UNIVERSITY OF SOUTHAMPTON

Dynamic discrete choice models: Forecasting of competitive events through optimal linear filtering of choice persistence effects

by

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A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Southampton Business School
Department of Decision Analytics and Risk

March 2020

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

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This thesis presents a new modelling framework for dynamic discrete decision-making problem settings, in which persistence in preferences, derived from previously made ranked choices, is taken into account. The endorsed framework leverages trends of the revealed preferences to model the evolution of the temporal persistence of unobserved attributes of alternatives, and it effectively incorporates changing choice sets and irregular time durations between the repeated availability of alternatives in consecutive decision events. The new model structure eliminates effect-confounding problems inherent in incumbent models, and it highlights the effects of time duration bias and the unreliability of lower ranked choices on the probabilities of future choice selections. Following a *post-positivistic* research paradigm, empirical validation of the models in a naturalistic market environment (UK horse-betting markets), which integrates behavioural (decision-maker-related) and economic (betting-market-related) information sets, is carried out. The proposed methodology centres around a two-stage model structure, which includes elements of the classical Conditional Logit approach, revealed order of preferences, and the Kalman filtering of the latent states, aimed at providing forecasts of choice probabilities. These probabilities are subsequently used for implementation of a Kelly betting strategy, which, together with standard statistical tests of significance, assesses the merits of the modelling approach. In particular, it is shown that a novel Kalman filter algorithm, developed for filter divergence mitigation, outperforms traditional Kalman filtering algorithms.

The empirical results and the associated analysis confirm that forecasted trend variables add statistically significant information over public market information (betting odds) and that incorporating trend variable forecasts in a betting strategy yields above-average monetary gains. Analysis of the evidence collected in the study leads to the conclusion that persistence in preference effects are significant and have to be controlled for, in order to mitigate the effects of the considered biases. In a wider context, obtained evidence confirms the propensity of vested decision makers to time duration bias in a revealed preference setup and that importance weighting of the ranked choice data may be used to mitigate the effects of lower ranked alternatives' unreliability.

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Nomenclature

$\mathbb{1}(x)$	The indicator function
$\prec$ or $\succ$	The preference/dominance symbol
α	The ageweighting factor
α_t	A state vector
$\alpha_{i,j}$	A decision maker's intrinsic unobservable preference for the choice j
$\hat{\alpha}_t$	The a-posteriori estimate of α_t
$\hat{\alpha}_{t\|t-1}$	The a-priori estimate of α_t
$\alpha_{0\|0}$	The initial (unconditional) state vector
α_j	The probit classification boundaries
β	A linear weighting of alternative attributes or other exogenous variables
$\boldsymbol{\beta}$	The Conditional Logit weighting of exogenous variables
β_y	The Conditional Logit weighting of an endogenous trend estimate
β_t	The trend state variable
b_i^j	The Kelly betting strategy bet sizes for every runner i in the race j
γ	The utility effects persistence weighting
c_t	The deterministic state drift vector
$\boldsymbol{D}$	The vectorised result of a race
δ_i	The time between i-th and $i+1$-th observation
d_t	The deterministic output offset vector
d_{ij}	The selected alternative i in a decision event j
$d_{i,j,t-1}$	A lagged dummy choice variable
ϵ_{nj}	The unobserved portion of the utility U_{nj} in a decision event n
ε	A stacked vector of all random disturbances
ε_t	The measurement noise in a state space model
$\varepsilon_{i,j,t}$	An idiosyncratic (alternative specific) taste shock
E	The expectation operator
eig	The matrix eigenvalue operator
η_t	The state noise vector in a state space model (unmodelled dynamics)
$\eta_{i,j,t}$	The fundamental disturbance in an AR model of $\varepsilon_{i,j,t}$
$F(\varepsilon)$	The joint CDF of a random vector ε
F_t	The output error covariance matrix
$f(\varepsilon)$	The joint PDF of a random vector ε

f_t	The output variable error covariance
ϕ_i	The parameters of the AR of an ARMA filter
$\boldsymbol{\theta}$	The vector of true model parameters
$\hat{\boldsymbol{\theta}}$	The vector of estimated model parameters
$\boldsymbol{\theta_{KF}}$	The vector of all State Space/Kalman Filter model parameters
$\boldsymbol{\theta_{CL}}$	The vector of all Conditional Logit model parameters
H_t	The measurement noise covariance matrix
h	A potential winner in a race
χ^2_κ	The chi-squared distribution with κ degrees of freedom
i	A decision event or maker i
j	An alternative in a choice set
J	A choice set
K_t	The Kalman Gain vector
κ	The degrees of freedom
ℓ	The likelihood function
λ	The forgetting factor in a smoothing algorithm
λ_i	The parameters of the MA part of an ARMA filter
λ	The Likelihood Ratio statistic
μ_t	A level state vector
$\mu_{i,j,t}$	A latent utility state variable
$\mathcal{N}$	The normal probability density function
N	The number of races in a data set
N_i	The length of a time series
N_{Rmax}	The total number of the alternatives in all decision events
N_{Emax}	The total number of decision events
NaN	Not a Number
ν_t	Innovation (the prediction output error)
Ω	The order of the state space model (i.e. the number of states)
ω	The total number of exogenous variables in a model
Φ	The standard normal cumulative distribution function
Pr	The probability function
P_{ni}	The probability that alternative i is selected in a decision event n
P_t	The state error covariance matrix
$P_{t\|t-1}$	The a-priori estimate of P_t
$P_{t\|t}$	The a-posteriori estimate of P_t
$P_{0\|0}$	The initial (unconditional) state error covariance matrix
p_t	The state error covariance
p^j_h	The actual winning probability of the runner h
Q_t	The state noise covariance matrix
q	The ratio of disturbance variances in structural time series models
$R_{xx}(\tau)$	The autocorrelation function

r_h^j	The decimal odds on a runner in a race
$\mathrm{se}(\hat{\theta})$	The standard error of parameter estimates
σ_x^2	The variance of a univariate random variable x
T	The length of a time series
T_t	The state transition matrix
t	A discrete decision event (time)
U_{nj}	The utility of alternative j obtained in a decision event n
V_{nj}	The representative utility of alternative j in a decision event n
W	The Wald statistic
$\mathbf{w_{i,t}}$	Exogenous explanatory and conditional heteroscedeastic variables
$\boldsymbol{X}$	A stacked vector of the exogenous variables in a race
$X_{i,j,t}$	The attributes of alternative j evaluated by a decision maker i
$x_{i,t}$	Exogenous explanatory variables of the intrinsic value of an asset i
Y_t	A stacked vector of output values of a state space models
$\hat{\boldsymbol{Y}}$	A stacked vector of the predicted endogenous trend variables
$\check{Y}$	A stacked vector of the ex-post rankings
y_t	The output of a state space model
$y_{i,t}$	The observable price of an asset i
$y_{i,j}$	The ordered alternative y_i in a decision event j
$y_{i,t}^*$	A latent continuous time random variable underlying an asset at time t
$y_{i,j}^c$	The continuous equivalent of the ordered alternative $y_{i,j}$
$\hat{y}_{i,j}^c$	A conditional estimate of $y_{i,j}^c$
Z	The standard normal distribution
Z_t	The measurement (sensor) matrix
ζ_t	The trend state noise variable

Acknowledgements

Doctoral studies and the preparation of this thesis were certainly the most significant academic challenges I have encountered to date. This endeavour would not have been possible without the guidance and support of a number of people, to whom I am extremely grateful.

First, I would like to express my deepest gratitude to my PhD supervisory team, Prof. Dr. Ming-Chien Sung and Dr. Tiejun Ma, for their excellent guidance and support in theoretical and practical aspects of this research, in spite of their many other academic and professional commitments. They gave me a lot of freedom to follow my research ideas whilst taking care to ensure that the research goals and agreed schedule remained the guiding stars. In addition, they frictionlessly managed all administrative aspects of the thesis' submittal and examination.

A very special thanks goes to my academic advisor Emeritus Prof. Dr. Johnnie Johnson, who guided me through the PhD journey through a combination of inspiring knowledge, optimism, and a positive attitude. He gave me the first impetus to consider the dynamic aspects of decision-making, which was very much aligned with my scientific, quantitative, and stochastic view of the world, and nurtured the discovery of new ideas through numerous discussions and suggestions about mathematical modelling and the interpretation of results.

I owe a great deal of gratitude to research fellow Dr. David McDonald for many general discussions on horse-betting, and Matlab code for Kelly betting strategy evaluation.

Furthermore, I want to thank my mother, Darinka Rajković, for all the years of support and love, and to my late father, Prof. Zvonimir Rajković, for nurturing my academic interests and foundations. I will always be greatly indebted to them.

Next, I would like thank my two gorgeous girls at home: Karla and Sonja. They had to put up with a lot of lost Daddy time during five long years of my PhD candidature.

Finally, and most importantly, I would like to thank my beloved wife, Raluca, for her continued faith in me, her serenity, encouragement, and a lot of patience. She was always there cheering me up whilst balancing numerous issues and supported me through the challenges of transcontinental movements and life management.

To Raluca, Karla, and Sonja for all your love and support.

Academic Thesis: Declaration Of Authorship

I, **Ivan Rajković**

declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Dynamic discrete choice models:
Forecasting of competitive events through optimal linear filtering of choice persistence effects

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;

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3. Where I have consulted the published work of others, this is always clearly attributed;

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6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;

7. None of this work has been published before submission

Signed: ……………………………………..

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Chapter 1

Introduction

In his 2000 Nobel Prize lecture, Daniel McFadden presented the theory and methods for analysing discrete choices, posited as an extension to the representative agent theory (McFadden, 2001). This theoretical extension has proven to be important for economists and other social scientists seeking to study many important settings in which choices involve discrete rather than continuous alternatives. His seminal model – in its original formulation named the Conditional Logit (CL) model – captures decision behaviour as the conditional distribution of demand (or desirability) amongst a set of non-overlapping alternatives in a standalone (i.e. static) decision event. Practical applications of discrete choice models in the fields of economics, psychology, and marketing expanded the original scope considerably to form aggregate (or market-level) models and include those that could account for different choice substitution patterns (Train, 2009).

The need to extend discrete choice models capable of analysing persistence and/or the variability effects of 'unexplainable' choice variance in repeated decision events realised by the same individual or households was identified more than three decades ago (Cherchi, 2012). For example, these problem settings occur naturally when repeated everyday purchases are captured for analysis (e.g. through shop loyalty programme data collection), when consumer attempt to evaluate the attractiveness of different transportation and leisure opportunities based on their previous experiences (Cherchi et al., 2017), policy makers estimates of job utility preferences of welfare receipts (Bhuller et al., 2017), and in behavioural sciences and consumer psychology (Lee et al., 2015). Models designed to analyse and/or predict persistence in preferences over time are referred to as 'dynamic discrete choice models'.

Inclusion of dynamic aspects expands the general structure of classical decision-making models, reportedly used by: (1) marketing researchers seeking to investigate the effects of different parameters, such as price changes and promotions on loyalty (Dubé et al., 2010) and brand value (Guhl, 2014) on consumer loyalty, (2) psychologists seeking to describe the process of learning and adaptation of Decision Makers (DMs) on first encountering a new product, the subsequent building of stable preferences and the usage of heuristics and habits instead of actually repeating the decision process during every decision event (Hoeffler and Ariely, 1999), (3) labour economists investigating, for example, the conditions and probability of re-entering into the labour force (Heckman, 1981), (4) environmental economists in quantisation of temporal impacts on welfare measures caused by an environmental policy (Swait et al., 2004), (5) public transportation planners evaluating levels of service (Hirobata and Kawakami, 1990) and the benefits of building of a new railway line (Bradley, 1997), and (6) economists interested in the demand-building process (Williams, 1977), to mention but a few.

The following section describes the problem statement, i.e. limitations of the incumbent approaches, and identifies gaps in the literature.

1.1 Problem Statement and Contribution

All of the academic studies listed above share common theoretical elements, used either individually or in combination, to capture behavioural dynamics effects in discrete choice settings: state dependence, habit persistence, and heterogeneity (Ramadurai and Srinivasan, 2006). State dependence is defined as a decision-maker's propensity to select an alternative because he selected it in a past decision event (i.e. the choice made). Some of the reasons for state dependence, for example, include reluctance to re-evaluate choices over time because of inertia, intrinsic satisfaction with the choice, and avoidance of switching costs (Hyslop, 1999). Habit persistence captures the lagged impact of past utility values on current choice, and it aims to determine the influence of habit and the temporal persistence of any unobserved attributes of the alternatives on the choice behaviour. Heterogeneity aims to explain variations in choice behaviour across multiple decision-makers having the same observed characteristics (explanatory attributes) or during repeated decision events realised by an individual DM. These variations can be caused either by unobserved static intrinsic attributes (i.e. idiosyncratic for each DM) or transitory behavioural shocks, represented through serial correlation of the unobserved characteristics affecting previous choices (Kitamura, 1990; Nerlove, 2014).

The above-identified components act concurrently, and the determination of their true relative contribution is not trivial, i.e. the unequivocal estimation of the parameters of an underlying model structure is prone to confounding effects (Cherchi, 2012). In addition, if unobservable and serially correlated characteristics (i.e. heterogeneity) are not controlled properly, spurious state dependence and habit persistence may be inferred and could result in a biased model (Heckman, 1981; Heiss, 2008; Hsiao, 2014; Tavassoli and Karlsson, 2015).

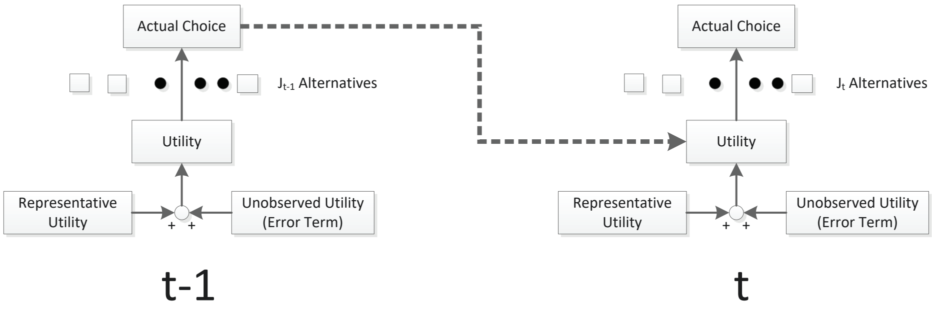

FIGURE 1.1: Schematic representation of State Dependence

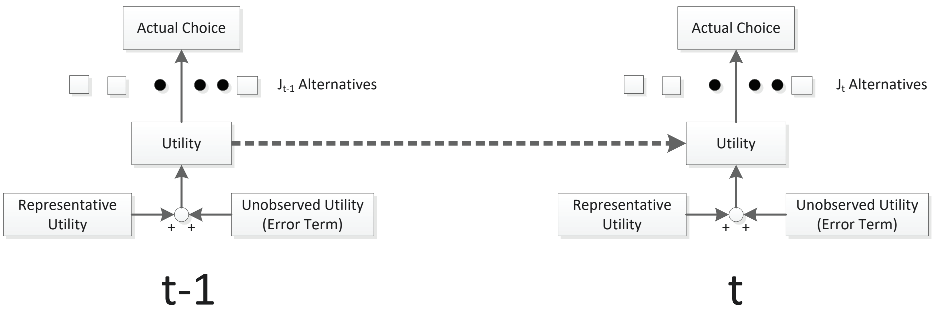

FIGURE 1.2: Schematic representation of Habit Persistence

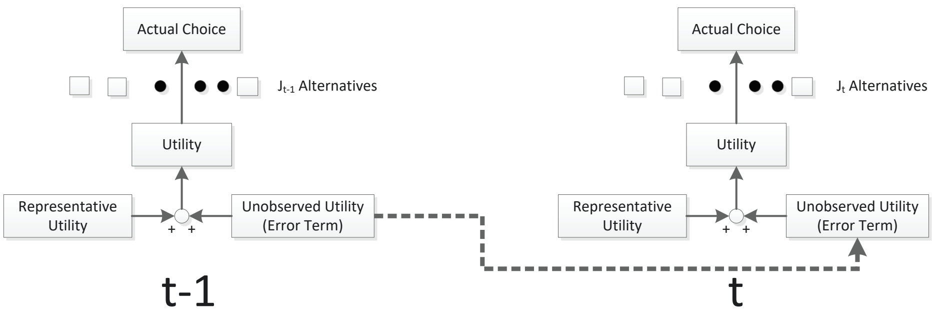

FIGURE 1.3: Schematic representation of Heterogeneity

The fundamental underpinning of Discrete Choice Models (DCMs) is mapping from a latent utility function, structured as an additive combination of an externally observable component and a random component, which contains all unobservable decision attributes, including heterogeneity effects, relating to a discrete choice. Models with such a structure are called Random Utility Models (RUMs). Decision-making based on RUM is typically derived by applying the principle of utility maximisation (Train, 2009). A schematic representation of the described dynamic effects are given in Figure 1.1 (State Dependency), Figure 1.2 (Habit Persistence), and Figure 1.3 (Heterogeinity).

Incorporating dynamic effects in a choice model typically follows one of three approaches: (1) additive correction of the alternative-specific utility functions constructed in the form of an autoregressive error process, (2) an adjustment to the latent utility function by taking into account information from the past, in the form of state dependence, inertia (through lagged choice information), habit persistence (through a lagged utility function and/or other exogenous variables) and heterogeneity/variety-seeking (through autocorrelation of the unobserved factors), and (3) discrete choice models in which functional parameters are time-variable and their evolution is based on distributed lag specifications. Several authors (Lachaab et al., 2006; Lee, 2014; Guhl, 2014) have endorsed the usage of State Space Models (SSMs) to model dynamic effects in combination with different Bayesian methods for parameter estimation, including a classical method borrowed from control engineering, namely the Kalman Filter (KF) algorithm, when the assumed functional form of the utility is linear in parameters.

A common characteristic of all dynamic models with lagged effects of previous choices, as discussed in the literature, is that all of them have been tested on equidistantly sampled and balanced panel type data, typically using short to medium length panels, with an implicit assumption that there are no time (forgetting) effects beyond the time between the current and the preceding decision event and that the same choice set is always available (Keane, 2015). Lack of support for changing choice sets (GAP 1) and different times between availability of alternatives (GAP 2) are identified as gaps in the knowledge in terms of discrete choice modelling.

Furthermore, in some of the mentioned disciplines, available data may also include the ranking of alternatives in the choice set evaluated in a decision event, in the form of ordered data. Examples of inherently ordered data are bond ratings, consumption preferences, and political and health satisfaction surveys (Hensher et al., 2005), to mention but a few. Ordered data contain additional information compared to the standard discrete one-out-of-many alternatives (i.e. multinomial) case and dynamic choice models can be adapted to exploit it in the form of lagged rankings. Lagged ranking may model the effects of state dependence and heterogeneity (in the form of autocorrelation of the unobserved variables).

Historically, dynamic models leveraging ordered data have been studied in finance. In these instances, for example, the Bank Rate (BR) adjustment policy of the Bank of England (Park, 2011), discrete stock price movements (Tsay, 2005) and high-frequency trading returns (Müller and Czado, 2005) were modelled, as well as in medical socio-economic studies aiming to explain the development of the perceived severity of headaches (Czado et al., 2005) or health satisfaction (Contoyannis and Jones, 2004). Dependent on the type of data used in these applications – time series type (i.e. single DM participating in repeated decision events) or panel type (i.e. multiple DMs) – the studies focused either on the serial correlation of alternative-specific utility function errors or state dependency types of models, assuming very similar model structures as seen in multinomial choice models (Kitamura and Bunch, 1990). However, due to the similarities between latent variable structures, dynamic ordered choice models suffer from similar limitations and exhibit the same gaps as presented previously, namely lack of support for changing choice sets and non-uniform sampling times.

The main goal of this research is to contribute a novel model structure that captures the evolution of preferences for alternatives over time in dynamic discrete choice problem settings. In order to accomplish this aim, a dynamic SSM that uses *weighted* revealed order of preference data from previous decision events is used to extract residual information beyond that captured by static DCMs. In essence, the method builds on the Kalman state filtering algorithm and hence initiates a noticeable departure from the classical approaches to dynamic discrete choice models leveraging either the lagged outcome variable (Keane, 2015) or continuous latent states to control for the impact of previous (unknown) utility functions (Lee, 2014). Furthermore, a new model fitting methodology, using a two-step approach for separating the linear state space and CL model parameters, is endorsed and its theoretical viability explained.

This research should contribute to understanding of DCMs with explicit modelling of the choice persistence effects (i.e. the effects of inertia, which influences the probability of repeated choice if the same choice was already made in the past), and how they can be used for predicting the behaviour of a DM. Postulating the lagged effects of a choice causes us to depart from the classical modelling assumptions used regularly in numerous marketing studies since McFadden (1974) endorsed the *logit* modelling structure and the associated statistical methods for parameter estimation.

The distinctive feature of the proposed approach is that it allows for separating multiple sources of preference and choice dynamics (persistence in preferences), i.e. it overcomes the problem of effect-confounding, inherent in other models (especially in state dependence models). Furthermore, it shows how the state space approach can model unobserved heterogeneity and temporal variability in preferences (variance seeking), using an ordered outcome as information. Finally, an empirical example of the derived approach, related both to behavioural (decision-maker related) and economic (betting market related) information sets, using recent data obtained from UK horse-wagering markets, is used to illustrate the model's usefulness and the effects of model bias if persistence in preferences is not taken into account.

Moreover, since the ultimate result of the described modelling approach is a probability of choosing an alternative from a given choice set, which occur regularly in several relevant problem settings, there are tangible academic and practical benefits of closing the identified gaps in the literature. In principle, all discrete choice modelling applications bound to the persistence in preferences can be classified either as behavioural-descriptive or direct prediction/forecast focused. The behavioural-descriptive category is bound to analysis and explanations of mechanisms related to persistence in preferences affecting the decisions made by a particular DM and derivation of decision-making strategies leveraging the obtained information. For example, studies designed to assess effects of some exogenous events, e.g. policies, on settings related to persistence in preferences under the conditions of changing choice sets combined with irregular times between availabilities of alternatives in decision events will directly benefit from the methodology that closes the mentioned gaps. There are numerous practical settings that can be related to the gap, such as credit risk scoring (Florez-Lopez and Ramon-Jeronimo, 2014), real options evaluations (Damaraju et al., 2015), transportation decisions evaluations (Mahmassani and Liu, 1999), political science (Berg et al., 2008), insurance (Florez-Lopez, 2007), labour economics (Dostie and Sahn, 2008), immigrant welfare (Hansen and Lofstrom, 2009), and corporate insolvency predictions (Khoja et al., 2016), to mention but a few.

On the other hand, applications that use direct prediction/forecast information may benefit from the methodology that closes the identified gaps in the literature in a very straightforward way. Indeed, a resulting model of persistence in preferences that would offer a 'better' model of probabilities of outcomes of competitive sporting or political events traded in bookmakers or exchange betting markets, could yield above average profit at the expense of less skilful market participants. The horse-wagering markets, used in this thesis as an empirical test setting for model evaluations, represent the most intensively studied market for forecasts of competitive events (Bruce and Johnson, 2000) and, arguably, the most important application domain. Beyond that, participants in forecasting markets for other competitive events, such as election results markets (Berg et al., 2008), or even market makers of general speculative markets (Ghosh, 2002) may profit from modelling strategy, methodology, and analyses used to close the identified knowledge gaps. The importance of both focus categories and the associated scope of research underpin the scientific relevance of the gaps' closing endeavours.

In summary, this thesis makes a significant contribution to understanding dynamic discrete choice models and persistence in preferences, applied in a naturalistic market environment, which can be extended easily to forecasting outcomes in financial and other competitive markets.

The next section discusses the research objectives derived from the problem statement and correlates them with potential application areas bound to models developed in the course of this research.

1.2 Research Objectives and Research Application Areas

Two interrelated gaps in the literature help formulate the formal research objectives bound to the persistence effects in changing choice sets and with irregular sampling. The first objective addresses the changing choice sets in sequential choice events for one decision-maker by taking the information contained in the revealed ranking of the alternatives, i.e. modelling of the changing choice sets part of the identified gap.

> **Research Objective 1.** To design a decision-making model that feeds back ordered choice information obtained from previous decision events, and to predict the behaviour of a decision-maker facing changing choice sets both in size (number of alternatives) and the actual selection of alternatives, based on the trend of the alternative-specific proxy of the preference (or worthiness). The trend shall be considered endogenous to the model.

The problem setting derived from changing choice sets is not artificial, even if it may seem to be the case at the first glance. For example, in a political context, representatives in a democratic parliament face constantly changing alternatives during voting, even though some of the alternative categories may be similar. Another example from the same context are municipal elections, in which some parties do not have representatives in all municipalities (Yamamoto, 2011). Furthermore, varying choice sets can model the results of sporting events where the probability of a winner from many teams/athletes is of interest. In such cases, the result of the sporting event can be modelled as the result of a choice process of an abstract DM ('nature'), which acts rationally through weighting of the underlying utility functions. Probably the best example of this approach is found in horseracing, as first posited by Bolton and Chapman (1986).

In order to close the non-uniform sampling portion of the identified gaps in theoretical coverage of dynamic discrete decision-making settings, a unified model which utilises ordered choice information from irregularly occurring previous decision events is required.

The second objective of the work is:

> **Research Objective 2.** To close the gap between uniform and non-uniform (irregular) sampling times in the context of changing choice sets in a dynamic discrete decision model that unifies support for non-uniform sampling, changing choice sets, and ordered choice information on preferences revealed in previous decision events. If available, information on the relative importance of past decision events can be used to enhance the informational content of the revealed preference and to mitigate effects bound to uncertain reliability of lower ranked alternatives. In particular, those interested in this research should appreciate the KF as a powerful and flexible method for estimating latent states, especially in the framework of dynamic decision-making derived from RUM and the utility maximisation principle. A model parameter identification methodology, which combines both the linear SSMs and non-linear structures of the CL in a likelihood functional form, shall be derived and its limitations explained.

Irregular times between the presence of alternatives in a choice set can occur in cases of real options, such as opening and closing of mines (Moel and Tufano, 2002) and divestment alternatives (Damaraju et al., 2015), market studies of differentiable (luxury) commodities, such as luxury wines (Wolf et al., 2018), and different prediction markets of irregular events (e.g. extraordinary elections). Moreover, many longitudinal surveys may lead to incomplete and/or corrupt data (Florez-Lopez, 2010), which can be treated as irregularly spaced dynamic decision models (Millimet and McDonough, 2017).

Moreover, empirical studies should validate that the endogenous trend variable extracts statistically significant additional information, which in turn allows for better decision prediction, compared to the case when only exogenous static variables are included. The example has a twofold intention: (1) it should present a proof of concept for the proposed methodology and (2) it should outline an approach that can be used to model competitive sporting or political events as well as some special financial and competitive markets.

1.3 Potential Challenges

Most of the applications mentioned above postulate, either explicitly or implicitly, the availability of balanced panel data for parameter fitting and inference, in spite of the fact that unbalanced panels are the norm (Baltagi, 2008). In a balanced panel, the number of time periods (i.e. the number of decision events) is the same for all decision makers. In addition, decision makers face the same alternatives in every decision event. Balanced panel applications do not pose any principal difficulties as long as the number of unknown model parameters is relatively small compared to the number of data points available.

Models developed under the assumption that the one decision-maker maker participates repeatedly in decision events, with changing choice sets and irregular times between two subsequent instances in which a particular alternative is part of the choice set, have to overcome two principal difficulties, assuming that the irregular times are *a-priori* known (Van Heerde et al., 2004): (1) capturing the correlation of choice attributes that exists over time and (2) different choice attribute sequence lengths with a significant portion of very short sequences attributes' evolution.

Resolving the first challenge is bound to the correct (or at least reasonably good) specification of the underlying dynamic model, which has to be selected based on some *a-priori* considerations. Obviously, since the actual Data Generating Process (DGP) of attributes' evolution is unknown, every selected model will generate either a smaller or a larger approximation error. Hence, the modelling strategy has to make explicit or implicit provisions for error compensation. For the purposes of this study, there are two ways of addressing this challenge. First, the trend estimation performance of some of traditional model-error compensation algorithms (c.f. Jazwinski (1970)), known from aerospace applications, is evaluated. Second, for the cases when the traditional algorithms do not yield a sufficient improvement of out-of-sample performance, derivation of a novel algorithm based on *a-priori* knowledge of the model structure and the noise propagation properties between the states and the measurement noise and minimisation of an estimation error functional with exponential ageweighting of the observed data is given instead.

The second challenge is of a statistical (data-fitting) nature and has two main consequences: first, correct initialisation of the dynamic model is essential for the short alternative attribute sequence lengths considered herein, since large initial condition errors could render any forecasted attribute values for the whole (short) sequence useless. From the practical point of view this translates to the maximum model order requirement and model initialisation with a diffuse prior. These requirements are addressed through *a-priori* model structure definition and selection of a simulation tool function that supports diffuse initialisation procedure (MATLAB).

Second, the resulting likelihood function, constructed over all alternatives for purposes of parameter fitting, may become very flat, so that standard optimisation algorithms may fail to converge and, consequently, optimal parameter values cannot be found. The problem of flatness of the likelihood function can be overcome through substitution of classical, gradient-based, optimisation algorithms with their modern evolutionary based counterparts, such as particle swarm or simulated annealing (Simon, 2013), because of their reduced sensitivity to local minima.

Furthermore, an additional aggravating effect is the non-stationarity of any *a-priori* selected DGP capable of incorporating irregular sampling times. As shown in this study, the KF with deterministic and known system matrices, as an optimal linear estimator, can be used to address all challenges inherent in dynamic discrete choice models aiming to close the identified gaps in the literature. In fact, the proposed model structure generates a behavioural proxy variable that is statistically significant over the best single attribute of choice in horse-racing markets (starting price), which is notoriously difficult to achieve (Bruce and Johnson, 2000).

1.4 Research Contributions – Preliminary Assesments

Guided by the stated research objectives, the thesis provides a gradual build-up and integration of theoretical and methodological elements needed to close the identified knowledge gaps. During the research endeavour, theoretical and methodological discussions, analyses, rationales, and evidence contribute, both incrementally and cumulatively, to knowledge in general and to understanding of dynamic decision-making and modelling of persistence in preferences in particular. Moreover, the study develops new insights relating to behavioural biases bound to time duration misperception and unreliability of lower ranked preferences, by developing several innovative models and carrying out empirical tests on recent data drawn from UK betting markets. Furthermore, it assesses the economic implications of the biases discussed.

This section provides a preliminary assessment of the theoretical and methodological aspects of the conducted research and its contribution to academic knowledge. In should be noted though, that the details of the contribution are deliberated throughout the thesis, and that this preliminary assessment pre-fetches some of the results and conclusions that can be put in the correct sequence only as the thesis develops. A concluding summary (see 6.2) reiterates the contributions and provides the final outlook on the evidence provided that this thesis makes a significant contribution towards understanding of the nature of dynamic decision-making in naturalistic environments and the associated economic importance of the obtained findings.

1.4.1 Theoretical Contributions

The stated research objectives call for formalisation of several effects observable in dynamic decision-making, modelled through variables proxying the effects and their integration aimed to explain and forecast the extent of persistence in preferences. The research objectives are formulated to close the identified gaps in literature and are bound to definition of a model framework that embeds support for evolving choice sets (GAP 1), ranked preference data, and irregular duration times between decision events (GAP 2).

Research endeavours presented in this thesis yield five significant theoretical contributions, derived and implemented in order to obtain an operational modelling framework, founded on the *post-positivistic* research paradigm. Explicitly, the first contribution is bound to estimation of trends of the 'attractiveness' of the alternatives, that may be arbitrary combined in a decision event, thus offering the needed flexibility to model changing choice sets. The second theoretical contribution is related to the definition of the stochastic trend dynamics, thus parting from known model structures without satisfactory provisions against possible confounding of dynamic effects, which in turn may lead to modelling biases. Treatise on the research-philosophical underpinning of the study, founded on author's understanding of the modelling process as incremental reduction of elements of randomness in existing dynamic decision-making models, comprises the third theoretical contribution. The fourth contribution of the research is bound to the research design which allows integration of the persistence in preferences effects, the unreliability of lower ranked preferences, and the time duration bias. Finally, the fifth theoretical contribution results from measures of economic significance of the decision-making biases, reflecting the evidence that trends in horse performances contain residual information over the market.

1.4.2 Methodological Contributions

Research objectives, as defined in section 1.2, are predominantly oriented towards definition of a novel modelling framework and far-reaching expansions of the incumbent methods. Hence, several methodological contributions resulted from the research endeavours. The first major methodological contribution is inherent in the designed model structure, that combines trends of revealed preferences for observed alternatives with the CL discrete decision-making structure capable of incorporating changing choice sets. The second major methodological contribution is closely related to the capability of the designed model structure to use the linear KF including the parameter estimation and filter initialisation in spite of the nonlinear model setting as a whole. The third major contribution is bound to the novel error-correcting algorithm, derived in order to mitigate filter divergence effects caused by unknown statistical properties of the trend DGP and over-fitting effects, which outperforms both standard linear and incumbent error-filtering algorithms used in different engineering applications.

In summary, outlined theoretical and methodological contributions are significant and relevant for researchers and practitioners who apply DCMs with choice persistence as a modelling approach, since it compensates for effects of some heuristic biases, significantly improves out-of-sample forecasts of alternative selections, mitigates the effects of over-fitting, and may offer insights undisclosed by incumbent modelling approaches.

1.5 Thesis Structure

The structure of the remaining chapters in the thesis is as follows:

Chapter 2 begins by introducing the general theory of discrete choice modelling and the connection between RUMs and modelling assumptions put forward by Thurstone (1927) and Luce and Suppes (1965). Next, a probabilistic model of decision-making for a model postulating the utility function, consisting of an observable and an unobservable random part derived from an expected value over all alternatives, is discussed without explicitly specifying the stochastic characteristics of the random variables describing the unobservable parts of the utility function. Specification of a particular form of the Probability Density Function (PDF) (Gumbel) leads to a *logit* model definition, which yields a closed-form representation of the probability of selecting an alternative. Comments on the usability of the *logit* model family, and subsequent critique and limitations for use, conclude the section, which aims at explaining the underlying theory required to understand the fundamental tools used in classical (static) DCMs.

The same chapter (2.1.2) develops further the structures of DCMs, aiming at capturing the behaviour of an adaptive DM, which uses information from decisions actually made and/or the perceived utility elicitation from the past (backward-looking information) and/or evaluates the consequences of decisions made today on a future accumulated pay-off (forward-looking models – Discrete Choice Dynamic Programming Models (DCDPs)). Two seminal dynamic models suggested by Heckman (1981) and Guadagni and Little (1983) endorse the usage of information on previous choices in the form of an additive dummy variable as a proxy of the state dependence effect. Limitations of the dummy variable approaches are discussed, together with a recent approach presented by Lee (2014), who includes direct feedback on previously made decisions, with the aim of mitigating some of the problems inherent in older models, such as the initialisation and overlapping of state dependence and habit persistence effects. Comments on the virtues of this model, and a critique regarding inflexibility with respect to time interval irregularity between the decision events, conclude this section, which sets the stage for delineating the gap in theoretical understanding of the dynamic DCMs.

Section 2.2 reviews the fundamental structure and equations of linear SSMs as a particularly convenient tool for analysing dynamic systems regularly used in control engineering. It provides the theoretical underpinning of a model structure capable of capturing the state dependence and heterogeneity effects defined above. Moreover, it identifies a link between the Markov property of the latent states vector and the stochastic characteristics of the two endogenous types of noise, namely the state and output noise. Furthermore, the concept of state estimation, using a recursive filtering algorithm, KF, is explained, together with details relating to recursive computation techniques. The intuition behind correcting the *a-priori* estimations of latent states and the magnitude of the Kalman gain, defined as the ratio of variances of the state and output noise, is given. Considerations regarding the optimality of the algorithm under certain assumptions follow. Two particular model structures, based on state space structural time series models, commonly used in econometrics, are put forward for extracting endogenous latent trends, namely a first-order model (Local Level Model (LLM)) and a second-order model (Local Linear Trend (LLT)), which are then compared and related to a classical smoothing algorithm – Exponentially Weighted Moving Average (EWMA). The KF is the central tool of the envisioned methodology and can be used to satisfy both research objectives.

Section 2.2.4 summarises the capabilities of KF when dealing with irregular and missing observations, i.e. providing a framework needed for the Research Objective 2, as well as ways to incorporate them into an operational model. In continuation, model-error compensation techniques, used when uncertainties regarding the correct model structure arise, are presented and commented upon.

Finally, Chapter 2 concludes with the formulation of the research questions derived from the research objectives, which, if analysed and answered, meet the set objectives of the research and close the identified gaps in the literature.

Chapter 3 describes the overall methodology selected to meet the research objectives. It starts with the research-philosophical considerations leading to the selected *post-positivistic* research paradigm. Next, the chapter turns to defining the necessary conditions for using KF as a data filtering, smoothing and prediction algorithm, and the concept of linearisation (inflation) of ordered discrete data, which allows its application. In 3.4, the tailoring of two general SSMs for trend estimation, with provisions for irregular sampling, is discussed. The operational algorithmic form of the models with the diffuse prior initialisation approach is explored, together with a definition of the likelihood function that needs to be maximised for model parameter identification. In order to demonstrate the viability of the selected modelling approach, an empirical validation of the first- and second-order linear SSMs is conducted, using a dataset capturing six years of Betfair UK horse-racing data. The dataset description includes the basic descriptive statistics of the invoked data and an explanation of the split into three data subsets for model parameter fitting, time series burn-in, and model quality evaluation.

The research design and an associated methodology for model parameter fitting and model quality evaluation are discussed next. An empirical illustration of the model leverages horse-betting data collected from UK racing tracks for the years 2007-2012 as a real-world decision-making setting. Testing the prediction power of the selected model structure, both in terms of the pseudo-R^2 metric and profit from a betting strategy, allows for an analysis that yields an answers to the research questions asked herein. Methods and criteria for model quality evaluation, both in-sample and out-of-sample, are deliberated next.

Empirical results in Chapter 4 present the results obtained from the parameter-fitting efforts and model evaluation based on the model setup and performance criteria both in- and out-of-sample. Results from univariate, bivariate, and trivariate models are used to analyse the statistical significance and economic merits of forecasting models. The results of the analysis are then used to answer all primary and secondary research questions.

Chapter 5 ties together all of the main findings from the conducted study to discuss the effects of persistence in preferences in a broader scientific context. Starting from the empirical results and the evidence collected during their analysis, sources and effects of the biases are put into the perspective of forecasting performance in the naturalistic decision-making environment used in this study. The discussion confirms some of the known effects in the selected setting and shows the way to mitigate the effects of the highlighted biases.

The conclusions (Chapter 6) reiterate the research in a condensed form whilst highlighting the theoretical and methodological contributions to knowledge. Internal and external reflection on the thesis chart the limitations of the study and indicate possibilities for future research paths that could address some of these limitations and allow for the generalisation of the results obtained herein. A research accomplishment summary recaps the major findings stemming from the results of this study and concludes the thesis.

Figure 1.4 shows the structure of the thesis following Dunleavy (2003)'s model that describes different breadth levels expected in every chapter.

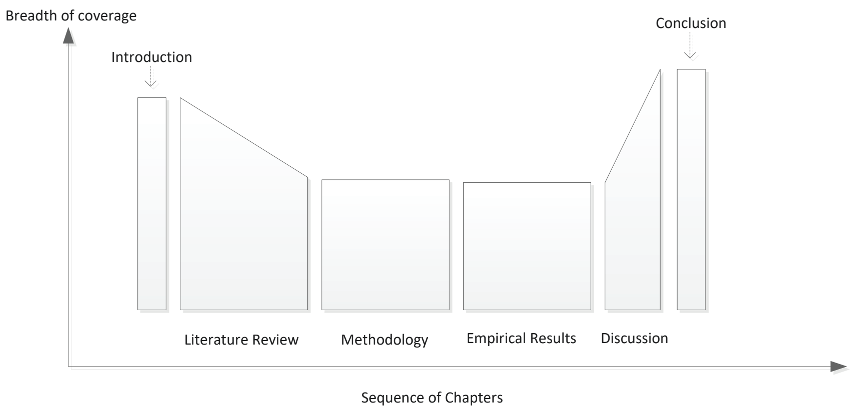

FIGURE 1.4: Opening out thesis structure

Chapter 2

Literature Review

Discrete choice models can be both static and dynamic. In section 2.1.1 the main ideas and conceptual basis behind classical DCMs *logit* and *probit*, applied on cross sectional (static) problem settings are discussed. This is used in section 2.1.2 to explore applications where a series of choices are made by a DM, who implicitly or explicitly modulates his behaviour over a sequence of choices or upon estimation of total future pay-off affected by decisions made now. In section 2.2, the theory of linear dynamic discrete SSMs together with some special forms used commonly in econometrics – structural time series models – are discussed. In section 2.2.1, the standard linear KF algorithm and its modification for models with irregular sampling are explained. Finally, in section 2.3, research questions designed to enlarge upon the research objectives and the identified gaps in the literature, as defined in Chapter 1, are formulated. Summary of the chapter concludes.

The chapter structure is selected with the aim to introduce and discuss the main theoretical and methodological components of the study. First, the historical development of the discrete decision making theory, together with the basic DCMs *logit* and *probit* is introduced. These models make up the basic building blocks, necessary for understanding of the dynamic DCMs (see 2.1.2).

