Pairwise Stable Networks in Homogeneous Societies with Weak Link Externalities

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Abstract

We study general properties of pairwise stable and pairwise Nash stable networks when players are ex-ante homogeneous. Rather than assuming a particular functional form of utility, we impose general link externality conditions on utility such as ordinal convexity and ordinal strategic complements. Depending on these rather weak notions of link externalities, we show that pairwise Nash stable networks of various structure exist. For stronger versions of the convexity and strategic complements conditions, we are even able to characterize all pairwise stable networks: they are nested split graphs. We illustrate these results with many examples from the literature, including utility functions that arise from games with strategic complements played on the network and utility functions that depend on centrality measures such as Bonacich centrality.

Keywords: Network Formation, Noncooperative Games, Convexity, Strategic Complements
JEL-Classification: A14, C72, D85

1. Introduction

A substantial literature has evolved in Operational Research and Economics modeling strategic network formation. Agents in these models have a preference ordering over the set of networks. Examples include firms' profit when forming R&D networks (Goyal and Joshi, 2003), individuals' utility from communicating with others (Harmsen-van Hout et al., 2013) or receiving information from others (Olaizola and Valenciano, 2014), agents' payoff from bargaining on a network (Gauer and Hellmann, 2017), or researchers' scientific output from collaborating with co-authors (Rêgo and dos Santos, 2019). Since the structure of interaction, i.e. the social network, affects outcomes such as profits of firms, the extent of communication or information sharing, bargaining outcomes, and scientific output, it is interesting to understand which kind of interaction structures emerge when links are formed strategically for both the OR community and economists. The most commonly used concepts of such equilibrium outcomes are the notions of *pairwise stability* (Jackson and Wolinsky, 1996) or *pairwise Nash stability* (Calvó-Armengol and Ilkiliç, 2009). A central question is then under which conditions stable networks exist and which structure they have.

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Symbol	Name	Property	Reference
SC	ordinal strategic complements	$\Delta u_i(g,ij) \geq 0 \Rightarrow \Delta u_i(g+l_{-i},ij) \geq 0$ $\forall g \in G,\ l_{-i} \subseteq L_{-i}(g^N - g),\ j \in N_i(g)$	Def. 4
CV	ordinal convexity	$\Delta u_i(g,ij) \geq 0 \Rightarrow \Delta u_i(g+l_i,ij) \geq 0$ $\forall g \in G,\ l_i \subseteq L_i(g^N - g),\ j \in N_i(g)$	Def. 4
$\kappa-$LM	κ-link monotonicity	$\Delta u_i(g,ij) \geq 0 \Rightarrow \Delta u_i(g+ik+l_{-i},ij) \geq 0$ $\forall g \in G, l_{-i} \subseteq L_{-i}(g^N - g) : \lvert l_{-i}\rvert = \kappa, ik \notin g, j \in N_i(g)$	Def. 5
WPP	weak preference for prominence	$\Delta u_i(g+ij,ij) \geq 0 \Rightarrow \Delta u_i(g+ik,ik) \geq 0$ $\forall g \in G,\ j,k \in N :\ N_j(g)\backslash\{k\} \subseteq N_k(g)\backslash\{j\}$	Def. 6
SPP	strong preference for prominence	$\Delta u_i(g,ij) \geq 0 \Rightarrow \Delta u_i(g+ik,ik) \geq 0$ $\forall g \in G,\ j \in N_i(g), k \in N \setminus N_i(g) :\ \lvert N_j(g)\rvert \leq \lvert N_k(g)\rvert$	Def. 6
AC	anonymous convexity	$\Delta u_i(g,ik) \geq 0 \Rightarrow \Delta u_j(g+jk,jk) \geq 0$ $\forall g \in G,\ i,j \in N, k \in N_j \setminus N_i(g) :\ \lvert N_i(g)\rvert \leq \lvert N_j(g)\rvert$	Def. 7
INP	independence of network posititon	$\Delta u_i(g,ij) \geq 0 \Rightarrow \Delta u_i(g+ik,ik) \geq 0$ $\forall g \in G,\ j \in N_i(g), k \in N$	Def. 11

Figure 1: Summary of link externality conditions. Notation: N the set of agents; G the set of undirected networks; $ij \in g$ a link between $i,j \in N$; $N_i(g) := \{j \in N \vert ij \in g\}$ the neighbors of $i \in N$; $L_i(g) := \{ij \in g \vert j \in N\}$ the set of links of agent $i \in N$; $g^N = \{ij \vert i,j \in N, i \neq j\}$ the complete network; $u_i(g)$ utility of agent i; $\Delta u_i(g,ij) := u_i(g) - u_i(g-ij)$ the marginal utility of the link ij.

In this paper, we approach this question from a very general point. Rather than assuming a particular functional form of utility, we simply look at settings where each agent's utility depends only on her network position but not on her name. In other words, the utility function from the network is as general as possible with the restriction that all agents are ex-ante homogeneous, i.e. the utility function satisfies an anonymity condition such that the names of agents do not matter which is a standard assumption in the network formation literature (see e.g. Jackson and Wolinsky (1996) or some surveys such as Jackson (2005); Goyal (2005); Hellmann and Staudigl (2014)). We then show that ordinal link externality conditions on the utility function are sufficient for the existence of stable networks of particular architecture. These ordinal link externality conditions define solely the impact that new links have on incentives to form own links. Since we will make use of several of these link externality conditions, we provide a summary of the definitions of link externalities that we will use in the paper in Figure 1. For future reference, we also include the formal definitions, although the full details of the model will only be introduced in Section 2. We later apply these concepts to specific utility functions which are adopted from the literature.

The two link externality notions that we will primarily use are the weak properties *ordinal strategic complements* (SC) and *ordinal convexity* (CV). SC describes a single crossing property of marginal utility in other agents' links, and CV represents a single crossing property of marginal utility in own links. These properties are single crossing in the sense that once the marginal utility of a given link is positive it stays positive when other links (SC) or own links (CV) are added. SC appears in network formation models where the incentive to form links is positively influenced by other agents forming links. Typical examples for such network utility functions include team production (Ballester et al., 2006), cost-reducing R&D collaborations with endogenous efforts (Hsieh et al., 2018; König et al., 2019) or provision of a public good (Goyal and Joshi, 2006). CV can be observed when the incentive to form links is positively influ-

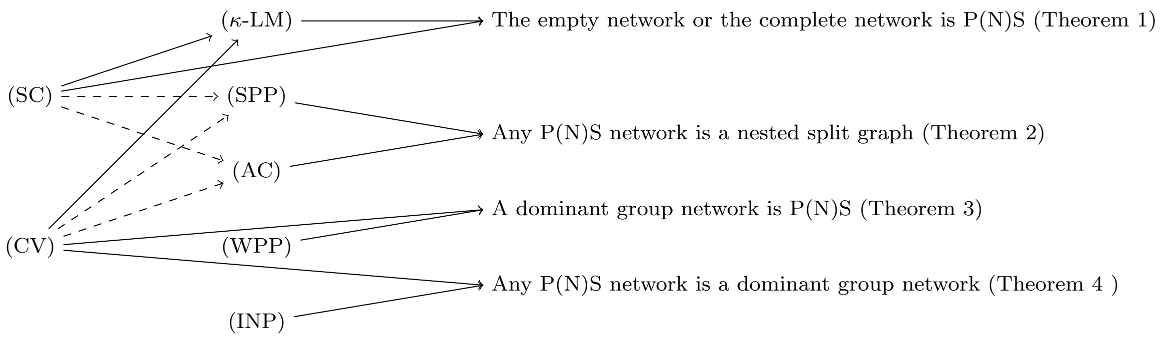


Figure 2: Results of this paper. A property in this diagram is implied if all conditions at the origin of the incoming arrows are satisfied. Dashed arrows hold for "Local Spillover" utility functions, see Proposition 1.

enced by additional own links. Examples, where this type of link externality condition is satisfied, are team production (Ballester et al., 2006)), cost-reducing R&D collaborations (Goyal and Joshi, 2003; Dawid and Hellmann, 2014; Hsieh et al., 2018; König et al., 2019), or market sharing agreements (Belleflamme and Bloch, 2004) among many others.

In general, SC and CV by themselves (i.e. if only one of them is assumed) are too weak to guarantee existence of a pairwise (Nash) stable network, even when agents are homogenous. Therefore, some regularity conditions are introduced. *κ-link monotonicity* (κ-LM) effectively bounds the externalities of own links. The preference for prominence notions express a general desire to connect to agents who have many links. *Weak preference for prominence* (WPP) applies only to agents whose neighbors are comparable by the set inclusion order such that if a link is desirable to a given node j, then it is also desirable to all nodes whose set of neighbors includes the set of neighbors of j (in the OR literature this is also known as the set of nodes who cover node j which plays an important role in the context of military networks, see Monsuur, 2007). *Strong preference for prominence* (SPP) applies to the size of the neighborhood and is therefore a more restrictive property. In this sense, *anonymous convexity* (AC) can be seen as a stronger notion of convexity since it means that, generally, agents with many links have a higher incentive to connect to a given agent. Finally, the most restrictive definition used is *independence of network position* (INP) which implies that either an agent wants to connect to all other agents or to no other agent at all, i.e. the network position of other agents plays only a negligible role. Examples of network formation games satisfying the various properties are provided in this paper.

We use combinations of these general link externality conditions to show existence and characterization results of *pairwise stable* (PS) and *pairwise Nash stable* (PNS) networks. A summary of the results in our paper is presented in Figure 2. As illustrated, SC together with the weak condition κ-LM, bounding the externalities from own links, guarantees that either the empty network or the complete network is always pairwise Nash stable if κ is not too large (Theorem 1). κ-LM is implied by SC and CV and therefore represents a much weaker condition for this existence result than imposing both SC and CV (Corollary 1). If, on the other hand, CV is satisfied together with the weak regularity assumption WPP, then there exists a dominant group network that is pairwise Nash stable (Theorem 3). Dominant group networks are such that there is a completely connected subset of agents while the remaining agents have no links (see also Definition 10). Since pairwise Nash stable networks are a subset of the pairwise stable networks, the existence results carry over to pairwise stable networks. We, therefore, use the notation P(N)S in Figure 2.

It is quite remarkable that the weak link externality conditions SC respectively CV together with the regularity conditions κ-LM respectively WPP yield such sharp existence results. However, they are not sufficient to characterize classes of networks that contain all pairwise stable or pairwise Nash stable networks. To achieve that, we impose the conditions SPP and AC (together with a no-indifference condition, see Definition 8) which imply that all pairwise stable networks are contained in the class of nested split graphs (Theorem 2). Nested split graphs are networks where the set of neighbors of any two agents can be ordered according to the set inclusion ordering. While the assumptions required for this characterization result are arguably stronger than the ordinal positive link externality conditions, we show that in some environments the reverse is also true, i.e. that SPP and AC are implied by SC and CV (Proposition 1), indicated by the dashed arrows in Figure 2. When we simplify utility functions further such that for linking decisions only the own network position matters but not the position of others, then the pairwise stable networks are only found in subclasses of the nested split graphs, the dominant group networks (Theorem 4). Again, since pairwise Nash stable networks are a subset of pairwise stable networks, the same characterization results hold for pairwise Nash stable networks.

We illustrate our general results with respect to several important applications. Among those is a model of network formation where the utility of agents is given by their Bonacich centrality (Bonacich, 1987). Such a utility function arises e.g. when individuals form costly links in the first stage and then engage in a second stage game of strategic complements between neighbors in the network. Indeed, Ballester et al. (2006) show that the unique pure strategy equilibrium of the second stage in such a game is determined by the Bonacich centrality. This measure of centrality counts the number of paths emanating from a given node which are discounted by the length of each path with a common discount factor. Utility functions which are increasing and convex in the Bonacich centrality give rise to SC and CV (Proposition 2) and, even more interestingly, for small discount factors, also SPP and AC are satisfied (Proposition 3). Hence applying our general results to network formation games where utility is a function of Bonacich centrality, we can conclude that either the empty network or the complete network is necessarily pairwise Nash stable (for any discount factor), while all pairwise stable networks are of nested split structure (for small discount factors). If rather than sequentially, choice of links and efforts are made simultaneously, then we show that a general class of games (as described in Hiller, 2017) satisfy SC and CV (Proposition 4).

General properties of stable networks are of high interest for several reasons. Our results may help characterize stable networks for future (maybe very complex) models of network formation, and they provide reasoning why certain stability structures emerge in existing models of network formation: the driving force are the link externality conditions. That our results are applicable to a variety of settings is due to the generality of our approach and the fact that the assumption of a homogeneous society is not restrictive as almost all models of strategic network formation share this property (cf. e.g. several surveys and textbooks including Jackson, 2003, 2005; Goyal, 2005, 2007; Vega-Redondo, 2007; Jackson, 2008; Easly and Kleinberg, 2010; Hellmann and Staudigl, 2014). Since the required properties are defined relative to link externalities, the application of our results does not depend on the model of network formation at hand. We show that our results hold both for simultaneous move games of network formation, as well as, multiple-stage games of network formation and action choice. A further extension to directed network formation or even weighted network formation should be possible, but we restrict our attention in this paper to bilateral network formation without weights.

Although the literature on strategic network formation is enormous, only few results concerning these general structural properties can be found. Exceptions are Jackson and Watts (2001) and Chakrabarti and Gilles (2007) who use the restrictive assumption of a potential function (Monderer and Shapley, 1996) to prove existence of stable networks, and Hellmann (2013) who – similar to our approach – uses link externality conditions to show existence and uniqueness of stable networks. A recent paper, Bich and Morhaim (2020) shows the existence of weighted pairwise stable networks. In light of their general approach, all these papers, however, are not able to show existence of pairwise stable networks of certain structure. We fill this gap with the help of the link externality conditions in a homogeneous society.

Assuming more structure on the functional form of utility, existence of particular stable network structures such as regular networks, and dominant group structures are shown in Goyal and Joshi (2006). Regular networks are such that all nodes have the same number of neighbors (degree). Throughout this paper, we only need ordinal notions of link externalities defined via a single crossing property of marginal utility compared to the stronger cardinal assumptions which additionally have to hold for strict inequalities in Goyal and Joshi (2006). Even more restrictively, in Goyal and Joshi (2006) two specific forms of utility are assumed which both depend only on a particular network statistic, the vector of agents' degrees. We introduce these assumptions in Sections 3.3.1 and 4.3 and show that some of their results can be generalized to hold for arbitrary utility functions in a homogeneous society, for cases where only ordinal versions of the link externality conditions are sufficient, and such that some of their sufficient conditions are not required. Thereby, our results are applicable to many examples of utility functions that are not captured in the framework of Goyal and Joshi (2006), Jackson and Watts (2001), and Chakrabarti and Gilles (2007). In these examples, our results contribute substantially more than the more general frameworks in Hellmann (2013) and Bich and Morhaim (2020) since existence of particular stable network structures and characterization results are provided in this paper. Among those examples is the aforementioned utility function given by Bonacich centrality.

