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Green operators in low regularity spacetimes and quantum field theory

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Abstract

In this paper we develop the mathematics required in order to provide a description of the observables for quantum fields on low-regularity spacetimes. In particular we consider the case of a massless scalar field ϕ on a globally hyperbolic spacetime M with $C^{1,1}$ metric g. This first entails showing that the (classical) Cauchy problem for the wave equation is well-posed for initial data and sources in Sobolev spaces and then constructing low-regularity advanced and retarded Green operators as maps between suitable function spaces. In specifying the relevant function spaces we need to control the norms of both ϕ and $\Box_g\phi$ in order to ensure that $\Box_g \circ G^{\pm}$ and $G^{\pm} \circ \Box_g$ are the identity maps on those spaces. The causal propagator $G = G^{+} - G^{-}$ is then used to define a symplectic form ω on a normed space $V(M)$ which is shown to be isomorphic to $\ker(\Box_g)$. This enables one to provide a locally covariant description of the quantum fields in terms of the elements of quasi-local C^{*}-algebras.

Keywords: low regularity, weak solutions, Green operators, quantum field theory

1. Introduction

This paper is concerned with developing the theory of quantum fields on low regularity spacetimes. We will follow the algebraic approach to quantisation as described in [1, 2]. In particular

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we will draw heavily on the detailed scheme given in the book [3]. The starting point is a smooth Lorentzian manifold (M, g) and a field equation $P\phi = f$ where P is a normally hyperbolic differential operator P acting on a vector bundle F. In this paper we will consider the scalar operator $P = \Box_g$ as there are no significant additional mathematical issues in dealing with the general case. Adding lower order terms to $\Box_g$ would require their coefficients to be globally bounded and Lipschitz continuous (as in theorem 3.7), while going from the case of a scalar operator to one acting on a vector bundle would require a change to vector valued Sobolev spaces. In terms of the proofs, the main change would be in the proof of uniqueness. For fields which satisfy the dominant energy condition, the proof of lemma 4.1 would be essentially unchanged, cf [4, lemma 7.4.4], while in the general case one would need to follow the lines of the proof of [5, lemma 12.8]. Thus the results obtained here would also apply to the case of a massive scalar field or an Abelian gauge field.

Almost all the literature dealing with quantum field theory on a curved spacetime assumes that the background metric is smooth. However from the physical point of view it is desirable to be able to deal with a much wider class of spacetimes. For example a spacetime with some kind of interface (such as at the surface of a star) or matching (e.g. at a domain wall) will only have finite differentiability. Other examples with finite differentiability include the Oppenheimer–Snyder model of a collapsing star and spacetimes with branes or cosmic strings. In braneworld scenarios cosmological perturbations should be described by QFT on corresponding backgrounds describing the collision of the branes [6] while recent discussions on entanglement entropy, compute this via the replica method which involves putting QFTs on backgrounds with conical singularities [7]. There is thus a strong physical motivation to establish a mathematical framework to deal with quantum field theories on low regularity spacetimes. In generalising the smooth results to the low regularity setting we need to choose an appropriate class of metrics. In the present paper we have chosen to work with metrics that are $C^{1,1}$. In many ways this choice is natural since it allows one to deal with spacetimes where the curvature remains finite while allowing for discontinuities in the energy–momentum tensor at, for example, an interface or the surface of a star. From the mathematical point of view $C^{1,1}$ is the minimal condition which ensures existence and uniqueness of geodesics and for which the standard results from smooth causality theory go through more or less unchanged [8]. It also guarantees that the solutions to the wave equation are in $H^2_{\mathrm{loc}}(M)$ (see appendix B for details of the function spaces we use) as shown in theorem 4.7 below, which ensures we have enough regularity to define the quantisation functors we need. Although one can define solutions to the wave equation for metrics of lower regularity [9, 10] (including the case of cosmic strings) there are additional technical difficulties that need addressing, so we will return to these cases in a later paper.

The essence of the algebraic approach as outlined in [3] is to first construct the advanced and retarded Green operators for P and use these to construct the causal propagator $G = G^+ - G^-$. Note that in order for the Green operators to be unique we require the spacetime to be globally hyperbolic. The causal propagator is then used to construct a skew symmetric bilinear map on the space of smooth functions of compact support by $\tilde{\omega}(\phi, \psi) := \langle G(\phi), \psi \rangle_{L^2(M,g)}$. This form is degenerate but gives rise to a symplectic form ω on the quotient space $\mathcal{D}(M)/\ker(G)$ of the test function space. The next step in the process is to use ω to construct representations of the canonical commutation relations (CCRs) as quasi-local C^*-algebras [3, theorem 4.4.11] which satisfy the Haag–Kastler axioms [11]. Each of these steps may be described in terms of a functor so that we have a functorial description of how to go to from the category of globally hyperbolic manifolds equipped with (formally self-adjoint) normally hyperbolic operators to the space of quasi-local C^*-algebras whose elements describe the observables of the field (see details below). Indeed this scheme gives rise to a *locally covariant quantum field* in the sense of [12]. Going from the rather abstract quantisation procedure described here to the more familiar Fock space representation requires one to pick out the physically relevant states. For the smooth case a mathematically appealing criterion (which corresponds to the standard answer in Minkowski space) is the micro-local spectrum condition of Radzikowski [13]. However this is not directly applicable in the low-regularity case. We return to this point and suggest a suitable modification in the discussion section at the end.

In section 3 we establish the results we need to prove existence and uniqueness of solutions to the forward (and backward) initial value problem for the wave equation on $\mathbb{R}^{n+1}$ for $C^{1,1}$ metrics. Rather than rework the entire theory of the wave equation for metrics of low-regularity we use a method of regularising the coefficients [14], using the smooth theory to obtain the corresponding solutions of the Cauchy problem and then using a compactness argument to show that this converges to a weak $H^2_{\mathrm{loc}}(\mathbb{R}^{n+1})$ solution of the original equation. This proceeds via the theory of Colombeau generalised functions [15] and is related to the work of [16] on very weak solutions. Furthermore by controlling the causal structure of the regularisation g_ε in such a way that $J^+_\varepsilon(U) \subset J^+(U)$ holds for any relatively compact subset U (cf appendix A) we obtain in lemma 3.5 that the forward solution u^+ with zero initial data satisfies the causal support condition $\mathrm{supp}(u^+) \subset J^+(\mathrm{supp}(f))$ (cf [5, theorem 2.6.4]).

In section 4 we introduce the notion of global hyperbolicity [17] and temporal functions for non-smooth metrics [18] as well as the other results from $C^{1,1}$ causality theory that we require [8]. We use the fact that even for $C^{1,1}$ spacetimes the temporal function can be chosen to be smooth so we can write M as $\mathbb{R} \times \Sigma$, where Σ is a Cauchy surface, and define function spaces where we make a split between space and time. However, as far as possible we formulate our final results in a way that is independent of the choice of temporal function, so that the particular choice of space-time split is not important. The remainder of the section shows how to go from existence and uniqueness results on $\mathbb{R}^{n+1}$ to global results on a globally hyperbolic $C^{1,1}$ spacetime. Our approach to this closely follows Ringström [5] and the causality results for $C^{1,1}$ metrics [8] ensure that the existence proof remains similar to the smooth case.

The next step is to define appropriate Green operators. In the smooth case the Green operator takes (compactly supported) smooth functions to smooth functions. However in the $C^{1,1}$ case it is crucial that the map is between suitable Sobolev type spaces. In fact, we find precise conditions on the regularity of the solutions and the causal support in order to define unique Green operators in globally hyperbolic spacetimes of limited differentiability. We need to control the (local) Sobolev norms of both ϕ and $\Box_g\phi$ in order to ensure that $\Box_g \circ G^\pm$ and $G^\pm \circ \Box_g$ are the identity maps on the corresponding spaces. The choice of function space is also relevant in the definition of $\ker(G)$ which is used to construct the factor space for the symplectic map ω. We end the section by considering the dependence of the construction on the choice of temporal function.

The passage from the symplectic space defined by ω to the canonical commutation relations proceeds almost identically to that of the smooth case [3] so we only sketch out the details although the analogue of [3, theorem 4.5.1] requires some work.

We end the paper with a summary of the results we have obtained and a discussion of the outstanding issues, including the choice of a Sobolev micro-local spectrum condition to single out the physical states in the low-regularity setting.

Appendix A briefly describes the basic properties of regularisation methods we use while appendix B gives a brief description of various Sobolev spaces we use in the paper.

Notation. The space of smooth functions of compact support on a manifold M will be denoted by $\mathcal{D}(M)$. A function f on an open subset $\mathcal{U}$ of $\mathbb{R}^n$ is said to be Lipschitz if there is some constant K such that for each pair of points $p, q \in \mathcal{U}$, $|f(p) - f(q)| \leqslant K|p - q|$. We denote by $C^{k,1}$ those C^k functions where the kth derivative is a Lipschitz continuous function. A function on a smooth manifold is said to be Lipschitz or $C^{k,1}$ if it has this property upon composition with any smooth coordinate chart. We gather basic notions and results concerning Sobolev and related function spaces in appendix B.

2. The smooth setting

In this section we briefly review the results in the smooth setting. The starting point is an existence and uniqueness result for solutions to a smooth second order hyperbolic equation on $\mathbb{R}^{n+1}$ (see e.g. [5, theorem 8.6]). This is used to obtain a corresponding result showing the existence of a unique smooth solution $u \in C^\infty(M)$ to the Cauchy initial value problem for a smooth normally hyperbolic operator P on a smooth globally hyperbolic manifold with smooth spacelike Cauchy surface Σ and normal vector field n given by

$$\begin{aligned} Pu &= f \quad \text{on } M \text{ where } f \in \mathcal{D}(M) \\ u|_\Sigma &= u_0 \quad \text{on } \Sigma \text{ where } u_0 \in \mathcal{D}(\Sigma) \\ \nabla_n u|_\Sigma &= u_1 \quad \text{on } \Sigma \text{ where } u_1 \in \mathcal{D}(\Sigma) \end{aligned}$$

which satisfies the causal support condition

$$\mathrm{supp}(u) \subset J(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)).$$

Note that the globally hyperbolic condition is essential in order to ensure that the solution is unique.

The next step is to show the existence of advanced and retarded Green operators.

Definition 2.1. Let M be a time-oriented connected Lorentzian manifold and let P be a normally hyperbolic operator. An *advanced Green operator* G^+ is a linear map $G^+ : \mathcal{D}(M) \to C^\infty(M)$ such that

(a) $P \circ G^+ = \mathrm{id}_{\mathcal{D}(M)}$
(b) $G^+ \circ P|_{\mathcal{D}(M)} = \mathrm{id}_{\mathcal{D}(M)}$
(c) $\mathrm{supp}(G^+\phi) \subset J^+(\mathrm{supp}(\phi))$ for all $\phi \in \mathcal{D}(M)$.

A *retarded Green operator* G^- satisfies (a) and (b) but (c) is replaced by $\mathrm{supp}(G^-\phi) \subset J^-(\mathrm{supp}(\phi))$ for all $\phi \in \mathcal{D}(M)$.

Corollary 3.4.3 of [3] shows that these exist and are unique on a globally hyperbolic manifold.

The advanced and retarded Green operators are then used to define the causal propagator $G := G^+ - G^-$ which maps $\mathcal{D}(M)$ to the space of spatially compact maps $C^\infty_{\mathrm{sc}}(M)$ i.e. the smooth maps ϕ such that there exists a compact subset $K \subset M$ with $\mathrm{supp}(\phi) \subset J(K)$. If M is globally hyperbolic then one has the following *exact sequence* [3, theorem 3.4.7]

$$0 \longrightarrow \mathcal{D}(M) \xrightarrow{P} \mathcal{D}(M) \xrightarrow{G} C^\infty_{\mathrm{sc}}(M) \xrightarrow{P} C^\infty_{\mathrm{sc}}(M),$$

in particular, $\ker(G) = P(\mathcal{D}(M))$.

We want to use G to construct a symplectic vector space to which one can apply the CCR functor. We first define a skew symmetric bilinear form on $\mathcal{D}(M)$ by $\tilde{\omega}(\phi, \psi) := \langle G(\phi), \psi\rangle_{L^2(M,g)}$. Unfortunately the bilinear form is degenerate so it fails to provide the required symplectic form. However we can rectify this by passing to the quotient space $V := \mathcal{D}(M)/\ker(G)$, which by the above is just $\mathcal{D}(M)/P(\mathcal{D}(M))$. Hence $\tilde{\omega}$ induces a symplectic form ω on V. One can go on to use this to construct representations of the canonical commutation relations (CCRs) on the space of quasi-local C^*-algebras [3, theorem 4.4.11] which satisfy the Haag–Kastler axioms [3, theorem 4.5.1].

3. The Cauchy problem on $\mathbb{R}^{n+1}$ for $C^{1,1}$ metrics

In this section we establish theorem 3.7 which gives the existence, uniqueness and causal support results we need concerning solutions to the Cauchy problem for the wave equation on $\mathbb{R}^{n+1}$ for a $C^{1,1}$ metric.

The proof follows from lemmas 3.1, 3.2, 3.3 below, which cover a slightly more general version of the Cauchy problem. The basic technique is to employ a Chruściel–Grant regularisation of the metric [19] (see details in appendix A) to obtain a family of smooth metrics $(g_\varepsilon)_{\varepsilon\in(0,1]}$ which converge to g in the C^1 topology, have uniformly bounded second derivatives on compact sets and satisfy $J^+_\varepsilon(K) \subset J^+(K)$ for every compact subset $K \subset M$ and $\varepsilon > 0$. This enables us to obtain the required solution of the wave equation by taking a suitable limit of solutions to the smooth equation as $\varepsilon \to 0$ while preserving the causal support properties. Lemma 3.1 provides a detailed form of the energy estimates which proves crucial in the transition to the non-smooth setting. The causal support properties follow from lemma 3.5. Note that although lemma 3.2 shows the existence of a generalised Colombeau solution [15] to a corresponding generalised Cauchy problem the main result of this section, theorem 3.7, concerns a classical weak solution and does not require explicitly referring to the Colombeau solution used in the proof.

As a basic setup, we consider a Lorentzian metric g of signature $(+, -, \ldots, -)$ on $\mathbb{R}^{n+1}$ with spatial components $(-h_{ij})_{1\leqslant i,j\leqslant n}$ and the corresponding wave operator

$$Pu := g_{00}\partial_t^2 u + 2\sum_{j=1}^{n} g_{0j}\partial_{x_j}\partial_t u - \sum_{i,j=1}^{n} h_{ij}\partial_{x_i}\partial_{x_j} u + \sum_{j=1}^{n} a_j\partial_{x_j} u + a_0\partial_t u + bu \tag{3.1}$$

where all coefficients are supposed to be real-valued, smooth, and bounded in all orders of derivatives. Moreover we assume that there exist positive constants $\tau_{\min}$, $\tau_{\max}$, $\lambda_{\min}$, $\lambda_{\max}$ such that

$$\tau_{\min} \leqslant g_{00}(t, x) \leqslant \tau_{\max} \tag{3.2}$$

$$\lambda_{\min}|\xi|^2 \leqslant \sum_{i,j} h_{ij}(t, x)\overline{\xi_i}\xi_j \leqslant \lambda_{\max}|\xi|^2 \tag{3.3}$$

for all $(t, x) \in \mathbb{R}^{n+1}$ and for all $\xi \in \mathbb{C}^n$. It follows from theorem 23.2.2 in [20] or theorem 2.6 in [21] that for initial data $u_0, u_1 \in H^\infty(\mathbb{R}^n)$ (see appendix A) and right-hand side

$f \in H^\infty((0,T)\times\mathbb{R}^n) \subset C^\infty([0,T]\times\mathbb{R}^n)$ the Cauchy problem

$$Pu = f \quad \text{on}\, \Omega_T := (0,T)\times\mathbb{R}^n, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \tag{3.4}$$

has a unique solution u which belongs to $C([0,T], H^{s+1}(\mathbb{R}^n)) \cap C^1([0,T], H^s(\mathbb{R}^n)) \cap H^{s+1}((0,T)\times\mathbb{R}^n)$ for every $s \in \mathbb{R}$. In the following lemma we provide an explicit energy estimate with $s = 1$.

Lemma 3.1. *Let $u \in C([0,T], H^2(\mathbb{R}^n)) \cap C^1([0,T], H^1(\mathbb{R}^n)) \cap H^2((0,T)\times\mathbb{R}^n)$ such that $Pu \in L^2([0,T], H^1(\mathbb{R}^n))$ and denote $\tilde{u}_0 := u(0,\cdot)$ and $\tilde{u}_1 := \partial_t u(0,\cdot)$. Then we have*

$$\|u\|_{C([0,T],H^2(\mathbb{R}^n))} + \|u\|_{C^1([0,T],H^1(\mathbb{R}^n))} \leqslant \mathrm{e}^{\beta T}\left(\|\tilde{u}_0\|_{H^2(\mathbb{R}^n)} + \|\tilde{u}_1\|_{H^1(\mathbb{R}^n)} + \|Pu\|_{L^2([0,T],H^1(\mathbb{R}^n))}\right).$$

where $\beta = C/\min(\tau_{\min}, \lambda_{\min})$ with the constant $C \geqslant 0$ depending only on the $L^\infty(\Omega_T)$-norms of the coefficients g, a, b and their first-order derivatives.

Proof. We show how to derive the energy estimate in terms of application of P to any function $u \in H^\infty(\mathbb{R}^{n+1})$: we may write

$$2\,\mathrm{Re}(Pu\,\partial_t\bar{u}) = \partial_t\mathcal{T} + \sum_{j=1}^n \partial_{x_j}X_j + Y \tag{3.5}$$

where

$$\mathcal{T}(u(t,x);t,x) = g_{00}(t,x)|\partial_t u(t,x)|^2 + \sum_{i,j=1}^n h_{ij}(t,x)\partial_{x_i}u(t,x)\,\partial_{x_j}\overline{u}(t,x)$$

$$X_j(u(t,x);t,x) = 2g_{0j}(t,x)|\partial_t u(t,x)|^2 - 2\,\mathrm{Re}\sum_{i=1}^n h_{ji}(t,x)\,\partial_{x_i}u(t,x)\,\partial_t\overline{u}(t,x)$$

and

$$\begin{aligned} Y(u(t,x);t,x) = &\left(-g_{00,0}(t,x) - 2\sum_{j=1}^n g_{0j,j}(t,x) + a_0(t,x)\right)|\partial_t u(t,x)|^2 \\ &+ \sum_{j=1}^n 2\,\mathrm{Re}\left(\sum_{i=1}^n h_{ij,i}(t,x) + a_j(t,x)\right)\partial_{x_j}u(t,x)\,\partial_t\overline{u}(t,x) \\ &- \mathrm{Re}\sum_{i,j=1}^n h_{ij,0}(t,x)\,\partial_{x_i}u(t,x)\,\partial_{x_j}\overline{u}(t,x) + b(t,x)u(t,x)\partial_t\overline{u}(t,x). \end{aligned}$$

Here we have used the notation $f_{i,\mu}$ for $\partial_{x^\mu} f_i$, where Greek indices range from 0 to n and $x^0 := t$. Considering time t as a parameter and integrating equation (3.5) over the spatial domain $\mathbb{R}^n$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\mathcal{T}(u(t,x);t,x)\mathrm{d}(x_1,\ldots,x_n) \\ &\quad = 2\,\mathrm{Re}\int_{\mathbb{R}^n}(Pu\partial_t\overline{u})(t,x)\mathrm{d}(x_1,\ldots,x_n) - \int_{\mathbb{R}^n}Y(u(t,x);t,x)\mathrm{d}(x_1,\ldots,x_n) \end{aligned} \tag{3.6}$$

where we have used that $\int_{\mathbb{R}^n}\partial_{x_j}X_j(u(t,x);t,x)\mathrm{d}(x_1,\ldots,x_n) = 0$ for all $j = 1,\ldots,n$ and for all $t \in [0,T]$, since $x \mapsto u(t,x)$ belongs to $H^1(\mathbb{R}^n)$ and the latter possesses $C_c^\infty(\mathbb{R}^n)$

as a dense subspace. Using the notation $g_{\mu\cdot}$ for the vector $(g_{\mu j})_{j=1}^{n}$ and $\mathrm{div} g_{\mu\cdot}$ for its divergence, we may estimate $Y(u(t,x);t,x)$ as follows:

$$\begin{aligned}
|Y(u(t,x);t,x)| &\leqslant \|g_{00,0} + 2 \text{ div } g_{0\cdot} - a_0\|_{L^\infty(\Omega_T)} |\partial_t u(t,x)|^2 \\
&\quad + \max_{1\leqslant i\leqslant n} \left(n \, \| \text{ div } h_{i\cdot}\|_{L^\infty(\Omega_T)} + \|a_i\|_{L^\infty(\Omega_T)}\right) \left(\sum_{j=1}^{n} |\partial_{x_j} u(t,x)|^2 + |\partial_t u(t,x)|^2\right) \\
&\quad + n \max_{1\leqslant i,j\leqslant n} \|h_{ij,0}\|_{L^\infty(\Omega_T)} \sum_{j=1}^{n} |\partial_{x_j} u(t,x)|^2 \\
&\quad + \frac{1}{2}\|b\|_{L^\infty(\Omega_T)} \left(|u(t,x)|^2 + |\partial_t u(t,x)|^2\right) \\
&\leqslant C_1 |\partial_t u(t,x)|^2 + C_2 \sum_{j=1}^{n} |\partial_{x_j} u(t,x)|^2 + C_3 |u(t,x)|^2.
\end{aligned}$$

We note that

$$\int_{\mathbb{R}^n} |Y(u(t,x);t,x)| \mathrm{d}(x_1,\ldots,x_n) \leqslant C_{g',a,b} \|u(t,\cdot)\|^2_{\tilde{H}^1(\mathbb{R}^n)}$$

with $C_{g',a,b} := \max(C_1, C_2, C_3)$ depending only on the $L^\infty(\Omega_T)$-norms of the coefficients g, a, b and of the first-order derivatives of g. (See appendix B for the definition of $\tilde{H}^1(M)$.) Moreover we have

$$\begin{aligned}
C_{\min} \left(\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right) &\leqslant \int_{\mathbb{R}^n} \mathcal{T}(u(t,x);t,x) \mathrm{d}(x_1,\ldots,x_n) \\
&\leqslant C_{\max} \left(\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)
\end{aligned}$$

where $C_{\min} := \min(\tau_{\min}, \lambda_{\min})$ and $C_{\max} := \max(\tau_{\max}, \lambda_{\max})$. We observe that $|2\,\mathrm{Re}(Pu\partial_t\overline{u})(t,x)| \leqslant |Pu(t,x)|^2 + |\partial_t u(t,x)|^2$. Using the fundamental theorem of calculus we may write

$$|u(t,x)|^2 = |u(0,x)|^2 + \int_0^t \partial_s |u(s,x)|^2 \,\mathrm{d}s \leqslant |u(0,x)|^2 + \int_0^t |\partial_s u(s,x)|^2 \,\mathrm{d}s + \int_0^t |u(s,x)|^2 \,\mathrm{d}s$$

and integration over the spatial variables yields

$$\begin{aligned}
&C_{\min} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \\
&\quad \leqslant C_{\max} \left(\|u(0,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|\partial_s u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}s + \int_0^t \|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} \,\mathrm{d}s\right) \quad (3.7)
\end{aligned}$$

where we have deliberately multiplied by $C_{\min} \leqslant C_{\max}$. Integrating equation (3.6) with respect to the time variable from 0 to t we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{T}(u(t,\cdot);t,\cdot)\mathrm{d}V \\
&\quad \leqslant \int_{\mathbb{R}^n} \mathcal{T}(u(0,\cdot);0,\cdot)\mathrm{d}V + 2\,\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} (Pu\partial_t\overline{u})(s,\cdot)\mathrm{d}V\mathrm{d}s + \int_0^t \int_{\mathbb{R}^n} |Y(u(s,\cdot);s,\cdot)|\mathrm{d}V\mathrm{d}s.
\end{aligned}$$

We will from now on write $\tilde{u}_0 := u(0,\cdot)$ and $\tilde{u}_1 := \partial_t u(0,\cdot)$. Upon adding inequality (3.7) we get

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{T}(u(t,\cdot);t,\cdot)\mathrm{d}V + C_{\min}\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \leqslant &\int_{\mathbb{R}^n} \mathcal{T}(u(0,\cdot);0,\cdot)\mathrm{d}V \\ &+ 2\,\mathrm{Re}\int_0^t \int_{\mathbb{R}^n} (Pu\,\partial_t\overline{u})(s,\cdot)\mathrm{d}V\mathrm{d}s + \int_0^t\int_{\mathbb{R}^n} |Y(u(s,\cdot);s,\cdot)|\mathrm{d}V\mathrm{d}s \\ &+ C_{\max}\left(\|\tilde{u}_0\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \|\partial_s u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}s + \int_0^t \|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}s\right).\end{aligned}$$

Employing the bounds $C_{\min}\left(\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right) \leqslant \int_{\mathbb{R}^n}\mathcal{T}(u(t,\cdot);t,\cdot)\mathrm{d}V$ and $\int_{\mathbb{R}^n}\mathcal{T}(u(0,\cdot);0,\cdot)\mathrm{d}V \leqslant C_{\max}\left(\|\tilde{u}_1\|^2_{L^2(\mathbb{R}^n)} + \|\nabla\tilde{u}_0\|^2_{L^2(\mathbb{R}^n)}\right)$ as well as the other estimates and rearranging terms, we arrive at

$$\begin{aligned}&C_{\min}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right) \\ &\quad\leqslant C_{\max}\left(\|\tilde{u}_0\|^2_{L^2(\mathbb{R}^n)} + \|\tilde{u}_1\|^2_{L^2(\mathbb{R}^n)} + \|\nabla\tilde{u}_0\|^2_{L^2(\mathbb{R}^n)}\right) + \int_0^t \|Pu(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\,\mathrm{d}s \\ &\quad+ (1+\max(C_1,C_2,C_3))\int_0^t \left(\|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\partial_s u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)\mathrm{d}s\end{aligned} \tag{3.8}$$

where

$$C_1 = 1 + \|g_{00,0} + 2\,\mathrm{div}\,g_{0\,\cdot} - a_0\|_{L^\infty(\Omega_T)} + \max_{1\leqslant i\leqslant n}\left(n\,\|\mathrm{div}\,h_{i\,\cdot}\|_{L^\infty(\Omega_T)} + \|a_i\|_{L^\infty(\Omega_T)}\right) + \frac{\|b\|_{L^\infty(\Omega_T)}}{2},$$

$$C_2 = \max_{1\leqslant i\leqslant n}\left(n\,\|\mathrm{div}\,h_{i\,\cdot}\|_{L^\infty(\Omega_T)} + \|a_i\|_{L^\infty(\Omega_T)}\right) + n\max_{1\leqslant i,j\leqslant n}\|h_{ij,0}\|_{L^\infty(\Omega_T)},$$

$$C_3 = \frac{1}{2}\|b\|_{L^\infty(\Omega_T)}.$$

Gronwall's inequality [22, chapter XVIII, sections 5 and 2.2] then yields the *a priori* energy estimate

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\left(\|u(t,\cdot)\|^2_{H^1(\mathbb{R}^n)} + \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right) \\ &\quad\leqslant C_{\min}^{-1}\left(C_{\max}\left(\|\tilde{u}_0\|^2_{L^2(\mathbb{R}^n)} + \|\tilde{u}_1\|^2_{L^2(\mathbb{R}^n)} + \|\nabla\tilde{u}_0\|^2_{L^2(\mathbb{R}^n)}\right) + \|Pu\|^2_{L^2(\Omega_T)}\right)\cdot \mathrm{e}^{\beta T}\end{aligned}$$

where $\beta := C_{\min}^{-1}(1+\max(C_1,C_2,C_3))$.