2.1 The Discrete Choice Model and Extensions

The famous management science academic Peter Drucker (1954) observed that the fundamental activity of management is decision making. Indeed, a manager, acting as an agent, regularly faces choices, or a series of choices over time, and exercises judgement by selecting one or multiple alternatives from a set of given options. Even though, theoretically, the agent might be facing an infinite number of alternatives, certainly the most relevant practical case is the one with a countable and finite number of alternatives (i.e. a discrete choice set). Furthermore, RUMs are based on the assumption that the decision maker acts rationally and causally, and identifies, assesses, and weighs decision factors which are unambiguously determining his choice. Unfortunately, in practical settings, not all decision factors can be identified, so that those elusive, latent, factors are considered to be unobservable to a scientist (observer) interested in formulating a mathematical model of decision making in the given setting (Train, 2009). From the scientist's point of view, the unobserved components are stochastic as they cannot be determined exactly. Hence, an approach based on probability theory and statistics is needed for any practical modelling of decision-making processes and the DM behaviour underpinning it.

2.1.1 Choice Models

Thurstone (1927), whilst trying to define a comparison model of the physical stimulus intensities and comparative qualitative judgements resulting from them, formulated the *law of comparative judgement*. This became the initial idea behind the DCMs. His model and its underlying assumptions yielded what is now known as *binary probit model* applied on resolution of ambiguity of the perception (i.e. perception variance) reported by the same observer exposed to repeated sequence of stimuli. He introduced the variable 'psychological state', which is taken implicitly into account by the decision maker, who would in turn choose the alternative with the highest value of the 'psychological state'. Obviously, this variable is very difficult to model through the first principles of physiology, neurology, sociology, and other behavioural sciences (Georgescu-Roegen, 1958) and should instead be modelled by a random variable.

The ambiguity, inherent in randomness, can be used to conveniently explain variability in responses, seen as unobserved and different realisations of the random variable proxying psychological state both between the individuals and when observing repeated decision outcomes from the same choice set due to time variability of the mentioned state. Thurstone (1927) assumed a simple additive utility model encompassing a deterministic and externally observable (i.e. known) variable V_{nj} and an unobservable random variable ϵ_{nj}. In other words, the utility U_{nj}, describing a decision event n where one amongst J alternatives is selected can be expressed as:

$$U_{nj} = V_{nj} + \epsilon_{n,j}, J \in [1, 2], \tag{2.1}$$

where V_{nj} is a function that relates the observed factors characterising the alternative and/or the decision maker to the decision maker's utility (termed 'representative utility') and ϵ_{nj} is a random variable, an 'error term', having some 'convenient' distributional properties. Since ϵ_{nj} depends implicitly on time (n), this may explain apparently inconsistent judgements made by the same decision maker facing repeated choices. This approach was termed RUM due to the stochastic (random) properties of the model.

Economic research swiftly embraced the theory of choice as a convenient simplification. This complements the theory of demand by offering a proxy of human cognition bound to biological and cultural desires of humans (Georgescu-Roegen, 1958). Marschak (1960) put forward an interpretation of physical stimulus intensities as random utility functions. Moreover, Holman and Marley (cited as an unpublished paper in Luce and Suppes (1965)), provided a decision making model derivation based on utility maximization, which reconciled Independence of Irrelevant Alternatives (IIA) axioms and RUMs for a specific (Gumbel) distribution of the error term variable in the additive utility model, paving the path for – arguably the most widespread discrete choice model – the *logit* model (Luce, 2005). Finally, in his seminal work, McFadden (1974) proved that the *logit* functional form for the choice probabilities inevitably yields that the unobserved error term has Gumbel (extreme value) PDF and is consistent with stochastic utility maximisation principle (Maddala, 1983; Train, 2009). Mathematical convenience of the *logit* models can be easily understood through derivation of probabilities of particular outcomes, as explained below.

In order to make a probabilistic statement regarding the decision maker's choice in a decision event n, the probability that the alternative i is selected over all other alternatives from the choice set $i \in (1, \ldots, J)$ has to be evaluated:

$$
\begin{aligned}
P_{ni} &= \Pr(U_{ni} > U_{nj}, \forall j \neq i) \\
&= \Pr(V_{ni} + \epsilon_{ni} > V_{nj} + \epsilon_{nj}, \forall j \neq i) \\
&= \Pr(\epsilon_{ni} - \epsilon_{nj} > V_{nj} - V_{ni}, \forall j \neq i),
\end{aligned}
\tag{2.2}
$$

using the same notation as in (2.1). This is equivalent to the probability that every pairwise difference of the disturbance terms $\epsilon_{ni} - \epsilon_{nj}$ is greater than the observed deterministic quantity $V_{nj} - V_{ni}$ and can be calculated as the expected value of the indicator function of the inequality expression in (2.2) over all possible values of the unobserved factors ϵ_{ni}. Indicator function, defined for a logical (binary) variable x, $\mathbb{1}_X : X \mapsto 0, 1$ as

$$
\mathbb{1}(x) = \begin{cases} 1 & \text{if } x \text{ is true} \\ 0 & \text{otherwise,} \end{cases}
\tag{2.3}
$$

allows simplification of (2.2) to

$$
\begin{aligned}
P_{ni} &= \Pr(\epsilon_{ni} - \epsilon_{nj} > V_{nj} - V_{ni}, \forall j \neq i) \\
&= \int_{\epsilon} \mathbb{1}(\epsilon_{ni} - \epsilon_{nj} > V_{nj} - V_{ni}, \forall j \neq i)\, f(\epsilon) \mathrm{d}\epsilon_n.
\end{aligned}
\tag{2.4}
$$

where $\epsilon = [\epsilon_{n1}, \epsilon_{n2}, \ldots, \epsilon_{nJ}]'$ is a stacked vector of all random disturbances and $f(\varepsilon)$ is the associated joint PDF. The multidimensional integral in (2.4) can be calculated analytically only for certain forms of $f(\varepsilon)$. Alternative specifications, e.g. multivariable normal (which defines the *probit* family of models) do not have a general closed form solution except for $J = 2$ alternatives. Distributional properties characteristic of a particular member of the exponential family of densities for each unobserved disturbance, namely type I extreme value (Gumbel), with the PDF in the form are as follows:

$$f(\epsilon_{n,j}) = \exp(-\epsilon_{n,j}) \exp(-\exp(-\epsilon_{n,j})). \tag{2.5}$$

and the corresponding Cumulative Density Function (CDF)

$$F(\epsilon_{n,j}) = \exp(-\exp(-\epsilon_{n,j})), \tag{2.6}$$

allow derivation of choice probabilities in closed form.

This PDF has two important characteristics: a constant variance ($\pi^2/6$) and a non-zero mean, which may appear as a nuisance because of the complete lack of flexibility possibly needed to model different absolute values of the utility. Fortunately, since only the difference in the utilities matter, the absolute scale of a utility is irrelevant. Moreover, the first property secures an implicit normalisation of the utility, i.e. the normalisation follows automatically from the assumptions regarding the underlying distribution. Indeed, since all disturbances appear pairwise in direct comparison to each other, i.e. as a difference, the absolute values of the utilities are irrelevant. It can be shown (Krishnamoorthy, 2016), that the difference between two Gumbel distributed random variables follows a logistic distribution. In other words, random variable $\epsilon^*_{n,j,i} = \epsilon_{n,j} - \epsilon_{n,i}, i \neq j$ has a logistic CDF

$$F(\epsilon^*_{n,j,i}) = \frac{\exp(\epsilon^*_{n,j,i})}{1 + \exp(\epsilon^*_{n,j,i})}. \tag{2.7}$$

Even though the choice of the Gumbel distribution for modelling the unobserved portion of the utility function may seem artificial, driven only by the numerical considerations, and that a choice of the normal distribution would be more logical, this is not necessarily the case. For example, the univariate normal and the logistic distributions are very close to each other in the wide range of probabilities (especially between 0.3 and 0.7). Furthermore, outside of that range, logistic distribution has slightly heavier tails, thus allowing more aberrant behaviour than the normal, i.e. it implies, conservatively, higher variability of choice probabilities. Hence, for practical purposes, the difference between the logistic and independent normal errors is negligible. On the other hand, when the correlation between the unobserved portion of utility of the alternatives is relevant from a statistical and/or domain specific standpoint, other assumptions regarding the distributional properties, (for example multivariate normal (*probit*) or Generalised Extreme Value (GEV)) are clearly preferable.

Evaluation of the integral in (2.4), yields a closed, readily interpretable, expression for a choice probability $P_{n,j}$ (Train, 2009; Pleskac, 2015):

$$P_{n,j} = \frac{\exp(V_{n,j})}{\sum_k \exp(V_{n,k})}. \tag{2.8}$$

Note that the functional form of the observable portion $V_{n,j}$ is not specified and it may also contain random variables. In that case, the probability $P_{n,j}$ becomes conditional on the realisations of the random variables and evaluation of the probability will involve multidimentional integration to evaluate the expected value of the sought probability. In addition, the observed portion of the utility function may be bound both to the characteristics of the choice and/or the preferences of the decision maker (Baltagi et al., 2016). For notational purposes, the selected choice, i.e. the result of the decision event i with respect to the alternative j is defined as follows:

$$d_{ij} = \begin{cases} 1 & \text{if the alternative j is selected in a decision event i} \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

In summary, *logit* is a very widespread and capable model, which can model a large variety of decision making problem settings. Building on the linear functional form of the observable utility function, which describes the characteristics of the alternatives (denominated CL by McFadden (1974)), a plethora of *logit*-based human decision-making models have been put forward in domains of transportation/carpooling (Neoh et al., 2018), economics (Martinelli et al., 2018), tax policy (Grigolon et al., 2018), credit rankings (Florez-Lopez and Ramon-Jeronimo, 2014), corporate insolvency (Khoja et al., 2016), insurance (Florez-Lopez, 2007), marketing (Elshiewy et al., 2017), financial education (Becchetti et al., 2018), gambling studies (Gainsbury et al., 2016), and others (Athey and Imbens, 2007; Hensher and Johnson, 2018). *Logit*-based models are appropriate when systematic taste variation (related to observed characteristics of the alternatives and/or decision maker) as opposed to random (uninformative) taste variation (differences in tastes that appear random and cannot be put in a functional relation to observed characteristics) are of interest. On the other hand, in cases when IIA does not hold, and more flexible forms of substitution between alternatives are inherent in the problem setting, alternative models may have to be invoked. Furthermore, if the effects of correlation in time are negligible, i.e. if disturbances as proxies for unobserved decision factors have constant statistical characteristics over time in repeated choice situations, then *logit* can capture the dynamics of repeated choice. In all other cases, where disturbances are correlated over time, the *logit* model structure, in the presented form, cannot be applied without risking significant bias. Consequently, an effective approach for modelling of dynamic choices has to be put forward (Hsiao, 2014).

All introduced rationales and models until now aim to explain the static decision behaviour of a single DM participating in a decision event. The next section discusses state-of-the-art models of evolution of preferences for alternatives over time, i.e. dynamic DCM models. Based on the critique of the properties of the available models, two gaps in the literature are identified. The gaps encompass lack of support for changing choice sets in repeated choices and non-uniform time distance between the successive availability of different alternatives. Fundamental characteristic of the studied dynamic models is the lagged information on the preferences revealed in the past.

2.1.2 Dynamic Discrete Choice Models

DCM discussed until now, certainly play a fundamental role in many sciences interested in the description of both aggregate and disaggregate decision making processes. However, especially in economics and marketing-driven studies of markets, a case for rigid structures and consistent behaviour of market participants can hardly be made (Hensher and Johnson, 2018). For example, demand preferences are in constant flux (e.g. shifting market conditions: appearance of new products and product substitutes, evolution of market participants, calendar effects, etc.), which indicates a need for an approach which takes the temporal dimension of decision making into account (Lachaab et al., 2006; Keane et al., 2011).

Economic decision making research was traditionally founded on the assumption that every discrete decision is based on a set of preferences for the given opportunity, evaluated using stochastic complete information. Considering, for example, a rational decision maker, aiming at utility maximisation resulting from his choice, a researcher could build a number of choices derived from the information set available. These might include product attributes, prices, and promotional activities (advertising, visual merchandising, etc.). Assuming that the attributes (under complete stochastic information), prices, and promotional activities are constant over time, the individual utility maximization principle will inevitably yield the same choice at every decision making instance (e.g. purchase) (Adamowicz and Swait, 2012).

On the other hand, psychologists have defended the stand that the preferences are constructed on the spot by an 'adaptive decision maker', based on the task and context factors present during choice evaluation (Dai et al., 2010). Casual observation suggests that the decisions made by an 'adaptive decision maker' may be temporally variable even in *ceteris paribus* cases (Swait et al., 2004; Leong and Hensher, 2012), caused possibly by learning (temporal preference evolution), habit persistence (current preferences being affected by previous preferences), consumption inertia (use of heuristics), and state dependence (current preferences being affected by previous choices).

Moreover, recent research in behavioural marketing endorses a family of models of decision making lifecycles (Cherchi and Manca, 2011). These start from initial (uninformed) preferences of a DM, defined as a initially favoured selection of attributes and follows their evolution from one decision event to the next, as the DM learn and acquire taste over time.

Furthermore, evolution of consumer preferences (translation of the initial decision into inertia of repeated choices) are of large practical interest, especially for marketeers and economists. In its essence, any model of taste evolution has to put forward a mechanism capable of capturing different levels of correlation between successive choices (Cherchi and Manca, 2011). As a consequence of the correlation, any DCM, which does not account for these temporal (dynamic) factors, is likely to be biased. In order to account for the dynamic behaviour, a representative utility function, which, in every time slice (i.e. between the decision events), depends on observed variables from previous periods and, possibly, characterize how will current decisions affect future choices and outcomes (resulting cumulative utilities). Models built around utility functions that depend both on forward and backward time horizons are called full dynamic models. Forward looking models (sometimes called DCDP in academic literature) postulate that the decision maker is solving a Bellman dynamic programming problem over the time period of interest. This is achieved by splitting the resulting cumulative utility (pay-off) into the component contemporaneous with the decision event and the future utility component, under the assumption that every future decision event instance will yield an optimal decision (Arcidiacono and Ellickson, 2011; Abbaszadeh, 2015). In order to formally define a DCDP model, both (1) an explicit expectation function for calculations of the present discounted value of lifetime utility or profits across all possible choices and (2) an optimal *ex-ante* decision rule for future decision events, have to be specified. In addition to the intrinsic complexity of such models and the computational burden for the optimal solution of the decision strategy (rule and pay-off evaluation), existence of multiple equilibria at decision event times can cause the indeterminacy of the solution for multiple-agents models (Aguirregabiria and Mira, 2010).

Full (forward) dynamic choice models are still relatively rare, in spite of numerous empirical problem settings in which total net future pay-off, derived from current decisions, plays a central role, caused predominantly by the computational burden and tractability of estimation of the parameters of the structural model. Several important studies of (1) occupational choices and job matching (Sullivan, 2010; Keane et al., 2011), (2) patent valuation (Pakes, 1986; Collan and Kyläheiko, 2013), (3) valuation of real options in mining business (Collan et al., 2016) (4) capital equipment replacement decision making (Rust, 1987; Schiraldi, 2011) and, (5) dynamic stochastic models of fertility and child mortality (timing and spacing of children) (Wolpin, 1984; Werding, 2014; Adda et al., 2017) demonstrated that, under certain restrictions, the full dynamic approach is mathematically tractable and economically feasible. However, due to the high modelling and computational effort ('the curse of dimensionality'), it is likely that fully dynamic discrete choice models will remain of interest only in cases that promise high net returns as a result of search for an optimal dynamic decision strategy and execution (Cantillo et al., 2007; Arcidiacono and Ellickson, 2011).

Backwards looking dynamic DCM are far more widespread. Ever since Guadagni and Little (1983) explored scanner data captured in supermarkets to model customer purchase dynamics and analyse effects of price and other marketing variables on repeated purchases, the application of choice models to panel type of data has been discussed in numerous studies by researchers from different social sciences (e.g. marketing, transportation research, agricultural, labour, and environmental economics (Keane, 2015)). Even before scanner data became a major source of empirical data, labour economist Heckman (1981), whilst studying employment dynamics, put forward a model of repeated choice. His model accounted for: (1) permanent unobserved heterogeneity in preferences (taste variation between decision makers) (2) state dependence (current preferences being affected by previous choices), (3) initial conditions (*a-priori* information about preferences before the observation period available to the researcher), and (4) serial correlation in idiosyncratic taste shocks (arising either from time variant unmodelled attributes of choice or from genuine random choice behaviour). He combined these four sources of dynamics in a 'canonical' additive model for the utility that a DM i receives from the choice j, at the choice event time t. This model is linear in known and exogenous (i.e. independent of the unobserved component of the utility choice attributes $X_{i,j,t}$):

$$U_{ijt} = \alpha_{i,j} + X_{i,j,t}\beta + \gamma d_{i,j,t-1} + \varepsilon_{i,j,t}; \qquad \varepsilon_{i,j,t} = \rho\varepsilon_{i,j,t-1} + \eta_{i,j,t} \tag{2.10a}$$

$$d_{i,j,t-1} = \begin{cases} 1 & \text{if} \quad U_{i,j,t-1} > U_{i,k,t-1} \quad \forall k \neq j \\ 0 & \text{else} \end{cases} \tag{2.10b}$$

The first term $\alpha_{i,j}$ in (2.10a) describes DM intrinsic unobservable time invariant preference for the choice j; β is a standard linear weighting of the attributes $X_{i,j,t}$ common across all DM. Utility state dependence is approximated through a lagged choice variable $d_{i,j,t-1}$. In its simplest form, it is only a dummy variable indicating whether the particular alternative has been selected during the last decision event (2.10b), named by Heckman (1981) 'structural' state dependence. The parameter γ reflects the effects of the selection made in the decision event at time $t-1$ on the utility function constructed for the event at time t. The literature also describes other variants of the state dependence, based either on classical weighted smoothing of the discrete time series of choices made (for example 'brand loyalty' by (Guadagni and Little, 1983; Vulcano et al., 2012)) or on a more sophisticated discrete or continuous state space model (Erdem and Keane, 1996; Lee, 2014). Defined in such way, state dependence is a proxy of combined psychological or microeconomic effects such as habit persistence, learning, variety seeking behaviour, switching barriers, growth of inventories, etc. The term $\varepsilon_{i,j,t}$ models time evolution of preferences through decision-maker-specific taste shocks (i.e. individual effects) following an autoregressive process (2.10a) with the fundamental disturbance $\eta_{i,j,t}$. Interpretation of the idiosyncratic taste shock $\varepsilon_{i,j,t}$ is dependent on the general modelling philosophy. In case of economic RUM theory, the taste shock models the time varying development preference part, as opposed to time invariant intrinsic preference portion $\alpha_{i,j}$ of a DM's heterogeneous and unobserved utility (Manzini and Mariotti, 2014; Fudenberg et al., 2015). On the other hand, in psychology-based models of choice, $\eta_{i,j,t}$ are genuinely random (erratic) portions of choice behaviour. For both models, a DM always maximizes his (externally unobservable) utility $U_{i,j,t}$ at time t (2.10b).

In spite of the relatively straight forward interpretation of the elements from (2.10a), lagged choices can generate a nuisance 'spurious state dependence' effect in models with serial correlation of the $\varepsilon_{i,j,t}$ shocks. This is caused by residual informational content in lagged choices (which can be used to predict the shocks), observed also in the econometric mover-stayer models (Singer and Spilerman, 1976; Cipollini et al., 2012; Shen and Cook, 2014). The same informational content can lead statistical tests to erroneously indicate the significance of the state dependence (Dubé et al., 2010) even when only serial correlation is present. In other words, serial correlation models explain persistence in individual unobserved differences between the decision makers, which (1) made an externally unexplainable preponderance in previous decision events and (2) do not affect the observed utility function of a decision maker. Obviously, the problem of distinguishing the true state dependence from serial correlation is difficult due to the unknown specification and theoretically infinite path dependence of the disturbance structure, and the fact that the conditional probability of a choice in not equal to the marginal probability, i.e. $\Pr(d_{i,j,t}|d_{i,j,t-s}, X_{i,j,t}) \neq \Pr(d_{i,j,t}|X_{i,j,t})$. Even though spurious state dependence was studied extensively in labour economics (Bell and Blanchflower, 2011; Mosthaf et al., 2014), lack of robustness against the disturbance functional form misspecification remains both a theoretical and a practical problem.

Another non-trivial problem bound to any model that contains lagged dependent variables is the problem of initial conditions, i.e. model fit reliance on dependent variable values $d_{i,j,0}, d_{i,j,-1}, \ldots$ realised before the beginning of the observation period $t = 1$. Lack of knowledge regarding the previous decisions made at $t \leq 0$ may lead to an inconsistent and biased model (Hsiao, 2014). Even though the problem of initial conditions has been investigated extensively in the past, for non-linear models one of the following, rather strong assumptions, is typically made; either (1) the initial conditions are assumed to be truly exogenous and can be treated similarly to a deterministic variable $X_{i,j,0}$, or (2) the initial conditions are random and the conditional distribution is in the steady state (in equilibrium). Neither of the approaches is really satisfactory. Both assumptions yield significantly biased estimators and are mathematically challenging with respect to derivation of the conditional likelihood function in the same time (Wooldridge, 2005;

Hsiao, 2014). Even though an approximate solution proposed by Heckman (1981) yields an effective bias correction (Miranda et al., 2007; Akay, 2012; Stegmueller, 2013), it is somewhat more complex to implement. In any case, any dynamic method proposed will have to put forward a method to address the initialisation and treatment of conditional probabilities in the model.

A recent PhD thesis by Lee (2014) endorsed a substantial modification of the model (2.10a), which starts from a discrete state Markov process towards more flexible (latent) continuous underlying utility state $\mu_{i,j,t}$. The expression for the utility function $U_{i,j,t}$ is additive, encompassing two latent variables: the underlying utility state $\mu_{i,j,t}$ and the serially correlated error term $w_{i,j,t} = \lambda_i w_{i,j,t-1} + \eta_{i,j,t}; \eta_{i,j,t} \sim \mathrm{NID}(0, \sigma_\eta^2)$ with the fundamental shock variable $\eta_{i,j,t}$:

$$U_{i,j,t} = \mu_{i,j,t} + w_{i,j,t} \tag{2.11a}$$

$$\mu_{i,j,t} = \phi_i \mu_{i,j,t-1} + \alpha_{i,j} + X_{i,j,t}\beta + (\lambda_i + \phi_i) w_{i,j,t-1} + \xi_{i,j,t}. \tag{2.11b}$$

Here, the dynamics is defined in terms of development of $\mu_{i,j,t}$, α_i (the intrinsic preference proxy of the decision maker i for the choice j) and the fixed utility weightings β for the (exogenous) attributes of the alternatives. Since $U_{i,j,t}$ is independent of its previous values (i.e. $U_{i,j,t-1}$), the current choice $d_{i,j,t}$ is independent of previous choices given the value of the current underlying state $\mu_{i,j,t}$. Parameters ϕ_i and λ_i are parameters of an equivalent ARMA(1,1) models of the latent state (habit persistence) with the input noise $w_{i,j,t}$.

For clarity, several comments regarding the model structure are in order: (1) Gumbel disturbance $\xi_{i,j,t}$ does not affect the utility function directly, but instead the latent state $\mu_{i,j,t}$, (2) the dynamic portion of the model (2.11b), together with the serially correlated error term builds a regression model with ARMA$(1,1)$ errors, and (3) in spite of the name of the model with terms 'state space' and 'habit' stressed, state dependence is not accounted for, since there is no direct feedback from the actually made decision $d_{i,j,t}$ to the utility function model.

Furthermore, Lee (2014) commented on insights regarding the decision making behaviour in the example of repeated purchases of fast-moving consumer goods in a major grocery store, captured as cashier scanner data. This demonstrated some improvement of the model fit compared to classical models with the lagged dummy variable capturing previous choices measured. Comparison of the fit criteria was derived from Bayesian Information Criteria (BIC). Somewhat surprisingly, his model is not affected by initiation problems and the autoregressive nature of the state space utility model provides better control of prior features and other shocks, thus allowing differentiation between habit and variety seeking behaviour. However, the modelling approach has some significant limitations; first, lack of the feedback on the actually made decision makes its usage in forecasting and out-of-sample studies (e.g. marketing applications aimed at estimating of price and promotion effects, and analyses across markets for hedonistic and utilitarian goods), problematic at best. In addition, the model is an event-dependent discrete time model (i.e. it does not take elapsed time between purchases into account), which could affect both habit and variety seeking. Furthermore, usage of Bayesian fitting in panel model setting requires long sequences of decision events on a single decision maker (typically more than 30), something which is available to researchers practically for utilitarian fast moving everyday goods only.

Addition of the revealed *ordered* preference information from previous decision events allows a generalisation of the proposed models. The next section discusses the informational gain given through inclusion of the ordered information in the model.

2.1.3 Dynamic Ordered Choice Models

All models discussed until now use binary or multinomial choice models, i.e. models which captured the decision made from a discrete choice set without specifying the order of preference for non-selected alternatives. In cases when the decision maker reveals this order the additional information may be taken into account to model dynamic changes of the underlying utility functions even in classical 'one-out-of-many' multinomial problem setting. Examples of ordered classification of choices include bond ratings, polls on political issues, product preferences, and happiness/health condition surveys (Boes and Winkelmann, 2006; Greene et al., 2010; MacKerron, 2012), to mention but a few.

Ordered choice dynamic effects in models have been studied in two settings (1) time series setting where behaviour of only one decision maker over a long time period is studied and (2) panel data setting, where the decision-making behaviour of multiple decision makers is addressed. The remainder of the section synthesises the taxonomy and 'show-case' examples of ordered choice dynamic effects models given in Greene et al. (2010).

Time series settings have been predominantly used for modelling tasks where behaviour of an 'abstract' decision maker – such as 'Nature' or 'Stock Market' is studied. For example, decisions made by 'stock market' determine stock prices movements, effects of monetary policy on unemployment rates and interest rate changes, foreign exchange rates, etc. Tsay (2005) describes a model with a latent continuous time random variable $y^*_{i,t}$ underlying an asset i at time t

$$y^*_{i,t} = \beta^{'}\mathbf{x_{i,t}} + \varepsilon_{i,t} \tag{2.12}$$

where $x_{i,t}$ are exogenous explanatory variables and $\varepsilon_{i,t}$ is heteroscedeastic disturbance with $\mathrm{E}(\varepsilon_{i,t}|\mathbf{x_{i,t}}, \mathbf{w_{i,t}}) = 0$ and $\mathrm{Var}(\varepsilon_{i,t}|\mathbf{x_{i,t}}, \mathbf{w_{i,t}}) = \sigma^2(\mathbf{w_{i,t}})$, where $\mathbf{w_{i,t}}$ is a vector of explanatory variables which includes the time between t and $t-1$ and 'some conditional heteroscedeastic variables' (Hensher et al., 2005). Under the assumption that the observed asset price change is discretised into a fixed set of intervals, an observable variable $y_{i,t}$ reflecting the associated return of the asset, has an ordered *probit* model structure

$$y_{i,t} = s_j \quad \text{if} \quad \alpha_{j-1} < y^*_{i,t} \leq \alpha_j, \quad j = 1, \dots, J, \tag{2.13}$$

with categorisation in J discrete groups and natural ordering with boundaries α_i. In the show-case application of the model, a panel of more than 100 New York Stock Exchange stock prices was initially presented by Hausman et al. (1992) and consolidated by Tsay (2005). A similar approach using the Australian Securities Exchange data was recently put forward by Yang and Parwada (2012). The received models allowed treatment of the irregular (and random) timing of transactions whilst accounting for the correlations between price changes and other exogenous macroeconomic and trading variables (e.g. trading volume). Model structure selection was very application oriented. Indeed, in the time of the study, the price changes were expressed in ticks, i.e. fractions of a dollar. Consequently, a discrete output variable was a natural choice and the heteroscedasticity of the disturbance dictated the usage of a *probit* model due to mathematical problems bound to the Gumbel distribution in the conditional probability concepts (Campbell et al., 1997; Gourieroux and Jasiak, 2018). However, in spite of the relatively good results on the studied data set, two modelling issues remained unsolved - selection of the the specification of the explanatory and conditional distribution density regressors and potential non-stationarity of the choice set.

Attempts to use an alternative specification of the error term (simple autoregressive structure $\varepsilon_{i,t} = \rho\varepsilon_{i,t-1} + u_t$, with normal fundamental disturbance u_t and serial autocorrelation coefficient ρ), complicates the parameter estimation considerably. This arises since the serial correlation in the model means that the time series is a path-dependent sequence of duration N, which, as a consequence, requires evaluation of a N-variate normal integral for parameter estimation (Eichengreen et al., 1985; Park, 2011). The models used the Bank of England BR adjustments policies data over a period of over six years (during the period of the inter war gold standard) with BR increments as a latent dependent variable and with absolute values of BR in the ordered *probit* evaluation. Their results revealed the violations of the stated Bank of England policies in a *ex-post* analysis and provided the benchmark for comparative research on central bank policies. However, this did not resolve the modelling issues inherent in the models described before – namely, the numerical complexity of model fitting caused by path dependence (due to the serial correlation) and lack of provisions for irregular sampling and out-of-sample forecasting.

A state dependency model, similar to (2.10b) without the disturbance correlation or heteroscedeasticity, was designed to study the perceived severity of migraine headaches of a single patient captured several times per day over nine months (Czado et al., 2005). The model implements a feedback on previous choice (state dependence), ordered by severity on a six-value scale in an autoregressive structure

$$y_t^* = \beta^{'}\mathbf{x_t} + \gamma y_{t-1}^* + \varepsilon_{i,t}, \tag{2.14}$$

where y_t^* is a continuous underlying state reflecting to migraine severity, recorded as a discrete value $y_t \in 0, 1, \ldots, 5$. Observed parameters, linearly weighted include the attributes such as weather conditions and day of the week. The parameter γ is the coefficient of autoregression (persistence of migraine severity compared to previous day) and $\varepsilon_{i,t}$ is the proxy of unobserved effects.

As expected, the state feedback component complicates the estimation and even though a fitting approach was proposed, the model suffers from the same problems as (2.10a). These problems include initialisation issues and lack of provisions for irregular sampling and changing choice set (Müller and Czado, 2005; Mizen and Tsoukas, 2012).

This thesis focuses on preference dynamics in choice models described by a time varying random utility model, where a choice decision has to be made through comparison of utilities U_{ijn} of j alternatives from the choice set J_k, all indexed on decision events n:

$$U_{ijn} = \beta x_{in} + \epsilon_{jn} + y_{ijn}, j = 1, \ldots, J_n; i = 1, \ldots, N_{Rmax}; n = 1, \ldots, N_{Emax} \tag{2.15}$$

Here, N_{Rmax} is the total number of alternatives in all decision events and N_{Emax} is the total number of decision events. The equation has the same basic structure as (2.1), with the specification of the representative utility as the linear combination of exogenous choice attributes and the split between two uncorrelated endogenous random variables ϵ_{nj} and y_{ijn} that can have different statistical properties, which can be understood as a decomposition of the latent error term in (2.1) in two independent random components. Idea behind this decomposition is that one of the endogenous random variables (y_{ijn}) can be an output of a random process that takes ordered information from previous decision events and constructs a random trend. The expected value of the trend at the time of the event n is then used as dynamic characteristic of an alternative and taken into account in the CL model (2.8). A conceptual representation of the introduced split of the unobserved utility is given in Figure 2.1. It should be noted that, even though there is no explicit time variable in (2.15), the dynamics of decision making is captured through the decision event index n. Furthermore, the decision set size J_n varies from one decision event n to another $n+1$. All possible alternatives are captured in a discrete pool and indexed by i. This is a decision event setup, in which the alternatives available in a decision event are *a-priori* known.

DCM with underlying unobservable, latent, states, which track dynamics of change and allow for estimation of choice preferences, temporal, and memory effects are not

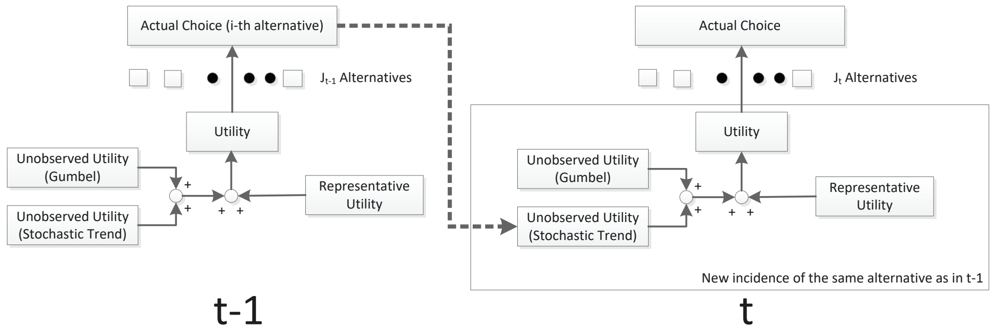

FIGURE 2.1: Error Components Decomposition

a new idea (as mentioned in Chapter 1). However, models with hidden states offer satisfactory results only in combination with an algorithm for estimation of the latent states following revelation of the actual choices. Unfortunately, the KF, as the optimal linear filter for state estimation, cannot be directly implemented in a decision making context when states represent a portion of a utility function. This arises because of the inherent nonlinearity of *logit* and *probit* functions used. Indeed, KF was used for adaptive adjustment of parameters of nonlinear models under the assumption that their evolution follows linear dynamic laws. A recent dissertation by Guhl (2014) discussed KF for demand and price estimation where the parameters follow a random walk. In spite of the sound theoretical treatise presented, the data sets used are seriously limited. In addition, there were no exogenous covariates available and the work only involved fixed choice sets. Moreover, only balanced and time equidistant panels were considered.

In an earlier publication, Edelman (2007b) applied KF on Hong Kong horse-racing panel type of data in which regression parameters of CL evolve race by race and are corrected after every race. Classical KF that tracks linear regression parameters of the *logit* function did not deliver good results. This may have arisen because the parameters of the KF were guessed and not estimated. A modified version of *logit* model, with KF using Radial Basis Functions instead of linear regression delivered viable Return on Investment (ROI) and pseudo-R^2 results. However, they mostly arose under selection of longshot horses implying highly risky betting patterns. In conclusion, a KF based model of underlying utility presented in this thesis is a novel approach and will be developed in section 3.2.

The next section discusses the fundamentals of SSM models and the adequacy of their usage in DCM, where the states underlying the utility function are modelled and where expectations about attributes are incorporated. In addition to the introduction on the intuition behind linear SSMs, the central topic of this section is the application of the KF for state estimation purposes, together with the modifications of the algorithms for non-uniform times between state updates and the model-error correction, implemented to mitigate effects of filter divergence.

2.2 State Space Models

Linear state space modelling approach is arguably one of the most important tools in a control engineer's toolbox – it is used for control of different systems, such as rocket propulsion vectoring, missile guidance, power plants, vehicle stability control, robotics, tracking applications (Kovacević and Djurović, 2008), etc. In spite of several simplifying assumptions that are practically never met, SSM are also used extensively in statistics, finance, and econometrics (Harvey, 1990; Durbin and Koopman, 2012; Schlittgen, 2015).

Fundamentally, every linear deterministic (stochastic) dynamic system described by a set of ordinary (stochastic) differential equations can be transformed into a SSM. State space models are extremely flexible and can be used to easily describe very general model settings with multiple inputs and/or multiple outputs, called multivariable models. Whilst multiple inputs alone do not principally cause any difficulties even in classical Autoregressive Moving Average (ARMA) settings, SSM are particularly convenient for definition of multivariate time series, i.e. systems with multiple cross-coupled inputs and outputs (Shumway and Stoffer, 2006). The model described below outlines the state space equations for the general multivariate case, which are than adjusted to a given problem setting.

The name 'state space' models reveals that it consists of states, which, contrary to the outputs y of the models, are hidden and cannot be observed directly. In physical systems the states model elements of the system having inertia (i.e. can store energy) or some kind of memory of the past which influences the present.

The relationship between unobserved states α_t and the output y_t is given through the *measurement equation*, yielding m observed output values stacked in a vector

$$y_t = Z_t\alpha_t + d_t + \varepsilon_t, t = 1, \ldots, T \tag{2.16}$$

with Z_t as an sensor matrix (which generates linear combination of the states to form the output) and d_t is a vector (output offset) and ε_t is a noise vector of normally distributed and serially uncorrelated disturbances with zero mean ($\mathrm{E}(\varepsilon_t) = 0$) and known variance ($\mathrm{Var}(\varepsilon_t) = H_t$). Furthermore, dynamics of the model is captured in the state equation which has a first order Markov property, i.e. states depend only on the immediately preceding state and have no memory of other past states (Anderson and Moore, 1979). The *transition equation*

$$\alpha_t = T_t\alpha_{t-1} + c_t + \eta_t, \mathrm{E}(\eta_t) = 0, \mathrm{Var}(\eta_t) = Q_t, t = 1, \ldots, T, \tag{2.17}$$

with T_t is a state transition matrix, c_t is a drift vector, and η_t is a state disturbance variable (called *process noise* in the further text). It should be noted that the vectors

d_t and c_t model additive effects caused by exogenous variables, such as economic policy changes or seasonal effects. Disturbances ε_t and η_t represent measuring uncertainties (*measurement noise*) and unmodelled dynamics (*process noise*). In the simplest form, the disturbances are neither cross correlated with each other nor correlated with the initial state vector α_0 which has a mean of a_0 and a covariance matrix P_0. The general term for the matrices T_t, Z_t, H_t, and Q_t is *system matrices* and they are assumed to be deterministic (i.e. non-stochastic) but possibly time-variant. This notation is kept throughout the thesis.

Tightly connected to the time variance of the system matrices is the notion of *stationarity* of the stochastic processes described by the model; a stochastic process $X(t)$ is said to be stationary if its properties do not change over time, i.e. if any joint PDF, constructed on a finite number of discrete time points $f(X(t_1), X(t_2), \dots, X(t_n))$ is invariant to time shifts $f(X(t_1), X(t_2), \dots, X(t_n)) = f(X(t_1 + \tau), X(t_2 + \tau), \dots, X(t_n + \tau)), \forall t_i, \tau, n$ of the realisation time points. *Weak stationarity*, i.e. the stationarity up to the second moment, is given for Gaussian processes when the autocorrelation function $R_{xx}(t_1, t_2)$ is dependent only on the time difference $\tau = t_2 - t_1$, $R_{xx}(t_1, t_2) = R_{xx}(\tau)$ and not on the actual times t_1 and t_2. Obviously, if the system matrices are time-variant, the model is non-stationary. Unfortunately, the opposite does not hold – even with time-invariant system matrices, a model is not necessarily stationary. Since the econometric models are seldom stationary, the ability of the SSM to handle non-stationarity is of crucial importance.

A fundamental characteristic of the model defined by (2.16) and (2.17) is that it describes a *linear* model, since all states and all variables are linear combinations of either previous or current states. In other words, the linearity allows that the output y_t at any time can be expressed by a linear combination of the initial state α_0 and the actual realisations of the disturbances ε_t and η_t. Finally, for the model (2.16) and (2.17) the whole set of mathematical tools inherited from control theory used to characterise the dynamic behaviour, such as stability, controllability, and observability criteria are readily available and well understood. For example, the linearity allows conclusions about the stationarity based on the stability criteria for systems with time-invariant system matrices and the observability criteria give indication regarding the identifiability of the model parameters.

Linear SSMs defined by (2.16) and (2.17) are very versatile, since (1) many nonlinear models can be approximated through linearisation and (2) even nonlinear effects of exogenous variables can be successfully modelled if the underlying functional form is known.

In addition, non-stationary time series models can easily be represented by SSMs, contrary to classical time series ARMA models. However, it should be noted that the models are capable of tracking dynamics of the states α_t (and, subsequently, y_t) only if all (eventually non-stationary) system matrices and exogenous inputs are predetermined (known), since only the states α_t are endogenous to the model (Harvey, 1990; Durbin and Koopman, 2012).

Definition of a SSM is achieved by construction (selection) of hidden state variables, whose dynamics is of interest. Moreover, the states may or may not have a physical (e.g. temperature, voltage) or substantive (e.g. trend, seasonality) interpretation, i.e. they can also be abstract constructs selected using some auxiliary criteria, such as numerical stability or value limitations. A SSM designed for a given problem is, as a rule, not unique, since for the same set of equations an infinite number of different models can be built through, for example, linear transformations. Hence, the model quality has to be evaluated based on general criteria, such as parsimony, on one side, and inclusion of all relevant information on the other. Models with the minimum system dynamics order (i.e. dimension of the state vector α_t), which can describe the dynamics satisfactory, have, in general, also the minimal number of parameter needed for system definition. In the most common practical case, the system matrices T_t, Z_t, H_t, and Q_t are unknown and have to be estimated from the available data set. There are two main estimation methods available, derived from Maximum Likelihood (ML) and from Expectation Maximisation (EM) approaches (Durbin and Koopman, 2012). However, the presence of two sources of disturbances (η_t and ε_t) complicates the estimation task considerably compared to 'classical' dynamic models such as, for example, ARMA models. In addition, two fundamentally different groups of unknown parameters can be defined - parameters describing the stochastic properties of the model - the system matrices T_t, Z_t, H_t, and Q_t concatenated into a hyperparameter vector $\boldsymbol{\theta_{KF}}$ and the parameters describing the time variant drift vectors c_t and d_t, which deterministically affect the expected values of the observations and the states.

