Towards closed strings as single-valued open strings at genus one

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We relate the low-energy expansions of world-sheet integrals in genus-one amplitudes of open- and closed-string states. The respective expansion coefficients are elliptic multiple zeta values in the open-string case and non-holomorphic modular forms dubbed "modular graph forms" for closed strings. By inspecting the differential equations and degeneration limits of suitable generating series of genus-one integrals, we identify formal substitution rules mapping the elliptic multiple zeta values of open strings to the modular graph forms of closed strings. Based on the properties of these rules, we refer to them as an elliptic single-valued map which generalizes the genus-zero notion of a single-valued map acting on multiple zeta values seen in tree-level relations between the open and closed string.

Contents

1 Introduction

One-loop amplitudes in string theories are computed from integrals over moduli spaces of punctured genus-one world-sheets. For open and closed strings, the punctures are integrated over a cylinder boundary and the entire torus, respectively, which is often done in a low-energy expansion, i.e. order by order in the inverse string tension α'. The coefficients of such α'-expansions involve special numbers and functions which have triggered fruitful interactions between number theorists, particle phenomenologists and string theorists. For instance, elliptic polylogarithms [1, 2] and *elliptic multiple zeta values* (eMZVs) [3] were identified to form the number-theoretic backbone of genus-one open-string integrals [4–6].

For the closed string, the analogous genus-one integrals involve non-holomorphic modular forms [7–9] dubbed *modular graph forms* (MGFs) [10,11] which inspired mathematical research lines [12–16]. As a unifying building block shared by open and closed strings, both eMZVs [3,17] and MGFs [10,11,18] can be reduced to iterated integrals over holomorphic Eisenstein series, or *iterated Eisenstein integrals*. Similar iterated integrals over holomorphic modular forms play a key role in recent progress on the evaluation of Feynman integrals [19–39]. As a main result of this work, we identify infinite families of closed-string integrals, where the appearance of iterated Eisenstein integrals is in precise correspondence with those in open-string α'-expansions.

More specifically, we give an explicit proposal for a single-valued map at genus one, mapping individual eMZVs to combinations of iterated Eisenstein integrals and their complex conjugates which should be contained in Brown's single-valued iterated Eisenstein integrals [13,14]. This generalizes the genus-zero result that the sphere integrals in closed-string tree amplitudes are single-valued versions of the disk integrals in open-string tree amplitudes [40–45]. The notion of single-valued periods [46,47] and single-valued integration [48,49] is very general, and in the case of multiple zeta values (MZVs) amounts to evaluating single-valued polylogarithms [50] at unit argument. While the single-valued map for the MZVs in tree-level α'-expansions has been pinpointed in [48,46], the genus-one studies of single-valued maps from mathematical [12–14] and physical [51,52] viewpoints[1] have not yet led to a consensus for the single-valued version of individual eMZVs.

Our proposal for single-valued eMZVs can be seen as a correspondence between integration cycles and antimeromorphic forms that is akin to Betti–deRham duality [57,58,49]. In a tree-level context, Betti–deRham duality relates the ordering of open-string punctures on a disk boundary to Parke–Taylor factors [40–45] – cyclic products of propagators $(\bar z_i - \bar z_j)^{-1}$ on the sphere. As a genus-one generalization, we spell out certain antielliptic (i.e. antimeromorphic and doubly-periodic) functions on the torus which will be referred to as the Betti–deRham duals[2] of integration cycles on a cylinder boundary.

It will be important to collect the various eMZVs and MGFs in generating series similar to those in [59–61, 18] as the genus-one single-valued map SV is most conveniently described

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[1]See [53–55, 16, 18, 56] for recent progress in identifying single-valued MZVs in the degeneration of dihedral MGFs from closed-string genus-one integrals at the cusp.

[2]We shall use this terminology at genus one even though we are not aware of any explicitly worked out notion of Betti–deRham duality beyond genus zero.
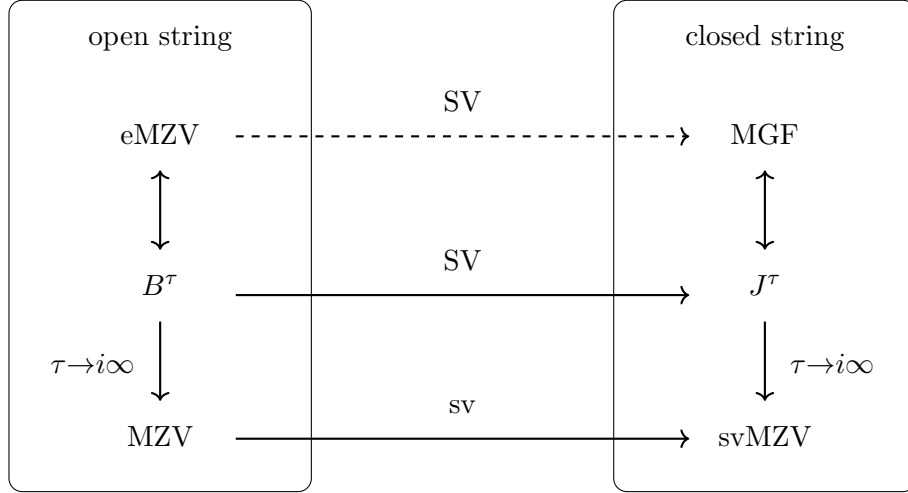
Figure 1: *Diagram illustrating the various pieces involved in constructing the proposal* SV *for an elliptic single-valued map with open-string quantities on its left-hand side and closed-string quantities on its right-hand side. The elliptic multiple zeta values (eMZV) of the open string are repackaged in a generating series B^τ whose degeneration limit $\tau \to i\infty$ contains multiple zeta values (MZV) as they also arise in tree-level open-string scattering. The closed-string modular graph forms (MGF) are organized in a generating series J^τ whose degeneration limit $\tau \to i\infty$ is expected to only contain single-valued multiple zeta values (svMZV) that are related to the MZV by the known single-valued map* sv. *Instead of attempting a direct construction of an elliptic* SV*-map from eMZV to MGF, we exploit the differential equations of the generating series B^τ and J^τ together with their boundary values from $\tau \to i\infty$ to describe the map* SV *at the level of generating series, see* (3.34). *From this one can extract the map* SV : eMZV → MGF *by inspecting individual orders in the α'-expansion.*

at the level of these generating series. The α'-expansion of genus-one closed-string integrals – using the techniques of [18] – yields an explicit form of the proposed single-valued map of the eMZVs in open-string integrals. The open-string punctures on a cylinder boundary are ordered according to the cycle which is Betti–deRham dual to the additional antielliptic functions in the closed-string integrand. For the purpose of this work, it will be sufficient to place all the open-string punctures on the same cylinder boundary which corresponds to planar genus-one amplitudes: As will be discussed in future work, single-valued non-planar open-string integrals yield the same collection of MGFs as the planar ones.

The main evidence for our proposal for an elliptic single-valued map stems from its consistency with holomorphic derivatives in the modular parameters τ of the surfaces and the degeneration $\tau \to i\infty$ of the torus to a nodal sphere. Compatibility with the holomorphic derivative is a simple consequence of recent results on the differential equations of genus-one open-string integrals [59, 60] and closed-string integrals [61] in τ. Our antielliptic integrands on the torus ensure that the closed-string differential equations match those of the open string apart from the disappearance of ζ_2 as expected from the single-valued map of MZVs. Moreover,

the antielliptic integrands are engineered such as to reproduce Parke–Taylor factors in the degeneration $\tau \to i\infty$. Hence, compatibility of the single-valued maps at genus zero and one is supported by the identification of sphere integrals as single-valued disk integrals [40–45]. The logic of our construction is illustrated in figure 1.