To obtain similar estimates for spatial derivatives $\nabla u = (\partial_{x_1}u,\ldots,\partial_{x_n}u)^{\mathsf{T}}$ of u, we first divide the equation by g_{00}:

$$\partial_t^2 u + 2\sum_{j=1}^n \frac{g_{0j}}{g_{00}}\partial_{x_j}\partial_t u - \sum_{i,j=1}^n \frac{h_{ij}}{g_{00}}\partial_{x_i}\partial_{x_j}u + \sum_{j=1}^n \frac{a_j}{g_{00}}\partial_{x_j}u + \frac{a_0}{g_{00}}\partial_t u + \frac{b}{g_{00}}u = \frac{Pu}{g_{00}}.$$

Differentiating this equation with respect to x^k yields

$$\partial_t^2\partial_{x_k}u + 2\sum_{j=1}^n \frac{g_{0j,k}g_{00} - g_{0j}g_{00,k}}{g_{00}^2}\partial_{x_j}\partial_t u + 2\sum_{j=1}^n \frac{g_{0j}}{g_{00}}\partial_{x_j}\partial_t\partial_{x_k}u - \sum_{i,j=1}^n \frac{h_{ij,k}g_{00} - h_{ij}g_{00,k}}{g_{00}^2}\partial_{x_i}\partial_{x_j}u$$
$$- \sum_{i,j=1}^n \frac{h_{ij}}{g_{00}}\partial_{x_i}\partial_{x_j}\partial_{x_k}u + \sum_{j=1}^n \frac{a_{j,k}g_{00} - a_j g_{00,k}}{g_{00}^2}\partial_{x_j}u + \sum_{j=1}^n \frac{a_j}{g_{00}}\partial_{x_j}\partial_{x_k}u + \frac{a_{0,k}g_{00} - a_0 g_{00,k}}{g_{00}^2}\partial_t u$$
$$+ \frac{a_0}{g_{00}}\partial_t\partial_{x_k}u + \frac{b_{,k}g_{00} - b g_{00,k}}{g_{00}^2}u + \frac{b}{g_{00}}\partial_{x_k}u = \frac{(Pu)_{,k}g_{00} - (Pu)g_{00,k}}{g_{00}^2}. \tag{3.9}$$

Multiplying by g_{00} we may write this as

$$P\partial_{x_k}u + 2\sum_{j=1}^n \left(g_{0j,k} - \frac{g_{0j}}{g_{00}}g_{00,k}\right)\partial_{x_j}\partial_t u - \sum_{i,j=1}^n \left(h_{ij,k} - \frac{h_{ij}}{g_{00}}g_{00,k}\right)\partial_{x_i}\partial_{x_j}u$$
$$+ \sum_{j=1}^n \left(a_{j,k} - \frac{a_j}{g_{00}}g_{00,k}\right)\partial_{x_j}u$$
$$= (Pu)_{,k} - \frac{Pu}{g_{00}}g_{00,k} - \left(a_{0,k} - \frac{a_0}{g_{00}}g_{00,k}\right)\partial_t u - \left(b_{,k} - \frac{b}{g_{00}}g_{00,k}\right)u \tag{3.10}$$

where we have brought all terms which do not contain spatial derivatives $\partial_{x_k}u$ to the right-hand side. Multiplying by $\partial_t\partial_{x_k}\overline{u}$ and summing over $1 \leqslant k \leqslant n$, we get

$$2\,\mathrm{Re}\sum_{k=1}^n P(\partial_{x_k}u)\partial_t\partial_{x_k}\overline{u} = \partial_t\sum_{k=1}^n \mathcal{T}(\partial_{x_k}u) + \sum_{k,j=1}^n \partial_{x_j}X_j(\partial_{x_k}u) + \sum_{k=1}^n Y(\partial_{x_k}u) + Z(\nabla u)$$
$$= \underbrace{2\sum_{k=1}^n \mathrm{Re}\left(\left(\left((Pu)_{,k} - \frac{Pu}{g_{00}}g_{00,k}\right) - \left(a_{0,k} - \frac{a_0}{g_{00}}g_{00,k}\right)\partial_t u - \left(b_{,k} - \frac{b}{g_{00}}g_{00,k}\right)u\right)\partial_t\partial_{x_k}\overline{u}\right)}_{=:F(u(t,x);t,x)} \tag{3.11}$$

where the whole expression depends on (t, x) and X and Y are defined exactly as before. The additional term Z collects all terms from (3.10) which are of lower order with respect to $\partial_{x_k}u$ (that is, they contain spatial derivatives of order 2 at most). We have

$$Z(\nabla u(t,x); t, x) = 2\,\mathrm{Re}\left(2\sum_{j,k=1}^n \left(g_{0j,k} - \frac{g_{0j}}{g_{00}}g_{00,k}\right)\partial_{x_j}\partial_t u\,\partial_{x_k}\partial_t\overline{u}\right.$$
$$- \sum_{i,j,k=1}^n \left(h_{ij,k} - \frac{h_{ij}}{g_{00}}g_{00,k}\right)\partial_{x_i}\partial_{x_j}u\,\partial_{x_k}\partial_t\overline{u}$$
$$\left.+ \sum_{j,k=1}^n \left(a_{j,k} - \frac{a_j}{g_{00}}g_{00,k}\right)\partial_{x_j}u\,\partial_{x_k}\partial_t\overline{u}\right)$$

and it is easy to see that Z can be estimated as follows:

$$
\begin{aligned}
|Z(\nabla u(t,x);t,x)| \leqslant{} & 2n\sum_{i=1}^{n} |\partial_t\partial_{x_i}u(t,x)|^2 \max_{1\leqslant j,k\leqslant n}\|g_{0j,k}-g_{0j}\partial_{x_k}\ \ln g_{00}\|_{L^\infty(\Omega_T)} \\
& + n^2\sum_{i=1}^{n}\left(\|\partial_{x_i}\nabla u(t,x)\|^2+\|\partial_t\nabla u(t,x)\|^2\right)\max_{1\leqslant j,k\leqslant n}\|h_{ij,k}-h_{ij}\partial_{x_k}\ln g_{00}\|_{L^\infty(\Omega_T)} \\
& + n\sum_{i=1}^{n}\left(|\partial_{x_i}u(t,x)|^2+\|\partial_t\nabla u(t,x)\|^2\right)\max_{1\leqslant k\leqslant n}\|a_{i,k}-a_i\partial_{x_k}\ \ln g_{00}\|_{L^\infty(\Omega_T)}
\end{aligned}
$$

and thus

$$
\begin{aligned}
& \int_{\mathbb{R}^n}|Z(\nabla u(t,x);t,x)|\,\mathrm{d}V \\
& \leqslant C_{g',a'}\left(\|\partial_t\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}+\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}+\sum_{j=1}^{n}\|\partial_{x_j}\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right) \\
& = C_{g',a'}\left(\|\partial_t\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}+\|\nabla u(t,\cdot)\|^2_{H^1(\mathbb{R}^n)}\right)
\end{aligned}
$$

where $\|\nabla u(t,x)\|^2 := \sum_{k=1}^{n}|\partial_{x_k}u(t,x)|^2$ and $C_{g',a'}$ depends only on the $L^\infty(\Omega_T)$-norms of the coefficients g, a and their first-order derivatives. Moreover, we have

$$
\begin{aligned}
|F(u(t,x);t,x)| \leqslant{} & \sum_{k=1}^{n}\left(|(Pu)_{,k}(t,x)|^2+|Pu(t,x)|^2\|\partial_{x_k}\ \ln g_{00}\|_{L^\infty(\Omega_T)}+2|\partial_t\partial_{x_k}u(t,x)|^2\right) \\
& + \max_{1\leqslant k\leqslant n}\|a_{0,k}-a_0\partial_{x_k}\ \ln g_{00}\|_{L^\infty(\Omega_T)}\sum_{k=1}^{n}\left(|\partial_t u(t,x)|^2+|\partial_t\partial_{x_k}u(t,x)|^2\right) \\
& + \max_{1\leqslant k\leqslant n}\|b_{,k}-b\,\partial_{x_k}\ \ln g_{00}\|_{L^\infty(\Omega_T)}\sum_{k=1}^{n}\left(|u(t,x)^2|+|\partial_t\partial_{x_k}u(t,x)|^2\right)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\int_{\mathbb{R}^n}|F(u(t,x);t,x)|\,\mathrm{d}V \leqslant{} & C_{g'_{00}}\|Pu(t,\cdot)\|^2_{H^1(\mathbb{R}^n)} \\
& + C_{g'_{00},a'_0,b'}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}+\|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}+\|\partial_t\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned}
$$

We may now proceed as in the proof for the basic energy estimate (3.8), following the steps from (3.6) to (3.8). Upon integrating equation (3.11) over the spatial domain $\mathbb{R}^n$, we get

$$
\begin{aligned}
\sum_{k=1}^{n}\int_{\mathbb{R}^n}\mathcal{T}(\partial_{x_k}u;t,\cdot)\mathrm{d}V \leqslant{} & \sum_{k=1}^{n}\int_{\mathbb{R}^n}\mathcal{T}(\partial_{x_k}u;0,\cdot)\mathrm{d}V+\sum_{k=1}^{n}\int_0^t\int_{\mathbb{R}^n}|Y(\partial_{x_k}u;t,\cdot)|\mathrm{d}V\mathrm{d}s \\
& +\int_0^t\int_{\mathbb{R}^n}|Z(\nabla u;s,\cdot)|\mathrm{d}V\mathrm{d}s+\int_0^t\int_{\mathbb{R}^n}|F(\nabla u;s,\cdot)|\mathrm{d}V\mathrm{d}s
\end{aligned}
$$

and by employing the estimates for Y, Z, and F, we arrive at an energy estimate for ∇u (recall that $\int_{\mathbb{R}^n} |Y(\partial_{x_k} u; t, \cdot)| \mathrm{d}V \leqslant C_{g',a,b} \|\partial_{x_k} u(t,\cdot)\|^2_{\tilde{H}^1(\mathbb{R}^n)}$):

$$C_{\min}\left(\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\partial_t \nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \sum_{k=1}^n \|\partial_{x_k}\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)$$
$$\leqslant C_{\max}\left(\|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u_1\|^2_{L^2(\mathbb{R}^n)} + \sum_{k=1}^n \|\partial_{x_k}\nabla u_0\|^2_{L^2(\mathbb{R}^n)}\right) + C_{g',a,b}\int_0^t \sum_{k=1}^n \|\partial_{x_k} u(s,\cdot)\|^2_{\tilde{H}^1(\mathbb{R}^n)}\,\mathrm{d}s$$
$$+ C_{g',a'}\int_0^t \left(\|\partial_s \nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \sum_{k=1}^n \|\partial_{x_k}\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)\mathrm{d}s$$
$$+ C_{g'_{00}}\int_0^t \|Pu(s,\cdot)\|^2_{H^1(\mathbb{R}^n)}\,\mathrm{d}s + C_{g'_{00},a'_0,b'}\int_0^t \left(\|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\partial_s u(s,\cdot)\|^2_{H^1(\mathbb{R}^n)}\right)\mathrm{d}s$$

which we may also write as

$$C_{\min}\left(\|\nabla u(t,\cdot)\|^2_{H^1(\mathbb{R}^n)} + \|\partial_t \nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)$$
$$\leqslant C_{\max}\left(\|\nabla u_0\|^2_{H^1(\mathbb{R}^n)} + \|\nabla u_1\|^2_{L^2(\mathbb{R}^n)}\right)$$
$$+ C_{g',a',b'}\int_0^t \left(\|\nabla u(s,\cdot)\|^2_{H^1(\mathbb{R}^n)} + \|\partial_s \nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)\mathrm{d}s$$
$$+ C_{g'_{00}}\int_0^T \|Pu(s,\cdot)\|^2_{H^1(\mathbb{R}^n)}\,\mathrm{d}s + C_{g'_{00},a'_0,b'}\int_0^T \left(\|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\partial_s u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)\mathrm{d}s$$

where $C_{g',a',b'}$ contains $L^\infty(\Omega_T)$-norms of at most first-order derivatives of the coefficients g, a, b. Applying Gronwall's inequality then results in an energy estimate for the spatial derivative vector ∇u,

$$\sup_{0\leqslant t\leqslant T}\left(\|\nabla u(t,\cdot)\|^2_{\tilde{H}^1(\mathbb{R}^n)} + \|\partial_t \nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)$$
$$\leqslant C^{-1}_{\min}\left(C_{\max}\left(\|\nabla u_0\|^2_{H^1(\mathbb{R}^n)} + \|\nabla u_1\|^2_{L^2(\mathbb{R}^n)}\right)\right.$$
$$\left. + C_{g'_{00}}\int_0^T \|Pu(s,\cdot)\|^2_{H^1(\mathbb{R}^n)}\,\mathrm{d}s + C_{g'_{00},a'_0,b'}\int_0^T \left(\|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\partial_s u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}\right)\mathrm{d}s\right)\mathrm{e}^{\widetilde{\beta}_1 T} \tag{3.12}$$

where $\widetilde{\beta}_1 := C^{-1}_{\min} C_{g',a',b'}$. Employing the basic energy estimate (3.8) to estimate $\|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}$ and $\|\partial_s u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)}$ in terms of the restrictions $\tilde{u}_0, \tilde{u}_1$ and Pu, we arrive at

$$\|u\|_{C^0([0,T],H^2(\mathbb{R}^n))} + \|u\|_{C^1([0,T],H^1(\mathbb{R}^n))} \leqslant \mathrm{e}^{\beta T}\left(\|\tilde{u}_0\|_{H^2(\mathbb{R}^n)} + \|\tilde{u}_1\|_{H^1(\mathbb{R}^n)} + \|Pu\|_{L^2([0,T],H^1(\mathbb{R}^n))}\right).$$

□

To prepare for the extension to the Colombeau solutions, we proceed by discussing higher-order energy estimates. Applying the operator ∂_{x_l} to equation (3.9) and afterwards multiplying by g_{00} we essentially get a system of the form $P(\nabla^2 u) = Q^{(2)}(\Sigma^2 u) + R^{(2)}(\Sigma^2 Pu)$, where Q^2 is a PDO of order 3, containing time derivatives of at most order 1, and R is a purely spatial PDO of order 2. Here we have used the notation

$$\begin{aligned} \nabla^1 u &:= (\partial_{x_1} u, \ldots, \partial_{x_n} u)^\mathsf{T}, \\ \nabla^{r+1} u &:= \nabla^1 \nabla^r u, \\ \Sigma^r u &:= (u, \ldots, u)^\mathsf{T}_{n^r}. \end{aligned}$$

Thus the terms produced by $Q^{(2)}(\Sigma^2 u)$ are either lower-order terms in $\nabla^2 u$ or terms with less than two spatial derivatives of u. In the first case, they can be dealt with just as the generic lower-order terms appearing in $P(\nabla^2 u)$. In the second case, they can be interpreted as source terms on the right-hand side, just as the term $R^{(2)}(\Sigma^2 Pu)$. More generally, we obtain

$$P(\nabla^r u) = Q^{(r)}(\Sigma^r u) + R^{(r)}(\Sigma^r Pu) \tag{3.13}$$

where $Q^{(r)}$ is a PDO of order $r+1$, containing time derivatives of at most order 1, and $R^{(r)}$ is a purely spatial PDO of order r. Moreover, the coefficients of $Q^{(r)}$ and $R^{(r)}$ depend on spatial derivatives of the metric g of at most order r. It is important to note that all principal coefficients of $Q^{(r)}$ depend only on $g_{\mu\nu}, a_\mu, b$ and on derivatives $g_{\mu\nu,\rho}$ $(0 \leqslant \rho \leqslant n)$. In particular, they can be viewed as lower-order terms of an operator P_r, which has the same principal symbol as P. The energy estimate following from (3.13) is of the form

$$\begin{aligned} \sup_{0\leqslant t\leqslant T} \|\nabla^r u(t,\cdot)\|^2_{\tilde{H}^1(\mathbb{R}^n)} &\leqslant C_{\min}^{-1} \Big(C_{\max} \left(\|\nabla^r u_0\|^2_{H^1(\mathbb{R}^n)} + \|\nabla^r u_1\|^2_{L^2(\mathbb{R}^n)} \right) \\ &+ C_{g'_{00}} \int_0^T \|Pu(s,\cdot)\|^2_{H^r(\mathbb{R}^n)}\, \mathrm{d}s + C_{g^{(r)},a^{(r)},b^{(r)}} \int_0^T \left(\|u(s,\cdot)\|^2_{H^{r-1}(\mathbb{R}^n)} + \|\partial_s u(s,\cdot)\|_{H^{r-1}(\mathbb{R}^n)} \right) \mathrm{d}s \Big)\, \mathrm{e}^{\beta_r T} \end{aligned} \tag{3.14}$$

where β_r depends only on $g_{\mu\nu}, a_\mu, b$ and on derivatives $g_{\mu\nu,\rho}$.

Summing up, we obtain energy estimates for $\nabla^r u$ and $\partial_t \nabla^r u$ for $r \in \mathbb{N}_0$. Equation (3.1) itself then immediately provides an estimate for $\partial_t^2 u$ as well. Similarly, equation (3.9) allows us to estimate $\partial_t^2 \partial_{x_k} u$ in terms of (mixed) derivatives of order $\leqslant 1$ in the time variable. More generally, equation (3.13) directly provides estimates for $\partial_t^2 \nabla^r u$ $(r \in \mathbb{N})$ in terms of (mixed) derivatives of order $\leqslant 1$ in the time variable. To get estimates for mixed derivatives $\partial_t^m \nabla^r u$ of order $m \geqslant 3$ in the time variable, we may apply the operator ∂_t^l to equation (3.13), yielding

$$P(\nabla^r \partial_t^l u) = Q^{(r)}(\Sigma^r \partial_t^l u) + R^{(r)}(\Sigma^r \partial_t^l Pu) + \tilde{Q}^{(r,l)}(\Sigma^r u) + \tilde{R}^{(r,l)}(\Sigma^r Pu) \tag{3.15}$$

where $\tilde{Q}^{(r,l)}$ is a PDO of order $r+l$ with time derivatives only up to order $l-1$ and $\tilde{R}^{(r,l)}$ is a PDO of order $r+l-1$ with time derivatives up to order $l-1$. The first term in $P(\nabla^r \partial_t^l u)$ is $g_{00} \nabla^r \partial_t^{l+2} u$. Thus, starting with $l=1$, we immediately obtain energy estimates for all mixed derivatives of the form $\nabla^r \partial_t^3 u$. In the next step, we can show energy estimates for $\nabla^r \partial_t^4 u$, and proceed iteratively to get energy estimates for all mixed derivatives $\nabla^r \partial_t^m u$ as well. These estimates are direct consequences of (3.15), once estimates for the terms with time derivatives of lower order have been established (there is no need for a Gronwall argument). However, a perhaps more elegant viewpoint is that for any $r, l \in \mathbb{N}$, equation (3.15) also implies $\tilde{H}^1$-estimates for $\nabla^r \partial_t^l u$ in terms of initial data and source terms via the energy estimate (3.8) for the operator P. Again, it is important to note that the principal coefficients of $\tilde{Q}^{(r,l)}$ and $\tilde{R}^{(r,l)}$ only depend on $g_{\mu\nu}, a_j, b$ and on first derivatives $g_{\mu\nu,\rho}$. Higher-order derivatives of the coefficients will only enter in the lower order terms and thus do not show up in the exponent after applying Gronwall's inequality. In the following lemma we refer to Colombeau theoretic notions which are summarised in appendix A.

Lemma 3.2. *We consider the initial value problem* (3.4) *with generalised functions as coefficients and data and assume that*

(*a*) *The components of g as well as the lower-order coefficients a_μ and b belong to $\mathcal{G}_{L^\infty}(\mathbb{R}^{n+1})$,*
(*b*) *There exist constants $\tau_{\min} > 0$ and $\lambda_{\min} > 0$ such that $g^\varepsilon_{00}(t,x) \geqslant \tau_{\min}(\log(1/\varepsilon))^{-1}$ and $\sum_{i,j} h^\varepsilon_{ij}(t,x)\overline{\xi_i}\xi_j \geqslant \lambda_{\min}(\log(1/\varepsilon))^{-1}|\xi|^2$ for all $(t,x) \in \mathbb{R}^{n+1}, \xi \in \mathbb{C}^n$ and for all $\varepsilon < \varepsilon_0$,*
(*c*) *All coefficients $g_{\mu\nu}$, a_μ, b as well as all derivatives $g_{\mu\nu,\rho}$ are of L^∞-log-type.*

Then, given initial data $u_0, u_1 \in \mathcal{G}_{L^2}(\mathbb{R}^n)$ and right-hand side $f \in \mathcal{G}_{L^2}(\Omega_T)$, the Cauchy problem (3.4) *has a unique solution $u \in \mathcal{G}_{L^2}(\Omega_T)$.*

Proof. We fix a symmetric representative of g and representatives of all lower-order coefficients, initial data, and right-hand side. As noted above, we have smooth solutions $u_\varepsilon \in C^\infty([0,T], H^\infty(\mathbb{R}^n))$ to the corresponding classical initial value problems for each $\varepsilon < \varepsilon_0$. To show moderateness, we apply lemma 3.1 and obtain

$$\|u_\varepsilon\|_{C^0([0,T],H^2(\mathbb{R}^n))} + \|u_\varepsilon\|_{C^1([0,T],H^1(\mathbb{R}^n))} \leqslant \mathrm{e}^{\beta^\varepsilon T}\left(\|u_{0\varepsilon}\|_{H^1(\mathbb{R}^n)} + \|u_{1\varepsilon}\|_{L^2(\mathbb{R}^n)} + \|f_\varepsilon\|_{L^2([0,T],H^1(\mathbb{R}^n))}\right).$$

where $(\mathrm{e}^{\beta^\varepsilon})$ is moderate thanks to the log-type condition on the coefficients and their first derivatives as well as the positivity condition (b). Thus the estimate shows that $(\|u_\varepsilon\|_{H^1(\Omega_T)})$ and $(\|\nabla u_\varepsilon\|_{H^1(\Omega_T)})$ are moderate. For $r \geqslant 2$, the higher-order estimates (3.14) for the spatial derivative vector $\nabla^r u_\varepsilon$ then imply that $(\|\nabla^r u_\varepsilon\|_{H^1(\Omega_T)})$ is also moderate (for any $r \in \mathbb{N}$), since all principal coefficients of the operators on the right-hand side of equation (3.13) only depend on $(g^\varepsilon_{\mu\nu})$, (a^ε_μ), (b^ε) and on first derivatives $(g^\varepsilon_{\mu\nu,\rho})$, all of which are of L^∞−log-type.

An iterative application of the higher-order estimates for spatial derivatives (3.14),

$$\sup_{0\leqslant t\leqslant T} \|\nabla^r u_\varepsilon(t,\cdot)\|^2_{\tilde{H}^1(\mathbb{R}^n)} \leqslant \Bigg(C^\varepsilon_{\max}\left(\|\nabla^r u_{0\varepsilon}\|^2_{H^1(\mathbb{R}^n)} + \|\nabla^r u_{1\varepsilon}\|^2_{L^2(\mathbb{R}^n)}\right) + C_{g^{\varepsilon\prime}_{00}}\int_0^T \|f_\varepsilon(s,\cdot)\|^2_{H^r(\mathbb{R}^n)}\,\mathrm{d}s + C_{g^{(r)}_\varepsilon,a^{(r)}_\varepsilon,b^{(r)}_\varepsilon}\int_0^T \|u_\varepsilon(s,\cdot)\|^2_{\tilde{H}^{r-1}(\mathbb{R}^n)}\,\mathrm{d}s\Bigg)\frac{\mathrm{e}^{\beta^\varepsilon_r T}}{C^\varepsilon_{\min}},$$

starting with $r \geqslant 2$ then establishes moderateness of $\|\nabla^r u_\varepsilon\|_{H^1(\Omega_T)}$ for all $r \in \mathbb{N}_0$, since $1/C^\varepsilon_{\min}$ is of logarithmic growth in ε (and thus moderate), $C^\varepsilon_{\max}$, $C^\varepsilon_{g'}$, $C_{g^{(r)}_\varepsilon,a^{(r)}_\varepsilon,b^{(r)}_\varepsilon}$ are of moderate growth and β^ε_r is of L^∞-log-type (where β^ε_r depends on $C^\varepsilon_{\min}$ and (first derivatives of) $g_\varepsilon, a_\varepsilon$, and b_ε). Finally, the corresponding energy estimates for mixed derivatives following from equation (3.15) yield moderateness of $\|\nabla^r\partial^l_t u_\varepsilon\|_{H^1(\Omega_T)}$ as well. Note that only the principal coefficients will be exponentiated in the Gronwall inequality. Thus the important observation in all these higher-order estimates is that the principal coefficients on both sides of equation (3.15) only depend on $g^\varepsilon_{\mu\nu}, a^\varepsilon_\mu, b^\varepsilon$ and on derivatives $g^\varepsilon_{\mu\nu,\rho}$, all of which are of L^∞-log-type.

In total this implies that for all $r, l \in \mathbb{N}_0$ there exists $m \in \mathbb{N}_0$ such that $\|\nabla^r\partial^l_t u_\varepsilon\|^2_{L^2(\Omega_T)} = O(\varepsilon^{-m})$ as $\varepsilon \to 0$ and thus $[(u_\varepsilon)_{\varepsilon>0}] \in \mathcal{G}_{L^2}(\Omega_T)$. To show uniqueness of the generalised solution in $\mathcal{G}_{L^2}(\Omega_T)$, we assume negligible initial data $(\widetilde{u_{0\varepsilon}}), (\widetilde{u_{1\varepsilon}}) \in \mathcal{N}_{L^2}(\mathbb{R}^n)$ and right-hand side $(\tilde{f}_\varepsilon) \in \mathcal{N}_{L^2}(\Omega_T)$. Then the same energy estimates we used for moderateness yield negligibility of the solution $[(\tilde{u}_\varepsilon)_{\varepsilon>0}]$. □

In the next lemma we want to identify conditions on the coefficients and data of a low-regularity Cauchy problem (3.4) such that the corresponding generalised Cauchy problem, obtained via regularisation, has a unique solution $u \in \mathcal{G}_{L^2}(\Omega_T)$ and, moreover, this solution admits a distributional shadow.

