.
\end{aligned} \tag{20}$$

Since $p \in \Omega$ we have $\langle Ap, y_n - p \rangle \geq 0$, from the pseudo-monotonicity of A we find

$$\langle A y_n, y_n - p \rangle \geq 0. \tag{21}$$

Combining (20) and (21) we obtain

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1 - \mu^2) \|x_n - y_n\|^2.$$

□

From Lemma 9, we have that for every $p \in \Omega$, $\lim_{n\to\infty} \|x_n - p\|$ exists. To obtain the weak convergence, following Lemma 6, it remains to prove that every weak limit point of $\{x_n\}$ belongs to Ω.

Lemma 10. *Every weak limit point of $\{x_n\}$ is a solution of $VI(C,A)$.*

Proof. Let z be a weak limit point of $\{x_n\}$ and let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converges weakly to z. From Lemma 9 we have that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Therefore, $y_{n_k} \rightharpoonup z$. Since $y_{n_k} \in C$ for all k and C is (weakly) closed we have $z \in C$. Since $y_{n_k} = P_C(x_{n_k} - \lambda_{n_k} A x_{n_k})$ it holds

$$\langle x_{n_k} - \lambda_{n_k} A x_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq 0 \;\; \forall x \in C.$$

or equivalently

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq \langle A x_{n_k}, x - y_{n_k} \rangle \;\; \forall x \in C.$$

This implies that

$$\frac{1}{\lambda_{n_k}} \langle x_{n_k} - y_{n_k}, x - y_{n_k} \rangle + \langle A x_{n_k}, y_{n_k} - x_{n_k} \rangle \leq \langle A x_{n_k}, x - x_{n_k} \rangle \;\; \forall x \in C. \tag{22}$$

Now, we show that

$$\liminf_{k\to\infty} \langle A x_{n_k}, x - x_{n_k} \rangle \geq 0. \tag{23}$$

Indeed, let us consider two possible cases. Suppose first that $\liminf_{k\to\infty} \lambda_{n_k} > 0$. We have $\{x_{n_k}\}$ is a bounded sequence, A is uniformly continuous on bounded subsets of H. By Lemma 5, we get that $\{Ax_{n_k}\}$ is bounded. Taking $k \to \infty$ in (22) since $\|x_{n_k} - y_{n_k}\| \to 0$, we get

$$\liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0.$$

Assume now that $\liminf_{k\to\infty} \lambda_{n_k} = 0$. Setting $z_{n_k} = P_C(x_{n_k} - \lambda_{n_k}.l^{-1}Ax_{n_k})$, we have $\lambda_{n_k} l^{-1} > \lambda_{n_k}$. Applying Lemma 3, we obtain

$$\|x_{n_k} - z_{n_k}\| \leq \frac{1}{l}\|x_{n_k} - y_{n_k}\| \to 0 \text{ as } k \to \infty.$$

Consequently, $z_{n_k} \rightharpoonup z \in C$, this implies that $\{z_{n_k}\}$ is bounded. From the uniformly continuity of the operator A on bounded subsets of H it follows that

$$\|Ax_{n_k} - Az_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{24}$$

By the Armijo line-search rule (9) we must have

$$\lambda_{n_k}.l^{-1}\|Ax_{n_k} - AP_C(x_{n_k} - \lambda_{n_k} l^{-1}Ax_{n_k})\| > \mu\|x_{n_k} - P_C(x_{n_k} - \lambda_{n_k} l^{-1}Ax_{n_k})\|.$$

That is,

$$\frac{1}{\mu}\|Ax_{n_k} - Az_{n_k}\| > \frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}}. \tag{25}$$

Combining (24) and (25) we obtain

$$\lim_{k\to\infty} \frac{\|x_{n_k} - z_{n_k}\|}{\lambda_{n_k} l^{-1}} = 0.$$

Furthermore, we have from the definition of z_{n_k} that

$$\langle x_{n_k} - \lambda_{n_k} l^{-1}Ax_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq 0 \quad \forall x \in C.$$

This implies that

$$\frac{1}{\lambda_{n_k} l^{-1}}\langle x_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle Ax_{n_k}, z_{n_k} - x_{n_k}\rangle \leq \langle Ax_{n_k}, x - x_{n_k}\rangle \quad \forall x \in C. \tag{26}$$