This work is organized as follows. We start by reviewing open- and closed-string integrals at genus zero and genus one as well as the basic definitions of single-valued MZVs, eMZVs and MGFs in section 2. Then, section 3 is dedicated to the modified open- and closed-string integrals as well as their relation through our proposed single-valued map at genus one. In particular, the central antielliptic integrands and the resulting proposal for an elliptic single-valued map can be found in sections 3.2 and 3.5, respectively. In section 4, we set the stage for generating explicit examples of single-valued eMZVs by introducing a new expansion method for open-string integrals over B-cycles and relating it to similar closed-string α'-expansions. This leads to the identifications of MGFs as single-valued eMZVs in section 5, where examples of the antielliptic integrands are related to earlier approaches to an elliptic single-valued map in the literature. In the concluding section 6, we comment on the relation of string amplitudes to the generating series of this work and further directions.

2 Review of genus-zero and genus-one integrals

In this section, we collect background material on world-sheet integrals at genus zero and one, including the genus-zero single-valued map, and review various definitions relevant to the single-valued map at genus one.

2.1 Genus-zero integrals

We briefly review the basic disk (open-string) and sphere (closed-string) integrals for genus-zero world-sheets and how they are related by the genus-zero single-valued map.

2.1.1 Definitions of disk and sphere integrals

Massless tree-level n-point amplitudes of the open superstring [62] and the open bosonic string [63] can be expanded in a basis of iterated integrals [64]

$$Z^{\text{tree}}(\gamma|\rho) = \int_{\mathfrak{D}(\gamma)} \frac{\left(\prod_{j=1}^{n} dz_j\right)}{\text{vol} \, \text{SL}_2(\mathbb{R})} \prod_{1 \le i < j}^{n} |z_{ij}|^{-s_{ij}} \text{PT}(\rho(1, 2, \ldots, n)) \tag{2.1}$$

over the boundary of a disk which we parametrize through the real line

$$\mathfrak{D}(\gamma) = \{z_j \in \mathbb{R}, \ -\infty < z_{\gamma(1)} < z_{\gamma(2)} < \ldots < z_{\gamma(n)} < \infty\}. \tag{2.2}$$

The disk integrands involve dimensionless Mandelstam invariants

$$s_{ij} = -\frac{\alpha'}{2} k_i \cdot k_j, \qquad k_j^2 = 0 \tag{2.3}$$

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and Parke–Taylor factors (with $z_{ij} = z_i - z_j$)

$$\text{PT}(\rho(1, 2, \ldots, n)) = \frac{1}{z_{\rho(1)\rho(2)} z_{\rho(2)\rho(3)} \cdots z_{\rho(n)\rho(1)}}. \tag{2.4}$$

The inverse $\text{vol}\,\text{SL}_2(\mathbb{R})$ in (2.1) instructs to set any triplet of punctures to $0, 1, \infty$, where the $\text{SL}_2(\mathbb{R})$ invariance of genus-zero integrands hinges on momentum conservation $\sum_{j=1}^n k_j = 0$. Both the domains and the Parke–Taylor integrands are indexed via permutations $\gamma, \rho \in S_n$ of the external legs $1, 2, \ldots, n$. One can arrive at smaller bases of $(n-3)!$ cycles γ and Parke–Taylor orderings ρ via monodromy relations [65, 66] and integration by parts [62, 64], respectively.

Closed-string tree amplitudes in turn can be reduced to sphere integrals

$$J^{\text{tree}}(\gamma|\rho) = \frac{1}{\pi^{n-3}} \int_{\mathbb{C}^{n-3}} \frac{\left(\prod_{j=1}^n \mathrm{d}^2 z_j\right)}{\text{vol}\,\text{SL}_2(\mathbb{C})} \prod_{1 \leq i < j}^n |z_{ij}|^{-2s_{ij}} \overline{\text{PT}(\gamma(1, 2, \ldots, n))} \text{PT}(\rho(1, 2, \ldots, n)) \tag{2.5}$$

involving permutations $\gamma, \rho \in S_n$ of meromorphic and antimeromorphic Parke–Taylor factors subject to the same integration-by-parts relations as in the open-string case.

2.1.2 Single-valued map between disk and sphere integrals

The disk and sphere integrals (2.1) and (2.5) converge for a suitable range of the $\text{Re}(s_{ij})$ and they admit a Laurent expansion in α', i.e. around the value $s_{ij} = 0$ of the dimensionless Mandelstam invariants (2.3). The coefficients in the α'-expansions of disk integrals Z^{tree} are MZVs [67, 68],

$$\zeta_{n_1, n_2, \ldots, n_r} = \sum_{0 < k_1 < k_2 < \ldots < k_r} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r}, \qquad n_r \geq 2 \tag{2.6}$$

whose weight $n_1 + n_2 + \ldots + n_r$ matches the order in α' beyond the low-energy limit (i.e. beyond the leading order in α'). The polynomial structure of the Z^{tree} in s_{ij} can for instance be generated from the Drinfeld associator [69] or Berends–Giele recursions [70], with explicit results available for download from [71, 72].

When applying the single-valued map [48, 46] of motivic [73] MZVs[3]

$$\text{sv}\,\zeta_{2k} = 0, \quad \text{sv}\,\zeta_{2k+1} = 2\zeta_{2k+1}, \quad \text{sv}\,\zeta_{3,5} = -10\zeta_3\zeta_5, \quad \text{etc.} \tag{2.7}$$

order by order in α', the disk and sphere integrals (2.1) and (2.5) are related by [40–45]

$$J^{\text{tree}}(\gamma|\rho) = \text{sv}\,Z^{\text{tree}}(\gamma|\rho). \tag{2.8}$$

The first permutation γ in Z^{tree} and J^{tree} refers to a disk ordering (2.2) and an antimeromorphic Parke–Taylor factor (2.4), respectively, which are connected by a Betti–deRham duality [57, 58, 49]. The key result of this work is to identify similar pairs of cycles and antimeromorphic functions at genus one.

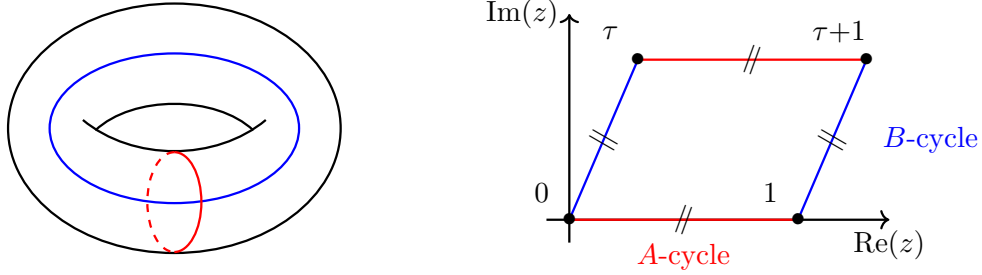[3] Strictly speaking, MZVs need to be replaced by their motivic versions to have a well-defined single-valued map.

Figure 2: *Parametrization of the torus $\mathfrak{T} = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ with identifications $z \cong z+1 \cong z+\tau$ marked by $\parallel$ along the A- and B-cycles. While the torus is drawn for non-vanishing $\mathrm{Re}(\tau)$ to accommodate closed-string amplitudes, the cylinder world-sheets for open-string amplitudes are derived from tori at $\tau \in i\mathbb{R}^+$ via suitable involutions [74].*

2.2 Genus-one integrals

As a preparation for our proposal of a genus-one single-valued map, we now introduce the basic genus-one world-sheet integrals and the objects appearing in their α'-expansion.