Lemma 3.3. *We consider the Cauchy problem* (3.4) *where all coefficients are supposed to be globally bounded and Lipschitz continuous on* $\mathbb{R}^{n+1}$ *and satisfy estimates according to* (3.2) *and* (3.3)*. Then, given initial data* $(u_0, u_1) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ *and right-hand side* $f \in L^2([0,T], H^1(\mathbb{R}^n))$*, we consider the corresponding generalised Cauchy problem obtained via convolution regularisation of all coefficients and data. Then the unique generalised solution according to lemma* 3.2 *has a distributional shadow* $\tilde{u} \in C^0([0,T], H^2(\mathbb{R}^n)) \cap C^1([0,T], H^1(\mathbb{R}^n)) \cap H^2(\Omega_T)$ *which is also the unique weak solution.*

Proof. We note that standard convolution regularizations of the coefficients and data provide ε-parametrized families of smooth approximations which guarantee the assumptions in lemma 3.2. Thus, we obtain a unique generalised solution. We aim at showing that any representative (u_ε) of the solution is a Cauchy net. To this end we apply the variant of the energy estimate in lemma 3.1, implicit in the proof (see also (3.8)), with order of spatial Sobolev norms reduced by one (except for the initial data): applying the operator P_ε to the difference $u_\varepsilon - u_{\tilde{\varepsilon}}$, we obtain

$$\begin{aligned} &\|u_\varepsilon - u_{\tilde{\varepsilon}}\|_{C^0([0,T],H^1(\mathbb{R}^n))} + \|u_\varepsilon - u_{\tilde{\varepsilon}}\|_{C^1([0,T],L^2(\mathbb{R}^n))} \\ &\quad \leqslant \mathrm{e}^{\beta_\varepsilon T} \left(\|u_{0\varepsilon} - u_{0\tilde{\varepsilon}}\|_{H^1(\mathbb{R}^n)} + \|u_{0\varepsilon} - u_{0\tilde{\varepsilon}}\|_{L^2(\mathbb{R}^n)} + \|P(u_\varepsilon - u_{\tilde{\varepsilon}})\|_{L^2(\Omega_T)} \right), \end{aligned} \tag{3.16}$$

where β^ε is bounded uniformly in ε thanks to the hypotheses on the non-smooth coefficients. We may write

$$P_\varepsilon(u_\varepsilon - u_{\tilde{\varepsilon}}) = f_\varepsilon - P_\varepsilon u_{\tilde{\varepsilon}} = f_\varepsilon - f_{\tilde{\varepsilon}} - (P_\varepsilon - P_{\tilde{\varepsilon}})u_{\tilde{\varepsilon}}.$$

Considering the regularity of the initial data and right-hand side, it is easy to see that (u_ε) is a Cauchy net, if for all $\eta > 0$ there exists ε_0 such that $\|(P_\varepsilon - P_{\tilde{\varepsilon}})u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} < \eta$ for all $\tilde{\varepsilon} < \varepsilon \leqslant \varepsilon_0$. We have

$$\begin{aligned} &\|(P_\varepsilon - P_{\tilde{\varepsilon}})u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} \\ &\quad \leqslant \|g^\varepsilon_{00} - g^{\tilde{\varepsilon}}_{00}\|_{L^\infty(\Omega_T)} \|\partial_t^2 u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} + 2\sum_{j=1}^n \|g^\varepsilon_{0j} - g^{\tilde{\varepsilon}}_{0j}\|_{L^\infty(\Omega_T)} \|\partial_{x_j}\partial_t u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} \\ &\quad + \sum_{i,j=1}^n \|h^\varepsilon_{ij} - h^{\tilde{\varepsilon}}_{ij}\|_{L^\infty(\Omega_T)} \|\partial_{x_i}\partial_{x_j} u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} + \sum_{j=1}^n \|a^\varepsilon_j - a^{\tilde{\varepsilon}}_j\|_{L^\infty(\Omega_T)} \|\partial_{x_j} u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} \\ &\quad + \|a^\varepsilon_0 - a^{\tilde{\varepsilon}}_0\|_{L^\infty(\Omega_T)} \|\partial_t u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} + \|b^\varepsilon - b^{\tilde{\varepsilon}}\|_{L^\infty(\Omega_T)} \|u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)} \\ &\quad \leqslant \sum_{\mu,\nu=0}^n \|g^\varepsilon_{\mu\nu} - g^{\tilde{\varepsilon}}_{\mu\nu}\|_{L^\infty(\Omega_T)} \|u_{\tilde{\varepsilon}}\|_{H^2(\Omega_T)} + \sum_{\mu=0}^n \|a^\varepsilon_\mu - a^{\tilde{\varepsilon}}_\mu\|_{L^\infty(\Omega_T)} \|u_{\tilde{\varepsilon}}\|_{H^1(\Omega_T)} \\ &\quad + \|b^\varepsilon - b^{\tilde{\varepsilon}}\|_{L^\infty(\Omega_T)} \|u_{\tilde{\varepsilon}}\|_{L^2(\Omega_T)}. \end{aligned}$$

Since all coefficients are bounded and Lipschitz continuous, we obtain converge of their regularising families in the L^∞-norm, hence it suffices to show that $\|u_{\tilde{\varepsilon}}\|_{H^2(\Omega_T)}$ is bounded uniformly in ε. First, observe that the energy estimate (3.8) implies that $\|u_{\tilde{\varepsilon}}\|_{H^1(\Omega_T)} = O(1)$ as $\varepsilon \to 0$. The energy estimate for ∇u, inequality (3.12),

$$\sup_{0\leqslant t\leqslant T} \|\nabla u_{\tilde{\varepsilon}}(t,\cdot)\|^2_{\tilde{H}^1(\mathbb{R}^n)} \leqslant C_{\min}^{-1} \Big(C_{\max} \Big(\|\nabla u^0_{\tilde{\varepsilon}}\|^2_{H^1(\mathbb{R}^n)} + \|\nabla u^1_{\tilde{\varepsilon}}\|^2_{L^2(\mathbb{R}^n)} \Big)$$

$$+ C_{g'_{00}} \int_0^T \|f_{\tilde{\varepsilon}}(s,\cdot)\|^2_{H^1(\mathbb{R}^n)}\,\mathrm{d}s + C_{g'_{00},a'_0,b'} \int_0^T \Big(\|u_{\tilde{\varepsilon}}(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|\partial_s u_{\tilde{\varepsilon}}(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} \Big)\,\mathrm{d}s \Big)\, \mathrm{e}^{\widetilde{\beta}_1^{\tilde{\varepsilon}} T}$$

then implies that $\|u_{\tilde{\varepsilon}}\|_{H^2(\Omega_T)} = O(1)$ as $\varepsilon \to 0$ as well. Going back to (3.16), we thus have shown that (u_ε) is indeed a Cauchy net in the norm $L^\infty([0,T],H^1(\mathbb{R}^n))$ and $(\partial_t u_\varepsilon)$ is a Cauchy net in the norm $L^\infty([0,T],L^2(\mathbb{R}^n))$. Hence the unique generalised solution $u \in \mathcal{G}_{L^2}(\Omega_T)$ has a distributional limit $\tilde{u} \in C([0,T],H^1(\mathbb{R}^n)) \cap C^1([0,T],L^2(\mathbb{R}^n))$. However, we can deduce even better regularity from the properties of the net (u_ε). For any $t \in [0,T]$ and any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$|\langle \partial_{x_j}\partial_{x_k}\tilde{u}(t,\cdot),\varphi\rangle| = |\lim_{\varepsilon\to 0} \langle \partial_{x_j}\partial_{x_k} u_\varepsilon(t,\cdot),\varphi\rangle| \leqslant \lim_{\varepsilon\to 0} |\langle \partial_{x_j}\partial_{x_k} u_\varepsilon(t,\cdot),\varphi\rangle| \leqslant C\|\varphi\|_{L^2(\mathbb{R}^n)}$$

since $\|u_\varepsilon(t,\cdot)\|_{H^2(\mathbb{R}^n)} = O(1)$ as $\varepsilon \to 0$ and we already have uniform convergence of $(u_\varepsilon(t,\cdot))$ in $H^1(\mathbb{R}^n)$ by the Cauchy net estimate and hence as a distribution. It follows that $\partial_{x_j}\partial_{x_k} u_0(t,\cdot) \in L^2(\mathbb{R}^n)$ and therefore $u_0 \in C^0([0,T],H^2(\mathbb{R}^n))$. Moreover we have

$$|\langle \partial_{x_j}\partial_t\tilde{u}(t,\cdot),\varphi\rangle| = |\lim_{\varepsilon\to 0} \langle \partial_{x_j}\partial_t u_\varepsilon(t,\cdot),\varphi\rangle| \leqslant \lim_{\varepsilon\to 0} |\langle \partial_{x_j}\partial_t u_\varepsilon(t,\cdot),\varphi\rangle| \leqslant \tilde{C}\|\varphi\|_{L^2(\mathbb{R}^n)}$$

and thus $\tilde{u} \in C^1([0,T],H^1(\mathbb{R}^n))$. Similarly we can show that $\partial_{x_\mu}\partial_{x_k}\tilde{u} \in L^2(\Omega_T)$ for $0 \leqslant \mu \leqslant n$ and $1 \leqslant k \leqslant n$. In addition, multiplying the equation $P_\varepsilon u_\varepsilon = f_\varepsilon$ by $(g^\varepsilon_{00})^{-1}$, it is easy to see that $\|\partial_t^2 u_\varepsilon\|_{L^2(\Omega_T)} = O(1)$ as $\varepsilon \to 0$, implying that $\partial_t^2\tilde{u} \in L^2(\Omega_T)$ as well. Summing up, we have obtained a distributional shadow $\tilde{u}$ of the generalised solution with

$$\tilde{u} \in C^0([0,T],H^2(\mathbb{R}^n)) \cap C^1([0,T],H^1(\mathbb{R}^n)) \cap H^2(\Omega_T).$$

In the last part of the proof we show that the distributional shadow $\tilde{u}$ of the generalised solution is the unique weak solution to the Cauchy problem. The proof follows the line of arguments in the proof of [14, corollary 4.6]. First note that both $\tilde{u}$ and $\partial_t\tilde{u}$ are continuous and thus, by construction of $\tilde{u}$, the initial conditions are satisfied. The Cauchy net estimate (3.16) implies that $u_\varepsilon \to \tilde{u}$ as $\varepsilon \to 0$ in the norm $H^1(\Omega_T)$. Our aim is to prove that $P\tilde{u} = f$ in a suitable weak sense. Due to boundedness and Lipschitz continuity of the coefficients we immediately obtain L^2-convergence of all first-order terms and H^1-convergence of all zero-order terms:

$$\sum_{\mu=0}^n a^\varepsilon_\mu \partial_{x_\mu} u_\varepsilon \xrightarrow{\varepsilon\to 0} \sum_{\mu=0}^n a_\mu \partial_{x_\mu}\tilde{u} \quad \text{in } L^2(\Omega_T) \quad \text{and} \quad b^\varepsilon u_\varepsilon \xrightarrow{\varepsilon\to 0} b\,\tilde{u} \quad \text{in } H^1(\Omega_T)$$

We claim that

$$P^{(2)}_\varepsilon u_\varepsilon := g^\varepsilon_{00}\partial_t^2 u_\varepsilon + 2\sum_{j=1}^n g^\varepsilon_{0j}\partial_{x_j}\partial_t u_\varepsilon - \sum_{i,j=1}^n h^\varepsilon_{ij}\partial_{x_i}\partial_{x_j} u_\varepsilon$$

$$\xrightarrow{\varepsilon\to 0} g_{00}\partial_t^2\tilde{u} + 2\sum_{j=1}^n g_{0j}\partial_{x_j}\partial_t\tilde{u} - \sum_{i,j=1}^n h_{ij}\partial_{x_i}\partial_{x_j}\tilde{u} =: P^{(2)}\tilde{u}$$

weakly* in $L^2(\Omega_T)$. Since there exists $C > 0$ such that $\|u_\varepsilon\|_{H^2(\Omega_T)} \leqslant C$ for all $\varepsilon < \varepsilon_0$, we know that the set $\{u_{1/n} | \, n \in \mathbb{N}\}$ is bounded in $H^2(\Omega_T)$. By the weak compactness theorem, this implies that there exists a weakly* convergent subsequence $(u_{1/n_k})_{k\in\mathbb{N}}$ with limit $\tilde{v} \in H^2(\Omega_T)$ (cf [23, theorem 6.64]) and indeed $\tilde{v} = \tilde{u}$ since we already know from the Cauchy net estimate that $u_{1/n_k} \to \tilde{u}$ in $H^1(\Omega_T)$ as $k \to \infty$. Therefore $\partial_\mu\partial_\nu u_{1/n_k}$ converges weakly* to $\partial_\mu\partial_\nu\tilde{u}$ in $L^2(\Omega_T)$ and we have for any $\varphi \in L^2(\Omega_T)$:

$$\begin{aligned}|\langle P^{(2)}_{1/n_k} u_{1/n_k}, \varphi\rangle - \langle P^{(2)}\tilde{u}, \varphi\rangle| &= |\langle (P^{(2)}_{1/n_k} - \tilde{P}^{(2)}) u_{1/n_k}, \varphi\rangle - \langle P^{(2)}(\tilde{u} - u_{1/n_k}), \varphi\rangle| \\ &\leqslant |\langle (P^{(2)}_{1/n_k} - P^{(2)}) u_{1/n_k}, \varphi\rangle| + |\langle P^{(2)}(\tilde{u} - u_{1/n_k}), \varphi\rangle| \to 0\end{aligned}$$

as $k \to \infty$. To see this, observe that the first term goes to zero because the coefficients of P_{1/n_k} converge to those of $\tilde{P}$ in $L^\infty(\Omega_T)$ as $k \to \infty$ and $(u_{1/n_k})_{k\in\mathbb{N}}$ is bounded in $H^2(\Omega_T)$; the second term vanishes as well in the limit $k \to \infty$ since u_{1/n_k} converges to $\tilde{u}$ in $H^2(\Omega_T)$ as $k \to \infty$. We provide the explicit calculation for the g_{00}-term (the others can be treated similarly):

$$\begin{aligned}|\langle g^{1/n_k}_{00} \partial_t^2 u_{1/n_k}, \varphi\rangle - \langle g_{00}\, \partial_t^2 \tilde{u}, \varphi\rangle| &= |\langle (g^{1/n_k}_{00} - g_{00}) \partial_t^2 u_{1/n_k}, \varphi\rangle - \langle g_{00}(\partial_t^2 \tilde{u} - \partial_t^2 u_{1/n_k}), \varphi\rangle| \\ &\leqslant \|g^{1/n_k}_{00} - g_{00}\|_{L^\infty(\Omega_T)} \|u_{1/n_k}\|_{L^2(\Omega_T)} \|\varphi\|_{L^2(\Omega_T)} \\ &\quad + \|g_{00}\|_{L^\infty(\Omega_T)} \|\partial_t^2 \tilde{u} - \partial_t^2 u_{1/n_k}\|_{L^2(\Omega_T)} \|\varphi\|_{L^2(\Omega_T)} \to 0 \ (k \to \infty).\end{aligned}$$

This shows that for any $\varphi \in L^2(\Omega_T)$,

$$\langle P\,\tilde{u}, \varphi\rangle = \lim_{\varepsilon\to 0} \langle P_\varepsilon\, u_\varepsilon, \varphi\rangle = \lim_{\varepsilon\to 0} \langle f_\varepsilon, \varphi\rangle = \langle f, \varphi\rangle$$

and thus $\tilde{u}$ is indeed a weak solution of the initial value problem.

To show uniqueness of the weak solution, we suppose that there exists another solution $\widetilde{w} \in H^2(\Omega_T) \cap C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$ such that $P\,\widetilde{w} = f$ and $(\widetilde{w}, \partial_t\widetilde{w})|_{t=0} = (u_0, u_1)$. We may regularise this solution so that $w_\varepsilon \to \widetilde{w}$ as $\varepsilon \to 0$ in $H^2(\Omega_T)$ and $(w_\varepsilon, \partial_t w_\varepsilon)|_{t=0} \to (u_0, u_1)$ in $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. The following estimate then shows that $P_\varepsilon w_\varepsilon \to P\,\widetilde{w} = f$ in $L^2(\Omega_T)$ as $\varepsilon \to 0$:

$$\begin{aligned}&\|P_\varepsilon w_\varepsilon - P\,\widetilde{w}\|_{L^2(\Omega_T)} \\ &\quad = \|P_\varepsilon w_\varepsilon - P_\varepsilon\,\widetilde{w} + P_\varepsilon\,\widetilde{w} - P\,\widetilde{w}\|_{L^2(\Omega_T)} \leqslant \|P_\varepsilon(w_\varepsilon - \widetilde{w})\|_{L^2(\Omega_T)} + \|(P_\varepsilon - P)\widetilde{w}\|_{L^2(\Omega_T)} \\ &\quad \leqslant C\|w_\varepsilon - \widetilde{w}\|_{H^2(\Omega_T)} + \sum_{\mu,\nu=0}^{n} \|g^\varepsilon_{\mu\nu} - g_{\mu\nu}\|_{L^\infty(\Omega_T)} \|\widetilde{w}\|_{H^2(\Omega_T)} \\ &\quad + \sum_{\mu=0}^{n} \|a^\varepsilon_\mu - a_\mu\|_{L^\infty(\Omega_T)} \|\widetilde{w}\|_{H^1(\Omega_T)} + \|b^\varepsilon - b\|_{L^\infty(\Omega_T)} \|\widetilde{w}\|_{L^2(\Omega_T)} \to 0 \quad (\varepsilon \to 0).\end{aligned}$$

Here we have only used the $H^2(\Omega_T)$-convergence of w_ε to $\widetilde{w}$ as $\varepsilon \to 0$. Denoting by $(u_\varepsilon)_\varepsilon$ a representative of the generalised solution and applying the basic energy estimate (3.8) to the difference $u_\varepsilon - w_\varepsilon$ then yields

$$\begin{aligned}\frac{1}{C} \sup_{0\leqslant t\leqslant T} \Big(\|u_\varepsilon(t,\cdot) - w_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R}^n)} + \|\partial_t u_\varepsilon(t,\cdot) - \partial_t w_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \Big) &\leqslant \|(u_\varepsilon - w_\varepsilon)|_{t=0}\|^2_{L^2(\mathbb{R}^n)} \\ + \|(\partial_t u_\varepsilon - \partial_t w_\varepsilon)|_{t=0}\|^2_{L^2(\mathbb{R}^n)} + \|\nabla(u_\varepsilon - w_\varepsilon)|_{t=0}\|^2_{L^2(\mathbb{R}^n)} &+ \|f_\varepsilon - P_\varepsilon\, w_\varepsilon\|^2_{L^2(\Omega_T)}.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $\tilde{u} = \widetilde{w}$ in $C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n))$. □

Remark 3.4. The required conditions for the existence of a distributional shadow (and weak solution) are weaker than those that would be required in a similar result based on transforming the equation (3.1) into a first-order system as in [24], since the lower-order coefficients of this system would contain derivatives of the principal coefficients of equation (3.1) and therefore $g_{\mu\nu} \in C^{1,1}(\mathbb{R}^{n+1})$ would be necessary (instead of mere Lipschitz continuity). However, for a wave equation derived from the Laplace–Beltrami operator of a Lorentzian metric g,

$$\Box_g u = \sum_{\mu,\nu=0}^{n} |\det g|^{-\frac{1}{2}} \partial_\mu \left(|\det g|^{\frac{1}{2}} g^{\mu\nu} \partial_\nu u \right) = g^{\mu\nu}(\partial_\mu \partial_\nu u + \Gamma^\rho_{\mu\nu} \partial_\rho u) = f,$$

where $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$, the lower-order coefficients contain derivatives of the metric and thus the metric has to be $C^{1,1}$ anyway in order to obtain a distributional shadow of the generalised solution.

Lemma 3.5. *Consider the Cauchy problem* (3.4) *where all coefficients are globally bounded and Lipschitz continuous on* $\mathbb{R}^{n+1}$ *and satisfy* (3.2) *and* (3.3)*. Then, given vanishing initial data and right-hand side* $f \in L^2([0,T]), H^1(\mathbb{R}^n))$ *with* $\mathrm{supp}(f) \subset J^+(K)$ *for some compact set K, the unique distributional weak solution* $\bar{u}^+$ *of the advanced problem given by lemma* 3.3 *satisfies the causal support condition*

$$\mathrm{supp}(\bar{u}^+) \subset J^+(\mathrm{supp}(f)).$$

Proof. Let u^+_ε be the unique solution of the corresponding advanced problem for the regularised metric g_ε where the right-hand side f is not necessarily smooth. Note that upon taking K slightly larger, we may assume that $\mathrm{supp}(f) \subset J^+_\varepsilon(K)$ for ε sufficiently small.

By lemmas 3.1 and 3.2 we can obtain u^+_ε as a limit $\alpha \to 0$ of $u^+_{\varepsilon,\alpha}$ where $(u^+_{\varepsilon,\alpha})_{\alpha>0}$ is a Colombeau representative of the unique generalised solution with fixed smooth g_ε-coefficients and right-hand side being the class of a convolution regularisation $(f_\alpha)_{\alpha>0}$. By the smooth theory we have for every ε and every α that

$$\mathrm{supp}(u^+_{\varepsilon,\alpha}) \subseteq J^+_\varepsilon(\mathrm{supp}(f_\alpha)). \tag{3.17}$$

At fixed ε and as $\alpha \to 0$ we have in terms of monotonically decreasing sets $\bigcap_{\alpha>0} \mathrm{supp}(f_\alpha) = \mathrm{supp}(f)$, i.e. $\mathrm{supp}(f_\alpha) \searrow \mathrm{supp}(f)$. By closedness of the causal relation [3, lemma A.5.5] we obtain that

$$J^+_\varepsilon(\mathrm{supp}(f_\alpha)) \searrow J^+_\varepsilon(\mathrm{supp}(f)). \tag{3.18}$$

Let ψ be a test function such that $\mathrm{supp}(\psi) \cap J^+_\varepsilon(\mathrm{supp}(f)) = \emptyset$. By (3.17) and (3.18) there exists some $\alpha_0 > 0$ such that $\mathrm{supp}(\psi) \cap J^+_\varepsilon(\mathrm{supp}(f_\alpha)) = \emptyset$ for all $\alpha < \alpha_0$. Therefore $\langle u^+_{\varepsilon,\alpha}, \psi \rangle = 0$ for every $\alpha < \alpha_0$; taking the limit $\alpha \to 0$ we obtain $\langle u^+_\varepsilon, \psi \rangle = 0$ and therefore

$$\mathrm{supp}(u^+_\varepsilon) \subseteq J^+_\varepsilon(\mathrm{supp}(f)).$$

Now let ψ have support disjoint from $J^+(\mathrm{supp}(f))$. Then, by the causal properties of the Chruściel–Grant regularisation (cf proposition A.1), we have that $\mathrm{supp}(\psi)$ is also disjoint from $J^+_\varepsilon(\mathrm{supp}(f))$ for every $\varepsilon > 0$.

Therefore, $\langle u^{+}_{\varepsilon}, \psi\rangle = 0$ for every such ψ and for every $\varepsilon > 0$, showing that $\mathrm{supp}(u^{+}) \subseteq J^{+}_{\varepsilon}(\mathrm{supp}(f))$ for every $\varepsilon > 0$, hence $\mathrm{supp}(\bar{u}^{+}) \subset J^{+}(\mathrm{supp}(f))$. □

Remark 3.6. More generally for non-zero initial data one has

$$\mathrm{supp}(\bar{u}^{+}) \subset J^{+}(\mathrm{supp}(f) \cup \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1)).$$

The proof of this follows from the above result by recasting the smoothed version of the problem as an equivalent inhomogeneous problem with zero initial data and a right-hand side incorporating u_0 and u_1.

Theorem 3.7. *Consider the Cauchy problem* (3.4) *where all coefficients are globally bounded and Lipschitz continuous on* $\mathbb{R}^{n+1}$ *and satisfy* (3.2) *and* (3.3)*. Then, given initial data* $(u_0, u_1) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ *and* $f \in L^2([0, T]), H^1(\mathbb{R}^n)$*, there exists a unique distributional weak solution* $u \in C([0, T], H^2(\mathbb{R}^n)) \cap C^1([0, T], H^1(\mathbb{R}^n)) \cap H^2(\Omega_T)$*. Furthermore u satisfies the causal support condition*

$$\mathrm{supp}(u) \subset J(\mathrm{supp}(f) \cup \mathrm{supp}(u_0) \cup \mathrm{supp}(u_1)).$$

Proof. The existence and uniqueness of the weak solution follow from lemma 3.3 and the causal support condition follows from lemma 3.5 together with remark 3.6 applied to both the past and future. □

4. The Cauchy problem for $C^{1,1}$ globally hyperbolic spacetimes

4.1. Causality results for $C^{1,1}$ spacetimes

In this paper we will be considering solutions of the wave equation on orientable spacetimes (M, g) endowed with a $C^{1,1}$ metric. Note that although the metric is only $C^{1,1}$ we will always assume that the manifold has a smooth structure. The concept of global hyperbolicity (for smooth metrics) was introduced by Leray [25] as a condition to ensure the existence of unique solutions to hyperbolic equations and in particular the Cauchy problem for the wave equation is well-posed for smooth globally hyperbolic spacetimes [3, theorem 3.2.11 p 84ff]. For our situation it is therefore natural to consider globally hyperbolic spacetimes with $C^{1,1}$ metrics. Global hyperbolicity is the strongest of the conditions on (M, g) in the causal hierarchy of spacetimes [26] and recently there has been considerable interest in looking at the causal properties of low-regularity spacetimes [8, 18]. It was shown explicitly by Chruściel [27] that essentially all of causality theory for smooth spacetimes goes through to the C^2 case. However in the proofs of these results an important role is played by the Gauss lemma and the existence of totally normal (convex) neighbourhoods whose existence is not at all obvious in the $C^{1,1}$ case. However it was shown in [8, 28, 29] that such neighbourhoods do exist and the exponential map gives a local lipeomorphism. This enables essentially the whole of the results of standard smooth causality theory to go through to the $C^{1,1}$ case (see [8, 29] for details).

For spacetimes with a C^2 metric there are four equivalent notions of global hyperbolicity (see for example [26, section 3.11 p 340ff.]). These are:

(a) Compactness of the causal diamonds and causality[4],

(b) Compactness of the space of causal curves connecting two points and causality [25],

[4] As shown by Bernal and Sánchez the requirement of strong causality in the classical definition of global hyperbolicity can be weakened to only require causality [30].