It is important to note that the state space model described by (2.16) and (2.17) is not the most general linear model. It is a discrete time model, i.e. it models system behaviour only at finite number of discrete points in time $t_i; i = 1, \ldots, T, T \in \mathbb{N}$, defining $T - 1$ time slices in between them. For the sake of simplicity, in the further text, the considered point in time t is denoted as t_i for any $i \in 1, \ldots, T$, and the immediately preceding point in time $t - 1$ is denoted t_{i-1} unless otherwise specified. A more fundamental generalisation, which allows representation continuous time behaviour can readily be constructed through conversion of the system of difference equations into a system of differential equations. However, particular care has to be given to non trivial conversion of the discrete random processes ε_t and η_t into their continuous counterparts (Jazwinski, 1970; Simon, 2006; Kovacević and Djurović, 2008), which require rigorous usage of generalised functions treated in functional analysis (Arsenović et al., 2012).

In the course of the research presented in this thesis, an adjustment for continuous time behaviour is needed because of the potentially irregular nature of the studied time series (as explained in 2.3). Didactically, it is far easier to understand the discrete (c.f. continuous) time SSM and, hence, the following sections focus only on them. The proper treatment of the irregularly sampled systems, which can be reduced to discrete models, is put forward in 2.2.4.

Estimation of the underlying latent states is one of the central topics in control theory. In this study, it is used to estimate unobserved portion of the utility. The next section provides general infrastructure for state estimation based on the KF, as required by Research Objective 2.

2.2.1 State Estimation and the Kalman Filter

As explained at the beginning of 2.2, the states, inherent in a SSM, are hidden, i.e. they are not directly observable from outside. Moreover, the exact realisations of the random disturbance variables are also not known. However, for many purposes, such as design of advanced control algorithms and statistical signal processing, estimates of the states are very beneficial and the efforts to develop an algorithm for that purpose was more than justified. Building on original Gauss ideas for estimating planetary orbits through recursive least squares and Wiener-Kolmogorov theory of stationary filtering (Sorenson, 1970; Simon, 2006), Hungarian mathematician and engineer Rudolf Kalman (orig. Kálmán) endorsed a recursive algorithm for estimation of (dynamically changing) states at the same time as new values (measurements) of the observed variables become available. Under the standard assumptions of linearity and normality of the defined random variables, his algorithm yields the first two moments of the conditional PDF, both when having the knowledge of all realisations of $y_i, i = 1, \ldots, t-1$ (stacked in a vector $Y_{t-1} = [y_1, y_2, \ldots, y_{t-1}]'$) - called *a-priori* estimate, as follows:

$$\begin{aligned} \hat{\alpha}_{t|t-1} &= \mathrm{E}(\alpha_t | Y_{t-1}) \\ P_{t|t-1} &= \mathrm{E}[(\alpha_t - \hat{\alpha}_{t|t-1})(\alpha_t - \hat{\alpha}_{t|t-1})'] \end{aligned} \tag{2.18}$$

and when having the knowledge of all realisations of $y_i, i = 1, \ldots, t$ (stacked in a vector $Y_t = [y_1, y_2, \ldots, y_t]'$) - called *a-posteriori* estimate, as follows

$$\begin{aligned} \hat{\alpha}_{t|t} &= \mathrm{E}(\alpha_t | Y_t) \\ P_{t|t} &= \mathrm{E}[(\alpha_t - \hat{\alpha}_{t|t})(\alpha_t - \hat{\alpha}_{t|t})']. \end{aligned} \tag{2.19}$$

The notation $\hat{\alpha}_{t|t-1}$ represents the estimated value of α at time t based on values known until and including the time $t-1$ (*a-priori* estimate), i.e. the state estimate conditional on realisations of the variable y until and including the time $t-1$. Similarly, $\hat{\alpha}_t$ represents the estimated α at time t conditional on the realisations of the variable y until and including the time t (*a-posteriori* estimate). In the engineering literature, the typical notation for *a-priori* ($\hat{\alpha}_t^-$) and *a-posteriori* ($\hat{\alpha}_t^+$) state values are different because of their merits for analysis of causality and easier interpretation of the results in non-equidistant discrete or continuous time models. Same convention can be applied to the *a-posteriori* $P_{t|t}$ and *a-priori* $P_{t|t-1}$ covariance matrices of the state estimation error.

Furthermore, practical applications of the Kalman filter require that the initial (*unconditional*) values for the states $\alpha_{0|0} = \hat{\alpha}_0$, the state estimate error covariance $P_{0|0} = P_0$, and the system matrices are known. For an arbitrary point in time t, assuming that an estimate of the states at time $t-1$ and the matrices T_t, Z_t, H_t, and Q_t are known and that (2.16) and (2.17) hold, the optimal *a-priori* estimator is given by:

$$\hat{\alpha}_{t|t-1} = T_t \hat{\alpha}_{t-1} + c_t \tag{2.20}$$

and the associated covariance matrix of the *a-priori* estimation error is

$$P_{t|t-1} = T_t P_{t-1} T_t{}' + Q_t. \tag{2.21}$$

Equations (2.20) and (2.21) are together called *prediction equations*. The prediction equations are based only on the matrices and vectors describing the state evolution, i.e. T_t (the state transition matrix), c_t (the deterministic drift vector), and the state error covariance matrix $P_{t|t-1}$, i.e. they do not directly depend on y_t.

Immediately after the newest measurement y_t becomes available, an updated estimate of the states can be made, as follows:

$$\hat{\alpha}_t = \hat{\alpha}_{t|t-1} + P_{t|t-1} Z_t^{'} F_t^{-1} (y_t - Z_t{}' \hat{\alpha}_{t|t-1} - d_t) \tag{2.22a}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} Z_t^{'} F_t^{-1} Z_t P_{t|t-1} \tag{2.22b}$$

The variables $\hat{\alpha}_t$, $\hat{\alpha}_{t|t-1}$, $P_{t|t}$, $P_{t|t-1}$, d_t, and y_t have the same meaning as in equations (2.16) to (2.21). Prediction output error (*innovation*) $\nu_t = y_t - Z_t \hat{\alpha}_{t|t-1} - d_t$ is a measure of the Kalman filter performance and its convergence. Innovation properties are crucial for proper understanding of the KF algorithm. First, innovation shows how much information can be extracted from the new measurement beyond the conditional information set known before the new measurement has been made available. Second, statistical properties derived from the whiteness of the innovation sequence can be used for model parameter identification (for example in procedures based on ML) and for model validity testing. In other words, this means that for a model with Gaussian disturbances and initiation in the infinitely remote past, innovation time series is a zero mean white noise stochastic process and it can be calculated through a linear transformation of time series y_t (Anderson and Moore, 1979; Simon, 2006). A statistical variable representing the output error covariance matrix defined as $F_t = \mathrm{E}(\nu\nu')$ can be brought in connection with other SSM matrices, as follows:

$$F_t = Z_t' P_{t|t-1} Z_t + H_t. \tag{2.23}$$

The last equation plays a significant role in considerations regarding different practical implementations of the Kalman filter as it combines the statistical properties of both *process* and *measurement* noise.

Equations (2.22a) and (2.22b) are together called *corrector* (or *updating*) equations. The corrector equations use the known relation between the states and the output (i.e. linear combination of the states to form the output), d_t (output offset), and measurement noise variance (H_t) to back-calculate the optimal correction of the *a-priori* states.

As can be seen in (2.22a) and (2.22b), F_t matrix has to be inverted in every step. This can pose a significant problem for practical implementations of the Kalman Filter algorithm since ill-conditioning of the matrix and/or error accumulation can cause negative semi-definiteness of the matrix. In practical implementations, in cases when the inverse F_t^{-1} does not exist, it is possible to replace it with a Penrose (pseudo) inverse (Jazwinski, 1970; Kovacević and Djurović, 2008) or, alternatively, to use some of the numerically more robust algorithm formulations, such as the Square Root Filter (Anderson and Moore, 1979; Simon, 2006). As a practical guideline when using an off-the-shelf Kalman filter algorithm, it is possible to enforce positive definitness of F_t^{-1} by substituting $P_{t|t}$ by a Frobenius-norm-nearest symmetric positive semi-definite matrix (Higham, 1988, 2002).

A further important statistical variable $K_t = P_{t|t-1} Z_t' F_t^{-1}$ is called Kalman Gain, here given in Joseph‘s stabilised form (Simon, 2006). The Kalman gain plays a crucial role in understanding the intuition behind the algorithm. As can be seen from (2.22a), the predicted value of the states $\hat{\alpha}_{t|t-1}$ is corrected by the product of the Kalman Gain and the error between the measured and the predicted output (*innovation*), after the new measurement arrives. This resembles a simple proportional (P) control algorithm, known from introductory control engineering theory. Moreover, as it would be expected, after the new measurement was made available, the uncertainty of the states P_t will become ‘smaller’[1] then $P_{t|t-1}$ due to the semi positive-definiteness of the state covariance matrix. For univariate models the following interpretation can be given; for small values of $K_t \approx 0$, there will be no correction of the predicted states, i.e. $\hat{\alpha}_t \approx \hat{\alpha}_{t|t-1}$, which means that the information in the dynamic model is superior to the information contained in the new measurement and it contributes very little to the state correction and the state estimate remains approximately the same. On the other hand, for large values of K_t ($K_t \to \infty$), the correction will be predominantly determined by the new measurement (*innovation*) $\hat{\alpha}_t \approx K_t \nu_t$, which means that the dynamics model is inferior to the informational content

[1] In a sense of some matrix norm, e.g. Frobenius or Max norm.

contained in the new observation. Algebraically, the magnitude of the Kalman gain $K_t = P_t Z_t' Q_t^{-1}$ is directly proportional to the variance P_t and inversely proportional to the measurement variance Q_t. Indeed, K_t can be small for either small P_t, which means that the model is adequate, or when H_t is large – equivalent to noisy measurement conditions. In both cases, only marginal corrections of the state estimates, due to the new measurement will have to be made. On the other hand, large P_t indicates that the model is rather poor (for fixed H_t) and small H_t characterises high fidelity measurement with high informational content, i.e. high signal-to-noise (S/N) ratio (Kovacević and Djurović, 2008, p. 298). Equations (2.22a) and (2.22b) stress the inherently adaptive nature of the KF. After every time slice, the algorithm calculates the optimal correction of the state estimates based on the known (albeit possibly time variant) system matrices and the actual measurement. Consequently, it can be said that the KF 'learns' from the data and automatically adjusts the model states to allow optimal prediction of the states in the next time slice.

Note that, irrespectively of the statistical properties of the disturbances, the Kalman Filter is an optimal linear estimator algorithm and it yields the smallest conditional error covariance matrix (Kailath et al., 2000), which in turn implies Minimum Mean Square Error (MMSE). This intuition is very useful for interpretation of the model quality obtained through estimation and the usefulness of Kalman filtering for a given problem setting.

This section introduced the general equations of the KF algorithm, which can be applied to linear models of arbitrary order. The next two sections describe two parsimonious and, at the same time quite powerful, structural time series models (LLM and LLT), that can be effectively used to study trends of random variables acting as a proxy of revealed previous choices. Construction of the endogenous trend is a partial requirement of Research Objective 1.

2.2.2 Local Level Model

One of the simplest and yet extraordinary useful models in time series analysis is the LLM. It describes a univariate *random walk* evolutionary time series with a state (level) μ_t, which randomly moves up or down at every discrete time sampling point, without having a steady upward or downward tendency. In other words, at every discrete time point t, the underlying level (state) of the process is constructed from the previous value (hence the term *Level* in the model name) and is shifted by an additive disturbance (shock) equal to the realisation of η_t:

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \mathrm{NID}(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + \eta_t, & \eta_t &\sim \mathrm{NID}(0, \sigma_\eta^2) \end{aligned} \tag{2.24}$$

Random variables ε_t and η_t represent *measurement* and *process* noise as in (2.16) and (2.17). Both noise variables are uncorrelated normally distributed variables having zero mean with known and time invariant variances. Obviously, the disturbances have different effect on the observed variable; whilst ε_t has only a temporary effect, η_t has a cumulative (persistent) effect on the observations y_t, which makes it the simplest of all *structural models*[2]. They make up a class of models predominantly used in econometrics – as opposed to classical Box et al. (2015) models preferred by statisticians and telecommunication engineers (Stoica et al., 2005).

From the modelling point of view, appealing simplicity of the model is bound to the fact that only two parameters (σ_ε^2 and σ_η^2) fully characterise the model, under the assumption that the initial state $\mu_{1|0} \sim \mathcal{N}(m_1, P_1)$ is known. A much less appealing characteristic is the non-stationary character[3] of the model and the non-separability of ε_t realisations from μ_t, making parameter estimation from a set of given observations $y_1, \dots, y_n$ non-trivial due to ill-posed definition of likelihood function (Durbin and Koopman, 2012). Any practical usage of the model is contingent on the knowledge of the variances, so that any application of the algorithm will have to address their estimation from the available data. The concrete estimation procedure used in this study is described in detail in 3.3.3.

Arguably the most important application of the LLM is forecasting (Harvey, 1990). In case of the model (2.24) the future forecasts of the variable y_t for $t \geq T$ will be equal to the estimate of $\hat{\mu}_T$. Algebraically, the estimate of the mean of the underlying random walk process is a weighted average of the past observations of y_t, with the discounting factor dependent on the ratio of the disturbance variances $q = \sigma_\eta^2/\sigma_\varepsilon^2$. Weighted moving average, inherent in the LLM model, resembles the classical forecasting workhorse – EWMA, a procedure proposed in the 1950s. EWMA yields a one-step ahead forecast of $\hat{y}_{t+1}|y_t, y_{t-1}, y_{t-2}, \dots$ based on input values y_t with the forgetting factor λ (smoothing constant)[4]

$$\hat{y}_{t+1} = (1-\lambda)\sum_{j=1}^{n} \lambda^j y_{t-j}, 0 < \lambda < 1, \tag{2.25}$$

which, if put into recursive form for computational purposes, is equivalent to

$$\hat{y}_{t+1} = (1-\lambda)y_t + \lambda\hat{y}_t = \hat{y}_t + (1-\lambda)(y_t - \hat{y}_t), t = 2, \dots T. \tag{2.26}$$

[2]Univariate structural models are a particular class of econometric models, which are build as additive models of distinctive, explicit, unobservable components having direct economic interpretation, such as trend, seasonality, cyclic behaviour, etc.

[3]For LLM $T_t = 1$ which means that the eigenvalue of the transition matrix does not meet the stationarity criteria $|\operatorname{eig}(T_t)| < 1$ and, consequently, indicates the non-stationary character of the output.

[4]There is a closed non-linear relationship between λ and q, as presented in (Harvey, 1990, p.175).

Choice of the initial condition $\hat{y}_2 = y_1$ secures that the weights in (2.25) sum to unity, even for finite sums. Muth (1960) has shown that the recursive procedure (2.26) can be expanded into the form of (2.24) and that EWMA yields minimum Mean Square Error (MSE) forecasts, which is in alignment with the KF theory. Indeed, from the second part of equation (2.26), it can be seen that with the Kalman Gain $K_t = 1-\lambda$, EWMA has the same steady state solution as the KF. It should be noted though, that for short time series results from EWMA and Kalman filtering can differ considerably due to different initial conditions. Obviously, since many of the econometric models have short time series, KF will likely allow more sound statistical inference in such cases. This is not to say that the KF initialisation is trivial, since, for non-stationary models, the unconditional distribution of the state m_t cannot be inferred from the system matrices and has to be selected based on some *a-priori* consideration or applying exact or approximate diffuse initialisation procedure (Durbin and Koopman, 2012).

2.2.3 Local Linear Trend Model

A slightly more complex model of second order, which aims to take the underlying linear trend movements into account and use them for future forecasting, is a natural extension of the LLM. In addition to the previously introduced level state μ_t, the state β_t implements the random slope, and also follows a random walk process. As before, y_t is a univariate output of the SSM. The name local underlines the difference between the *global linear trend*, which is a deterministic function of time (fitted usually using Ordinary Least Squares (OLS)), and the random trend of the most recent slope (*local trend*), which may change direction at any time point.

$$\begin{aligned} y_t = \quad & \mu_t + \varepsilon_t & \varepsilon_t \sim \mathrm{NID}(0, \sigma_\varepsilon^2) \\ \mu_{t+1} = \quad & \mu_t + \beta_t + \eta_t & \eta_t \sim \mathrm{NID}(0, \sigma_\eta^2), \\ \beta_{t+1} = \quad & \beta_t + \zeta_t & \zeta_t \sim \mathrm{NID}(0, \sigma_\zeta^2), \end{aligned} \tag{2.27}$$

Flexibility provided by three independent Gaussian disturbances (ε_t, η_t, ζ_t) have been criticised by econometricians for the lack of smoothness and, in practice, the parameter σ_η^2 is usually set to zero. Similarly to the connection between the EWMA forecasting procedure and the LLM model, application of the KF with this particular structure yields steady state results close to classical Holt-Winters (Downing et al., 2011) local linear trend forecasting procedure (albeit without the seasonal component (Harvey, 1990, p.38)).

Contrary to classical filtering algorithms, the KF can be easily modified to incorporate support for irregular and missing observations of a random variable under study. The next section explains the algorithm modification as required to support the Research Objective 2.

2.2.4 Irregular and Missing Observations

Description of the KF algorithm given so far is based on the assumption that observations for discrete points in time $t \in 1, \ldots, T$ are available. However, for dynamic choice model problem setting, a method has to be put forward which would support irregular observations, i.e. for the case when the time intervals between arriving measurements are non-uniform, $\exists i \neq j : t_{i+1} - t_i \neq t_{j+1} - t_j$. For the particular case, when irregular observations have an common divisor in the form of underlying time interval δ_{min}, so that a uniform grid of time can be generated, the problem of irregularity can be reduced to much simpler problem of the missing observations. In more formal terms, for univariate observations $t_i; i = 1, \ldots, T, T \in \mathbb{N}$, a subset of missing observations can be modelled with $y_i = \mathrm{NaN}$,[5] for the interval $i \in (\tau, \tau^*)$. A minor modification of the Kalman filter algorithm, obtained by setting the Kalman gain to zero during the correction steps when the observations are missing, can be used for both parameter identification and forecasting. By setting $K_t = 0$, equations (2.22a) and (2.22b) reduce to

$$\begin{aligned} \hat{\alpha}_t &= \hat{\alpha}_{t|t-1} \\ P_{t|t} &= P_{t|t-1}. \end{aligned} \tag{2.28}$$

This reflects the fact that no additional information can be added to the model when no new observation is available and, consequently, no state correction will be made. The same principle can be applied to a multi-step forecast and an arbitrary mix of the missing and available observations.

Furthermore, simple structural models such as LLM and LLT can be put in a particular form to account for the irregular sampling without any modification of the (discrete) KF algorithm, through adjustments of the process covariance matrices. In this particularly simple case the only difference between the regular LLT and its modification for the irregular sampling is that the process noise variance depends linearly on the time difference between successive observations (Durbin and Koopman, 2012, p.65)

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t, & \varepsilon_t &\sim \mathrm{NID}(0, \sigma_\varepsilon^2), \\ \mu_{t+1} &= \mu_t + \eta_t, & \eta_t &\sim \mathrm{NID}(0, \delta_i \sigma_\eta^2), \\ \delta_i &= t_{i+1} - t_i. \end{aligned} \tag{2.29}$$

The nomenclature here is the same as in (2.24) with the δ_i as the time between i-th and $i+1$-th observation. Similarly, it can be shown that LLT model (2.27) for irregular

[5] 'Not a Number' representation, used in computer engineering for a value which is undefined or cannot be represented (Committee et al., 2008)

sampling can be expressed as

$$
\begin{aligned}
y_t = \mu_t + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{NID}(0, \sigma_\varepsilon^2), \\
\begin{bmatrix} \mu_t \\ \beta_t \end{bmatrix} = \begin{bmatrix} 1 & \delta_t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{t-1} \\ \beta_{t-1} \end{bmatrix} + \begin{bmatrix} \eta_t \\ \zeta_t \end{bmatrix},
\end{aligned} \tag{2.30}
$$

keeping the same nomenclature as in (2.27), with δ_i as the time distance between the successive observations. The covariance matrix of the equivalent discrete time state disturbances (Harvey, 1990, p.487) becomes

$$
\mathrm{Var} \begin{bmatrix} \eta_t \\ \zeta_t \end{bmatrix} = \delta_t \begin{bmatrix} \sigma_\eta^2 + \frac{1}{3}\delta_t^2 \sigma_\zeta^2 & \frac{1}{2}\delta_t \sigma_\zeta^2 \\ \frac{1}{2}\delta_t \sigma_\zeta^2 & \sigma_\zeta^2 \end{bmatrix} \tag{2.31}
$$

Equations (2.29), (2.30), and (2.31) allow the application of the classical discrete time Kalman filter algorithm on irregularly sampled signals. However, this gain is not without cost – (1) whilst the state disturbances in the uniformly sampled model (2.30) are independent, the disturbances in (2.31) are correlated and (2) a particular care regarding the indices used for the dynamic part of the model in connection with the calculation of δ_i due to slope changes between the forward and backward difference calculation (Harvey, 1990, p.487) is needed.

As explained in the introduction of the KF algorithm, its optimality is given only in the case when the selected linear model corresponds to the DGP of the random trend variable. Unfortunately, this is practically never the case. In presence of the model mismatch, the KF can diverge, resulting in a poor filtering/forecasting performance. The next section explains how a deliberately suboptimal model can mitigate the divergence of KF.

2.2.5 Model Error Compensation with Kalman Filter

As pointed out in 2.2.1, the KF is an optimal linear filter in the sense, that it yields minimum MSE within the class of linear estimators independently of the statistical properties of the measurement and process noise (Harvey, 1990; Durbin and Koopman, 2012). This property, is, however, given only when the underlying model is exactly known. In the case when the model is unknown, a degradation of the filter performance is expected and in the worst case the estimated and the 'real' states can significantly diverge. This is particularly the problem when the assumed noise covariances are small and the KF 'learns the wrong state too well' (Jazwinski, 1970; Simon, 2006; Karvonen and Särkkä, 2014). Obviously, in the context of endogenous trends studied here, measurement noise is practically zero since the values are actually calculated based on known integer values (placement). Traditionally, the standard ways of dealing with divergence are to artificially increase the noise input to the system or to over-weigh the new data. It should be noted that (2.29) also imposes a linear growth of input noise in time. This is, however, the result of the exact conversion of continuous stochastic differential equations into a system of stochastic difference equation (discretisation). For the purposes of this study the state error covariance matrix is engineered based on characteristics of the selected model.

A natural way of giving more importance to new data is exponential ageweighting of old data. For LLM (2.29) the output variable y_t is equal to the state μ_t with some measurement noise. In this case, the measurement noise variance σ_ε^2 can be equated to the state covariance matrix $P_{t|t}$, i.e.

$$P_{t|t} = \sigma_\varepsilon^2 \tag{2.32}$$

and for the *a-priori* state covariance ($P_{t|t-1}$):

$$P_{t|t-1} = P_{t-1|t-1}(\exp(\alpha\delta_t) - 1), \tag{2.33}$$

with α as the ageweighting factor. All other SSM and KF equations remain the same as in LLM. The same adaptation can be used also for LLT models after the adjustment for dimensions of the system matrices. It can be shown that this result can be obtained through minimisation of a quadratic cost function that penalises state errors whilst taking exponential discounting of the information in the old data (Jazwinski, 1970; Simon, 2006; Karvonen and Särkkä, 2014). This modification of the classical KF is intended to be used on variables which certainly do not have the same data generating characteristics as the modelled stochastic process (Anderson, 1973; Kramer and Kandel, 2011; Murata et al., 2014; Goff et al., 2015). This is done to avoid the filter divergence and obtain reasonable forecasting performance.

At this point of the literature review, all components needed to meet the requirements of the Research Objectives have been identified and discussed. Conceptual overview (Figure 2.2) shows their logical content and interrelations, with the exception of the measures of model performance block, which are explained in the Methodology Chapter (3.3.4), due to their closeness to the empirical testing setup.

The next section develops four research questions from the research objectives in order to facilitate demonstration that the research objectives are fulfilled. In other words, answers to the set research questions gathered through analysis of empirical data are the logical components of the evidence that the Research Objectives are fully met and the theoretical and methodological aspects of this study effectively close the identified gaps in the literature.

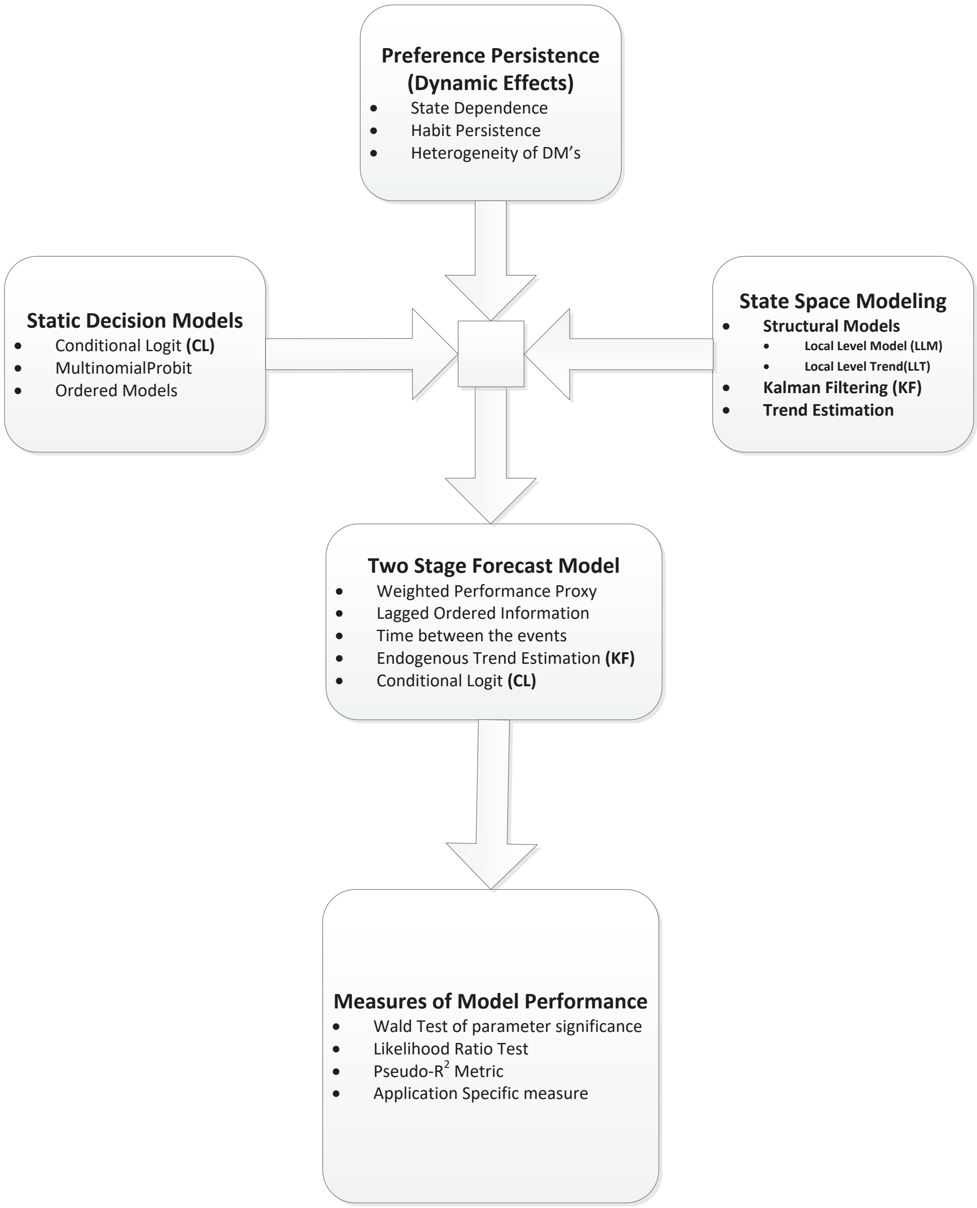

FIGURE 2.2: Conceptual Framework of the Research

2.3 Research Questions

Up to this point, this chapter has discussed relevant elements of the body of knowledge related to DCM and SSM, starting from RUM and IIA as theoretical underpinnings of the CL model as the basic building block for modelling single decision-level discrete situations. As a static model, CL is not adequate to model the dynamics of a choice process. The Research Objective 1 identifies feedback on information relating to ranking alternatives in previous choice sets, which is one of the major pillars of the modelling efforts in this study. Academic research contains a considerable number of models which take into account previous choices made and define several different mechanisms to embed them in a DCM. Examples include inertia, taste variation between decision-makers and, over time, state dependence (current preferences being affected by previous choices), *a-priori* information about preferences, and serial correlation in idiosyncratic taste shocks. In spite of the merits of these approaches, not a single one uses ordered information or is capable of taking into account choice sets changing over time. Instances of dynamic models taking ordered information are not very common, and this group of models also suffers from the same drawbacks. In addition, most of the dynamic models, irrespective of the availability of ordered data, exploit structural characteristics of panel data but offer no provision for irregular data collection. Hence, a behaviour model of a net-utility-maximising DM, who takes into account non-uniform time intervals between decisions, meets the provisions of the Research Objective 1.

Section 2.2 was dedicated to SSM and KF as a method of estimating latent states. SSM model structure resembles the fundamental separation of externally observable and non-observable portions of the RUM. The latent state model form can effectively be used to track and forecast linear and higher-order trends of different variables. SSM, together with KF can be easily adapted to account for non-uniform time intervals and missing observations, which, together with an appropriate method for model parameter fitting, directly correspond to the aims of the Research Objective 2.

The intrinsic value of such a model without an illustrative application is clearly very limited. Hence, an application which combines provisions for dynamic modelling, changing choice sets, and non-uniform times between the availability of alternatives in a choice set has to be proposed. For the purposes of this study, horse-wagering markets have been selected as an empirical test setup facilitating analysis of the research questions in order to meet the research objectives, which are used to guide the demonstration of fulfilment of the research objectives.

Horse-wagering markets are considered a very convenient test setup (Sauer, 1998) for many economic theories (e.g. Efficient Matket Hypothesis (EMH) testing), because of their similarity to conventional financial markets, including a large number of agents, rule-making and brokering by a market maker, competitive bidding, a vast amount of public information affecting trading, and the market micro-structure founded on zero-sum game strategies (Levitt, 2004). Horse race-wagering markets also offer further advantages derived from: (1) a well-defined, finite time horizon of uncertainty (end of a horse race), at which point all uncertainties regarding related trading are resolved, financial claims are settled, and transaction pay-offs are determined without any assumptions regarding the transaction stopping (Pham, 2009), (2) fixed odds on particular race outcomes, connoting simplified pricing (Sauer, 1998) of trading instruments (bets), and (3) deterministic short- and long-term rates on investment evaluation for wagered funds (Sung and Johnson, 2007). Consequently, these advantages allow for benchmarking the selected trading strategies (i.e. decisions made) over a finite time frame (Law and Peel, 2002; Johnson et al., 2006; Sung and Johnson, 2007).

The conventional approach to forecasting horse-racing outcomes, originally proposed by Bolton and Chapman (1986), leverages CL regression as a discrete choice modelling tool. The CL regression, either as a one-step or as a two-step procedure, discriminates the alternatives through the usage of layered fundamental and market-generated data (e.g. odds). Several studies have demonstrated that the two-step procedure yields substantially higher pseudo-R^2 values (Sung et al., 2005) – and hence higher predictive power then single-step approaches. Recently, some studies (Lessmann et al., 2009, 2012; MacDonald et al., 2013) have endorsed different modifications, borrowed from the contemporary expansion of machine learning algorithms, such as Support Vector Machines (SVM), Random Forests (RF), etc., to the two-step procedure – all with varying levels of success. In general, these algorithms have sought to estimate a complex non-linear mapping (regression) of the fundamentals to a variable proxying an absolute, or a relative runner's performance through a non-linear discriminant analysis. A fundamental drawback of this class of algorithm is that all examples are static, i.e. they do not model parameter changes through time. This is a considerable drawback, since the inclusion of dynamics may compensate for a number of model mis-specifications intrinsic to static models, such as improper functional form (e.g. linear vs. non-linear (Crisan and Rozovskii, 2011)), the omission of relevant exogenous or endogenous variables, and wrong assumptions regarding underlying stochastic characteristics (Raj and Ullah, 2013).

In order to demonstrate the forecasting capabilities of those models resulting from the presented research objectives, net utility-maximising single (i.e. not aggregate) decision-level models are designed, that take into account inertia and/or state dependence effects through linear or non-linear trends (time-varying preferences) starting from the previous valuation of ranked alternatives. Furthermore, *ex-ante* forecasted performance (in the form of winning probabilities used for betting) is used to reach a verdict regarding the merits of the approach. By combining characteristics of the choice process, in terms of aggregation of previous choices, trends, temporal discounting, and the relative importance of the decision on hand, the following research question, together with the associated secondary research question, can be formulated:

Research Question 1. *In repeated decision-making events with changing choice sets, do patterns of previous choices contain a (statistically) significant information set explaining an additional part of the unobserved portion of the utility?*

Research Question 1a. *To what level does the additional information set increase the competitive advantage of a savvy market agent aiming to forecast an outcome of a decision event over his uninformed counterparts (market)?*

These two questions address the simplest form of incorporation of the previous choices made, namely only their face value without any temporal considerations. In order for them to be answered, a model (or a set of models) which embed choice ranking information, in alignment with the Research Objective 1, has to be developed and evaluated (c.f. Figure 2.3).

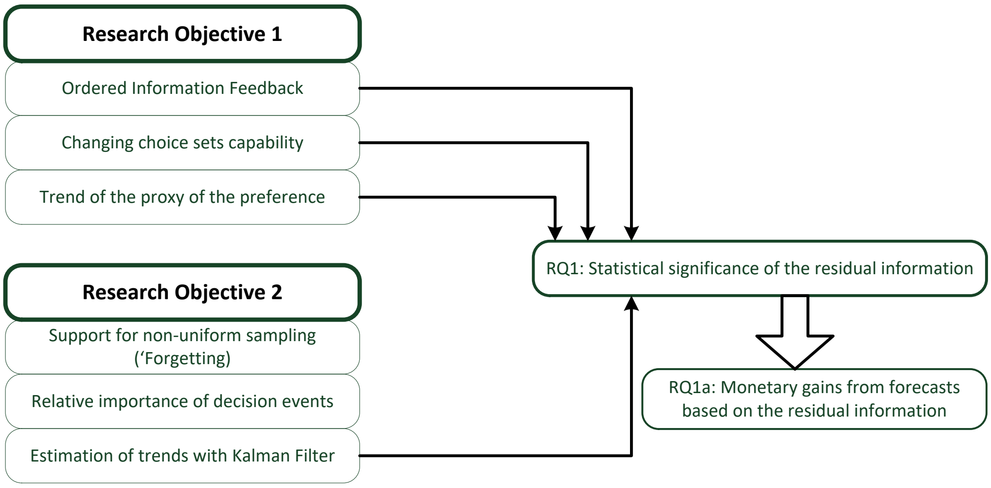

FIGURE 2.3: Mapping of Research Objectives to the Research Question 1

The next question, together with its associated secondary research question, evaluate the improvement of the model if the time dimension is added, corresponding to the portion of the Research Objective 2 requiring support for the non-uniform sampling (c.f. Figure 2.4).

Research Question 2. *In repeated decision-making events with changing choice sets, does 'forgetting', i.e. temporal distance between successive decision events, in patterns of previous choices made contain a (statistically) significant information set explaining an additional part of the unobserved portion of the utility?*

Research Question 2a. *To what level does the additional information set increase the competitive advantage of a savvy market agent, aiming to forecast an outcome of a decision event over his uninformed counterparts (market)?*

The third set of questions, also consisting of one main research question and its associated secondary research question, introduces an importance weighting to the previous choices, giving more weight to choices with higher net pay-off. In the case of horse-racing importance can, for example, be correlated to the race prize. These questions correspond to the importance weighting part of the Research Objective 2 (c.f. Figure 2.5).

Research Question 3. *In repeated decision-making events with changing choice sets, does importance weighting, i.e. the attribution of higher fidelity scores to decision events with a larger pay-off, in patterns of previous choices made, contain a (statistically) significant information set explaining an additional part of the unobserved portion of the utility?*

Research Question 3a. *To what level does the additional information set increase the competitive advantage of a savvy market agent aiming to forecast an outcome of a decision event over his uninformed counterparts (market)?*

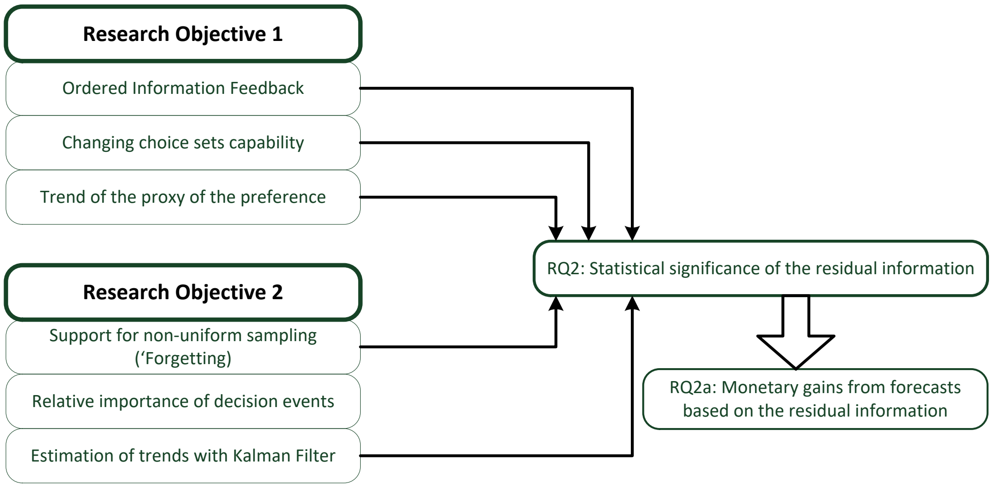

FIGURE 2.4: Mapping of Research Objectives to the Research Question 2

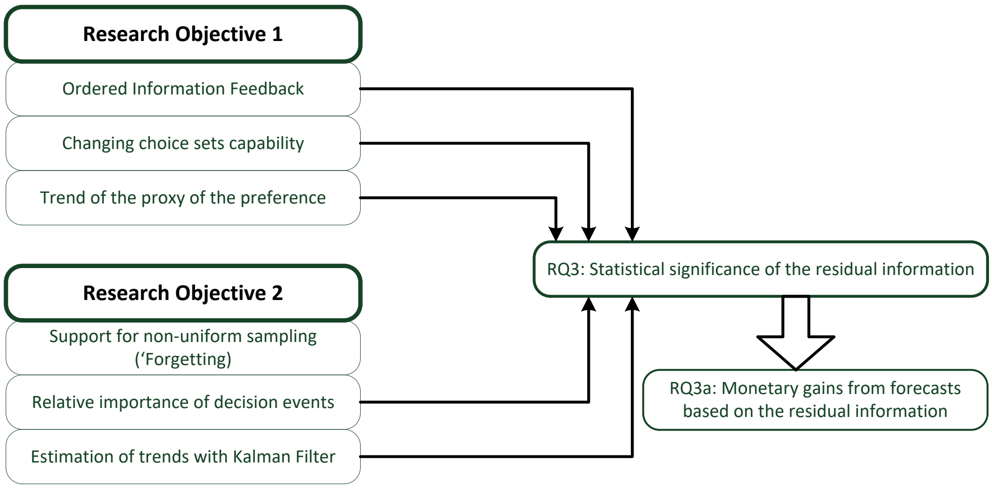

FIGURE 2.5: Mapping of Research Objectives to the Research Question 3

Finally, the fourth set of questions attempts to assess the merits of the complete modelling approach, predicting trends derived from previous choices weighted on both importance and time. They encompass all subcomponents of the Research Objective 2 (c.f. Figure 2.6).

Research Question 4. *In repeated decision-making events with changing choice sets, does* simultaneous *inclusion of importance, i.e. the attribution of higher fidelity scores to decision events with a larger pay-off, and 'forgetting', i.e. temporal distance between successive decision events weightings in patterns of previous choices made, contain a (statistically) significant information set explaining an additional part of the unobserved portion of the utility?*

Research Question 4a. *To what level does the additional information set increase the competitive advantage of a savvy market agent aiming to forecast an outcome of a decision event over his uninformed counterparts (market)?*

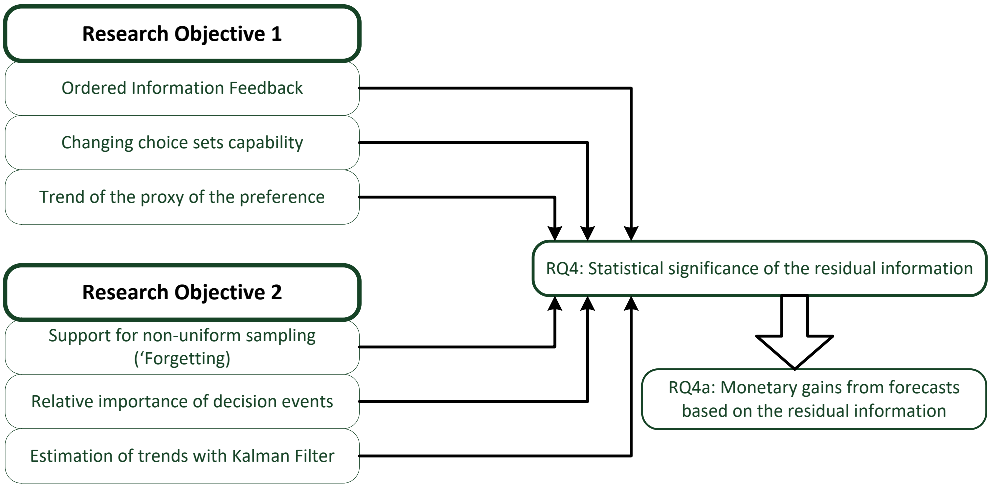

FIGURE 2.6: Mapping of Research Objectives to the Research Question 4

An integrative explanation that provides an overview of the interrelation between the identified gaps in the literature, the auxiliary information used to construct the outlined modelling framework, the research questions, and the research objectives is in order here. The gaps in the literature, as explained in 1.1, are identified as lack of support for changing choice sets (GAP 1) and different times between availability of alternatives (GAP 2) in incumbent DCMs. Moreover, inclusion of the ranked *ex-post* data and information on relative importance of previous decision events allows construction of dynamic decision models that have both behavioural-explanatory and economic significance. A schematic overview of the links between the identified gaps, the research questions, and the research objectives is depicted in Figure 2.7. The gradual buildup of the models starting with the simplest one linked to the Research Question 1, which includes only the changing choice sets and the ranked *ex-post* data, is expanded through addition of the time information (corresponding to Research Question 2) and through addition of the importance weighting (corresponding to Research Question 3). Integration of all available auxiliary information beyond the models is linked to the Research Question 4, aiming to cover both identified gaps.