Taking the limit $k \to \infty$ in (26) we get

$$\liminf_{k\to\infty} \langle Ax_{n_k}, x - x_{n_k}\rangle \geq 0.$$

Therefore, the inequality (23) is proved.
On the other hand, we have

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Ax_{n_k}, x - x_{n_k}\rangle + \langle Ax_{n_k}, x - x_{n_k}\rangle + \langle Ay_{n_k}, x_{n_k} - y_{n_k}\rangle. \tag{27}$$

From $\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0$ and the uniformly continuity of A we get

$$\lim_{k\to\infty} \|Ax_{n_k} - Ay_{n_k}\| = 0 \tag{28}$$

which, together with (23) and (27) implies that

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x-y_{n_k}\rangle \geq 0. \tag{29}$$

Next, we show that $z\in\Omega$. We choose a sequence $\{\varepsilon_k\}$ of positive numbers decreasing and tending to 0. We can construct a strictly increasing sequence $\{N_k\}$ of positive integers such that

$$\langle Ay_{n_j}, x-y_{n_j}\rangle + \varepsilon_k \geq 0 \ \ \forall j\geq N_k, \tag{30}$$

where the existence of N_k follows from (29). Furthermore, for each k setting

$$v_{N_k} = \frac{Ay_{N_k}}{\|Ay_{N_k}\|^2},$$

we have $\langle Ay_{N_k}, v_{N_k}\rangle = 1$. We deduce from (30) that for each k

$$\langle Ay_{N_k}, x+\varepsilon_k v_{N_k} - y_{N_k}\rangle \geq 0.$$

From the fact that A is pseudo-monotone on H, we get

$$\langle A(x+\varepsilon_k v_{N_k}), x+\varepsilon_k v_{N_k} - y_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Ax, x-y_{N_k}\rangle \geq \langle Ax - A(x+\varepsilon_k v_{N_k}), x+\varepsilon_k v_{N_k} - y_{N_k}\rangle - \varepsilon_k\langle Ax, v_{N_k}\rangle. \tag{31}$$

We show that $\lim_{k\to\infty}\varepsilon_k v_{N_k} = 0$. Indeed, since $x_{n_k}\rightharpoonup z$ and $\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\| = 0$, we obtain $y_{N_k}\rightharpoonup z$ as $k\to\infty$. Since A is sequentially weakly continuous on bounded subset of H, $\{Ay_{n_k}\}$ converges weakly to Az. We have that $Az\neq 0$ (otherwise, z is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|Az\| \leq \liminf_{k\to\infty}\|Ay_{n_k}\|.$$

Since $\{y_{N_k}\}\subset\{y_{n_k}\}$ and $\varepsilon_k\to 0$ as $k\to\infty$, we obtain

$$0 \leq \limsup_{k\to\infty}\|\varepsilon_k v_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\varepsilon_k}{\|Ay_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty}\varepsilon_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty}\varepsilon_k v_{N_k} = 0$. Letting $k\to\infty$, the right hand side of (31) tends to zero since A is uniformly continuous, $\{x_{N_k}\}$ is bounded and $\lim_{k\to\infty}\varepsilon_k v_{N_k} = 0$. Thus, we get

$$\liminf_{k\to\infty}\langle Ax, x-y_{N_k}\rangle \geq 0.$$

Hence, for all $x\in C$ we have

$$\langle Ax, x-z\rangle = \lim_{k\to\infty}\langle Ax, x-y_{N_k}\rangle = \liminf_{k\to\infty}\langle Ax, x-y_{N_k}\rangle \geq 0.$$

By Lemma 5 we obtain $z\in\Omega$ and the proof is complete. □

Remark 2. *As remarked in [3, 33], when the operator A is monotone, it is not necessary to impose the sequential weak-weak continuity on A.*

Combining Lemma 9 and Lemma 10 with Lemma 6 we obtain the weak convergence of Algorithm 1.

Theorem 1. *The sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a solution of $VI(C,A)$.*

4 Strong Convergence Method

In this section, we introduce our second method which is a combination of Tseng's extragradient method with Mann type method [25]. Through out this section, in addition to conditions 1-3 in Section 3 we also need the following condition.

Condition 4. *Let $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $(0,1)$ such that $\{\beta_n\} \subset (a, 1-\alpha_n)$ for some $a > 0$ and*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

The proposed algorithm is of the following form:

Algorithm 2.