(c) Existence of a Cauchy hypersurface,
(d) The metric splitting of the spacetime.

For $C^{1,1}$ spacetimes we will adopt the first definition. However for non-totally imprisoned [31] C^0 spacetimes (and hence in particular for globally hyperbolic $C^{1,1}$ spacetimes) these four definitions remain equivalent [17]. See also [18, theorem 2.45] for a more general notion formulated in terms of closed cone structures.

In our constructions below we will make use of time functions and temporal functions. A *time function* is a function that is strictly increasing along every causal curve while a *temporal function* has the additional property that its gradient is everywhere past-directed and timelike. It is shown by Minguzzi [18 theorem 2.30] (see also [32]) that for a stably causal closed cone structure (and hence in particular for a $C^{1,1}$ globally hyperbolic spacetime) there exists a *smooth* temporal function $t : M \to \mathbb{R}$. Furthermore every globally hyperbolic closed cone structure is the domain of dependence of a stable Cauchy surface (see definition below) Σ, so that $M = D(\Sigma)$ and that M is the topological product $\mathbb{R} \times \Sigma$ where the first projection is t and the level surfaces $\Sigma_\tau = \{x \in M : t(x) = \tau\}$ are *diffeomorphic* to Σ [18, theorem 2.42]. So that although the metric is only $C^{1,1}$ the topological splitting remains smooth.

In the case of a smooth metric Bernal and Sánchez [33] show that given a smooth spacelike Cauchy hypersurface Σ there exists a smooth temporal function t such that $\Sigma = t^{-1}(0)$. However in the case of a non-smooth metric the temporal function they construct will not be smooth. To generalise the results of [33] to the non-smooth case we need the concept of a *stable Cauchy hypersurface* introduced by Minguzzi in [18]. These are Cauchy hypersurfaces which are also Cauchy hypersurfaces for some metric $g' \succ g$ with strictly wider lightcones than g.

Bernard and Suhr [34, corollary 2.4] show that a smooth spacelike Cauchy hypersurface is a stable Cauchy hypersurface and that furthermore one can construct a smooth temporal function such that $\Sigma = t^{-1}(0)$ [34, theorem 1]. A full discussion of this issue is given in the paper by Minguzzi [35]. The approach in [35] is complementary to that in [34] and consists of using topological arguments to show that the causal cones can be widened while preserving the Cauchy property of the hypersurface. One may then use the methods of Bernal and Sánchez [33] to construct a smooth time function with $\Sigma = t^{-1}(0)$ which as shown in [18] is a smooth temporal function for the original spacetime (M, g). See [35, theorem 2.22] for details. Indeed given two smooth spacelike Cauchy hypersurfaces Σ_0 and Σ_1 with $\Sigma_1 \subset J^+(\Sigma_0)\backslash\Sigma_0$ one can find a smooth temporal function t that interpolates between them so that $\Sigma_0 \subset t^{-1}(0)$ and $\Sigma_1 \subset t^{-1}(1)$ [35, theorem 2.23].

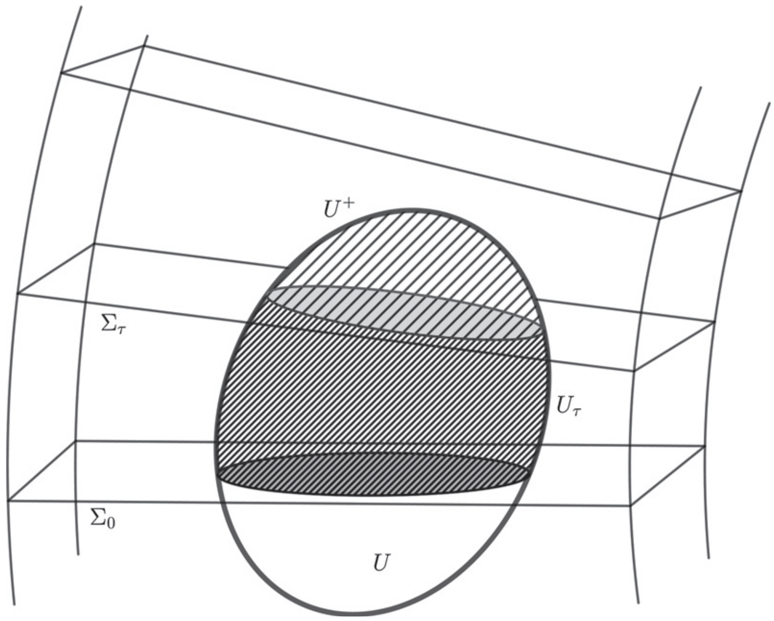
4.2. Existence and uniqueness

In this section we extend the results of section 3 to a globally hyperbolic $C^{1,1}$ spacetime (M, g). The main result is theorem 4.7 which establishes the existence and uniqueness of $H^2_{\mathrm{loc}}(M)$ solutions to the wave equation for a globally hyperbolic $C^{1,1}$ spacetime. We start by obtaining an energy inequality which we use to establish uniqueness and the causal support properties of solutions to the wave equation.

Lemma 4.1 (energy inequality) [4, *lemma 7.4.4*]. *Let (M, g) be an $(n+1)$-dimensional $C^{1,1}$ globally hyperbolic spacetime with Σ a smooth spacelike n-dimensional Cauchy surface and t a smooth temporal function such that $\Sigma = \Sigma_0 := t^{-1}(0)$. Let $U \subset M$ be an open set with* *compact closure and let* $U^+ := U \cap J^+(\Sigma)$ *be such that* $\partial U \cap \bar{U}^+$ *is achronal. Then if u is a* (*weak*) $H^2_{\rm loc}(M)$ *solution of* $\Box_g u = f$ *where* $f \in L^2_{\rm loc}(M)$ *then*

$$\|u\|_{\tilde{H}^1(\Sigma_\tau \cap U^+)} \leqslant K \left(\|u\|_{\tilde{H}^1(\Sigma_0 \cap U^+)} + \|f\|_{L^2(U_\tau)} \right) \tag{4.1}$$

where $U_\tau = \{q \in U : 0 \leqslant t(q) \leqslant \tau\}$.



Proof. To establish the energy inequality we follow Hawking and Ellis by applying the divergence theorem to an enhanced energy–momentum tensor [4, lemma 7.4.4]. Let

$$\overset{(1)}{T}_{\alpha\beta} := \nabla_\alpha u \nabla_\beta u - \frac{1}{2}\left(g^{\rho\sigma}\nabla_\rho u \nabla_\sigma u\right) g_{\alpha\beta}.$$

be the energy momentum tensor of the scalar field. Then $\overset{(1)}{T}_{\alpha\beta}$ has vanishing divergence and satisfies the dominant energy condition. We now follow [4] and modify this by adding on the term

$$\overset{(0)}{T}_{\alpha\beta} := -\frac{1}{2} g_{\alpha\beta} u^2$$

to obtain

$$S_{\alpha\beta} = \overset{(0)}{T}_{\alpha\beta} + \overset{(1)}{T}_{\alpha\beta}$$

which still satisfies the dominant energy condition. Let $\xi_\alpha = \nabla_\alpha t$, then we obtain the required inequality by applying the divergence theorem to $S^{\alpha\beta}\xi_\alpha$ over the region $U_\tau = \{q \in U : 0 \leqslant t(q) \leqslant \tau\}$. In order to do this we require that $\mathrm{div}(S^{\alpha\beta}\xi_\alpha)$ should be integrable with respect to the volume form ν_g, and this is guaranteed by the compactness of $\bar{U}$ and the fact that our solution is in $H^2_{\rm loc}(M)$. In fact it is enough that the weak solutions have two derivatives in $L^2_{\rm loc}(M, g)$ if the metric and the timelike vector field are in the space $C^{0,1}$ (see

[36]). The boundary of U_τ consists of three parts; the level surface $\Sigma_\tau \cap \bar{U}^+$, the level surface $\Sigma_0 \cap \bar{U}^+$ and the remainder which we denote $\mathcal{H}$. Because of the dominant energy condition and the fact that $\partial U \cap \bar{U}^+$ is achronal, the contribution to the surface integral from $\mathcal{H}$ is positive. We therefore obtain the following inequality:

$$\int_{\Sigma_\tau \cap \bar{U}^+} S^{\alpha\beta}\xi_\alpha\xi_\beta\mu_\tau - \int_{\Sigma_0 \cap \bar{U}^+} S^{\alpha\beta}\xi_\alpha\xi_\beta\mu_0 \leqslant \int_{U_\tau} \nabla_\alpha(S^{\alpha\beta}\xi_\beta)\nu_g \tag{4.2}$$

where ν_g is the volume form on U_τ given by g, and μ_τ is the volume form induced by g on Σ_τ.

We now define an energy type integral

$$E(\tau) = \int_{\Sigma_\tau \cap \bar{U}^+} S^{\alpha\beta}\xi_\alpha\xi_\beta\mu_\tau.$$

Then on $\bar{U}$ this is equivalent [37] to the restricted Sobolev norm (B.3)

$$C_1\|u\|_{\tilde{H}^1(\Sigma_\tau \cap \bar{U}^+)} \leqslant E(\tau) \leqslant C_2\|u\|_{\tilde{H}^1(\Sigma_\tau \cap \bar{U}^+)}.$$

Note that since the solution u is in $H^2_{\mathrm{loc}}(M)$ we have well-defined traces in $\tilde{H}^1(\Sigma)$. In terms of the energy norm we may write (4.2) in the form

$$E(\tau) \leqslant E(0) + \int_{U_\tau} ((\nabla_\alpha S^{\alpha\beta})\xi_\beta + S^{\alpha\beta}\nabla_\alpha\xi_\beta)\nu_g.$$

Repeated application of the Cauchy–Schwarz inequality and $\Box_g u = f$ then gives [37]

$$E(\tau) \leqslant E(0) + C_1\|f\|^2_{L^2(U_\tau)} + C_2\int_0^\tau E(s)\mathrm{d}s \tag{4.3}$$

which on applying Gronwall's inequality gives

$$E(\tau) \leqslant \left(E(0) + C_1\|f\|^2_{L^2(U_\tau)}\right)\mathrm{e}^{C_2\tau}.$$

In terms of the Sobolev type norms this gives

$$\|u\|_{\tilde{H}^1(\Sigma_\tau \cap U^+)} \leqslant K\left(\|u\|_{\tilde{H}^1(\Sigma_0 \cap U^+)} + \|f\|_{L^2(U_\tau)}\right).$$

□

We now use the energy inequality (4.1) to prove uniqueness of the solution as well as the causal support properties of the solution to the Cauchy problem.

Proposition 4.2 (uniqueness) *Let (M, g) be a (connected, oriented, time oriented,) globally hyperbolic $(n+1)$-dimensional Lorentzian manifold with $C^{1,1}$ metric and Σ a smooth n-dimensional spacelike Cauchy surface. Let u be a (weak) $H^2_{\mathrm{loc}}(M)$ solution of*

$$\Box_g u = f$$

with $f \in H^1_{\mathrm{loc}}(M)$, which satisfies the initial conditions

$$u|_\Sigma = u_0,$$
$$\nabla_n u|_\Sigma = u_1, \quad \text{where } n \text{ is the unit normal to } \Sigma$$

with $(u_0, u_1) \in H^2(\Sigma) \times H^1(\Sigma)$. Then u is unique.

Proof. Let $q \in M$ and without loss of generality suppose that $q \in I^+(\Sigma)$. Then since our spacetime is globally hyperbolic we may find a p such that $q \in I^-(p) \cap I^+(\Sigma) := U$ where by $C^{1,1}$ causality theory U has compact closure [8]. Now suppose there exist two solutions u and $\tilde{u}$ to the above initial value problem. Then applying lemma 4.1 to $\hat{u} := u - \tilde{u}$ over the region U^+ gives

$$\|\hat{u}\|_{\tilde{H}^1(\Sigma_\tau \cap U^+)} \leqslant K\|\hat{u}\|_{\tilde{H}^1(\Sigma_0 \cap U^+)} = 0.$$

Hence $\|\hat{u}\|_{\tilde{H}^1(\Sigma_\tau \cap U^+)} = 0$ so that $\hat{u}$ and $\nabla\hat{u}$ vanish in U^+. Since q is arbitrary the solution is unique in $D^+(\Sigma)$. A similar result applies to $D^-(\Sigma)$, so we have uniqueness in the whole of $M = D(\Sigma)$. □

Proposition 4.3 (causal support) *Let (M, g) be a connected, oriented, time oriented, globally hyperbolic $(n+1)$-dimensional Lorentzian manifold with $C^{1,1}$ metric, Σ_0 a smooth spacelike n-dimensional Cauchy surface and t a smooth temporal function such that $t^{-1}(0) = \Sigma_0$. Let $u \in C^0\left(\mathbb{R}, H^2(\Sigma_t)\right) \cap C^1\left(\mathbb{R}, H^1(\Sigma_t)\right) \cap H^2_{\mathrm{loc}}(M)$ be a (weak) solution to the initial value problem*

$$\begin{aligned} \Box_g u &= f \quad \text{on } M, \\ u &= u_0 \quad \text{on } \Sigma_0, \\ \nabla_n &= u_1 \quad \text{on } \Sigma_0. \end{aligned}$$

Then $\mathrm{supp}(u) \subset J\left(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)\right)$.

Proof. We prove the result for J^+. A similar proof holds for J^-. Let $V = J^+\left(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f)\right)$ and suppose $q \in M\backslash V$. Then we may find a point $p \in D^+(\Sigma_0)$ such that $q \in I^-(p) \cap I^+(\Sigma_0)$ and u and ∇u vanish on $J^-(p) \cap \Sigma_0$ and f vanishes on $J^-(p) \cap J^+(\Sigma_0)$. Now let $U = I^-(p) \cap I^+(\Sigma_0)$ and apply (4.3) on this region to obtain

$$\|u\|_{\tilde{H}^1(\Sigma_\tau \cap J^+(\Sigma_0) \cap I^-(p))} \leqslant K\left(\|u\|_{\tilde{H}^1(\Sigma_0 \cap J^-(p))} + \|f\|_{L^2(J^+(\Sigma_0) \cap J^-(p))}\right), \quad \text{for } 0 \leqslant \tau \leqslant t(p).$$

But by the choice of p the right-hand side vanishes, so u must also vanish in the region U with τ in the range $0 \leqslant \tau \leqslant t(p)$. Hence u vanishes on a neighbourhood of q. Since q was arbitrary, u vanishes on $M\backslash V$ which proves the result. □

To establish existence on M, we need the following two lemmas from Ringström [5]. In both cases the proof given in [5] for the smooth case goes through to that of a $C^{1,1}$ metric unchanged.

Lemma 4.4 (*Ringström [5, lemma 12.5]*). *Let (M, g) be an $(n+1)$-dimensional Lorentzian manifold with $C^{1,1}$ metric and Σ a smooth spacelike n-dimensional submanifold. If $p \in S$ there is a chart (U, x) with $p \in U$ and $x = (x^0, x^1, \ldots, x^n)$ such that $q \in U \cap \Sigma$ if and only if $q \in U$ and $x^0(q) = 0$. Furthermore we may choose x so that $\frac{\partial}{\partial x^0}$ is the future directed unit normal to Σ for $q \in \Sigma \cap U$.*

If we fix $\epsilon > 0$ and let $g_{\mu\nu} := g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$, then we can assume U to be such that $|g_{0i}| \leqslant \epsilon, i = 1, \ldots, n$ on U. If we let $a = g_{00}(p)$ and $b > 0$ be such that $g_{ij}(p)$ regarded as a positive definite matrix is bounded below by b (i.e. $g_{ij}(p)\xi^i\xi^j > b|\xi|^2$) then we may assume that $g_{00}(q) < a/2$ and $g_{ij}(q)$ regarded as a positive definite matrix is bounded below by $b/2$ for $q \in U$.

Lemma 4.5 (*Ringström [5, lemma 12.16]*). *Let (M, g) be a (connected, oriented, time oriented,) globally hyperbolic $(n+1)$-dimensional Lorentzian manifold with $C^{1,1}$ metric and Σ a smooth spacelike n-dimensional Cauchy surface. Let t be a (smooth) temporal function (as* *given by* [34, 35]) *with* $t^{-1}(0) = \Sigma_{t_0}$*. If* $p \in \Sigma_{t_0}$ *there is an* $\epsilon > 0$ *and open neighbourhoods U, W of p such that*

(*a*) *The closure of W is compact and contained in U;*
(*b*) *If* $q \in W$ *and* $\tau \in [t_0 - \epsilon, t_0 + \epsilon]$*, then* $J^+(\Sigma_\tau) \cap J^-(q)$ *is compact and contained in U;*
(*c*) *There is a chart* (U, ϕ) *with* $\phi = (x^0, \ldots, x^n)$ *and* $x^0 = t$*, such that there exist* $a, b > 0$ *with* $g_{00}(q) < -a$ *and* $g_{ij}(q)\xi^i\xi^j \geqslant b\delta_{ij}\xi^i\xi^j$ *for* $q \in U$*;*
(*d*) *For any compact* $K \subset U$ *there is a* $C^{1,1}$ *matrix valued function h on* $\mathbb{R}^{n+1}$ *such that* $h_{\mu\nu} = g_{\mu,\nu} \circ \phi^{-1}$ *on* $\phi^{-1}(K)$ *and such that there are positive constants* a_1*,* $b_1 c_1$ *with* $h_{00} \leqslant -a_1$*,* $h_{ij}\xi^i\xi^j \geqslant b_1\delta_{ij}\xi^i\xi^j$ *and* $|h_{\mu\nu}| \leqslant c_1$ *on all of* $\mathbb{R}^{n+1}$*.*

Note that although the proof is identical to that in [5] it relies on the $C^{1,1}$ causality results of [8] and [34, theorem 1]. Note also that in point (d) above the matrix valued function is only $C^{1,1}$ rather than smooth as it is in [5, lemma 12.16].

We are now in a position to establish existence.

Proposition 4.6 (existence for compactly supported source and initial data) *Let* (M, g) *be a time oriented* $(n+1)$*-dimensional Lorentzian manifold with* $C^{1,1}$ *metric and* Σ *a smooth spacelike n-dimensional hypersurface. Let t be a smooth temporal function with* $t^{-1}(0) = \Sigma$ *and let n be the future directed timelike unit normal to* Σ*. Given initial data* $(u_0, u_1) \in H^2_{\mathrm{comp}}(\Sigma) \times H^1_{\mathrm{comp}}(\Sigma)$ *and source* $f \in H^1_{\mathrm{comp}}(M)$*, then there exists a (weak) solution* $u \in C^0\left(\mathbb{R}, H^2(\Sigma_t)\right) \cap C^1\left(\mathbb{R}, H^1(\Sigma_t)\right) \cap H^2_{\mathrm{loc}}(M)$ *to the initial value problem*

$$\begin{aligned} \Box_g u &= f \quad \text{on } M, \\ u &= u_0 \quad \text{on } \Sigma, \\ \nabla_n u &= u_1 \quad \text{on } \Sigma. \end{aligned}$$

Proof. We again closely follow Ringström [5, theorem 12.17]. Let $K_1 \subset \Sigma$ be a compact set such that $\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \subset K_1$ and $K_2 \subset M$ a compact set such that $\mathrm{supp}(f) \subset K_2$. Let $t_1 > 0$ and define R_{t_1} to be the set of q such that $0 \leqslant t(q) \leqslant t_1$. Then R_{t_1} is closed and $K_3 = K_2 \cap R_{t_1}$ is compact. The union of $I^+(p)$ for $p \in I^-(\Sigma)$ is an open cover of $K_1 \cup K_3$, so there is a finite number of points $p_1, \ldots, p_\ell$ such that the $I^+(p_i)$ are a finite subcover of $K_1 \cup K_3$. Note that the set $F = \bigcup_{i=1}^{\ell} J^+(p_i) \cap J^-(\Sigma_{t_1})$ is compact and that if there is a solution in R_{t_1} it has to be zero in $R_{t_1} \backslash F$ by proposition 4.3. We now show that there is a solution in the compact set $R_{t_1} \cap F$.

Let $F_\tau := F \cap \Sigma_\tau$. Let $0 \leqslant \tau < t_1$ and assume we have a solution in the function space specified in the proposition up to time τ, i.e. on R_τ or for every R_s with $0 \leqslant s < \tau$. We now extend the solution to the future of τ. For every $p \in F_\tau$ there are neighbourhoods U_p, W_p and ϵ_p with the properties of lemma 4.5. By compactness there is a finite number of points $\tilde{p}_i, \ldots, \tilde{p}_N$ such that the $W_{\tilde{p}_i}$ cover F_τ. Let $0 < \epsilon \leqslant \min\{\epsilon_{\tilde{p}_1}, \ldots, \epsilon_{\tilde{p}_N}\}$ be such that

$$F_s \subset \bigcup_{i=1}^{N} W_{\tilde{p}_i} \tag{4.4}$$

for all $s \in [\tau - \epsilon, \tau + \epsilon]$. Now let $s_1 \in [\tau - \epsilon, \tau]$ be such that there is a solution, in the function space specified in the proposition up to and including s_1 and let $p \in F_s$ for any $s \in [s_1, \tau + \epsilon]$. Then $K_p := J^-(p) \cap J^+(\Sigma_{s_1})$ is compact and contained in one of the charts, say $(U_{\tilde{p}_k}, \phi)$. Let $\chi \in C_0^\infty(U_{\tilde{p}_k})$ be such that $\chi(q) = 1$ for all $q \in K_p$. Then we use our solution up to time s_1 to define new initial data on Σ_{s_1} given by $\tilde{u}_0 := (\chi u)|_{\Sigma_{s_1}} \in H^2(\Sigma_{s_1})$ and $\tilde{u}_1 := (\chi \nabla_n u)|_{S_{s_1}} \in$

$H^1(\Sigma_{s_1})$ and source $\tilde{f} := \chi f$. These all have their support within $U_{\tilde{p}_k}$ so we may use the chart $(U_{\tilde{p}_k}, \phi)$ to regard these as data and source on the whole of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively. We may also extend the Lorentz metric $g_{\mu\nu} \circ \phi^{-1}$ to a Lorentz matrix-valued function $h_{\mu\nu}$ on the whole of $\mathbb{R}^{n+1}$ which coincides with $g_{\mu\nu} \circ \phi^{-1}$ on K_p. We may therefore regard the tildered version as an initial value problem on $\mathbb{R}^{n+1}$. The third condition of lemma 4.5 and the fact that the solution is in $C^0\left(\mathbb{R}, H^2(\Sigma_t)\right) \cap C^1\left(\mathbb{R}, H^1(\Sigma_t)\right) \cap H^2_{\mathrm{loc}}(M)$ ensures that we may apply lemma 3.3 to obtain a solution on $\mathbb{R}^{n+1}$ which on $\phi(U_{\tilde{p}_k})$ may be transferred back to give a solution on K_p. In the region $V_p := I^-(p) \cap J^+(\Sigma_{s_1})$ we define u to be this solution. In the region $V_p \cap V_q$ then uniqueness ensures that the two potential solutions coincide. We now define O_1 to be the union of the V_p for $p \in F_s$, $s \in [s_1, \tau + \epsilon]$ then the above construction defines a unique solution in O_1. Note that the interior of O_1 contains F_s for all $s \in (s_1, \tau + \epsilon)$. Now define O_2 to be the set of points for which $s_1 \leqslant t(q) < \tau + \epsilon$ and for which $q \notin F$. We want to define the solution to be zero in this set, however we need to check that there is no contradiction for points in both O_1 and O_2. If $q \in O_2 \cap O_1$ with $t(q) > s_1$ then both u and ∇u vanish at $J^-(q) \cap S_{s_1}$ and f vanishes in $J^-(q) \cap J^+(\Sigma_{s_1})$. Furthermore there is an r such that $q \in V_r \subset O_1$. So by uniqueness the solution defined on O_1 has to vanish for a sufficiently small neighbourhood at q.

In summary we have shown that if there exists a solution for all $s < \tau$ or up to time τ in the required function space, we get a solution in the same space on the larger region $R_{\tau+\epsilon}$ for some $\epsilon > 0$. Let $\mathcal{A}$ be the set of $s \in [0, \infty)$ such that there is a solution up to time s. Taking $\tau = 0$ in the above we have a solution on R_ϵ so $\mathcal{A}$ is not empty. We have also shown that if $\tau \in \mathcal{A}$ then a solution exists for $[0, \tau + \epsilon)$, so that for any $\tau > 0$ we may find an open interval containing τ in which a solution exists. Thus $\mathcal{A}$ is open in the relative topology of $[0, \infty)$. Finally we note that by definition $\mathcal{A}$ is also closed in $[0, \infty)$ because it contains its limit points. Then this set is open, closed and non-empty, so it must be the whole of $[0, \infty)$ and we have a solution for all future times. By an analogous argument interchanging past and future we also have a solution for all past times and hence on the whole of M. □

Theorem 4.7 (global existence and uniqueness) *Let (M, g) be a connected, oriented, time oriented $(n+1)$-dimensional Lorentzian globally hyperbolic manifold with $C^{1,1}$ metric and Σ a smooth spacelike n-dimensional Cauchy hypersurface. Let t be a smooth temporal function with $t^{-1}(0) = \Sigma$ and let n be the future directed timelike unit normal to Σ.*

Given initial data $(u_0, u_1) \in H^2(\Sigma) \times H^1(\Sigma)$ and source $f \in C^0\left(\mathbb{R}, H^1(\Sigma_t)\right) \cap H^1_{\mathrm{loc}}(M)$ then there exists a unique (weak) solution $u \in C^0\left(\mathbb{R}, H^2(\Sigma_t)\right) \cap C^1\mathbb{R}, H^1(\Sigma_t) \cap H^2_{\mathrm{loc}}(M)$ to the initial value problem

$$\Box_g u = f \quad \text{on } M,$$
$$u = u_0 \quad \text{on } \Sigma,$$
$$\nabla_n u = u_1 \quad \text{on } \Sigma.$$

Moreover $\mathrm{supp}(u) \subset J\left(\mathrm{supp}(u_0) \cup \mathrm{supp}\,(u_1) \cup \mathrm{supp}(f)\right)$.