2.2.1 Genus-one open-string A-cycle integrals

In the same way as disk integrals can be cast into a Parke–Taylor-type basis (2.1), the basis integrals for massless genus-one open-string amplitudes are claimed to be generated by [59,60]

$$Z_{\vec{\eta}}^{\tau}(\gamma|\rho) = \int\limits_{\mathfrak{A}(\gamma)} \Big(\prod_{j=2}^n \mathrm{d}z_j \Big) \varphi_{\vec{\eta}}^{\tau}(1, \rho(2, \ldots, n)) \prod_{1 \leq i < j}^n e^{s_{ij}\mathcal{G}_{\mathfrak{A}}(z_{ij},\tau)} , \tag{2.9}$$

where we have set $z_1 = 0$ by translation invariance. In this work we restrict to planar amplitudes with all state insertions on a single cylinder boundary (as opposed to non-planar amplitudes with punctures on both boundaries of the cylinder). We do not impose momentum conservation in a genus-one context and treat all the s_{ij} with $1 \leq i < j \leq n$ as independent. The ordering of the open-string punctures on a cylinder boundary is encoded in an integration domain on the A-cycle of a torus (see figure 2 for the standard parametrization) with $\tau \in i\mathbb{R}^+$ [74]

$$\mathfrak{A}(\gamma) = \{z_j \in \mathbb{R}, \ 0 < z_{\gamma(2)} < z_{\gamma(3)} < \ldots < z_{\gamma(n)} < 1\}, \tag{2.10}$$

with similar integration domains [75] for the non-planar open-string integrals.

The integrand of (2.9) features the open-string Green function on an A-cycle (which is chosen to enforce $\mathcal{G}_{\mathfrak{A}}(z,\tau) = \mathcal{G}_{\mathfrak{A}}(-z,\tau)$ and $\int_0^1 \mathrm{d}z\, \mathcal{G}_{\mathfrak{A}}(z,\tau) = 0$ [51,76])

$$\mathcal{G}_{\mathfrak{A}}(z,\tau) = -\log\Big(\frac{\theta_1(|z|,\tau)}{\eta(\tau)} \Big) + \frac{i\pi\tau}{6} + \frac{i\pi}{2} , \qquad z \in (-1,1) \tag{2.11}$$

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and the following combination of the doubly-periodic Kronecker–Eisenstein series [77]

$$\Omega(z, \eta, \tau) = \exp\left(2\pi i \eta \frac{\text{Im } z}{\text{Im } \tau}\right) \frac{\theta_1'(0, \tau)\theta_1(z+\eta, \tau)}{\theta_1(z, \tau)\theta_1(\eta, \tau)} \tag{2.12}$$

$$\varphi_{\vec{\eta}}^{\tau}(1, 2, \ldots, n) = \Omega(z_{12}, \eta_{23\ldots n}, \tau)\Omega(z_{23}, \eta_{3\ldots n}, \tau)\cdots\Omega(z_{n-1,n}, \eta_n, \tau) \tag{2.13}$$

with $\eta_{ij\ldots k} = \eta_i + \eta_j + \ldots + \eta_k$.[4] The permutation $\rho \in S_{n-1}$ in $\varphi_{\vec{\eta}}^{\tau}(1, \rho(2, \ldots, n))$ is taken to act on both the z_j and the formal expansion variables $\eta_j \in \mathbb{C}$ in (2.13). The conjectural basis (2.9) is a generating function of the world-sheet integrals over the Kronecker–Eisenstein coefficients $f^{(w)}$

$$\Omega(z, \eta, \tau) = \sum_{w=0}^{\infty} \eta^{w-1} f^{(w)}(z, \tau) \tag{2.14}$$

that occur in the integrands of genus-one open- and closed-string amplitudes [78, 4, 52], e.g.

$$f^{(0)}(z, \tau) = 1, \quad f^{(1)}(z, \tau) = \partial_z \log \theta_1(z, \tau) + 2\pi i \frac{\text{Im } z}{\text{Im } \tau}. \tag{2.15}$$

While the massless four-point genus-one amplitude of the open superstring [79] is proportional to the most singular η_j^{-3}-order of $Z_{\vec{\eta}}^{\tau}(\cdot|1, 2, 3, 4)$, the analogous amplitude of the open bosonic string additionally involves contributions of $Z_{\vec{\eta}}^{\tau}(\cdot|1, 2, 3, 4)$ (and its permutations in $2, 3, 4$) at the orders of $\eta_j^{\pm 1}$ [75][5]. The short-distance behavior $f^{(1)}(z, \tau) = \frac{1}{z} + \mathcal{O}(z)$ introduces kinematic poles into the α'-expansion of (2.9), and the remaining $f^{(w \neq 1)}(z, \tau)$ are regular for any $z \in \mathbb{C}$.

2.2.2 Genus-one closed-string integrals

In the same way as (2.9) is claimed to be a universal basis of genus-one open-string integrals, the integrals over the torus punctures for massless genus-one amplitudes in type II, heterotic and bosonic string theories should be generated by [61]

$$Y_{\vec{\eta}}^{\tau}(\gamma|\rho) = (2i)^{n-1} \int_{\mathfrak{T}^{n-1}} \left(\prod_{j=2}^{n} \mathrm{d}^2 z_j\right) \prod_{1 \leq i < j}^{n} e^{s_{ij}\mathcal{G}_{\mathfrak{T}}(z_{ij}, \tau)} \overline{\varphi_{\vec{\eta}}^{\tau}(1, \gamma(2, \ldots, n)|\tau)} \varphi_{(\tau-\bar{\tau})\vec{\eta}}^{\tau}(1, \rho(2, \ldots, n)) \tag{2.16}$$

with $z_1 = 0$. The remaining z_j are integrated over the torus $\mathfrak{T} = \frac{\mathbb{C}}{\tau\mathbb{Z}+\mathbb{Z}}$ with modular parameter $\tau \in \mathbb{H} = \{\tau \in \mathbb{C}, \text{ Im } \tau > 0\}$. The closed-string Green function

$$\mathcal{G}_{\mathfrak{T}}(z, \tau) = -\log\left|\frac{\theta_1(z, \tau)}{\eta(\tau)}\right|^2 + \frac{2\pi(\text{Im } z)^2}{\text{Im } \tau} \tag{2.17}$$

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[4]Our conventions for the standard odd Jacobi theta function are

$$\theta_1(z, \tau) = q^{1/8}(e^{i\pi z} - e^{-i\pi z})\prod_{n=1}^{\infty}(1 - q^n)(1 - e^{2\pi i z}q^n)(1 - e^{-2\pi i z}q^n)$$

and $\eta(\tau)$ is the Dedekind eta function. In order to avoid confusion with the later expansion parameters η_j, we always spell out the argument τ of the Dedekind eta function. Representations of the open-string Green function in terms of elliptic polylogarithms are discussed in [4, 5, 51], and we follow the conventions of [4] for regularizing endpoint divergences.

[5]By using Fay identities and integration by parts, the massless four-point genus-one amplitude of open bosonic strings in section 8.1.1 of [75] can be rewritten in terms of the coefficients in the η_j-expansion of $Z_{\vec{\eta}}^{\tau}(\cdot|1, 2, 3, 4)$.

is chosen to be modular invariant and to obey $\int_{\mathfrak{T}} \mathrm{d}^2 z\, \mathcal{G}_{\mathfrak{T}}(z, \tau) = 0$, and its holomorphic derivatives parallel those of the open-string Green function $\mathcal{G}_{\mathfrak{A}}(z, \tau)$ in (2.11),

$$\partial_z \mathcal{G}_{\mathfrak{T}}(z, \tau) = -f^{(1)}(z, \tau), \qquad 2\pi i \partial_\tau \mathcal{G}_{\mathfrak{T}}(u\tau + v, \tau) = -f^{(2)}(u\tau + v, \tau) \qquad \text{(fixed } u, v)$$

$$\partial_v \mathcal{G}_{\mathfrak{A}}(v, \tau) = -f^{(1)}(v, \tau), \qquad 2\pi i \partial_\tau \mathcal{G}_{\mathfrak{A}}(v, \tau) = -f^{(2)}(v, \tau) - 2\zeta_2, \tag{2.18}$$

where $u, v \in \mathbb{R}$ parametrize the covering space of the torus and the $f^{(w)}(z, \tau)$ with $z = u\tau + v$ are defined by (2.15). The second arguments $(\tau - \bar{\tau})\eta_j$ and $\bar{\eta}_j$ of the Kronecker–Eisenstein series and their complex conjugates in (2.16) have been chosen such that each order in the η_j- and α'-expansion gives rise to modular forms of purely antiholomorphic modular weight[6].