Initialization: *Given $\gamma > 0, l \in (0,1), \mu \in (0,1)$. Let $x_1 \in C$ be arbitrary*

Iterative Steps: *Given the current iterate x_n, calculate x_{n+1} as follows:*

Step 1. *Compute*

$$y_n = P_C(x_n - \lambda_n A x_n)$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma l^m \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|.$$

If $x_n = y_n$ or $Ay_n = 0$ then Stop, y_n is a solution of VI(C,A). Otherwise
Step 2. *Compute*

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n(y_n - \lambda_n(Ay_n - Ax_n)).$$

*Set $n := n+1$ and go to **Step 1**.*

Lemma 11. *The sequence $\{x_n\}$ generated by Algorithm 2 is bounded.*

Proof. Setting $z_n = y_n - \lambda_n(Ay_n - Ax_n)$. Thanks to Lemma 9, for every $p \in \Omega$ we have

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - (1-\mu^2)\|x_n - y_n\|^2. \tag{32}$$

This implies that

$$\|z_n - p\| \leq \|x_n - p\|. \tag{33}$$

We have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n-\beta_n)x_n + \beta_n z_n - p\| \\ &= \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p) - \alpha_n p\| \\ &\leq \|(1-\alpha_n-\beta_n)(x_n - p) + \beta_n(z_n - p)\| + \alpha_n\|p\|. \end{aligned} \tag{34}$$

From (33) we obtain

$$
\begin{aligned}
&\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(z_n-p)\|^2 \\
&\quad =(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+2(1-\alpha_n-\beta_n)\beta_n\langle x_n-p,z_n-p\rangle+\beta_n^2\|z_n-p\|^2 \\
&\quad \le(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+2(1-\alpha_n-\beta_n)\beta_n\|z_n-p\|\|x_n-p\|+\beta_n^2\|z_n-p\|^2 \\
&\quad \le(1-\alpha_n-\beta_n)^2\|x_n-p\|^2+2(1-\alpha_n-\beta_n)\beta_n\|x_n-p\|^2+\beta_n^2\|x_n-p\|^2 \\
&\quad =(1-\alpha_n)^2\|x_n-p\|^2.
\end{aligned}
$$

This implies that

$$\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(z_n-p)\| \le (1-\alpha_n)\|x_n-p\|. \tag{35}$$

Combining (34) and (35) we get

$$
\begin{aligned}
\|x_{n+1}-p\| &\le (1-\alpha_n)\|x_n-p\|+\alpha_n\|p\| \\
&\le \max\{\|x_n-p\|,\|p\|\} \\
&\le ... \le \max\{\|x_0-p\|,\|p\|\}.
\end{aligned}
$$

i.e., the sequence $\{x_n\}$ is bounded and so is $\{z_n\}$. □

Lemma 12. *For every $p\in\Omega$ we have*

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2+\alpha_n[2\beta_n\|x_n-z_n\|\|x_{n+1}-p\|+2\langle p,p-x_{n+1}\rangle]. \tag{36}$$

Proof. Using (2) we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &=\|(1-\alpha_n-\beta_n)x_n+\beta_n z_n-p\|^2 \\
&=\|(1-\alpha_n-\beta_n)(x_n-p)+\beta_n(z_n-p)+\alpha_n(-p)\|^2 \\
&=(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|z_n-p\|^2+\alpha_n\|p\|^2-\beta_n(1-\alpha_n-\beta_n)\|x_n-z_n\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)\|x_n\|^2-\alpha_n\beta_n\|z_n\|^2 \\
&\le(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|z_n-p\|^2+\alpha_n\|p\|^2, \qquad (37)
\end{aligned}
$$

which, together (32) implies

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le(1-\alpha_n-\beta_n)\|x_n-p\|^2+\beta_n\|x_n-p\|^2-\beta_n(1-\mu^2)\|x_n-y_n\|^2+\alpha_n\|p\|^2 \\
&=(1-\alpha_n)\|x_n-p\|^2-\beta_n(1-\mu^2)\|x_n-y_n\|^2+\alpha_n\|p\|^2 \\
&\le\|x_n-p\|^2-\beta_n(1-\mu^2)\|x_n-y_n\|^2+\alpha_n\|p\|^2. \qquad (38)
\end{aligned}
$$

Therefore,

$$\beta_n(1-\mu^2)\|x_n-y_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|p\|^2. \tag{39}$$