In summary, a set of models that takes into account the importance and temporal weighting of previous choices made in a setup where a single decision-maker faces choice sets changing both in size and alternatives is needed, in order to close the identified gaps in current decision-making theory. These models are derived from the research design, which connects the research questions and the empirical dataset used for inference and forecasting evaluations. The logical connection between the research objectives and the research questions is given through completeness and consistency coverage of the key characteristics of the models capabilities:

1. Inclusion of feedback on ordered choice information from previous decision events (Research Questions 1, 2, 3, and 4)

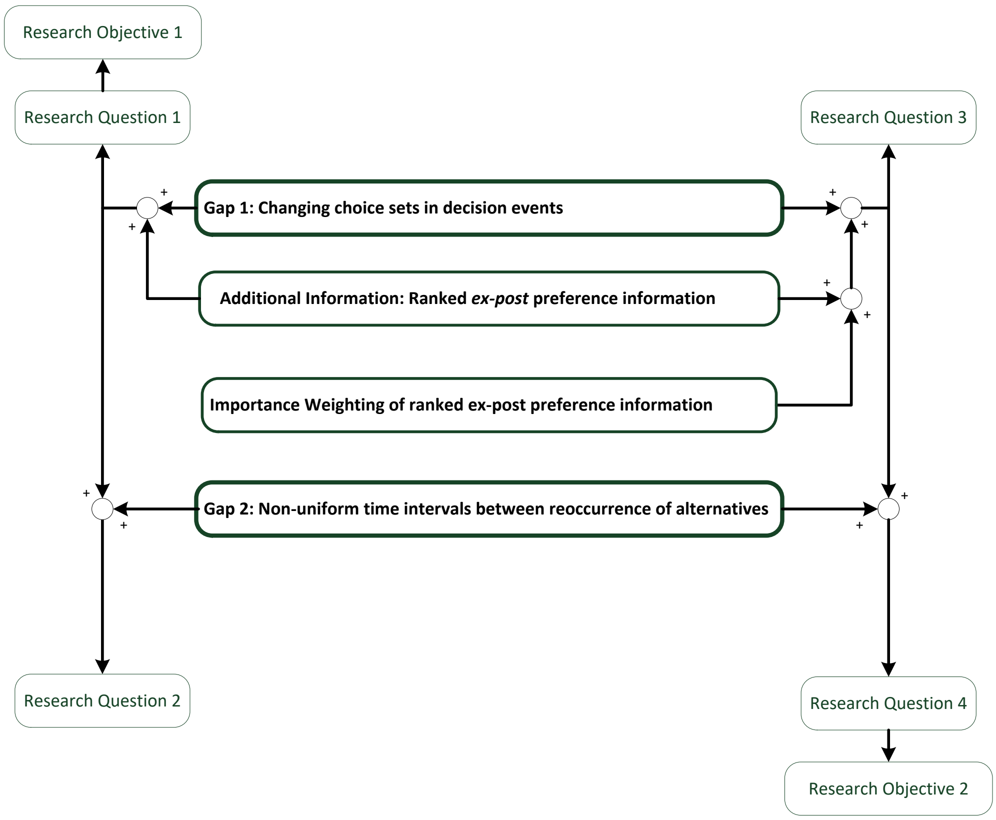

FIGURE 2.7: Linking the identified gaps in the literature and the research questions

2. Support for changing choice sets (both in number and the selection of alternatives) (Research Questions 1, 2, 3, and 4)
3. Inclusion of trends in the alternative-specific proxies (Research Questions 1, 2, 3, and 4)
4. Support for non-uniform time distances between recurring appearances of the same alternative in two different choice sets (Research Questions 2 and 4)
5. Inclusion of the KF, together with the associated model parameters identification procedure, given as a methodology guidance (c.f. Sections 3.2 and 3.3.3.1)

Chapter Summary

The literature review chapter describes the underlying state-of-the-art theory of the different models and concepts needed to close the identified gaps in the literature and meet the research objectives set in the introduction. The focus of the chapter is to conceptualise and critically evaluate two groups of models, the DCM and SSM. These model groups put together, offer a powerful framework adequate for modelling of dynamic DCM with irregular times between the subsequent appearance of a particular alternative as a member of a choice set.

The chapter begins with a general description of ideas that led to development of static DCM and the properties of *logit* and *probit* models. The strengths, especially the property of *logit* that yields closed analytic solutions for choice probabilities, and the inherent weaknesses of the models are discussed in depth. As a further development of decision making concepts, available state dependence and inertia models are critiqued. Here, the concept of latent states in DCMs is introduced as an elegant model structure to describe the dynamic behaviour of DMs.

Next, the chapter turns to SSM and the KF as an optimal (under certain conditions) estimator of the latent model states. Basic theory of linear KF for regularly sampled models is explained and drawbacks shown. Two fundamental structural models LLM and LLT are introduced and their role in forecasting presented. Important extensions of the KF, discrete time adaptation for irregular sampling times and constructed state covariance matrices for model-error compensation, are added in order to allow for appropriate research design, which is further developed in Chapter 3.

Finally, as specified in 1.2, the identified literature gaps are used to define the objectives of this particular research. In order to meet the objectives, a conceptual research framework design, supporting the formulation of research questions is developed and explained. Results from empirical testing of the models and their analysis will yield definite answers to research questions, which will close the chain of arguments and evidence needed to address all aspects of the research objectives. In other words, the modelling framework, together with the evidence of the achieved modelling performance and answers to research questions, will show that the declared objectives of the research are met and that the identified gaps in the literature are effectively closed.

Chapter 3

Methodology

This chapter depicts the research paradigm and the methodology evaluated and selected to meet the research objectives. The first section of the chapter explains the applied research philosophy and the rationale for the selected research paradigm. Second, derivation of a fully operational two-stage model, based on the decomposition of the error term (Figure 2.1), and CL, SSM, and KF equations is given. Finally, research design, which includes empirical test setup and planning description, model parameter identification based on ML principles, statistical and application specific model evaluation criteria, and the characteristics of the data set used for the empirical analysis, concludes the chapter.

3.1 Research Philosophy

The research objectives of this work, as defined in Chapter 1, are bound to the development of a novel dynamic DCM geared to show a computationally efficient way to account for the effects of ranked choices made in the past. In order to be able to define the research design in a coherent and consistent way, an appropriate research paradigm has to be identified and its adequacy defended against alternative approaches.

Decision-making behaviour is inherently a social phenomenon, regardless of the nature of the decision maker, which may be a person, an animal, an aggregate group of humans or animal herds or, to the extreme, an abstract, non-living entity. As such, decision-making behaviour cannot be seen as a context-free, timeless, unconditional, and immutable natural law, which, once discovered and formalised, perfectly describes the true reality (Guba et al., 1994). In addition, at least in the case of some research settings (e.g. managers or agents in studies of decision-making in organisational settings), decision-makers can modify their behaviour as a result of the interference caused by the research process. Hence, the study of dynamics in decision-making cannot be seen as *reductionist* or *deterministic* (Creswell and Creswell, 2017). Moreover, it can be argued that the evolving, non-deterministic, and not completely tractable abstract nature of decision-making behaviour cannot be comprehended completely, due to basically limited intellectual capacity and cognition of human researchers (Todd and Gigerenzer, 2000). Consequently, the focus of the decision-making research should not be the quest to find the perfect, ultimate and prehensile reality but to find a model or series of models that sufficiently well describe the reality for the declared purpose, whilst serving the perennial aim of converging and generalising the results, reflecting what is known as the 'ontological argument' put forward by *critical realists* (Kemp, 2005). For example, the merit of the proposed models can be assessed through evaluating predictions of the behaviours based on them. Furthermore, different models (theories) have to be compared based on their performance and parsimony bound to relevant applications and, over time, withstand scrutiny of the scientific community, evolve, and adapt (Popper, 1972; Guba et al., 1994), in order to enhance our understanding of non-deterministic (probability-laden) reality. It should be noted that *critical realism* is not reserved purely for social sciences research – even in physics, which is arguably the flagship of natural and empirical sciences, the field of quantum mechanics is inherently probabilistic (i.e. non-deterministic), and the proposed models do not necessarily aspire to provide a description of the reality which is reconcilable with all other branches of physics (Myrvold, 2018; Harrigan and Spekkens, 2010). This reinforces the plausibility of a similar paradigm in the modelling of the most complex systems of all – social systems. The presented argumentation herein reflects the author's *ontological* stance adopted in this study, which, together with the epistemological considerations described below, underpins the research process and reflects the *post-positivistic* perspective of the work (Crotty, 1998).

The second aspect of the research paradigm, the *epistemological* stance with respect to the objectives of this study, cannot be based on *objectivist* axioms, since, at least in the case of some decision-maker types (e.g. managers studying decision-making in organisational settings), decision-makers can modify their behaviour as a result of the interference caused by the process of research-forming learning/adaptation feedback, especially in cases when such adaptation offers important incentives. Hence, even in cases where careful precautions are taken, theories, inferences, and conclusions may end up being contaminated by the results. An example of this can be observed when increasing the efficiency of different financial and betting markets in which some heuristics (e.g. disappearance of the 'January Effect' (Szakmary and Kiefer, 2004) or the efficiency of a bookmaker after introducing betexchanges horse-betting markets (Sung et al., 2016a)) ceased to be relevant. Hence, any decision-making theory of behaviour may face diminishing validity over time, thereby calling for the *modified dualist/objectivist*-driven selection of research methods.

The adopted *post-positivistic* research paradigm does not reject the *hypotetico-deductive* method per se; rather, it shifts the focus more towards the *falsification* of the stated hypotheses, as opposed to their acceptance, since proof of social 'laws' remains elusive due to the non-deterministic nature of the reality. Note that *quantitative and deductive* method selection can perfectly align with the selected research paradigm if no hypotheses are proposed and research questions are used to guide the elimination or reduction of elements of randomness in existing theories and models. This elimination/reduction of randomness is intrinsically reductionist, as it aims to break down individual theory contributors into a small, discrete set of ideas under testing, expressed as statistically defined random variables (Creswell and Creswell, 2017). Due to the necessity to obtain a statistical characterisation of the variables, a careful quantitative research design is needed, which, contrary to classical *positivist* paradigms, does not have to follow the classical path theory→test based on the instruments derived from the theory. Moreover, as emphasised by the *post-positivist* school of thought, the data needed for the characterisation of the instruments are prone to biases and fallacies, affecting both the perspectives of researcher's observations (which are possibly theory-laden) and the *a-priori* selection of the relevant theories used for the (quantitative) design of the experimental research. An experimental design setup has to address the selection of a manipulative methodology (controlled experiment) or observation of the instruments by enquiring into natural settings. Obviously, confounding effects have to be controlled in both setups.

For the purpose of this study, this translates into the fundamental decision as to whether an Revealed Preference (RP) or an Stated Preference (SP) methodology and experimental design should be pursued. The RP methodology collects and analyses data obtained by capturing actual choices made by DM in their natural environment and then made public (i.e. revealed) to an interested researcher/observer. In contrast, SP data are captured during controlled experiments defining hypothetical situations (sociological simulations), where the DM state their preferences explicitly. A skilful researcher can leverage the inherent advantages of an SP such as pre-specification of choice sets presented to the respondents, firm control of the choice attributes and the possibility of including 'abstract', not easily quantified attributes such as measures of environmental policies or leisure time. This also means that the attributes may be selected in such wise as to minimise confusion and be measured without error.

On the other hand, the SP research design has to withstand scrutiny regarding the reliability of the observed data. Data reliability in an SP context has to be assessed from two standpoints: (1) validity (level of alignment between the stated preference and the preference that the DM would have in a real decision event (i.e. similar to model bias in statistics) and (2) stability – an inversely proportional measure of random error (noise) in preference information. Noise represents uninformative data resulting from irrelevant considerations not captured by pre-specified attributes. Obviously, stability depends strongly on the relevance of the experiment setting and the clarity of the questionnaire used to state the preference (Ben-Akiva et al., 1991). The flexibility of the hypothetical setup proves to be a double-edged sword, in that subjects under simulated conditions tend to focus on the most important attribute only (in alignment with the 'prominence effect' (Tversky, 1972; Simmons and Nelson, 2006)), which in addition to the risks of misinterpretation or even complete ignorance of attributes, if the presented decision scenarios are deemed unrealistic, may lead to practically useless data. Moreover, such scenarios usually fail to replicate constraints imposed in reality. However, the most serious drawback of the SP methodology is that the participants do not have a natural incentive to replicate a behaviour that they would exhibit in a real setting, as they do not experience any consequences or benefits of their decisions (cf. validity). As an extreme example, some subjects may even use the questionnaires as a way to express their stance regarding the broad survey context. Berg et al. (2010) demonstrated that incentives increase the validity of the captured data, i.e. yield more economically consistent behaviour. From the modelling point of view, the truth-revealing incentives result in superior statistical and explanatory properties of fitted decision-making models (Berg et al., 2010).

Discrete choice analyses based on RP data, which describe the actual behaviour of decision-makers, represent a far more traditional approach to discrete choice modelling research (Ben-Akiva et al., 1994). As might be expected, these data do not suffer from validity issues, and they are particularly well-suited to modelling aggregate market behaviour based on objectively measured variables (Ben-Akiva et al., 1985). These variables constitute a selection of observations collected in real life, which may be made in error. This is not relevant here, however, since the research objectives call for a trend analysis of the (revealed and hence known) past ranked choices which are known exactly (being integer ranks). Moreover, the model incorporates some market variables, which are public and reflect public opinion regarding the probabilities of choices (in form of prices). In addition, the relationship with the studied market provides a considerable incentive for truth-revealing behaviour. Furthermore, research designs, which are bound to decision-making related to some kind of exchange or contingent market mechanisms (i.e. micro structure), fail to emulate the numerous constraints and underlying market movements that decision-makers could take into account. For example, Ziegelmeyer et al. (2004) conclude their SP-based study with the observation that they found only weak support for one of the most robust biases (Bruce and Johnson, 2000) observed in studying asset markets and behaviour – the favourite longshot bias – and that the lack of liquidity (noise) bettors impedes the replication of actually reported marked dynamics.

In conclusion, for the selected *post-positivist* research paradigm, the key element driving the methodology selection is the random error reduction and testing of its significance. By using the RP data, quantitative characterisation of the random error can be effectively achieved. In addition, it can be used for discovering and quantifying the effects of selected cognitive biases through statistical testing (Tversky and Kahneman, 1974). Moreover, the combination of the changing choice sets and, especially, the time dependence of decisions favouring large datasets with high validity, are far easier to obtain with RP data, when the choice sets are a priori known. Taking all of the aspects of SP and

RP into account, it can be concluded that the RP approach is more adequate to meet the research objectives, due to the inherent truth-revealing incentives (monetary pay-off from correct decisions made), the likelihood of better data-fitting results and the validity of the choice ranking data, and hence it has been selected for the overarching research paradigm and methodology of the study, in order to guide the research design (see 3.3).

The next section explains the model-building methods that combine elements from the conceptual framework (cf. Figure 2.2) into one operational modelling landscape. The relative simplicity of the model structure and parameter identification from empirical RP datasets, as compared to a previously proposed discrete SSM in the context of decision-making, makes the new approach conceptually more appealing than incumbent approaches and provides enhanced flexibility and interpretability in applications where monetary incentives bound to the correctness of the choices made are paramount.

3.2 Operational State Space Discrete Choice Model

Conceptually, all dynamic DCM are non-linear models, due to the fact that the underlying latent utility values are effectively discretised (collapsed) to make a choice in a certain decision making event. Non-linearity of the models implies that, except for very special models, such as *logit*, only numerical methods can be used for model fitting and inference. Hence, the dynamic DCM are built around different methods leaning on a Bayesian approach to statistical inference, such as Markov Chain Monte Carlo (MCMC) or particle filtering (Crisan and Rozovskii, 2011).

On the other hand, particle filtering can be seen as a generalisation of the KF, which, provided that the assumption of inherent linearity of the model is reasonable, offers a straight forward approach for statistical inference in dynamic DCMs. Limitations caused by the linearity assumption is not that severe as it may seem on the first glance. In the vast majority of engineering applications, dynamic models are routinely linearised around some kind of equilibrium – a steady state operating point (Goodwin and Sin, 2014), using, for example, Taylor series expansion. A similar idea can be applied on ordered type of data. In particular, it can be linearised (*inflated*) by using the following function (Lessmann et al., 2012)

$$y_{i,j}^c = -0.5 + \frac{y_{i,j} - \min_j(y_{i,j})}{\max_j(y_{i,j}) - \min_j(y_{i,j})}, \forall i \in J_j, \tag{3.1}$$

which represents the continuous equivalent $y_{i,j}^c$ of the ordered alternative y_i in a decision event j.

This approximation is possibly violating the implicit assumption of equal value distance between alternatives. However, this is often ignored in practice and many researchers treat ordered dependent variables as if they were measured on an interval scale for integer data coded sequentially (Long, 1997). Consequently, variables approximated in this way should be used in statistical modelling with due caution. In particular, this arises because of the questionable reliability of the stated and revealed ordering of lower ranked alternatives (Fok et al., 2012). A possible explanation for that effect is that some decision makers only care about the first few alternatives, either because of the lack of ability to make a distinction amongst the less-preferred alternatives or because the available choice set is too large for fine differentiation and, hence, ranking. If not accounted for, this may lead to a substantial bias in the parameter identification (Chapman and Staelin, 1982). Several methods have been put forward to leverage the available information and increase the reliability of the rankings. For example, Chapman and Staelin (1982) defined, utilising the Ranking Choice Theorem, an 'explosion process', which decomposes the rank ordered data in several independent data sets. The number of the different data sets is determined by the explosion 'depths' calculated based on testing of statistical equivalence of the data pooled in a different ways (Bolton and Chapman, 1986). Fok et al. (2012) introduced a model which grades the decision capabilities of the DM, which are endogenously identified through a latent ranking quality variable. Lessmann et al. (2012) adopted the Normalized Discounted Cumulative Gain (NDCG) criterion inspired by internet search engine benchmarking derived from Edelman (2007a) study, which (similarly to Lessmann et al. (2009)) endorses the usage of binary classifiers in horse-betting context because of their higher robustness against low reliability of ordered data.

Intuitively, if a larger pay-off is associated with an outcome of a decision event, the reliability of the ranking should increase (Sung and Johnson, 2007). Hence, if the inflated variable is weighted by some *a-priori* information about the decision event, it would result in increased reliability. This strategy for increasing the ordered data reliability has been applied in this thesis (see also 5.1 for wider context).

Instead of using lagged ordered output variables in modelling of the dynamic dependency, as endorsed by Kitamura and Bunch (1990), the following model of utility for an alternative i in a decision event j can be constructed (c.f. Figure 2.1):

$$U_{ij} = V_{i,j} + f(\hat{y}^c_{i,j}) + \epsilon_{i,j} \tag{3.2}$$

where $y^c_{i,j}$ is the inflated (continuous) ranking of the alternative during the decision event j and $V_{i,j}$ and $\epsilon_{i,j}$ have the same meaning as in (2.1), namely the observed and unobserved portion of the utility function. Note that this decomposition implies a separation of the unobserved part (which is traditionally modelled as a purely random variable), in two parts. The first represents the utility contribution derived from a prediction of the normalised choice ranking based on the dynamic development of the previous choice rankings. The second represents the stochastic properties of the decision maker's utility function. In alignment with the equation (2.15), the representative utility is a linear combination of linearly (through β) regressed exogenous components $x_{i,j}$ and the trend prediction, i.e. $V_{i,j} = \beta x_{i,j}$ and $f(\hat{y}^c_{i,j}) = \beta_y \hat{y}^c_{i,j}$.

Obviously, the endogenous trend variables modelling the dynamics of the previous choice rankings of the entire choice set, can be specified as an output of J parallel and uncorrelated linear SSM,

$$y^c_{i,j} = Z_{i,j}\alpha_{i,j} + \varepsilon_{i,j}, \mathrm{E}(\varepsilon_{i,j}) = 0, \mathrm{Var}(\varepsilon_j) = \sigma^2_\varepsilon I_{J_j \times J_j}, \tag{3.3}$$

with the aggregated disturbance vector $\varepsilon_j = [\varepsilon_{1,j}, \varepsilon_{2,j}, \ldots, \varepsilon_{J_j,j}]'$ and J_j separate state vectors $\alpha_{i,j}$ having the model with the same structure

$$\alpha_{i,j} = T_{i,j}\alpha_{i,j-1} + \eta_{i,j}, \mathrm{E}(\eta_{i,j}) = 0, \mathrm{Var}(\eta_{i,j}) = Q_{i,j}, \tag{3.4}$$

whilst using the same nomenclature as in (2.16) and (2.17).

It should be noted that this model does not directly correspond to the state dependency model (2.10a), where the complete utility $U_{i,j}$ is seen as a dynamic state, as opposed to (3.3) where only the trend portion of the utility is modelled as a latent state vector. However, the approach presented here can be easily adopted to include state dependency, if the considerations regarding the causality outlined below are considered. A second characteristic of the model worth mentioning, is that there are three independent sources of disturbances, two describing the linear dynamic part and one describing the unmodelled factors of the decision making. For the purposes of this work, standard assumptions regarding stochastic properties of the disturbances are followed. In particular, SSM disturbance components $\varepsilon_{i,j}$ and $\eta_{i,j}$ are uncorrelated and Gaussian and $\epsilon_{i,j}$ follows the Gumbel distribution.

Careful analysis of equations (3.2) to (3.4) reveals a potential causality problem if the variable $y^c_{i,j}$ is intended to be used for prediction. The variable $y^c_{i,j}$ is not known before the decision j is made, so that the value has to be substituted by an estimate of the value using the information set that includes all the values known before the decision event j. Since $y^c_{i,j}$ is conditional on $y^c_{i,j-1}$ under Markov condition, due to the assumed trend effect, the conditional expected value of the $y^c_{i,j}$, $\mathrm{E}(y^c_{i,j}|y^c_{i,j-1}) = \hat{y}^c_{i,j}$ represents the natural choice for the estimate in the context of linear models with normal distribution of the disturbances. In the further text, the designation $()^c$ will be omitted for simplicity.

The linear functional form of the dependency $f()$, can be understood as a weighting of an estimate of the contribution of the residual information derived from the trend (i.e. inertia). Otherwise, if $y^c_{i,j}$ were known with certainty, there would be no need for any other explanatory effects $V_{i,j}$.

In the context of linear prediction, KF is an optimal MMSE estimator and an optimal conditional estimator in the case of independent and normally distributed disturbances.

Consequently, the conditional expected value of the latent state vector can be expressed in the form of the *a-priori* dynamic estimate (see (2.20)), as follows:

$$\hat{\alpha}_{i,j|j-1} = T_{i,j}\hat{\alpha}_{i,j-1}. \tag{3.5}$$

Combined with the measurement equation

$$\hat{y}_{i,j} = Z_{i,j}\hat{\alpha}_{i,j|j-1}, \tag{3.6}$$

this rounds up the general model definition structure.

For practical applications, the order of the state space vector has to be postulated in advance. Since it cannot be expected that a simple linear model actually constitutes a DGP of the inflated output of a dynamic decision making process, *a-priori* specification of the SSM model structure (order) has to be put forward. This approach resembles modelling postulation in the spline smoothing problem setting, where order and smoothing parameters have to be chosen as a trade-off between the complexity, smoothness, and accuracy of the regression function. Moreover, it was shown by Wecker and Ansley (1983) that LLT can be made equivalent to cubic splines smoothing, stemming from the fact, that the KF recursion effectively performs incremental Cholesky decomposition of the state vector variance-covariance matrix (2.22a) (Eubank and Wang, 2002). The equivalence between the KF algorithm and smoothing splines, can be generalised to other spline orders from the first order (corresponding to LLM) up to the theoretically infinite order (Durbin and Koopman, 2012). In a further treatise, Eubank et al. (2003) endorsed a KF based algorithm for automatic selection of both the order and the smoothing parameters. This method, whilst effective, is appropriate only for long time series. In other settings, e.g. for time sequences with heterogeneous durations and different statistical parameters it would require optimisation for all alternatives in the alternative pool, so that possibly every alternative would require a separate optimisation – clearly a large computational burden. What is even more important is the fact the algorithm fails for time series of short duration since the model parameters cannot be identified. Hence, a model of the endogenous trend is postulated to be of first or second order to ensure (1) identifiability of the parameters and (2) that the number of decision events needed to initialise the model stated does not significantly reduce the number of available data points for parameter identification and model validation. The first order model (2.24), is versatile and parsimonious at the same time, since it requires only a few parameters to be identified. After setting time invariant values for system matrices $T_{i,j} = 1$, $Z_{i,j} = 1$, $H_t = \sigma_\varepsilon^2 = 1$, and $Q_t = \sigma_\eta^2$ in (3.3) and correcting the variance (2.29) for irregular sampling δ_t, a parameter vector that has to be identified can be expressed as $\boldsymbol{\theta_{KF}} = [q \quad \sigma_\varepsilon^2]$, with signal to noise (S/N parameter) ratio $q = \frac{\sigma_\eta^2}{\sigma_\varepsilon^2}$. Figure 3.1 illustrates the process of state updating for LLM.

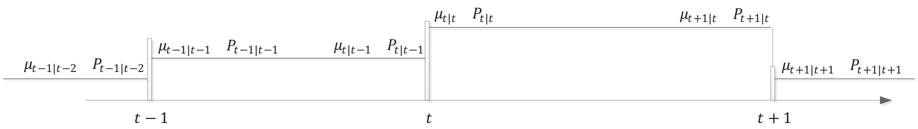

FIGURE 3.1: LLM Timing Diagram

A structural second order model (LLT) is only slightly more complicated. Under the same assumptions as for LLM, equations (2.30) and (2.31) require identifications of exactly the same number of parameters $\boldsymbol{\theta_{KF}} = [q \quad \sigma_{\zeta}^2]$. It should be noted that the variance σ_{η}^2 is zero, making the level component μ_t deterministic. Moreover, it can be shown that cubic splines smoothing and the KF algorithm are equivalent, if controlled for particular filter initialisation (Wecker and Ansley, 1983; Harvey and Koopman, 2000; Koopman and Harvey, 2003). After consolidation, the equations of the model of dynamic decision making, which incorporate the information on previous ordered choice and extracts the trend for decision forecasting can be split in two stages, KF stage and CL stage. Figure 3.2 illustrates the process of state updating for LLT.

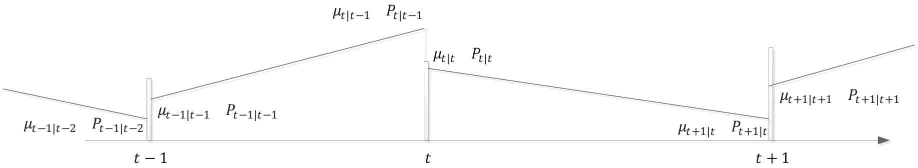

FIGURE 3.2: LLT Timing Diagram

The measurement equation is identical for both structural models, LLM and LLT, as follows:

$$y_{i,j}^c = \mu_{i,j} + \varepsilon_{i,j}, \mathrm{E}(\varepsilon_{i,j}) = 0, \mathrm{Var}(\varepsilon_j) = \sigma_{\varepsilon}^2 I_{J_j \times J_j} \tag{3.7}$$

The KF stage with the first order structure LLM can be expressed as follows:

$$\mu_{i,j} = \mu_{i,j-1} + \eta_{i,j}, \mathrm{E}(\eta_{i,j}) = 0, \mathrm{Var}(\eta_t) = q\delta_{i,j,j-1}\sigma_{\varepsilon}^2 \tag{3.8a}$$

$$\hat{y}_{i,j} = \hat{\mu}_{i,j-1} \tag{3.8b}$$

$$p_{i,j|j-1} = p_{i,j-1} + q\delta_{i,j,j-1}\sigma_{\varepsilon}^2. \tag{3.8c}$$

Here, the same notation as in (3.3) and (3.4) is applied with tailoring of the process noise covariance $\mathrm{Var}(\eta_t) = q\delta_{i,j,j-1}\sigma_{\varepsilon}^2$, which encompasses the S/N ratio q as the noise scaling parameter and $\delta_{i,j,j-1}$ as the time distance between the decision events $j-1$ and j in which the alternative i was available. Alternatively, following model error compensation approach, (2.33), the adjusted *a-priori* covariance matrix becomes:

$$p_{i,j|j-1} = p_{i,j-1|j-1}(\exp(\alpha\delta_{i,j,j-1}) - 1) + q\delta_{i,j,j-1}\sigma_\varepsilon^2, \tag{3.9}$$

with α as the ageweighting factor. In addition, for the first order SSM, somewhat model error correction simpler algorithm for calculation of the *a-posteriori* covariance matrix can be found in literature, (e.g. (Jazwinski, 1970, p.307-311)) for the continuous time $(\delta_{i,j,j-1} \neq 1)$ can be formulated

$$K_{i,j} = p_{i,j-1}(\exp(-\alpha\delta_{i,j,j-1}))/(p_{i,j-1} + \sigma_\varepsilon^2) \tag{3.10a}$$

$$p_{i,j|j} = \exp(\alpha\delta_{i,j,j-1})/(1 - K_{i,j}p_{i,j-1}), \tag{3.10b}$$

substituting (3.8c).

Second order (LLM) equations become only slightly more complicated through inclusion of $\beta_{i,j}$, as follows:

$$\begin{aligned} \mu_{i,j} &= \mu_{i,j-1} + \delta_{i,j,j-1}\beta_{i,j-1} + \eta_{i,j} \\ \beta_{i,j} &= \beta_{i,j-1} + \zeta_{i,j} \end{aligned} \tag{3.11}$$

Based on (3.8c) and (2.31) the covariance matrix becomes

$$P_{i,j|j-1} = T_t P_{i,j-1} T_t' + \delta_{i,j,j-1} \begin{bmatrix} \sigma_\eta^2 + \frac{1}{3}\delta_{i,j,j-1}^2\sigma_\zeta^2 & \frac{1}{2}\delta_{i,j,j-1}\sigma_\zeta^2 \\ \frac{1}{2}\delta_{i,j,j-1}\sigma_\zeta^2 & \sigma_\zeta^2 \end{bmatrix}. \tag{3.12}$$

In case the model error compensation variant of the covariance matrix is specified (3.12) is modified as follows:

$$P_{i,j|j-1} = P_{i,j-1}(\exp(\alpha\delta_{i,j,j-1}) - 1) + \delta_{i,j,j-1} \begin{bmatrix} \sigma_\eta^2 + \frac{1}{3}\delta_{i,j,j-1}^2\sigma_\zeta^2 & \frac{1}{2}\delta_{i,j,j-1}\sigma_\zeta^2 \\ \frac{1}{2}\delta_{i,j,j-1}\sigma_\zeta^2 & \sigma_\zeta^2 \end{bmatrix}. \tag{3.13}$$

As before, α as the ageweighting factor.

Equations (3.8) to (3.13) are together called *prediction* equations.

Immediately after the newest observation of $y_{i,j}^c$ is available, an updated estimate of the level state $\hat{\mu}_{i,j}$ can be made for LLM (*correction* equations)

$$\hat{\mu}_{i,j} = \hat{\mu}_{i,j|j-1} + \frac{p_{i,j|j-1}}{f_{i,j}}(y_{i,j}^c - \hat{\mu}_{i,j|j-1}) \tag{3.14a}$$

$$p_{i,j} = p_{i,j|j-1} - \frac{p_{j|j-1}^2}{f_{i,j}} \tag{3.14b}$$

$$f_{i,j} = p_{i,j|j-1} + \sigma_\varepsilon^2. \tag{3.14c}$$

Univariate variables $p_{i,j|j-1}$ and $p_{i,j}$ are *a-priori* and *a-posteriori* state covariance values. The output error covariance values associated with the model are denoted as $f_{i,j}$. For the model error compensation variant, (2.32) applied to LLM yields a modified state error covariance matrix:

$$p_{i,j} = \sigma_\varepsilon^2. \tag{3.15}$$

Similarly, if the discrete time approximation is used ($\delta_{i,j,j-1} = 1$), a discrete error correction equations can be formulated (Jazwinski, 1970):

$$\mathrm{Var}(\eta_j) = \alpha^{-2(j-1)} q \delta_{i,j,j-1} \sigma_\varepsilon^2 \tag{3.16a}$$

$$\mathrm{Var}(\varepsilon_j) = \alpha^{-2(j-1)} \delta_{i,j,j-1} \sigma_\varepsilon^2 I_{J_j \times J_j}, \tag{3.16b}$$

which superseed the corresponding covariance parameters from (3.8). Note that the equation (3.10) uses recursive calculations for the Kalman Gain $K_{i,j}$, which is beneficial in terms of algorithm implementation (Simon, 2006). For the LLT model, using $Z_t = \begin{bmatrix} 1 & 0 \end{bmatrix}$ as time invariant output vector, the state correction equations become:

$$\hat{\mu}_{i,j} = \hat{\mu}_{i,j|j-1} + \frac{P_{i,j|j-1} Z_t'}{f_{i,j}} (y_{i,j}^c - \hat{\mu}_{i,j|j-1}). \tag{3.17a}$$

$$P_{i,j} = P_{i,j|j-1} - \frac{P_{i,j|j-1} Z_t' Z_t P_{i,j|j-1}}{f_{i,j}} \tag{3.17b}$$

$$f_{i,j} = Z_t P_{i,j|j-1} Z_t' + \sigma_\varepsilon^2, \tag{3.17c}$$

having the same notational convention as in (3.14) having $P_{i,j|j-1}$ and $P_{i,j}$ as *a-priori* and *a-posteriori* state covariance matrices. Similarly, in case of model error compensation approach, (2.32) applied to LLT yields a modified state error covariance matrix:

$$P_{i,j} = \sigma_\varepsilon^2 I_{2\times 2}. \tag{3.18}$$

For purposes of ordering and easier understanding of the model results in the further text, abbreviations of trend model names are put forward in Table 3.1, together with the system of equations defining the recursive algorithm for their implementation. It should be noted that the abbreviations can be used both as model names and time series results from forecasting using those models, interchangeably.

TABLE 3.1: Kalman Filter Model Mapping

Model	**Description**	**Equations**
LLM_XXX_c	Continuous time LLM	(3.7)(3.8)(3.14)
LLM_XXX_m	Modified continuous time LLM with engineered state state error covariance matrix KF	(3.7)(3.8a)(3.8b) (3.9)(3.15)
LLT_XXX_c	Continuous time LLT	(3.7)(3.11)(3.12) (3.17)
LLT_XXX_m	Modified continuous time LLM with engineered state error covariance matrix KF	(3.7)(3.11)(3.13)(3.17a)(3.17c)(3.18)
LLM_XXX_dm	Modified discrete time LLM with exponentially weighted state error covariance matrix KF	(3.7)(3.8)(3.14) (3.16)
LLM_XXX_mod1	Modified continuous time LLM with exponentially weighted state error covariance matrix KF	(3.7)(3.8a)(3.8b) (3.10)

[1] _*XXX* identifies the (measured) trend output variable of the KF model

The CL stage uses $\hat{y}_{i,j}$ as a known, albeit conditional, variable in the following utility function $U_{i,j}$, together with the vector of exogenous attributes $x_{i,j}$. This yields an expression for the utility function used under normal CL assumptions regarding the statistical properties of the disturbance $\epsilon_{i,j}$:

$$U_{i,j} = \boldsymbol{\beta_{ex}} x_{i,j} + \beta_y \hat{y}_{i,j} + \epsilon_{i,j}; \epsilon_{i,j} \sim \text{Gumbel}(0,1), \tag{3.19}$$

with the linear parameters vector β and the endogenous trend coefficient β_y defining the representative utility part, so that the probability to select an alternative becomes:

$$P_{i,j} = \frac{\exp(\boldsymbol{\beta_{ex}} x_{i,j} + \beta_y \hat{y}_{i,j})}{\sum\limits_k \exp(\boldsymbol{\beta_{ex}} x_{k,j} + \beta_y \hat{y}_{k,j})}. \tag{3.20}$$

Revealed ranking of choices $y_{i,j}$ after the decision event j is taken into account through the inflated continuous feedback variable $y^c_{i,j}$. Obviously, the choice $y^c_{i,j}$ is not known before the decision event j takes place. Consequently, $\hat{y}_{i,j}$ is actually an estimate of the trend of the revealed ordered preference at the time of the decision event j, based on the information on revealed preference known at the time of the decision event $j-1$. Moreover, this means that the denominator in (3.20) includes estimates of the trends of all alternatives in the choice set J_i, which have different previous decision events as a basis for the estimate of the value $\hat{y}_{k,j}$.

Careful study of equations (3.7) to (3.19) shows that a two-step calculation procedure can be applied to simplify the algorithm implementation, since there is no direct feedback from the CL results (3.20) to the KF stage. In other words, that the KF results feed directly the CL stage and can be calculated separately, whilst the actual results of the DM process are treated as known for states correction after the race. This approach is in alignment with previous studies conducted in similar setups (Sung and Johnson, 2007). The major reason for a two-step approach is the inherent linearity of the KF step and the inherent non-linearity of the CL step. A single stage strategy based on KF prediction (i.e. forecast of the inflated ranking) would be clearly inappropriate because of three reasons. First, the forecasts do not take properties of other runners in a race, i.e. they do not incorporate competition and (wrongly) assume that the performance of the runners is independent (Lessmann et al., 2009). Second, there is no good way to compensate for the variances of the forecasts for the ranking. Finally, the KF does not provide an estimate of the ranking probabilities needed for formulation of any sensible betting strategy.

On the other hand, the classical CL naturally models within-race competition and it is expected that it dominates the single stage KF approach. However, a property of the CL regression is that if the independent variables of vastly different relative contribution to prediction (prediction power) are used together, the variables with the smaller prediction power will be over proportionally diminished compared to the dominating variables. This masks the subtle relationships amongst them and effectively reduces the information set (Sung and Johnson, 2007). In the horse-racing context, the market variables, such as winning odds, clearly dominate other track variables (e.g. horse and/or jockey related fundamentals) and in order to counter that, a two-step approach KF/CL is endorsed.

It should be noted, that the model defined in this section represents a definition of a dynamic discrete choice model which takes the trends of revealed ordered preferences into account, as required by Research Objective 1. However, the model, as it stands here, is not operational, since an initialisation procedure for the recursion and a method of parameter fitting have to be specified for model completeness. Both missing elements are explained in Section 3.3.

3.3 Research Design

This section provides a link between the model structure defined by equations (3.1) – (3.20), Research Objectives as defined in Chapter 1, types of data needed to answer the Research Questions as defined in Section 2.3, and the statistical and empirical testing of the resulting model performance leveraged to provide answers to Research Questions. The research design outlines the approach on data collection and data preprocessing needed for empirical testing of the obtained models and conclusions that can be drawn from the test results.

3.3.1 Research Design and Horse Racing as Empirical Test Setup

In the context of discrete decision-making model definition and testing, a set of horse races can be interpreted as a sequence of decision events in which decisions are made by one abstract decision-maker, namely ‘nature’. In every race, ‘nature’ specifies an order of runners from the field based on a (externally unobserved) utility. This, in turn, yields the winner, the runner-up, and the finishing positions of all other runners in a race. In this constellation, ‘nature’ acts rationally and chooses the ‘best’ horse at the time of the race (Bolton and Chapman, 1986). Obviously, each race has a different choice set (different runners), and the number of runners (size of the choice set) in a race, which may vary from two to well above twenty. Moreover, the role of the researcher that observes the decisions made, as defined in Chapter 1, is given to betting market participants (i.e. bettors) engaging in transactions based on expectations (forecasts). Obviously, all market participants base their predictions regarding the decisions to be made on explicit or implicit models behind ‘behavioural’ process underlying the selection. The variables of the mentioned prediction models can be split into two groups: (1) market-based variables, reflecting *ex-ante* public opinion about the outcome of the race and (2) physical variables, reflecting the environmental influences and intrinsic characteristics of the runners. Bettors establish their models by weighting different predictive variables and are acting in accordance with the predicted probabilities of particular race results. It should be noted that every agent freely selects the variables and their weightings to formulate mathematical tools and/or models relevant to market activities. In order to compare the usefulness of a particular decision-making tool, which would facilitate the prediction of the race results, it is assumed that all bettors utilise the same set of variables for modelling, albeit they may differ in their information extraction skills. In that case, a particularly well-informed bettor, capable of extracting additional information from the 'behaviour' of the 'nature' as a decision-maker, would have a decisive advantage over other market participants. This is equivalent to the ability to refine the statistical properties of the unobserved portion of the underlying utility, as seen in the model (2.1), and use it as an additional variable. However, such an advantage is not easy to obtain, and fairly sophisticated methods are therefore needed to 'beat' the market, which underscores the suitability of the horse-wagering markets as a test setup for different statistical modelling methods. It is therefore obvious, that the DCM described in section 3.2 directly meets the Research Objective 1, since it considers the ordered preference information from previous decision events, in order to construct an endogenous trend, and the horse-racing setting intrinsically contains changing choice sets (i.e. 'no race is the same') with different numbers of runners in each race.