Proof. Let p be any point to the future of Σ. Then $K_p = J^-(p) \cap J^+(\Sigma)$ is a compact set. Let $\chi \in C_0^\infty(M)$ be such that $\chi(q) = 1$ for all $q \in K_p$. Now define $f' = \chi f$, $u_0{}' = \chi u_0$ and $u_1{}' = \chi u_1$. Then by proposition 4.6 there is a unique solution u' to the primed initial value problem. Now set $u = u'$ in $V_p := I^-(p) \cap J^+(\Sigma)$. Similarly given some other point $\tilde{p}$ to the future of Σ there is a unique solution $\tilde{u}'$ to the initial value problem in $V_{\tilde{p}} := I^-(\tilde{p}) \cap J^+(\Sigma)$ and we may set $u = \tilde{u}'$ in $V_{\tilde{p}}$. If we now consider some further point $q \in V_p \cap V_{\tilde{p}}$ then by uniqueness the two potential solutions, given by u' and $\tilde{u}'$ respectively, agree in $V_p \cap V_{\tilde{p}}$, so there is no contradiction in setting $u = u'$ in V_p and $u = \tilde{u}'$ in $V_{\tilde{p}}$ since $u' = \tilde{u}'$ in the intersection. Thus we have a solution on the whole of $D^+(\Sigma)$. A similar argument gives us a solution on $D^-(\Sigma)$ and hence on the whole of M. The solution is unique by proposition (4.2) and satisfies the causal support condition by proposition (4.3).

□

We also want to show that the initial value problem is well-posed in the sense of corollary 4.11 below. For solutions in $H^1_{\mathrm{loc}}(M)$ this follows immediately from the energy estimate (4.1). But well-posedness in $H^2_{\mathrm{loc}}(M)$ requires a higher order estimate which we now establish.

Lemma 4.8. *For each compact subset $K \subset M$ there exists a $\delta > 0$ with the following property: if $(u_0, u_1) \in H^2(\Sigma) \times H^1(\Sigma)$ with $\mathrm{supp}(u_j) \subset K \cap \Sigma$ $(j = 1, 2)$ and $f \in H^1(M)$ with $\mathrm{supp}(f) \subset K$, then the (weak) solution $u \in C^0(\mathbb{R}, H^2(\Sigma_t)) \cap C^1(\mathbb{R}, H^1(\Sigma_t)) \cap H^2_{\mathrm{loc}}(M)$ of $\Box_g u = f$ with initial data (u_0, u_1) satisfies the energy inequality*

$$\|u\|_{C^0([0,\delta],H^2(\Sigma_t))} + \|u\|_{C^1([0,\delta],H^1(\Sigma_t))} \leqslant C\left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{H^1(M)}\right). \tag{4.5}$$

Proof. We use a similar approach to that in the existence proof given in [3, theorem 3.2.11].

For every $p \in K$ there are neighbourhoods U_p, W_p and $\epsilon_p > 0$ with the properties of lemma 4.5. By compactness there is a finite number of points $p_1, \ldots, p_N$ such that the corresponding W_{p_j} cover K. Now let $\{\chi_j\}_{j=1}^N$ be a partition of unity of K subordinate to the W_{p_j}.

We now define

$$u_{0,j} := \chi_j u_0, \quad u_{1,j} := \chi_j u_1, \quad f_j := \chi_j f,$$

so that

$$u_0 = \sum_{j=1}^N u_{0,j}, \quad u_1 = \sum_{j=1}^N u_{1,j}, \quad f = \sum_{j=1}^N f_j.$$

We also define

$$K_j := \mathrm{supp}(u_{0,j}) \cup \mathrm{supp}(u_{1,j}) \cup \mathrm{supp}(f_j) \subset K \cap \mathrm{supp}(\chi_j) \subset W_{p_j}.$$

Let u_j be the (weak) solution of the initial value problem

$$\Box u_j = f_j, \quad u_j|_\Sigma = u_{0,j}, \quad \nabla_n u_j|_\Sigma = u_{1,j}.$$

We will employ (an implicit choice of a temporal function in terms of) the diffeomorphism $M \cong \mathbb{R} \times \Sigma$ and slightly abuse notation from now on by considering all functions u, u_j etc to be defined already on products $I \times \Sigma$, where I is some open interval, thus suppressing the transfers of functions via restrictions of the underlying global diffeomorphism.

By point two of lemma 4.5 there exists an $\epsilon_{p_j} > 0$ such that

$$\left((-2\epsilon_{p_j}, 2\epsilon_{p_j}) \times \Sigma\right) \cap J(K_j) \subset U_{p_j}. \tag{4.6}$$

Let $\delta = \min\{\epsilon_{p_1}, \ldots, \epsilon_{p_N}\}$. Given the solutions u_j of the local problem we may extend them by zero on all of $(-2\delta, 2\delta) \times \Sigma$ and sum them to give our *unique* solution

$$u = \sum_{j=1}^N u_j, \quad \text{on } (-2\delta, 2\delta) \times \Sigma.$$

Since by (4.6) each of the u_j lie entirely within some chart (U_{p_j}, ϕ_{p_j}) we may regard the initial value problem as one on $I \times \mathbb{R}^n$ where I is an interval chosen sufficiently large such that the images of $(-2\delta, 2\delta) \times \Sigma$ under all the ϕ_{p_j} are contained in $I \times \mathbb{R}^n$.

Then the third condition of lemma 4.5 enables us to transfer the basic energy estimate according to lemma 3.1 from $I \times \mathbb{R}^n$ to ones for u_j on $U_j \subset M$ to give

$$\|u_j\|_{C^0([0,\delta],H^2(\Sigma))} + \|u_j\|_{C^1([0,\delta],H^1(\Sigma))} \leqslant C_j \left(\|u_{0,j}\|_{H^2(\Sigma)} + \|u_{1,j}\|_{H^1(\Sigma)} + \|f_j\|_{L^2([0,\delta],H^1(\Sigma))}\right).$$

Now $u = \sum_j u_j$ so that

$$\begin{aligned}
\|u\|_{C^0([0,\delta],H^2(\Sigma))} + \|u\|_{C^1([0,\delta],H^1(\Sigma))} &\leqslant \sum_{j=1}^{N} \left(\|u_j\|_{C^0([0,\delta],H^2(\Sigma))} + \|u_j\|_{C^1([0,\delta],H^1(\Sigma))}\right) \\
&\leqslant \sum_{j=1}^{N} C_j \left(\|u_{0,j}\|_{H^2(\Sigma)} + \|u_{1,j}\|_{H^1(\Sigma)} + \|f_j\|_{L^2([0,\delta],H^1(\Sigma))}\right) \\
&\leqslant \sum_{j=1}^{N} C_j \left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{L^2([0,\delta],H^1(\Sigma))}\right) \\
&\leqslant C \left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{L^2([0,\delta],H^1(\Sigma))}\right),
\end{aligned}$$

where we may replace $\|f\|_{L^2([0,\delta],H^1(\Sigma))}$ by the larger value $\|f\|_{H^1(M)}$, since $f \in H^1_{\text{comp}}(M)$. □

We remark that in the above proof (and formulation of the result) we have replaced the norm $\|f\|_{L^2([0,\delta],H^1(\Sigma))}$ by $\|f\|_{H^1(M)}$, which is valid for $f \in H^1_{\text{comp}}(M)$, to avoid the need to specify a particular choice of a temporal function.

Proposition 4.9 (global higher energy estimates) *Let (M, g) be a time oriented $(n+1)$-dimensional Lorentzian manifold with $C^{1,1}$ metric and Σ a smooth spacelike n-dimensional hypersurface. Let t be a smooth temporal function with $\Sigma = t^{-1}(0)$ and let n be the future directed timelike unit normal to Σ.*

Given initial data $(u_0, u_1) \in H^2_{\text{comp}}(\Sigma) \times H^1_{\text{comp}}(\Sigma)$ and source $f \in H^1_{\text{comp}}(M)$, then the (weak) solution $u \in C^0\left(\mathbb{R}, H^2(\Sigma_t)\right) \cap C^1\left(\mathbb{R}, H^1(\Sigma_t)\right) \cap H^2_{\text{loc}}(M)$ satisfies

$$\|u\|_{C^0([0,T],H^2(\Sigma))} + \|u\|_{C^1([0,T],H^1(\Sigma))} \leqslant C\left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{H^1(M)}\right)$$

for any interval $[0, T]$.

Proof. We first use lemma 4.8 to obtain an estimate for the data $\hat{u}_0 := u|_{\Sigma_\delta}$ and $\hat{u}_1 := \nabla_n u|_{\Sigma_\delta}$ induced by u on Σ_δ. It follows from (4.5) and the fact that $u \in C^0(\mathbb{R}, H^2(\Sigma_t)) \cap C^1(\mathbb{R}, H^1(\Sigma_t)) \cap H^2_{\text{loc}}(M)$ that

$$\|\hat{u}_0\|_{H^2(\Sigma_\delta)} + \|\hat{u}_1\|_{H^1(\Sigma_\delta)} \leqslant \tilde{C}\left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{H^1([0,\delta]\times\Sigma)}\right).$$

Now applying lemma 4.8 to the initial surface Σ_δ we obtain a $\hat{\delta} > \delta$ such that

$$\begin{aligned}
&\|u\|_{C^0([\delta,\hat{\delta}],H^2(\Sigma))} + \|u\|_{C^1([\delta,\hat{\delta}],H^1(\Sigma))} \leqslant \hat{C}\left(\|\hat{u}_0\|_{H^2(\Sigma_\delta)} + \|\hat{u}_1\|_{H^1(\Sigma_\delta)} + \|f\|_{H^1([\delta,\hat{\delta}]\times\Sigma)}\right) \\
&\leqslant \hat{C}\left(\tilde{C}\left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{H^1([0,\delta]\times\Sigma)}\right) + \|f\|_{H^1([\delta,\hat{\delta}]\times\Sigma)}\right) \\
&\leqslant C_1\left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{H^1([0,\hat{\delta}]\times\Sigma)}\right).
\end{aligned}$$

Combining the two energy inequalities on $[0, \delta]$ and $[\delta, \hat{\delta}]$ we have

$$\begin{aligned}
&\|u\|_{C^0([0,\hat{\delta}],H^2(\Sigma))} + \|u\|_{C^1([0,\hat{\delta}],H^1(\Sigma))} \\
&\leqslant \|u\|_{C^0([0,\delta],H^2(\Sigma))} + \|u\|_{C^1([0,\delta],H^1(\Sigma))} + \|u\|_{C^0([\delta,\hat{\delta}],H^2(\Sigma))} + \|u\|_{C^1([\delta,\hat{\delta}],H^1(\Sigma))} \\
&\leqslant C \left(\|u_0\|_{H^2(\Sigma)} + \|\tilde{u}_1\|_{H^1(\Sigma)} + \|f\|_{H^1([0,\delta]\times\Sigma)}\right) + C_1 \left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{H^1([0,\hat{\delta}]\times\Sigma)}\right) \\
&\leqslant C_2 \left(\|u_0\|_{H^2(\Sigma)} + \|\tilde{u}_1\|_{H^1(\Sigma)} + \|f\|_{H^1([0,\hat{\delta}]\times\Sigma)}\right).
\end{aligned}$$

This shows that we may extend the energy inequality from $[0, \delta]$ to the larger time interval $[0, \hat{\delta}]$ by repeatedly applying lemma 4.8.

Similarly, for every $\tau \in [0, T]$ we may find a $\delta(\tau) > 0$ such that lemma 4.8 applies to the time interval $(\tau - \delta(\tau), \tau + \delta(\tau))$ with initial data given on Σ_τ. By compactness, finitely many intervals $(\tau_k - \delta(\tau_k), \tau_k + \delta(\tau_k))$ $(k = 1, \ldots, m)$ cover $[0, T]$ and the energy inequalities on these may be combined. □

We now use the above proposition to obtain a spacetime energy inequality.

Proposition 4.10 (higher order spacetime energy estimates) *Let (M, g) be a time oriented $(n + 1)$-dimensional Lorentzian manifold with $C^{1,1}$ metric and Σ a smooth spacelike n-dimensional hypersurface. Given initial data $(u_0, u_1) \in H^2_{\mathrm{comp}}(\Sigma) \times H^1_{\mathrm{comp}}(\Sigma)$ and source $f \in H^1_{\mathrm{comp}}(M)$, then the (weak) solution u satisfies the energy-inequality*

$$\|u\|_{H^2(K)} \leqslant C \left(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + \|f\|_{H^1(M)}\right). \tag{4.7}$$

for any compact $K \subset M$.

Proof. Without loss of generality we may assume that $K \subset [0, T] \times \Sigma$. Due to the regularity of $u \in C^0(\mathbb{R}, H^2(\Sigma_t)) \cap C^1(\mathbb{R}, H^1(\Sigma_t))$ we have control of the second order spatial derivatives, the second order mixed derivatives and the lower order terms

$$\max(\|\partial_i\partial_j u\|_{L^2(K)}, \|\partial_j u\|_{L^2(K)}) \leqslant C \left(\|u\|_{C^0([0,T],H^2(\Sigma))}\right), \tag{4.8}$$

$$\max(\|\partial_i\partial_t u\|_{L^2(K)}, \|\partial_t u\|_{L^2(K)}) \leqslant C \left(\|u\|_{C^1([0,T],H^1(\Sigma))}\right). \tag{4.9}$$

In order to obtain the required estimate we also need to control $\partial_{tt}u$ in the $L^2(K)$ norm. Using $\Box_g u = f$ we have

$$\|g^{00}\partial_{tt}u\|_{L^2(K)} = \|f + (-g^{ti}\partial_t\partial_i - g^{ij}\partial_j\partial_i + g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma)u\|_{L^2(K)}$$

From (4.8) and (4.9), the regularities of f and g, we obtain an L^2 estimate for $\partial_{tt}u$,

$$\begin{aligned}
\|\partial_{tt}u\|_{L^2(K)} &= C_1\|f + (-g^{ti}\partial_t\partial_i - g^{ij}\partial_j\partial_i + g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\partial_\gamma)u\|_{L^2(K)} \\
&\leqslant C_2 \left(\|u\|_{C^0([0,T],H^2(\Sigma))} + \|u\|_{C^1([0,T],H^1(\Sigma))} + \|f\|_{L^2(K)}\right).
\end{aligned}$$

Combining the above with proposition 4.9 completes the proof. □

Equation (4.7) implies the following result.

Corollary 4.11. *The solution to the Cauchy problem described in theorem 4.7 is well-posed in the sense that the* solution map

$$Sol : H^2(\Sigma) \times H^1(\Sigma) \times H^1_{\mathrm{comp}}(M) \to H^2_{\mathrm{loc}}(M), (u_0, u_1, f) \mapsto u,$$

is continuous in the topologies coming from the respective Sobolev spaces (see appendix B).

5. Green operators for $C^{1,1}$ spacetimes

In this section we will define Green operators for $\Box_g$ on globally hyperbolic manifolds M with $C^{1,1}$ metrics. We will show existence and uniqueness of Green operators via the existence of solutions to the wave equation with appropriate regularity and causal support. We define below the notion of generalised hyperbolicity which will give us the required conditions in this situation.

A spacetime (M, g) is said to satisfy the condition of *generalised hyperbolicity* if the inhomogeneous wave equation for zero Cauchy data is well-posed and causal. The precise choice of function spaces in the definition of well-posedness depends upon the regularity of the metric. In our case we give the following definition:

Definition 5.1 (generalised hyperbolicity). A $C^{1,1}$ spacetime (M, g) satisfies the condition of generalised hyperbolicity if the following conditions hold

(a) **Existence**: for every $f \in H^1_{\mathrm{comp}}(M)$ there exists a unique future solution $u^+ \in H^2_{\mathrm{loc}}(M, g)$ such that

$$\Box_g u^+ = f \quad \text{on } M$$

which satisfies the causal support condition $\mathrm{supp}(u^+) \subset J^+(\mathrm{supp}(f))$. Note that this condition implies that on a Cauchy surface to the past of $\mathrm{supp}(f)$ one must have zero initial data. However the particular choice of such a Cauchy surface makes no difference to the solution (see proof of theorem 5.2 below for more details).

(b) **Uniqueness of causal solution**: for every $f \in H^1_{\mathrm{comp}}(M)$ there exists a unique past solution $u^- \in H^2_{\mathrm{loc}}(M, g)$ such that

$$\Box_g u^- = f \quad \text{on } M$$

which satisfies $\mathrm{supp}(u^-) \subset J^-(\mathrm{supp}(f))$ where we can choose any Cauchy surface to the future of $\mathrm{supp}(f)$ and solve the Cauchy problem going back in time.

(c) **Well-posedness**: we require that the maps $f \mapsto u^+$ and $f \mapsto u^-$ are continuous maps from $H^1_{\mathrm{comp}}(M) \to H^2_{\mathrm{loc}}(M)$ in the sense that $\|u^\pm\|_{H^2(K)} \leqslant C_\pm \|f\|_{H^1(M)}$ holds for any compact subset $K \subset M$ with suitable constants $C_\pm > 0$.

Theorem 5.2. *Let (M, g) be a globally hyperbolic time oriented $(n+1)$-dimensional Lorentzian manifold with $C^{1,1}$ metric and Σ a smooth spacelike n-dimensional Cauchy surface. Then (M, g) satisfies the condition of generalised hyperbolicity.*

Proof. Theorem 4.7 shows that a globally hyperbolic $C^{1,1}$ spacetime satisfies the condition of generalised hyperbolicity to the future by considering the forward initial value problem

$$\Box_g u^+ = f \quad \text{on } M, \quad u^+(\Sigma_+) = 0, \quad \nabla_n u^+(\Sigma_+) = 0,$$

where $f \in H^1_{\mathrm{comp}}(M)$ and Σ_+ is a smooth spacelike Cauchy hypersurface such that $J^+(\mathrm{supp}(f)) \cap \Sigma_+ = \emptyset$.

Note: if we were to choose some other smooth spacelike Cauchy hypersurface $\tilde{\Sigma}_+$, which also satisfies $J^+(\mathrm{supp}(f)) \cap \tilde{\Sigma}_+ = \emptyset$, then the corresponding solution is the same, since the divergence theorem arguments used in lemma 4.1 apply and yield that the solution must vanish in the region between Σ_+ and $\tilde{\Sigma}_+$.

Similarly, theorem 4.7 shows it satisfies the condition of generalised hyperbolicity to the past by considering the backwards initial value problem

$$\Box_g u^- = f \quad \text{on}\, M, \quad u^-(\Sigma_-) = 0, \quad \nabla_n u^-(\Sigma_-) = 0.$$

where $f \in H^1_{\text{comp}}(M)$ and Σ_- is a smooth spacelike Cauchy hypersurface such that $J^-(\text{supp}(f)) \cap \Sigma_- = \emptyset$. Again the solution is independent of the choice of Cauchy surface as long as it satisfies the causal support condition. □

5.1. Green operators

The definition of the Green operators in the non-smooth setting will require us to choose suitable spaces of functions as domain and range (see theorem 5.9). We therefore define the following spaces:

$$\begin{aligned} V_0 &= \{\phi \in H^2_{\text{comp}}(M)\, \text{s.t.}\, \Box_g \phi \in H^1_{\text{comp}}(M)\} \\ U_0 &= H^1_{\text{comp}}(M) \\ V_{sc} &= \big\{\phi \in H^2_{\text{loc}}(M)\, \text{s.t.}\, \Box_g \phi \in H^1_{\text{loc}}(M) \\ &\quad \text{and}\ \text{supp}(\phi) \subset J(K)\ \text{where}\ K\ \text{is a compact subset of}\ M\big\} \end{aligned} \tag{5.1}$$

Remark 5.3. Note that none of the spaces defined above depend upon the choice of background metric used in the definition of the Sobolev spaces.

Definition 5.4. A linear map

$$G^+ : H^1_{\text{comp}}(M) \to H^2_{\text{loc}}(M)$$

satisfying the properties

(a) $\Box_g G^+ = \text{id}_{H^1_{\text{comp}}(M)}$,
(b) $G^+ \Box_g|_{V_0} = \text{id}_{V_0}$,
(c) $\text{supp}(G^+(f)) \subset J^+(\text{supp}(f))$ for all $f \in H^1_{\text{comp}}(M)$,

is called an *advanced Green operator* for $\Box_g$. A *retarded Green operator* G^- is defined similarly.

Remark 5.5.

(a) Clearly, the regularity condition in the definition of the space V_0 was chosen to guarantee that $\Box_g f$, given $f \in V_0$ belongs to the domain $H^1_{\text{comp}}(M)$ of the Green operators.
(b) In several proofs below we will show the identities in properties (a) and (b) to hold weakly, i.e., when evaluated on test functions in $\mathcal{D}(M)$. However it can be shown using the results of Hörmander [38, theorem 1.25] that if two such Sobolev functions have the same effect on test functions then they are actually equal as Sobolev functions.
(c) The function spaces $H^2_{\text{loc}}(M)$ and $H^1_{\text{comp}}(M)$ used as target space and domain for the Green operators are in perfect accordance with the theory of so-called *regular fundamental solutions* for hyperbolic operators with constant coefficients[5] (cf [39, section 12.5]) as we sketch briefly in the following: let M be $(n+1)$-dimensional

[5] Of course, in case of the wave operator we even have explicit representations for the advanced and retarded fundamental solutions E_+ and E_-, e.g., in [38, sections 6.2 and 7.4], but these are not required here.

Minkowski space so that $\square$ has the symbol $p(\tau,\xi) = \tau^2 - |\xi|^2$, which is hyperbolic with respect to the directional vectors $(\pm 1, 0)$ and produces the temperate weight function $\tilde{p}(\tau,\xi) := \sqrt{\sum_{|\alpha|\geqslant 0} |\partial^\alpha p(\tau,\xi)|^2} = \sqrt{(\tau^2 - \xi^2)^2 + 4(\tau^2 + \xi^2 + 2)} \geqslant \sqrt{1 + \tau^2 + \xi^2} =: w_1(\tau,\xi)$. The unique fundamental solution $E_\pm$ with support in the half space where $\pm t \geqslant 0$ belongs to $B^{\mathrm{loc}}_{\infty,\tilde{p}}$, i.e., for every test function ϕ the Fourier transform $\mathcal{F}(\phi E_\pm)$ times $\tilde{p}$ is measurable and bounded. The advanced and retarded Green operators are then given by convolution $G^\pm f = E_{\pm *} f$ for every $f \in H^1_{\mathrm{comp}}(\mathbb{R}^{n+1}) = \mathcal{E}'(\mathbb{R}^{n+1}) \cap B_{2,w_1}$, where $B_{2,w_1} = \{u \in \mathcal{S}' | w_1 \cdot \mathcal{F}u \in L^2\} = H^1(\mathbb{R}^{n+1})$. Finally, we may apply [39, theorem 12.5.3 or theorem 10.1.24] to obtain $G^\pm f \in B^{\mathrm{loc}}_{2,\tilde{p}\cdot w_1} \subseteq B^{\mathrm{loc}}_{2,w_1^2} = H^2_{\mathrm{loc}}(\mathbb{R}^{n+1})$, since $\tilde{p}(\tau,\xi) w_1(\tau,\xi) \geqslant w_1^2(\tau,\xi) = 1 + \tau^2 + \xi^2$.

We next show that the advanced and retarded Green operators (if they exist) are adjoints of one another. To do this we use the following lemma.

Lemma 5.6. *Given $\chi, \varphi \in H^2_{\mathrm{loc}}(M,g)$ and $\mathrm{supp}(\chi) \cap \mathrm{supp}(\varphi)$ compact. Then, we have*

$$\int_M \square_g \chi \varphi \nu_g = \int_M \chi \square_g \varphi \nu_g. \tag{5.2}$$

The proof of the lemma follows from using integration by parts twice and the support properties given in the hypothesis. Note that the specified regularity of the metric g and of the functions is needed in order to use the L^2 inner product. We may now prove the following theorem.

Theorem 5.7. *Given Green operators satisfying conditions* (*a*) *and* (*c*) *of definition 5.4 and $\chi, \varphi \in H^1_{\mathrm{comp}}(M)$ we have that*

$$\int_M G^+(\chi) \varphi \nu_g = \int_M \chi G^-(\varphi) \nu_g.$$

Proof. First, notice that if $\chi, \varphi \in H^1_{\mathrm{comp}}(M)$ we have that $G^+(\chi), G^-(\varphi) \in H^2_{\mathrm{loc}}(M,g)$. Moreover, $G^+(\chi) \cap G^-(\varphi) \subset J^+(\mathrm{supp}(\chi)) \cap J^-(\mathrm{supp}(\varphi))$ is compact by the global hyperbolicity condition.

Hence,

$$\begin{aligned} \int_M G^+(\chi)\varphi\nu_g &= (G^+(\chi), \varphi)_{L^2(M,g)} = (G^+(\chi), \square_g G^-(\varphi))_{L^2(M,g)} \\ &= (\square_g G^+(\chi), G^-(\varphi))_{L^2(M,g)} = (\chi, G^-(\varphi))_{L^2(M,g)} = \int_M \chi G^-(\varphi)\nu_g. \end{aligned}$$

$\square$

We are now in a position to prove the main result about existence of Green operators.

Theorem 5.8. *Let (M,g) be a spacetime that satisfies the definition of generalised hyperbolicity* (*definition 5.1*). *Then there exist unique continuous advanced and retarded Green operators for $\square_g$ on M.*

Proof. We will only discuss the advanced Green operator, the existence and the properties of the retarded Green operator follow from an analogous argument with the roles of future and past interchanged.

Existence: we define the linear map

$$G^+ : H^1_{\mathrm{comp}}(M) \to H^2_{\mathrm{loc}}(M)$$

which sends a source function f to the (unique) advanced weak solution u^+. That such a u^+ exists and is unique is a consequence of generalised hyperbolicity. Property (a) in definition 5.4 is immediate. In addition, the energy estimate (4.7) shows that G^+ is a continuous operator. It remains to prove properties (b) and (c) in definition 5.4.

Property (b): let $f \in V_0$ and $v \in \mathcal{D}(M)$, then

$$(G^+(\Box_g f), v)_{L^2(M,g)} = (\Box_g f, G^-(v))_{L^2(M,g)} = (f, \Box_g G^-(v))_{L^2(M,g)} = (f, v)_{L^2(M,g)},$$

where we have used theorem 5.7 and property (a) for the retarded Green operator G^-. Thus the weak form of the required identity holds, which implies $G^+\Box_g(f) = f$ for every $f \in V_0$ (see remark 5.5(b)).

Property (c) follows because $\mathrm{supp}(G^{+(\mathrm{f})}) = \mathrm{supp}(u^+) = \mathrm{supp}(u^+) \subset J^+(\mathrm{supp}(f))$ by proposition 4.3.

Uniqueness: let $\tilde{G}^+$ be another linear operator satisfying definition 5.4. Given $f \in H^1_{\mathrm{comp}}(M)$ we have that $v := \tilde{G}^+(f)$ satisfies $\Box_g v = f$ and $\mathrm{supp}(v) \subset J^+(\mathrm{supp}(f))$. Since $f \in H^1_{\mathrm{comp}}(M)$, $\mathrm{supp}(v) \subset J^+(\mathrm{supp}(f))$ and M is globally hyperbolic, there is a smooth timelike Cauchy surface Σ to the past of the support of f where the Cauchy data vanishes, i.e., $v = 0$ and $\nabla_n v = 0$. Hence, v is a solution to the zero initial data forward Cauchy problem on Σ. By uniqueness we must have $v = u^+$ so we can conclude that $\tilde{G}^+(f) = G^+(f)$ for all $f \in H^1_{\mathrm{comp}}(M)$. □

We now show that the low-regularity Green operators satisfy an exact sequence result similar to that in the smooth case [3, theorem 3.4.7].