When assembling genus-one amplitudes of open and closed strings from the series $Z_{\vec{\eta}}^\tau$ and $Y_{\vec{\eta}}^\tau$, it remains to dress the component integrals in their η_j-expansions with kinematic factors that carry the dependence on the external polarizations. The latter are determined from the conformal-field-theory correlators of the vertex operators, see e.g. [80,81], and unaffected by our proposal for the single-valued map at genus one.

2.2.3 Differential equations in τ

Based on the differential equations (2.18) of the Green functions and integration by parts in the z_j, the open- and closed-string integrals (2.9) and (2.16) were shown in [60] and [61] to obey the differential equations

$$2\pi i \partial_\tau Z_{\vec{\eta}}^\tau(\gamma|\rho) = \sum_{k=0}^\infty (1-k) \mathrm{G}_k(\tau) \sum_{\alpha \in S_{n-1}} r_{\vec{\eta}}(\epsilon_k)_\rho{}^\alpha Z_{\vec{\eta}}^\tau(\gamma|\alpha) \tag{2.19}$$

$$2\pi i \partial_\tau Y_{\vec{\eta}}^\tau(\gamma|\rho) = \sum_{k=0}^\infty (1-k)(\tau-\bar{\tau})^{k-2} \mathrm{G}_k(\tau) \sum_{\alpha \in S_{n-1}} R_{\vec{\eta}}(\epsilon_k)_\rho{}^\alpha Y_{\vec{\eta}}^\tau(\gamma|\alpha),$$

respectively. The right-hand sides involve holomorphic Eisenstein series $\mathrm{G}_0 = -1$ and

$$\mathrm{G}_k(\tau) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}, \qquad k \geq 4 \tag{2.20}$$

as well as $(n-1)! \times (n-1)!$ matrices $r_{\vec{\eta}}(\epsilon_k), R_{\vec{\eta}}(\epsilon_k)$ independent of τ that vanish for $k = 2$ and $k \in 2\mathbb{N}-1$. This means in particular that $\mathrm{G}_2(\tau)$ does not appear in (2.19).

The two-point instances are

$$r_{\eta_2}(\epsilon_0) = s_{12} \left(\frac{1}{\eta_2^2} + 2\zeta_2 - \frac{1}{2}\partial_{\eta_2}^2 \right)$$

$$r_{\eta_2}(\epsilon_k) = R_{\eta_2}(\epsilon_k) = s_{12}\eta_2^{k-2}, \qquad k \geq 4 \tag{2.21}$$

$$R_{\eta_2}(\epsilon_0) = s_{12} \left(\frac{1}{\eta_2^2} - \frac{1}{2}\partial_{\eta_2}^2 \right) - 2\pi i \bar{\eta}_2 \partial_{\eta_2}.$$

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[6]Functions $F(\tau)$ on the upper half plane with transformations $F(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}) = (\gamma\tau + \delta)^w (\gamma\bar{\tau} + \delta)^{\bar{w}} F(\tau)$ under $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ are said to carry holomorphic and antiholomorphic modular weight w and $\bar{w}$, respectively.

The notation ϵ_k reflects the expectation that the $r_{\vec{\eta}}(\epsilon_k), R_{\vec{\eta}}(\epsilon_k)$ are matrix representations of Tsunogai's derivation algebra [82] and obey relations such as (see [83,84,17] for similar relations at higher weight and depth)

$$[r_{\vec{\eta}}(\epsilon_{10}), r_{\vec{\eta}}(\epsilon_4)] - 3[r_{\vec{\eta}}(\epsilon_8), r_{\vec{\eta}}(\epsilon_6)] = 0 \,. \tag{2.22}$$

The all-multiplicity formulae for these $(n-1)! \times (n-1)!$ representations in [60,61] manifest that $r_{\vec{\eta}}(\epsilon_k)$ is linear in the s_{ij}, i.e. proportional to α', and their closed-string analogues $R_{\vec{\eta}}(\epsilon_k)$ additionally involve terms $\sim \bar{\eta}_j \partial_{\eta_j}$ independent of α' (with $s_{12...n} = \sum_{1 \leq i < j}^{n} s_{ij}$):

$$R_{\vec{\eta}}(\epsilon_k) = \begin{cases} r_{\vec{\eta}}(\epsilon_k) & : \ k \geq 4 \\ r_{\vec{\eta}}(\epsilon_0) - 2\zeta_2 s_{12...n} - 2\pi i \sum_{j=2}^{n} \bar{\eta}_j \partial_{\eta_j} & : \ k = 0 \end{cases} \,. \tag{2.23}$$

2.2.4 Basic definitions of eMZVs and MGFs

We shall now review the definitions of the eMZVs and MGFs that occur as the expansion coefficients of the above genus-one integrals. The η_j- and α'-expansion of the open-string integrals $Z_{\vec{\eta}}^{\tau}(\gamma|\rho)$ in (2.9) gives rise to A-cycle eMZV [4–6]

$$\omega(n_1, n_2, \ldots, n_r|\tau) = \int\limits_{0 < z_1 < z_2 < \ldots < z_r < 1} dz_1 \, f^{(n_1)}(z_1, \tau) \, dz_2 \, f^{(n_2)}(z_2, \tau) \, \ldots \, dz_r \, f^{(n_r)}(z_r, \tau) \tag{2.24}$$

introduced by Enriquez [3] which are said to carry weight $n_1 + n_2 + \ldots + n_r$ and length r. Endpoint divergences in case of $n_1 = 1$ or $n_r = 1$ are shuffle-regularized as in section 2.2.1 of [4]. The specific eMZVs at a given order of $Z_{\vec{\eta}}^{\tau}(\gamma|\rho)$ in s_{ij} and η_j can be obtained from the differential equations (2.19) along with the initial values $Z_{\vec{\eta}}^{\tau \to i\infty}(\gamma|\rho)$ in [60] or from matrix representations of the elliptic KZB associator [85,86].

The closed-string integrals $Y_{\vec{\eta}}^{\tau}(\gamma|\rho)$ in (2.16) in turn introduce multiple sums over the momentum lattice of a torus [52,61]

$$\Lambda = \mathbb{Z} + \tau\mathbb{Z}\,, \qquad \Lambda' = \Lambda \setminus \{0\} \tag{2.25}$$

that are known as MGFs [10,11]. With the removal of $p = 0$ from Λ, they can be thought of as infrared-regulated and discretized versions of Feynman integrals on a torus. The MGFs associated with Feynman graphs of dihedral topology are defined by[7]

$$\mathcal{C}\begin{bmatrix} a_1 & a_2 & \ldots & a_r \\ b_1 & b_2 & \ldots & b_r \end{bmatrix} = \sum_{p_1, p_2, \ldots, p_r \in \Lambda'} \frac{\delta(p_1 + p_2 + \ldots + p_r)}{p_1^{a_1} \bar{p}_1^{b_1} p_2^{a_2} \bar{p}_2^{b_2} \ldots p_r^{a_r} \bar{p}_r^{b_r}}\,, \tag{2.26}$$

and more general topologies are for instance discussed in [11,87]. The simplest examples of dihedral MGFs (2.26) have two columns and are associated with one-loop graphs on the world-sheet

$$\mathcal{C}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \sum_{p \in \Lambda'} \frac{1}{p^a \bar{p}^b}\,. \tag{2.27}$$

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[7]Note that the definition of $\mathcal{C}[\ldots]$ in this work follows the conventions of [61,18,87] but differs from those in [11,88,52,54] by factors of $\mathrm{Im}\,\tau$ and π.