In addition, over a racing career consisting of a number of professional races, a runner will have a number of involvements with varying time periods between participation in subsequent races, thus defining a setup appropriate for fulfilling the irregular sampling portion of the Research Objective 2. Furthermore, if the model involves relatively efficient market variables (in the sense of prediction power regarding the likelihood of a particular horse winning) a statistically significant improvement obtained from data processing publicly available data is challenging. In this case, it can be claimed that the model extracts 'residual' information from publicly available data, which is not incorporated in the runner's odds for winning a race. The selected research design is deployed to demonstrate that the latent state information in SSMs, updated by the KF, can be used for extracting residual information not fully discounted by market participants, which directly meets the both proposed research objectives.

Finally, residual information should provide a competitive edge to a savvy bettor and allow him to make a profit (i.e. to 'beat' the market) through the consequent application of those betting strategies that take into account the additional information.

The outlined research design and methodology indicate the broad types of variables needed to establish logical conclusions leading to answers to the research questions. The most important type of data are related to publicly available horse-racing market information and are split into two groups – race-level data and runner-level data. Race-level data have to include information regarding the decision event itself (i.e. cross-sectional data), such as the race date, identification of the runners participating in the race and, for the *ex-post* evaluation, the result of the race as an ordered list. In addition, the best single predictor regarding the runner's probability of winning (betting odds) has to be available both for comparison of different variables and for evaluation of betting strategies. As expected, these kinds of data are easily obtainable, since they are in the public domain and are traditionally of interest to bookmakers, professional bettors, and the general public alike. The second type of race-level data includes the endogenous variables (fundamentals), which are seen as predictors of the performance of runners in the race at hand, such as prize money, race category, type of ground, etc. This type of data can be used effectively either as fundamental values or as weightings of previous outcomes. Obviously, all of these variables are common to all runners in the considered race.

Runner-level data, on the other hand, are bound to current and previous characteristics of a horse, developed and tracked either throughout its career or derived from the circumstances of a particular race (e.g. weight, draw, gender, sire, jockey rating, beaten lengths, etc.). All data types identified until now are in principle commercially available, even if some pre-processing may be necessary before they can be used for modelling.

The last category of data encompasses data generated through the statistical processing of *ex-post* data, i.e. ordered data available after a race (placement). Here, race placement data can be seen as an alternative-specific proxy of the preference. In the context of horse-racing, the revealed ordered preference can be also seen as a proxy of the performance of a runner in the previous race. Obviously, the simplest alternative-specific proxy of the performance is the continuous (*inflated*) race finish order *NFP*, c.f. (3.1). This proxy has a simple interpretation: in every race, independently of the number of runners, the winning horse has the performance proxy of 0.5 and the last horse has the performance proxy of -0.5. The performance of all other horses sits between these two values. In races with odd numbers of runners, the performance 0.0 marks the middle field.

As specified in Research Objective 1, the proxy of the preference has to be fed back to the model in order to be used to predict the behaviour of a decision-maker, i.e. to predict the outcome of the next decision event (race). Moreover, the same research objective requires that a proxy trend has to be constructed and incorporated as an endogenous prediction variable. An endogenous trend can be generated either through simple data pre-processing (e.g. Moving Average (MA)) or through the application or more sophisticated statistical data processing algorithms (e.g. classical linear/non-linear filtering or KF). Applying these algorithms to the inflated performance proxy, either individually or in combination with other market- and runner-level data, can potentially yield predictors of the performance of a runner in the considered race.

In addition, more complex performance proxies are imaginable which may incorporate other variables into one improved performance proxy. For example, including beaten lengths information may indicate that a win was possibly after a 'dead heat', thus witnessing minimal performance advantage over the runner-up, or it may show that a runner outclassed all other runners, thereby witnessing a very high level of racing performance in that race. Moreover, the monetary prize bound to a particular race may indicate the relative attractiveness of a race compared to other races. Arguably, races with higher monetary compensation are more attractive for owners and jockeys, resulting in stronger competition in the field (participation of better horses/jockeys), and a win in such a high stakes race is therefore an indicator of better relative performance. This weighting of the performance proxy is derived from the relative importance of a race as a decision event. Naturally, the weightings of the *NFP* can be combined and build a complex proxy. In this study, prize moneys and beaten length weighting of *NFP* are combined to build an 'ultimate performance' *ex-post* proxy (*ULTIPERF*, also abbreviated *UP*) which, together with other runner- or race-level variables, can be used to generate *ex-ante* forecasts of performance proxies.

Trend-building, as specified in the Research Objective 1, takes the time evolution of performance into account. Translated into a countable number of decision events, the time is equivalent to (recency) weighting of a performance proxy, i.e. how much time has passed from the last captured performance, expressed through the time distance between a runner's successive races. Intuitively, a recent performance proxy should be more reliable than a performance proxy from the distant past, simply because of the greater uncertainty regarding intrinsic form (performance) development over time.

Starting from the simplest performance proxy *LAGGED_NFP*, the following endogenous variables are designed in order to provide an infrastructure that will help address all elements of the research objectives. Combinations and comparisons of the generated variables allow for inference regarding the performance of different models generating trends used for predicting discrete decision outcomes. The auxiliary variables are as follows:

(a) Lagged *NFP* from the last race in which the runner participated – *LAGGED_NFP*

(b) Cumulative moving average of *NFP* of a runner – *MA_NFP*

(c) The KF produced trend prediction of *NFP* with or without recency weighting – *KF_NFP_xxx*

(d) Lagged *UP* from the last race in which the runner participated – *LAGGED_UP*

(e) Cumulative moving average of *UP* of a runner – *MA_UP*

(f) The KF produced trend prediction of *ULTIPERF* with or without recency weighting – *KF_UP_xxx*

Note that *_xxx* stands for a particular KF model structure, as mapped in Table 3.1.

Combinations of designed proxy variables with the racing odds allow the development of a range of models for statistical and empirical testing used to answer research questions, posed for the purpose of the easier demonstration of how the research objectives are met and that the identified gaps in the literature are closed. These models and the verification methodology designed to test them are summarised in the next section on empirical test planning (cf. Table 3.2, Table 3.3, Table 3.4, and Table 3.5).

3.3.2 Empirical Test Planning

A fundamental part of any research design is the methodology, which when applied to the empirical data yields results, evidence, and finally conclusions, needed to answer the set research questions. This chapter outlines the methodology used for testing/analysis and the logic applied to answer primary and secondary research questions.

Research Question 1 called for a statistical evaluation of the informational content embedded in patterns of previous choices, that could explain an additional part of the unobserved portion of the utility. Starting from a static CL model, using the best single predictor of a win in a race (i.e. winning odds) as a baseline, the statistical significance of the contribution of endogenous trend variables will be confirmed or rejected through the Wald test (i.e. the Z-test) of significance in the univariate case and Likelihood Ratio (LR) testing in the bivariate case (c.f. 3.3.4). If any of the statistical tests confirms the statistical significance of any endogenous trend variable, an affirmative answer to the Research Question 1 can be given. Furthermore, if the tests reject the statistical significance, a conclusion can be reached that the designed endogenous trend variables do not extract any additional information on the uninformed portion of the utility. This, however, does not allow for a conclusion that there are no endogenous trend variables with the embedded information, but rather that the designed trend variables perform poorly when extracting embedded information not reflected in the winning odds.

The associated secondary Research Question 1a called for determining the level of the competitive advantage increase (gain over uninformed bettors) that a savvy market agent, aiming to forecast an outcome of a decision event, experiences in form of increased gains from the additional information set, as investigated in the Research Question 1. To answer this question, the financial performance of the Kelly betting strategy (see (3.33)) is evaluated, especially in terms of total profit and ROI. If the model including a (significant) trend variable outperforms the model including only the winning odds, the difference in the total profit and ROI yields a proxy of the competitive advantage increase. Obviously, the answer to this question is not a dichotomous yes/no statement, but a set of continuous values. Table 3.2 summarises the empirical testing with respect to the Research Question 1 and 1a.

TABLE 3.2: Research Design - Summary of variable comparison for RQ1

VUT	Base	Goal	Evidence
Research Question 1 - Normalised Finishing Position Tests			
LAGGED_NFP	-	Significance Testing	zTest and p-R^2
MA_NFP	-	Significance Testing	zTest and p-R^2
LLM_NFP_dm	-	Significance Testing	zTest and p-R^2
LAGGED_NFP	*LOGPRICE*	Significance Testing	LR Test and p-R^2
MA_NFP	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_NFP_dm	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LAGGED_NFP	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
MA_NFP	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLM_NFP_dm	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit

[1] Statistical significance is assessed both in- and out-of-sample. Kelly betting results are assessed only out-of-sample.

Research Question 2 called for a statistical evaluation of the informational content embedded in patterns of previous choices, which additionally take into account the temporal distance between successive decision events (i.e. a ‘forgetting’ factor) that could explain an additional part of the unobserved portion of the utility. Starting from a static CL model, using the best single predictor of a win in a race (i.e. winning odds) as a baseline, the statistical significance of the contribution of the endogenous trend variables will be confirmed or rejected through the Wald test (i.e. the Z-test) of significance in the univariate case and LR testing in the bivariate case (c.f. 3.3.4). If any of the tests confirms the statistical significance of any endogenous trend variable, an affirmative answer to the Research Question 2 can be given. Furthermore, if the tests reject statistical significance, a conclusion can be drawn that the designed endogenous trend variables do not extract any additional information on the uninformed portion of the utility. This does not allow for a conclusion that there are no endogenous trend variables with the embedded information, but rather that the designed trend variables perform poorly in extracting embedded information not reflected in the winning odds.

The associated secondary Research Question 2a called for determining the level of competitive advantage increase (gain over uninformed bettors) that a savvy market agent aiming, to forecast an outcome of a decision event, experiences in form of increased gains from the additional information set, as investigated in Research Question 2. To answer this question, the financial performance of the Kelly betting strategy (see (3.33)) is evaluated, especially in terms of total profit and ROI. If the model including a (significant) trend variable outperforms the model including only winning odds, the difference in the total profit and the ROI yields a proxy of competitive advantage increase. Obviously, the answer to this question is not a dichotomous yes/no statement but a set of continuous values. Table 3.3 summarises the empirical testing with respect to the Research Question 2 and 2a.

Table 3.3: Research Design - Summary of variable comparison for RQ2

VUT	**Base**	**Goal**	**Evidence**
Research Question 2 - Recency Weighted Normalised Finishing Position Tests			
LLM_NFP_c	-	Significance Testing	zTest and p-R^2
LLM_NFP_m	-	Significance Testing	zTest and p-R^2
LLT_NFP_c	-	Significance Testing	zTest and p-R^2
LLT_NFP_m	-	Significance Testing	zTest and p-R^2
LLM_NFP_mod1	-	Significance Testing	zTest and p-R^2
LLM_NFP_c	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_NFP_m	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLT_NFP_c	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLT_NFP_m	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_NFP_mod1	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_NFP_c	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLM_NFP_m	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLT_NFP_c	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLT_NFP_m	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLM_NFP_mod1	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit

[1] Statistical significance is assessed both in- and out-of-sample. Kelly betting results are assessed only out-of-sample.

Research Question 3 called for a statistical evaluation of the informational content embedded in patterns of previous choices and which additionally consider importance weighting, i.e. the attribution of higher fidelity scores to decision events having larger pay-offs, in order to explain an additional part of the unobserved portion of the utility. Starting from a static CL model, using the best single predictor of a win in a race (i.e. winning odds) as a baseline, the statistical significance of the contribution of endogenous trend variables will be confirmed or rejected through the Wald test (i.e. the Z-test) of significance in the univariate case and LR testing in the bivariate case (c.f. 3.3.4). If any of the statistical tests confirms the statistical significance of any endogenous trend variable, an affirmative answer to the Research Question 3 can be given. Furthermore, if the statistical tests rejects the statistical significance, a conclusion can be drawn that the designed endogenous trend variables do not extract any additional information on the uninformed portion of the utility. This does not allow for a conclusion that there are no endogenous trend variables with the embedded information, but rather that the designed trend variables perform poorly in extracting embedded information not reflected in the winning odds.

The associated secondary Research Question 3a called for determining the level of competitive advantage increase (gain over uninformed bettors) that a savvy market agent, aiming to forecast an outcome of a decision event, experiences in form of increased gains from the additional information set, as investigated in Research Question 3. To answer this question, the financial performance of the Kelly betting strategy (see (3.33)) is evaluated, especially in terms of total profit and ROI. If the model including a (significant) trend variable outperforms the model including only winning odds, the difference in the total profit and the ROI yields a proxy of the competitive advantage increase. Obviously, the answer to this question is not a dichotomous yes/no statement but a set of continuous values. Table 3.4 summarises the empirical testing with respect to the Research Question 3 and 3a.

TABLE 3.4: Research Design - Summary of variable comparison for RQ3

VUT	**Base**	**Goal**	**Evidence**
Research Question 3 - Importance Weighted Normalised Finishing Position Tests			
LAGGED_UP	-	Significance Testing	zTest and p-R^2
MA_UP	-	Significance Testing	zTest and p-R^2
LLM_UP_dm	-	Significance Testing	zTest and p-R^2
LAGGED_UP	*LOGPRICE*	Significance Testing	LR Test and p-R^2
MA_UP	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_UP_dm	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LAGGED_UP	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
MA_UP	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLM_UP_dm	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit

[1] Statistical significance is assessed both in- and out-of-sample. Kelly betting results are assessed only out-of-sample.

Research Question 4 called for a statistical evaluation of the informational content embedded in patterns of previous choices, which additionally consider the importance of weighting and the temporal distance between successive decision events, and which could explain an additional part of the unobserved portion of the utility. Starting from a static CL model, using the best single predictor of a win in a race (i.e. winning odds) as a baseline, the statistical significance of the contribution of endogenous trend variables will be confirmed or rejected through the Wald test (i.e. the Z-test) of significance in the univariate case and LR testing in the bivariate case (c.f. 3.3.4). If any of statistical tests confirms the statistical significance of any endogenous trend variable, an affirmative answer to the Research Question 4 can be given. Furthermore, if the tests reject the statistical significance, a conclusion can be reached that the designed endogenous trend variables do not extract any additional information on the uninformed portion of the utility. This does not allow for a conclusion that there are no endogenous trend variables with the embedded information, but rather that the designed trend variables perform poorly in extracting embedded information not reflected in the winning odds.

The associated secondary Research Question 4a called for determining the level of competitive advantage increase (gain over uninformed bettors) that a savvy market agent, aiming to forecast an outcome of a decision event, experiences in form of increased gains from the additional information set, as investigated in Research Question 4. To answer this question, the financial performance of the Kelly betting strategy (see (3.33)) is evaluated, especially in terms of total profit and ROI. If the model including a (significant) trend variable outperforms the model including only the winning odds, the difference in the total profit and the ROI yields a proxy of the competitive advantage increase. Obviously, the answer to this question is not a dichotomous yes/no statement, but a set of continuous values. Table 3.5 summarises the empirical testing with respect to the Research Question 4 and 4a.

TABLE 3.5: Research Design - Summary of variable comparison for RQ4

VUT	**Base**	**Goal**	**Evidence**
Research Question 4 - Recency and Importance Weighted Normalised Finishing Position Tests			
LLM_UP_c	-	Significance Testing	zTest and p-R^2
LLM_UP_m	-	Significance Testing	zTest and p-R^2
LLT_UP_c	-	Significance Testing	zTest and p-R^2
LLT_UP_m	-	Significance Testing	zTest and p-R^2
LLM_UP_mod1	-	Significance Testing	zTest and p-R^2
LLM_UP_c	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_UP_m	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLT_UP_c	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLT_UP_m	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_UP_mod1	*LOGPRICE*	Significance Testing	LR Test and p-R^2
LLM_UP_c	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLM_UP_m	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLT_UP_c	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLT_UP_m	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit
LLM_UP_mod1	*LOGPRICE*	Kelly Betting Strategy	ROI and Profit

[1] Statistical significance is assessed both in- and out-of-sample. Kelly betting results are assessed only out-of-sample.

It should be noted, that the research design is robust against the non-uniqueness of how the trend variables are designed, and any designed variable which incorporates additional information in a statistically significant way is deemed sufficient to answer a research question in an affirmative way. On the other hand, designed variables which do not yield an affirmative answer can be discarded without negative conclusion and alternatives could be sought.

Since all research questions can be answered in full by the research design outlined here and the research questions cover all aspects of the research objectives and closure of the identified gaps in the literature, it can be concluded that the research strategy is adequate to meet the research objectives.

Figure 3.3 shows step-by-step algorithm of the quantitative data generation, prediction, test, and analysis, defined as a sequential procedure applicable to every model under test.

Algorithm Input

Learn Data Set (**LDS**)
Burn-In Data Set (**BDS**)
Validation Data Set (**VDS**)
All data subsets contain:
- *Horse ID*
- *RaceID*
- Date of the race (*t*)
- Log of final odds of the race (*LOGODDS*)
- Starting Price (*DECIMALODDS*)
- Ranked finishing order of runners
- Normalised Finishing Position (*NFP*) (after the race)
- *ULTIPERF* – Importance weighted *NFP* (based on Prize money and Beaten Lengths)

Step1: Kalman Filter (KF) Stage

Make time series data for every runner based on the HORSEID and the date of the race (t) over the length of their careers
Estimate Θ_{KF} parameters of all KF defined trend model structures by maximisation of the Log Likelihood over **LDS** and all runners within. (Eq. 3.23 – 3.25)

Aposteriori data for KF used for trend correction are *NFP* and *ULTIPERF*
Set diffuse prior initialisation for every runner and every model
Calculate KF based prediction (trend) of *NFP* and *ULTIPERF* for every runner over **FDS** using (Eq. 3.3 – 3.18).
Reorder the time series causal predictions according to races (ordered by *RaceID*) and dates

Step 2: Conditional Logit (CL) Stage

Substitute Kalman Filter trend predictions of the first two races with odds implied winning probabilities (Apendix A)
Estimate Θ_{CL} parameters of all CL models defined in research design section (Eq. 3.26 - 3.27)

Calculate winning probabilities (Eq. 3.28) for all races in **VDS**

Model Evaluation

Eliminate tainted (Sec. 3.3.5) races in **VDS**
Perform statistical tests of significance – Wald and Likehod Ratio tests (Eq. 3.29-3.32)
Evaluate Kelly betting strategies on all models using *DECIMALODDS* variable **VDS** data set (Eq. 3.33)

FIGURE 3.3: Algorithm of the quantitative data prediction and analysis

3.3.2.1 Kalman Filter Initialisation

Until now, all discussed SSM models were defined under the assumption that the initial distribution of the states is *a-priori* known or set to some arbitrary values. Indeed, in the vast majority of engineering applications, due to the fast sampling of the observations, errors due to the wrong initialisation diminish fast and can typically be neglected. On the other hand, *a-priori* knowledge of the initial distribution of the states, especially for non-stationary models, is rarely given in econometrics. This led some statisticians to conclusion, that KF is of less practical value unless a Bayesian approach, which specifies the proper prior distribution for the initial state $\alpha_{0|0}$, is applied. For the horse racing setting it has to be assumed that there is no prior knowledge and that every runner has its own improper prior, called also a *diffuse* prior.

Diffuse prior is defined through the assumption that the improper prior $\alpha_{0|0} \sim \mathcal{N}(0, \kappa\mathbf{I}_n)$ is a result of the limit process $\kappa \to \infty$. In the context of horse-racing, and taking for example *NFP* as the univariate observed variable, this prior implies that the horse imaginary past performance is equal to 0.0 (i.e. middle of the field) and that that there is infinite uncertainty (variance) regarding the past performance - i.e. complete lack of information. Moreover, the prior is improper, since its PDF does not integrate to one (Harvey, 1990). In general, classical KF algorithm cannot be initiated with infinite variance. An approximation with a large κ, for example $\kappa \simeq 10^7$ may be used, albeit with caution, as it may lead to considerable numerical problems. Alternatively, the exact initialisation algorithm described by Durbin and Koopman (2012), which effectively constructs a proper prior from the initial sequence of observations with the length equal to the order of the SSM and afterwards switches to the classical algorithm may be used. Obviously, this sets a limit for the minimum sequence length for initialisation, which is equal to the order of the SSM. MATLAB function `dssm` implements the diffuse model (The MathWorks Inc., 2017) and it is used for model initialisations in this study.

3.3.3 Model Parameter Estimation

Estimation of the parameters of the selected CL utility structure, together with the underlying KF model, is the final step in operationalisation of the DCM model. The following subsection depicts the Maximum Likelihood Estimation (MLE) approach of parameters estimation conditional on the observations from Learn Data Set (LDS). In alignment with the racing context, model parameter estimation is achieved through the maximisation (hence the name MLE) of the likelihood that the decision maker (‘nature’) made the observed series of choices, taking into account all races and all runners’ careers. In the case of DCM, the likelihood function is given as a product of the probabilities that particular observed outcomes have occurred taking all races into account, i.e. the likelihood is calculated over all N races in the LDS. The parameter estimates, resulting from the (numerical) maximisation of the likelihood function is than used in Burn-in Data Set (BDS) and Validation Data Set (VDS) for out-of-sample model validation. The fact that mixed discrete and continuous (and conditional) distributions are used is of no importance for the derivation (Merkle, 2002).

In mathematical terms, the maximisation with respect to sought parameters $\boldsymbol{\theta}$:

$$\boldsymbol{\theta} = \arg\max_{\boldsymbol{\theta}}(\ell(\boldsymbol{\theta}|\boldsymbol{D^1},\ldots,\boldsymbol{D^N},\boldsymbol{\hat{Y}^1},\boldsymbol{\hat{Y}^2},...,\boldsymbol{\hat{Y}^N},\boldsymbol{X^1},\ldots,\boldsymbol{X^N}) \tag{3.21}$$

with $\hat{Y}_j = [\hat{y}_{1,j}, \hat{y}_{2,j}, \ldots \hat{y}_{J_j,j}]$ as vector of predicted endogenous trend variables (3.3) for each runner i from the choice set of size J_j in race j. The vectorised result of the race $D^j = [d_{1,j}, \ldots, d_{i,j}, d_{J_j,j}]$ is constructed with the convention $d_{i,j} = 1$ if runner i actually won the race j and zero otherwise. Independent and deterministic exogenous variables affecting the runner i in the race j $X_i^j = [x_{1,i,j}, x_{2,i,j}, \ldots x_{\omega,i,j}]$ are encapsulated in race level exogenous matrices $X^j = [X_1^j X_2^j \ldots X_{J_j}^j]$. The parameter ω denotes the total number of exogenous variables in the model.

A key element of MLE is the product of probabilities that the observed sample actually occurs. This is calculated as functional values of the joint PDF $f(\boldsymbol{D},\boldsymbol{X},\boldsymbol{\hat{Y}};\boldsymbol{\theta})$ for the given observation data point, under the assumption that the PDF of the sample $(\boldsymbol{D},\boldsymbol{X},\boldsymbol{\hat{Y}})$ is a member of a family of functions parametrised by $\boldsymbol{\theta}$. In addition, the fact that the values $\boldsymbol{\hat{Y}}$ are actually realisations of random variables allows the separation of the joint density in product of a conditional and a marginal density

$$f(\boldsymbol{D},\boldsymbol{X},\boldsymbol{\hat{Y}};\boldsymbol{\theta}) = f(\boldsymbol{D},\boldsymbol{X}|\boldsymbol{\hat{Y}};\boldsymbol{\theta_{CL}})f(\boldsymbol{\hat{Y}};\boldsymbol{\theta_{KF}}), \tag{3.22}$$

where $\boldsymbol{\theta_{CL}}$ is the subset of the parameter vector $\boldsymbol{\theta}$ that parametrise the conditional density function of the CL stage and $\boldsymbol{\theta_{KF}}$ is the subset parametrising the marginal density function bound to KF stage, i.e. $\boldsymbol{\theta} = [\boldsymbol{\theta_{CL}}, \boldsymbol{\theta_{KF}}]$, as defined in discussion after equation (3.6). In addition, under the assumption that there is no functional relationship between $\boldsymbol{\theta_{CL}}$ and $\boldsymbol{\theta_{KF}}$, MLE of $\boldsymbol{\theta}$ is achieved through separate maximisation of the conditional likelihood built around the product of $f(\boldsymbol{D},\boldsymbol{X}|\boldsymbol{\hat{Y}};\boldsymbol{\theta_{CL}})$ with respect to $\boldsymbol{\theta_{CL}}$ and maximisation of the product of marginal likelihoods $(\boldsymbol{\hat{Y}};\boldsymbol{\theta_{KF}})$ with respect to $\boldsymbol{\theta_{KF}}$ (Hayashi, 2000). This property follows directly from Fubini's theorem (Pilipović and Seleši, 2012).

Furthermore, it should be appreciated that the marginal density function of the KF stage is conditional on inflated rankings from the previous races. Henceforth, the following convention is introduced; every random variable $\hat{y}_i^j$ is considered conditional on previous observations derived from the *ex-post* rankings $\check{Y}_i^{N_i} = [y_i^{N_i}, y_i^{N_i-1}, \ldots, y_i^1]'$ throughout the whole racing career of the runner i, i.e. all N_i races before the current race j. Note that this does not state anything about the actual time passed between two successive races in the past. In addition, notation y_i^0 does not imply that an actual 'zero-th' race took place, rather it represents unconditional *a-priori* ranking in a virtual race before the actual début.

3.3.3.1 Maximum likelihood estimation of Kalman Filter parameters

Estimation of the parameters of the KF stage follows a logical extension of the classical prediction error decomposition and is taking into account that, due to the selected dynamic structure, the samples are not independent, so that the likelihood function for the runner i, over all N_i career races in the data set can be expressed as

$$\ell_i(y;\boldsymbol{\theta_{KF}}) = \prod_{k=1}^{N_i} p(y_k|\check{Y}_i^{k-1}), \tag{3.23}$$

where $p(y_k|\check{Y}_i^{k-1})$ is the PDF of y_k, conditional on previous realisations known at $k-1$, i.e. $\check{Y}_i^{k-1} = [y_i^{k-1}, y_i^{k-2}, \ldots, y_i^1]$.

Based on the properties of multivariable normal distribution and the Markov property regarding the states, the prediction error log likelihood functional form follows immediately from the KF formulas for the conditional mean and covariance (Harvey, 1990):

$$log(\ell_i) = -\frac{N_i - \Omega}{2}\log(2\pi) - \frac{1}{2}\sum_{k=\Omega+1}^{N_i}\log f_k - \frac{1}{2}\sum_{k=\Omega+1}^{N_i}\frac{\nu_k^2}{f_k}, \tag{3.24}$$

with ν_k as scalar innovation (prediction output error) equivalent to the term in bracket in (2.22a) and f_k as the output error covariance. Equation (3.24) is derived taking the diffuse prior initialisation into account. In absence of any prior information, the KF for (3.7) and (3.11) has to be initialised with a diffuse prior, as explained in Section 3.3.2.1. Fundamentally, this approach generates exact proper conditional distributions after Ω observations, with Ω being equal to the system order of the selected non stationary models. It should be noted that (3.24) is not the only way to specify the likelihood function. The exact type of the function is dependent on the assumptions regarding the initial condition. Fortunately, the process of parameter identification is not too sensitive to the actual form of the likelihood function for large samples, since all specifications have the same asymptotic properties. Small sample properties can, obviously, differ considerably.

In order to extract the maximum data content for the parameter estimation, all N runners have to be included in the following likelihood function

$$\ell(\boldsymbol{\theta_{KF}}) = \sum_{i=1}^{N} log(\ell_i). \tag{3.25}$$

In addition, this expression has to be maximised with respect to the hyper parameters $\boldsymbol{\theta_{KF}}$ using some appropriate numerical optimisation procedure, capable of imposing constraints on the parameters and preferably insensitive to relatively 'flat' likelihood functions. Both classical gradient based with numerical approximations for the first and second derivatives (e.g. MATLAB function `fminunc` with additional nonlinear transformations for constraints handling) and modern evolutionary optimisation algorithms (e.g. MATLAB function `particleswarm`) can be used sensibly.

After the successful parameter identification, KF can be run on all available sequences of the endogenous trend variables and construct the predictions $\hat{y}_i^j$, which will be used for the purpose of the identification of $\boldsymbol{\theta_{CL}}$, as *a-priori* known realisations.

3.3.3.2 Maximum Likelihood Estimation of Conditional Logit parameters

With all $\hat{y}_i^j$ known, the conditional likelihood function $f(\boldsymbol{D}, \boldsymbol{X}|\boldsymbol{\hat{Y}}; \boldsymbol{\theta_{CL}})$ can be minimised with respect to $\boldsymbol{\theta_{CL}}$ considering all data as exogenous. Following the definition of the likelihood function as a product of probabilities, it yields for N decision events, all with different choice sets J_j and the corresponding choice variable $d_{i,j}$ (equivalent to (2.9)):

$$\ell(\boldsymbol{\theta_{CL}}|\boldsymbol{D^1}, \ldots, \boldsymbol{D^N}, \boldsymbol{\hat{Y}^1}, \boldsymbol{\hat{Y}^2}, ..., \boldsymbol{\hat{Y}^N}, \boldsymbol{X^1}, \ldots, \boldsymbol{X^N}) = \prod_{j=1}^{N} \prod_{i=1}^{J_j} (P_{i,j})^{d_{i,j}}, \tag{3.26}$$

The logarithm of the likelihood function is globally concave for a linear combination of exogenous variables, which facilitates easier numerical optimisation (Train, 2009):

$$\underset{\boldsymbol{\theta_{CL}}}{\arg\max}(\log \ell(\boldsymbol{\theta_{CL}})) = \underset{\boldsymbol{\theta_{CL}}}{\arg\max}(\sum_{j=1}^{N} \sum_{i=1}^{J_j} d_{i,j} \log P_{i,j}). \tag{3.27}$$

In the above equation, $P_{i,j}$ is the associated conditional probability of runner i to win the race j and $\boldsymbol{\theta_{CL}} = [\boldsymbol{\beta'_{ex}} \quad \beta_y]$ is the vector of sought parameter, i.e. has the same meaning as in (3.20). In particular, since all non chosen alternatives are raised to the power of zero, only the conditional win probability of the runner that actually won the race remains in the equation, thus accelerating numerical calculations associated with the likelihood optimisation.

When considering the trends of the proxies as input variables, the equation for probabilities of a *logit* decision model (3.20) can still be used directly. However, since the trends of the proxies are random variables, the probability expressions become conditional. For the assumed linear functional form of ω independent exogenous variables $X_i^j = [x_{1,i,j}, x_{2,i,j}, \ldots x_{\omega,i,j}]'$ and the trend estimate variables $\hat{y}_i^j$ this yields:

$$P_{i,j}|\hat{y}_i^j, \hat{y}_2^j, ..., \hat{y}_{J_j}^j; \boldsymbol{\theta_{CL}} = \frac{\exp(\boldsymbol{\beta_{ex}} \boldsymbol{X_i^j} + \beta_y \hat{y}_i^j)}{\sum_{k=1}^{J_j} \exp(\boldsymbol{\beta_{ex}} \boldsymbol{X_k^j} + \beta_y \hat{y}_k^j)}. \tag{3.28}$$

Even though this equation does not look particularly complex, it should be noted that every random variable $\hat{y}_i^j$ is conditional on the previous placements in the races $y_i^{k_i}; k_i \in$

$[1, N_i]$ and the KF parameters $\boldsymbol{\theta_{KF}}$ (i.e. $\hat{y}_i^j | y_i^{N_i}, y_i^{N_i - 1}, \ldots, y_i^1; \boldsymbol{\theta_{KF}}$). This means that the KF step model parameter identification and subsequent evaluation of endogenous trend estimates $\hat{y}_i^j$ have to be carried out first.

Actual parameter optimisation can be performed using the same algorithm as in the KF stage, using conventional gradient-based or evolutionary routines. For the first type of algorithms, analytic derivations of the associated gradients and Hessians are available and can be used for acceleration of convergence (Train, 2009).

Parameters obtained following the MLE procedures in this section, can be readily used on out-of-sample data. This step is described in Chapter 4. Finally, parameter estimation using MLE in the context of two selected stages is asymptotically efficient and asymptotic normal, when sample size increases to infinity (Mladenović, 2005), thus promising reasonable performance.

3.3.4 Model Evaluation Strategy

In order to provide an objective evaluation of the relative merits of the proposed models, a metric, a set of measures, of the achieved performance are needed. Selection of summary metrics of performance of a non-linear DCMs is not a trivial task. Whilst for linear models some variant of MSE, implying a squared error loss function, is typically sufficient to provide a quick and robust measure of accuracy, both in-sample and out-of-sample, for DCM, a different approach is needed.

For example, McFadden (1974) pseudo-R^2 measures the improvement in data fit accuracy between the likelihoods of the null (uninformative or naïve) probability calculation model in which one or more parameters are set to zero ($\ell(0)$) and the full model under evaluation – $\ell(\hat{\theta})$ (Amemiya, 1981; Bolton and Chapman, 1986; Sung et al., 2016b, 2019). McFadden (1974) pseudo-R^2 is defined as follows:

$$\tilde{R}^2 = 1 - \frac{\ell(\hat{\theta})}{\ell(0)} \tag{3.29}$$

with ℓ defined as the maximised likelihood of the model (3.27) taking all available data into account. Even though $\tilde{R}^2$ somewhat resembles the R^2 measure from ordinary regression analysis it has a different interpretation. In the OLS setting, R^2 measures how good the independent variables explain the variance of the observed data around the mean or, as an alternative interpretation, measures the degree to which the dependent variable can be predicted by a linear combination of the independent variables (Hoel, 1984). For example, $R^2 = 0.69$ implies that 69% of the variability of the observed variable is explained by the model, with 31% of the remaining unexplained variance. Similarly, pseudo-R^2 values around zero indicate no improvement over the uninformative model (i.e. the model where all choice probabilities are equal). In other words, the model has no predictive power in that case. Conversely, if the maximised model likelihood $\ell(\hat{\theta})$ is much closer to zero than $\ell(0)$ (note that both likelihoods are always negative, since the marginal likelihood contribution from each alternative in a race is a value between zero and one), the fitted model outperforms the uninformative model. In practice, large values of pseudo $\tilde{R}^2$ are usually not to be expected, and "... values of 0.2 to 0.4 for ρ^2 represent an excellent fit"[1] (McFadden, 1979). Obviously, evaluation of $\tilde{R}^2$ in LDS and VDS will have different interpretation. $\tilde{R}^2_{LDS}$ reveals the measure of the model fit, whilst $\tilde{R}^2_{VDS}$ indicates the predictive power in out-of-sample data subset. In spite of the fact that pseudo-$\tilde{R}^2$ are often used in academic studies, they are dependent both on the exact definition of the measure and on the choice of model. In this study, only the combination of the pseudo-$\tilde{R}^2$ definition (3.29) and CL model structure is considered.

Another statistical test, which complements the pseudo-$\tilde{R}^2$ metric, is the LR test (Neyman and Pearson, 1928), derived from the principles of ML. The LR test is especially powerful when an asymmetrical contribution of independent variables is assumed. In its essence, it compares the likelihoods of two models, for example, M_a with the parameters $\hat{\theta}_a$ and M_b with the parameters $\hat{\theta}_b$, after created of M_a through augmentation of the M_b with additional explanatory variables. The test evaluates whether the log-likelihood of M_a is appreciably greater than the log-likelihood of M_b over the same sample, when controlling for the total number of additional independent variables κ – Degrees of Freedom (DOF). Statistics of the difference of the likelihoods has the asymptotic χ^2_κ distribution:

$$\lambda = 2[\ell(\hat{\theta}_a) - \ell(\hat{\theta}_b)] \ \sim \chi^2_\kappa. \tag{3.30}$$

The statistical rationale behind this test is that, if the maximum of the log-likelihood function of the base model does not differ too much from the value of the log-likelihood function maximised using the augmented model structure, than the contribution of the additional variables is not statistically significant. In other words, the base model can be seen as a constrained (restricted) sub-model of M_a (Maddala, 1977; Greene, 2008), i.e. the models are nested. Finally, the criterion for evaluating the statistical significance is met if $\lambda \geq c_{\alpha_0}$, where α_0 is a constant, dependent on DOF κ of the χ^2 distribution, defining the statistical power of the test. Note that the likelihood of the augmented model on the same sample is always greater, i.e. it will fit at least as well as the base model, possibly due to overfitting. In the context of horse-racing, the application is straight forward – the basic model includes the variables which are known to be good predictors of the runner's performance (e.g. *LOGPRICE*), which are then augmented with the variables under test.

In cases where univariate models have to be tested for significance, LR testing cannot be applied and an alternative statistic has to be sought. The Wald test (Dinardo et al., 1997) has a similar testing power as LR, with the advantage that the evaluation of significance can be made immediately after parameter fitting and can be applied in situations where

[1] In the original text ρ is used instead of R

practically no other statistical test of variable significance can be applied. The Wald test evaluates the null hypothesis, that a parameter is equal to some specified value, most frequently zero. The idea behind the test is, that if the test fails to reject the null hypothesis it indicates that eliminating the tested variable from the model will not substantially affect the model fit. In other words, if a variable has a weighting parameter (in this application – β_y) which is very small compared to its standard error is practically an error (noise) and, consequently, does not contribute significantly to prediction of the dependent variables (Davidson and MacKinnon, 1984).

The Wald test statistic is based on the weighted distance between the unrestricted parameter estimate ($\hat{\theta}$) and the value specified in the null hypothesis ($\hat{\theta}_0$). It can be shown that if the weighting is selected to be the inverse of the variance of the unrestricted parameter estimate ($\mathrm{Var}(\hat{\theta})$), the statistic asymptotically follows the χ^2 distribution, i.e.

$$W = \frac{(\hat{\theta} - \hat{\theta}_0)^2}{\mathrm{Var}(\hat{\theta})} \sim \chi^2_1. \tag{3.31}$$

However, for the single variable case and with $\hat{\theta}_0 = 0$, a simpler statistic defined as the square root of W can be used, becoming an equivalent of the Z-test. Indeed, based on the fact that the MLE are approximately Gaussian, and that their asymptotic variance is the Fisher information (Davidson and MacKinnon, 1984), the standard error of the parameter estimate ($\mathrm{se}(\hat{\theta})$) can be set as a denominator

$$\sqrt{W} = \frac{\hat{\theta}}{\mathrm{se}(\hat{\theta})} \sim \mathcal{N}(0, 1), \tag{3.32}$$

which in turn means that the statistic $\sqrt{W}$ asymptotically follows the Z distribution (i.e. the standardised Gauss distribution) (Davidson and MacKinnon, 1993). For the purposes of significance testing of CL parameters the value of $\sqrt{W} > 1.96$ to confirm the null hypothesis that the parameter estimate $\hat{\theta}$ is statistically different from zero with $p < 0.05$.

It can be shown that the Wald test statistic is greater than the LR test statistic taken for two nested models, which means that the superior test power resides with the LR test, even though the tests are asymptotically equivalent (Godfrey, 1991; Kmenta, 1997). Hence, for the purposes of this study, LR test is favoured in all cases when nested models can be assessed, i.e. in cases where more than one variable are included in a model, and when out-of-sample considerations are beneficial. Finally, an application specific metric of model quality, the Kelly (1956) wagering strategy, is used to ascertain profitability of the forecasts provided by the model. Moreover, Clements and Hendry (1998) argue that in cases when the forecasting errors evaluation is strongly non-linear – for example like in horse betting, where correct forecasts regarding top three placements is far more important than correct forecasts of horses which are 'out-of-the-money'– application specific forecasting performance measures dominate standard forecasting measures like MSE. The Kelly betting strategy proposes bet sizes b_i^j for every runner i in terms of the fraction of wealth before the race j, with the aim to maximise the expected log-payoff and, consequently, increase of wealth after the race. The following maximisation yields a set of bet sizes b_h^j:

$$\max_{b_h^j} \sum_{h=1}^{J_j} p_h^j \log(1 - \sum_{i=1}^{J_j} b_i^j + b_h^j r_h^j), \tag{3.33}$$

where h denotes a possible winner with the associated return r_h^j in the case of win (i.e. decimal odds) and p_h^j denotes the actual winning probability of the runner h. Increase of the wealth after the race is equal to the term in the brackets. Consequent application of the Kelly strategy on a large set of races is optimal in the sense that it maximizes the asymptotic rate of accumulated wealth growth (Johnson et al., 2006; McDonald, 2012). At the same time, due to the adaptive size of the wagers related to the level of accumulated wealth before the race, this strategy has zero probability of bankruptcy.