Theorem 5.9. *Let M be a connected time-oriented globally hyperbolic Lorentzian manifold that satisfies definition 5.1. Define the causal propagator as*

$$G = G^+ - G^- : H^1_{\mathrm{comp}}(M) \to H^2_{\mathrm{loc}}(M)$$

Then the image of G is contained in V_{sc} and the following complex is exact:

$$0 \longrightarrow V_0 \xrightarrow{\Box_g} U_0 \xrightarrow{G} V_{\mathrm{sc}} \xrightarrow{\Box_g} H^1_{\mathrm{loc}}(M).$$

Proof of theorem 5.9. First we show that the sequence is a complex: we have from the definitions $G^+\Box_g|_{V_0} = G^-\Box_g|_{V_0} = \mathrm{id}_{V_0}$ and $\Box_g G^+ = \Box_g G^- = \mathrm{id}_{H^1_{\mathrm{comp}}(M)}$, therefore $G\Box_g\phi = 0$ for all $\phi \in V_0$ and $\Box_g G\psi = 0$ for all $\psi \in U_0$.

- Exactness at V_0, i.e., injectivity of $\Box_g$: let $\phi \in V_0$ be such that $\Box_g\phi = 0$. By compactness of the support there is a smooth spacelike Cauchy hypersurface Σ such that $\phi = 0$ and $\nabla_n\phi = 0$ on Σ. Therefore, ϕ is a solution to the Cauchy problem with vanishing initial data and source. Uniqueness of the solution implies $\phi = 0$.
- Exactness at U_0: let $\phi \in U_0$ be such that $\phi \in \ker(G)$, i.e., $G^+(\phi) = G^-(\phi)$. Define $\psi := G^+(\phi) = G^-(\phi)$, hence $\psi \in H^2_{\mathrm{loc}}(M)$ and $\Box_g\psi = \phi$. Moreover, ψ is compactly supported in M because $\mathrm{supp}(\psi) \subset \mathrm{supp}(G^+(\phi)) \cap \mathrm{supp}(G^-(\phi)) \subset J^+(\mathrm{supp}(\phi)) \cap J^-(\mathrm{supp}(\phi))$ and the latter is compact due to global hyperbolicity. Thus there exists $\psi \in H^2_{\mathrm{loc}}(M)$ such that $\Box_g\psi = \phi \in U_0$ and $\mathrm{supp}(\psi)$ is compact. Hence, $\psi \in V_0$ and $\phi \in \mathrm{Im}(G)$.
- Exactness at V_{sc}: let $\phi \in \ker(\Box_g)$ and $\phi \in V_{\mathrm{sc}}$. Without loss of generality we may assume that $\mathrm{supp}(\phi) \subset I^+(K) \cup I^-(K)$ for some compact set[6] K of M. Using a partition of unity

[6] We may take $I^+(K)$ rather than $J^+(K)$ by replacing an initial choice of compact set $\tilde{K}$ with a slightly larger K for which $J^+(\tilde{K}) \subset I^+(K)$. Specifically we can take $K = \bar{U}$ where U is any open set containing $\tilde{K}$ with compact closure. Similar remarks apply to $I^-(K)$.

$\{\chi_-, \chi_+\}$ subordinate to $\{I^-(K), I^+(K)\}$ we let $\phi_1 = \chi_-\phi$ and $\phi_2 = \chi_+\phi$, thus $\phi = \phi_1 + \phi_2$. Then $\mathrm{supp}(\phi_1) \subset J^-(K)$ and $\mathrm{supp}(\phi_2) \subset J^+(K)$, hence $\phi_1, \phi_2 \in V_{\mathrm{sc}}$.

Define $\psi := -\Box_g\phi_1 = \Box_g\phi_2$. Then $\mathrm{supp}(\psi)$ is compact because $\mathrm{supp}(\psi) \subset J^-(K) \cap J^+(K)$. Moreover, $\psi \in H^1_{\mathrm{loc}}(M)$ since $\phi \in V_{\mathrm{sc}}$. Combining these two observations, we conclude that $\psi \in H^1_{\mathrm{comp}}(M)$ and therefore $G^+(\psi)$ is defined.

For arbitrary $\chi \in \mathcal{D}(M)$ we have

$$(\chi, G^+(\psi))_{L^2(M,g)} = (\chi, G^+(\Box_g\phi_2))_{L^2(M,g)} = (\Box_g G^-\chi, \phi_2)_{L^2(M,g)} = (\chi, \phi_2)_{L^2(M,g)}$$

which shows $G^+(\psi) = \phi_2$, where we have made use of the fact that the supports of $G^-\chi$ and ϕ_2 intersect in a compact set due to global hyperbolicity. Similarly, $G^-(\psi) = -\phi_1$ and therefore $G(\psi) = G^+(\psi) - G^-(\psi) = \phi_2 + \phi_1 = \phi$. In summary, we may conclude that there exists $\psi \in U_0$ satisfying $G(\psi) = \phi$. □

5.2. Restrictions

We briefly discuss the restriction of Green operators to *causally compatible* subsets $\Omega \subset M$, that is, sets such that

$$J_\Omega(x) = J_M(x) \cap \Omega \quad \forall x \in \Omega.$$

We have the following theorem (cf [3, proposition 3.5.1]).

Theorem 5.10. *Let M be a time oriented connected globally hyperbolic manifold with a $C^{1,1}$ Lorentzian metric, G^+ be the advanced Green operator for $\Box_g$ and $\Omega \subset M$ be a causally compatible open subset. Then we may define an advanced Green operator for the restriction of $\Box_g$ to Ω by*

$$\tilde{G}^+(\varphi) := G^+(\varphi_{\mathrm{ext}})|_\Omega, \quad \textit{for } \varphi \in H^1_{\mathrm{comp}}(M) \textit{ with } \mathrm{supp}(\varphi) \subseteq \Omega$$

where φ_{ext} denotes the extension of ϕ by zero. Similar results hold for G^-.

Remark 5.11. We denote the restriction of $\Box_g$ to Ω by $\widetilde{\Box}_g$. Notice that for all $u \in H^2_{\mathrm{loc}}(M)$ we have $\widetilde{\Box}_g(u|_\Omega) = \Box_g|_\Omega(u|_\Omega) = (\Box_g u)|_\Omega$ and for all $u \in H^2(\Omega)$ with $\mathrm{supp}(u) \subseteq \Omega$ we have $(\widetilde{\Box}_g u)_{\mathrm{ext}} = \Box_g(u_{\mathrm{ext}})$.

Proof of theorem 5.10. Property (a): let $f \in H^1_{\mathrm{comp}}(M)$ with $\mathrm{supp}(f) \subseteq \Omega$, then

$$\widetilde{\Box}_g\tilde{G}^+(f) = \widetilde{\Box}_g(G^+(f_{\mathrm{ext}})|_\Omega) = \Box_g(G^+(f_{\mathrm{ext}}))|_\Omega = f_{\mathrm{ext}}|_\Omega = f.$$

Property (b): let $f \in V_0$ with $\mathrm{supp}(f) \subset \Omega$, then

$$\tilde{G}^+(\widetilde{\Box}_g f) = (G^+((\widetilde{\Box}_g f)_{\mathrm{ext}})|_\Omega) = (G^+(\Box_g f_{\mathrm{ext}}))|_\Omega = f_{\mathrm{ext}}|_\Omega = f.$$

Property (c): for $f \in H^1_{\mathrm{comp}}(M)$ with $\mathrm{supp}(f) \subseteq \Omega$ we have

$$\begin{aligned} \mathrm{supp}(\tilde{G}^+(f)) &= \mathrm{supp}\left(G^+(f_{\mathrm{ext}})|_\Omega\right) = \mathrm{supp}(G^+(f_{\mathrm{ext}})) \cap \Omega \subset J^+_M(\mathrm{supp}(f_{\mathrm{ext}})) \cap \Omega \\ &= J^+_M(\mathrm{supp}(f)) \cap \Omega = J^+_\Omega(\mathrm{supp}(f)). \end{aligned}$$

□

6. Quantisation functors

In this section we discuss suitable categories and functors as in the smooth case that will allow us to construct the algebra of observables of the quantum theory.

6.1. The functor SYMPL and the categories GENHYP and SYMPLVECT

This subsection defines a category based on the analytic results in the previous sections and a functor assigning to each object a symplectic space.

Definition 6.1. Let GENHYP denote the category whose objects are 3-tuples (M, G^+, G^-) where M is a time oriented connected generalised globally hyperbolic manifold as in definition 5.1 and G^+, G^- are the unique Green operators of $\Box_g$. Let $X = (M_1, G_1^+, G_1^-)$ and $Y = (M_2, G_2^+, G_2^-)$ be two objects in GENHYP, then $\mathrm{Mor}(X, Y)$ consists of all smooth maps $\iota : M_1 \to M_2$ which are time-orientation preserving isometric embeddings such that $\iota(M_1) \subset M_2$ is a causally compatible open subset.

Remark 6.2. Theorem 5.2 shows that time oriented globally hyperbolic spacetimes with $C^{1,1}$ metrics are objects in this category.

Before considering quantisation we prove the following result on compatibility of Green operators.

Theorem 6.3. *Let M_1 and M_2 be as in definition 6.1, then the following diagram commutes:*

$$\begin{array}{ccc} H^1_{\mathrm{comp}}(M_1) & \xrightarrow{\mathrm{ext}} & H^1_{\mathrm{comp}}(M_2) \\ {\scriptstyle G_1^\pm}\downarrow & & \downarrow{\scriptstyle G_2^\pm} \\ H^2_{\mathrm{loc}}(M_1) & \xleftarrow{\mathrm{res}} & H^2_{\mathrm{loc}}(M_2) \end{array}$$

Proof. Theorem 5.10 shows that $\tilde{G}^\pm(\phi) := G_2^\pm(\phi_{\mathrm{ext}})|_{M_1}$ is a Green operator. By uniqueness, this operator has to be equal to $G_1^\pm$ and the result follows. □

Remark 6.4. In the smooth setting [3] the category LORFUND is defined as the category with objects being 5-tuples (M, F, G^+, G^-, P), where M is a Lorentzian manifold, F is a real vector bundle over M with non-degenerate inner product, P is a formally self-adjoint normally hyperbolic operator acting on sections in F and G^+, G^- are the advanced and retarded Green operators for P. The morphisms consist of maps ι such that $\iota : M_1 \to M_2$ is a time-orientation preserving isometric embedding such that $\iota(M_1) \subset M_2$ is a causally compatible open subset [3]. Moreover, given the condition of global hyperbolicity one can form the category GLOBHYP where objects are 3-tuples (M, F, P), where M is a Lorentzian manifold, F is a real vector bundle over M with non-degenerate inner product, P is a formally self-adjoint normally hyperbolic operator acting on sections in F. The morphisms are then given by maps ι such that $\iota : M_1 \to M_2$ is a time-orientation preserving isometric embedding such that $\iota(M_1) \subset M_2$ is a causally compatible open subset. The existence and uniqueness of Green operators allow us to form a functor from GLOBHYP to LORFUND [3].

We now use the Green operators in order to construct a symplectic vector space. Let (M, G^+, G^-) be an object of GENHYP and define

$$\tilde{\omega} : H^1_{\mathrm{comp}}(M) \times H^1_{\mathrm{comp}}(M) \to \mathbb{R}$$

by

$$\tilde{\omega}(\phi, \psi) = \int_M G(\phi)\psi\nu_g$$

where $G = G^+ - G^-$ is the causal propagator (see theorem 5.9). Then $\tilde{\omega}$ is bilinear and skew-symmetric by theorem 5.7. However, $\tilde{\omega}$ is degenerate because $\ker(G)$ is nontrivial. Moreover, using theorem 5.9 we have that

$$\ker(G) = \Box_g V_0.$$

Therefore on the quotient space $V(M) = U_0/\ker(G) = U_0/\Box_g V_0$ the degenerate form $\tilde{\omega}$ induces a symplectic form which we denote by ω.

Remark 6.5. It follows from corollary 4.11 that G is continuous so that $\ker G$ is a closed subspace and hence $V(M)$ is a normed space (and in particular, Hausdorff). See the discussion section for more details on this point.

Finally, we need a functor SYMPL : GENHYP $\to$ SYMPLVECT, where SYMPLVECT is the category whose objects are symplectic vector spaces with morphisms given by symplectic maps, i.e., linear maps A such that $\omega_1(f, g) = \omega_2(Af, Ag)$. The following theorem shows the existence of such a functor.

Theorem 6.6. *Let* $X = (M_1, G_1^+, G_1^-)$ *and* $Y = (M_2, G_2^+, G_2^-)$ *be two objects in GENHYP and* $f \in \mathrm{Mor}(X, Y)$ *be a morphism. Then ext* : $H^1_{\mathrm{comp}}(M_1) \to H^1_{\mathrm{comp}}(M_2)$ *maps the null space* $\ker(G_1)$ *into the null space* $\ker(G_2)$ *and hence induces a continuous symplectic linear map* $V(M_1) \to V(M_2)$.

Proof. Let $\phi \in \ker(G_1)$ then $\phi = \Box_{g_1}\psi$ for some $\psi \in V_0(M_1)$ where we have used theorem 5.9.

From the fact that $G_2(\phi_{\mathrm{ext}}) = G_2((\Box_{g_1}\psi)_{\mathrm{ext}}) = G_2\Box_{g_2}\psi_{\mathrm{ext}} = 0$ we see that $\mathrm{ext}(\ker(G_1)) \subset \ker(G_2)$. Hence, ext induces a linear map from $V(M_1) \to V(M_2)$. Moreover, for $\phi, \psi \in H^1_{\mathrm{comp}}(M_1)$ we have on taking representatives

$$\omega_1(\phi, \psi) = \int_{M_1} G_1(\phi)\psi\nu_{g_1} = \int_{M_1} G_2(\phi_{\mathrm{ext}})|_{M_1}\psi\nu_{g_1} = \int_{M_2} G_2(\phi_{\mathrm{ext}})\psi_{\mathrm{ext}}\nu_{g_2} = \omega_2(\phi_{\mathrm{ext}}, \psi_{\mathrm{ext}}).$$

Therefore, ext induces a symplectic map from $V(M_1)$ to $V(M_2)$. This induced map is also continuous (with the respective quotient topologies), because it is the composition $\pi_2 \circ E$ of two continuous maps, namely $E : V(M_1) = H^1_{\mathrm{comp}}(M_1)/\ker(G_1) \to H^1_{\mathrm{comp}}(M_2)$, the factor map of ext, and π_2, the quotient map $H^1_{\mathrm{comp}}(M_2) \to H^1_{\mathrm{comp}}(M_2)/\ker(G_2) = V(M_2)$. □

6.2. The functor CCR and the categories C^*-ALG and QUASILOCALALG

In this section we closely follow [3] and define the algebraic structures that will be required to represent the observables of the quantum theory. The definitions are algebraic in nature and do not require any further analytical considerations with respect to the regularity of solutions to the Cauchy problem. Nevertheless, the $C^{1,1}$ causality theory is required and will be mentioned below when it is used. Another modification with respect to the smooth case is that when considering the symplectic space (V, ω) in the smooth theory one has $V(M) = \mathcal{D}(M)/\ker G$ where $G = G^+ - G^-$ is a map $G : \mathcal{D}(M) \to C^\infty_{\mathrm{sc}}(M)$. Employing the short-hand notation $[f]$ for the class $f + \ker(G)$ in $U_0/\ker(G)$, the symplectic form is given by $\omega([f], [h]) = (f, Gh)_{L^2(M,g)}$

whereas in this section we will have $V(M) = H^1_{\mathrm{comp}}(M)/\ker G$ where now $G : H^1_{\mathrm{comp}}(M) \to V_{\mathrm{sc}}$ and the symplectic form is given by $\omega([f],[h]) = (f, Gh)_{L^2(M,g)}$.

We now introduce the definition of a Weyl system and a CCR-representation of (V, ω).

Definition 6.7. A *Weyl system* of the symplectic vector space (V, ω) consists of a C^*-algebra $\mathcal{A}$ with unit and a map $W : V \to \mathcal{A}$ such that for all $\varphi, \psi \in V$,

(a) $W(0) = 1$,
(b) $W(-\varphi) = W(\varphi)^*$,
(c) $W(\varphi) \cdot W(\psi) = \mathrm{e}^{-\mathrm{i}\omega(\varphi,\psi)/2} W(\varphi + \psi)$.

A Weyl system $(\mathcal{A}, W)$ of a symplectic vector space (V, ω) is called a *CCR-representation* of (V, ω) if $\mathcal{A}$ is generated as a C^*-algebra by the elements $W(\varphi)$, $\varphi \in V$. In this case we call $\mathcal{A}$ a CCR-algebra of (V, ω) and write it as $\mathrm{CCR}(V, \omega)$.

It is always possible to construct a CCR-representation $(\mathrm{CCR}(V, \omega), W)$ for any symplectic vector space (V, ω) (see [3, example 4.2.2]). Moreover, the construction is categorical in the sense that if (V_1, ω_1) and (V_2, ω_2) are two symplectic vector spaces and $S : V_1 \to V_2$ is a symplectic linear map, then there exists a unique injective $*$-morphism $\mathrm{CCR}(S) : \mathrm{CCR}(V_1, \omega_1) \to \mathrm{CCR}(V_2, \omega_2)$.

The proof can be found in corollary 4.2.11 in [3].

From uniqueness of the map $\mathrm{CCR}(S)$ it is possible to define a functor

$$\mathrm{CCR} : \mathrm{SYMPL} \to C^* - \mathrm{ALG}$$

where $C^* - \mathrm{ALG}$ is the category whose objects are C^*-algebras and whose morphisms are injective unit preserving $*$-morphisms.

A set I is called a directed set with orthogonality relation, if it carries a partial order $\leqslant$ and a symmetric relation $\perp$ between its elements such that:

(a) For all $\alpha, \beta \in I$ there exists a $\gamma \in I$ with $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$,
(b) For every $\alpha \in I$ there is a $\beta \in I$ with $\alpha \perp \beta$,
(c) If $\alpha \leqslant \beta$ and $\beta \perp \gamma$, then $\alpha \perp \gamma$,
(d) If $\alpha \perp \beta$ and $\alpha \perp \gamma$, then there exists a $\delta \in I$ such that $\beta \leqslant \delta, \gamma \leqslant \delta$ and $\alpha \perp \delta$.

Sets of this type allow to define the objects and morphisms of the category QUASILOCALALG.

Definition 6.8. The objects of the category QUASILOCALALG are bosonic quasi-local C^*-algebras which are pairs $(\mathcal{U}, \{\mathcal{U}_\alpha\}_{\alpha \in I})$ of a C^*-algebra $\mathcal{U}$ and a family $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of C^*-subalgebras, where I is a directed set with orthogonality relation such that the following holds:

(a) $\mathcal{U}_\alpha \subset \mathcal{U}_\beta$ whenever $\alpha \leqslant \beta$,
(b) $\mathcal{U} = \overline{\bigcup_\alpha \mathcal{U}_\alpha}$,
(c) The algebras $\mathcal{U}_\alpha$ have a common unit 1,
(d) If $\alpha \perp \beta$, then the commutators of elements from $\mathcal{U}_\alpha$ with those of $\mathcal{U}_\beta$ are trivial.

A morphism between two quasi-local C^*-algebras $(\mathcal{U}, \{\mathcal{U}_\alpha\}_{\alpha \in I})$ and $(\mathcal{V}, \{\mathcal{V}_\beta\}_{\beta \in J})$ is defined as a pair (φ, Φ) where $\Phi : \mathcal{U} \to \mathcal{V}$ is a unit-preserving C^*-morphism and $\varphi : I \to J$ is a map such that

(a) φ is monic, i.e., if $\alpha_1 \leqslant \alpha_2$ in I then $\varphi(\alpha_1) \leqslant \varphi(\alpha_2)$ in J,
(b) φ preserves orthogonality, i.e., if $\alpha_1 \perp \alpha_2$ in I, then $\varphi(\alpha_1) \perp \varphi(\alpha_2)$,

(c) $\Phi(\mathcal{U}) \subset \mathcal{V}_{\varphi(\alpha)}$ for all $\alpha \in I$.

In the remainder of this section we discuss a functor from GENHYP to QUASILOCALALG. Let (M, G^+, G^-) be an object in GENHYP and

$$I =: \{O \subset M | O \text{ is open, rel. compact, causally compatible, glob. hyperbolic}\} \cup \{\emptyset, M\}.$$

The relation $O \perp O'$ means that O and O' are causally independent, i.e., there is no causal curve connecting a point in $\overline{O}$ to a point in $\overline{O'}$.

Remark 6.9. The proof that the set I is a directed set with orthogonality relation requires results from causality theory in a low regularity setting [8, 17, 19] to obtain lemma A.5.11 in [3] and the existence of smooth time functions [34, 35] to obtain proposition A.5.13 in [3]. Properties (a) and (b) follow upon taking $\alpha = M$, $\beta = \emptyset$. Property (c) follows from the observation that $O \subset O'$ implies $J(O) \subset J(O')$, and Property (d) is implied by lemma 4.4.8 in [3] with the appropriate modifications of lemma A.5.11 and proposition A.5.13 therein.

For any non-empty set $O \in I$ take the restriction of the operator $\Box_g$ to the region O. Due to causal compatibility of $O \subset M$ the restriction of Green operators G^+, G^- to the region O yields Green operators G^+_O, G^-_O. Therefore, we get an object (O, G^+_O, G^-_O) for each $O \neq \emptyset \in I$. For $\emptyset \neq O_1 \subset O_2$ the inclusion induces a morphism ι_{O_2,O_1} in the category GENHYP. This morphism is given by the embedding $O_1 \to O_2$. Let α_{O_2,O_1} denote the morphism CCR $\circ$ SYMPL(ι_{O_2,O_1}) in C^*-ALG and recall that α_{O_2,O_1} is an injective unit preserving $*$-morphism.

We set for $\emptyset \neq O \in I$,

$$(V_O, \omega_O) := \mathrm{SYMPL}(O, G^+_O, G^-_O),$$

and for $O \in I, O \neq \emptyset, M$,

$$\mathcal{U}_O := \alpha_{M,O}(\mathrm{CCR}(V_O, \omega_O)),$$

for $O = M$ define

$$\mathcal{U}_M := C^* \left(\mathcal{U}_{O \in I, O \neq \emptyset, M}\right)$$

which is the algebra of CCR(V_M, ω_M) generated by all the $\mathcal{U}_O$; for $O = \emptyset$, set $\mathcal{U}_\emptyset = \mathbb{C}$.

Now we assign to any morphism in GENHYP a morphism between quasi-local algebras in QUASILOCALALG: consider a morphism $\iota : (M, G^+, G^-) \to (N, \tilde{G}^+, \tilde{G}^-)$ in GENHYP. Let I_1, I_2 denote the index sets associated to M, N respectively. We define a map $\varphi : I_1 \to I_2$ by $M \to N$ and $O_1 \to \iota(O_1)$ if $O_1 \neq M$. Since ι is an embedding such that $\iota(M) \subset N$ is causally compatible, the map φ is monotonic and preserves causal independence. Therefore, (φ, Φ) with $\Phi = \mathrm{CCR} \circ \mathrm{SYMPL}(\iota)$ is the required morphism. To be precise we have the following result.

Theorem 6.10. *The assignment $(M, G^+, G^-) \to (\mathcal{U}_M, \{\mathcal{U}_O\}_{O \in I})$ and $\iota \to (\varphi, \Phi)$ yields a functor QUANT from GENHYP to QUASILOCALALG.*

Proof. A detailed proof can be found in [3, lemma 4.4.10, theorem 4.4.11 and lemma 4.4.13]. □

Remark 6.11. In the low regularity setting the proof above requires one to consider elements $\phi \in H^1_{\mathrm{comp}}(O)$ rather than $\phi \in \mathcal{D}(O)$ in lemma 4.4.10 and the low regularity quotient $H^1_{\mathrm{comp}}(M)/\ker(G)$ instead of $\mathcal{D}(M)/\ker(G)$ with $C^{1,1}$ causality theory in lemma 4.4.13 in [3].

6.3. The Haag–Kastler axioms

In this subsection we show that the functor QUANT given by theorem 6.10 satisfies the Haag–Kastler axioms.

Theorem 6.12. *The functor* QUANT : GENHYP → QUASILOCALALG *satisfies the Haag–Kastler axioms, i.e., for every object* (M, G^+, G^-) *in GENHYP the corresponding quasi-local* C^**-algebra* $(\mathcal{U}_M, \{\mathcal{U}_\mathcal{O}\}_{\mathcal{O}\in I})$ *satisfies:*

(*a*) *If* $O_1 \subset O_2$ *then* $\mathcal{U}_{\mathcal{O}_1} \subset \mathcal{U}_{O_2}$ *for all* $O_1, O_2 \in I$.
(*b*) $\mathcal{U}_M = \overline{\bigcup_{O\in I, O\neq M,\varnothing} \mathcal{U}_\mathcal{O}}$.
(*c*) $\mathcal{U}_M$ *is simple.*
(*d*) *The* $\mathcal{U}_O$*'s have a common unit 1.*
(*e*) *For all* $O_1, O_2 \in I$ *with* $J(\overline{O_1}) \cap \overline{O_2} = \emptyset$ *the subalgebras* $\mathcal{U}_{O_1}, \mathcal{U}_{O_2}$ *commute.*
(*f*) (*Time-slice axiom*) *let* $O_1 \subset O_2$ *be nonempty elements of I admitting a common smooth spacelike Cauchy hypersurface, then* $\mathcal{U}_{O_1} = \mathcal{U}_{O_2}$.
(*g*) *Let* $O_1, O_2 \in I$ *and let the Cauchy development* $D(O_2)$ *be relatively compact in M. If* $O_1 \subset D(O_2)$*, then* $\mathcal{U}_{O_1} \subset \mathcal{U}_{O_2}$.

The proof will be based on the following two lemmas.

Lemma 6.13. *Let O be a causally compatible globally hyperbolic open subset of a* $C^{1,1}$ *globally hyperbolic manifold M. Assume there exists a smooth spacelike Cauchy hypersurface* Σ *of O which is also a Cauchy hypersurface of M, let h be a smooth Cauchy time function on O and* $K \subset M$ *be compact. Assume that there exists* $t \in \mathbb{R}$ *with* $K \subset I^+(h^{-1}(t))$*. Then there is a smooth function* $\rho : M \to [0, 1]$ *such that*

(*a*) $\rho = 1$ *on a neighbourhood of K,*
(*b*) $\mathrm{supp}(\rho) \cap J^-(K) \subset M$ *is compact, and*
(*c*) $\overline{\{x \in M | 0 < \rho(x) < 1\}} \cap J^-(K)$ *is compact and contained in O.*

The proof of lemma 6.13 in the $C^{1,1}$ setting can be carried out following that of [3] with suitable modifications using results of low regularity causality theory [8, 17, 19].