As long as the entries $a+b > 2$, the lattice sums are absolutely convergent and the MGFs (2.27) are expressible in terms of non-holomorphic Eisenstein series $\mathrm{E}_k(\tau)$ and their Cauchy–Riemann derivatives

$$\mathrm{E}_k(\tau) = \left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^k \mathcal{C}\left[\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right]\,, \qquad \nabla^m \mathrm{E}_k(\tau) = \frac{(\mathrm{Im}\,\tau)^{k+m}}{\pi^k} \frac{(k+m-1)!}{(k-1)!} \mathcal{C}\left[\begin{smallmatrix} k+m & 0 \\ k-m & 0 \end{smallmatrix}\right]\,,$$

$$\overline{\nabla}^m \mathrm{E}_k(\tau) = \frac{(\mathrm{Im}\,\tau)^{k+m}}{\pi^k} \frac{(k+m-1)!}{(k-1)!} \mathcal{C}\left[\begin{smallmatrix} k-m & 0 \\ k+m & 0 \end{smallmatrix}\right]\,, \qquad (2.28)$$

where $\nabla = 2i(\mathrm{Im}\,\tau)^2 \partial_\tau$ and $\overline{\nabla} = -2i(\mathrm{Im}\,\tau)^2 \partial_{\bar\tau}$. As will be detailed below, both eMZVs (2.24) and MGFs such as (2.26) can be represented via iterated integrals of holomorphic Eisenstein series $\mathrm{G}_k = \mathcal{C}\left[\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right]$ defined by (2.20). Both eMZVs [17] and MGFs [89, 11, 90, 88, 87] exhibit a multitude of relations over rational combinations of MZVs, all of which are automatically exposed in their iterated-Eisenstein-integral representation.[8] A computer implementation for the decomposition of a large number of eMZVs and MGFs into basis elements is available in [92] and [87], respectively.

3   New types of genus-one integrals

The goal of this paper is to relate the α'-expansions of suitable generating functions of genus-one open- and closed-string integrals. The $Z_{\vec\eta}^\tau(\gamma|\rho)$ and $Y_{\vec\eta}^\tau(\gamma|\rho)$ in (2.9) and (2.16) can be anticipated to not yet furnish the optimal building blocks for this purpose since

(i) The τ-dependence $\sim \mathrm{G}_k(\tau)$ and $\sim (\tau-\bar\tau)^{k-2}\mathrm{G}_k(\tau)$ of the open- and closed-string differential equations (2.19) does not match, even in absence of $\bar\tau$.

(ii) The contributions $\sim \bar\eta_j \partial_{\eta_j}$ to the closed-string derivations $R_{\vec\eta}(\epsilon_0)$ in (2.21) and (2.23) do not have any open-string counterpart in $r_{\vec\eta}(\epsilon_0)$.

Both of these shortcomings will be fixed by the improved open- and closed-string generating functions $B_{\vec\eta}^\tau(\gamma|\rho)$ and $J_{\vec\eta}^\tau(\gamma|\rho)$ to be introduced in this section.

3.1   Genus-one open-string B-cycle integrals

Instead of parametrizing the cylinder boundary through the A-cycle of a torus as in (2.9), one can perform a modular S transformation

$$B_{\vec\eta}^\tau(\gamma|\rho) = Z_{\vec\eta}^{-1/\tau}(\gamma|\rho) = \int_{\mathfrak{B}(\gamma)} \left(\prod_{j=2}^n \mathrm{d}z_j\right) \varphi_{\tau\vec\eta}^\tau(1, \rho(2, \dots, n)) \prod_{1 \leq i < j}^n e^{s_{ij}\mathcal{G}_{\mathfrak{B}}(z_{ij}, \tau)} \qquad (3.1)$$

to attain a parametrization through the B-cycle (recalling that $z_1 = 0$ and $\tau \in i\mathbb{R}^+$)

$$\mathfrak{B}(\gamma) = \bigoplus_{j=1}^n \mathfrak{B}_j(\gamma) \qquad (3.2)$$

$$\mathfrak{B}_j(\gamma) = \left\{z_i = \tau u_i, \quad -\tfrac{1}{2} < u_{\gamma(j+1)} < u_{\gamma(j+2)} < \dots < u_{\gamma(n)} < 0 < u_{\gamma(2)} < u_{\gamma(3)} < \dots < u_{\gamma(j)} < \tfrac{1}{2}\right\},$$

---

[8]This relies on the linear-independence result of [91] on holomorphic iterated Eisenstein integrals.
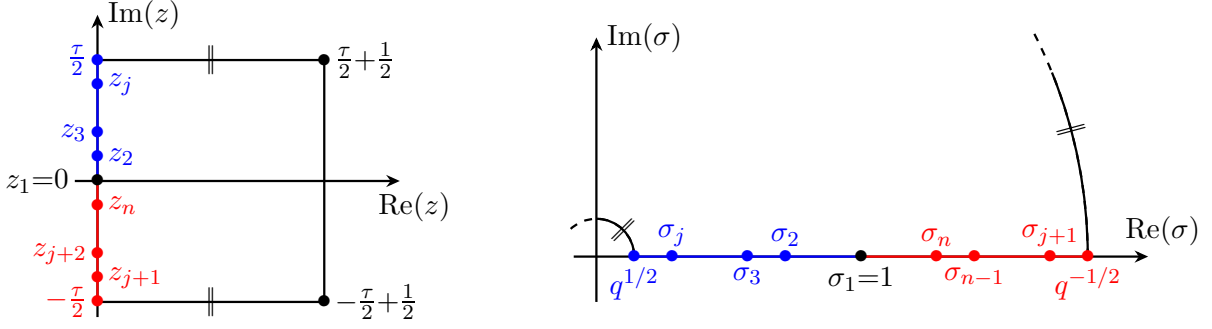
Figure 3: *The parametrization (3.2) of the B-cycle is mapped to the positive real axis in the $\sigma_j = e^{2\pi i z_j}$ variables which exhausts all of $\mathbb{R}^+$ as $\tau \to i\infty$ and $q = e^{2\pi i \tau} \to 0$. The line segments in the z-coordinate and the semicircles in the σ-coordinate marked by $\|$ are identified by the periodic direction of the cylinder, i.e. the B-cycle of the parental torus.*

where $u_i \in \mathbb{R}$, and the B-cycle Green function $\mathcal{G}_{\mathfrak{B}}(z, \tau)$ is constructed in two steps: First, we define $\mathcal{G}_{\mathfrak{B}}(z, \tau)$ for z on the line $(0, \tau)$ by [76]

$$\mathcal{G}_{\mathfrak{B}}(z, \tau) = \mathcal{G}_{\mathfrak{A}}\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -\frac{i\pi z^2}{\tau} - \log\left(\frac{\theta_1(z, \tau)}{\eta(\tau)}\right) - \frac{i\pi}{6\tau} + i\pi, \qquad z \in (0, \tau). \tag{3.3}$$

Then, we extend this to $z \in (-\tau, 0)$ by imposing $\mathcal{G}_{\mathfrak{B}}(z, \tau) = \mathcal{G}_{\mathfrak{B}}(-z, \tau)$ for compatibility with (2.11) under modular S transformations, leading to the combined expression

$$\mathcal{G}_{\mathfrak{B}}(u\tau, \tau) = -i\pi u^2 \tau - \log\left(\frac{\theta_1(|u|\tau, \tau)}{\eta(\tau)}\right) - \frac{i\pi}{6\tau} + i\pi, \qquad u \in (-1, 1). \tag{3.4}$$

Instead of integrating over $z_i = \tau u_i$ with $u_i \in (0, 1)$, we have chosen the representative $u_i \in (-\frac{1}{2}, \frac{1}{2})$ of the B-cycle in order to facilitate the comparison with genus-zero integration cycles as $\tau \to i\infty$. Figure 3 illustrates the integration cycle (3.2) in both the z_j and $\sigma_j = e^{2\pi i z_j}$ variables (the latter becoming the coordinates on the sphere as $\tau \to i\infty$), where $z_j \in i\mathbb{R}$ and $\sigma_j \in \mathbb{R}^+$ for purely imaginary choices of τ. Note that non-planar versions of the B-cycle integrals involve additional punctures at $z_j \in \frac{1}{2} + i\mathbb{R}$ or negative $\sigma_j \in (-q^{-1/2}, -q^{1/2})$.