The rest of the paper is organized as follows. Section 2 defines the model and presents the important assumptions and definitions used throughout the paper. Sections 3 and 4 present the results ordered by the externalities that are respectively assumed. Section 5 concludes.

2. The model

Let $N = \{1, 2, ..., n\}$ be a finite set of agents with $n \geq 3$. Depending on the application these can be firms, countries, individuals, etc. These agents strategically form links and are henceforth called players. Throughout this paper we will assume network formation to be undirected. A connection or link between two players $i \in N$ and $j \in N$, $i \neq j$ will be denoted by $\{i, j\}$ which we abbreviate for simplicity by $ij = ji := \{i, j\}$. We then define the complete network by $g^N = \{ij \mid i, j \in N, i \neq j\}$ and we define the set of all networks by $G = \{g \mid g \subseteq g^N\}$. We further denote the set of links of some player i in a network $g \in G$ by $L_i(g) = \{ij \in g \mid j \in N\}$, and all other links $L_{-i}(g) := g - L_i(g)$, where $g - g' := g \setminus g'$ denotes the network obtained by deleting the set of links $g' \cap g$ from network g. Analogously, $g + g' := g \cup g'$. The set of player i's neighbors in a network $g \in G$ is given by $N_i(g) = \{j \in N \mid ij \in g\}$ and $\eta_i(g) := |L_i(g)| = |N_i(g)|$ is called the degree of player i. Analogously we denote $\eta_{-i}(g) := |L_{-i}(g)| = |g| - \eta_i(g)$.

Players have preferences over networks. The profile of utility functions is denoted by $u(g) = (u_1(g), u_2(g), ..., u_n(g))$, where u_i is a mapping from G to $\mathbb{R}$ for all $i \in N$. Since each player wants to maximize own utility, the decision of adding or deleting links is based on the marginal utility of each link. We denote the marginal utility of a set of links $l \subseteq g$ currently in g by $\Delta u_i(g, l) := u_i(g) - u_i(g - l)$. Using this notation, the marginal utility of a set of links $l \subseteq g^N - g$ outside of g is given by $\Delta u_i(g + l, l) = u_i(g + l) - u_i(g)$. Altogether, we call $\mathbb{G} = (N, G, u)$ a society.

2.1. Network Formation and Stability

The study of equilibrium/stability of networks has been a subject of interest in many models of network formation. Depending on the rules of network formation which are assumed in a given model, there are many definitions of equilibrium at hand. A game-theoretic foundation and a comparison of the several definitions of stability can be found in Bloch and Jackson (2006). Here, we only present the well-known concept of pairwise stability introduced by Jackson and Wolinsky (1996) and pairwise Nash stability.

Definition 1 (Pairwise Stability)**.** *A network g is pairwise stable (PS) if*

(i) $\forall ij \in g, \quad \Delta u_i(g, ij) \geq 0$ *and* $\Delta u_j(g, ij) \geq 0$;

(ii) $\forall ij \notin g, \quad \Delta u_i(g + ij, ij) > 0 \;\Rightarrow\; \Delta u_j(g + ij, ij) < 0$.

This approach to stability defines desired properties directly on the set of networks. The implicit assumption of network formation is that players are in control of their links and each link is considered one by one; any player can unilaterally delete a given link, but to form a link both involved players need to agree.

It should be noted that this definition of stability is rather a necessary condition of stability as it is fairly weak. It can be refined to account for multiple link deletion, called Pairwise Nash stability (Bloch and Jackson, 2006) or pairwise Nash equilibrium (Calvó-Armengol and Ilkiliç, 2009).

Definition 2 (Pairwise Nash Stability)**.** *A network g is pairwise Nash stable (PNS) if* *(i)* $\quad \forall l_i \subseteq L_i(g), i \in N\colon \quad \Delta u_i(g, l_i) \geq 0$

(ii) $\quad \forall ij \notin g\colon \quad \Delta u_i(g+ij, ij) > 0 \;\Rightarrow\; \Delta u_j(g+ij, ij) < 0.$

The only difference between these two definitions of network stability is property (i). PS considers each link one by one, while PNS has the interpretation that each player can unilaterally delete as many own links as desired. It can be easily verified that this definition of PNS is equivalent to those in the literature defining PNS via Nash equilibria of the link announcement game due to Myerson (1991) (for a formal proof and a discussion, see also Hellmann, 2013). The networks which satisfy property (i) of Definition 1 are called *link deletion proof*, the networks which satisfy (ii) are called *link addition proof* and the networks which satisfy property (i) of Definition 2 are called *Nash deletion proof*. In the following analysis, many results hold for both PS and PNS networks. We, therefore, use the convention that we write that a network is P(N)S if it is PNS (and hence also PS) and that a network is not P(N)S if a network is not PS (and therefore also not PNS).

2.2. Anonymity

We assume all players to be ex-ante equal in order to assure that differences in utility solely depend on players' respective network positions but not on their name.

Definition 3 (Anonymity)**.** *Let $g_\pi := \{\pi(i)\pi(j) \mid ij \in g\}$ be the network obtained from a network g by some permutation of players $\pi\colon N \to N$. A profile of utility functions is anonymous if $u_i(g) = u_{\pi(i)}(g_\pi)$ for every permutation $\pi\colon N \to N$.*

A society $\mathbb{G}$ with a profile of utility functions satisfying anonymity will be called homogeneous. In other words, this definition implies that players in symmetric network positions receive the same utility. Here, two players $i, j \in N$, $i \neq j$ are in a symmetric position in a network $g \in G$ if there exists a permutation of the set of players $\pi : N \to N$ such that $\pi(i) = j$, $\pi(j) = i$, and $g_\pi = g$ (i.e. after applying the permutation the same set of links emerges). This is most trivially satisfied for two players $i, j \in N$, $i \neq j$ sharing the same set of neighbors (disregarding a possible common link), i.e. , $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$ since applying the permutation π^{ij}, defined by $\pi^{ij}(i) = j$, $\pi^{ij}(j) = i$, and $\pi^{ij}(k) = k$ for all $k \in N$, $k \neq i, j$, delivers $g_\pi = g$. Thus, $u_i(g) = u_j(g)$ by Definition 3. For players in symmetric positions, in particular those who share the same neighborhood, more general statements can be made.

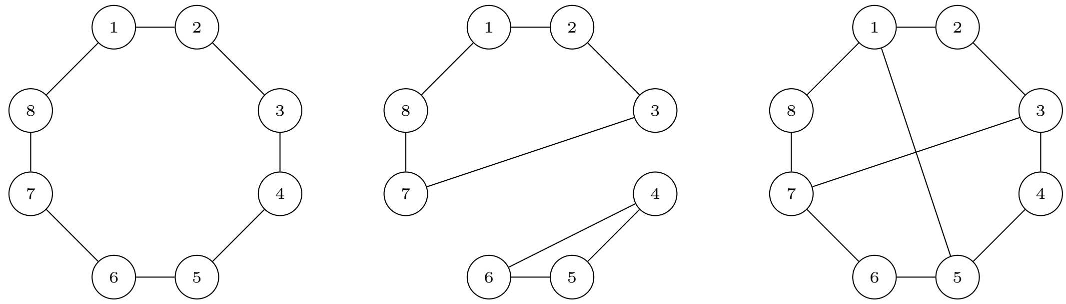
Lemma 1. *Let some profile of utility functions u satisfy anonymity. Then the following statements are true:*

(i) $\quad u_i(g) = u_j(g),$ *if i and j are in a symmetric network position,*

(ii) $\quad \Delta u_i(g+ik, ik) = \Delta u_j(g+jk, jk) \;\; \forall k \in N \backslash N_i(g), \;$ *if $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$,*

(iii) $\quad \Delta u_k(g+ik, ik) = \Delta u_k(g+jk, jk) \;\; \forall k \in N \backslash N_i(g), \;$ *if $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$.*

The proof of Lemma 1 together with the proofs of the other results can be found in the appendix. To illustrate Lemma 1, note that Figure 3(a) is a *symmetric network*, i.e. a network in which all players are in symmetric positions. In graph theory, this



(a) Symmetric network (b) Non-symmetric network (c) Non-symmetric network

Figure 3: Networks (a) and (b) are regular, but only (a) is symmetric. In network (b), two players of different components are not in symmetric positions. In network (c), players 1, 3, 5, and 7, respectively players $2, 4, 6$, and 8 are in symmetric positions, while the network is obviously not.

notion of symmetric networks is called vertex-transitive graphs. Note that a necessary condition for a network to be symmetric is that all players have the same number of neighbors which defines a *regular network*. Hence by (i) of Lemma 1, all players $i \in N = \{1, 2, \ldots, 8\}$ receive the same utility. However, for no two players $i, j \in N$ in this network we have $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$. This means that (ii) and (iii) of Lemma 1 do not apply here. Indeed, incentives to create links may differ considerably. If, for instance, utility is distance-based (consider e.g. the symmetric connections model in Jackson and Wolinsky, 1996), then for Player 1 the link to Player 5 might be a lot more desirable than the link to Player 3. Instead, consider the regular network in Figure 3(b) which is not symmetric. There, players 4, 5, and 6, are not only in symmetric positions and, thereby, receive the same utility by Lemma 1(i), but they also share the same neighbors, i.e. $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$ holds for $i, j \in \{4, 5, 6\}$. Thus, by Lemma 1(ii), they receive the same marginal utility from the connection to any player from the other component $k \in N \setminus \{4, 5, 6\}$. Vice versa, all other players $k \in N \setminus \{4, 5, 6\}$ have equal incentives to connect to players $i, j \in \{4, 5, 6\}$ by Lemma 1(iii). Note that both of these properties do not depend on which functional form of utility we apply, but only on the anonymity condition. As a final example, note that the network in Figure 3(c) is not symmetric and no two players share the same neighborhood, but players with odd numbers (respectively even numbers) are in symmetric positions.

Even if the utility functions satisfy anonymity, PS networks and therefore PNS networks may fail to exist. The following example proves this.

Example 1. *Let $N = 5$ and utility for all $i \in N$ be such that*

$$u_i(g) = \begin{cases} \eta_i(g) & \text{if } g \in \{\{ij\}, \{ij, ik\}, \{ij, jk\}, \{ij, kl, lm\}\} \text{ for } i \neq j \neq k \neq l \neq m, \\ -\eta_i(g) & \text{else} \end{cases}$$

This utility function is well defined and satisfies anonymity since names of players do not matter. To see that there does not exist a PS network, consider Figure 4 where up to any permutation of players the only networks where a subset of players receives positive utility are shown. First, the empty network is not PS since any two players

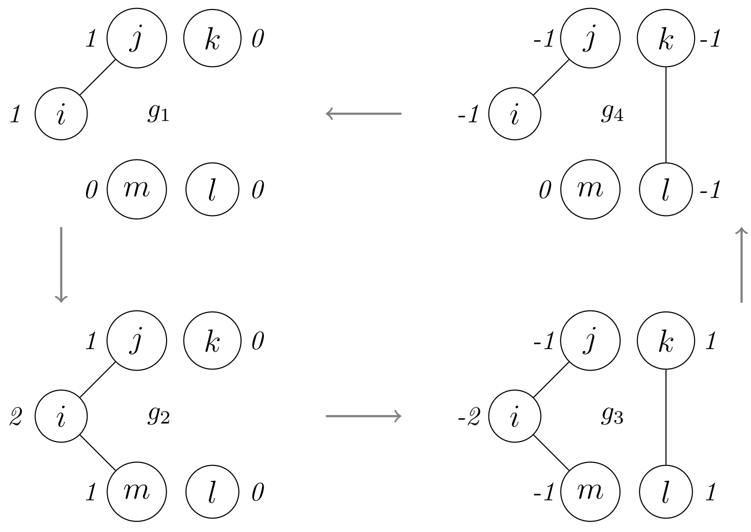


Figure 4: The only possible network structures which may yield positive utility for some players (up to permutation) in Example 1 form a cycle with improvement paths indicated by gray arrows.

have an incentive to form a link receiving utility of 1 *each. Thus, there exists an improvement path from the empty network to e.g.* g_1. *None of the networks which yield positive utility for a subset of players is PS either, since an improvement cycle as the one displayed in Figure 4 exists, where the arrows indicate an improvement, i.e. two players adding a link which is mutually beneficial or a player deleting a link which yields negative marginal utility. In any network not of the form displayed in Figure 4 (up to permutation), players receive* $u_i(g) = -\eta_i(g)$ *and hence have incentives to delete links until one of the networks in the cycle is reached. Hence there does not exist a PS network (and hence no PNS network) although the utility function is anonymous.*

2.3. Link externalities

Additionally to existence problems, it is impossible to say anything about stability of particular network structures without any assumptions on the utility function. In the literature on network formation, however, many utility functions admit certain link externality conditions. By link externalities we mean conditions on how marginal utility is affected when links are added to or deleted from a network. Hence, without losing much of the generality of our approach, we will examine whether PNS networks of certain structure exist if various combinations of link externalities in the context of homogeneous societies are satisfied. We will consider the weakest version of link externalities in the literature, namely the ordinal versions presented in Hellmann (2013).

Definition 4 (Ordinal link externalities)**.** *A utility function* u_i *satisfies* ordinal convexity *(CV) if for all* $g \in G$, *for all* $l_i \subseteq L_i(g^N - g)$ *and for all* $ij \in g$ *it holds that*

$$\Delta u_i(g, ij) \geq (>) 0 \;\Rightarrow\; \Delta u_i(g + l_i, ij) \geq (>) 0. \tag{1}$$

A utility function u_i *satisfies* ordinal strategic complements *(SC) if for all* $g \in G$, *for all* $l_{-i} \subseteq L_{-i}(g^N - g)$ *and for all* $ij \in g$ *it holds that*

$$\Delta u_i(g, ij) \geq (>) 0 \;\Rightarrow\; \Delta u_i(g + l_{-i}, ij) \geq (>) 0. \tag{2}$$

The ordinal versions of link externality conditions are given by single crossing properties of marginal utility with respect to additional own or other players' links. If the utility function of a player is such that once a given link yields positive marginal utility, the marginal utility of this link always stays positive when this player adds some other links, then CV is satisfied. In this case, we also speak of positive externalities from own links. In the same sense, SC captures positive externalities of other players' links.