We reproduce the main argument of the proof as done in [3].

Proof. By assumption there are real numbers t_-, t_+ in the range of h such that $K \subset I^\pm(S_{t_\pm})$ where $S_t = h^{-1}(t)$. Since S_t is a Cauchy surface of O and since O and M admit a common Cauchy hypersurface, from $C^{1,1}$ causality theory it follows that $S_\pm$ are also Cauchy hypersurfaces of M. Since $J^\pm(S_\pm)$ are disjoint closed subsets of M there exists a smooth function $\rho : M \to [0, 1]$ such that $\rho|_{J^+(S_{t_+})} = 1$ and $\rho|_{J^-(S_{t_-})} = 0$.

The function ρ satisfies the first property stated in the lemma because $K \subset I^+(S_{t_+})$.

Since $\rho|_{J^-(S_{t_-})} = 0$, we have $\mathrm{supp}(\rho) \subset J^+(S_{t_-})$. Moreover, in the $C^{1,1}$ setting the causal relationship is still closed and implies that $J^+(S_{t_-}) \cap J^-(K)$ is compact and therefore $\mathrm{supp}(\rho) \cap J^-(K) \subset M$ is compact. This shows that the second property is satisfied.

The last property follows from two observations. The first one is that the closed set $\overline{\{x \in M | 0 < \rho(x) < 1\}} \cap J^-(K)$ is contained in the compact set $\mathrm{supp}(\rho) \cap J^-(K) \subset M$. The second observation is that $J^+(S_{t_-}) \cap J^-(S_{t_+}) \subset O$ which implies $\overline{\{x \in M | 0 < \rho(x) < 1\}} \cap J^-(K)$ is contained in O. The statement that $J^+(S_{t_-}) \cap J^-(S_{t_+}) \subset O$ follows from the characterisation of Cauchy hypersurfaces as surfaces that are met exactly once by every inextendible causal curve. This characterisation also holds in the $C^{1,1}$ setting. □

Lemma 6.14. *Let (M, G^+, G^-) be an object of GENHYP and O be a causally compatible globally hyperbolic open subset of M. Assume that there exists a Cauchy hypersurface Σ which is also a Cauchy hypersurface of M. Let $\varphi \in U_0$, then there exist $\chi \in V_0$ and $\psi \in U_0$ such that* $\mathrm{supp}(\psi) \subset O$ *and* $\varphi = \psi + \Box_g \chi$.

Proof. Let h be a Cauchy time function on O. Fix $t_- \leqslant t_+$ in the range of h. Then the subsets $\Sigma_- = h^{-1}(t_-), \Sigma_+ = h^{-1}(t_+)$ are Cauchy hypersurfaces of M. Hence every inextendible timelike curve in M meets Σ_-, Σ_+. Since $t_- \leqslant t_+$, the set $\{I^+(\Sigma_-), I^-(\Sigma_+)\}$ is a finite open cover of M. Let $\{f_+, f_-\}$ be a smooth partition of unity subordinated to this cover. In particular, $\mathrm{supp}(f_\pm) \subset I^\pm(\Sigma_\mp)$. Set $K_\pm := \mathrm{supp}(f_\pm \varphi) = \mathrm{supp}(\varphi) \cap \mathrm{supp}(f_\pm)$. Then $K_\pm$ is a compact subset of M satisfying $K_\pm \subset I^\pm(\Sigma_\mp)$. Applying lemma 6.13 we obtain two smooth functions $\rho_\pm : M \to [0, 1]$ satisfying

(a) $\rho_\pm = 1$ on a neighbourhood of $K_\pm$,
(b) $\mathrm{supp}(\rho_\pm) \cap J^\mp(K_\pm) \subset M$ is compact, and
(c) $\overline{\{x \in M | 0 < \rho_\pm(x) < 1\}} \cap J^\mp(K_\pm)$ is compact and contained in O.

Set $\chi_\pm := \rho_\pm G^\mp(f_\pm \varphi), \chi := \chi_+ - \chi_-$ and $\psi := \varphi - \Box_g \chi$. Since $\mathrm{supp}(G^\mp(f_\pm \varphi)) \subset J^\mp(K_\mp)$, the support of $\chi_\pm$ is contained in $\mathrm{supp}(\rho_\pm) \cap J^\mp(K_\pm)$ which is compact by the second property of $\rho_\pm$. Since $\rho_\pm$ and $f_\pm$ are smooth by construction, we have $\chi_\pm \in H^2_{\mathrm{comp}}(M)$. Moreover,

$$\begin{aligned}\Box_g \chi_\pm &= G^\mp(f_\pm \varphi) \Box_g \rho^\pm + g^{\alpha\beta} \partial_\alpha \rho^\pm \partial_\beta G^\mp(f_\pm \varphi) + \rho^\pm \Box_g G^\mp(f_\pm \varphi) \\ &= G^\mp(f_\pm \varphi) \Box_g \rho^\pm + g^{\alpha\beta} \partial_\alpha \rho^\pm \partial_\beta G^\mp(f_\pm \varphi) + \rho^\pm(f_\pm \varphi),\end{aligned}$$

which implies $\Box_g \chi_\pm \in H^1_{\mathrm{loc}}(M)$. Notice that $\Box_g \rho^\pm$ is not smooth but $C^{0,1}$.

Now ψ is the difference of $H^1_{\mathrm{loc}}(M)$ functions so it remains to show that $\mathrm{supp}(\psi)$ is compact and contained in O. By the first property of $\rho_\pm$, one has $\chi_\pm := G^\mp(f_\pm \varphi)$ in a neighbourhood of $K_\pm$. Moreover, $f_\pm \varphi = 0$ on $\{\rho_\pm = 0\}$. Hence, $\Box_g \chi_\pm = f_\pm \varphi$ on $\{\rho_\pm = 0\} \cup \{\rho_\pm = 1\}$. Therefore, $f_\pm \varphi - \Box_g \chi_\pm$ vanishes outside $\overline{\{x \in M | 0 < \rho_\pm(x) < 1\}}$, i.e., $\mathrm{supp}(f_\pm - \Box_g \chi_\pm) \subset \overline{\{0 < \rho_\pm(x) < 1\}}$. By the definitions of $\chi_\pm, f_\pm$ one also has $\mathrm{supp}(f_\pm \varphi - \Box_g \chi_\pm) \subset J^\mp(K_\pm)$, hence $\mathrm{supp}(f_\pm \varphi - \Box_g \chi_\pm) \subset J^\mp(K_\pm) \cap \overline{\{0 < \rho_\pm(x) < 1\}}$ which is compact and contained in O by the third property of $\rho_\pm$. Therefore, $\psi \in U_0$ with $\mathrm{supp}(\psi) \subset O$. Moreover, $\varphi - \psi = \Box_g \chi \in U_0$ which gives $\chi \in V_0$. □

Proof of theorem 6.12. The first, fourth and fifth axiom follow from the definition of the quasi-local C^*-algebra, the definition of the set I and [3, lemma 4.4.10]. The second axiom follows from [3, lemma 4.4.13] and the third axiom follows from [3, remark 4.5.3]. Remark 6.11 mentions the necessary modifications of those lemmas in the $C^{1,1}$ setting.

It therefore remains to prove the time-slice axiom. Let $O_1 \subset O_2$ be nonempty causally compatible globally hyperbolic subsets of M admitting a common smooth spacelike Cauchy hypersurface Σ. Let $[\phi] \in V(O_2)$. Then lemma 6.14 applied to $M := O_2$ and $O = O_1$ yields $\chi \in V_0, \psi \in U_0$ such that $\phi = \psi_{\mathrm{ext}} + \Box_g \chi$ with $\mathrm{supp}(\psi) \subset O_1$. Since, $\Box_g \chi \in \ker(G_{O_2})$ we have $[\phi] = [\psi_{\mathrm{ext}}]$, that is, $[\phi]$ is the image of the symplectic linear map $V(O_1) \to V(O_2)$ induced by the inclusion $\iota : O_1 \to O_2$. Therefore, the map is surjective, and hence an isomorphism of symplectic topological vector spaces. This isomorphism functorially induces an isomorphism of C^*-algebras, hence $\mathcal{U}_{O_1} = \mathcal{U}_{O_2}$. This proves the time-slice axiom. Finally, the seventh axiom can be deduced from the first and the sixth axiom [3, theorem 4.5.1]. □

7. Discussion

In this paper we have constructed Green operators for spacetime metrics of regularity $C^{1,1}$. The function spaces for the domain and range of the Green operators play a fundamental role in low regularity spacetimes and our choices for these spaces were motivated by the following two requirements: global well-posedness of the Cauchy problem and employing Sobolev spaces, such as $H^k_{\mathrm{loc}}(M)$ and $H^k_{\mathrm{comp}}(M)$ ($k \in \mathbb{N}_0$), that do not depend on a Riemannian background metric. We have shown that the quotient space $V(M) = U_0/\Box_g V_0$ can be used to construct quasi-local C^*-algebras that satisfy the Haag–Kastler axioms, so that in a quantum theoretic setting the self-adjoint elements in these C^*-algebras can be associated with the observables of the theory.

7.1. Topological issues

Let us describe the quotient vector space $U_0/\Box_g V_0$ in some more detail for the *globally hyperbolic case*, where we have $\ker(G) = \mathrm{Im}(\Box_g)$ as a consequence of the spectral sequence given in theorem 5.9, thus $U_0/\Box_g V_0 = U_0/\ker(G)$ in this case. Recall that G is a linear map $U_0 \to V_{\mathrm{sc}}$ and let G_0 denote the associated map from the quotient $U_0/\ker(G)$ to $\mathrm{Im}(G) \subseteq V_{\mathrm{sc}}$, defined by $G_0(\phi + \ker(G)) := G\phi$ for every $\phi \in U_0$. Therefore, G_0 is linear and bijective by construction and we arrive at the following chain of (algebraic) isomorphisms of vector spaces

$$U_0/\Box_g V_0 = U_0/\ker(G) \cong \mathrm{Im}(G) = \ker(\Box_g) \subseteq V_{\mathrm{sc}}. \tag{7.1}$$

Recall that the analogue of (7.1) in the smooth globally hyperbolic case, as discussed in [3], is

$$\mathcal{D}(M)/\Box_g\mathcal{D}(M) = \mathcal{D}(M)/\ker(G) \cong \mathrm{im}(G) = \ker(\Box_g) \subseteq C^\infty_{\mathrm{sc}},$$

showing also that the quotient is isomorphic to the space of solutions to the homogeneous wave equation.

The question arises whether the isomorphism in the middle part of (7.1), obtained via the factored map G_0, is topological, where the quotient $U_0/\ker(G)$ is equipped with the finest topology such that the canonical surjection $\pi : U_0 \to U_0/\ker(G)$, $\phi \mapsto \phi + \ker(G)$ is continuous. Note that by continuity of G we have that $\ker(G)$ is closed in the normed space U_0, hence $U_0/\ker(G)$ is a normed space (in particular, Hausdorff). Furthermore, G_0 is continuous by construction and the continuity of G, thus it remains to be checked whether the inverse of G_0 is continuous, or, equivalently, whether G_0 is an open map.

Remark 7.1. We note that by [40, chapter III, proposition 1.2], the factored map G_0 is a topological isomorphism if and only if G is open as a map from U_0 to $\mathrm{Im}(G)$ (with the relative topology on the latter). In case of Fréchet spaces such a property for G could be deduced conveniently via an open mapping principle or from a closed image criterion, but observe that neither $U_0 = H^1_{\mathrm{comp}}(M)$ nor $\mathrm{Im}(G)$ is complete (with respect to the metric inherited from the Banach space $H^1(M)$ and the Fréchet space $H^2_{\mathrm{loc}}(M)$, respectively).

We choose a finer topology σ on V_{sc} to make $\Box_g : (V_{\mathrm{sc}}, \sigma) \to H^1_{\mathrm{loc}}(M)$ continuous by adding the semi-norms $p_\chi(\phi) := \|\chi \cdot \Box_g\phi\|_{H^1}$ ($\chi \in \mathcal{D}(M)$) to those on V_{sc} inherited from $H^2_{\mathrm{loc}}(M)$. Note that this has no effect on the subspace $\mathrm{Im}(G) \subseteq V_{\mathrm{sc}}$, since $\mathrm{Im}(G) \subseteq \ker(\Box_g)$ in the complex of maps in theorem 5.9 (even equality holds due to global hyperbolicity). In fact, σ is precisely the coarsest topology that is finer than the $H^2_{\mathrm{loc}}(M)$-topology on V_{sc}, which we denote by τ_2, and renders $\Box_g$ continuous as a map $V_{\mathrm{sc}} \to H^1_{\mathrm{loc}}(M)$, i.e., σ is the supremum (in the lattice of topologies on V_{sc}) of τ_2 and the initial (projective) topology τ_1 with respect to $\Box_g$.

Therefore, we have continuity of $G : U_0 \to (V_{\mathrm{sc}}, \sigma)$, since G is continuous $U_0 \to (V_{\mathrm{sc}}, \tau_2)$ by corollary 4.11 and also continuous $U_0 \to (V_{\mathrm{sc}}, \tau_1)$ due to the obvious continuity of $\Box_g \circ G = 0$ from U_0 into $H^1_{\mathrm{loc}}(M)$.

Lemma 7.2. *The inverse of* $G_0 : U_0/\ker(G) \to \mathrm{Im}(G)$, $\phi + \ker(G) \mapsto G\phi$, *is continuous.*

Proof. We will show that G_0^{-1} can be written as the composition $G_0^{-1} = \pi \circ P \circ Z$ of three continuous linear maps. The map $\pi : U_0 \to U_0/\ker(G)$ is the canonical surjection, which is continuous by construction. It remains to construct suitable continuous maps P and Z with $P \circ Z : \mathrm{Im}(G) \to U_0$ and such that $G_0 \circ \pi \circ P \circ Z = \mathrm{id}_{\mathrm{Im}(G)}$ and $\pi \circ P \circ Z \circ G_0 = \mathrm{id}_{U_0/\ker(G)}$.

Let $V^{\pm}_{\mathrm{sc}} := \{\phi \in V_{\mathrm{sc}} | \mathrm{supp}(\phi) \subseteq J^{\pm}(K) \text{ for some compact subset } K \subseteq M\}$ and define the subspace

$$W := \{(\phi_-, \phi_+) \in V^-_{\mathrm{sc}} \times V^+_{\mathrm{sc}} | \phi_- + \phi_+ \in \ker(\Box_g)\} \subseteq V_{\mathrm{sc}} \times V_{\mathrm{sc}},$$

which we equip with the trace of the product topology stemming from σ.

Construction of P: we consider $P : W \to U_0$, given by $P(\phi_-, \phi_+) := (\Box_g\phi_+ - \Box_g\phi_-)/2$. Note that *a priori*, $P(\phi_-, \phi_+)$ is only in $H^1_{\mathrm{loc}}(M)$ and we have to show that $P(\phi_-, \phi_+)$ has compact support, thus belongs to $U_0 = H^1_{\mathrm{comp}}(M)$. To prove this, observe that $\phi = \phi_- + \phi_+ \in \ker(\Box_g)$ implies $\Box_g\phi_- = -\Box_g\phi_+$, hence $P\phi = \Box_g\phi_+ = -\Box_g\phi_-$. Let K_- and K_+ be compact subsets of M with $\mathrm{supp}(\phi_\pm) \subseteq J^{\pm}(K_\pm)$, then we have $\mathrm{supp}(P\phi) \subseteq \mathrm{supp}(\phi_-) \cap \mathrm{supp}(\phi_+) \subseteq J^-(K_-) \cap J^+(K_+)$, where $J^-(K_-) \cap J^+(K_+)$ is compact by global hyperbolicity ([3], lemma A.5.7]). The continuity of P is clear by construction of the topology σ.

Construction of Z: as a preparation we will first construct two continuous maps $S^{\pm} : V_{\mathrm{sc}} \to V^{\pm}_{\mathrm{sc}}$, such that $\phi = S^-\phi + S^+\phi$ holds for every $\phi \in V_{\mathrm{sc}}$ and, moreover,

$$G\,\Box_g S^{\pm}\phi = \pm\phi \quad \forall \phi \in \mathrm{Im}(G). \tag{7.2}$$

Choose a smooth spacelike Cauchy surface $\Sigma \subseteq M$ and let $t : M \to \mathbb{R}$ be a smooth temporal function such that $\Sigma = t^{-1}(0)$ (compare the earlier discussion on causality in $C^{1,1}$-spacetimes). We obtain an open covering of M by the two sets $O_- := \{x \in M | t(x) < 1\}$ and $O_+ := \{x \in M | t(x) > -1\}$ and choose a subordinate partition of unity $\chi_-, \chi_+ \in C^\infty(M)$, i.e., $\mathrm{supp}(\chi_\pm) \subseteq O_\pm$ and $\chi_- + \chi_+ = 1$. We define $S^{\pm}\phi := \chi_\pm\phi$, then the relation $\phi = S^-\phi + S^+\phi$ holds by construction and the continuity of $S^{\pm}$ is clear from continuity of multiplication by fixed smooth functions with respect to (localised) Sobolev norms. It remains to show that $S^{\pm} \in V^{\pm}_{\mathrm{sc}}$ for every $\phi \in V_{\mathrm{sc}}$ and equation (7.2) is true.

Let $\phi \in V_{\mathrm{sc}}$ and $K \subseteq M$ be compact such that $\mathrm{supp}(\phi) \subseteq J^-(K) \cup J^+(K)$. Then $\mathrm{supp}(\chi_+\phi) \subseteq O_+ \cap (J^-(K) \cup J^+(K)) \subseteq (O_+ \cap J^-(K)) \cup J^+(K)$. Note that $O_+ \cap J^-(K)$ is relatively compact by [3, corollary A.5.4], since $O_+ \subseteq J^+(\Sigma_-)$ holds with $\Sigma_- := t^{-1}(-1)$ (note that the time function is strictly increasing along causal curves). Therefore, with some compact set K_+ containing K as well as $O_+ \cap J^-(K)$ we obtain $\mathrm{supp}(\chi_+\phi) \subseteq J^+(K_+)$, thus $S^+\phi \in V^+_{\mathrm{sc}}$. The reasoning for $S^-\phi \in V^-_{\mathrm{sc}}$ is analogous.

For the proof of (7.2) we start by noting that $\phi \in \mathrm{Im}(G) = \ker(\Box_g)$ implies $0 = \Box_g\phi = \Box_g S^-\phi + \Box_g S^+\phi$, so that the part with S^- in (7.2) follows once the equation for S^+ is shown. Recall that we have $\Box_g S^+\phi = -\Box_g S^-\phi \in H^1_{\mathrm{comp}}(M)$ from the reasoning in the construction of P above. Moreover, for every test function ψ on M we have that $\mathrm{supp}(G^{\mp}\psi) \cap \mathrm{supp}(S^{\pm}\phi) \subseteq J^{\mp}(\mathrm{supp}(\psi)) \cap J^{\pm}(K)$ for some compact set K, hence global hyperbolicity guarantees that the supports of $G^-\psi$ and $S^+\phi$ as well as those of $G^+\psi$ and $S^-\phi$ always have compact intersection. To summarise, we may apply theorem 5.7 and lemma 5.6 to obtain the following chain of weak equalities

$$(\psi, G^{\pm}\Box_g S^{+}\phi)_{L^2(M,g)} = (G^{\mp}\psi, \Box_g S^{+}\phi)_{L^2(M,g)} = (G^{\mp}\psi, \pm\Box_g S^{\pm}\phi)_{L^2(M,g)}$$
$$= (\Box_g G^{\mp}\psi, \pm S^{\pm}\phi)_{L^2(M,g)} = (\psi, \pm S^{\pm}\phi)_{L^2(M,g)},$$

which implies $G^{\pm}\Box_g S^{+}\phi = \pm S^{\pm}\phi$ and therefore $G\Box_g S^{+}\phi = G^{+}\Box_g S^{+}\phi - G^{-}\Box_g S^{+}\phi = S^{+}\phi - (-S^{-}\phi) = S^{+}\phi + S^{-}\phi = \phi$. Thus, equation (7.2) is proved and concludes the preparatory construction of $S^{\pm}$.

Finally, we turn to the definition of the map Z. Observe that $\phi \in \mathrm{Im}(G) = \ker(\Box_g) \subseteq V_{\mathrm{sc}}$ implies $(S^{-}\phi, S^{+}\phi) \in W$, which allows to set $Z\phi := (S^{-}\phi, S^{+}\phi)$ for every $\phi \in \mathrm{Im}(G)$ and obtain a continuous linear map $Z : \mathrm{Im}(G) \to W$.

We complete the proof by showing that $\pi \circ P \circ Z$ is the inverse of G_0.

- The relation $G_0 \circ \pi \circ P \circ Z = \mathrm{id}_{\mathrm{Im}(G)}$ holds, since for every $\phi \in \mathrm{Im}(G)$ we have

$$G_0(\pi(P(Z\phi))) = G_0(\pi(P(S^{-}\phi, S^{+}\phi))) = \frac{1}{2}G_0(\pi(\Box_g S^{+}\phi - \Box_g S^{-}\phi))$$
$$= \frac{1}{2}G_0(\Box_g S^{+}\phi - \Box_g S^{-}\phi + \ker(G)) = \frac{1}{2}(G\,\Box_g S^{+}\phi - G\,\Box_g S^{-}\phi),$$

  where we may apply (7.2) to rewrite the last term as $\frac{1}{2}(\phi + \phi) = \phi$.
- We finally show that the equation $\pi \circ P \circ Z \circ G_0 = \mathrm{id}_{U_0/\ker(G)}$ is true. Let $f \in U_0$, then

$$\pi\,(P\,(Z\,(G_0\,(f + \ker(G))))) = \pi(P(Z(Gf))) = \pi(P(S^{-}Gf, S^{+}Gf))$$
$$= \pi(\frac{1}{2}(\Box_g S^{+}Gf - \Box_g S^{-}Gf)) = \frac{1}{2}(\Box_g S^{+}Gf - \Box_g S^{-}Gf) + \ker(G)$$

  and in the last term we may replace $f_1 := \frac{1}{2}(\Box_g S^{+}Gf - \Box_g S^{-}Gf)$ by f, since thanks to (7.2) the difference is in the kernel: $G(f_1 - f) = \frac{1}{2}(Gf + Gf) - Gf = Gf - Gf = 0$.

□

Proposition 7.3. *For a globally hyperbolic $C^{1,1}$ spacetime (M, g), we obtain a topological isomorphism $U_0/\ker(G) \cong \mathrm{Im}(G)$ according to (7.1), where V_{sc} carries the topology σ.*

Remark 7.4. We are not using an inductive limit construction for the topology on V_{sc} as, e.g., in [41], because we preferred to stay with questions of convergence and continuity in the simpler realm of local Sobolev norms. Moreover, in the above context, we would otherwise not have a topological isomorphism of $\mathrm{Im}(G)$ with $U_0/\ker(G)$, since we decided coherently that U_0 should inherit the norm topology from $H^1(M)$, thus rendering $U_0 = H^1_{\mathrm{comp}}(M)$ normed, but incomplete. However, the basic constructions of quantisation for the associated symplectic (quotient) vector spaces do not require completeness.

7.2. An equivalent symplectic structure

An analogous construction of the CCR representation can be achieved using a symplectic structure on the vector space of solutions to the homogeneous problem parametrised by their initial data [1]. In that context, one defines a symplectic structure Ξ on $\ker(\Box_g)$ given by $\Xi(\phi, \psi) = \int_\Sigma (u_1 v_0 - v_1 u_0)\mu_h$ where $(u_0, u_1), (v_0, v_1)$ are compactly supported smooth initial data induced by the smooth solutions ϕ, ψ respectively on the Cauchy hypersurface Σ. Moreover, the Weyl system generated by the symplectic space $(\ker(\Box_g), \Xi)$ is isomorphic to the Weyl system generated by $(U_0/\ker(G), \omega)$ [1, 42]. In the $C^{1,1}$ setting this isomorphism remains true with suitable modifications.

To be precise, using theorem 5.9 we know that $\ker(\Box_g) = \mathrm{Im}(G)$. Moreover, for any smooth spacelike Cauchy hypersurface Σ, if $\phi = G(f)$ then $\phi|_\Sigma \in H^2_{\mathrm{comp}}(\Sigma)$ and $\nabla_n\phi|_\Sigma \in H^1_{\mathrm{comp}}(\Sigma)$. This follows from the observation that $\phi \in V_{\mathrm{sc}}$ and is the difference of two solutions to the Cauchy problem with zero initial data, which by theorem 4.7 belong to the space $C^0(\mathbb{R}, H^2(\Sigma_t)) \cap C^1(\mathbb{R}, H^1(\Sigma_t))$. Therefore, given any smooth spacelike Cauchy hypersurface Σ, we define for $\phi, \psi \in \ker\Box_g$ with $u_0 := \phi|_\Sigma$, $u_1 := \nabla_n\phi|_\Sigma$, $v_0 := \psi|_\Sigma$, $v_1 := \nabla_n\psi|_\Sigma$, hence $(u_0, u_1), (v_0, v_1) \in H^2_{\mathrm{comp}}(\Sigma) \times H^1_{\mathrm{comp}}(\Sigma)$, the skew-symmetric bilinear form

$$\Xi_\Sigma(\phi, \psi) = \int_\Sigma (u_1 v_0 - v_1 u_0)\mu_h.$$

It follows from linearity, the uniqueness of solutions to the Cauchy problem, and direct computations that Ξ is symplectic where to show non-degeneracy one tests with elements of the form $(0, u_1)$ and $(v_0, 0)$, i.e., with $u_0 = 0$ and $v_1 = 0$ and employs uniqueness in the Cauchy problem (cf [42]).

We show that Ξ_Σ does not depend on Σ: this follows from the divergence theorem in a region bounded by two Cauchy hypersurfaces Σ_1, Σ_2 and the conservation of the current $j^\mu(\phi, \psi) = g^{\mu\nu}\,(\phi\nabla_\nu\psi - \psi\nabla_\nu\phi)$. Explicitly we have for any $\phi, \psi \in \ker(\Box_g)$

$$\int_{J^-(\Sigma_1)\cap J^+(\Sigma_2)} \mathrm{div}(j^\mu(\phi, \psi))\nu_g = 0$$

and

$$\int_{J^-(\Sigma_1)\cap J^+(\Sigma_2)} \mathrm{div}(j^\mu(\phi, \psi))\nu_g = \int_{\Sigma_2} j^\mu(\phi, \psi) n_\mu \mu_{h_1} - \int_{\Sigma_1} j^\mu(\phi, \psi) n_\mu \mu_{h_2} = 0.$$

Therefore,

$$\Xi_{\Sigma_1}(\phi, \psi) = \Xi_{\Sigma_2}(\phi, \psi),$$

so we will drop the Σ from the notation of Ξ. Notice that the H^2_{loc} regularity is required in order to make sense of the divergence of the current.