The modular transformation $\Omega(z, \eta, -\frac{1}{\tau}) = \tau \Omega(\tau z, \tau \eta, \tau)$ of the doubly-periodic Kronecker–Eisenstein series (2.12) leads to the rescaling $\eta_j \to \tau \eta_j$ in the subscript of the ρ-dependent integrand $\varphi_{\tau\vec{\eta}}^\tau$ of (3.1).

3.2 Dual closed-string integrals

The doubly-periodic integrands $\varphi_{\vec{\eta}}^\tau$ in (2.13) are non-holomorphic

$$\partial_{\bar{z}_j} \varphi_{\vec{\eta}}^\tau(\rho(1, 2, \dots, n)) = \frac{2\pi i \eta_j}{\tau - \bar{\tau}} \varphi_{\vec{\eta}}^\tau(\rho(1, 2, \dots, n)) \tag{3.5}$$

leading to the terms $\sim \bar{\eta}_j \partial_{\eta_j}$ in the closed-string derivations $R_{\vec{\eta}}(\epsilon_0)$ in (2.23). This introduces a tension between the open- and closed-string differential equations (2.19) such that the $\varphi_{\vec{\eta}}^\tau$ do

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not qualify as Betti–deRham duals of open-string integration cycles. In order to generalize the interplay of Parke–Taylor factors (2.4) with single-valued integration [48, 49] to genus one, the factor of $\overline{\varphi_{\bar{\eta}}^{\tau}(\ldots)}$ in the $Y_{\bar{\eta}}^{\tau}$ integrals (2.16) needs to be replaced by an antimeromorphic function that is still well-defined on the torus, i.e. the complex conjugate of an elliptic function in all of $z_1, z_2, \ldots, z_n$.

Such elliptic functions of n punctures can be generated by cycles of Kronecker–Eisenstein series [78]

$$\Omega(z_{12}, \xi, \tau)\Omega(z_{23}, \xi, \tau)\ldots\Omega(z_{n1}, \xi, \tau) =: \xi^{-n} \sum_{w=0}^{\infty} \xi^{w} V_{w}(1, 2, \ldots, n|\tau) \, , \tag{3.6}$$

where V_w has holomorphic modular weight w. Even though the individual Kronecker–Eisenstein series Ω are not meromorphic in the z_j, the V_w are elliptic functions since the non-holomorphic phase factors in (2.12) cancel from the cyclic product in (3.6). As will be detailed below, these elliptic functions degenerate to suitable combinations of Parke–Taylor factors when forming the linear combinations

$$V(1, 2, \ldots, n|\tau) = \sum_{w=0}^{n-2} \frac{V_w(1, 2, \ldots, n|\tau)}{(2\pi i)^w (n-w-1)!} \tag{3.7}$$

such as

$$V(1, 2|\tau) = 1 \, , \qquad\qquad V(1, 2, 3|\tau) = \frac{1}{2} + \frac{V_1(1, 2, 3|\tau)}{2\pi i}$$
$$V(1, 2, 3, 4|\tau) = \frac{1}{6} + \frac{1}{2}\frac{V_1(1, 2, 3, 4|\tau)}{2\pi i} + \frac{V_2(1, 2, 3, 4|\tau)}{(2\pi i)^2} \tag{3.8}$$
$$V(1, 2, 3, 4, 5|\tau) = \frac{1}{24} + \frac{1}{6}\frac{V_1(1, 2, 3, 4, 5|\tau)}{2\pi i} + \frac{1}{2}\frac{V_2(1, 2, 3, 4, 5|\tau)}{(2\pi i)^2} + \frac{V_3(1, 2, 3, 4, 5|\tau)}{(2\pi i)^3} \, .$$

To lend credence to this definition of the V-function, let us see how their properties parallel those of the genus-zero case: The Betti–deRham duality at genus zero relies on the simple-pole residues

$$\mathrm{Res}_{z_j=z_{j\pm1}}\mathrm{PT}(1, 2, \ldots, j, \ldots, n) = \pm\mathrm{PT}(1, 2, \ldots, j{-}1, j{+}1, \ldots, n) \tag{3.9}$$

of the Parke–Taylor factors (2.4). These residues correspond to the situation when two neighboring points of the disk ordering (2.2) at $z_j = z_{j\pm1}$ come together, which is crucial for sphere integrals being single-valued disk integrals [45, 43].

Similarly, at genus one, the generating function (3.6) of the elliptic V_w functions exposes the recursive structure of their simple-pole residues

$$\mathrm{Res}_{z_j=z_{j\pm1}}V_w(1, 2, \ldots, j, \ldots, n) = \pm V_{w-1}(1, 2, \ldots, j{-}1, j{+}1, \ldots, n) \tag{3.10}$$

and the absence of higher poles in $z_j{-}z_{j\pm1}$. Consequently, the pole structure of the elliptic combinations (3.7)

$$\mathrm{Res}_{z_j=z_{j\pm1}}V(1, 2, \ldots, j, \ldots, n) = \pm\frac{1}{2\pi i}V(1, 2, \ldots, j{-}1, j{+}1, \ldots, n) \tag{3.11}$$

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mirrors the boundaries of the open-string integration cycles as $z_j = z_{j\pm 1}$, i.e. one recovers mutually consistent V-functions and cycles at multiplicity $n-1$ in both cases.[9]

The absence of V_w with $w \geq n-1$ in (3.7) can be understood from

- the vanishing of $V_{n-1}(1, 2, \ldots, n|\tau)$ since (3.6) would otherwise be an elliptic function of ξ with a simple pole at the origin [78]

- the breakdown of uniform transcendentality when integrating Koba–Nielsen integrals involving $V_n(1, 2, \ldots, n|\tau)$ over the z_j [52] (which is in tension with the transcendentality properties of open-string integrals [60])

- the fact that $V_{w \geq n+1}(1, \ldots, n|\tau)$ is expressible in terms of $G_k V_{w-k}(1, \ldots, n|\tau)$ [78]

Similar to the closed-string integrals $Y_{\vec\eta}^\tau$, we define an $(n-1)! \times (n-1)!$ matrix of torus integrals

$$J_{\vec\eta}^\tau(\gamma|\rho) = (2i)^{n-1} \int_{\mathfrak{T}^{n-1}} \Big(\prod_{j=2}^n \mathrm{d}^2 z_j \Big) \prod_{1 \leq i < j}^n e^{s_{ij}\mathcal{G}_{\mathfrak{T}}(z_{ij},\tau)} \overline{V(1, \gamma(2, \ldots, n)|\tau)} \varphi_{(\tau-\bar\tau)\vec\eta}^\tau(1, \rho(2, \ldots, n))$$

(3.12)

indexed by permutations $\gamma, \rho \in S_{n-1}$ of (3.7) and (2.13). Note that the cyclic symmetry

$$V_w(2, \ldots, n, 1|\tau) = V_w(1, 2, \ldots, n|\tau), \qquad V(2, \ldots, n, 1|\tau) = V(1, 2, \ldots, n|\tau) \tag{3.13}$$

exposed by the generating function (3.6) has been used to bring the integrand of (3.12) into the form of $\overline{V(1, \ldots |\tau)}$.

3.3 Asymptotics at the cusp

The modular S transformation in (3.1) maps the A-cycle eMZVs (2.24) in the η_j- and α'-expansion of $Z_{\vec\eta}^\tau$ to B-cycle eMZVs [3] in the analogous expansion of $B_{\vec\eta}^\tau$. As detailed in [51,93,76], the asymptotic expansion of B-cycle eMZVs as $\tau \to i\infty$ is governed by Laurent polynomials in $T = \pi\tau \in i\mathbb{R}^+$ whose coefficients are $\mathbb{Q}$-linear combinations of MZVs, for instance

$$\omega(0, 0, 2|-\tfrac{1}{\tau}) = -\frac{T^2}{180} - \frac{\zeta_2}{2} + \frac{i\zeta_3}{T} + \frac{3\zeta_4}{2T^2} + \mathcal{O}(e^{2iT})$$

$$\omega(0, 0, 1, 0|-\tfrac{1}{\tau}) = \frac{iT}{120} - \frac{i\zeta_2}{4T} - \frac{3\zeta_3}{4T^2} + \frac{3i\zeta_4}{4T^3} + \mathcal{O}(e^{2iT}) \tag{3.14}$$

$$\omega(0, 0, 3, 0|-\tfrac{1}{\tau}) = \frac{iT^3}{1260} - \frac{3i\zeta_4}{4T} - \frac{9\zeta_5}{2T^2} + \frac{15i\zeta_6}{2T^3} + \mathcal{O}(e^{2iT}).$$

The suppressed terms $\mathcal{O}(e^{2iT})$ are series in $q = e^{2\pi i\tau} = e^{2iT}$ with Laurent polynomials in T as their coefficients.