Ordinal link externalities as first defined by Hellmann (2013) are implied by the more commonly used but stronger cardinal link externalities (see e.g. Bloch and Jackson, 2006, 2007; Goyal and Joshi, 2006, where marginal utility is assumed to be non-decreasing instead of single crossing), as well as, by several related concepts such as α-submodularity (Calvó-Armengol and Ilkiliç, 2009).

3. Ordinal Strategic Complements

To gain some insights into the structure of PS and PNS networks, we assume in this section that the profile of utility functions satisfies SC. Hence, links -once profitable- stay profitable when links between *other players* are added. When, moreover, externalities from *own* links can be bounded from below such that some weak link monotonicity conditions are satisfied, existence of a stable network is guaranteed and some additional structural properties concerning the empty and the complete network are implied (Section 3.1). Using more restrictive versions of these link externalities such that for marginal utility only link positions in terms of degree matter, we are able to show that all PS networks are contained in the set of nested split graphs (Section 3.2). We also provide applications for our results in Section 3.3.

3.1. Link Monotonicity

When the incentives to form links are increasing in both own and other players' links, then the utility function is supermodular (see Calvó-Armengol and Ilkiliç, 2009). In game-theory, it is possible to prove the existence of a pure strategy Nash equilibrium under supermodularity assumptions (see Milgrom and Roberts, 1990). For this reason, we could expect to get similar results for the existence of a pairwise stable network when utility satisfies CV and SC. Note, however, that a direct application of the results of Milgrom and Roberts (1990) is not possible since the structure of stable networks is different from Nash equilibria. We show here that we do not need to assume positive link externalities from both own and other players' links to arrive at an analogous result. Instead, these are relaxed in two ways: first, strategic complements only need to hold in ordinal terms, and second, externalities from own links may not even satisfy the single crossing property, but instead shall not be "too negative". To account for the latter, we introduce a general link monotonicity condition.

Definition 5 (κ-Link Monotonicity)**.** *A utility function u_i satisfies* κ-link monotonicity *(κ-LM) if for all $g \in G$, for all $l_{-i} \subseteq L_{-i}(g^N - g)$ with $|l_{-i}| = \kappa$, for all $ik \in L_i(g^N - g)$, and for all $j \in N_i(g)$:*

$$\Delta u_i(g, ij) \geq (resp. >)\ 0 \quad \Rightarrow \quad \Delta u_i(g + ik + l_{-i}, ij) \geq (resp. >)\ 0, \tag{3}$$

If some player i's utility function satisfies κ-LM and if i has an incentive to add some link to a network, then i still wants to add the link even after i added another link and other players added κ links. In the context of SC which imply that i keeps the desire to add a link after other players added links anyway, Definition 5 puts an additional restriction on the externalities from own links which cannot be too negative as to dominate the (positive) externalities from κ links of other players. Hence, in the case of SC, κ-LM implies κ'-LM for all $\kappa' > \kappa$. Thus, the larger κ, the weaker is the restriction on the externalities from own links in the presence of SC. In this case, κ-LM is weaker than the CV assumption which (together with SC) requires κ-LM to hold for all $\kappa \geq 0$. Examples, where κ-LM and SC are satisfied, but CV is not, are easy to construct (see e.g. Example 2). With these assumptions we are able to derive necessary conditions for networks to be PS in Lemma 2 for the two cases where the empty network is not stable and the complete network is not stable. Note that these conditions are also necessary for networks to be PNS and that both stability notions are equivalent for the empty network (recall that in this case, we write P(N)S), while for the complete network, PNS is obviously stronger than PS.

Lemma 2. *Suppose the profile of utility functions u satisfies ordinal strategic complements, anonymity and κ-link monotonicity. Then the following are necessary conditions for a network $g \in G \setminus \{g^{\emptyset}, g^N\}$ to be PS (and thus to be PNS):*

(i) If the empty network is not P(N)S, then the set of players $E(g) := \{i \in N : \kappa\eta_i(g) \leq \eta_{-i}(g)\}$ has to be completely connected for g to be PS.

(ii) If the complete network is not PS, then for all $i \in N$, it must be that $\kappa(n-1-\eta_i(g)) > \frac{(n-1)(n-2)}{2} - \eta_{-i}(g)$ or $\eta_i(g) = 0$ for g to be PS.

While the smaller κ, the stronger the assumptions on the utility function, as argued above, note that at the same time both necessary conditions in Lemma 2 also become stronger, the smaller κ.

In the first case, all those players with $\kappa\eta_i(g) \leq \eta_{-i}(g)$ have a desire to add any link if the empty network is not P(N)S. Thus, for a network to be PS (and therefore also to be PNS), these players have to be completely connected. The smaller κ, the more players must then be completely connected.

The second condition requires that the network is not too unbalanced, i.e. that there is no player who has excessively many links relative to the number of links that other players have if the complete network is not PS. For small enough κ, condition (ii) of Lemma 2 then implies $\eta_i(g) = 0$ for all players if the complete network is not PS since there always exists $i \in N$ with $\eta_i(g) \geq \frac{\eta_{-i}(g)}{n-1}$ for all $g \in G$.

In Theorem 1, we make use of the necessary conditions in Lemma 2 to derive upper bounds on κ such that either the empty network or the complete network is always PNS, respectively PS. For the result, as usual, the floor function $\lfloor x \rfloor$ denotes the largest integer $z \in \mathbb{Z}$ such that $z \leq x$, which is used since κ can only be an integer.

Theorem 1. *Suppose the profile of utility functions u satisfies ordinal strategic complements and anonymity.*

(i) If u satisfies $\hat{\kappa}(n)$-link monotonicity for $\hat{\kappa}(n) := \lfloor n/2 \rfloor - 1$, then the following holds: if the complete network is not PS, then the empty network is uniquely P(N)S, while if the empty network is not P(N)S, then the complete network is PNS (and thus also PS).

(ii) If u satisfies $\bar{\kappa}(n)$-link monotonicity for $\bar{\kappa}(n) := \left\lfloor \sqrt{2(n-1)(n-2)} \right\rfloor - (n-1)$, then the following holds: if the empty network is not P(N)S, then the complete network is uniquely PS (and thus uniquely PNS).

The result consists of two parts. First, if externalities from a single own link do not dominate the externalities from $\hat{\kappa}(n) = \lfloor n/2 \rfloor - 1$ other players' links, then either the empty or the complete network is guaranteed to be PNS (the same holds for PS). Note that the case where the complete network is PS but not PNS is not explicitly mentioned in the result, however, from reversing statement (i) it is clear that in this case, the empty network must be P(N)S.

If we restrict the externalities from own links further such that $\bar{\kappa}(n)$-LM is satisfied, then also the complete network is uniquely PS if the empty network fails to be P(N)S. Since $\bar{\kappa}(n) \leq \hat{\kappa}(n)$, we get by part (i) that the complete network is PNS which is also unique since no other network can be PS. The reason for the asymmetry between the uniqueness of stability of the empty network vs the complete network is that link deletion can be done unilaterally, while it takes two players with positive marginal utility to form a link. In the second case of Theorem 1, the structure of PS networks is very reminiscent of the structure of Nash equilibria in a supermodular game: if multiple networks are PS, then there always exists a smallest and a largest stable network in the sense of the set inclusion ordering, namely the empty and the complete network. To the contrary, if one of these networks (empty network or complete network) fails to be PS, then the other network must be uniquely PS, i.e. the least and maximal stable network coincide. For this result, however, we do not require supermodularity of the utility function, as both strategic complements and convexity have been relaxed. To illustrate Theorem 1, consider Example 2 first.

Example 2. *Suppose utility only depends on own degree and the number of other players' links such that*

$$u_i(g) = (a + \eta_{-i}(g))\eta_i(g) - \tfrac{c}{2}\left(\eta_i(g)\right)^2. \tag{4}$$

for some $a, c \in \mathbb{R}$, $c > 0$, for all $i \in N$. This utility function falls into the class of playing the field games defined in Goyal and Joshi (2006) which are discussed in Section 4.3. Calculating marginal utility for $ij \in g$, we get from (4)

$$\Delta u_i(g, ij) = a + \eta_{-i}(g) - c\left(\eta_i(g) - \tfrac{1}{2}\right). \tag{5}$$

Note that since $\Delta u_i(g + l_i, ij) - \Delta u_i(g, ij) = -c|l_i| < 0$, CV cannot be satisfied (it can be checked that, concavity is satisfied, see Hellmann, 2013, which implies that all PS networks are also PNS). Further, SC hold, as $\Delta u_i(g + l_{-i}, ij) - \Delta u_i(g, ij) = |l_{-i}| > 0$. Although the externalities from own links are negative on marginal utility, it is easy to see that κ-LM is satisfied for all $\kappa \geq c$, since $\Delta u_i(g + l_{-i} + ik, ij) - \Delta u_i(g, ij) = |l_{-i}| - c$.

From (5) *we further get that the empty network is P(N)S if and only if* $\Delta u_i(g^{\emptyset} + ij, ij) \leq 0$ *which simplifies to* $c \geq \frac{a}{2}$ *and the complete network is P(N)S if and only if* $\Delta u_i(g^N, ij) \leq 0$ *which simplifies to* $c \leq \frac{(n-1)(n-2)+2a}{2n-3}$*. Thereby, we can easily verify the first part of Theorem 1 for this example. Suppose that the empty network and the complete network are both not P(N)S. Then we must have* $\frac{(n-1)(n-2)+2a}{2n-3} < c < \frac{a}{2}$ *which can only be satisfied for* $a > \frac{2(n-1)(n-2)}{2n-7}$*. Plugging in the lower bound of* a *into the lower bound of* c *gives* $c \geq \frac{(n-1)(n-2)+2a}{2n-3} > \frac{(n-1)(n-2)}{2n-7}$ *implying* $\frac{n-1}{2} < c$*. But then, for* $g \in G$ *and* $l_{-i} \in L_{-i}(g^N - g)$ *with* $|l_{-i}| = \hat{\kappa}(n) = \left\lfloor \frac{n}{2} \right\rfloor - 1$ *we get* $\Delta u_i(g + l_{-i} + ik, ij) - \Delta u_i(g, ij) = |l_{-i}| - c < 0$*. This implies that* $\hat{\kappa}(n)$*-LM cannot be satisfied since* $\Delta u_i(g^{\emptyset} + ij, ij) > 0$ *and* $\Delta u_i(g^N, ij) < 0$ *implying that there must exist a network* $g \in G$ *where a switch of signs of marginal utility occurs when adding one own link and* $\hat{\kappa}(n)$ *other players' links. The uniqueness parts of Theorem 1 are a bit more difficult to illustrate as all possible networks have to be checked, so we refer directly to the proof of Theorem 1.*

Let us further elaborate on the interpretation of link monotonicity and the odd-looking condition on $\kappa(n)$ in combination with SC which basically implies that the externalities from own links cannot be "too negative". Since the larger κ, the weaker is the restriction on the externalities from own links and $\bar{\kappa}(n), \hat{\kappa}(n) \to \infty$ for $n \to \infty$, the restriction on the externalities from own links gets smaller for larger societies. We can conclude that in large homogeneous societies ($n \to \infty$), SC alone is sufficient for the result.

For small n, instead, $\hat{\kappa}(n)$-LM and $\bar{\kappa}(n)$-LM become more restrictive. If e.g. $3 \leq n \leq 5$, then 0-link monotonicity is required for the second part of Theorem 1. Note that 0-link monotonicity requires the externalities from own links to satisfy a single crossing property and is, therefore, equivalent to CV. As a direct consequence of Theorem 1, we, therefore, get the same result in case of ordinal positive externalities since CV and SC imply κ-LM for $\kappa = 0$. No additional restrictions on n are hence required.

Corollary 1. *Suppose the profile of utility functions* u *satisfies ordinal strategic complements, ordinal convexity and anonymity. If the empty network is not P(N)S, then the complete network is uniquely PS (and thus uniquely PNS), and vice versa.*

3.2. Prominence-based Utility Functions

Although it is possible to gain some insights into the structure of PS and PNS networks in a homogeneous society when ordinal link externalities are not too negative, these assumptions are not sufficient to characterize all PS networks (and therefore all PNS networks). In particular, it would be interesting to examine which stable structures emerge when the empty and complete network are both PS such that multiple stable networks exist. However, in the general framework that we imposed so far, there is little hope to say more about the structure of PS networks without putting stronger assumptions on the utility function.

We therefore focus attention on the relative sizes of the link externalities. By that we mean the following. Consider a player $i \in N$ and a network $g \in G$ and suppose there are two players $k, l \in N$ with $kl, ik \notin g$ who form the link kl (see Figure 5). Then

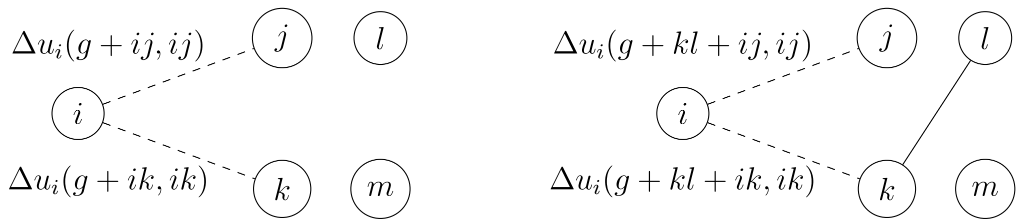


Figure 5: The marginal utilities of the links ij and ik in the networks $g^{\emptyset}$ and $g^{\emptyset}+kl$.

by SC, if $\Delta u_i(g+ik,ik)\geq 0$ then $\Delta u_i(g+ik+kl,ik)\geq 0$ and if $\Delta u_i(g+ij,ij)\geq 0$ then $\Delta u_i(g+ij+kl,ij)\geq 0$ for some $j\neq k,l$, $ij\notin g$. However, SC does not specify on which links the effect of other players' links is stronger (and this cannot be captured by the cardinal notion of either). In other words, does the addition of the link kl increase the incentive for player $i\notin\{k,l\}$ more to link to k (resp. l) than to $j\notin\{k,l\}$, or vice versa?

In this setting, it would be quite natural to say that the externality of the link kl on the incentive for player i to form a link to k is larger than the externality of the link kl on the incentive for player i to form a link to j. Coupled with the cardinal notion of SC which means that the externalities are non-decreasing in own links, this implies that the addition of the link kl increases the marginal utility of the link ik more than the link ij.

First, we only apply this idea to players $k,l\in N$ who are in completely symmetric positions such that $N_j(L_{-k}(g))=N_k(L_{-j}(g))$. Thus, for any $i\in N$ by Lemma 1, $\Delta u_i(g+ij,ij)=\Delta u_i(g+ik,ik)$ and hence by above reasoning, $\Delta u_i(g'+ij,ij)\leq\Delta u_i(g'+ik,ik)$ for any $g'\in G$ such that $N_j(L_{-k}(g'))\subseteq N_k(L_{-j}(g'))$. Using only the ordinal version we receive the property of Weak Preference for Prominence in Definition 6. The stronger notion of Strong Preference for Prominence in Definition 6 goes beyond that by applying the logic also to players with different degrees.