Finally, we show that the linear bijective factor map G_0 of G, as defined before (7.1), provides a symplectic map from $(U_0/\ker(G), \omega)$ to $(\ker(\Box_g), \Xi)$.

Proposition 7.5. *Let the symplectic vector spaces* $(\ker(\Box_g), \Xi)$, $(U_0/\ker(G), \omega)$ *and the factor map* G_0 *be defined as above. Then we have for every* $f, f' \in U_0$ *with* $\phi = G_0([f']) = Gf'$, $\psi = G_0([f]) = Gf \in \ker(\Box_g)$,

$$\Xi(\phi, \psi) = \omega([f'], [f]).$$

Proof. Without loss of generality we may consider $M \cong \mathbb{R} \times \Sigma$ and suppose that $\mathrm{supp}(f) \subset (t_1, t_2) \times \Sigma$ for some real $t_1 < t_2$. Then we have for every $\phi \in \ker(\Box_g)$ upon integrating by parts twice,

$$\int_{(t_1,t_2)\times\Sigma} \phi \Box_g G^+(f)\nu_g = \int_{(t_1,t_2)\times\Sigma} \Box_g\phi G^+(f)\nu_g$$
$$- \int_{\Sigma_{t_2}} \left(\phi\nabla_n G^+(f) - G^+(f)\nabla_n\phi\right)\mu_{h_2} + \int_{\Sigma_{t_1}} \left(\phi\nabla_n G^+(f) - G^+(f)\nabla_n\phi\right)\mu_{h_1}.$$

Using the fact that $\Box_g \phi = 0$ and that Σ_{t_1} is disjoint[7] from $\mathrm{supp}(G^+ f)$ we obtain

$$\int_{(t_1,t_2)\times\Sigma} \phi \Box_g G^+(f) \nu_g = -\int_{\Sigma_{t_2}} \left(\phi \nabla_n G^+(f) - G^+(f) \nabla_n \phi \right) \mu_{h_2}.$$

Similarly, from the causal properties again we have that Σ_{t_2} and $\mathrm{supp}(G^-(f))$ are disjoint. Therefore $G^+ f|_{\Sigma_{t_2}} = Gf|_{\Sigma_{t_2}}$ and $\nabla_n G^+ f|_{\Sigma_{t_2}} = \nabla_n Gf|_{\Sigma_{t_2}}$ which gives

$$\int_{(t_1,t_2)\times\Sigma} \phi \Box_g G^+(f) \nu_g = -\int_{\Sigma_{t_2}} \left(\phi \nabla_n G(f) - G(f) \nabla_n \phi \right) \mu_{h_2}.$$

Recalling that $\psi = G(f)$ and $t_1 < t < t_2$ in $\mathrm{supp}(f)$ we obtain

$$\Xi(\phi, \psi) = \int_{(t_1,t_2)\times\Sigma} \phi f \nu_g = \int_M \phi f \nu_g.$$

Here, we use also the assumption $G(f') = \phi$ to proceed with

$$\int_M \phi f \nu_g = \int_M G(f') f \nu_g = \tilde{\omega}(f', f) = \omega([f'], [f]).$$

$\Box$

We have established a symplectomorphism between the spaces $(\ker(\Box_g), \Xi)$ and $(U_0/\ker(G), \omega)$. This implies that the functor CCR will give isomorphic C^*-algebras in the quantisation. Therefore, the result shows that one can use either the elements of $U_0/\ker(G)$ or those of $\ker(\Box_g)$ to construct the algebra of quantum observables.

7.3. The physical quantum states

Finally, in order to construct a full quantum field theory in a low regularity spacetime, a suitable choice of quantum states must be made. In the algebraic quantisation method, a quantum state Λ is a normalised positive linear functional on the quasi-local C^*-algebra. In particular, given a real scalar product $\mu : \ker(\Box_g) \times \ker(\Box_g) \to \mathbb{R}$ satisfying $|\Xi(\phi, \psi)|^2 \leqslant \frac{1}{4}\mu(\phi, \psi)\mu(\phi, \psi)$ for all $\phi, \psi \in \ker(\Box_g)$, we define a quasi-free state by $\Lambda_\mu(W(\phi)) = \mathrm{e}^{\frac{1}{2}\mu(\phi,\phi)}$. As a consequence of the GNS construction, a quasi-free state on a C^*-algebra possesses a natural Fock space structure [1]. Moreover, if additional symmetries exist such as the existence of a timelike Killing vector field one can define ground states and thermal equilibrium states at finite temperature as quasi-free states that can be represented as elements of a suitable Fock space.

In the absence of symmetries a common candidate for the physical quantum states in the smooth case are the quasi-free states that satisfy the microlocal spectrum condition. This condition allows via a renormalization procedure to define the energy momentum tensor of the state [1].

To specify the microlocal spectrum condition, we need to define appropriate subsets of $T^*(M \times M)\backslash 0$, i.e., the cotangent bundle with the zero section removed, and the two-point function of the state Λ_μ, which is a distribution on $M \times M$. Let

$$C = \left\{ (x_1, \eta, x_2, \tilde{\eta}) \in T^*(M \times M)\backslash 0; g^{ab}(x_1)\eta_a \eta_b = 0, g^{ab}(x_2)\tilde{\eta}_a \tilde{\eta}_b = 0, (x_1, \eta) \sim (x_2, \tilde{\eta}) \right\},$$

[7] Because $\mathrm{supp}(G^+ f) \subseteq J^+(\mathrm{supp}(f)) \subseteq (t_1 + \varepsilon, \infty) \times \Sigma$ for some $\varepsilon > 0$.

and $C^+ = \left\{(x_1,\eta,x_2,\tilde{\eta}) \in C; \eta^0 \geqslant 0, \tilde{\eta}^0 \geqslant 0\right\}$,

where $(x_1,\eta) \sim (x_2,\tilde{\eta})$ means that $\eta, \tilde{\eta}$ are cotangent to the null geodesic γ at x_1, x_2 respectively, and parallel transports of each other along γ. The value of the two-point function of a state Λ_μ acting on the elements of the algebra defined by ϕ and ψ is

$$\langle \Lambda_2, \phi \otimes \psi \rangle := -\frac{\partial^2}{\partial s \partial t} \Lambda_\mu(W(t\phi)W(s\psi))|_{s=t=0} = -\frac{\partial^2}{\partial s \partial t}\left(\Lambda_\mu[W(s\phi+t\phi)]\mathrm{e}^{\frac{\mathrm{i}st\Xi(\phi,\psi)}{2}}\right)|_{s=t=0}.$$

Using the isomorphism between $\ker(\Box_g)$ and $V(M)$ the two-point function can be seen to induce a bidistribution on spacetime, i.e., $\Lambda_2 \in \mathcal{D}'(M \times M)$.

Definition 7.6. A quasi-free state Λ_H on the algebra of observables satisfies the microlocal spectrum condition if its two-point function Λ_{2H} is a distribution $\mathcal{D}'(M \times M)$ and satisfies the following wavefront set condition

$$WF'(\Lambda_{2H}) = C^+,$$

where $WF'(\Lambda_{2H}) := \{(x_1,\eta;x_2,-\tilde{\eta}) \in T^*(M \times M); (x_1,\eta;x_2,\tilde{\eta}) \in WF(\omega_{2H})\}$.

The states that satisfy the microlocal spectrum condition[8] are called Hadamard states and their class includes ground states and KMS states ([13, 44, 45]). An alternative method of constructing a Fock space makes use of the *S–J* vacuum states. These are quasi-free states that in general do not satisfy this condition [46].

In the low regularity setting we require a generalisation of Hadamard states. A larger class of states, called adiabatic states of order N and characterised in terms of their Sobolev wavefront set, has been obtained by Junker and Schrohe [47]. These states are natural candidates to replace the Hadamard states in spacetimes with limited regularity. In particular, quantum ground states have been constructed in static spacetimes using semigroup techniques [48] and they can be described as adiabatic states [49]. We briefly recall the definition of this class of states and of the Sobolev wavefront set.

Definition 7.7. A quasi-free state Λ_N on the algebra of observables is called an adiabatic state of order $N \in \mathbb{R}$ if its two-point function Λ_{2N} is a bidistribution that satisfies for every $s \leqslant N + \frac{3}{2}$ the H^s-wavefront set condition

$$WF^{s\prime}(\Lambda_{2N}) \subset C^+,$$

where WF^s denotes the refinement of the notion of the wavefront set in terms of Sobolev regularity ([43]), i.e., $(x,\xi) \notin WF^s(u)$ if and only if $u = u_1 + u_2$ with $u_1 \in H^s$ and $(x,\xi) \notin WF(u_2)$.

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[8] The singular structure of these states is equal to the singular structure of the Feynman parametrix minus the retarded parametrix [13]. The Feynman and retarded parametrix are examples of the distinguished parametrices of the wave equation [43]. In the smooth setting one can construct them using geodesically convex domains. Therefore, for this type of constructions in the non-smooth setting the regularity of the exponential map is crucial [8].

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Appendix A. Regularisation methods and generalised functions

In this section we gather a minimum of notions required from the theory of Colombeau generalised functions and regularisation methods for Lorentzian metrics. For a comprehensive introduction to the theory of Colombeau algebras we refer to [15, 50], the details about the approximation results for Lorentzian metrics can be found in [19, 26, 51].

Let E be a locally convex topological vector space whose topology is given by the family of semi-norms $\{p_j\}_{j\in J}$. The elements of

$$\mathcal{M}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall j \in J \exists N \in \mathbb{N}_0 \;\; p_j(u_\varepsilon) = O(\varepsilon^{-N}) \quad \text{as}\, \varepsilon \to 0\}$$

and

$$\mathcal{N}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall j \in J \forall q \in \mathbb{N}_0 \;\; p_j(u_\varepsilon) = O(\varepsilon^{q}) \quad \text{as}\, \varepsilon \to 0\}$$

are called *E-moderate* and *E-negligible*, respectively. Defining operations component-wise turns $\mathcal{N}_E$ into a vector subspace of $\mathcal{M}_E$. We define the *generalised functions based on E* as the quotient $\mathcal{G}_E := \mathcal{M}_E/\mathcal{N}_E$. If E is a differential algebra, then $\mathcal{N}_E$ is an ideal in $\mathcal{M}_E$ and $\mathcal{G}_E$ is a differential algebra as well, called the Colombeau algebra based on E.

Let Ω be an open subset of $\mathbb{R}^n$. By choosing $E = C^\infty(\Omega)$ with the topology of uniform convergence of all derivatives one obtains the standard Colombeau algebra $\mathcal{G}_{C^\infty(\Omega)} = \mathcal{G}(\Omega)$; here we will mainly use $E = H^\infty(\Omega) = \{h \in C^\infty(\overline{\Omega}) : \partial^\alpha h \in L^2(\Omega)\, \forall \alpha \in \mathbb{N}_0^n\}$ with the family of semi-norms

$$\|h\|_{H^k} = \left(\sum_{|\alpha|\leqslant k} \|\partial^\alpha h\|_{L^2}^2\right)^{1/2} \qquad (k \in \mathbb{N}_0)$$

or $E = W^{\infty,\infty}(\Omega) = \{h \in C^\infty(\overline{\Omega}) : \partial^\alpha h \in L^\infty(\Omega)\; \forall \alpha \in \mathbb{N}^n\}$ with the family of semi-norms

$$\|h\|_{W^{k,\infty}} = \max_{|\alpha|\leqslant k} \|\partial^\alpha h\|_{L^\infty} \quad (k \in \mathbb{N}_0).$$

We employ the notation

$$\mathcal{G}_{L^2}(\Omega) := \mathcal{G}_{H^\infty(\Omega)} \quad \text{and} \quad \mathcal{G}_{L^\infty}(\Omega) := \mathcal{G}_{W^{\infty,\infty}(\Omega)}.$$

Colombeau algebras contain the distributions as a linear subspace, though not every element of a Colombeau algebra is a regularisation of a distribution. Their elements are equivalence classes of nets of smooth functions, $\mathcal{G}(\Omega) \ni u = [(u_\varepsilon)_\varepsilon]$. We say that a Colombeau function u is *associated with a distribution* $w \in \mathcal{D}'(\Omega)$ if some (and hence every) representative $(u_\varepsilon)_\varepsilon$ converges to w in $\mathcal{D}'(\Omega)$. The distribution w represents the macroscopic behaviour of u and is called the *distributional shadow* of u.

A generalised function $u \in \mathcal{G}(\Omega)$ is said to be of *L^∞-log-type* if

$$\|u_\varepsilon\|_{L^\infty(\Omega)} = O(\log(1/\varepsilon)) \quad \text{as}\, \varepsilon \to 0.$$

Logarithmic growth conditions on the coefficients of a differential equation are typical in statements on existence and uniqueness of generalised solutions. These results are usually derived from a detailed analysis of regularisation techniques and Colombeau solutions often lead to very weak solutions in the sense of [16].

A related methodology of regularisation is used in the approximation results of [51] which show how to approximate a globally hyperbolic $C^{1,1}$ metric by a smooth family of globally hyperbolic metrics while controlling the causal structure. We recall from [26, sections 3.8.2], [19, section 1.2] that for two Lorentzian metrics g_1, g_2, we say that g_2 has *strictly wider light cones* than g_1, denoted by

$$g_1 \prec g_2, \quad \text{if for any tangent vector } X \neq 0,\ g_1(X,X) \leqslant 0 \text{ implies that } g_2(X,X) < 0. \quad \text{(A.1)}$$

Thus any g_1-causal vector is g_2-timelike. The key result is [19, proposition 1.2], which we give here in the strengthened version of [51, proposition 2.3].

Proposition A.1. *Let (M, g) be a C^0-spacetime and let h be some smooth background Riemannian metric on M. Then for any $\varepsilon > 0$, there exist smooth Lorentzian metrics $\check{g}_\varepsilon$ and $\hat{g}_\varepsilon$ on M such that $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ and $d_h(\check{g}_\varepsilon, g) + d_h(\hat{g}_\varepsilon, g) < \varepsilon$, where*

$$d_h(g_1, g_2) := \sup_{p\in M, 0\neq X,Y\in T_pM} \frac{|g_1(X,Y) - g_2(X,Y)|}{\|X\|_h \|Y\|_h}. \quad \text{(A.2)}$$

Moreover, $\hat{g}_\varepsilon(p)$ and $\check{g}_\varepsilon(p)$ depend smoothly on $(\varepsilon, p) \in \mathbb{R}^+ \times M$, and if $g \in C^{1,1}$ then letting g_ε be either $\check{g}_\varepsilon$ or $\hat{g}_\varepsilon$, we additionally have:

(*a*) *For any compact subset $K \Subset M$ there exists a sequence $\varepsilon_j \searrow 0$ such that $\hat{g}_{\varepsilon_{j+1}} \prec \hat{g}_{\varepsilon_j}$ on K (resp. $\check{g}_{\varepsilon_j} \prec \check{g}_{\varepsilon_{j+1}}$ on K) for all $j \in \mathbb{N}_0$.*

(*b*) *If g' is a continuous Lorentzian metric with $g \prec g'$ (resp. $g' \prec g$) then $\hat{g}_\varepsilon$ (resp. $\check{g}_\varepsilon$) can be chosen such that $g \prec \hat{g}_\varepsilon \prec g'$ (resp. $g' \prec \check{g}_\varepsilon \prec g$) for all ε.*

(*c*) *There exist sequences of smooth Lorentzian metrics $\check{g}_j \prec g \prec \hat{g}_j$ $(j \in \mathbb{N})$ such that $d_h(\check{g}_j, g) + d_h(\hat{g}_j, g) < 1/j$ and $\check{g}_j \prec \check{g}_{j+1}$ as well as $\hat{g}_{j+1} \prec \hat{g}_j$ for all $j \in \mathbb{N}$.*

(*d*) *If g is $C^{1,1}$ and globally hyperbolic then the $\hat{g}_\varepsilon$ (and $\check{g}_\varepsilon$) can be chosen to be globally hyperbolic.*

(*e*) *If g is $C^{1,1}$ then the regularisations can in addition be chosen such that they converge to g in the C^1-topology and such that their second derivatives are bounded, uniformly in ε on compact sets.*

Remark A.2. In our application the main point we will need compared to [19, section 1.2] is property (d) which guarantees that for globally hyperbolic metrics there exist approximations with strictly narrower (wider) lightcones that are themselves globally hyperbolic. Extending methods of [52], it was shown in [53] that global hyperbolicity is stable in the interval topology. Consequently, if g is a smooth, globally hyperbolic Lorentzian metric, then there exists some smooth globally hyperbolic metric $g' \prec g$ (resp. $g' \succ g$). Constructing $\hat{g}_\varepsilon$ resp. $\hat{g}_j$ as in (b) then automatically gives globally hyperbolic metrics (cf [53], section II]).

Appendix B. Function spaces

The (real) Hilbert space $L^2(M, g)$ is used in the section on Green operators to formulate adjointness properties and is defined as follows: recall that for any Lorentzian manifold (M, g) we have a unique positive density μ_g on M [54, proposition 2.1.15], which has the local coordinate expression $\sqrt{|\det(g_{ij})|}\,|dx^0 \wedge \cdots \wedge dx^n|$; in case of a Lipschitz continuous metric g the density μ_g is continuous and induces a positive Borel measure on M, which we employ to define the corresponding L^2 space and denote it by $L^2(M, g)$. If M is orientable, then we have a global volume form ν_g on M [55, chapter 7] from which the density μ_g can be obtained.

We consider the (real) Sobolev spaces $H^m(M)$ for a nonnegative integer m to be defined with respect to some chosen smooth Riemannian background metric on M as described in [56], i.e., by completion of the space of (real) smooth functions whose covariant derivatives up to order m are square integrable with respect to the positive Borel measure on M associated with the Riemannian metric (cf [57, section III.3]; it can be written in terms of a global Riemannian volume form, if M is orientable). Recall ([56], theorems 2.7 and 2.8]) that the space $\mathcal{D}(M)$ (of smooth compactly supported test functions) is dense in $H^1(M)$, if M is complete with respect to the chosen Riemannian metric, and also in $H^m(M)$ for $m \geqslant 2$, if, in addition, Riemannian curvature bounds hold as well.

On a compact manifold M, the definition of $H^m(M)$ is independent of the chosen Riemannian background metric ([56], proposition 2.3]) and, similarly, one concludes that Sobolev norms induced by two different Riemannian metrics on functions with support contained in a fixed compact subset are equivalent. This observation guarantees that the following two spaces are independent of the chosen background metric, namely the compactly supported Sobolev functions $H^m_{\mathrm{comp}}(M) := \{f \in H^m(M) | \mathrm{supp}(f) \,\text{is compact in}\, M\}$ and the local space $H^m_{\mathrm{loc}}(M) := \{f : M \to \mathbb{R} \,\text{measurable} | \forall \varphi \in \mathcal{D}(M) : \varphi f \in H^m(M)\}$ (in fact, $\varphi f \in H^m_{\mathrm{comp}}(M)$ in the latter case).

In the context of the function space topologies for the current paper, we simply consider $H^m_{\mathrm{comp}}(M)$ as a subspace of the Banach space $H^m(M)$, hence it is normed and not complete. One could equip $H^m_{\mathrm{comp}}(M)$ with a complete (non-metrizable) locally convex vector space topology, e.g., as in [41] or [58, part II, chapter 31], via a strict inductive limit construction which turns it into a so-called (LF)-space, but we prefer to formulate our results more directly in terms of the inherited Sobolev norm.

For $H^m_{\mathrm{loc}}(M)$ we have the family of semi-norms $f \mapsto \|\varphi f\|_{H^m(M)}$, parametrised by $\varphi \in \mathcal{D}(M)$, which provides us with a Fréchet space topology on $H^m_{\mathrm{loc}}(M)$ (cf [58, part II, chapter 31] or [41]). We clearly have $H^m(M) \subseteq H^m_{\mathrm{loc}}(M)$ (with continuous embedding).

If K is a compact subset of M and $f \in H^m_{\mathrm{loc}}(M)$ we occasionally abuse the notation and write $\|f\|_{H^m(K)}$ to mean the value obtained when the integrals defining $\|\varphi f\|_{H^m(M)}$ are only evaluated on K and $\varphi \in \mathcal{D}(M)$ is a cut-off such that $\varphi = 1$ on K. (No cut-off is required if $f \in H^m(M)$.)

In case of $M = (0, T) \times \Sigma$ we may choose the Riemannian background metric in the form $\mathrm{d}t \otimes \mathrm{d}t + \gamma$, where γ is a Riemannian metric on Σ. We will then often consider a function $v \in L^2((0, T) \times \Sigma)$ as a map $t \mapsto v(t)$ from the interval into the Hilbert space $L^2(\Sigma)$ in the sense that $v(t)(x) = v(t, x)$ holds pointwise for continuous v. Thanks to Fubini's theorem, we may then write

$$\|v\|^2_{L^2((0,T)\times\Sigma)} = \int_0^T \|v(t)\|^2_{L^2(\Sigma)}\,\mathrm{d}t. \tag{B.1}$$

For general constructions with measurable functions valued in Banach spaces we refer to [59, 60]; in particular we will make use of the isomorphism $L^2((0, T) \times \Sigma) \cong L^2((0, T), L^2(\Sigma))$ [59, theorem 8.28]. If v is differentiable and interpreted as a function $t \mapsto v(t)$, we will occasionally denote the partial derivative $\partial_t v$ by $\dot{v}$ and write ∂_t for the corresponding vector field on $(0, T) \times \Sigma$. The space $C^k([0, T]; H^m(\Sigma))$ (k a nonnegative integer) consists of all k times continuously (strongly) differentiable functions (if $k = 0$, simply continuous functions) $v : [0, T] \to H^m(\Sigma)$

with finite norm

$$\|v\|_{C^k([0,T],H^m(\Sigma))} := \max_{0\leqslant j\leqslant k} \sup_{t\in[0,T]} \|\partial_t^j v(t)\|_{H^m(\Sigma)} < \infty. \tag{B.2}$$

We have $C^k([0,T];H^m(\Sigma)) \subseteq L^2((0,T),H^m(\Sigma))$ (with continuous embedding).

In place of a bounded time interval we will occasionally consider the basic spacetime to be $\mathbb{R}\times\Sigma$ and deal with function spaces of Bochner measurable maps from $\mathbb{R}$ to some of the Sobolev-type Hilbert spaces (cf [59, chapter 8]), in particular, $L^2(\mathbb{R},H^1(\Sigma))$. We will then use the notation $L^2_{\mathrm{loc}}(\mathbb{R},H^1(\Sigma))$ for the set of all Bochner-measurable functions $v:\mathbb{R}\to H^1(\Sigma)$ such that for every compact subinterval $I\subset\mathbb{R}$ the restriction $v|_I$ belongs to $L^2(I,H^1(\Sigma))$.

In looking at energy estimates on $\mathbb{R}\times\Sigma$ we will also need versions of the Sobolev norms where the derivatives are taken in both the space and time directions but the integration and volume form are confined to the $t=\tau$ level hypersurfaces $S_\tau := \{\tau\}\times\Sigma$. These norms will be denoted by

$$\|u\|_{\tilde{H}^m(S_\tau)} = \left(\sum_{j=1}^{m}\int_{S_\tau}(u^2+(\partial_t^j u)^2+|\tilde{\nabla}^j u|^2)\mathrm{d}\mu_\tau\right)^{\frac{1}{2}}, \tag{B.3}$$

where $\tilde{\nabla}$ is the covariant derivative with respect to the spatial background metric γ and μ_τ is the Riemannian measure on S_τ which is just that given by the spatial metric γ.

Finally let us adapt the basic function space structures to the situation of a general globally hyperbolic $C^{1,1}$ spacetime (M,g) with Cauchy hypersurface Σ, where we suppose that—according to the discussion in the subsection on $C^{1,1}$ causality theory—we have chosen a smooth temporal function $t:M\to\mathbb{R}$ such $\Sigma=t^{-1}(0)$ and a corresponding diffeomorphism $\Phi:M\to\mathbb{R}\times\Sigma$. For $\tau\in\mathbb{R}$ denote the corresponding level surface by $\Sigma_\tau := t^{-1}(\tau) = \Phi^{-1}(\{\tau\}\times\Sigma)$, hence $\Sigma_0=\Sigma$, and consider again a background Riemannian metric of the form $h=\mathrm{d}t\otimes\mathrm{d}t+\gamma$ on the product manifold $\mathbb{R}\times\Sigma$, which in turn provides us with the convenient background metric Φ^*h on M. In the sequel, all Sobolev spaces on submanifolds of M or $\mathbb{R}\times\Sigma$ will be considered to be defined via Riemannian metrics induced by Φ^*h or h, respectively. Let Φ_τ denote the induced diffeomorphism $\Sigma_\tau\to\Sigma$, i.e., $\Phi(x) := (\tau,\Phi_\tau(x))$ for every $x\in\Sigma_\tau$.

We will commit another abuse of notation and a somewhat naive simplification in defining now the spaces $C^k(I,H^m(\Sigma_t))$ for the case of a compact interval $I=[0,T]$ or for $I=\mathbb{R}$ without using the full theory of more sophisticated constructions in terms of sections, e.g., as in [41]. Let $B_m(I)$ denote the set of all maps $u:I\to\bigcup_{\tau\in I}H^m(\Sigma_\tau)$ such that $u(\tau)\in H^m(\Sigma_\tau)$ for every $\tau\in I$. Then we have that $u\in B_m(I)$ implies (equivalently) $u(\tau)\circ\Phi_\tau^{-1}\in H^m(\Sigma)$ for every $\tau\in I$. We define $C^k(I,H^m(\Sigma_t))$ to be the subset of those elements $u\in B_m(I)$ such that the map $\tau\mapsto u(\tau)\circ\Phi_\tau^{-1}$ belongs to $C^k(I,H^m(\Sigma))$.

For elements $u\in C^k(I,H^m(\Sigma_t))$ we can then also define the norms over spatial domains, but involving derivatives in space and time directions, such as $\|u\|_{\tilde{H}^m(\Sigma_\tau)}$ via the corresponding $\|.\|_{\tilde{H}^m(S_\tau)}$-norm evaluated for the associated map $\tau\mapsto u(\tau)\circ\Phi_\tau^{-1}$ in $C^k(I,H^m(\Sigma))$.

Note that the definition of the spaces $C^k(I,H^m(\Sigma_t))$ depends on the splitting $M\cong\mathbb{R}\times\Sigma$ and on the choice of temporal function. However, the reasoning in the main text tries to use the temporal function only in intermediate calculations and afterwards gives formulations of results essentially in 'pure' spacetime terms without recourse to the splitting.

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