The underlying mechanism of the Kelly wagering strategy is that it compares the difference between the real p_h^j and the odds-implied winning probabilities of the runners and, subsequently, determines the size of the bets in a race through optimisation. Careful study of (3.33) reveals that the Kelly strategy may yield wagers on outcomes with negative expected returns in order to hedge bets in a given race. Obviously, this can be seen as a problematic proposition since the real probability of wins are not known in practice. Instead, they are proxied by estimates provided by the model under investigation. Hence, a modification of the strategy is implemented instead, in order to mitigate the inaccuracies in probability estimates (McDonald, 2012). In particular, the actual bets are calculated under the assumption that the estimated probabilities are correct, but only the 'value' bets, having positive expected returns, are actually placed. Furthermore, in spite of the optimal properties regarding the asymptotic growth of wealth, there are three possible caveats bound to the applied wagering strategy (MacLean et al., 1992, 2011): (1) the high volatility of the accumulated wealth caused by large bets on horses with high estimated probability of win (similar to small Sharpe ratio in finance (Sharpe, 1994)), (2) unfortunate loosing streaks of high stake bets can skew the evaluation of the strategy on short samples, and (3) the presence of tainted races (races in which the odds implied probability is greater than one), caused by one or more 'non-runners' can yield completely meaningless results. In order to counter these potential problems the stakes are limited to 10% of the aggregate wealth before the race and all out-of-sample tainted races are eliminated for the evaluation of the ROI.

For the purpose of the study, all above mentioned evaluation criteria are applied and compared, taking into account the data fit aspects and the economic importance of the model predictions.

3.4 Data Set

Empirical data used to test the validity and the contribution of the endorsed models was imported from the Betfair database of public UK racetrack information on horse races run from January 1st, 2007 until December 31st, 2012 on all-weather and turf racing courses. Betfair data are considered to have higher degree of public information discounted in the final odds compared to classical bookmaker odds since it reflects settled 'wisdom of crowds', obtained from many informed market participant, such as bookmakers, betting syndicates, and professional bettors (Sung and Johnson, 2007). They are shown to be more efficient predictors of the winning probabilities that the bookmaker odds (Smith and Williams, 2010).

The complete data set (i.e. FDS Data Set (FDS)) covers 42768 races with 43424 runners with a total of 436681 data points. The model identification and evaluation strategy leverages three non-overlapping data subsets; (1) LDS: with 21051 races between January 1st, 2007 to December 31st, 2009 for parameter estimation, both of KF and CL stages, (2) BDS: with 7302 races between January 1st, 2010 to December 31st, 2010 for data *'burn-in'*, inserted as a middle layer to allow the error of unconditional KF initialisations to converge to zero, and (3) VDS: with 14415 races between January 1st, 2011 to December 31st, 2012, for out-of-sample model validations.

The first data subset – LDS – is treated as a training set used to estimate parameters of the KF and CL stages. The second data subset – BDS – is introduced to allow at least partial decay of errors in initial conditions of the state machines through effective extension of the race prehistory before using the trend information for forecasting. Both subsets are considered as in sample data. Finally, the third data subset – VDS – is used for out-of-sample evaluation of the forecasting model. For model performance comparison, both the McFadden (1974) pseudo-R^2 and cumulative profit achieved through Kelly betting strategy in VDS are evaluated. In order to have unbiased evaluation of the betting strategy, 'tainted' races (races in which the sum of implied winning probabilities is less than one, thus artificially increasing probability of a profitable wagering strategy) have to be eliminated from the considerations. Table 3.6 gives an overview of the distribution of valid and 'tainted' races in the data subsets, together with the number of data points.

Data set variables (Table 3.7) are split in two groups – variables known before each race (*ex-ante* variables) and the variables realised after the order of the runners is revealed (*ex-post* variables), corresponding in principle to independent and (stochastically) dependent variables. The first group consists of three types of variables (1) data identifiers (e.g. *RACEID*, *NUMBEROFRUNNERS*) used for queries and data aggregation, (2) fundamental variables, related to runner's ability and past performance (e.g. transformed beaten lengths, gender, moving average of previous finishing positions, etc.) and the conditions of the current race (e.g. draw advantage, weight carried) and (3) market variables, reflecting market opinion on probability of win (e.g. *DECIMALODDS*), upon which bettors define their strategies and claim pay-offs. Even though the fundamental variables follow a similar logic to that applied in Bolton and Chapman (1986), the variables in the data set are selected based on the expectations that their contribution to the utility varies considerably from runner to runner and from race to race. The first reason for this, potentially unintuitive selection, is that variables endorsed by Bolton and Chapman are in the public domain for a long time and it is to be expected that they are already incorporated in final odds. The second reason is that with the larger variance it becomes increasingly difficult for a human to recognize an underlying trend of a variable, as it will appear erratic and noisy. Without any knowledge on statistical properties of the noisy signal, is virtually impossible to filter them out. Hence, human bettors will likely fail to correctly discount the utility contribution of the selected fundamental variables. A stochastic filter, such as the KF, is arguably better suited to extract the residual trend information from the data than static and/or heuristic methods, and hence should be able to allow formulation of the strategies which would 'beat' the market.

Note that out of four *ex-post* variables, only the variable *FINISHINGPOSITION* is actually captured, being the ordered result of the decision event (race). As explained in 3.3.1, the variable *NFP* is linearly inflated finishing position calculated using equation (3.1). In order to account for unreliability of the lower ranking choices the performance proxy variable *ULTIPERF* is built as a weighted sum of *NFP*, *BEATENLENGTHS*[2], and prize money. By selecting this approach, more weighting is given to results stemming from high stakes (importance) races and to the runners with higher relative performance in the previous race (see discussion in 3.3.1).

TABLE 3.6: Statistics on Races in Data Subsets

	Races	**Tainted Races**	**Runners**	**Data Points**
LDS	21051	108	26923	222847
BDS	7302	53	14121	72741
VDS	14415	103	19440	141093

Analysis of the data set reveals that the informational content in the defined subsets is quite heterogeneous, in the sense of prehistory (i.e. pedigree) of the runners in the data set (Table 3.8). The table differentiates between the **Newcomers** and **Newcommers**

[2]Beaten lengths is a measure of time rather than distance (Kerr, 2017)

TABLE 3.7: Data Set Variables

Variable Name	Explanation
	ex-ante variables
DATE	date of the race
HORSEID	unique horse identifier
RACEID	unique race identifier
NUMBEROFRUNNERS	number of runners in the race
DECIMALODDS	decimal odds for winning
LOGPRICE	$\log(1/DECIMALODDS)$
DRAW	draw advantage in this race
MA_NFP	moving average of past *NFP* (derived)
BEATENLENGTHS	transformed beaten lengths in the previous race
	ex-post variables
FINISHINGPOSITION	finishing position (observed ranking)
NFP	Normalised Finishing Position (derived)
ULTIPERF(UP)	Importance Weighted *NFP* (derived)

MA, based on the inclusion of the information on past beaten lengths. **Newcommers MA** are identified based solely on the cumulative moving average calculated before the time period of LDS. **New Appearances** are horses appearing for the first time in in the data set considered. In particular, if considering the initialisation of the SSM, a large difference between the number of first appearances in LDS is far larger (26923) than the number of débutantes (i.e. runners having no professional racing records – **Newcomers** (17945)), identified through the filtering query $BEATENLENGTHS == 0$ $AND\ MA\_NFP == 0$. This means that 8798 horses (33%) have a previous history of unknown length not taken into consideration during KF initialisations. Moreover, for the purposes of model parameter estimation, especially for the stability of numerical optimisation, débutantes need a special treatment. This is explained in Appendix A.

TABLE 3.8: Data Loss due to initialisation

Data Subset	NewApperances	Newcomers	Newcomers MA
LDS	26923	17945	19156
BDS	5657	5621	6036
VDS	10844	10823	11623

Table 3.9: Statistics on Races in Data Subsets

Data Subset	**Time Between Races**		**Races Run**		**Data Loss**	
	avg.	max.	avg.	max.	$N = 1$	$N \leq 2$
LDS	41.0	1016	8.28	84	2900	5637
BDS + LDS	43.4	1379	9.07	115	3327	6361
FDS	45.8	2118	10.06	163	4078	7743

Finally, a comment regarding the suitability of the data set for meeting the research objectives is in order here. The horse racing setup is very well suited for meeting the research objectives with respect to heterogeneous choice sets regarding size (number of alternatives) and the actual selection of alternatives. First, there is a large variance in number of runners in a race (Figure 3.4) going from two to more that 25, whilst the majority of races have between seven and eleven runners. Moreover, length of runners' careers (i.e. number of races run in the data set) is variable with less than seven races for the majority of runners (see Figure 3.5), with a high maximum value (163) and average career length of around 10 races (see Table 3.9). It is important to note that the runners with only one (both in KF and CL stages) or two races (KF stage) in the data set are not useful in the analysis (**Data Loss** column) due to large state errors caused by the unknown initial values and provisions have been made in the algorithm to incorporate those cases. Those instances have been eliminated from the considerations and the data set has been restructured using the approach described in Appendix A to obtain consistent race level data. Table 3.9 shows also the irregularity of the time duration between the races, which varies between one day and 2118 days between two consecutive races of a runner, with the average value of approximately 45 days.

Chapter Summary

This chapter depicts the research philosophy, operational state space discrete choice model structure, research design developed for this study, and the empirical data used to evaluate the merits of the modelling approach and answer the research questions. First, the *post-positivistic* research paradigm, embedding the *critical realism* as the ontological argument together with the modified *dualist/objectivist* driven selection of research methods has been put forward. The research paradigm was used to provide a scientific rationale for RP methodology and experimental design of the study positioned in the horse-racing setup. Next, the modelling elements discussed in the literature review are put together to provide several operational dynamic discrete choice models based on LLM, LLT, and the Kalman filtering. Research design primarily identifies types of variables needed to answer the research questions and the statistical and economic

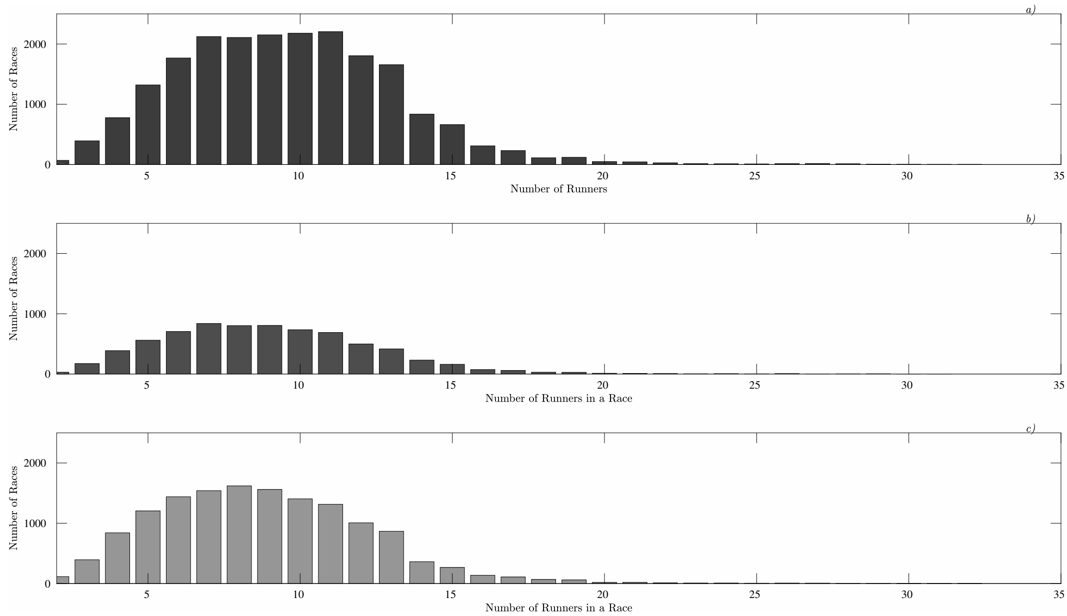

FIGURE 3.4: Distribution of Field Size (Number of Runners in a Race)
a.) LDS; b.) BDS; c.) VDS

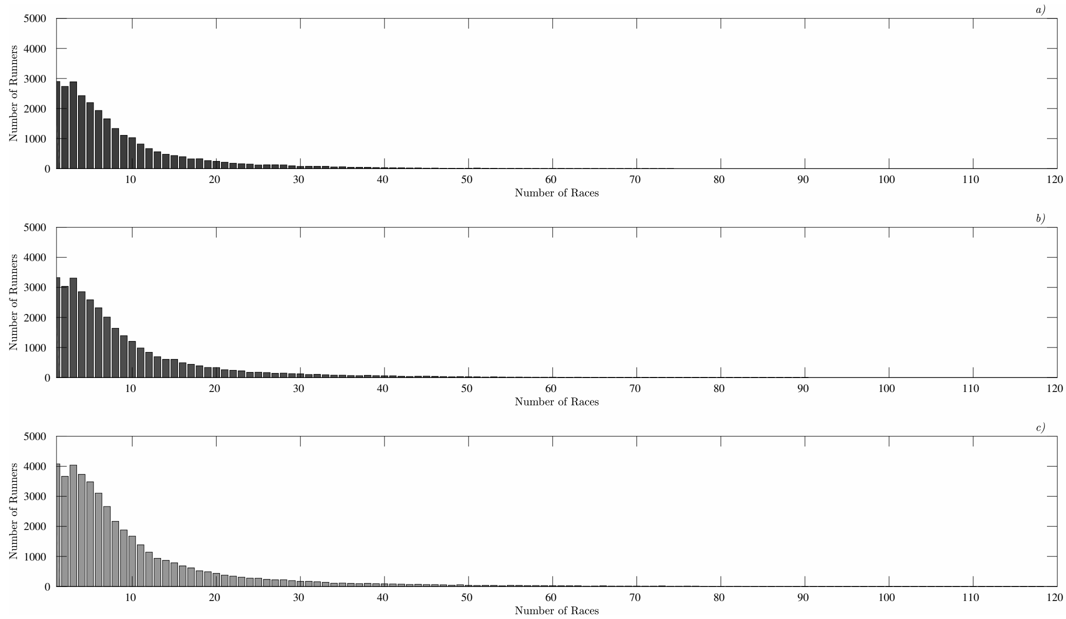

FIGURE 3.5: Distribution of Career Lengths of a Runner
a.) LDS; b.)BDS + LDS; c.) FDS

model evaluation criteria, namely Wald and LR tests of significance and the Kelly betting strategy evaluation. Furthermore, the research design includes estimation of model parameters and initialisation of trend dynamics states in the selected test setup. Next, the research design and the model test setup are presented and the way forward how to answer the research questions and close the gaps in the literature is given. Finally, relevant descriptive statistics and the meaning of the used model variables in the horse wagering context conclude the chapter.

Chapter 4

Empirical Results

This chapter presents the results obtained from the parameter-fitting efforts and model evaluations based on the model setup and performance criteria described in section 3.3.4. As explained in the previous chapter, the dataset is split into three subsets, which are used either cumulatively, on the runner level (for the KF model runs), or separately, on the race level, for the CL portion. After explaining the empirical results, stemming from univariate, bivariate, and trivariate models on in-sample data, the statistical and economic merits of forecasting models on out-of-sample (i.e. holdout) datasets are compared and discussed in detail. To conclude the chapter, out-of-sample results act as the basis of answers to the research questions.

It should be noted that throughout the chapter and until the end of the thesis, the symbol $A \succ B$ is used to express the dominance of model A over the model B, based on the pseudo-R^2 criterion, unless otherwise specified.

4.1 In-Sample Results

Parameters of the stand-alone KF trend models (cf. Table 3.1), obtained through the MLE procedure, are grouped for comparison and presented in Table 4.1 for the *NFP* variable and in Table 4.2 for the *UP* variable. Parameters α and q minimise the aggregated forecasting log-likelihood of all runs in LDS, as defined in equation (3.25).

A comparison of MSE forecasting performance derived from *NFP* trends reveals that classical error-correcting models – both in discrete (3.16) and continuous time (3.10) versions with exponential weighting of the state error covariance matrix – dominate all other trend models, including the model *LLM_NFP_m* with an engineered state error covariance matrix (2.33) and the model *LLM_NFP_c* based on a linear extension of measurement covariance (2.29). Both LLT models yield less than satisfactory results.

Table 4.1: Model Parameters of *NFP* Kalman Filtering

Model	**Parameter** q	**Parameter** α	**MSE**	**LL**	**Rank**
LLM_NFP_c	1.112e-3	-	17673.94	−82009.85	3
LLM_NFP_m	1.415e-2	1.015e-5	18830.74	−94622.48	4
LLT_NFP_c	4.900e-8	-	20236.41	−236786.45	5
LLT_NFP_m	1.519e-4	1.113e-4	30564.07	−297956.25	6
LLM_NFP_dm	1.619e-6	1.123e+0	2151.46	−76666.08	2
LLM_NFP_mod1	8.916e-4	4.095e-3	2147.91	−76539.58	1

[1] Due to the linear property of the implemented KF models maximisation of the likelihood is equivalent to minimisation of the MSE criterion.

Similarly, a comparison of MSE forecasting performance derived from *UP* trends shows that the models' pecking order has been kept. However, it should be noted that the relative distance between the best and the second best model is much larger, which is an indication of possible over-fitting.

Table 4.2: Model Parameters of *UP* Kalman Filtering

Model	**Parameter** q	**Parameter** α	**MSE**	**LL**	**Rank**
LLM_UP_c	1.689e-3	-	334430.09	−654567.38	3
LLM_UP_m	1.505e-2	4.470e-6	365536.87	−942731.66	4
LLT_UP_c	5.100e-8	-	379847.54	−356474.87	5
LLT_UP_m	1.564e-4	1.111e-4	584394.92	−497283.49	6
LLM_UP_dm	1.374e-1	9.720e-1	41606.933	−403423.42	2
LLM_UP_mod1	1.004e-3	4.964e-3	41528.46	−805211.91	1

[1] Due to the linear property of the implemented KF models maximisation of the likelihood is equivalent to minimisation of the MSE criterion.

Table 4.3 and Table 4.4 present the results of CL univariate parameter fitting and the associated statistics on the in-sample data subset LDS. Due to the initialisation effects bound to KF, a data restructuring algorithm was applied to all races in LDS, after eliminating every runner's first two races whilst keeping the pecking order in the race results (see Appendix A for an explanation). Based on the IIA property of CL, this restructuring allows for a fair comparison of the fitting results between fundamental variables without bias, albeit at the cost of a slight reduction in the number of data points.

In order to compare the predictive powers of the trend variables constructed from performance proxies, models *N1* – *N8* and *U1* – *U8* have been fitted and evaluated. In addition, the reference model *R*, which is built around the market type predictor variable *LOGPRICE*, is used as a proxy for the probability (i.e. log of final odds-implied probability) of selecting an alternative (winner) projected by the general public.

A statistical assessment of the significance of the univariate models *N1* – *N8* (Table 4.3) reveals that all models are significant with $p < 0.01$, which results from the Wald test of the hypothesis that the coefficients $\boldsymbol{\beta}$ (i.e. column **Beta**) are different from zero (Hayashi, 2000). As expected, the Wald test statistics (column **zValue**) and the obtained pseudo-R^2 values reveal that the *LOGPRICE* variable (Model R: $zValue = 95.74$) has by far the strongest numerical predictive power pseudo-$R^2 = 0.14981969$; $\beta_{ex} = 1.1319$).

For models based on *NFP* trends, the next two predictors are *LLM_NFP_dm*, with in-sample pseudo-$R^2 = 0.063409$ and *LLM_NFP_mod1* having in-sample pseudo-$R^2 = 0.061496$. Moreover, it can be observed that all LLT models perform rather poorly, with the pseudo-R^2 worse by one order of magnitude compared to LLM models. Simple non-parametric models *N1* (pseudo-$R^2 = 0.053475$; $\beta_{ex} = 3.400$) and *N2* (pseudo-$R^2 = 0.0452240$; $\beta_{ex} = 0.602$) are dominated by the LLM models, with the exception of the model *N4*, which, as expected, under-performs in the learning dataset due to the (deliberately suboptimal) covariance law (see 2.2.5), engineered for model error compensation. The model *N4* is expected to show its strength in an out-of-sample (holdout) dataset. Finally, all models add significant predictive power over the naïve model, in which all runners have the same probability of winning.

An interesting observation is that $N7 \succ N8$, which is contrary to the pecking order of the stand-alone KF trend variables ranking, caused by the fact that MSE ranking is based on averaging over different career lengths of the runners and hence smooths out the differences caused by converging KF from an initial uninformed prior.

Similarly, the models *U1* – *U8* (Table 4.4) are all significant with $p < 0.01$, according to the Wald test of the coefficients β_{ex}, and the models add significant predictive power over the naïve model. The models derived from trends in the importance-weighted outcomes of previous races (*U1* – *U9*) clearly dominate the models *N1* – *N8*. Two best predictors of race win probability after the market variable *LOGPRICE* (model R) are given through *U7* (in-sample pseudo-$R^2 = 0.087289$; $\beta_{ex} = 0.866$) and *U8* (in-sample pseudo-$R^2 = 0.085485$; $\beta_{ex} = 0.8917$). Non-parametric models *U1* (pseudo-$R^2 = 0.071374$; $\beta_{ex} = 0.842$) and *U2* (pseudo-$R^2 = 0.062940$; $\beta_{ex} = 0.429$) are positioned in the middle of the relative ranking. The model error compensation approach defined through an engineered covariance law (*U4*) is in alignment with the relative performance indicated by KF ranking (Table 4.2). LLT models have relatively low predictive power, because of the imposed linear trend between the races. The linear trend is inadequate for modelling due to its 'noisy' behaviour, observable through the large variance in the performance proxy variables from one race to the other.

Another distinctive result is that the models that take the time between races explicitly into account do not clearly dominate the discrete time models, which ignore the times between the races, due to the over-fitting and uniform weighting of all performance proxy errors, as opposed to the focus on winners only.

Bivariate models *NB1* – *NB8* and *UB1* – *UB8*, which represent the combinations of the univariate models described until now (i.e. stand-alone trend variables) with the model

Table 4.3: Model Parameters Conditional Logit (*NFP* in-sample)

Model	Variables	Beta	zTest	pseudo-R2	LogL	Rank
Naïve	-	-	-	0	−40184.17	-
R	*LOGPRICE*	1.132 (0.012)	95.74^{***}	0.149820	−34163.79	1
N1	*MA_NFP*	3.400 (0.055)	61.51^{***}	0.053475	−38035.32	5
N2	*LAGGED_NFP*	0.602 (0.028)	57.80^{***}	0.045224	−38366.89	7
N3	*LLM_NFP_c*	3.468 (0.053)	65.41^{***}	0.060560	−37750.63	4
N4	*LLM_NFP_m*	2.723 (0.045)	61.02^{***}	0.051241	−38125.10	6
N5	*LLT_NFP_c*	0.290 (0.012)	23.51^{***}	0.007739	−39873.18	8
N6	*LLT_NFP_m*	0.169 (0.008)	20.82^{***}	0.005830	−39949.91	9
N7	*LLM_NFP_dm*	3.432 (0.051)	67.05^{***}	0.063409	−37636.14	2
N8	*LLM_NFP_mod1*	3.475 (0.053)	65.83^{***}	0.061496	−37713.01	3

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

R, are presented in Table 4.5 and Table 4.6. They are used to evaluate the residual informational content of the trend variables over the market variable representing the settled 'wisdom' of the market regarding the odds of a particular runner winning. Since all of the models are nested, the LR test of significance can be effectively applied (3.30).

A study of Table 4.5 reveals that not a single model derived from the *NFP* trend variables is significant over *LOGPRICE*. Moreover, the LR statistics are so far from the borderline value for $p = 0.05$ ($\lambda_1 = 3.84$) that it can be concluded that models *NB1* – *NB8* do not extract statistically significant information from the defined trends in-sample.

On the other hand, five bivariate models based on *UP* trends (Table 4.6) contain statistically significant information over *LOGPRICE*. Indeed, not only are KF-based models significant, even the simple cumulative moving average (model *UB1*) is significant with ($p < 0.01$) and the model *UB2*, based on a simple one-race lag of *UP* being significant with ($p < 0.1$). This indicates far higher reliability of the ordered data models compared to those derived from *NFP* trends. Furthermore, it can be observed that the best model is *UB8* (pseudo-$R^2 = 0.149998$), which represents a considerable improvement over the univariate reference model *R* (pseudo-$R^2 = 0.149820$). LLT models (*UB5* and *UB6*)

TABLE 4.4: Model Parameters Conditional Logit (*UP* in-sample)

Model	Variables	Beta	zTest	pseudo R2	LogL	Rank
Naïve	-	-	-	0	-40184.17	-
R	*LOGPRICE*	1.132 (0.012)	95.74^{***}	0.149820	-34163.79	1
U1	*MA_UP*	0.842 (0.012)	69.53^{***}	0.071374	-37316.09	5
U2	*LAGGED_UP*	0.429 (0.006)	67.49^{***}	0.062940	-37654.98	7
U3	*LLM_UP_c*	0.888 (0.012)	75.01^{***}	0.084126	-36803.62	4
U4	*LLM_UP_m*	0.696 (0.010)	69.96^{***}	0.069966	-37372.64	6
U5	*LLT_UP_c*	0.083 (0.003)	28.69^{***}	0.011762	-39711.54	8
U6	*LLT_UP_m*	0.048 (0.004)	25.10^{***}	0.008561	-39840.16	9
U7	*LLM_UP_dm*	0.866 (0.011)	76.69^{***}	0.087289	-36676.53	2
U8	*LLM_UP_mod1*	0.8917 (0.012)	75.50^{***}	0.085485	-36749.03	3

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

are not as statistically significant as evidenced in *NFP* trend-derived models. The engineered model error correction model *UB4* is statistically significant over *LOGPRICE* with only mediocre relative performance (Rank 5), due to the data over-fitting in LDS.

In-sample trivariate model analysis (Table 4.7) confirms the dominance of *UP* trend-derived models. Models *T1* and *T2* have a sign reversal on the *LLM_NFP_m* whilst achieving $p < 0.01$ significance. A group of models which should assess the contribution of selected KF-generated trend variables (*T3* – *T6*) cannot confirm that the recency weighted variables are significant. The fact that only the model *T5* is significant reinforces the notion that the in-sample data analysis based on the MLE optimised KF models is affected by over-fitting and non-discrimination between the data points. To reiterate, the *LLM_UP_dm* variable does not include time as a factor at all. Instead, the weighting (i.e. fidelity) of the values is expected to increase as the career of a runner progresses, so that early results are treated as being less indicative than the results that come later in the runner's career. Obviously, this might be true for short careers, but it remains questionable for those with long careers, since a reversal to the mean is expected (Ma et al., 2016). This effect is more pronounced in the parameter-fitting datasets, since data fitting happens typically at the beginning of the data sets, due to the causality reasons.

Table 4.5: Model Parameters Conditional Logit (*NFP* bivariate in-sample)

Model	Variables	Beta	LR Test	pseudo R2	LogL	Rank
NB1	*LOGPRICE*	1.113	0.64	0.149828	−34163.47	2
		(0.014)				
	MA_NFP	0.054				
		(0.068)				
NB2	*LOGPRICE*	1.129	0.11	0.149821	−34163.74	6
		(0.014)				
	LAGGED_NFP	0.011				
		(0.033)				
NB3	*LOGPRICE*	1.128	0.20	0.149822	−34163.69	5
		(0.014)				
	LLM_NFP_c	0.299				
		(0.067)				
NB4	*LOGPRICE*	1.126	0.78	0.149829	−34163.40	1
		(0.014)				
	LLM_NFP_m	0.054				
		(0.013)				
NB5	*LOGPRICE*	1.131	0.08	0.149821	−34163.75	7
		(0.012)				
	LLT_NFP_c	0.004				
		(0.013)				
NB6	*LOGPRICE*	1.132	0.01	0.149820	−34163.79	8
		(0.012)				
	LLT_NFP_m	0.001				
		(0.009)				
NB7	*LOGPRICE*	1.126	0.40	0.149825	−34163.59	3
		(0.014)				
	LLM_NFP_dm	0.042				
		(0.066)				
NB8	*LOGPRICE*	1.128	0.23	0.149823	−34163.68	4
		(0.014)				
	LLM_UP_mod1	0.032				
		(0.067)				

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Table 4.6: Model Parameters Conditional Logit (*UP* bivariate in-sample)

Model	Variables	Beta	LR Test	pseudo R2	LogL	Rank
UB1	*LOGPRICE*	1.103	11.46^{***}	0.149962	−34158.06	4
		(0.015)				
	MA_UP	0.054				
		(0.016)				
UB2	*LOGPRICE*	1.116	3.51^{*}	0.149863	−34162.04	6
		(0.014)				
	LAGGED_UP	0.015				
		(0.008)				
UB3	*LOGPRICE*	1.096	11.50^{***}	0.149829	−34158.04	3
		(0.016)				
	LLM_UP_c	0.056				
		(0.008)				
UB4	*LOGPRICE*	1.103	11.37^{***}	0.149961	−34158.11	5
		(0.015)				
	LLM_UP_m	0.043				
		(0.013)				
UB5	*LOGPRICE*	1.130	0.15	0.149822	−34163.72	7−8
		(0.012)				
	LLT_UP_c	0.001				
		(0.003)				
UB6	*LOGPRICE*	1.130	0.15	0.149822	−34163.72	7−8
		(0.012)				
	LLT_UP_m	0.012				
		(0.002)				
UB7	*LOGPRICE*	1.094	11.63^{***}	0.149964	−34157.98	2
		(0.016)				
	LLM_UP_dm	0.055				
		(0.016)				
UB8	*LOGPRICE*	1.092	14.30^{***}	0.149998	−34156.64	1
		(0.016)				
	LLM_UP_mod1	0.062				
		(0.016)				

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

Table 4.7: Model Parameters Conditional Logit (Trivariate in-sample)

Model	**Variables**	**Beta**	**LR Test**	**pseudo R2**	**LogL**
T1	*LOGPRICE*	1.103	20.88^{***}	0.150089	−34152.96
		(0.015)			
	LLM_NFP_m	−0.317			
		(0.094)			
	LLM_UP_m	0.105			
		(0.022)			
T2	*LOGPRICE*	1.106	11.51^{***}	0.149973	−34157.65
		(0.015)			
	LLM_NFP_m	−0.057			
		(0.063)			
	MA_UP	0.062			
		(0.018)			
T3	*LOGPRICE*	1.097	2.20	0.149990	−34156.96
		(0.015)			
	MA_UP	0.032			
		(0.021)			
	LLM_UP_m	0.025			
		(0.017)			
T4	*LOGPRICE*	1.093	2.83^{*}	0.149997	−34156.65
		(0.016)			
	MA_UP	0.054			
		(0.034)			
	LLM_UP_mod1	0.054			
		(0.036)			
T5	*LOGPRICE*	1.095	11.51^{***}	0.149973	−34157.65
		(0.078)			
	MA_UP	0.028			
		(0.027)			
	LLM_UP_dm	0.031			
		(0.031)			
T6	*LOGPRICE*	1.093	1.28	0.149980	−34157.34
		(0.016)			
	LLM_UP_dm	0.039			
		(0.025)			
	LLM_UP_m	0.019			
		(0.020)			

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

In summary, in-sample results indicate that the importance-weighting of the performance proxy values improves the reliability of the trend variables. On the other hand, the question as to whether the inclusion of the time between the races in the models improves the data fit accuracy remains inconclusive when only in-sample results are considered.

4.2 Out-of-Sample results

Out-of-sample tests of the accuracy of forecasting algorithms are generally considered to be superior indicators of model performances over goodness-of-fit to past data. There are two main arguments for this point. First, forecasting errors are likely to be affected by the method selection and parameter estimation, since they incorporate implicit nuances about past history which might not persist in the future. Hence, over-fitting to historical data will fail to maintain reasonable post-sample performance, due to structural changes inherent in the modelling environment. Second, model structures and the associated model parameters selected and estimated based on the best in-sample fit do not necessarily yield good post-sample data predictions (Fildes and Makridakis, 1995; Tashman, 2000). Consequently, for the purpose of answering the research questions of this study, arguments based on the out-of-sample data are postulated as *prīmā facīe* evidence of model performance.

Compared to in-sample considerations, univariate models *N1* – *N8* (Table 4.8) demonstrate very similar out-of-sample (i.e. in VDS) pecking orders of performance, with the only difference that *N4* and *N1* swap places, with the reference model *R* (*LOGPRICE* variable) retaining the strongest numerical predictive power (pseudo-R^2 = 0.158189). For models based on *UP* (*U1* – *U8* Table 4.9), a similar observation can be made, since the change in the pecking order is reflected only in the models *U1*, *U2*, and the *U4* group. It can be stressed that the models *N4* and *U4*, engineered for model error compensation, improved the placement as expected. All *UP* derived models have higher pseudo-R^2 values than *NFP* derived models. Note that the significance of the models based on the Wald test is unchanged, since the test evaluates the ratio of the coefficients to their standard error, which does not change out-of-sample.

Bivariate out-of-sample model results are presented in Table 4.10 and Table 4.11. Both tables show the LR significance confirmation of trend variables over *LOGODDS*, representing the settled market opinion regarding the winning probability of a particular runner, which in turn demonstrates the extraction of the residual informational content inherent in the trend variables. Contrary to in-sample results, where not a single trend variable derived from the *NFP* is even close to being significant over *LOGPRICE*, model *NB4* (pseudo-R^2 = 0.158232, $\lambda = 2.73$) is significant with $p < 0.10$ ($\chi^2_1(0.1) = 2.706$), showing that KF can extract residual information from the trend. Furthermore, the bivariate models *UB1* – *UB8* demonstrate different levels of significance for five models.

Indeed, the models *UB4* (pseudo-$R^2 = 0.158379$) and *UB1* (pseudo-$R^2 = 0.158301$) are significant with $p < 0.01$, *UB8* (pseudo-$R^2 = 0.158290$) with $p < 0.05$, and *UB7* (pseudo-$R^2 = 0.158247$) and *UB3* (pseudo-$R^2 = 0.158241$) with $p < 0.10$. It is important to note that discrete models (*UB1*, *UB2*, and the *UB7*) do not take the time between races into account, and their assumed unit (one-day) time distances between the races are capable of extracting significant residual information over the market. This, however, does not imply that the models extract the same portion of the residual information. This can be assessed through a trivariate comparison of the models, combining those using time information and those not doing so.

Trivariate models (c.f. Table 4.12), constructed to compare models that consider different types of information (time vs. no time information and importance-weighted vs. non-weighted), offer the final insights needed for answering the posed research questions. Selection of the base models for the comparison (taken from the bivariate pool) follows the following convention. As a first resort, significant out-of-sample bivariate models are taken as the basis, if they actually exist. If there is no significant model with the desired characteristics, the next best model is taken for comparison.

The first group of trivariate models compares the informational content of *NFP* trend-derived models with the *UP* trend-derived models. In models *T1* and *T2*, the best *NFP* model (*NB4*) is enhanced with the best recency weighted variable (*LLM_UP_m*) and the best non-recency weighted variable (*MA_UP*) in combination with *LOGPRICE* out-of-sample. It is notable that both *NFP* variables experience a sign reversal, meaning that the added *UP* variables override *NFP* contribution.

The second group of trivariate models compares the contribution of different variables over discrete *UP* trend-derived bivariate models to assess the significance of time information. As illustrated, the variable *LLM_UP_m* extracts statistically significant ($p < 0.05$) information over models *UB1* ($\lambda = 5.53$) and *UB7* ($\lambda = 4.41$). On the other hand, the variables *LLM_UP_dm* and *LLM_UP_mod1* do not contribute significantly to the discrete models, in spite of their relative in-sample ranking (i.e. the first and the second after the reference model *R*). This confirms the dominance of the error correction model outside of the fitting dataset (LDS) over all other studied models, irrespective of their nature, i.e. embedding of time and importance-weighting information.

4.3 Economic Significance of Persistence in Preferences

In order to answer the secondary research questions, the economic significance of the constructed models is evaluated by developing a betting strategy based on out-of-sample model predictions (forecasts). As explained in 3.3.4, a Kelly (1956) wagering strategy is employed, based on out-of-sample probabilities. The Kelly strategy aims at the optimal exponential growth of total wealth in the long run, whilst securing zero probability of

TABLE 4.8: Model Parameters Conditional Logit (*NFP* out-of-sample)

Model	Variables	Beta	pseudo-R2	LogL	Rank
R	*LOGPRICE*	1.132 (0.012)	0.158189	−26725.72	1
N1	*MA_NFP*	3.400 (0.055)	0.077472	−29288.30	6
N2	*LAGGED_NFP*	0.602 (0.028)	0.073487	−29414.83	7
N3	*LLM_NFP_c*	3.468 (0.053)	0.084769	−29056.65	4
N4	*LLM_NFP_m*	2.723 (0.045)	0.082395	−29131.99	5
N5	*LLT_NFP_c*	0.290 (0.012)	0.048481	−30208.69	8
N6	*LLT_NFP_m*	0.169 (0.008)	0.046612	−30268.04	9
N7	*LLM_NFP_dm*	3.432 (0.051)	0.086510	−29001.38	2
N8	*LLM_NFP_mod1*	3.475 (0.053)	0.086241	−29009.89	3

TABLE 4.9: Model Parameters Conditional Logit (*UP* out-of-sample)

Model	Variables	Beta	pseudo-R2	LogL	Rank
R	*LOGPRICE*	1.132 (0.012)	0.158189	−26725.72	1
U1	*MA_UP*	0.842 (0.012)	0.082997	−29112.89	7
U2	*LAGGED_UP*	0.429 (0.006)	0.087173	−28980.30	6
U3	*LLM_UP_c*	0.888 (0.012)	0.101170	−28535.93	4
U4	*LLM_UP_m*	0.696 (0.010)	0.096187	−28694.14	5
U5	*LLT_UP_c*	0.083 (0.003)	0.052309	−30087.18	8
U6	*LLT_UP_m*	0.048 (0.004)	0.049082	−30189.64	9
U7	*LLM_UP_dm*	0.866 (0.011)	0.103508	−28461.71	2
U8	*LLM_UP_mod1*	0.8917 (0.012)	0.103222	−28470.78	3

Table 4.10: Model Parameters Conditional Logit (*NFP* bivariate out-of-sample)

Model	Variables	Beta	LR Test	pseudo-R2	LogL	Rank
NB1	*LOGPRICE*	1.113	0.95	0.158204	−26725.24	2
		(0.014)				
	MA_NFP	0.054				
		(0.068)				
NB2	*LOGPRICE*	1.129	−0.25	0.158185	−26725.84	9
		(0.014)				
	LAGGED_NFP	0.011				
		(0.033)				
NB3	*LOGPRICE*	1.128	0.12	0.158191	−26725.65	5
		(0.014)				
	LLM_NFP_c	0.299				
		(0.067)				
NB4	*LOGPRICE*	1.126	2.73^{*}	0.158232	−26724.35	1
		(0.014)				
	LLM_NFP_m	0.054				
		(0.013)				
NB5	*LOGPRICE*	1.131	−0.22	0.158185	−26725.83	8
		(0.012)				
	LLT_NFP_c	0.004				
		(0.013)				
NB6	*LOGPRICE*	1.132	−0.06	0.158188	−26725.75	6
		(0.012)				
	LLT_NFP_m	0.001				
		(0.009)				
NB7	*LOGPRICE*	1.126	0.20	0.158192	−26725.75	4
		(0.014)				
	LLM_NFP_dm	0.042				
		(0.066)				
NB8	*LOGPRICE*	1.128	0.43	0.158196	−26725.50	3
		(0.014)				
	LLM_NFP_mod1	0.032				
		(0.067)				

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 4.11: Model Parameters Conditional Logit (*UP* bivariate out-of-sample)

Model	Variables	Beta	LR Test	pseudo R2	LogL	Rank
UB1	*LOGPRICE*	1.103	7.12^{***}	0.158301	−26722.15	2
		(0.015)				
	MA_UP	0.054				
		(0.016)				
UB2	*LOGPRICE*	1.116	0.89	0.158203	−26725.27	6
		(0.014)				
	LAGGED_UP	0.015				
		(0.008)				
UB3	*LOGPRICE*	1.096	3.30^{*}	0.158241	−26724.06	5
		(0.016)				
	LLM_UP_c	0.056				
		(0.008)				
UB4	*LOGPRICE*	1.103	12.06^{***}	0.158379	−26719.69	1
		(0.015)				
	LLM_UP_m	0.043				
		(0.013)				
UB5	*LOGPRICE*	1.130	−0.01	0.158189	−26725.72	7−8
		(0.012)				
	LLT_UP_c	0.001				
		(0.003)				
UB6	*LOGPRICE*	1.130	−0.01	0.158189	−26725.72	7−8
		(0.012)				
	LLT_UP_m	0.012				
		(0.002)				
UB7	*LOGPRICE*	1.094	3.67^{*}	0.158247	−26723.88	4
		(0.016)				
	LLM_UP_dm	0.055				
		(0.016)				
UB8	*LOGPRICE*	1.092	6.45^{**}	0.158290	−26722.49	3
		(0.016)				
	LLM_UP_mod1	0.062				
		(0.016)				

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

TABLE 4.12: Model Parameters Conditional Logit (Trivariate out-of-sample)

Model	Variables	Beta	LR Test	pseudo R2	LogL
T1	*LOGPRICE*	1.103	7.36^{***}	0.158348	−26720.67
		(0.015)			
	LLM_NFP_m	−0.317			
		(0.094)			
	LLM_UP_m	0.105			
		(0.022)			
T2	*LOGPRICE*	1.106	1.07	0.158249	−26723.82
		(0.015)			
	LLM_NFP_m	−0.057			
		(0.063)			
	MA_UP	0.062			
		(0.018)			
T3	*LOGPRICE*	1.097	5.53^{**}	0.158385	−26719.49
		(0.015)			
	MA_UP	0.032			
		(0.021)			
	LLM_UP_m	0.025			
		(0.017)			
T4	*LOGPRICE*	1.093	−0.23	0.158297	−26722.27
		(0.016)			
	MA_UP	0.054			
		(0.034)			
	LLM_UP_mod1	0.054			
		(0.036)			
T5	*LOGPRICE*	1.095	−0.50	0.158293	−26722.40
		(0.078)			
	MA_UP	0.028			
		(0.027)			
	LLM_UP_dm	0.031			
		(0.031)			
T6	*LOGPRICE*	1.093	4.41^{**}	0.158316	−26721.67
		(0.016)			
	LLM_UP_dm	0.039			
		(0.025)			
	LLM_UP_m	0.019			
		(0.020)			

$^{***}p < 0.01$, $^{**}p < 0.05$, $^{*}p < 0.1$

bankruptcy. Note that a non-reinvesting Kelly strategy, limiting the stakes to 10% of aggregate wealth, is applied, in order to avoid bias that may occur due to a streak of large wins towards the end of the out-of-sample dataset. Economic significance is assessed by evaluating the total profit and rates of return obtained by applying a Kelly strategy for betting on the out-of-sample races. All betting results are presented in Table 4.13,

Table 4.13: Kelly betting strategy (Univariate *NFP* Models)

Model	Profit	ROI	Total Bets	Winning Bets	Bet Capital
R	−1080.19	−1.73	651	268	62449.98
N1	−99998.05	−29.47	3597	281	339315.31
N2	−99998.89	−29.94	3579	294	333977.51
N3	−99997.71	−25.77	4090	337	387985.49
N4	−99997.65	−28.14	3713	305	355302.32
N5	−99998.39	−33.51	3129	224	298449.22
N6	−100000.00	−32.59	3205	235	306885.74
N7	−99997.10	−27.28	3899	328	366494.15
N8	−99996.77	−25.77	4118	342	387961.49

Table 4.14, Table 4.15, Table 4.16 and Table 4.17, assuming the same initial capital of £100000.