Definition 6 (Weak and Strong Preference for Prominence)**.** *A utility function u_i satisfies* weak preference for prominence (WPP) *if for all $g\in G$ and for all $j,k\in N$ such that $N_j(L_{-k}(g))\subseteq N_k(L_{-j}(g))$ it holds that*

$$\Delta u_i(g+ij,ij)\geq(>)0\quad\Rightarrow\quad\Delta u_i(g+ik,ik)\geq(>)0,\tag{6}$$

A utility function u_i satisfies strong preference for prominence (SPP) *if for all $g\in G$ and for all $j\in N_i(g)$, $k\in N\setminus N_i(g)$ such that $\eta_j(g)\leq\eta_k(g)$ it holds that*

$$\Delta u_i(g,ij)\geq 0\quad\Rightarrow\quad\Delta u_i(g+ik,ik)\geq 0\tag{7}$$

The notions of weak and strong preference of prominence, as the names suggest, have a quite intuitive interpretation, expressing a preference for nodes with many neighbors. This property is often satisfied, if players want to be central in the network (which holds for centrality notions where a node is central if neighboring nodes are central, e.g. eigenvector based centrality notions, or the Bonacich centrality, see Equation (10)). WPP is an extremely weak notion of preference for prominence. It simply requires that if a link to a node is desirable then a link should be also desirable to a more prominent node. In this case, the prominence relation is a partial ordering which orders two nodes whose sets of neighbors are related by the set inclusion order (see also Monsuur, 2007, where this is called node covering). When we consider, instead, a prominence relation such that a node is more prominent if and only if it has more neighbors, we receive a complete ordering on the set of nodes making the notion of preference for prominence more demanding and which is defined as strong preference for prominence (SPP).

Although the latter notion of SPP seems demanding at first sight, it may be very naturally satisfied in societies where SC is given. We elaborate on this in Section 3.3.1 for the framework of Goyal and Joshi (2006) where SPP is implied by SC as the externalities from other players' links act homogeneously on marginal utility since the Goyal and Joshi (2006) utility functions depend on fewer network statistics (see Proposition 1). Hence, assuming SPP instead of SC can also be seen as reducing the network statistics that enter the utility function when SC is satisfied.

Similarly as SPP represents SC in a society with utilities reduces to fewer network statistics, we can also consider externalities from own links. We call this stronger notion anonymous convexity.

Definition 7 (Anonymous Convexity)**.** *A utility profile u satisfies* anonymous convexity (AC) *if for all $g \in G$ and all $i, j \in N$ such that $\eta_i(g) \leq \eta_j(g)$ we have for any $k \in N_i \setminus N_j$:*

$$\Delta u_i(g, ik) \geq 0 \;\Rightarrow\; \Delta u_j(g + jk, jk) \geq 0. \tag{8}$$

Similar to above, AC implicitly assumes a higher degree of homogeneity compared to CV: if a player i likes the connection to k then any player with more links also has an incentive to keep the connection to k. In a more homogeneous society where players with same degree have the same incentives, this formulation reflects the idea of CV since once the marginal utility of a link is positive, it stays positive if own links are added. Hence, AC translates the CV notion to other players. We show in Section 3.3.1 that AC is very naturally implied by CV in homogeneous societies by the example of the Goyal and Joshi (2006) utility functions (Proposition 1).

Finally, since all assumptions so far are assuming weak inequalities (see SPP and AC) we need to rule out indifferences in order to characterize all PS networks.

Definition 8 (No-Indifference)**.** *A utility function u_i satisfies* no-indifference (NI) *if for all $g \in G$ and all $ij \in g$: $\Delta u_i(g, ij) \neq 0$*

This definition is taken from Jackson and Wolinsky (1996). We need to assume NI in order to rule out that due to indifference of two players, a network outside the characterizing class of networks can be PS. This is not a strong assumption since it holds for a generic subset of payoff functions, i.e. a subset with open and full measure. It is also a fairly standard assumption in the literature for uniqueness or characterization results (see also Hellmann, 2013). Other papers instead directly assume strict cardinal link externalities in order to characterize all PS networks such that NI is built into the link externality conditions (see e.g. Goyal and Joshi, 2006). The set of networks that we will need in order to characterize all PS networks is given by the following definition.

Definition 9 (Nested Split Graph). *A network $g \in G$ is a* nested split graph *(NSG) if for all players $i, j \in N$: $\eta_i(g) \leq \eta_j(g) \Rightarrow N_i(g) \subseteq N_j(g) \cup \{j\}$*

In a nested split graph the neighborhood structure of all players is nested in the sense that for any two players $i, j \in N$ the set of their neighbors can be ordered according to the set inclusion order, i.e. $N_i(L_{-j}(g)) \subseteq N_j(L_{-i}(g))$ or $N_i(L_{-j}(g)) \supseteq N_j(L_{-i}(g))$. Definition 9 is taken from Olaizola and Valenciano (2019). For equivalent definitions, see Cvetković and Rowlinson (1990), Mahadev and Peled (1995), and Simić et al. (2006).

More importantly for our purposes, the set of nested split graphs contains all PS networks when the profile of utility functions satisfies SPP and AC.

Theorem 2. *Suppose a profile of utility functions satisfies strong preference for prominence, anonymous convexity, and no-indifference. Then any PS network is a nested split graph.*

Although the utility functions are not specified in our framework, we learn a lot about the structure of PS networks when SPP and AC are satisfied: any two players' neighborhoods can be ordered with respect to the set inclusion order. This reduces the set of possible candidates for PS networks considerably as the set of nested split graphs only makes up a very small fraction of the set of all possible networks G. Since any PNS network is also PS, Theorem 2 also characterizes the PNS networks.

Further, note that for this result anonymity is not explicitly required. Instead, a different kind of homogeneity is implicitly captured by the assumptions AC and SPP. These require the externalities from own links (AC) and other players' links (SPP) to act homogeneously across all players on the incentives to form links.

3.3. Applications

The assumptions in previous results may seem demanding at first sight, in particular for Theorem 2. In this section, we want to show that there exist models in the literature on network formation that are captured by our approach. Not only the Local Spillovers utility functions from Goyal and Joshi (2006) are captured and therefore generalized by our approach (Section 3.3.1), we are moreover able to apply the results to models which do not fall into that class such as utility functions depending on Bonacich centrality (Section 3.3.2), and network formation games with effort choices under strategic complementarities (Section 3.3.3), to name only a few examples.

3.3.1. Local Spillovers

In Goyal and Joshi (2006), two utility functions with a particular structure –called playing the field and local spillovers– are studied with respect to existence and characterization of stable networks. Both utility functions reduces the network to only one characteristic: the vector of degrees $\eta(g) = (\eta_1(g), \ldots, \eta_n(g))$. We introduce playing the field utility functions in Section 4.3 and consider here only local spillover utility functions. A utility function is of *local spillover* type if there exist functions $f_1, f_2, f_3 : \{0, ..., n-1\} \rightarrow \mathbb{R}$ such that with these functions applied to own degree, neighbors' degrees, and non-neighbors' degrees, respectively, utility can be expressed as the sum of these functions net of costs,

$$u_i^{LS}(g) := f_1(\eta_i(g)) + \sum_{j \in N_i} f_2(\eta_j(g)) + \sum_{k \notin N_i \cup \{i\}} f_3(\eta_k(g)) - c\eta_i(g). \tag{9}$$

To establish existence of stable networks, Goyal and Joshi (2006) additionally assume various combinations of strict cardinal notions of link externalities. The seemingly restrictive assumptions of SPP and AC are implied by our weak ordinal link externality conditions if the utility function is of above type which is formally stated in the following result.

Proposition 1. *If u^{LS} satisfies ordinal convexity and ordinal strategic complements, then u^{LS} satisfies anonymous convexity and strong preference for prominence.*

If the strict cardinal versions of SC and CV are satisfied, Goyal and Joshi (2006) show that in local spillovers games (9) all PNS networks are of interlinked star architecture. A network g is of interlinked star architecture if there exists $M \subset N$ such that $i \in M$, $i \neq j \Rightarrow ij \in g$ and $i, j \in N \setminus M$ $i \neq j \Rightarrow ij \notin g$. In other words, one group of players is completely connected while the remaining players have links to all players in the completely connected group but do not connect among themselves. Interlinked stars are a subset of the set of nested split graphs which is confirmed by Theorem 2 since SPP and AC are satisfied for local spillover utility functions if SC and CV hold by Proposition 1. Thus we are able to show that in very homogeneous societies, the properties SC and CV are the driving force for the emergence of nested split graphs which particularly contain interlinked stars. Since PNS networks are also PS, our characterization result is more general not only with respect to the functional form of utility but also with respect to the stability notion used. Furthermore, a result that either the empty or complete network is always P(N)S cannot be found in Goyal and Joshi (2006) although their utility functions and the link externality conditions are far less general.

3.3.2. Bonacich Centrality

With our general approach, we are able to study interesting utility functions that do not fall into the class of games in Goyal and Joshi (2006). One such example where more than the degree distribution matters for utility is given by the important class of utility functions which depend on players' Bonacich centrality.

Bonacich (1987) introduced a parametric family of centrality measures to formulate the intuitive idea that the centrality of a single node in a network should depend on the centrality of its neighbors. This self-referential definition of centrality leads to an eigenvector-based measure, which can be defined as follows: Let $\mathbf{A}(g)$ be the $n \times n$ adjacency matrix of a given network g, $\mathbf{I}$ be the $n \times n$ identity matrix, and $\mathbf{1}$ be the $n \times 1$ vector with all entries equal to 1. The adjacency matrix $\mathbf{A}(g)$ of a network g is a matrix with entries $a_{ij}(g) = 1$ if $ij \in g$ and $a_{ij}(g) = 0$ otherwise. Then $(\mathbf{A}(g))^k\mathbf{1}$ counts the total number of walks of length k where a walk is a sequence of adjacent links in g. Letting $0 < \delta < \lambda_1(\mathbf{A})^{-1}$ be a given parameter discounting for walk length, where $\lambda_1(\mathbf{A})$ is the eigenvalue of $\mathbf{A}$ having largest modulus, we get that $[\mathbf{I} - \delta\mathbf{A}(g)]^{-1}$ exists, and the centrality index proposed by Bonacich (1987) is then given by,

$$\mathbf{b}(g, \delta) = \sum_{n=0}^{\infty} \delta^n \mathbf{A}^n \mathbf{1} = [\mathbf{I} - \delta\mathbf{A}]^{-1}\mathbf{1}. \tag{10}$$

This centrality measure is actually a Nash equilibrium of an interesting class of non-cooperative games: Suppose there are N agents who are involved in a team production problem (for an in-depth introduction of this game, see Ballester et al., 2006). Each player chooses a non-negative quantity $x_i \geq 0$, interpreted as efforts invested in the team production. Efforts are costly, and the level of effort invested by the other players affects the utility of player i. To capture these effects, player i's payoff from an effort profile $\mathbf{x} = (x_i, \mathbf{x}_{-i})$ is given by

$$\pi^{BC}(g, x_i, \mathbf{x}_{-i}) = x_i - \frac{1}{2}x_i^2 + \delta \sum_{j \in N_i(g)} x_i x_j. \tag{11}$$

Ballester et al. (2006) show that this game has a unique Nash equilibrium $\mathbf{x}^* = \mathbf{b}(g, \delta)$.

Given network g, and discount factor $\delta \in \mathbb{R}$, so that (10) is well defined, the equilibrium payoff of player i can be computed as

$$\pi_i^{BC}(g, \mathbf{x}^*) = \frac{1}{2}b_i(g, \delta)^2. \tag{12}$$

where $b_i(g, \delta)$ is the i-th component of the vector $\mathbf{b}(g, \delta)$. There are many other examples of games where equilibrium is given by a function of the Bonacich centrality. Among those are models of R&D cooperation (König et al., 2019; Hsieh et al., 2018), local public goods (Allouch, 2015; Bramoullé et al., 2014), and trade (Bosker and Westbrock, 2014).

In a stage game where players can first form the network prior to engaging in such a game, equilibrium payoffs as a function of Bonacich centrality $f\left(b_i(g, \delta)\right)$ in the second stage are anticipated when forming links. If utility is given by (11), we have $f(x) = \frac{1}{2}x^2$ by (12), but more generally we allow f to be *increasing* and *convex* for the results in this section. Assuming cost of link formation to be linear in the number of links, we then arrive at a more general class of utility functions,

$$u_i^{BC}(g) = f\left(b_i(g, \delta)\right) - \eta_i c. \tag{13}$$

When considering link formation with the utility function $u_i^{BC}(g)$ as the objective, we have to make sure that $b_i(g, \delta)$ is well defined for any network. Since the largest eigenvalue $\lambda_1(g)$ is maximized for the complete network g^N, and we need $\delta < \frac{1}{\lambda_1(g)}$ for $b_i(g, \delta)$ to exist, we assume $\delta < \frac{1}{\lambda_1(g^N)} = \frac{1}{n-1}$ in order to define a consistent model of network formation. Further, we restrict here to the case where $\delta > 0$ as a special case of the Ballester et al. game such that efforts are complements.

The profile of utility functions u^{BC} obviously satisfies anonymity. Moreover, the following result states that u^{BC} also satisfies positive link externalities, i.e. SC and CV.

Proposition 2. *If f is increasing and convex, then u_i^{BC} as defined by* (13) *satisfies strategic complements and convexity.*

The result is intuitive since more own or other players' links increase the number of paths that an additional link creates. An increasing and convex transformation does not change this fact and since linking costs are linear, marginal utility is increasing in own and other players' links.

Thus, we can apply Corollary 1, to conclude that either the empty network or the complete network is uniquely P(N)S, or both are P(N)S when utility net of costs is given by an increasing and convex function of the Bonacich centrality. As an example, it can be checked that for $f(x) = x$ in (13), the empty network is P(N)S iff $c \geq \frac{\delta}{1-\delta}$ and the complete network is PNS iff $c \leq n\frac{\delta}{1-\delta}$.

It is worth noting that to the best of our knowledge, there is so far only one result from the literature that can be applied to shed some light on the structure of P(N)S networks when individuals form links according to u_i^{BC}. From Hellmann (2013) it is known that a PS network exists. Other models are not applicable, since u_i^{BC} does not fall in the category of games in Goyal and Joshi (2006), and does not allow for a network potential (cf. Jackson and Watts, 2001; Chakrabarti and Gilles, 2007). We go beyond showing existence since Corollary 1 is applicable.