Setting $t_n=(1-\beta_n)x_n+\beta_n z_n$ we obtain

$$
\begin{aligned}
\|t_n-p\| &=\|(1-\beta_n)(x_n-p)+\beta_n(z_n-p)\| \\
&\le(1-\beta_n)\|x_n-p\|+\beta_n\|z_n-p\| \\
&\le(1-\beta_n)\|x_n-p\|+\beta_n\|x_n-p\| \\
&=\|x_n-p\|, \qquad (40)
\end{aligned}
$$

and

$$\|t_n - x_n\| = \beta_n \|x_n - z_n\|. \tag{41}$$

Combining (40) and (41) we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 =& \|(1-\alpha_n - \beta_n)x_n + \beta_n z_n - p\|^2 \\
=& \|(1-\beta_n)x_n + \beta_n z_n - \alpha_n x_n - p\|^2 \\
=& \|(1-\alpha_n)(t_n - p) - \alpha_n(x_n - t_n) - \alpha_n p\|^2 \\
\le& (1-\alpha_n)^2\|t_n - p\|^2 - 2\langle \alpha_n(x_n - t_n) + \alpha_n p, x_{n+1} - p\rangle \\
=& (1-\alpha_n)^2\|t_n - p\|^2 + 2\alpha_n\langle x_n - t_n, p - x_{n+1}\rangle + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
\le& (1-\alpha_n)\|t_n - p\|^2 + 2\alpha_n\|x_n - t_n\|\|x_{n+1} - p\| + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
\le& (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[2\beta_n\|x_n - z_n\|\|x_{n+1} - p\| + 2\langle p, p - x_{n+1}\rangle].
\end{aligned}$$

□

We are now in the position to establish the main results of this section. To this end, we assume that Algorithm 2 does not terminate at any step n, i.e., it generates an infinite sequence $\{x_n\}$.

Theorem 2. *Suppose that Algorithm 2 generates an infinite iterative sequence $\{x_n\}$ then $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = \arg\min\{\|z\| : z \in \Omega\}$.*

Proof. Since Ω is closed and convex [18], there exists an unique element $p \in \Omega$ such that $p = P_\Omega(0)$. We will show that the sequence $\{\|x_n - p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n - p\|^2\}$.

Case 1: There exists an $n_0 \in \mathbb{N}$ such that $\|x_{n+1} - p\|^2 \le \|x_n - p\|^2$ for all $n \ge n_0$. This implies that $\lim_{n\to\infty}\|x_n - p\|^2$ exists. It implies from Claim 2 that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{42}$$

We also have

$$\begin{aligned}
\|z_n - x_n\| &= \|y_n - \lambda_n(Ay_n - Ax_n) - x_n\| \\
&\le (1+\mu)\|x_n - y_n\|.
\end{aligned} \tag{43}$$

Combining (42) and (43) we get

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Using this we find

$$\|x_{n+1} - x_n\| \le \alpha_n\|x_n\| + \beta_n\|x_n - z_n\| \to 0 \text{ as } n \to \infty.$$

Since $\{x_n\}$ is bounded we assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle p, p - x_n\rangle = \lim_{j\to\infty}\langle p, p - x_{n_j}\rangle = \langle p, p - q\rangle.$$

We have $x_{n_j} \rightharpoonup q$ and $\|x_n - y_n\| \to 0$, using the same technique as in Lemma 10 we get $q \in \Omega$. On the other hand, since $p = P_\Omega 0$, we obtain

$$\limsup_{n\to\infty}\langle p, p - x_n\rangle = \langle p, p - q\rangle \le 0.$$

By $\|x_{n+1}-x_n\| \to 0$ we get

$$\limsup_{n\to\infty}\langle p, p-x_{n+1}\rangle \le 0.$$

From Lemma 12 and Lemma 7 we have $\lim_{n\to\infty}\|x_n-p\|^2=0$, i.e., $x_n \to p$.