The MGFs (2.26) in the η_j- and α'-expansion of (2.16) admit similar expansions around the cusp, where the leading term is a Laurent polynomial in $y = \pi \operatorname{Im} \tau$ instead of T. The coefficients in the Laurent polynomials of MGFs were shown to be $\mathbb{Q}$-linear combinations of MZVs[10] [15]

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[9]On the closed-string side of the 'genus-one Betti–deRham duality' we note that, by double-periodicity of the V-functions, additional poles with identical residues occur as $z_j \to z_{j\pm 1} + m\tau + n$ ($m, n, \in \mathbb{Z}$). On the open-string side in turn, the delimiters of the integration cycles in the B-cycle parametrization of figure 3 are separated by τ.

[10]See [12] for an earlier proof of the weaker statement that the Laurent polynomials of modular graph functions are $\mathbb{Q}$-linear combinations of cyclotomic MZVs.

and are conjectured to be single-valued MZVs [12, 10]. Simple examples of the asymptotics of MGFs include

$$
\begin{aligned}
\mathrm{E}_2(\tau) &= \frac{y^2}{45} + \frac{\zeta_3}{y} + \mathcal{O}(e^{-2y}) \\
\frac{\pi \overline{\nabla} \mathrm{E}_2(\tau)}{y^2} &= \frac{2y}{45} - \frac{\zeta_3}{y^2} + \mathcal{O}(e^{-2y}) \\
\mathrm{E}_3(\tau) &= \frac{2y^3}{945} + \frac{3\zeta_5}{4y^2} + \mathcal{O}(e^{-2y}),
\end{aligned}
\tag{3.15}
$$

see (2.28) for the lattice-sum representations of the non-holomorphic Eisenstein series.

In a variety of examples, the Laurent polynomials of MGFs and B-cycle eMZVs have been related by an extension of the single-valued map (2.7) to [51, 52, 16]

$$
\mathrm{sv}\, T = 2iy \quad \Leftrightarrow \quad \mathrm{sv}\, \log(q) = \log|q|^2.
\tag{3.16}
$$

By (3.14) and (3.15), for instance, the Laurent polynomials of $\omega(0, 0, 2|-\frac{1}{\tau}) \longrightarrow \mathrm{E}_2(\tau)$ as well as $\omega(0, 0, 1, 0|-\frac{1}{\tau}) \longrightarrow -\frac{3}{8}\frac{\pi \overline{\nabla} \mathrm{E}_2(\tau)}{y^2}$ and $\omega(0, 0, 3, 0|-\frac{1}{\tau}) \longrightarrow 3\mathrm{E}_3(\tau)$ are related by (3.16).

The A-cycle eMZVs in $Z_{\vec{\eta}}^\tau$, by contrast, enjoy a Fourier expansion in $q = e^{2\pi i \tau}$ whose coefficients are $\mathbb{Q}[(2\pi i)^{-1}]$ combinations of MZVs [3, 17] and do not feature any analogues of the Laurent polynomials in the expansion of $B_{\vec{\eta}}^\tau$. This is yet another indication besides their differential equations that the B-cycle integrals (3.1) are a more suitable starting point for comparison with closed-string integrals than their A-cycle counterparts (2.9).

3.4 Single-valued correspondence of the Laurent polynomials

As a particular convenience of the elliptic combinations (3.7) in the integrands of $J_{\vec{\eta}}^\tau(\gamma|\rho)$, their degeneration at the cusp gives rise to Parke-Taylor factors in $n+2$ punctures ($\sigma_1 = 1$ by $z_1 = 0$)

$$
\sigma_j = e^{2\pi i z_j}, \qquad \sigma_+ = 0, \qquad \sigma_- \to \infty.
\tag{3.17}
$$

Since the non-holomorphic exponentials of $\Omega(z, \xi, \tau) = \exp(2\pi i \xi \frac{\mathrm{Im}\, z}{\mathrm{Im}\, \tau}) F(z, \xi, \tau)$ cancel from the cyclic products in (3.6), one can determine the asymptotics of $V(\ldots|\tau)$ as $\tau \to i\infty$ by using the degeneration of the holomorphic Kronecker–Eisenstein series

$$
F(z_{ij}, \xi, \tau) = \pi \cot(\pi \xi) + i\pi \frac{\sigma_i + \sigma_j}{\sigma_i - \sigma_j} + \mathcal{O}(q).
\tag{3.18}
$$

The relative factors of the V_w in (3.7) have been engineered to obtain the following cyclic combinations of Parke–Taylor factors at the cusp,

$$
\lim_{\tau \to i\infty} \frac{V(1, 2, \ldots, n|\tau)}{\sigma_1 \sigma_2 \ldots \sigma_n} = (-1)^{n-1} \lim_{\sigma_- \to \infty} |\sigma_-|^2 \big[\mathrm{PT}(+, n, n-1, \ldots, 2, 1, -) + \mathrm{cyc}(1, 2, \ldots, n) \big],
\tag{3.19}
$$

which have featured in the context of one-loop gauge-theory amplitudes in ambitwistor string theories [94]. The denominators on the left-hand side of (3.19) arise from $\mathrm{d}z_j = \frac{\mathrm{d}\sigma_j}{2\pi i \sigma_j}$, and the

factor of $|\sigma_-|^2$ on the right-hand side identifies functions on a degenerate torus with SL_2-fixed expressions at genus zero [64]. Given that Parke–Taylor factors are Betti–deRham dual to disk orderings $\mathfrak{D}(\ldots)$ in (2.2), the $\tau \to i\infty$ asymptotics of $J_{\vec{\eta}}^{\tau}(\gamma|\rho)$ should yield the single-valued map of suitably chosen disk integrals. In fact, upon rewriting the B-cycle ordering in terms of the σ_j variables (3.17), each of the contributions $\mathfrak{B}_j(\gamma)$ in (3.2) and figure 3 degenerates to a single disk ordering

$$\mathfrak{B}_j(\gamma)\big|_{\tau\to i\infty} = (-1)^{n-1}\mathfrak{D}(+,\gamma(j,j-1,\ldots,3,2),1,\gamma(n,n-1,\ldots,j+1),-)\,, \tag{3.20}$$

such that the overall B-cycle ordering $\mathfrak{B}(\gamma) = \bigoplus_{j=1}^{n} \mathfrak{B}_j(\gamma)$ at the cusp becomes the Betti–deRham dual to the cyclic combination of Parke–Taylor factors in (3.19),

$$\mathfrak{B}(2,3,\ldots,n)\big|_{\tau\to i\infty} = (-1)^{n-1}\bigoplus_{j=1}^{n}\mathfrak{D}(+,j,j-1,\ldots,3,2,1,n,n-1,\ldots,j+1,-)\,. \tag{3.21}$$