Further by restricting to low discount factors, we show in Proposition 3 that u^{BC} satisfies SPP and AC and therefore all PS networks are of nested split architecture.

Proposition 3. *If f is increasing and convex and $\delta < \frac{1}{(n-1)^2}$, then u_i^{BC} as defined by* (13) *satisfies strong preference for prominence and anonymous convexity.*

Although the utility function given by the Bonacich centrality seems to be quite a complex object since it considers the infinite discounted sum of all possible paths in the networks, it is possible to characterize the set of PS (and therefore also PNS) networks at least for low enough discount factors. This is because u^{BC} satisfies SPP and AC for these low discount factors since the benefits from second-order connections (degree of neighbors) dominate any benefits from higher-order connections which is shown in the proof of Proposition 3. Note that this does not mean that the network itself, i.e. connections beyond second-order do not play a role for gross payoff. Hence, although our results hold for general utility functions, they are still applicable to interesting classes of utility functions and help characterize the structure of PS and PNS networks, even where no results are available so far. One notable exception is König et al. (2014) who show that in a dynamic model of network formation only the nested split graphs are absorbing for a more restrictive functional form of Bonacich centrality (see also Hsieh et al., 2018). This however does not imply that any PS network is a nested split graph, even in their setting.

3.3.3. Simultaneous Choice of Links and Efforts under Strategic Complementarities

In Section 3.3.2, we presented a two-stage game where the network is formed prior to action choice in a game between neighbors in the network. Suppose, instead, that action choice and link formation are done simultaneously. Such a framework is employed in two recent papers Baetz (2015) and Hiller (2017). The assumption of simultaneous choices of network and actions simplifies analysis a lot compared to a two-stage game.

The reason is that when network formation takes place before action choice as in the previous section, then the effects of forming links on the equilibrium of the second stage have to be taken into account. For instance, the resulting utility function from the second stage equilibrium outcome of the Ballester et al. (2006) game is a function of the Bonacich centrality and is quite a complex object. We needed additional assumptions on δ in Proposition 3 to characterize all PS networks. For equilibria in games of simultaneous choice of links and efforts, instead, only single-player deviations have to be considered taking other players' equilibrium effort choices as given.

Both frameworks of Baetz (2015) and Hiller (2017) are almost identical differing only in the curvature assumption on the value function and the type of network formation (directed vs undirected). We discuss here briefly the model due to Hiller (2017). Adapting Hiller (2017)'s notation and setup to our framework and letting $x_i \in \mathbb{R}$, utility is given by

$$u_i(g, \mathbf{x}) = \pi(x_i, \sum_{k \in N_i(g)} x_k) - \eta_i c \tag{14}$$

such that $\partial \pi(x,y)/\partial y, \partial^2 \pi(x,y)/(\partial x \partial y) > 0$ and $c > 0$. Hiller (2017) further assumes that for all $i \in N$ best reply effort choices satisfy $\bar{x}_i(g, \mathbf{x}_{-i}) = \bar{x}(\sum_{k \in N_i(g)} x_k)$ with $\bar{x}(0) > 0$, $0 < \lim_{y \to \infty} \bar{x}'(y) < 1/(n-1)$ and either $\bar{x}''(y) < 0$ or $\bar{x}''(y) = 0$ for all $y \in \mathbb{R}$. Moreover, gross payoffs π evaluated at best reply can be written as $\pi(\bar{x}_i(g, \mathbf{x}_{-i}), \sum_{k \in N_i(g)} x_k) = v(\sum_{k \in N_i(g)} x_k)$ with $v(0) \geq 0$, $v' > 0$, and $v'' \geq 0$. One example, where all these assumptions are satisfied, is given by the Ballester et al. (2006) utility function, see (11).

Given these assumptions, Hiller (2017) finds that for each network $g \in G$ there exists a unique Nash equilibrium of effort choices (see Hiller (2017), Proposition 1) denoted by $\mathbf{x}^*(g)$. To account for pairwise nature of network formation also deviations by two players are allowed for equilibrium considerations. When players $i, j \in N$ connect in network g, we denote the vector of deviation effort levels by $\mathbf{x}^{ij}(g+ij)$ with entries $x_k^{ij}(g+ij) = \bar{x}\left(g+ij, \mathbf{x}_{-k}^{ij}(g+ij)\right)$ for $k = i, j$, and $x_k^{ij}(g+ij) = x_k^*(g)$ for $k \neq i, j$. Marginal utility of such a deviation can then be defined by

$$\Delta^d u_i(g+ij, ij) := u_i(g+ij, \mathbf{x}^{ij}(g+ij)) - u_i(g, \mathbf{x}^*(g)) \tag{15}$$

Similarly, when player i deletes links $l_i \subseteq L_i(g)$ denote the vector of deviation effort choice levels by $\mathbf{x}^i(g-l_i)$ with entries $x_i^i(g-l_i) = \bar{x}\left(g-l_i, \mathbf{x}_{-i}^*(g)\right)$ and $x_k^i(g-l_i) = x_k^*(g)$ for all $k \neq i$. Marginal utility of such a deviation is hence given by

$$\Delta^d u_i(g, l_i) := u_i(g, \mathbf{x}^*(g)) - u_i(g-l_i, \mathbf{x}^i(g-l_i)) \tag{16}$$

Assuming that the unique equilibrium effort levels are obtained, a network $g^* \in G$ is then PNS if for all $l_i \subseteq L_i(g^*)$: $\Delta^d u_i(g^*, l_i) \geq 0$ and for all $ij \notin g^*$: $\Delta^d u_i(g^*+ij, ij) > 0$ $\Rightarrow \Delta^d u_j(g^*+ij, ij) < 0$. In other words, a network-efforts pair $(g^*, \mathbf{x}^*)$ is an equilibrium if no two players can profitably deviate by forming a link in g^* (and adjusting efforts) and no single player can benefit by deleting a link in g^* (and adjusting efforts) while the unique equilibrium in efforts $\mathbf{x}^* = \mathbf{x}^*(g)$ obtains.

Considering the so defined marginal utility of deviations, we find that SC and CV are satisfied under the assumptions imposed by Hiller (2017).

Proposition 4. *In the simultaneous move game of links and efforts given by Hiller (2017), marginal utility of deviations satisfies ordinal strategic complements and ordinal convexity.*

Because PNS networks in this context are solely defined via marginal utility of deviations $\Delta^d u$ and this type of marginal utility satisfies the conditions of Theorem 1, we can immediately apply the result to conclude that either the empty network of the complete network is uniquely PNS or both are PNS. Not surprisingly, Hiller (2017) finds the same result in his paper (see Hiller, 2017, Proposition 2). Hiller (2017) continues to show that all PNS networks are nested split graphs. Although SPP and AC are not necessarily satisfied it is possible to show that quite similar properties hold i.e. increasing marginal utility with respect to the effort exerted by the players (instead of increasing marginal utility with respect to the degree) which is the driving force for Hiller (2017)'s result.

Finally, note that while CV and SC are always satisfied in Hiller (2017) by Proposition 4 which is due to the assumptions of the best replies and convexity of gross payoff function evaluated at best reply $v'' \geq 0$, the same is generally not true for Baetz (2015) since there it is assumed that the gross payoff function evaluated at best reply is concave, i.e. $v'' \leq 0$.

4. Ordinal Convexity

We finally want to study the structure of PS networks in homogeneous societies when SC is not necessarily satisfied. To obtain results we will assume that at least the externalities from own links satisfy a single crossing property such that CV is satisfied. Recall that CV, as given in Definition 4, orders the externalities from own links on marginal utility in a way that, once positive, it will stay positive whenever own links are added to the network. In presence of this form of complementarity between own links, the intuition is that players that already have links are likely to strive for more. Notice, however, that due to ambiguous marginal effects of other links, cycling behavior may still arise such that PS (and thus PNS) networks may fail to exist.

4.1. Weak Preference for Prominence

We show in the following that with the additional assumption of WPP in Definition 6 stable networks exist. To do so we define the following class of networks.

Definition 10 (Dominant Group Networks)**.** *A network $g \in G$ is a* dominant group network *if there exists $S \subseteq N$ such that $ij \in g \Leftrightarrow i, j \in S$, $i \neq j$.*

In other words, a network is of dominant group architecture if a subset of players S is completely connected, while the remaining players stay isolated. Now, by the anonymity assumption if for some $S \subseteq N$ a dominant group network is P(N)S, then any dominant group network of same size is P(N)S for all $\tilde{S} \subset N$ with $|\tilde{S}| = |S|$. Since, therefore, stable dominant group networks are completely characterized by the size of their dominant group in a homogeneous society, we also write g_s^{dg} to denote dominant group networks of size s with $1 \leq s \leq n$ where for $s = n$, g_s^{dg} is the complete network, while for $s = 1$, g_s^{dg} is the empty network.

Theorem 3. *Suppose the profile of utility functions satisfies ordinal convexity, weak preference for prominence, and anonymity. Then, there exists $s \in \{1, \dots, n\}$ such that all dominant group networks g_s^{dg} are PNS.*

Since PNS networks are also PS, existence of a PS dominant group network is also guaranteed. The intuition for Theorem 3 is as follows. First, as CV holds, players' incentive to form a link is not destroyed by additional own links. Second, players tend to connect to others that already have more links, due to WPP. Both effects together point to networks where players either have many or no links. In Theorem 3, we then naturally find existence of a stable network in the extreme case, namely one completely connected subset of players and the remaining players being isolated.

It must be noted that Theorem 3 is weaker than Theorem 1 since the empty network and the complete network are among the dominant group networks. However, let us emphasize that WPP is also a very weak assumption. Recall that the only restriction imposed by the WPP assumption is that the desire to form links stays positive when connecting to more prominent nodes where prominence is meant with respect to the set inclusion order of the neighborhoods. As the set inclusion order is only a partial order, the assumption is not binding in networks where no two players' neighborhood structures can be ordered. Further, it is very naturally satisfied in many utility functions where players have a desire to be central in the network. As an example, consider some self-referential definition of centrality where a player is central if her neighbors are central. Then, clearly, the connection to a player j such that any of j's neighbors is also a neighbor of some other player k increases centrality by a smaller amount as the connection to k. For instance, u^{BC} given by (13) satisfies WPP.

4.2. Independence of Network Position

WPP and convexity imply the existence of PS networks in a homogeneous society as shown in Section 4.1. However, to characterize all PS networks, these conditions are not sufficient. The main reason is that WPP is too weak to exclude other network structures from being PS. Instead, consider the following stronger condition.

Definition 11. *A utility function u_i satisfies* independence of the network position of other players (INP) *if for all $g \in G$, whenever there exist $j \in N_i(g)$ such that $\Delta u_i(g, ij) \geq 0$, then $\Delta u_i(g + ik, ik) \geq 0$ for all $k \in N$.*

If player i's utility function satisfies INP, then the network position of players with whom i can form a link cannot play a big role. In particular, it must be that if i wants to connect to *some* player, then i wants to connect to *any* player. In this sense, the marginal utility is independent (in an ordinal sense) of the network position of other players. Clearly, if a utility function satisfies INP, then it also satisfies WPP (and even stronger: SPP, see Definition 6).

Now, in combination with the convexity assumption in this section, INP has also strong implications. When the network position of other players does not matter for the willingness to form links, the convexity assumption then implies that a player either wants to form no links or all possible links. Straightforwardly, we then get that only dominant group networks can be PS (up to no indifferences). Further, existence is still guaranteed since INP implies WPP, and hence Theorem 3 still applies.

Theorem 4. *Suppose a profile of utility functions satisfies ordinal convexity, independence of network position, and no-indifference. Then, any PS network is of dominant group architecture.*

Since PNS networks are also PS, Theorem 4 also applies to PNS networks. INP is a quite strong assumption. We may compare this result with Theorem 2. There, we used SPP and AC to show that any PS network is a nested split graph. Note that dominant group networks are in fact nested split graphs of special structure. Thus, characterizing PS networks by the dominant group architecture is a stronger result than characterizing them by a nested split graph. The conditions required cannot be compared in the same way since INP implies SPP, but AC is not implied by INP and CV. Finally note that both Theorem 2 and Theorem 4 do not require the anonymity assumption explicitly.

While the conditions used in Theorem 4 may seem demanding, these only have to hold in ordinal terms. We also show that there exist some applications where these are satisfied.

4.3. Applications: Playing the Field Games

While we discussed local spillovers utility functions in Section 3.3.1 and showed how our general results extend the characterization results by Goyal and Joshi (2006) for this type of utility function, we are left to discuss the second type introduced by Goyal and Joshi (2006), called playing the field games which can be seen to even satisfy INP. According to Goyal and Joshi (2006) a network utility function is of *playing the field* type if benefits can be written as a function $f : \{0, 1, \ldots, n-1\} \times \{0, 1, \ldots, (n-1)^2\} \to \mathbb{R}$ of $\eta_i(g)$ and $\eta_{-i}(g)$ net of per unit link formation costs $c \in \mathbb{R}_+$ such that,

$$u_i^{PF}(g) = f(\eta_i(g), \eta_{-i}(g)) - c\eta_i(g). \tag{17}$$

If strict cardinal convexity is satisfied, Goyal and Joshi (2006) show for playing the field utility functions that all PNS networks belong to the class of dominant group networks. For a PNS network to exist, however, SC is required.

When computing marginal utility in the Playing the Field utility function we get from (17),

$$\Delta u_i^{PF}(g, ij) = f(\eta_i, \eta_{-i}(g)) - f(\eta_i - 1, \eta_{-i}(g)) - c \tag{18}$$

Since the right-hand side of (18) does not depend on j, playing the field type utility functions always satisfy INP and, hence, also WPP. Goyal and Joshi (2006) show that all PNS networks are characterized by the dominant group architecture if utility is of playing the field type and strict cardinal convexity is satisfied (Goyal and Joshi, 2006, Proposition 3.1). With Theorems 3 and 4, we thereby complement and extend the results from Goyal and Joshi (2006) in the following way: we show in Theorem 3 that PNS networks in fact exist without additional assumptions and that the utility structure of playing the field is not required for the existence of a dominant group PNS network. Instead, the result holds for general functional forms of utility as long as the weak properties WPP and CV are satisfied. Theorem 4, moreover, is a true generalization of Goyal and Joshi's characterization result as we show that all PS networks are characterized by the dominant group structure. This includes, trivially also the pairwise Nash stable networks. The characterization result hence not only holds for utility functions which are of playing the field type and satisfy CV, but instead for all utility functions as long as CV and INP are satisfied.