As expected, all univariate models (Table 4.13 and Table 4.15) offer very poor economic performance – with the exception of the reference model (R) with a relatively small total loss of £1080.19 and ROI of −1.73%. This confirms that *LOGPRICE* is the best single predictor of a winning horse (i.e. decision made by 'nature'), but it does not allow for positive profit.

Bivariate models build on *LOGPRICE*, which in combination with a trend variable aim to extract residual information not discontinued by the market participants, and are commented next. Since none of the derived discrete *NFP* is significant, model *NB1* is selected as being representative, due to the highest LR statistic ($\lambda = 0.95$), with a total loss of £1710.11 and ROI of −2.81%, which is below of the yield provided by the reference model (R). Furthermore, the single statistically significant continuous time model *NB5* does not offer any improvement over the reference model with £1911.01 total loss and ROI of −3.04%, and it is even dominated by the discrete time model *NB1*. Note that other models offer slightly better yield (e.g. model *NB5*); however, without statistical significance, those results are not relevant.

Contrary to the *NFP* trend-based models, bivariate models embedding *UP* trends are considerably better. Several models yield positive total profits and ROI. Discrete model *UB1* yields £901.84 total profit and ROI of 1.22 with 323 wins out of 766 total bets. Continuous time bivariate models are even better – *UB8* with £3518.24 total profit and an ROI of 4.21 (a 371/865 wins-to-bets ratio) and *UB4* with £1020.61 total profit, an ROI of 1.25 (a 352/856 wins-to-bets ratio).

Trivariate models provide mixed messages in terms of economic significance. Out of three statistically significant models (*T1*, *T3* and *T6*), only *T6* yields positive £158.33 total

TABLE 4.14: Kelly betting strategy (Bivariate *NFP* Models)

Model	Profit	ROI	Total Bets	Winning Bets	Bet Capital
NB1	-1710.11	-2.81	633	258	60873.28
NB2	-1072.68	-1.71	654	269	62729.29
NB3	-1894.88	-3.05	648	263	62074.31
NB4	-1911.01	-3.04	652	264	62797.11
NB5	-875.93	-1.40	648	270	62532.95
NB6	-1141.28	-1.83	648	268	62427.02
NB7	-1817.39	-2.95	644	262	61698.10
NB8	-1984.77	-3.19	651	265	62172.30

TABLE 4.15: Kelly betting strategy (Univariate *UP* Models)

Model	Profit	ROI	Total Bets	Winning Bets	Bet Capital
R	−1080.19	−1.73	651	268	62449.98
U1	−99998.09	−34.93	2989	224	286286.97
U2	−99998.72	−28.13	3760	332	355474.86
U3	−99996.87	−26.06	4086	344	383681.33
U4	−99998.10	−26.29	4049	334	380308.02
U5	−99999.99	−32.55	3237	234	307218.32
U6	−100000.00	−32.79	3195	231	304976.98
U7	−99997.56	−21.99	5001	445	454673.48
U8	−99997.16	−26.58	4022	339	376243.36

profit and an ROI of 0.19 (a 358/882 wins-to-bets ratio), which is smaller than the gain made from a betting strategy built on *UB8*.

Time series representations of wealth growth for several selected bivariate and all evaluated trivariate models are presented in Figure 4.1 and Figure 4.2. Bivariate model selection encompasses the model *NB4*, which is the sole significant model derived from the variable *NFP* and the models *UB1*, *UB4* and *UB8*, which have the highest out-of-sample LR statistics of all bivariate models. Betting information based on the non-significant models is inconclusive, as it results from pure chance and the eventual order of 'lucky' wins and/or losses, and hence it should be omitted from scientific considerations.

4.4 Answers to Research Questions

The empirical results presented herein allow for answering the research questions posed in Chapter 2.3 and seen through the lens of a horse-racing setting. First, the answers to primary and secondary research questions are explicitly formulated, and then a discussion

TABLE 4.16: Kelly betting strategy (Bivariate *UP* Models)

Model	Profit	ROI	Total Bets	Winning Bets	Bet Capital
UB1	901.84	1.22	766	323	73955.87
UB2	1413.71	2.04	720	305	69150.37
UB3	−105.75	−0.14	804	334	77569.58
UB4	1020.61	1.25	856	352	81846.69
UB5	−467.58	−0.75	651	274	62416.88
UB6	−894.07	−1.44	649	270	62290.47
UB7	−255.55	−0.33	821	337	78601.89
UB8	3518.24	4.21	865	371	83558.13

TABLE 4.17: Kelly betting strategy (Trivariate Models)

Model	Profit	ROI	Total Bets	Winning Bets	Bet Capital
T1	−1514.10	−1.28	1234	485	118367.92
T2	−1827.96	−2.91	652	267	62800.96
T3	−270.07	−0.34	819	341	79082.04
T4	3289.91	3.99	855	368	82406.43
T5	1651.69	2.17	792	332	75944.68
T6	158.33	0.19	882	358	84864.85

on the underlying effects and their connection to market-influencing cognitive biases follows in Chapter 5.

4.4.1 Discussion on Research Question 1

Research Question 1 sought to answer whether patterns of past performance (race ranking) add statistically significant information to publicly available market data. For the selected empirical setup, it translates to the interpretation of whether the patterns of past race rankings are reflected in performance trends, which can be used for forecasting the future performance of a runner and, ultimately, its placement in future races. Expectations bound to informational content inherent to performance trends, derived solely from the ordered placement information, have to be modest at best. First, the data are very noisy from race to race, and the trend alone cannot account for the strength of the competition from race to race. For example, a higher placement in a race with weaker competition can be followed by finishing in last place in the next race when facing a formidable field of competitors. Second, as explained in Section 3.2, the reliability of the rank-ordered information for lower placements can be quite low, since some of the runners could be made to deliberately diminish their performance due to tactical (e.g. sparing of a horse or a jockey for subsequent races) or even fraudulent (e.g. feigning of

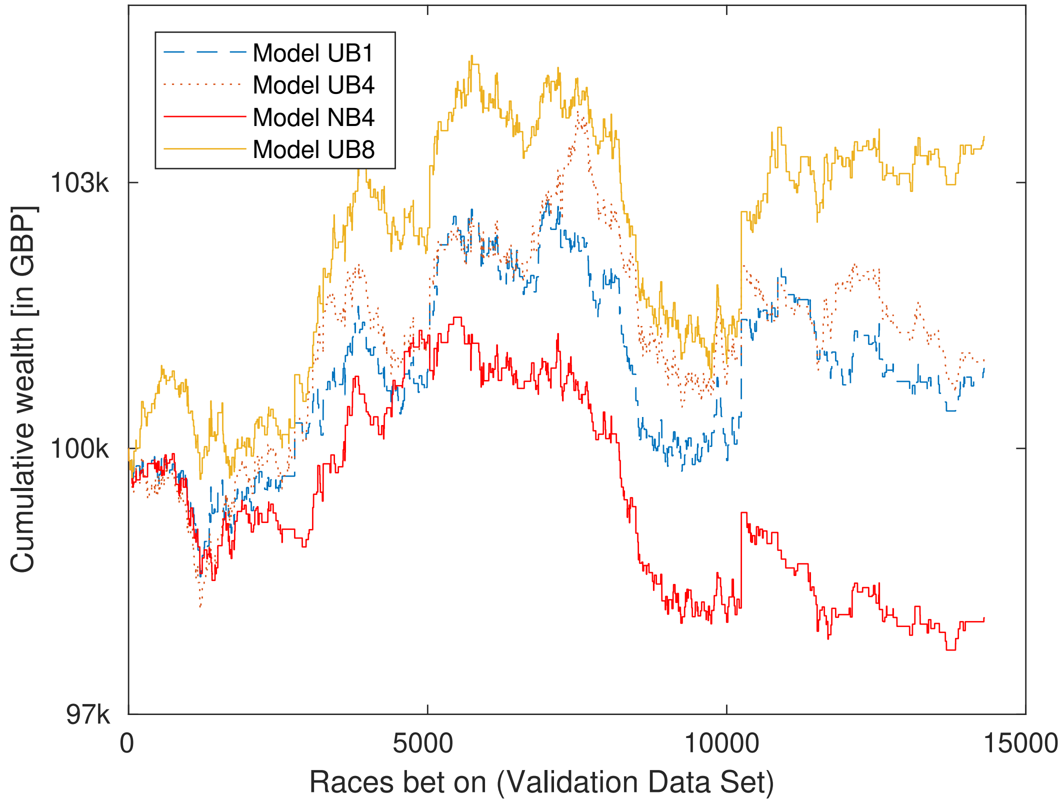

FIGURE 4.1: Cumulative wealth from Kelly betting strategy (Bivariate Models)

poor performance in low-key races to manipulate odds/handicap in a subsequent high-stakes race) motives. Third, a horse is a living organism, a physical animal that evolves over the life cycle of its career, following stages and cycles of the competitive form, which affects perceived trends in performance.

In order to provide an answer regarding the informational content of the trend derived from ordered placement information, the statistical significance of additional information has been assessed by using two types of unobserved utility: with and without market information containing public opinion on the winning probability of a horse (reflected in the variable *LOGODDS*). In the absence of market information, as evidenced in Table 4.3, several discrete trend variables (as captured through models *N1*, *N2* and *N7*) are significant, based on the confirmed hypothesis that the β_{ex} coefficients statistically differ from zero, as confirmed through the Wald test. On the other hand, the informational content of these trend variables does not explain an additional part of the unobserved utility when market information is included, taken out-of-sample (Table 4.8), based on an LR test of significance for nested models.

These results indicate that trend information, on the one hand, has some (albeit weak) explanatory power when viewed in isolation, whilst on the other hand, if viewed within the market environment, it has no additional explanatory power. In other words, the evidence of the statistical significance yields an *affirmative* answer to Research Question 1, in spite of the fact that the betting public properly discounts this information in full.

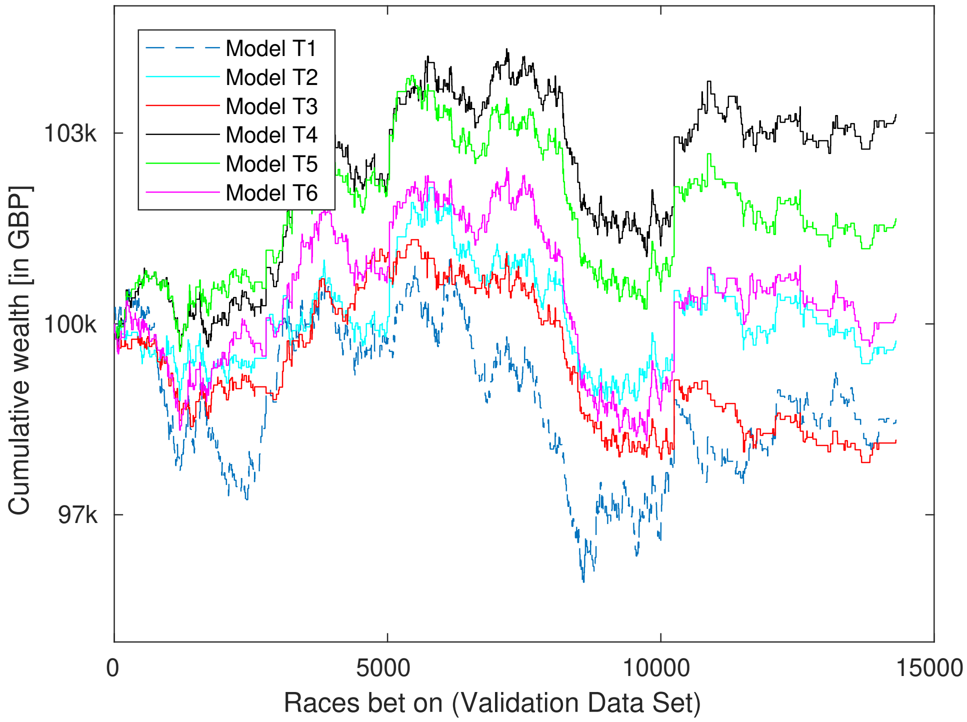

FIGURE 4.2: Cumulative wealth from Kelly betting strategy (Trivariate Models)

The associated secondary Research Question 1a addresses the economic implications of persistence in preferences through monetary gains, which could potentially be realised through the application of additional information. Monetary gains from betting, using the *NFP* derived trend variables, are presented in Table 4.13 and Table 4.14. Obviously, betting activities without public odds cause significant losses, and following them is not advisable. In addition, since no discrete *NFP* trend-derived variable is significant, the model with the highest out-of-sample LR statistic has been selected to illustrate economic significance – the *NB1* model. Betting strategy based on the Kelly algorithm yields in the case of the *NB1* model a -2.81% ROI with a £1710.11 loss, compared to the reference model R with a -1.73% ROI and a £1080.19 loss. This indicates that a bettor applying the approach from this study would experience additional monetary loss over a naïve bettor using only the closing odds – which, per definition, yields average gains. Hence, the answer to the secondary Research Question 1a is that the absence of statistically significant residual information means that its monetary value is negative, as it fails to even match the average (negative) profit from betting.

A summary of the supporting evidence to Research Questions 1 and 1a is given in Figure 4.3. The top part of the figure schematically depicts in- and out-of-sample results of the 'best' models in each category, selected based on the highest pseudo-R^2 value. Consequently, the model *N7* achieves in-sample statistical significance ($zValue = 67.05 > 1.96$ for $p < 0.05$), whilst the bivariate model *NB1* fails to achieve out-of-sample statistical significance over *LOGODDS* ($\lambda = 0.95 < 3.84$ for $p < 0.05$). Based on the definition of the Research Question 1, the associated answer is *affirmative*, even with only one statistically significant model category (in this case in-sample). This, in turn, means that the answer to Research Question 1 directly closes the first identified gap in the literature – the lack of theoretical and methodological support for changing choice sets (c.f. Figure 2.7).

The bottom part of Figure 4.3 presents a summary of the economic significance of the best, albeit in this case not statistically significant, out-of-sample model that includes ranked *ex-post* preference information combined with the changing choice sets (*NB1*). In alignment with the remark on the inconclusiveness of betting results derived from non-significant models given in section 4.3, it can be seen that the model fails to outperform the market reflected in *LOGODDS* variable and yields an additional loss (−1.08% ROI and a £629.92 loss of capital) over a naïve betting strategy based on market odds. The answers to Research Questions 1 and 1a fulfil directly the Research Objective 1 (c.f. Figure 2.3).

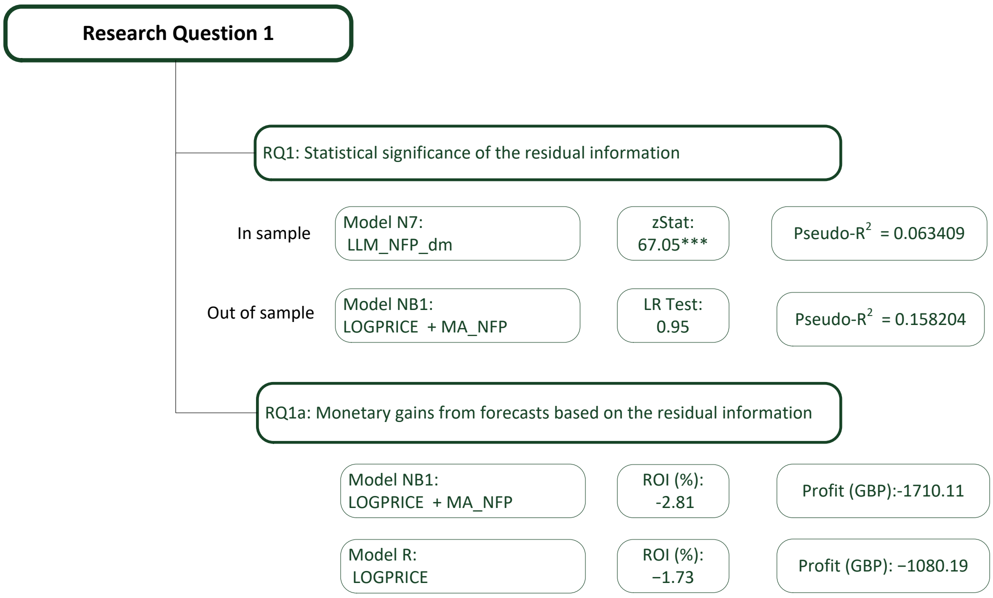

FIGURE 4.3: Research Question 1 Test Evidence Structure

4.4.2 Discussion on Research Question 2

Research Question 2 requires an answer as to whether patterns relating to past performance (race ranking) data, combined with information on the temporal distance between successive decision events in changing choice sets, add statistically significant information over publicly available market data. In the particular case of empirical testing defined in this study, information on time passed between decision events (races) should account, at least partially, for the evolution of a runner regarding its competitive form since the last race. The most straightforward way to incorporate that evolution is stochastic – the variance (i.e. uncertainty) of the previous performance grows monotonously over time, following a linear or a nonlinear functional form, thus indicating that the informational content derived from the available performance value diminishes over time. Obviously, issues with race-to-race noise and the reliability of the trends derived from ranked data are not likely to be resolved by adding time information.

The reduction of informational content over time can be incorporated into trend building and forecasting, so that the relative weighting of the information realised during the last decision event is taken into account. As above, an assessment of the informational content of trends derived from the available ranking information is carried out, based on the statistical significance of the additional information explaining a part of the unobserved utility, constructed with and without market information reflected in the variable *LOGODDS*.

In the absence of market information, as evidenced in Table 4.3, all continuous time variables (embedded in models *N3*, *N4*, *N5*, *N6* and *N8*) are significant, based on the confirmed hypothesis that the β_{ex} coefficients statistically differ from zero, as confirmed by the Wald test). In the bivariate case, the informational content of the trend variable *LLM_NFP_m* (model *NB4*) explains an additional part of the unobserved utility out-of-sample (Table 4.8) when the market information is included, as confirmed by the LR test of significance ($p < 0.1$).

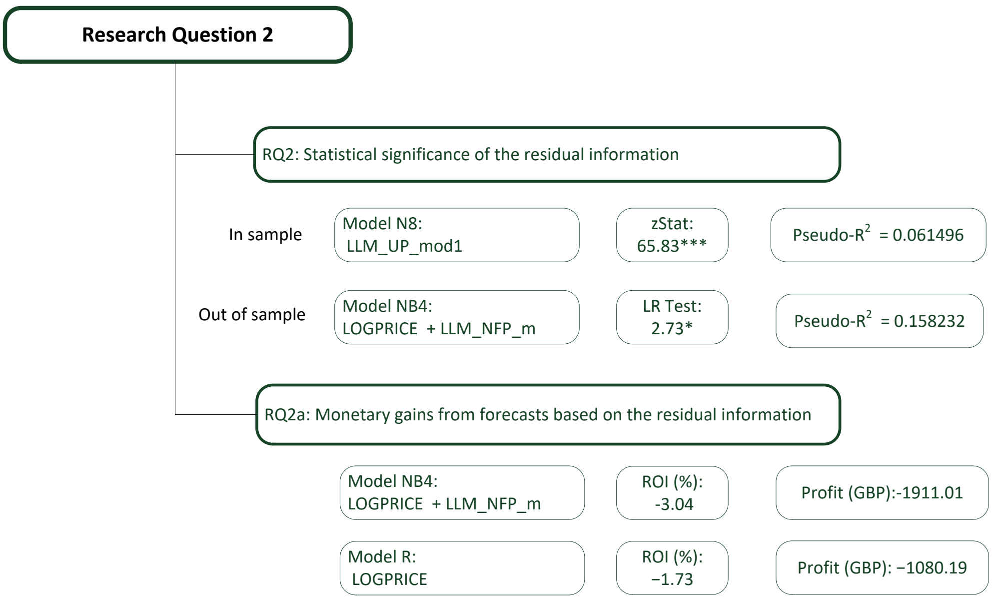

FIGURE 4.4: Research Question 2 Test Evidence Structure

Hence, the answer to Research Question 2 is *affirmative*, since the time ('forgetting') factor embedded in time distance between the two successive decision events adds statistically significant information over the market variable *LOGODDS*. Furthermore, this implies that the temporal development of the attributes of an alternative (trend of performance) contains statistically significant information, which, if omitted, results in model bias. This results in the conclusion that in similar DCM applications, feedback regarding the previous (ordered) preference should be included in any modelling effort.

The economic importance of the persistence of preference effects, reflected in monetary gains which could potentially be realised through the exploits of additional information, builds the focus of the secondary Research Question 2a. Monetary gains from betting, using the *NFP*-derived continuous trend variables (models *N3*, *N4*, *N5*, *N6* and *N8*), are presented in Table 4.13 and Table 4.14. Obviously, betting activities without public odds cause significant losses, and following them is not recommended. On the other hand, the model *NB4*, being statistically significant, can be used for betting. Betting strategy based on a Kelly algorithm yields, in the case of the *NB4* model, a -3.04% ROI and a £1911.01 loss, compared to the reference model R with a -1.73% ROI and a £1080.19 loss. This indicates, in answer to Research Question 2a, that a bettor applying the approach from this study would experience below-average monetary gains (i.e. in this case a loss). Moreover, the cumulative wealth growth for the model *NB4* is presented in Figure 4.1. Hence, the answer to the secondary Research Question 2a is that even though statistically significant residual information is given, its monetary value is negative, as it fails to even match the average (negative) profit from betting. It should be noted, though, that the significance level is less rigorous than the typically defined level ($p < 0.05$).

A summary of the supporting evidence to Research Questions 2 and 2a is given in Figure 4.4. The top part of the figure schematically depicts in- and out-of-sample results of the 'best' models in each category, selected based on the highest pseudo-R^2 value. Consequently, the model *N8* achieves in-sample statistical significance ($zValue = 65.83 > 1.96$ for $p < 0.05$), whilst the bivariate model *NB4* achieves out-of-sample statistical significance over *LOGODDS* ($\lambda = 2.73 > 2.71$ for $p < 0.10$). Based on the definition of the Research Question 2, the associated answer is *affirmative*. Note that the answer to Research Question 2 comprises an intermediate result with respect to the closure of the identified gaps – it embeds non-uniform time intervals between reoccurrence of alternatives and ranked *ex-post* preference information in order to assess the incremental improvement of the model accuracy before a claim for gap closure can be made (c.f. Figure 2.3). In addition, this particular intermediate step has been selected to enable an analysis of theoretically important effects of misperceptions (biases) in temporal discounting in decision-making (Ma et al., 2016), discussed in detail in Chapter 5.

The bottom part of Figure 4.4 presents a summary of the economic significance of the best out-of-sample model that includes ranked *ex-post* preference information, the changing choice sets and non-uniform time intervals between reoccurrence of alternatives (*NB4*). It can be seen that the model fails to outperform the market reflected in *LOGODDS* variable and yields an additional loss (-1.31% ROI and a £830.82 loss of capital) over a naïve betting strategy based on market odds. A rationale for this result is that even though the inclusion of the time discounting in the model improved the model accuracy considerably and reached (weak) statistical significance, the effects bound to the reliability of the ranked data impede its economic performance. Theoretical aspects of reliability of lower ranked alternatives are discussed in the next chapter.

4.4.3 Discussion on Research Question 3

Research Question 3 sought an answer to whether patterns of importance weighted past performance data can add statistically significant information over publicly available market data. Importance-weighting is designed to mitigate the effects of low reliability of non-selected ordered choice data. In the horse-racing context, the importance of the event is modelled through prize money, which indicates its relative attractiveness compared to other races and its intrinsic incentive to perform according to the inherent performance ability of the horse and jockey. In addition, races with higher prize money are correlated with generally better horses causing stronger in-race competition, which, combined with the inclusion of information on lagging time behind the winner (*BEATENLENGTHS*), additionally calibrates the relative performance used for trend prediction. This, in turn, promises the better predictive power of the resulting variable *UP*.

As with the previous two research questions, statistically testing the significance of informational content of the importance weighted trend variable derived from the ordered placement information is carried out using two types of unobserved utility – with and without the market information. For univariate models, i.e. when market information is not included, all discrete trend variables (as captured through models *U1*, *U2*, and *U7*) are significant ($p < 0.05$), based on the Wald test (see Table 4.4). Furthermore, the evaluation of bivariate models that include market information reveals noticeable improvements in the reliability of trend variables – two discrete models are out-of-sample statistically significant over *LOGODDS* (Table 4.11), with $p < 0.10$ (*UB7*) and $p < 0.05$ (*UB1*).

These results indicate that weighted trend information, both in univariate and bivariate model settings, contains additional explanatory power, and the betting public has not properly discounted that information in full. This yields an *affirmative* answer to Research Question 3.

The secondary Research Question 3a calls for an evaluation of the economic importance of preference persistence effects, reflected through monetary gains that could potentially be realised through the exploits of additional information. Monetary gains from betting, using the *UP*-derived discrete trend variables (models *U(B)1*, *U(B)2*, and *U(B)7*), are presented in Table 4.15 and Table 4.16. As above, betting activities without market information yield significant losses and should not be pursued. On the other hand, with the inclusion of the market variable *LOGODDS*, the model *UB1* allows profitable betting, i.e. a betting strategy based on Kelly staking yields a 1.22% ROI with a £901.84 profit, compared to the reference model *R* with a −1.73% ROI and a £1080.19 loss. The cumulative wealth growth for model *UB1* is included in Figure 4.1. This indicates that, contrary to the discrete unweighed case, even a simple moving average of the variable *UP* can yield a positive profit. In summary, the answer to the secondary Research Question 3a is that it is possible to achieve above-average profit (i.e. an ROI improvement of 2.95%) from betting, based on trend information derived from the importance weighted ranked order information.

A summary of the supporting evidence to Research Questions 3 and 3a is given in Figure 4.5. The top part of the figure schematically depicts in- and out-of-sample results of the 'best' models in each category, selected based on the highest pseudo-R^2 value. Consequently, the model *U7* achieves in-sample statistical significance ($zValue = 76.69 > 1.96$ for $p < 0.05$), whilst the bivariate model *UB1* achieves out-of-sample statistical significance over *LOGODDS* ($\lambda = 7.12 > 3.84$ for $p < 0.05$). Based on the definition of the Research Question 3, the associated answer is *affirmative*. Note that the answer to Research Question 3 comprises an intermediate result with respect to the closure of the identified gaps – it embeds changing choice sets and importance weighting of ranked *ex-post* preference information in order to assess the incremental improvement of the model accuracy before a claim for gap closure can be made (c.f. Figure 2.3). In addition, this particular intermediate step has been selected to enable an analysis of theoretically important effects of diminishing reliability of ranked information (Fok et al., 2012), discussed in detail in Chapter 5.

The bottom part of Figure 4.5 presents a summary of the economic significance of the best out-of-sample model that includes importance weighted ranked *ex-post* preference information, the changing choice sets (*UB1*). It can be seen that the model outperforms the market reflected in *LOGODDS* variable and yields a profit (2.95% ROI and a £1982.02 capital gain) over a naïve betting strategy based on market odds. Theoretical aspects of reliability of lower ranked alternatives are discussed in the next chapter.

4.4.4 Discussion on Research Question 4

Research Question 4 integrated both aspects of persistence in preference effects, that is, importance-weighting and the temporal distance between the successive decision events in changing choice sets. It sought an answer whether patterns of importance weighted past performance data, combined with the information of temporal distance between successive decision events, add statistically significant information to publicly available market data.

Following the same approach as in the previous three set-ups, statistical testing of the significance of residual informational content, in the form of trends derived from past performance data for two cases of unobserved utility (with and without market information), is carried out. As expected, models that embed both considered aspects of

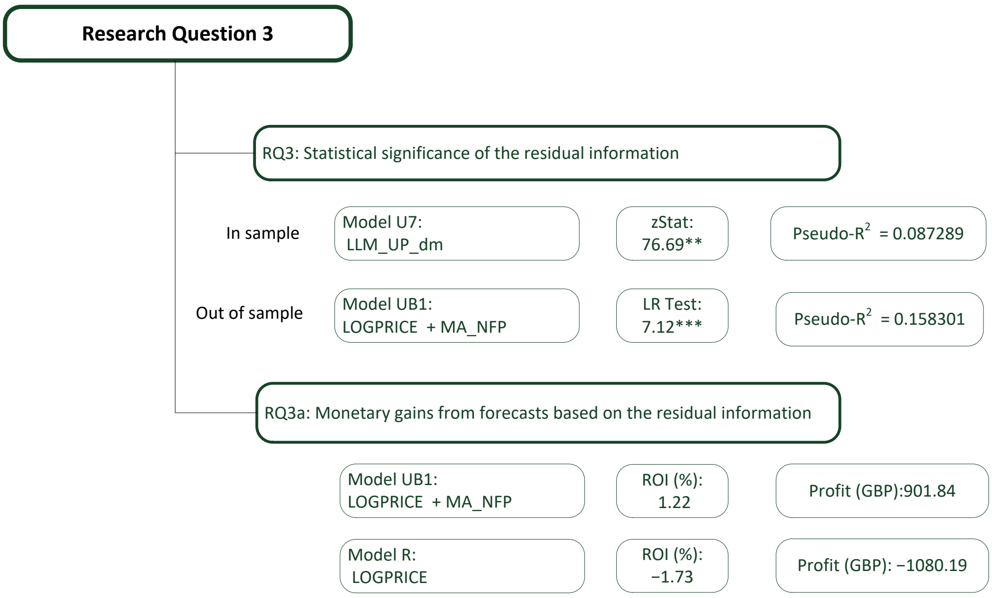

FIGURE 4.5: Research Question 3 Test Evidence Structure

persistence yield very good results, i.e. the models are statistically significant both in- and out-of-sample. The Wald test reveals (see Table 4.4) that all univariate continuous time trend models (as captured in models *U3*, *U4*, *U5*, *U6* and *U8*) are significant at $p < 0.05$. Furthermore, the evaluation of bivariate models that include market information demonstrate out-of-sample significance over *LOGODDS* (Table 4.11), with $p < 0.10$ (*UB3*) and $p < 0.05$ (*UB4* and *UB8*).

These results demonstrate that the residual information obtained from the trends of past performance data, enhanced with importance-weighting and the temporal distance between the successive decision events in changing choice sets, contains additional explanatory power in both the univariate and the bivariate model settings, and that the betting public has not properly discounted that information in full. This yields an *affirmative* answer to Research Question 4.

The economic aspects of residual information of the persistence in preference effects are assessed next, in order to answer the secondary Research Question 4a. Monetary gains that could potentially be realised through the exploits of additional information contained in the *UP*-derived continuous trend variables (models *U3*, *U4*, *U5*, *U6* and *U8*) are presented in Table 4.15 and Table 4.16. As before, betting activities without market information do not build a successful investment endeavour, as they result in significant losses. Conversely, the inclusion of the market variable *LOGODDS* allows profitable betting for *UB4*, which yields a 1.25% ROI with a £1020.61 profit, compared to the reference model R with a -1.73% ROI and a £1080.19 loss. The cumulative wealth growth for the model *UB4* is included in Figure 4.1. Note that the *UB8* has even better financial performance – a 4.21% ROI with a £3518.61 profit, but a lower out-of-sample pseudo-R^2 value than the model *UB4*, and thus it was not used for a representative comparison. In summary, the answer to the secondary Research Question 4a is that it is possible to achieve above-average profit (i.e. a ROI improvement of 2.98%) from betting based on trend information derived from importance-weighted ranked order information. Figure 4.1 includes cumulative wealth growth for the models *UB4* and *UB8*.

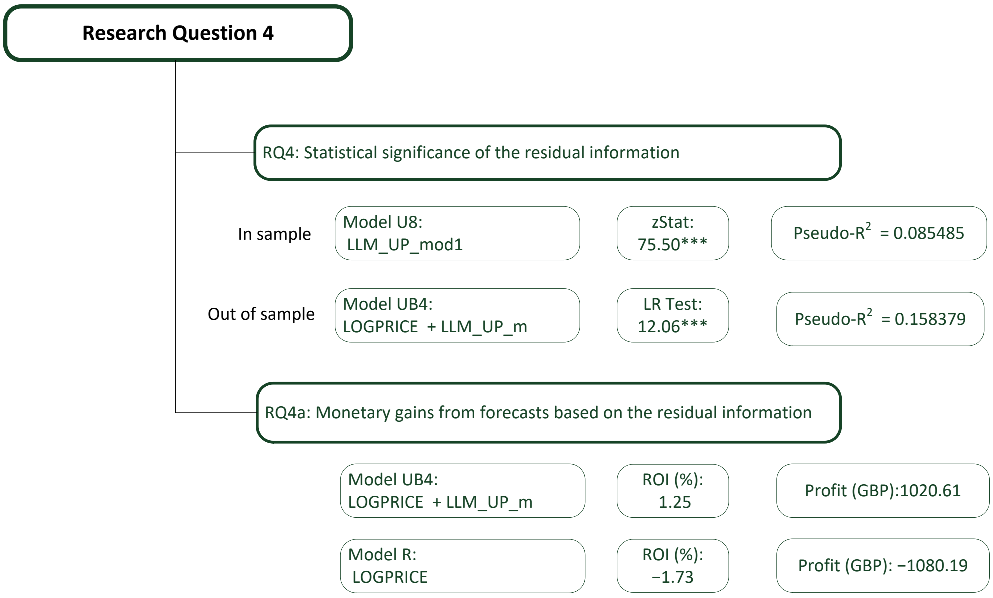

FIGURE 4.6: Research Question 4 Test Evidence Structure

A summary of the supporting evidence to Research Questions 4 and 4a is given in Figure 4.6. The top part of the figure schematically depicts in- and out-of-sample results of the 'best' models in each category, selected based on the highest pseudo-R^2 value. Consequently, the model *U8* achieves in-sample statistical significance ($zValue = 75.50 > 1.96$ for $p < 0.05$), whilst the bivariate model *UB4* achieves out-of-sample statistical significance over *LOGODDS* ($\lambda = 12.06 > 3.84$ for $p < 0.05$). Based on the definition of the Research Question 4, the associated answer is *affirmative*. This, in turn, means that the answer to Research Question 4 directly closes both identified gap in the literature – the lack of theoretical and methodological support for changing choice sets and non-uniform time intervals between reoccurrence of alternatives, together with the importance weighted ranked *ex-post* preference information (c.f. Figure 2.7).

The bottom part of Figure 4.6 presents a summary of the economic significance of the best out-of-sample model that includes importance weighted ranked *ex-post* preference information and non-uniform time intervals between reoccurrence of alternatives information (*UB4*). It can be seen that the model outperforms the market reflected in *LOGODDS* variable and yields a profit (2.98% ROI and a £2100.79 capital gain) over a naïve betting strategy based on market odds. The answers to Research Questions 4 and 4a fulfil directly the Research Objective 2 (c.f. Figure 2.6).

The next chapter expands on the salient theoretical points regarding persistence in preferences effects, i.e. the effects of the reliability of the ranked data and misperceived temporal discounting, using the data and the analysis presented in this chapter.

Chapter Summary

The empirical results chapter describes the results obtained following the approach described in the research design and the evidence leveraged to answer the research questions.

The chapter started with results from the KF stage parameter fitting and the observation that the pecking order in terms of model quality obtained in this stage does not necessarily correspond to the expected informational contribution in the CL stage. Next, the chapter turned to CL stage results by evaluating the in-sample univariate, bivariate, and trivariate models. The most important results from this evaluation are that importance-weighting considerably improves the reliability of the trend variables and that the improvement in residual information, resulting from including data on times between the decision events, remains inconclusive.

Out-of-sample results evaluation revealed that the univariate models' pecking order does not change too much, with the notable exception that those models designed for model error compensation models improved their placement due to their robustness against over-fitting. Bivariate models, however, witnessed a significant change – the inclusion of time information yielded evidence of the statistical significance of several models (even on one *NFP*-based model). In order to confirm that time information actually contains residual information, a set of trivariate models was generated to compare the statistical significance of trend models, including temporal- and importance-weighting over the models based on importance weighted trends only. The trivariate comparison confirmed the significance of the time information.

Analysis of the economic implications of persistence in preference effects showed that, albeit far from trivial, it is possible to achieve above-average profits in combination with the market variable *LOGPRICE* for models that include importance and temporal weighting, which proved that the market did not completely discount the available information.

Evidence collected during empirical testing, analysed in-sample and out-of-sample, provides sufficient information to answer all four research questions posed in Section 2.3 and discussed in the final portion of this chapter. Answers to all four research questions meet the research objectives (1.2) and close the identified gaps in literature in full.

Chapter 5

Discussion

The main findings from the previous two chapters are that the effects of persistence in preferences explain part of the unobserved utility in a repeated decision-making setup. In particular, the research questions posed herein allowed for a gradual model build-up where combinations of importance weighting and temporal discounting effects were assessed in an naturalistic empirical test setup. In this study, importance weighting has been introduced to mitigate the effects of the low reliability of ranked choice data as set out in Research Objective 2. Similarly, the modelling approach for non-uniform sampling has been included in the same research objective to account for 'forgetting' or the evolution of preferences over time, which are logically bound to the ageing of available information and the cognitive biases of DM related to the perception of time.

This chapter discusses the findings obtained in this study in a broader context of previous research on these effects.

5.1 Reliability of ordered data

One of the fundamental results of research efforts on decision-making modelling is that the inclusion of ranked alternatives yields more efficient estimations of the preferences of a subject or a set of subjects compared to when a DM reveals or states their most preferred option only (Fok et al., 2012; Beresteanu and Zincenko, 2018). One of the earliest models is Rank-Ordered Logit (ROL) (Beggs et al., 1981), used to analyse the preferences of individuals over multi-alternative choice sets, provided that either the revealed or the stated rank data is available. Empirical use of this model has been demonstrated in many fields, such as accident research (Bogue et al., 2017), transportation studies (Zhang et al., 2005) and automotive marketing (Zheng, 2010), to mention but a few.

However, the ROL model is based on the assumption that a DM under study consistently assigns a utility value to each alternative independently, and subsequently it ranks the alternatives based on (perceived) utility values. This is, however, quite a strong assumption and, as it turns out, not very realistic, since, in practice, respondents may be unable and/or unwilling to perform the ranking task or a part thereof, due to numerous reasons. First, the DM may face limitations in terms of the cognitive capabilities needed to perform this task, such as an inability to perform the ranking due to too many, and possibly overly complex, alternatives in the choice set or a lack of distinctive differences between the less-preferred alternatives. Of course, a partial fulfilment of the ranking task is also possible – a DM may find it easier and more worthwhile to choose the most preferable alternative first than assign ranks to less desirable alternatives. Moreover, it should be noted that the task of ranking multiple alternatives simultaneously is not necessarily equivalent to the sequential task of repeated preference choices from a choice set, such as the most preferred, the next most preferred, etc., until the last remaining pair of alternatives is resolved (Louviere et al., 2008). Furthermore, the DM may be less careful in ranking inferior alternatives (since the impact of false reasoning is lower) or even suffer from a 'response bias', and it can be reasonably concluded that the reliability of ranking information deteriorates in line with decreasing rank. This obviously may produce empirically unreliable data, which, if not corrected for, may lead to a substantial modelling bias (Chapman and Staelin, 1982).