Case 2: Assume that there is no $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|\}_{n=n_0}^{\infty}$ is monotonically decreasing. Following the technique in [22] we define $\Gamma_n = \|x_n-p\|^2$ for all $n \ge 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\},$$

i.e. $\tau(n)$ is the largest number k in $\{1,...,n\}$ such that Γ_k increases at $k=\tau(n)$; note that, in view of Case 2, this $\tau(n)$ is well-defined for all sufficiently large n. From [22], τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$

$$\begin{aligned} &0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \\ &0 \le \Gamma_n \le \Gamma_{\tau(n)+1}. \end{aligned} \tag{44}$$

Since $\beta_n \ge a \; \forall n \in \mathbb{N}$, from (39) we have

$$\begin{aligned} a(1-\mu^2)\|x_{\tau(n)}-y_{\tau(n)}\|^2 \le& \beta_{\tau(n)}(1-\mu^2)\|x_{\tau(n)}-y_{\tau(n)}\|^2 \\ \le& \|x_{\tau(n)}-p\|^2 - \|x_{\tau(n)+1}-p\|^2 + \alpha_{\tau(n)}\|p\|^2 \\ \le& \alpha_{\tau(n)}\|p\|^2. \end{aligned}$$

Therefore

$$\lim_{n\to\infty}\|x_{\tau(n)}-y_{\tau(n)}\| = 0. \tag{45}$$

As proved in the first case, we have

$$\|x_{\tau(n)+1}-x_{\tau(n)}\| \to 0$$

and

$$\limsup_{n\to\infty}\langle p, p-x_{\tau(n)+1}\rangle \le 0.$$

From Lemma 12 and $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1} \;\; \forall n \ge n_0$ we have

$$\begin{aligned} \|x_{\tau(n)+1}-p\|^2 \le& (1-\alpha_{\tau(n)})\|x_{\tau(n)}-p\|^2 \\ &+ \alpha_{\tau(n)}[2\beta_{\tau(n)}\|x_{\tau(n)}-z_{\tau(n)}\|\|x_{\tau(n)+1}-p\| + 2\langle p, p-x_{\tau(n)+1}\rangle] \\ \le& (1-\alpha_{\tau(n)})\|x_{\tau(n)+1}-p\|^2 \\ &+ \alpha_{\tau(n)}[2\beta_{\tau(n)}\|x_{\tau(n)}-z_{\tau(n)}\|\|x_{\tau(n)+1}-p\| + 2\langle p, p-x_{\tau(n)+1}\rangle]. \end{aligned}$$

This implies that

$$\|x_{\tau(n)+1}-p\|^2 \le 2\beta_{\tau(n)}\|x_{\tau(n)}-z_{\tau(n)}\|\|x_{\tau(n)+1}-p\| + 2\langle p, p-x_{\tau(n)+1}\rangle,$$

which implies that $\limsup_{n\to\infty}\|x_{\tau(n)+1}-p\|^2 \le 0$, that is $\lim_{n\to\infty}\|x_{\tau(n)+1}-p\| = 0$. The conclusion is follows from (44). □

Remark 3. *Comparing with Theorem 3.1 in [37], Theorem 3.2 in [29] and Theorem 3.2 in [35], Theorem 2 has two major advantages.*

1. *We weaken the Lipschitz continuity of A to uniform continuity on bounded subsets.*
2. *We replace the monotonicity by pseudo-monotonicity and sequentially weakly continuous of A.*

5 Numerical Illustrations

In this section we present some numerical examples illustrating the behavior of our proposed schemes. We consider the classical Hilbert space $H = l_2$ and the $VI(C,A)$ with

$$C := \{x = (x_1, x_2, ..., x_i, ...) \in H \mid |x_i| \leq \frac{1}{i}, i = 1, 2, ..., n, ...\}$$

and

$$Ax := \left(||x|| + \frac{1}{||x|| + \alpha} \right) x,$$

for some $\alpha > 0$. It is easy to see that $\Omega = \{0\}$ and moreover, A is pseudo-monotone on H, uniformly continuous and sequentially weakly continuous on C but not Lipschitz continuous on H. Indeed, let $x, y \in H$ be such that $\langle Ax, y - x \rangle \geq 0$. This implies that $\langle x, y - x \rangle \geq 0$. Consequently,

$$\begin{aligned}
\langle Ay, y - x \rangle &= \left(||x|| + \frac{1}{||x|| + \alpha} \right) \langle y, y - x \rangle \\
&\geq \left(||x|| + \frac{1}{||x|| + \alpha} \right) (\langle y, y - x \rangle - \langle x, y - x \rangle) \\
&= \left(||x|| + \frac{1}{||x|| + \alpha} \right) ||y - x||^2 \geq 0.
\end{aligned}$$