Another classical example in the literature, where the assumptions of CV and INP are satisfied, is a Cournot oligopoly model where firms can form bilateral collaboration links lowering marginal costs before competing in quantities (Goyal and Joshi, 2003, 2006; Dawid and Hellmann, 2014). In these models, equilibrium quantities are given by

$$q_i(g) = \frac{(a-\gamma_0) + (n-1)\gamma\eta_i(g) - \gamma\sum_{j\neq i}\eta_j(L_{-i}(g))}{n+1}, \quad i \in N.$$

With Cournot profits given by $\pi_i(g) = q_i^2(g)$, this results in marginal profit of an additional link $ij \notin g$ being equal to

$$\Delta\pi_i(g+ij, ij) = \frac{\gamma(n-1)}{(n+1)^2}\Big[2(\alpha-\gamma_0) + \gamma(n-1) + 2\gamma n\eta_i(g) - 2\gamma\sum_{j\neq i}\eta_j(g)\Big]^2 - c.$$

Clearly, as $\eta_{-i}(g) = \frac{1}{2}\left(\sum_{j\neq i}\eta_j(g) - \eta_i\right)$, marginal utility is then just a function of own and other players' number of links and, hence, the associated utility function is of playing the field type. In particular, WPP, INP and anonymity are satisfied. Moreover, Dawid and Hellmann (2014) show that CV is satisfied and also conclude that all PS networks are of dominant group structure. Theorem 4 could have worked as a shortcut for this result.

5. Conclusion

We have shown that in a very general environment of network formation, it is possible to derive results on the structural properties of PS and PNS networks by exploiting the ordinal link externality conditions in a homogeneous society. While almost all models in the literature (that we can think of) share the anonymity assumption, we have shown that the link externality conditions are also quite often satisfied. This paper hence contributes to a better understanding of what the driving force for the structure of PS and PNS networks in those models is: the link externality conditions. The results in this paper may, moreover, be used to characterize PS and PNS networks in future models of network formation that satisfy the link externality conditions (which e.g. arise from multistage games).

For the results on existence of PS and PNS networks, we do not rely on the assumptions of a potential function that is very restrictive or on the assumption of supermodularity of the utility function. Instead, supermodularity can be weakened such that the externalities from own links only satisfy a boundary condition while the externalities from other players' links only need to satisfy a single crossing property for the PS and PNS networks to exhibit a structure like the set of Nash equilibria in supermodular games (cf. Milgrom and Roberts, 1990). An interesting question for future research could be to apply the ideas developed in this paper to pure strategy Nash equilibria in non-cooperative games, e.g. in the context of directed network formation where the formation of links does not require the consent of other players.

We have thereby generalized some results in the literature that also rely on link externality conditions to derive structural properties of stable networks. Compared to Hellmann (2013), we were able to show existence of PS networks of specific structures like the empty and the complete network or the dominant group structure. The only additional assumption made is that of a homogeneous society while some other assumptions are relaxed (like the externality conditions of either own or other players' links). On the other hand, we have generalized some of the results in Goyal and Joshi (2006): they hold for arbitrary functional forms of utility, they require only ordinal versions of externalities, and some assumptions are not even needed. Ordinal link externalities are robust in the sense that they still hold when utilities are perturbed by small error terms, e.g. as in Harmsen-van Hout et al. (2016).

While the present work exhibits a focus on positive link externalities (convexity and strategic complements) it would be interesting for future research to show similar results in case of negative link externalities (i.e. concavity and strategic substitutes). Our conjecture for the case of both concavity and strategic substitutes however is that existence of PS networks is not always guaranteed. Second, a full characterization of PS networks if utility profiles are functions of Bonacich centrality remains an open question. While we provide a first contribution to this goal, proving existence of a PS or PNS network for any discount factor and characterizing stable networks for low discount factors, it remains a challenge to characterize stable networks for the rest of the set of admissible discount factors.

6. Appendix

Proof of Lemma 1. Let the profile of utility functions u satisfy anonymity.

(i) Suppose that $i, j \in N$ are symmetric. Then, by definition there exists a permutation π with $\pi(i) = j$ and $g_\pi = g$. By anonymity, we get

$$u_i(g) = u_{\pi(i)}(g_\pi) = u_j(g).$$

(ii) Now let $i, j \in N$ such that $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$. Recall that π^{ij} was defined as the permutation where players i and j switch positions, that is

$$\pi^{ij} \colon N \to N, \quad \pi^{ij}(i) = j,\ \pi^{ij}(j) = i,\ \pi^{ij}(k) = k\ \ \forall k \in N \setminus \{i, j\}.$$

Then since $N_i(L_{-j}(g)) = N_j(L_{-i}(g))$ we have $(L_i(g) + L_j(g))_{\pi^{ij}} = L_i(g) + L_j(g)$. Moreover, since $\pi^{ij}(k) = k\ \ \forall k \in N \setminus \{i, j\}$, we have $(g - (L_i(g) + L_j(g)))_{\pi^{ij}} = g - (L_i(g) + L_j(g))$. Hence,

$$\begin{aligned} g_{\pi^{ij}} &= [(g - (L_i(g) + L_j(g))) + (L_i(g) + L_j(g))]_{\pi^{ij}} \\ &= [g - (L_i(g) + L_j(g))]_{\pi^{ij}} + [L_i(g) + L_j(g)]_{\pi^{ij}} \\ &= [g - (L_i(g) + L_j(g))] + [L_i(g) + L_j(g)] = g \end{aligned}$$

Take now any $k \in N \backslash \{i, j\}$ and define $\tilde{g} = g + ik$. Anonymity then yields

$$u_i(g + ik) = u_{\pi^{ij}(i)}(g_{\pi^{ij}} + \{ik\}_{\pi^{ij}}) = u_j(g + jk).$$

Then it directly follows that

$$\Delta u_i(g + ik, ik) = u_i(g + ik) - u_i(g) = u_j(g + jk) - u_j(g) = \Delta u_j(g + jk, jk).$$

(iii) By the same arguments as in *(ii)* we get

$$u_k(g + ik) = u_{\pi^{ij}(k)}(g_{\pi^{ij}} + \{ik\}_{\pi^{ij}}) = u_k(g + jk).$$

and consequently

$$\Delta u_k(g + ik, ik) = u_k(g + ik) - u_k(g) = u_k(g + jk) - u_k(g) = \Delta u_j(g + jk, jk).$$

□

Proof of Lemma 2. (i) Suppose the empty network $g^\emptyset$ is not PS (otherwise there is nothing to show). Then by Definition 1, there exist $i, j \in N$ such that $0 < \Delta u_i(g^\emptyset + ij, ij)$ implying by Lemma 1, $0 < \Delta u_i(g^\emptyset + ij, ij)$ for all $i, j \in N$ since anonymity is assumed.

Let $g \in G \setminus \{g^\emptyset, g^N\}$, recall that $\eta_i(g) = |L_i(g)|$ and $\eta_{-i}(g) = |L_{-i}(g)|$ and suppose there exist a player $i \in N$ with $\kappa \eta_i(g) \leq \eta_{-i}(g)$ for some $\kappa \in \mathbb{N}$. This player has an incentive to add any link $ij \notin g$, if the empty network is not PS and strategic complements and κ-link monotonicity hold. To see this, partition the set $L_{-i}(g)$ into $\eta_i(g)+1$ disjoint subsets $l_0^{-i}, l_1^{-i}, l_2^{-i}, \dots l_{\eta_i(g)}^{-i} \subset L_{-i}(g)$ such that $l_0^{-i} \dot\cup l_1^{-i} \dot\cup \dots \dot\cup l_{\eta_i(g)}^{-i} = L_{-i}(g)$, and $|l_k^{-i}| = \kappa$ for all $k \in \{1, 2, \dots, \eta_i(g)\}$ which is possible since $\kappa\eta_i(g) \leq \eta_{-i}(g)$ implying $|l_0^{-i}| = \eta_{-i}(g) - \kappa\eta_i(g) \geq 0$ (l_0^{-i} is the empty set if $\eta_{-i}(g) = \kappa\eta_i(g)$). Labelling i's links by $L_i(g) = \{ij_1, ..., ij_{\eta_i(g)}\}$ and letting $ij \in L_i(g^N - g)$, we get by strategic complements:

$$\begin{aligned}
&0 < \Delta u_i(g^\emptyset + ij, ij) \\
&\Rightarrow 0 < \Delta u_i(g^\emptyset + l_0^{-i} + ij, ij)
\end{aligned}$$

By repeatedly applying κ-link monotonicity, we then receive:

$$\begin{aligned}
&\Rightarrow 0 < \Delta u_i(g^\emptyset + l_0^{-i} + ij_1 + l_1^{-i} + ij, ij) \\
&\Rightarrow 0 < \Delta u_i(g^\emptyset + l_0^{-i} + ij_1 + ij_2 + l_1^{-i} + l_2^{-i} + ij, ij) \\
&\Rightarrow \dots \Rightarrow 0 < \Delta u_i\left(g^\emptyset + l_0^{-i} + \cup_{k=1}^{\eta_i(g)}\left(ij_k + l_k^{-i}\right) + ij, ij\right) = \Delta u_i\left(g + ij, ij\right)
\end{aligned}$$

Thus, if the empty network is not PS, then for every $g \in G$, any player $i \in N$ with $\kappa\eta_i(g) \leq \eta_{-i}(g)$ has an incentive to add any link (and, analogously, has no incentive to delete a link). This implies that for a network $g \in G \setminus \{g^\emptyset, g^N\}$ to be PS, the set of players $E^\kappa(g) := \{i \in N | \kappa\eta_i(g) \leq \eta_{-i}(g)\}$ has to be completely connected.

(ii) Suppose the complete network g^N is not PS (otherwise there is nothing to show). Then by Definition 1, there exist $i, j \in N$ such that $\Delta u_i(g^N, ij) < 0$ implying by Lemma 1, $\Delta u_i(g^N, ij) < 0$ for all $i, j \in N$, $i \neq j$, since anonymity is assumed.

Let $g \in G \setminus \{g^\emptyset, g^N\}$ and suppose there exist a player $i \in N$ with $\eta_i(g) > 0$ and $\kappa\eta_i(g) \geq \eta_{-i}(g) - (n-1)(\frac{n-2}{2} - \kappa)$ for some $\kappa \in \mathbb{N}$. This player has an incentive to delete any link $ij \in g$, if the complete network is not PS and strategic complements and κ-link monotonicity hold. To see this, we can (analogously to above) partition the set $L_{-i}(g^N - g)$ into $(n - \eta_i(g))$ disjoint subsets $l_0^{-i}, l_1^{-i}, l_2^{-i}, \dots l_{n-1-\eta_i(g)}^{-i} \subset L_{-i}(g^N - g)$ such that $l_0^{-i} \dot\cup l_1^{-i} \dot\cup \dots \dot\cup l_{n-1-\eta_i(g)}^{-i} = L_{-i}(g^N - g)$, and $|l_k^{-i}| = \kappa$ for all $k \in \{1, 2, \dots, n-1-\eta_i(g)\}$ which is possible since

$$\eta_{-i}(g^N) - \eta_{-i}(g) = \tfrac{(n-1)(n-2)}{2} - \eta_{-i}(g) \geq \kappa(n - 1 - \eta_i(g)) = \kappa(\eta_i(g^N) - \eta_i(g))$$

where the inequality follows from $\kappa\eta_i(g) \geq \eta_{-i}(g) - (n-1)(\frac{n-2}{2} - \kappa)$. Moreover, $|l_0^{-i}| = \eta_{-i}(g) - (n-1)(\frac{n-2}{2} - \kappa) - \kappa\eta_i(g) \geq 0$ (l_0^{-i} is the empty set if $\kappa\eta_i(g) \geq \eta_{-i}(g) - (n-1)(\frac{n-2}{2} - \kappa)$ is satisfied with equality). Labelling i's links outside of g by $L_i(g^N - g) = \{ij_1, ..., ij_{n-1-\eta_i(g)}\}$ and letting $ij \in L_i(g)$, we get by strategic complements:

$$\begin{aligned}
&0 > \Delta u_i(g^N, ij) \\
&\Rightarrow 0 > \Delta u_i(g^N - l_0^{-i}, ij)
\end{aligned}$$

By repeatedly applying κ-link monotonicity, we then receive:

$$\begin{aligned}
&\Rightarrow 0 > \Delta u_i(g^N - l_0^{-i} - ij_1 - l_1^{-i}, ij) \\
&\Rightarrow 0 > \Delta u_i(g^N - l_0^{-i} - ij_1 - ij_2 - l_1^{-i} - l_2^{-i}, ij) \\
&\Rightarrow \ldots \Rightarrow 0 > \Delta u_i\left(g^N - l_0^{-i} - \cup_{k=1}^{n-1-\eta_i(g)}\left(ij_k + l_k^{-i}\right), ij\right) = \Delta u_i(g, ij)
\end{aligned}$$

Thus if the complete network is not PS, then for every $g \in G$, any player $i \in N$ with $\kappa\eta_i(g) \geq \eta_{-i}(g) - (n-1)(\frac{n-2}{2} - \kappa)$ has an incentive to delete any link (and -analogously- has no incentive to add a link). This implies that for a network $g \in G \setminus \{g^\emptyset, g^N\}$ to be PS, either $\eta_i(g) = 0$ or $\kappa\eta_i(g) < \eta_{-i}(g) - (n-1)(\frac{n-2}{2} - \kappa)$. □

Proof of Theorem 1. To show the first part of the Theorem, let anonymity, strategic complements, and $\hat{\kappa}$-link monotonicity be satisfied with $\hat{\kappa} := \lfloor n/2 \rfloor - 1$. For some $i \in N$ and $l_i \subseteq L_i(g^N)$ order the links arbitrarily $l_i = \{ij_1, \ldots, ij_{|l_i|}\}$ and denote by $l_i^k = \{ij_1, \ldots, ij_k\}$ the first k links in this order, $0 \leq k \leq |l_i|$ with the convention that $l_i^0 = \emptyset$. We then get,

$$\begin{aligned}
\Delta u_i(g^N, l_i) = u_i(g) - u_i(g - l_i) &= \sum_{k=0}^{|l_i|-1} \left(u_i(g - l_i^k) - u_i(g - l_i^{k+1})\right) \\
&= \sum_{k=0}^{|l_i|-1} \left(u_i(g - l_i^k) - u_i(g - l_i^k - ij_{k+1})\right) = \sum_{k=0}^{|l_i|-1} \Delta u_i(g^N - l_i^k, ij_{k+1}) \quad (19)
\end{aligned}$$

Note that in the complete network g^N, we have $\eta_i(g^N) = n-1$ and $\eta_{-i}(g^N) = \frac{(n-1)(n-2)}{2}$ and thus in the network $g^N - l_i^k$ we have $\eta_i(g^N - l_i^k) = n - k - 1$ and $\eta_{-i}(g^N - l_i^k) = \frac{(n-1)(n-2)}{2}$ implying $\hat{\kappa}\eta_i(g^N - l_i^k) \leq \eta_{-i}(g^N - l_i^k)$ for all $k \geq 0$. Thus by the first part of the proof of Lemma 2, if the empty network is not P(N)S, then every summand of (19) is non-negative implying that $\Delta u_i(g^N, l_i) \geq 0$, i.e. the complete network is PNS since $l_i \subset L_i(g^N)$ and $i \in N$ where chosen arbitrarily. Thus, the complete network is also PS.