Decision-making researchers have put forward numerous more or less successful ways to overcome reliability bias. The first and the simplest approach was to use only the first few ranks, which immediately begs the question how to determine the appropriate number of ranks to be used for the given application. Chapman and Staelin (1982) proposed a method based on the LR test statistic which evaluates the statistical equivalence of parameters of data pooled over several 'explosion levels'. Their procedure allows for the decomposition of alternative rankings into a series of conventional (i.e. unranked) and statistically independent choice observations (cf. IIA), in order to meet the preconditions for applying conventional CL modelling. In spite of the mathematical appeal of the procedure, Hausman and Ruud (1987) observed that noise in information, i.e. levels of uncertainty, increases in line with growth in the number of included levels, and the authors tried to model the increase in uncertainty with a rank-heteroscedeastic unobserved utility function. Both approaches, however, assume that the ranking capabilities are the the same for every participating DM or that the capabilities do not change from one to the next decision event, in the case of repeated decision-making (Louviere et al., 2008). This is obviously not the case when different decision protocols govern the ranking process at different depths, which is typically the case for horse-racing placements (Sung and Johnson, 2007). Furthermore, and this is an even more limiting assumption, is that identical choice sets are postulated for each decision event, which may lead to a considerable loss of model efficiency.

In a further development, Fok et al. (2012) proposed a model capable of incorporating the fact that lower rankings may not completely reflect the true preferences of an observed DM. The approach introduces a latent variable that endogenously identifies the unobserved ranking capabilities of the heterogeneous (or evolving) DM, which may be used to design more efficient SP ranking surveys, proven to be of interest in marketing research applications. Whilst they could demonstrate a clear efficiency gain compared to standard CL models, and non-bias in spite of ranking inabilities of some of the DM, their model does not provide support for changing choice sets.

The problem of unreliable ranking appears naturally in the horse-racing context and is particularly pronounced amongst those runners finishing in the minor placings. It has been observed that the lower rankings do not necessarily carry informational value, and rank order finish data beyond the winner and the runner-up are of questionable value and cannot be relied upon (Sung and Johnson, 2007). Even though the racing protocol calls for sportsmanship and requires jockeys to continue riding to the best achievable performance standard, there is little incentive to follow it if a profitable win or a placement is out of reach. Moreover, as described in 4.4, there are (fraudulent and/or tactical) incentives to secure a finishing position under the actual placement potential, such as promoting the lengthening of betting odds or reducing the weight handicap in the next race, aimed at increasing betting gains or the probability of a win.

Lessmann et al. (2012), whilst investigating a two-step forecasting approach (SVM/CL) of horse race outcomes, came to the conclusion that the MSE criterion applied in the first step (SVM) leads to over-fitting and generally suboptimal performance, since prediction errors in the finishing order of the two last places have far smaller monetary importance than prediction errors in the finishing order of the winner and the runner-up, albeit they can have the same MSE. The authors pointed out also that a similar problem setting for ranked information has been studied extensively in machine learning-based optimisation of internet search engines, with the aim of maximising accuracy within those results ranked highest (Cao et al., 2006; Le and Smola, 2007).

The conclusion regarding potential inadequacy of the MSE criterion is confirmed in this study, following two main rationales – first, the pecking order of the trend models in-sample is completely turned over when compared to predictions of out-of-sample results, and second, even though the KF trend modelling was invoked, it should yield a theoretically minimum MSE, and only the error compensating model including the time information (c.f. *NB4*) can provide statistically significant out-of-sample results. A way to mitigate unreliability in the ranking data through importance weighting has been demonstrated in this thesis. Based on a priori information regarding the importance of the decision event at hand, and a proxy of relative distance between the revealed choices, an improved continuous ranking variable was created to compensate for reliability issues. The improvement is so high over the non-weighted models that even those models without time information outperform non-weighted models with time information. Importance weighting, as applied in this thesis, has several benefits over the modelling approaches discussed above: (1) simplicity of calculation, since variable conditioning for reliability is done exclusively during preprocessing, which, in turn, eliminates the need for multilayered model fitting (Fok et al., 2012), defining assumptions regarding the distributional properties of 'latent segments' (e.g. Mixed Logit) for reliability estimation (Train, 2009) or learning postulated scoring functions with desirable properties (Cao et al., 2006), (2) supports 'non-designed' changing choice sets and (3) can be used both in SP and RP problem settings. In the context of horse-racing, the results presented in this thesis confirm that the questionable reliability of the ranking information, in combination with the MSE criterion of fit, does not provide the desired out-of-sample performance. The methodology presented herein mitigates the effects of the unreliability of lower ranks through importance weighting associated with the particular race and, at the same time, reduces the the susceptibility of the models to in-sample over-fitting noise. This example of domain-based heuristics in using a priori data can be replicated in decision-making problem settings in other sciences. It should be noted that the presented approach, even though it mimics the general idea of weighted distance-based models for ranking data (Alvo and Philip, 2014), is far more flexible, as it does not require any formal consideration of the rationality levels of the DM. However, this discussion still begs the question about underlying rationality, which is addressed next.

5.2 Rationality

In the last three decades, the assumption of rationality postulated in the vast majority of decision-making models has been increasingly challenged. One of the reasons for this increased scrutiny is the dissimilarity of objectives exhibited by researchers from different sciences. For example, psychologists are mostly interested in replicating the individual behaviour of decision-makers as human subjects, whilst economists and transportation researchers tend to concentrate on demand and policy effectiveness forecasts. Notably, there is a large body of evidence of departures from rationality, observed in practically all fields of human decision-making (Cherchi, 2012). Attempts to explain the sources of (temporary) departure from the rationality principle are typically based on dedicated structural components added to a base rationality-based model, such as state dependence, habit persistence, and taste heterogeneity, as explained in Chapter 1. However, all of these approaches face the problem of unequivocal differentiation between the contributions of different components, i.e. effects are mutually confounded, and only a careful research design can help separate these effects and allow for conclusions regarding their relative contribution to the decision process.

In this study, namely a decision-making modelling approach, which aimed to fill the identified gap in the literature with respect to changing choice sets, support for irregular durations between the recurring availability of alternatives in a choice set, and feedback on the relative preferences for alternatives in previous decision events, has been developed. Moreover, the selected setup of empirical testing allows for the effective separation of confounding effects that may otherwise introduce model bias.

Horse-wagering markets used for empirical test setup effectively eliminate some of the confounding effects. As explained previously, the outcomes of horse races are modelled as decisions made by an abstract DM 'nature' based on rational evaluations of utility function bound to the virtues of a runner and idiosyncratic race attributes (Bolton and Chapman, 1986). In addition, it is postulated that the principle of causality applies, i.e. 'nature' faces repeated decisions, as time linearly progresses from race to race. First, state dependence effects are excluded as dynamic components by virtue of the changing choice sets, since choosing a winner in a race does not alter the preferences, prices and/or constraints (Hsiao, 2014) affecting the next immediate race, because, as a rule, they do not contain a single runner from the previous race. In other words, the choice made in the previous race does not increase the probability of experiencing the same choice in the next race.

Second, the effect of the heterogeneity which captures differences between different decision-makers and variety-seeking is equally not applicable in this setting. Repeated choices made by a single DM do not exhibit static differences between decision-makers. Moreover, an implied autocorrelation in the unobserved characteristics of the DM is suppressed through the random order of the races on any given day, which involves different racing tracks and a large number of races for consideration, which averages out variety-seeking behaviour.

Consequently, the remaining effect is habit persistence, which aims to capture the temporal persistence of the unobserved attributes of the available alternatives. The empirical setup based on the horse-racing betting markets is adequate to assess the results of the revealed preference as a proxy of the desirability of an alternative, and it considers its temporal persistence, i.e. forgetting in time, to isolate the effects of habit persistence. Furthermore, a comparison with the market-level variable *LOGPRICE*, revealing the market assessment of the probability of selecting an alternative, demonstrates the significance of the approach.

Ironically, as Gowdy (2008) pointed out, *'homo economicus'*, who follows the axiomatic rational choice model, negates the idea of human individuals who act with highly developed intelligence and understanding of social behaviour. He also concludes that "First, social animals, such as primates, also have a sense of fairness and a tendency to cooperate, even at a cost to themselves. Second, 'lower' animals do appear to behave in accordance with the rational actor model", which leads to the conclusion that an 'abstract', non-living DM tends to achieve higher levels of rationality than humans. Furthermore, the main goal of the traditional decision-making modelling for a given application is typically focused on improving the representative utility, in order to explain as much as possible the variance of the behaviour and to minimise the variance attributed to the unobserved utility (error term). In this study, model improvements are geared towards a particular decomposition of the error term, postulating the statistical properties of the error term components, in order to achieve better forecasting performance. Hence, this approach does not consider representative utility as a confirmation of the neoclassical economic theory; however, in alignment with the *post-positivistic* paradigm, it aims at reducing uncertainty and improving prediction of behaviours, which can no longer be seen as irrational (Ariely, 2008; Cherchi, 2012).

5.3 Time duration bias

One of the less explored heuristics that DM use to balance between inherently limited cognitive capabilities, time pressure, and the ever-growing complexity of the modern world is *duration neglect*, whereby the human brain abstracts time information and focuses its perception of experience on 'peak-and-end' events whilst neglecting both the total duration of the experience and the duration of different phases thereof (Kahneman and Frederick, 2002). However, ignorance or misestimation of the duration may lead to a cognitive bias generating systematic errors that affect the correctness of any decisions made. Duration neglect has been investigated in several controlled experimental studies mimicking organisational and business problem settings, e.g. the inventory management setting 'Beer Distribution Game'. Sterman (1989) found that scheduling tasks, especially involving long-lead tasks affecting each other, are regularly biased towards underestimating time delays between them. This is analogue to one of the fundamental results from dynamic system control theory, indicating that a time delay in a linear feedback system reduces stability margins and may lead to system oscillations (Franklin et al., 2014). A related study by Sterman and Diehl (1993), in a somewhat more complex simulation of an inventory management problem setting, investigated the effects of closed feedback loops, confirming that humans generally adopt a linearly causal (e.g. open loop) view of system behaviour, which in combination with time delay misperceptions leads to the poor performance of all subjects participating in the simulation.

Furthermore, several studies have demonstrated that the general population faces great difficulties in differentiating between stock (measured at a particular point of time, e.g. voltage) and flow variables (measured over a defined period of time, e.g. frequency), and they find relationships between them confusing (Kainz and Ossimitz, 2002). Sweeney and Sterman (2000) reported that even subjects from the population of arguably numerically well-calibrated decision-makers (a cohort of MBA students) do not anticipate bias caused by difficulties in recognising delays and understanding their impact; rather, they rely on pattern-matching heuristics which yield catastrophic performance in scenarios involving long delays, further aggravated when irrelevant information is included (Lafond et al., 2012). All of the cognitive difficulties bound to time delays, feedback, and dynamic behaviour in the observed decision scenarios beg the question, whether and to what extent do DMs fail to account fully for information inherent in (discrete or continuous) times between observed events or variables. Ma et al. (2016) investigated whether individuals engaging in a naturalistic decision making environment (horse-wagering market) are subject to biases resulting from duration misperception. The authors identified that (1) market odds do not fully discount temporal information in spite of the well-known fact that form cycles of runners and jockeys affect winning probabilities and (2) cognitive bias bound to misperception of the time-period (duration) since the last 'significant' event related to a horse (i.e. since its last win) creates betting market inefficiencies of more than 20% when controlled for using the same information set, including the age and gender of runners, the last distance run, together with prize money and previous winning odds market variables. The modelling approach put forward in their study was based on an adaptation of an Survival Analysis (SA) interval model (Prentice et al., 1981; Harrell Jr, 2015), which predicts the conditional event probabilities in progressive time intervals since the last 'significant' event considered (i.e. win) for every horse in the dataset. Obviously, the selected time intervals are selected as being identical with the time between actual races. Obtained conditional probabilities are then used as an input into a CL model to account for within-race competition. The results indicate the presence of time duration bias, even in settings where significant monetary rewards for accurate judgements are available.

However, events based on which the temporal misperception is measured are in fact quite rare relative to the total number of runners participating in the races. Indeed, the mean number of wins in one season (2005), with those horses with at least one win in their previous career at 2.81, with the standard deviation of 2.45, indicates that a very large dataset of runners covering their careers before the out-of-sample season is needed. Moreover, no additional information can be gained for runners without previous wins.

While the results presented in this thesis also confirm the existence of a significant time duration bias amongst bettors, the presented methodology allows for the evaluation of temporal effects on every horse with more than two runs without limitation bound to at least one win in the previous career. This, in turn, means that the temporal discounting method put forward in this thesis allows for a broader adoption of proposed methodologies in other fields residing within operational research. The method developed herein, for instance, may effectively be used in political science and project management, where either only a few political parties have ever won elections or only a limited amount of work packages of similar scope are long-lasting, thus having the largest impact in a case of misperception.

Chapter Summary

The discussion chapter builds on the empirical results from the previous chapter and closes the circle of the theoretical framework presented in the introduction and literature review. The structure allows for a better understanding of two sources of biases which occur in dynamic discrete decision-making contexts, together with the modelling framework on how to overcome them in practical settings. In particular, the effects of the studied biases in a naturalistic decision-making environment demonstrate that even a well-calibrated and incentivised DM is prone to bias, resulting in suboptimal decisions.

The problem of the unreliability of the rankings in a dynamic setting where the DM faces changing choice sets in every subsequent decision event was discussed in light of previous approaches presented in the literature. Since unreliability is particularly strong for lower ranks, most of the incumbent approaches are geared towards identifying ‘acceptable’ rank levels, either through breaking down datasets into reliable and unreliable parts or through a latent variable determining the threshold of reliability. The approach presented herein uses *a-priori* knowledge of the monetary importance of the decision event on hand to assign relative importance weighting, and it could be demonstrated that importance weighting contains statistically significant residual information on the general betting market. In addition, assumptions regarding the rationality of the DM in the selected setting, and its implications, were discussed.

Finally, temporal bias affecting persistence in preferences or, in general, any decision-making setup, which includes dynamic system behaviour, is addressed. Previous research focused on time duration bias mostly in SP research settings, concluding that misperceptions of time duration are prevalent even for mathematically competent subjects. A notable exception to SP research was the study conducted by Ma et al. (2016), based on a two-step modelling approach (SA/CL). The main drawback of that approach was that the time interval bias considered therein was evaluated by taking the last win of every single runner as the time baseline for conditional probability calculation, which eliminates the vast majority of runners in the dataset and requires rather large datasets extending over many competitive seasons. The modelling framework introduced in this thesis avoids this limitation, since the time duration between all successive races is taken into account directly. This engenders a significant advantage in applications where repeated choices are rare, but nevertheless reoccurring, and it is important for predicting outcomes.

Chapter 6

Summary and Conclusions

6.1 Introduction

The final chapter of the thesis summarises the research endeavours undertaken in this study. The thesis started with a preliminary review of the discrete choice modelling literature with a focus on behavioural dynamics effects in discrete choice settings, namely state dependence, habit persistence, and heterogeneity. Next, a common characteristic of incumbent dynamic models with lagged effects of previous choices was identified, i.e. reliance on balanced panel-type data, equidistant in time, under an (implicit) assumption that the DM faces the same choice set in every decision event. Furthermore, a potential increase in model fitting efficiency, if the ranked *ex-post* event data are included, was discussed. These considerations, which emerged from the limitations of the incumbent approaches and identified gaps in the literature, yielded a research problem statement (1.1) with clearly delineated research objectives (1.2).

These research objectives postulated a definition of discrete decision-making models which could be used for predicting the behaviour of a DM facing changing choice sets and irregular but known times between decision events. Endogenous trends of alternative-specific proxies of the preferences, constructed from ranked data are seen as input to a SSM model with latent states capturing dynamic effects. Dynamics of the trends evolution is estimated through the KF algorithm. These trends are interpreted as conditional random variables partially explaining an unobserved utility in CL-based RUM models. An estimation of the model parameters completed the conceptual modelling framework outlined in the research objectives.

An extensive literature review of static and dynamic DCM approaches founded on RUM, including an analysis of unobserved properties of the utility functions, and the interplay between dynamic latent variable modelling of learning, habit persistence, consumption inertia, and state dependence effects, was given in Chapter 2. This was then followed by an introduction to a linear SSM as an appropriate model structure for estimating latent states (variables), proxying the dynamic effects listed above. For the conditional estimation of the latent states, the optimal linear filter, namely the KF algorithm, was put forward, and its properties and the intuition behind it were explained. Two parsimonious and simultaneously versatile dynamic models (LLM and LLT) were presented next, together with the measurement noise covariance adjustments for non-equidistant sampling. A discussion on their adequacy for modelling the unobserved portion of the utility function followed, with the conclusion that because of the unknown DGP underlying the random portion of utility, ageweighting model error-correcting algorithms are more likely to provide better out-of-sample results than the standard KF algorithm.

The research objectives and the proposed modelling structure allowed for the development of research questions, which, when analysed and answered based on empirical data, provide the basis for evidence that the research objectives are met and the identified gaps in the literature are closed. The formulation of four broad research questions in an empirical setup (horse-wagering markets) which combines provisions for dynamic modelling, changing choice sets, and non-uniform times between the availability of alternatives in a choice set delineates the boundaries of the research. Organisation of the research questions was implicitly based on two cognitive biases, observable as (1) the unreliability of ordered data for less preferred alternatives and (2) misperceptions of time duration. In the research questions, controls and mitigating measures for the two biases are progressively introduced for easier results analysis. For each of the four primary research questions, a secondary research question, addressing the economic significance of the residual information obtained from bias control and mitigation measures, assessed through monetary gain that could be achieved through betting activities utilising that information, was posed. The first research question was concerned with the informational content of patterns of previous ordered choices and its statistical significance over publicly available market data whilst ignoring the time duration between decision events (races). The second research question queried the informational content of patterns of previous ordered choices and its statistical significance over publicly available market data whilst controlling for temporal bias. The third research question addressed the effectiveness, i.e. statistical significance gain over publicly available market data, of controls for the unreliability of lower ranked ordered choices in the same context. Finally, the fourth research question integrated proposed controls for both biases and sought to evaluate the statistical significance of the residual information content derived from previous choices weighted on both importance (unreliability bias) and time (time duration bias).

The methodology chapter (3) started by endorsing the ontological stance founded on *critical realism*, with the target to define a model or series of models that sufficiently well describe the reality within the framework required in the research objectives. This, in combination with the epistemological view of the research, rooted in *modified dualism/objectivism*, positioned the study within the *post-positivistic* research paradigm. Moreover, the *quantitative and deductive* methodologies required the careful selection of data capture and analysis methods. In this study, an RP was selected in order to avoid biased decision-making, inherent in SP research designs, where no monetary incentives for correct decisions are present. On the other hand, the selected empirical setup framed for assessing the research questions, namely horse-wagering markets, consisted of well-calibrated decision-makers with considerable monetary incentives for selecting alternatives to the best of their knowledge. Next, the mathematical foundation of state space discrete choice models geared towards persistence in preference effects was derived, and key details of the practical implementation of the algorithms, such as initialisation with diffuse priors, were given. Finally, a research design description, encompassing (1) empirical test planning, (2) model parameter fitting procedure, (3) statistical and application relevant model evaluation criteria, and (4) the used dataset explicitly closed the circle of the conceptual modelling framework, explicitly or implicitly needed for every orderly academic study (Trafford and Leshem, 2008).

The last section of the methodology chapter describes the dataset used – 42,768 races with 43,424 runners imported from the Betfair UK horse-racing database covering the time frame from January 1st, 2007, until December 31st, 2012, including both all-weather and turf racing courses. For the purposes of understanding variances in the choice of set sizes, distribution of time durations between the races and an evaluation of models, the dataset has been split into three non-overlapping datasets: LDS, BDS, and VDS. All data types, i.e. runner-level, race-level, and market variables, used and the associated relevant descriptive statistics have been presented and discussed.

Empirical results for the study were captured in Chapter 4. The results include all of the main and intermediate results of the steps captured in the overarching quantitative algorithm integrating data generation, prediction and analysis (cf. Figure 3.3). In particular, the results, divided into two groups (in-sample and out-of-sample), underscored the analysis, based on which all four primary research questions could be discussed and answered in full. Finally, the economic significance of the residual information extracted from the trends of performance proxies was evaluated, in order to provide answers to all four research questions.

Chapter 5 picked up a broader context of two biases bound to the unreliability of the lower ranked alternatives, if ordered preference data are available, and the time duration bias and framed the research results in the context of their influence. The biases are particularly relevant in dynamic decision-making settings, in that a successful modelling approach which would allow convergence and generalisation of the results in adjacent disciplines builds a considerable contribution to knowledge. Finally, the modelling framework, which was one of the outcomes of this research, was linked to the understanding of the biases and the ways of mitigating them in an empirical setting.

The remainder of this chapter highlights the theoretical and methodological contribution of the research (6.2), signposts the limitations (6.3), suggestions for future research (6.4), and potential impact of the research (6.5) related to dynamic discrete decision models. Finally, an accomplishment summary (6.6) wraps up the thesis in a form of summary of the major findings realised during the research endeavour.

6.2 Contribution of the Research

This section reiterates and discusses the contributions made by the thesis to knowledge and our understanding of dynamic decision-making and modelling of persistence in preferences. Overall, this thesis makes a significant contribution to understanding the effects and underlying elements of choice persistence effects, i.e. the effects of inertia, which influence the probability of repeated choice, if the same choice was already made in the past, and how they can be used for predicting the behaviour of a DM. In order to underscore the extent of the contribution to knowledge, two aspects of the research are discussed – the theoretical and methodological contributions to academic knowledge.

6.2.1 Theoretical Contributions

The foremost aspect of the theory of dynamic decision-making addressed in this thesis was the identification of different effects and their modelling proxies, and how these may be combined to explain and predict persistence in preferences. A detailed study of the existing literature revealed that the incumbent models lack provisions for evolving choice sets and irregular duration times between decision events. Furthermore, it was concluded that including ranked data allows for the extraction of additional information that may improve the efficiency of estimators of the probabilities bound to particular alternatives.

The first theoretical contribution involved the construction of a model that estimates (non-linear) trends in the 'attractiveness' of the alternatives, to allow arbitrary combinations of alternatives in every decision event and hence overcome the problem relating to changing choice sets. A direct consequence of the new proposed structure is the elimination of effect-confounding problems, inherent in incumbent models – the second theoretical contribution. The thesis presents a convenient mathematical interpretation of the prediction of the 'attractiveness' of alternatives as random variables conditional on trend information, which constitute a part of the unobserved portion of the utility in the RUM structure underlying the CL. The interpretation is closely coupled with the postulated *post-positivistic* research paradigm, which supports the incremental reduction of elements of randomness in existing models – in this case founded on RUM. This *post-positivistic* reflection on interpretation defines the third significant theoretical contribution, which can readily be used in all dynamic decision-making problem settings based on RUM, such as CL, *probit* or GEV models.

Furthermore, the *post-positivistic* research paradigm dictates the prudent selection of an empirical methodology that can mitigate biases and fallacies, both from the researcher's observations and the *a-priori* selection of the relevant underlying theories. Selecting the RP research design in a naturalistic environment (UK horse-wagering markets), in which participants have a vested interest in correct (optimal) decisions, allows for capturing and analysing datasets related both to behavioural (decision-maker related) and economic (betting market-related) information. It should be noted that this study is certainly not the first to utilise betting markets to illustrate discrete decision-making model performance. However, it is the first one in which persistence in preferences and associated biases, time duration bias, and the unreliability of lower ranked preferences have been taken into account in combination – the fourth contribution. The results presented in this thesis demonstrate that the biases affect modelling performance and that it is possible to obtain above-average financial returns if controls for the mentioned biases are implemented. These results demonstrate evidence of two economically relevant biases in decision-making and that trends in horse performances contain residual information over the market, thus marking the fifth significant contribution to the literature.

6.2.2 Methodological Contributions

The methodological contribution of the thesis effectively complements the theoretical contributions outlined above. Including dynamic effects in discrete decision-making settings in the traditional literature is typically achieved through the inclusion of lagged (and dummy) outcome variables (Keane, 2015), continuous latent states that model previous (unknown) utility functions (Lee, 2014) or autoregressive processes explaining the autocorrelation of the unobserved parts of the utility, with each of the mechanisms targeting one of the fundamental effects of decision-making – state dependence, habit persistence, and spurious dependence (cf. Figure 1.1, Figure 1.2, and Figure 1.3). In this thesis, a new modelling strategy was applied, which simultaneously includes several methodological contributions.

The first major methodological contribution lies in the model structure, which introduces the latent states that track revealed preferences for a certain alternative in the form of (nonlinear) trends (evolution) of attractiveness (performance) proxies, and these are combined in a CL structure during any decision event in which the alternative is available. The modelling approach offers support for changing choice sets automatically. Moreover, the estimation of the latent states is accomplished with the KF, which was used only in very few DM studies (Edelman, 2007b; Guhl, 2014) in which the KF was used in a different context of the adaptive estimation of (static) model parameters.

The second major methodological contribution is closely related to the two-step model approach. As explained in the section on assumptions on KF, cascading KF and CL invalidates the fundamental assumption regarding the linearity of dynamic system behaviour. In other words, parameter estimation of the KF, based on MLE, cannot be carried out, due to the inherent non-linearity of a CL alternative selection. Section 3.3.3 resolves the problem of parameter estimation and establishes how the model parameters of both KF and CL can be obtained through the separate maximisation of the conditional likelihood built around the CL whilst using KF predicted trends and the maximisation of the product of marginal likelihoods.

The third major contribution is bound to the actual KF algorithm. Standard, linear KF equations were originally developed under the assumption that the stochastic model of underlying dynamics is fully known. This is, unfortunately, practically never the case, and filters can exhibit unacceptable performance caused by filter divergence. In this study, a novel error-correction filtering structure was derived, which outperforms both standard linear and known error-correcting algorithms used in aerospace navigation and robotics applications (Jazwinski, 1970; Skelton and Likins, 2012). In fact, it has been shown that the new model yields by far the best out-of-sample predictions of winning probabilities in spite of rather mediocre in-sample ranking, thus highlighting effective mitigation against over-fitting.

The contributions of the study outline opportunities to expand the theoretical underpinnings and associated methodologies in other academic and industrial decision-making applications. However, some limitations which might impede the generalisation of the results have to be highlighted and explained. These are discussed in the next section.

6.3 Limitations of the Study

Every research is based on a set of explicit and implicit assumptions which reinforce and delineate both the main strengths and the main limitations of a study. The new modelling structure, whilst powerful and versatile, contains assumptions regarding the random nature of unobserved utility functions, model uncertainty, and measurement noise. Postulation of the Gumbel distribution for the CL model stage, and the normal distribution for the KF stage, corresponds to the standard assumptions typically used with these models because of their 'convenient' numerical or interpretational properties. However, no proof is available that this is actually true for the given empirical application, in spite of the good performance achieved.

The next limitation is bound to the selected empirical setup – horse-wagering markets with an abstract decision-maker. The advantages of an RP empirical setup are obvious (see 3.1), albeit the inherent narrowness of the research scope has to be pointed out. Furthermore, the assumption of rationality of a DM in a decision-making context has been increasingly scrutinised in the literature. Even though the approach presented in this thesis is considered to be robust against (temporary) departures from rationality (see 5.2), transferability of the conclusions made on other empirical settings has to be validated and checked for consistency in the new settings.

Finally, two methodological limitations bound to the KF can be identified: (1) the inability to predict the performance of runners, with only one or two races in their career, due to the initialisation of the filter and (2) the deterministic linear and exponential growth of the state and measurement noise covariances in time to account for irregular time durations between decision events. These methodological limitations are similar to those related to the modelling structure, in that they are functionally effective but without a formal proof of truth – exclusion of the runners with short careers slightly reduces the available datasets for both model fitting and evaluation purposes and the covariance time growth law, whilst a reasonable assumption remains only an approximation of an unknown growth law.

6.4 Suggestion for Future Research

The models and the methodology put forward in this research could be easily adopted in a variety of research studies in decision-making and operational research. The presented approach should prove to have a particular appeal in marketing and political science settings, where time between events, competition, i.e. relative attractiveness of choices, and ranking information are of interest. In particular, all research positioned in a naturalistic setting could assess the transferability and consistency of the methods used to predict the choices made by different types of DM having potentially different rationality levels. Furthermore, the inclusion of some other cognitive biases, such as anchoring (Furnham and Boo, 2011) or preference reversal (Tversky et al., 1990), should offer interesting results.

From the methodological point of view, further interesting research topics may be found in researching hypotheses and investigating characteristics of the underlying stochastic properties of the models, which could possibly improve forecasting performance, as a continuation of the model improvements inherent in the *post-positivistic* research paradigm. Finally, the linear and exponential laws defining the covariances of the state and measurement noise may be abandoned in favour of more sophisticated stochastic volatility modelling, known from financial econometrics (Andersen and Benzoni, 2014).

6.5 Potential Impact of the Research

As expected, multidisciplinary research which embraces problem settings and methods from different social sciences, statistics, econometrics, and control systems theory is likely to create methodology that might benefit other domains. This section outlines how the results of this thesis may be used in other domains/applications and which potential academic impact they may have. Obviously, the domains, facing either irregular sampling times in the context of changing choice sets in a dynamic discrete decision model or challenges, similar to ones explained in 1.3, may find opportunities to improve modelling results.

The first domain that faces changing choice sets can be found in political sciences, in particular, in modelling of electoral results. Classical forecasts of electoral results are built around the expectation that the parties participating in elections should converge to the electoral mean (Gallego et al., 2014). However, the recent research rarely confirms such convergence, especially in geographically large countries with different regional, transregional, and national parties competing in the elections, which may be irregular and offset in time. As a result of such electoral setting, voters face different choices of parties in different regions and the choice set may vary from one elections to another. Similarly, multi-candidate and multi-party elections settings can be used to investigate regional heterogeneity in voter behaviour (Glasgow, 2001). In terms of modelling, methodology put forward in this thesis could be replicated to yield a *logit* relationship between party position, i.e. the probability of win in the considered election event, and the KF trend estimation of voter positions in the policy space. Obviously, additional idiosyncratic variables, such as competence and socio-demographic valence, could be added to the CL part for further model improvement. Due to the dynamic characteristics of the model it is expected that the gain in modelling accuracy would be the greatest in countries with unstable party systems, i.e. with regular appearing of new parties and loose coalitions, that changes from election to election. Finally the results of electoral modelling could be used either for calibration of probabilities in prediction market for election results (Berg et al., 2008) or for electoral campaign planning.

The next domain that may profit from the methodology presented herein are behavioural change management studies, particularly of adoption of new technologies or products. As easily recognised, resistance to change is reflected in failure of a user to switch from an incumbent technology or product to a newly introduced one. Psychologists relate the inertia, or persistence in preferences, to behavioural, cognitive, and affective effects (Polites and Karahanna, 2012). All these effects have different psychological explanations, for example continuous repetition of old patterns of behaviour without much consideration for new technology, product, or information (behavioural effects), deliberate ignorance of new information that challenges 'old' beliefs (cognitive/mental inertia), and change avoidance because of the expected stress bound to change or the emotional attachment to the incumbent technology/product (affective-based inertia) (Barnes et al., 2004; Rumelt, 1995; Kim, 2009). Whilst the methodology presented in this thesis does not differentiate amongst these effects but rather focuses on the statistical decomposition of the error term, any methodology attempting to provide prediction of change behaviour may use the methodology proposed here as a application specific benchmark of the model quality (c.f. general forecasting performance criteria).

Traditionally, exploration of travel habits has been one of the most widespread application domains of decision making theories (Neoh et al., 2018). Numerous studies of travel habits confirmed that rational static and quasi-static day-to-day models of route choices have a very limited prediction power. Indeed, habit, the related reinforcement learning, and risk attitude of the travellers are important components of their decision making process. Moreover, travellers take readily available information regarding the en-route traffic conditions (e.g. expected travel time, congestion, delay, etc.) into account (Bogers et al., 2005). Similarly to the methodology presented in this thesis, *ex-post* information on previous individual trips contains the information that allows adaptation of the persistence of preferences, and its incorporation will likely improve the model performance. In the same time, it should offer a way to define policies that may lead to more efficient traffic planning and control. Finally, travellers differentiate themselves in their attitude towards uncertainty, either in *ex-ante* or *ex-post* en-route traffic conditions and travel durations. As presented in the methodology section, KF trend estimation can easily be adapted to account for the variance of the available information and hence mitigate the effects of uncertain (noisy) information on traffic conditions. Finally, eventual non-linear risk averseness, e.g. strong dislike for being late, could be modelled through moderate modifications of the KF which can incorporate non-linear functional forms in the trends estimation – Extended Kalman Filter (EKF) and Unscented Kalman Filter (UKF) (Simon, 2006).

Another large application area, that may readily incorporate the methodology presented in this thesis, are general speculation markets. Candelon et al. (2014) endorsed dynamics considerations in predictions of currency crises in form of Early Warning System (EWS), as opposed to more traditional static panel *logit* (Bussiere and Fratzscher, 2006) and *probit* (Berg and Pattillo, 1999) models based on the reference utility as linearly weighting of concurrently implemented economic and monetary policies captured in form of macroeconomic variables, with moderate success. As an *ex-post* proxy of performance, either the probability of crisis (for binary models) or ranking of probabilities can be effectively constructed. The idea of the residual information decomposition, endorsed in this thesis, shows a way to increase efficiency of estimation through importance weighting based, for example, on the size of a country (e.g. USA has much higher monetary base than Greece) and economic relationships between the countries. Furthermore, (Emekter et al., 2012) studied the time duration bias in terms of persistence of the over-valuation periods in which rational agents remain and fuel the growth of the speculative bubbles in 28 commodity markets. His model assesses the probability of bubble bursts in terms of the hazard function representing the probability of hazard rate is defined as the probability of obtaining a negative excess return after a streak of prior positive above average returns captured as a logarithm of number of time periods with strictly above average returns. These probabilities can be enhanced with volumes of trading in order to assess the public time duration bias and the estimated trends following the methodology presented here, which in turn, may propel commodity future market regulators to aim to reduce informational asymmetries in order to inhibit speculative financial bubbles.

Further implications of the presented findings are that, in financial and other prediction markets, trading rules may be imposed to minimise the impact of irregularly spaced or event-driven information updates on transient pricing, in order to allow sufficient time for the market to discount the updated information and reach desired efficiency levels after a market disturbance (Berg et al., 2008)

6.6 Accomplishment Summary

This section summarises the major findings stemming from the results of this study. Starting from two research objectives, cascaded down to four primary and four secondary research questions, driving the generation of the conceptual research framework and the empirical test setting and tied together by the postulated research paradigm, empirical results were collected and analysed in order to answer the set research questions. Based on the empirical findings, all research questions have been answered. The results of the analysis – the empirical findings – together with the development of the modelling framework and the detailed methodology, provide evidence that the research objectives have been met and that the identified gaps in the literature have been closed in full.

Findings presented in this thesis contribute new knowledge on dynamic discrete choice models designed to predict the effects of temporal persistence in preferences. The proposed research setup, aiming to eliminate confounding effects bound to different dynamic model components, allows for the proper separation of the possible causes of persistence in preferences, through extended support for changing choice sets, irregular durations between the recurring availability of alternatives in a choice set and feedback on the relative preferences for alternatives revealed in previous decision events. It has been found that incorporation of the irregular durations between the availability of alternatives in decision events, in the form of endogenous trends forecasts, adds statistically relevant information to the market data information set, which may be used by informed bettors to achieve above-average profits from betting. In addition, it has been shown that the standard KF algorithm is dominated by the error-compensating implementation, in which the covariance matrix was specially engineered for model stability (i.e. mitigation of filter divergence). This indicates that the standard structural LLM model is not an appropriate data-generating model underlying persistence state variables. This is not surprising, since the decision behaviour of an abstract DM (‘nature’), whose absolute rationality has been postulated, has been tested. Degrees of rationality of DM are not known *a-priori*, since the agents do not have algorithmic (i.e. consistent or immutable) preferences. That is to say that even objectively irrelevant contextual factors, such as a preference for a blue over a red public transportation vehicle (Train, 2009), can (and do) affect individual decision-making behaviour in systematic ways (Cherchi, 2012). However, if the main goal of decision-making modelling efforts is forecasting, these factors have to be taken into account, to avoid possible model bias. In other words, contextual factors leading to persistence in preferences explain a portion of unobserved utility, interpreted as an additive random variable (i.e. noise) to observed reference utilities. Discovering systematic preference regularities in the form of endogenous trends can be used to obtain better models for different applied problem settings, such as transportation planning or marketing, etc. (Brailsford et al., 2014).

Based on the summary of research accomplishments given above, this thesis has made significant contributions to the existing literature on biased decision-making in an naturalistic empirical setting, and it has shown a number of ways for the interpretation and expansion of models on other academic and industrial decision-making applications.

Appendix A

Conditional Logit variable filtering

In horse racing context, probabilities in CL models may be affected by numerical difficulties when some runners have variables and some do not. This is a standard case for débutant races and for some lagged variables early in runner's career. In such cases, following filtering procedure is applied under the assumption that all runners in all races have *LOGPRICE*.

1. Estimate a model containing only *LOGPRICE* as exogenous variable over all horses and races in LDS - denominated as **M1**. As a guidance regarding the validation β should be around 1.15.
2. Select all horses that have all relevant parameter available. Estimate a model containing only selected runners over all races - denominated as **M2**. Parameter β for *LOGPRICE* should be slightly lower than in **Model 1**.
3. For out-of-sample probability calculation, each race is considered separately. Some horses will have variables (selected - $\boldsymbol{s}$), some won't (non-selected - $\boldsymbol{ns}$).
4. Over every runner, probabilities according to **M1** are calculated ($p_{i,j}^{\boldsymbol{M1}}$). Probabilities will sum to 1, i.e. $\sum p_{i,j}^{\boldsymbol{M1}} = 1$.
5. Sum of the probabilities over non-selected runners will be less than one ($x = \sum_{\boldsymbol{ns}} p_{i,j}^{\boldsymbol{M1}} \leq 1$).
6. Over all selected runners **Model 2** is applied resulting in sum of probabilities equal 1.
7. To merge the probabilities different weighting of the calculated probabilities is applied
   (a) if runner is non-selected, $p_{i,j} = p_{i,j}^{\boldsymbol{M1}}$
   (b) if runner is selected, $p_{i,j} = (1 - x)p_{i,j}^{\boldsymbol{M2}}$

Final (weighted) probabilities will sum to 1 since

$$\sum_{\boldsymbol{ns}} p_{i,j}^{\boldsymbol{M1}} + \sum_{\boldsymbol{s}} (1-x) p_{i,j}^{\boldsymbol{M2}} = x + (1-x) = 1. \tag{A.1}$$

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Acronyms

ARMA Autoregressive Moving Average. 31–33

BDS Burn-in Data Set. 73, 81–85, 125

BIC Bayesian Information Criteria. 26

BR Bank Rate. 6, 28

CDF Cumulative Density Function. 20

CL Conditional Logit. 3, 7, 9, 13, 21, 29, 30, 43, 45, 51, 59, 62, 63, 67–70, 73, 74, 78, 79, 81, 84, 87, 88, 113, 116, 117, 121–123, 126–128, 130, 135

DCDP Discrete Choice Dynamic Programming Model. 14, 23

DCM Discrete Choice Model. 5, 7, 13, 14, 17, 18, 22–24, 29, 30, 43, 48–51, 55, 64, 73, 77, 108, 123

DGP Data Generating Process. 10, 11, 13, 41, 58, 124

DM Decision Maker. 3, 4, 6–9, 13, 17, 18, 22–25, 44, 50, 53, 56, 62, 115–123, 126–129, 132

DOF Degrees of Freedom. 78

EKF Extended Kalman Filter. 131

EM Expectation Maximisation. 33

EMH Efficient Matket Hypothesis. 44

EWMA Exponentially Weighted Moving Average. 14, 38, 39

EWS Early Warning System. 131

FDS FDS Data Set. 81, 84, 85

GEV Generalised Extreme Value. 20, 126

IIA Independence of Irrelevant Alternatives. 19, 21, 43, 88, 116

KF Kalman Filter. 6, 9, 11, 13–15, 17, 30, 34, 35, 37, 39–42, 44, 49–51, 55, 57–59, 61–66, 73–77, 81–84, 87–91, 95, 113, 117, 123, 124, 127–132

LDS Learn Data Set. 73, 78, 81–85, 87, 88, 91, 96, 125, 135

LLM Local Level Model. 14, 37–40, 42, 50, 58–61, 84, 89, 124, 132

LLT Local Linear Trend. 14, 37, 40, 42, 50, 58, 59, 61, 84, 87, 89, 90, 124

LR Likelihood Ratio. 67–70, 78, 79, 85, 90, 95, 101, 102, 104, 105, 107, 116

MA Moving Average. 65

MCMC Markov Chain Monte Carlo. 55

ML Maximum Likelihood. 33, 35, 51, 78

MLE Maximum Likelihood Estimation. 73, 74, 77, 79, 87, 91, 128

MMSE Minimum Mean Square Error. 37, 57

MSE Mean Square Error. 39, 41, 77, 80, 87–89, 117, 118

NDCG Normalized Discounted Cumulative Gain. 56

OLS Ordinary Least Squares. 39, 77

PDF Probability Density Function. 13, 19, 20, 32, 34, 73–75

RF Random Forests. 45

ROI Return on Investment. 30, 67, 68, 70, 71, 80, 101, 102, 105, 106, 108–112

ROL Rank-Ordered Logit. 115

RP Revealed Preference. 53–55, 84, 118, 125, 127, 128

RUM Random Utility Model. 5, 9, 13, 18, 19, 25, 43, 44, 123, 126

SA Survival Analysis. 121, 122

SP Stated Preference. 53, 54, 117, 118, 122, 125

SSM State Space Model. 6, 7, 9, 14, 15, 17, 30–34, 36, 39, 42–44, 49–51, 55, 57, 58, 60, 64, 72, 73, 83, 123, 124

SVM Support Vector Machines. 45, 117

UKF Unscented Kalman Filter. 131

VDS Validation Data Set. 73, 78, 81–83, 85, 95, 125