meaning that A is pseudo-monotone. Moreover, since C is compact, the operator A is uniformly continuous and sequentially weakly continuous on C. Finally we show that A is not Lipschitz continuous on H. Assume to the contrary that A is Lipschitz continuous on H, i.e., there exists $L > 0$ such that

$$||Ax - Ay|| \leq L||x - y|| \quad \forall x, y \in H.$$

Let $x = (L, 0, ..., 0, ...)$ and $y = (0, 0, ..., 0, ...)$ then

$$||Ax - Ay|| = ||Ax|| = \left(||x|| + \frac{1}{||x|| + \alpha} \right) ||x|| = \left(L + \frac{1}{L + \alpha} \right) L.$$

Thus, $||Ax - Ay| \leq L||x - y||$ is equivalent to

$$\left(L + \frac{1}{L + \alpha} \right) L \leq L^2,$$

leading to

$$\frac{1}{L + \alpha} \leq 0,$$

which is a contraction and thus A is not Lipschitz continuous on H. In the following figure, we present the numerical behavior of Algorithm 1 and Algorithm 2 when $\alpha = 1$, $H = \mathbb{R}^m$ for different values of m. In this case, the feasible set C is a box

$$C := \{x \in \mathbb{R}^m \mid \frac{-1}{i} \leq x_i \leq \frac{1}{i}, i = 1, 2, ..., m\},$$

for which we have explicit formula of the projection onto C. We choose $\gamma = 0.1, l = 0.5, \mu = 0.8$ for both Algorithms and

$$\alpha_n = \frac{1}{\sqrt{n}+2}, \beta_n = \frac{1-\alpha_n}{2} \quad \forall n \in \mathbb{N},$$

for Algorithm 2. This choice of parameters implies that (9) is satisfied with $m_n = 0$ for all iterations. The starting point is chosen as $x_0 = (1,1,\ldots 1) \in \mathbb{R}^m$. Experiments were generated with Matlab R2017a on a Linux OS with a 2.39 Ghz processor and 16 GB of memory. When $m = 20$, Algorithm 1 provides the solution after 149 iterations and 0.001746 seconds of CPU time, while Algorithm 2 needs 71 iterations and 0.001022 seconds of CPU time. When $m = 50$, Algorithm 1 provides the solution after 150 iterations and 0.001784 seconds of CPU time, while Algorithm 2 needs 73 iterations and 0.001191 seconds of CPU time.

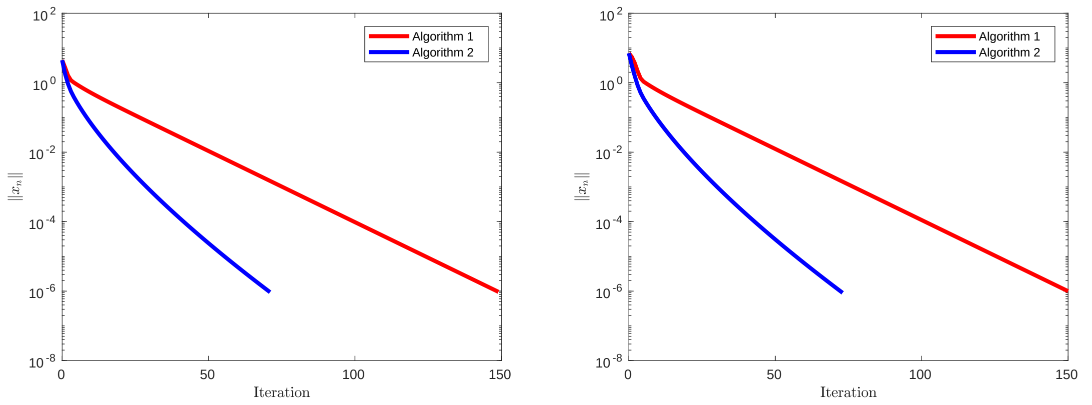

Figure 1: Performance of Algorithm 1 and Algorithm 2 when $m = 20$ (left) and $m = 50$ (right).

6 Conclusions

In this paper we proposed two Tseng's extragradient extensions for solving non-Lipschitzian pseudo-monotone variational inequalities in real Hilbert spaces. Under suitable and standard conditions we establish weak and strong convergence theorems of the proposed schemes. Our results extend and generalize some existing results in the literature and numerical illustrations demonstrate the behavior and potential applicability of the proposed methods.

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