If, on the other hand, the complete network is not PS, then take $g \in G \setminus \{g^\emptyset, g^N\}$ arbitrarily and let $\zeta \in \arg\max_{j \in N} \eta_j(g)$. Since $g \neq g^\emptyset$, we have $\eta_\zeta(g) > 0$. Recall that $\eta_{-\zeta}(g) = |L_{-\zeta}(g)| = |g| - \eta_\zeta(g) = \sum_{j \in N} \frac{\eta_j}{2} - \eta_\zeta$. By construction, we therefore get

$$\begin{aligned}
\eta_{-\zeta}(g) - (n-1)(\tfrac{n}{2} - 1 - \hat{\kappa}) &= \sum_{j \in N} \tfrac{\eta_j(g)}{2} - \eta_\zeta(g) - (n-1)(\tfrac{n}{2} - \lfloor \tfrac{n}{2} \rfloor) \\
&\leq \left(\tfrac{n}{2} - 1\right)\eta_\zeta(g) - (n-1)(\tfrac{n}{2} - \lfloor \tfrac{n}{2} \rfloor) \\
&= \left(\lfloor \tfrac{n}{2} \rfloor - 1\right)\eta_\zeta(g) - (n - 1 - \eta_\zeta(g))(\tfrac{n}{2} - \lfloor \tfrac{n}{2} \rfloor) \\
&\leq \left(\lfloor \tfrac{n}{2} \rfloor - 1\right)\eta_\zeta(g) = \hat{\kappa}\eta_\zeta(g).
\end{aligned}$$

This contradicts the necessary condition and, hence, by Lemma 2(ii), g is not link deletion proof and therefore cannot be PS (and hence also not PNS). Stability of the empty network then follows from the fact that no one-link network is link deletion proof. Therefore the empty network is uniquely P(N)S, if the complete networks fails to be PS.

For the second part, let $\hat{\kappa}$-link monotonicity be satisfied (and strategic complements and anonymity) with $\bar{\kappa} := \left\lfloor \sqrt{2(n-1)(n-2)} \right\rfloor - (n-1)$ and suppose that the empty network is not P(N)S. Take $g \in G \setminus \{g^{\emptyset}, g^N\}$ arbitrarily. To show that g cannot be PS, it suffices to show by Lemma 2 that $E := E^{\bar{\kappa}}(g) = \{i \in N | \bar{\kappa}\eta_i(g) \leq \eta_{-i}(g)\}$ cannot be completely connected (for ease of notation we drop any dependence on g when the reference is clear and simply write η_i instead of $\eta_i(g)$). Suppose to the contrary that E is completely connected and let $E^C := N \setminus E = \{j \in N | \bar{\kappa}\eta_j(g) > \eta_{-j}(g)\}$ be the complement of E. By definition, $\eta_i(g) < \eta_j(g)$ for all $i \in E$, $j \in E^C$.

Since E is completely connected and $\eta_i(g) < \eta_j(g)$ for all $i \in E$, $j \in E^C$ we first get that $|E| < |E^C|$. To see this, let $\iota \in \arg\min_{j \in E^C} \eta_j(g)$ and $\zeta \in \arg\max_{i \in E} \eta_i(g)$ which implies $\eta_\zeta(g) < \eta_\iota(g)$. Note that $\sum_{j \in E^C} \eta_j(g) \geq |E^C|\eta_\iota(g)$ implying that there are at least $(\eta_\iota(g) - (|E^C| - 1))|E^C| \leq L(E, E^C)$ links from players in E^C to players in E where $L(S, S') = \{ij \in g | i \in S, j \in S'\}$ for $S, S' \subseteq N$. Since E is completely connected, there are at most $(\eta_\zeta(g) - (|E| - 1))|E| \geq L(E, E^C)$ links from players in E to players in E^C. Since $|E^C| = n - |E|$, this implies

$$(\eta_\iota(g) - ((n - |E|) - 1)(n - |E|) - (\eta_\zeta(g) - (|E| - 1))|E| \leq 0 \tag{20}$$

Note that for $|E| = \frac{n}{2}$ the left-hand side of (20) equals $\frac{n}{2}(\eta_\iota(g) - \eta_\zeta(g)) > 0$ and the derivative of the left-hand side of (20) with respect to $|E|$ can be calculated to be $2(n-1) - (\eta_\iota(g) + \eta_\zeta(g)) \geq 0$ (since $\eta_i(g) \leq n - 1$ for all $i \in N$). Hence, we must have $|E| < \frac{n}{2}$ for (20) to hold and therefore $|E^C| > \frac{n}{2} > |E|$.

Using again the notation $L(S, S')$ for $S, S' \subseteq N$, note that the set of links in g can be divided into the set of links within E, denoted by $L(E, E)$, the set of links within E^C, denoted by $L(E^C, E^C)$, and the set of links across both sets, denoted by $L(E, E^C)$. Hence,

$$|g| = |L(E,E)| + \big(|L(E, E^C)| + |L(E^C, E^C)|\big) = \tfrac{|E|(|E|-1)}{2} + \left(\sum_{j \in E^C} \eta_j(g) - |L(E^C, E^C)|\right)$$

where the last equality follows since E is completely connected and the links $ij \in g$ with $i, j \in E^C$ are counted twice in $\sum_{j \in E^C} \eta_j(g)$. Letting, as above, $\iota \in \arg\min_{j \in E^C} \eta_j(g)$, and recalling that $\eta_{-\iota}(g) = |g| - \eta_\iota(g)$, we get that

$$\frac{\eta_{-\iota}(g)}{\eta_\iota(g)} = \frac{1}{\eta_\iota(g)} \left(\frac{|E|(|E|-1)}{2} + \sum_{j \in E^C} \eta_j(g) - |L(E^C, E^C)| - \eta_\iota(g) \right)$$

By construction, $\eta_\iota(g) \leq \eta_j(g)$ for all $j \in E^C$. Hence,

$$\frac{\eta_{-\iota}(g)}{\eta_\iota(g)} \geq \frac{1}{\eta_\iota(g)} \left(\frac{|E|(|E|-1)}{2} + |E^C|\eta_\iota(g) - |L(E^C, E^C)| - \eta_\iota(g) \right) \tag{21}$$

Fixing $\eta_\iota(g)$, the right-hand side of (21) is clearly minimal if $|L(E^C, E^C)|$ is maximal. Thus for $\eta_\iota(g) \geq |E^C| - 1$, a lower bound for the right-hand side of (21) is obtained if E^C is completely connected, i.e. $|L(E^C, E^C)| = \frac{|E^C|(|E^C|-1)}{2}$. Thus for $\eta_\iota(g) \geq |E^C| - 1$,

$$\frac{\eta_{-\iota}(g)}{\eta_\iota(g)} \geq \frac{1}{\eta_\iota(g)} \left(\frac{|E|(|E|-1)}{2} - \frac{|E^C|(|E^C|-1)}{2} \right) + (|E^C| - 1) \tag{22}$$

If on the other hand, we have $\eta_\iota(g) \leq |E^C| - 1$, a lower bound for the right-hand side of (21) is again obtained if E^C is maximal meaning in this case that $|L(E^C, E^C)| = \frac{|E^C|\eta_\iota(g)}{2}$ (and $|L(E, E^C)| = 0$). Hence for $\eta_\iota(g) \leq |E^C| - 1$,

$$\frac{\eta_{-\iota}(g)}{\eta_\iota(g)} \geq \frac{1}{\eta_\iota(g)} \left(\frac{|E|(|E|-1)}{2} + \eta_\iota(g) \frac{|E^C| - 2}{2} \right) \tag{23}$$

Note that the right-hand side of (22) is increasing in $\eta_\iota(g)$ since $0 < |E| < |E^C|$ and the right hand side of (23) is decreasing in $\eta_\iota(g)$. Thus, setting $\eta_\iota(g)$ minimal in the case $\eta_\iota(g) \geq |E^C| - 1$ and setting $\eta_\iota(g)$ maximal in the case $\eta_\iota(g) \leq |E^C| - 1$ such that both are achieved for $\eta_\iota(g) = \left|E^C\right| - 1 = n - |E| - 1$ obtains,

$$\frac{\eta_{-\iota}(g)}{\eta_\iota(g)} \geq \frac{1}{2\eta_\iota(g)} \left((n - \eta_\iota(g) - 1)(n - \eta_\iota(g) - 2) + (\eta_\iota(g) - 1)\eta_\iota(g) \right) \tag{24}$$

Now, minimizing the right-hand side of (24) with respect to $\eta_\iota(g)$ delivers $\eta^*_\iota(g) = \sqrt{\frac{n^2-3n+2}{2}}$ as the global minimizer. Thus,

$$\begin{aligned} \frac{\eta_{-\iota}(g)}{\eta_\iota(g)} &\geq \tfrac{1}{2\eta^*_\iota(g)} \left((n - \eta^*_\iota(g) - 1)(n - \eta^*_\iota(g) - 2) + \eta^*_\iota(g)(\eta^*_\iota(g) - 1) \right) \\ &= \sqrt{2(n-1)(n-2)} - (n-1) \\ &\geq \left\lfloor \sqrt{2(n-1)(n-2)} \right\rfloor - (n-1) = \bar{\kappa}. \end{aligned}$$

Which contradicts our assumption that $\iota \in E^C$. Thus E cannot be completely connected which means there exist two players $i, j \in E$ which have a strict incentive to form a link contradicting pairwise stability. We conclude that no network other than the complete network can be PS. Note that $\hat{\kappa}(n) = \lfloor \frac{n}{2} - 1 \rfloor \leq \lfloor \sqrt{2(n-1)(n-2)} - (n-1) \rfloor = \bar{\kappa}(n)$ for all $n \in \mathbb{N}$ (simply compare the arguments of the floor functions), implying with part (i) that g^N is PNS. We have hence shown that if the empty network is not P(N)S then the complete network is uniquely PS and therefore uniquely PNS. □

Proof of Corollary 1. First note that by convexity and strategic complements,

$$\Delta u_i(g, ij)(\geq) > 0 \;\Rightarrow\; \Delta u_i(g + ik, ij)(\geq) > 0 \;\Rightarrow\; \Delta u_i(g + l_{-i} + ik, ij)(\geq) > 0$$

for all $l_{-i} \in L_{-i}(g^N - g)$ with $0 \leq |l_i|$, i.e. 0-link monotonicity is satisfied. The statement is then directly implied by Theorem 1 □

Proof of Theorem 2. Suppose to the contrary that there exists a PS network g which is not a nested split graph. Then by definition there exists players i, j with $\eta_i(g) \leq \eta_j(g)$ but $N_i(g) \not\subseteq N_j(g) \cup \{j\}$. Hence, there exists $k \in N$ with $ik \in g$ and $jk \notin g$. Since g is assumed to be stable, we have $\Delta u_i(g, ik) \geq 0$ and $\Delta u_k(g, ik) \geq 0$. Then however by SPP and NI,

$$\Delta u_k(g, ik) \geq 0 \;\Rightarrow\; \Delta u_k(g + jk, jk) \geq 0 \;\Rightarrow\; \Delta u_k(g + jk, jk) > 0,$$

and further by AC,

$$\Delta u_i(g, ik) \geq 0 \;\Rightarrow\; \Delta u_j(g + jk, jk) \geq 0,$$

contradicting pairwise stability. □

Proof of Proposition 1. First note that for the local spillover utility function u^{LS}, marginal utility of player i from the link $ij \in g$ in a network $g \in G$ is given by

$$\Delta u_i^{LS}(g, ij) = f_1(\eta_i(g)) - f_1(\eta_i(g) - 1) + f_2(\eta_j(g)) - f_3(\eta_j(g) - 1) - c$$

If u_i satisfies ordinal convexity (CV), then we must have

$$\begin{aligned} &0 \leq f_1(\eta_i) - f_1(\eta_i - 1) + f_2(\eta_j) - f_3(\eta_j - 1) - c \\ \Rightarrow\, &0 \leq f_1(\eta_i') - f_1(\eta_i' - 1) + f_2(\eta_j) - f_3(\eta_j - 1) - c \quad \text{for all } \eta_i' \geq \eta_i \end{aligned}$$

Similarly, ordinal strategic complements (SC) implies

$$\begin{aligned} &0 \leq f_1(\eta_i) - f_1(\eta_i - 1) + f_2(\eta_j) - f_3(\eta_j - 1) - c \\ \Rightarrow\, &0 \leq f_1(\eta_i) - f_1(\eta_i - 1) + f_2(\eta_j') - f_3(\eta_j' - 1) - c \quad \text{for all } \eta_j' \geq \eta_j,\ j \neq i \end{aligned}$$

To show that strong preference for prominence (SPP) is implied by CV and SC, consider $g \in G$ and let $j, k \in N$ such that $\eta_j(g) \leq \eta_k(g)$, $ij \in g$, and $ik \notin g$. Then, we get by SC and CV

$$\begin{aligned} 0 \leq \Delta u_i^{LS}(g, ij) \quad &\Leftrightarrow \quad 0 \leq f_1(\eta_i(g)) - f_1(\eta_i(g) - 1) + f_2(\eta_j(g)) - f_3(\eta_j(g) - 1) - c \\ &\overset{CV}{\Rightarrow} \quad 0 \leq f_1(\eta_i(g) + 1) - f_1(\eta_i(g)) + f_2(\eta_j(g)) - f_3(\eta_j(g) - 1) - c \\ &\overset{SC}{\Rightarrow} \quad 0 \leq f_1(\eta_i(g) + 1) - f_1(\eta_i(g)) + f_2(\eta_k(g) + 1) - f_3(\eta_k(g)) - c \\ &\Leftrightarrow \quad 0 \leq \Delta u_i^{LS}(g + ik, ik) \end{aligned}$$

Thus, u_i^{LS} also satisfies SPP if CV and SC are satisfied.

To show that anonymous convexity (AC) is also implied by CV and SC, consider $g \in G$ and let $i, j, k \in N$ such that $\eta_i(g) \leq \eta_j(g)$, $ik \in g$, and $jk \notin g$. Then, we get by SC and CV

$$\begin{aligned} 0 \leq \Delta u_i^{LS}(g, ik) \quad &\Leftrightarrow \quad 0 \leq f_1(\eta_i(g)) - f_1(\eta_i(g) - 1) + f_2(\eta_k(g)) - f_3(\eta_k(g) - 1) - c \\ &\overset{CV}{\Rightarrow} \quad 0 \leq f_1(\eta_j(g) + 1) - f_1(\eta_j(g)) + f_2(\eta_k(g)) - f_3(\eta_k(g) - 1) - c \\ &\overset{SC}{\Rightarrow} \quad 0 \leq f_1(\eta_j(g) + 1) - f_1(\eta_j(g)) + f_2(\eta_k(g) + 1) - f_3(\eta_k(g)) - c \\ &\Leftrightarrow \quad 0 \leq \Delta u_j^{LS}(g + jk, jk) \end{aligned}$$

Thus, u_i^{LS} also satisfies AC if CV and SC are satisfied. □

Proof of Proposition 2. First, we show (more generally) by induction over $k \in \mathbb{N}$ that $(\mathbf{A}' + \mathbf{B})^k - (\mathbf{A}')^k \geq (\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k$ if for all nonnegative $n \times n$ matrixes $\mathbf{A}, \mathbf{A}', \mathbf{B}$ with $\mathbf{A} \leq \mathbf{A}'$, where for matrices $\mathbf{A}$ and $\mathbf{B}$ we write $\mathbf{A} \leq \mathbf{B}$, if and only if the entries satisfy $a_{ij} \leq b_{ij}$ for all $i, j \in N$.

For $k = 1$, we have the assertion satisfied with equality,

$$(\mathbf{A}' + \mathbf{B})^1 - (\mathbf{A}')^1 = \mathbf{B} = (\mathbf{A} + \mathbf{B})^1 - (\mathbf{A})^1$$

Now suppose that the assertion holds for some $k \in \mathbb{N}$. Then,

$$\begin{aligned}
&(\mathbf{A}' + \mathbf{B})^k - (\mathbf{A}')^k \geq (\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k \\
\Rightarrow\ &(\mathbf{A}' + \mathbf{B}) \left((\mathbf{A}' + \mathbf{B})^k - (\mathbf{A}')^k \right) \geq (\mathbf{A} + \mathbf{B}) \left((\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k \right) \\
\Leftrightarrow\ &(\mathbf{A}' + \mathbf{B})^{k+1} - (\mathbf{A}')^{k+1} - \mathbf{B} (\mathbf{A}')^k \geq (\mathbf{A} + \mathbf{B})^{k+1} - (\mathbf{A})^{k+1} - \mathbf{B} (\mathbf{A})^k \\
\Rightarrow\ &(\mathbf{A}' + \mathbf{B})^{k+1} - (\mathbf{A}')^{k+1} \geq (\mathbf{A} + \mathbf{B})^{k+1} - (\mathbf{A})^{k+1}
\end{aligned}$$

where we repeatedly used that $\mathbf{A}' \geq \mathbf{A}$. Thus for $g, g' \in G$, with $g \subset g'$ and $ij \notin g'$ and recalling that $\mathbf{A}(g)$ denotes the adjacency matrix of g, we can set $\mathbf{A} := \mathbf{A}(g)$, $\mathbf{A}' := \mathbf{A}(g')$ and $\mathbf{B} := \mathbf{A}(\{ij\})$ implying $\mathbf{A} \leq \mathbf{A}'$. We then obtain

$$\begin{aligned}
\mathbf{b}(g' + ij, \delta) - \mathbf{b}(g', \delta) &= \sum_{k=0}^{n} \delta^k \left((\mathbf{A}' + \mathbf{B})^k - (\mathbf{A}')^k \right) \mathbf{1} \\
&\geq \sum_{k=0}^{n} \delta^k \left((\mathbf{A} + \mathbf{B})^k - (\mathbf{A})^k \right) \mathbf{1} = \mathbf{b}(g + ij, \delta) - \mathbf{b}(g, \delta).
\end{aligned}$$

Since f is an increasing and convex function and $b_i(g' + ij, \delta) \geq b_i(g', \delta) \geq 0$ and $b_i(g + ij, \delta) \geq b_i(g, \delta) \geq 0$, we then have,

$$\begin{aligned}
u_i^{BC}(g' + ij) - u_i^{BC}(g') &= f(b_i(g' + ij, \delta)) - f(b_i(g', \delta)) - c \\
&\geq f(b_i(g + ij, \delta)) - f(b_i(g, \delta)) - c \\
&= u_i^{BC}(g + ij) - u_i^{BC}(g),
\end{aligned}$$

Letting g' and g being such that $g' - g \subset L_{-i}(g^N - g)$ we obtain the (cardinal) strategic complements property and letting g' and g being such that $g' - g \subset L_i(g^N - g)$ we obtain (cardinal) convexity. □

Proof of Proposition 3. Remember that

$$u_i^{BC} = f(b_i(g)) - \eta_i(g)c = f(\mathbf{e}_i'(\sum_{t=0}^{\infty} \delta^t \mathbf{A}^t)\mathbf{1}) - \eta_i(g)c,$$

with $\mathbf{A}$ being the adjacency matrix of network g and $\mathbf{e}_i'$ the transpose of the i-th unit vector. Take some players $i, j, k \in N$ and a network g such that $ij \in g$, $ik \notin g$ and $\eta_j(g) \leq \eta_k(g)$. We get,

$$\begin{aligned}\Delta u_i^{BC}(g+ik, ik) =& f(b_i(g+ik)) - f(b_i(g)) - c \\ =& f(b_i(g) + \delta + \delta^2(\eta_k(g)+1) + \mathbf{e}_i'(\sum_{t=3}^{\infty} \delta^t((\mathbf{A}+\mathbf{A}(ik))^t - \mathbf{A}^t)\mathbf{1}) \\ & - f(b_i(g)) - c, \\ \geq & f(b_i(g) + \delta + \delta^2(\eta_k(g)+1)) - f(b_i(g)) - c,\end{aligned}$$

We can find an upper bound for the marginal utility of deleting j by considering utility of the complete network from order 3 on,

$$\begin{aligned}\Delta u_i^{BC}(g, ij) \leq & f(b_i(g) + \delta + \delta^2(\eta_j(g)) + \sum_{t=3}^{\infty} \delta^t \eta_j(g)(n-1)^{t-2}) - f(b_i(g)) - c, \\ =& f(b_i(g) + \delta + \delta^2\eta_j(g) + \delta^2\eta_j(g)(\sum_{t=0}^{\infty} \delta^t (n-1)^t - 1)) - f(b_i(g)) - c \\ =& f(b_i(g) + \delta + \tfrac{\delta^2\eta_j(g)}{1-\delta(n-1)}) - f(b_i(g) - c.\end{aligned}$$

Now, from $0 < \delta < \frac{1}{(n-1)^2}$, we get $\frac{\eta_j(g)}{1-\delta(n-1)} < \frac{\eta_j(g)}{1-\frac{1}{(n-1)}} = \eta_j + \frac{\eta_j}{n-2}$. Since k is not connected to i in g, we have $\eta_j(g) \leq \eta_k(g) \leq n-2$. Thus, $\eta_j + \frac{\eta_j}{n-2} \leq \eta_k + \frac{n-2}{n-2} = \eta_k + 1$, implying

$$f(b_i(g) + \delta + \delta^2 \tfrac{\eta_j(g)}{1-\delta(n-1)}) - f(b_i(g)) - c < f(b_i(g) + \delta + \delta^2(\eta_k(g)+1)) - f(b_i(g)) - c,$$

since f is an increasing function. We conclude that for $0 < \delta < \frac{1}{(n-1)^2}$ the following holds

$$\eta_j(g) \leq \eta_k(g) \;\Rightarrow\; \Delta u_i^{BC}(g, ij) < \Delta u_i^{BC}(g+ik, ik).$$

implying that u^{BC} satisfies SPP.

Letting $\eta_i(g) \leq \eta_j(g)$ and $\Delta u_i^{BC}(g, ik) \geq 0$, we get for $0 < \delta < \frac{1}{(n-1)^2}$ the same bounds on third order terms, implying analogously,

$$\begin{aligned}0 \leq \Delta u_i^{BC}(g, ik) \leq & \delta + \delta^2\eta_k(g) + \sum_{t=3}^{\infty} \delta^t \eta_k(g)(n-1)^{t-2}) - c \\ < & \delta + \delta^2(\eta_k(g)+1) - c \leq \Delta u_j^{BC}(g+jk, jk),\end{aligned}$$

thus u^{BC} also satisfies AC. □

Proof of Proposition 4. First note that by Hiller (2017) Proposition 1, $\mathbf{x}^*(g) \leq \mathbf{x}^*(g')$ for all $g \subseteq g'$. We get $x_i^{ij}(g+ij)$ and $x_j^{ij}(g+ij)$ as a solution to the system of two equations $x_i^{ij}(g+ij) = \bar{x}\left(g+ij, (x_j^{ij}(g+ij), \mathbf{x}_{-ij}^*(g))\right)$ and $x_j^{ij}(g+ij) =$

$\bar{x}\left(g+ij,(x_i^{ij}(g+ij),\mathbf{x}^*_{-ij}(g))\right)$ where $\mathbf{x}^*_{-ij}(g)$ is the vector obtained by deleting the entries i and j from $\mathbf{x}^*$. Now, since the best reply function $\bar{x}$ is strictly increasing and for all $g\subseteq g'$ and $\mathbf{x}^*_{-ij}(g)\le \mathbf{x}^*_{-ij}(g')$, we also get $x_j^{ij}(g+ij)<x_j^{ij}(g'+ij)$ for all $g\subseteq g'$. Since the value function v is increasing and convex we then get for $g\subseteq g'$,

$$\begin{aligned}
\Delta^d u_i(g'+ij,ij) &= u_i(g'+ij,\mathbf{x}^{ij}(g'+ij))-u_i(g,\mathbf{x}^*(g'))\\
&= v(\sum_{k\in N_i(g')} x_k^*(g'+ij)+x_j^{ij}(g'+ij))-v(\sum_{k\in N_i(g')} x_k^*(g'))\\
&\underset{v''\ge 0}{\ge} v(\sum_{k\in N_i(g)} x_k^*(g+ij)+x_j^{ij}(g'+ij))-v(\sum_{k\in N_i(g)} x_k^*(g))\\
&\underset{v'>0}{\ge} v(\sum_{k\in N_i(g)} x_k^*(g+ij)+x_j^{ij}(g+ij))-v(\sum_{k\in N_i(g)} x_k^*(g))\\
&= \Delta^d u_i(g+ij,ij).
\end{aligned}$$

Since $g,g'\in G$ with $g\subseteq g'$ were chosen arbitrarily, we obtain the strategic complements property by restricting to g,g' with $g'-g\subseteq L_{-i}(g')$ and we obtain the convexity property by restricting to g,g' with $g'-g\subseteq L_i(g')$. □

Proof of Theorem 3. Consider a dominant group network g_s^{dg} and denote the set of completely connected players by S (of size $|S|=s$). Suppose that g_s^{dg} is not Nash deletion proof, i.e. there exists a player $i\in S$ such that $\Delta u_i(g_s^{dg},l_i)<0$. for some $l_i\subseteq L_i(g_s^{dg})$. Denote $l_i^c := L_i(g_s^{dg})-l_i$.

Let $g_{s-1}^{dg} := g_s^{dg}-L_i(g_s^{dg})$ be the network obtained after deleting all of player i's links in g_s^{dg} which is again a dominant group network with dominant group $S\setminus\{i\}$ of size $s-1$. Suppose $\Delta u_i(g_{s-1}^{dg}+ij,ij)\ge 0$ for some $j\in N$. Then, by WPP, $\Delta u_i(g_{s-1}^{dg}+ij,ij)\ge 0$ for all $j\in S\setminus\{i\}$ (or equivalently: for all $ij\in L_i(g_s^{dg})$). Convexity implies, $\Delta u_i(g_{s-1}^{dg}+l_i^c+im,im)\ge 0$ for all $im\in l_i$ since $l_i\subseteq L_i(g_s^{dg})$. Order the links arbitrarily $l_i=\{ij_1,\dots,ij_{|l_i|}\}$ and denote by $l_i^k=\{ij_1,\dots,ij_k\}$ the first k links in this order, $0\le k\le |l_i|$ with the convention that $l_i^0=\emptyset$. We then get (similarly to the proof of the proof of Theorem 1), $\Delta u_i(g_s^{dg},l_i)=\sum_{k=0}^{|l_i|-1}\Delta u_i(g_s^{dg}-l_i^k,ij_{k+1})$. Now, since $\left(g_{s-1}^{dg}+l_i^c\right)\subseteq\left(g_s^{dg}-l_i^k\right)$ for all $k=0,\dots|l_i|$ and $\Delta u_i(g_{s-1}^{dg}+l_i^c+im,im)\ge 0$ for all $im\in l_i$, convexity implies $\Delta u_i(g_s^{dg},l_i)=\sum_{k=0}^{|l_i|-1}\Delta u_i(g_s^{dg}-l_i^k,ij_{k+1})\ge 0$ hence contradicting the assumption that g_s^{dg} is not Nash deletion proof.

Thus we cannot have that there exists a $j\in N$ such that $\Delta u_i(g_{s-1}^{dg}+ij,ij)\ge 0$. Applying anonymity, we then get for any $k\in S^C$, $\Delta u_k(g_{s-1}^{dg}+kl,kl)<0$ for all $l\in N$ since $N_i(g_{s-1}^{dg})=N_k(g_{s-1}^{dg})=\emptyset$. Therefore, no isolated player has an incentive to form a link, implying that the network g_{s-1}^{dg} is link addition proof.

We have, hence, shown that if g_s^{dg} is not Nash deletion proof, then g_{s-1}^{dg} is link addition proof. Now, since the complete network ($s=n$) is trivially link addition proof and the empty network ($s=1$) is trivially (Nash) deletion proof, there must exists a $1\le s\le n$ such that g_s^{dg} is link addition proof and Nash deletion proof and, hence, PNS. □

Proof of Theorem 4. Suppose a profile of utility functions satisfies ordinal convexity (CV), independence of network position (INP), and no-indifference (NI). Consider a network $g \in G$ which is not of dominant group structure meaning that there exists two players $i, j \in N$ with $N_i(g), N_j(g) \neq \emptyset$ such that $ij \notin g$. Suppose to the contrary that g is PS. For $k \in N_i(g) \neq \emptyset$ we then get $0 \leq \Delta u_i(g, ik)$. CV then implies $0 \leq \Delta u_i(g + ij, ik)$. Together with INP, this implies $0 \leq \Delta u_i(g + ij, ij)$ and by NI, we get $0 < \Delta u_i(g + ij, ij)$. Analogously we get $0 < \Delta u_j(g + ij, ij)$, contradicting that g is PS. $\square$

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