Hence, the tree-level result (2.8) provides evidence for our central conjecture

$$J_{\vec{\eta}}^{\tau}(\gamma|\rho)\big|_{\mathrm{LP}} = \mathrm{sv}\,B_{\vec{\eta}}^{\tau}(\gamma|\rho)\big|_{\mathrm{LP}}\,, \tag{3.22}$$

where the notation $\big|_{\mathrm{LP}}$ instructs to only keep the Laurent polynomials in τ and $\mathrm{Im}\,\tau$ while discarding any contribution $\sim q, \bar{q}$. The conjectural part of (3.22) concerns the non-constant parts of the Laurent polynomials, i.e. corrections $\sim (\log q)^{\pm 1}$ to the expansion around the cusp $q = 0$, so it is not implied by the Betti–deRham duality of (3.2) and (3.19) which only holds at the cusp. That is why we support (3.22) by extensive tests at low orders in η_j, α' as detailed below, and by the fact that the asymptotic expansions of the Green functions are related by the single-valued map with $\mathrm{sv}\,\log\sigma_{ij} = \log|\sigma_{ij}|^2$,

$$G_{\mathfrak{B}}(z_{ij},\tau)\big|_{\mathrm{LP}} = -\frac{iT}{6} - \frac{i\zeta_2}{T} + \frac{1}{2}(\log\sigma_i + \log\sigma_j) - \log|\sigma_{ij}| + \frac{i(\log\sigma_i - \log\sigma_j)^2}{4T} \tag{3.23}$$

$$G_{\mathfrak{I}}(z_{ij},\tau)\big|_{\mathrm{LP}} = \frac{y}{3} + \log|\sigma_i| + \log|\sigma_j| - 2\log|\sigma_{ij}| + \frac{\big(\log|\sigma_i| - \log|\sigma_j|\big)^2}{2y}$$

$$= \mathrm{sv}\,G_{\mathfrak{B}}(z_{ij},\tau)\big|_{\mathrm{LP}}\,. \tag{3.24}$$

Note that the absolute value in (3.23) is due to the argument $|u|\tau$ of θ_1 in (3.4). For the two-point instances $B_{\vec{\eta}}^{\tau}$ and $J_{\vec{\eta}}^{\tau}$ of the open- and closed-string integrals (3.1) and (3.12), the Laurent polynomials in the asymptotics at the cusp can be determined [95] by mild generalizations of

the techniques in [55, 16] (also see [56] for an alternative approach to the closed-string case):

$$B_\eta^\tau(2|2)\big|_{\mathrm{LP}} = \exp\left(-\frac{is_{12}\zeta_2}{T} - \frac{is_{12}T}{6}\right) \tag{3.25}$$

$$\times\left\{\left[i\cot(\eta_2 T)+1\right]\exp\left(\frac{is_{12}}{4T}\partial_{\eta_2}^2\right)\frac{1}{s_{12}+2\eta_2}\left[\frac{\Gamma(1+\frac{s_{12}}{2}+\eta_2)\Gamma(1-s_{12})}{\Gamma(1-\frac{s_{12}}{2}+\eta_2)} - e^{iT(\frac{s_{12}}{2}+\eta_2)}\right]\right.$$

$$+\left[i\cot(\eta_2 T)-1\right]\exp\left(\frac{is_{12}}{4T}\partial_{\eta_2}^2\right)\frac{1}{s_{12}-2\eta_2}\left[\frac{\Gamma(1+\frac{s_{12}}{2}-\eta_2)\Gamma(1-s_{12})}{\Gamma(1-\frac{s_{12}}{2}-\eta_2)} - e^{iT(\frac{s_{12}}{2}-\eta_2)}\right]$$

$$\left.+\frac{1}{s_{12}}\exp\left(\frac{is_{12}}{4T}\partial_{\eta_2}^2\right)\left[\frac{\Gamma(1+\frac{s_{12}}{2}-\eta_2)\Gamma(1-s_{12})}{\Gamma(1-\frac{s_{12}}{2}-\eta_2)} - \frac{\Gamma(1+\frac{s_{12}}{2}+\eta_2)\Gamma(1-s_{12})}{\Gamma(1-\frac{s_{12}}{2}+\eta_2)}\right]\right\}$$

$$J_\eta^\tau(2|2)\big|_{\mathrm{LP}} = \exp\left(\frac{s_{12}y}{3}\right) \tag{3.26}$$

$$\times\left\{\left[i\cot(2i\eta_2 y)+1\right]\exp\left(\frac{s_{12}}{8y}\partial_{\eta_2}^2\right)\frac{1}{s_{12}+2\eta_2}\left[\frac{\Gamma(1+\frac{s_{12}}{2}+\eta_2)\Gamma(1-s_{12})\Gamma(1+\frac{s_{12}}{2}-\eta_2)}{\Gamma(1-\frac{s_{12}}{2}+\eta_2)\Gamma(1+s_{12})\Gamma(1-\frac{s_{12}}{2}-\eta_2)} - e^{-y(s_{12}+2\eta_2)}\right]\right.$$

$$\left.+\left[i\cot(2i\eta_2 y)-1\right]\exp\left(\frac{s_{12}}{8y}\partial_{\eta_2}^2\right)\frac{1}{s_{12}-2\eta_2}\left[\frac{\Gamma(1+\frac{s_{12}}{2}+\eta_2)\Gamma(1-s_{12})\Gamma(1+\frac{s_{12}}{2}-\eta_2)}{\Gamma(1-\frac{s_{12}}{2}+\eta_2)\Gamma(1+s_{12})\Gamma(1-\frac{s_{12}}{2}-\eta_2)} - e^{-y(s_{12}-2\eta_2)}\right]\right\}$$

These two-point expressions are easily seen to line up with the all-multiplicity claim (3.22) since

$$\mathrm{sv}\left[\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-a-b)}\right] = \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(1+a+b)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1-a-b)}, \tag{3.27}$$

and the last line of (3.25) therefore vanishes under sv. Moreover, we have checked the three-point Laurent polynomials to obey (3.22) to the orders in the s_{ij}- and η_j-expansions where MGFs such as (2.26) of total modular weight 10 occur[11]. Finally, we have checked (3.22) to hold at four points to the orders where MGFs of total modular weight 8 occur, at least for contributions from $\varphi_{\vec{\eta}}^\tau$ in (2.13) without any singular factors of $f^{(1)}(z_{ij}, \tau)$.[12] These checks are based on Enriquez' methods [3] (also see appendix B of [51]) to determine the Laurent polynomials of B-cycle eMZVs. The Laurent polynomials for all B-cycle eMZVs with (length+weight) ≤ 16 obtained from a FORM implementation [96] of these methods are available for download [97].

While the two-point Laurent-polynomials generated by (3.25) and (3.26) only involve Riemann zeta values, higher-point examples also introduce irreducible MZVs of depth ≥ 2. The appearance of $\zeta_{3,5}$ in B-cycle Laurent polynomials is later on exemplified in (4.27) and (5.36). Moreover, the appearance of $\zeta_{3,5,3}$ in open- and closed-string calculations at three points in agreement with (3.22) was observed in section 3.3.5 of [51], based on earlier closed-string computations [12].

3.5 Single-valued correspondence of the differential equations

The holomorphic derivatives of the $B_{\vec{\eta}}^\tau$ and $J_{\vec{\eta}}^\tau$-integrals (3.1) and (3.12) can be easily deduced from (2.19): In the open-string case, the modular S transformation relating $B_{\vec{\eta}}^\tau = Z_{\vec{\eta}}^{-1/\tau}$ and

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[11]This amounts to performing the α'- and η-expansion to order 10 in the terminology of section 3.4.2 of [18].

[12]We have excluded the singular functions $f^{(1)}(z_{ij}, \tau) = \frac{1}{z_{ij}} + \mathcal{O}(z_{ij})$ in the integrand from our checks to avoid the tedious treatment of the resulting kinematic poles in the α'-expansion. For the contributions of V_0 and V_1 to the integrand $\overline{V(1,2,3,4|\tau)}$ in (3.8), we have checked the Laurent polynomials from up to one factor of $f^{(1)}(z_{ij}, \tau)$ in the integrand to obey (3.22), see section 3.6 for the disentanglement of different V_w entering $\overline{V(1,2,\dots,n|\tau)}$.

the modular weight $(k,0)$ of the holomorphic Eisenstein series G_k give rise to

$$2\pi i \partial_\tau B_{\vec{\eta}}^\tau(\gamma|\rho) = \sum_{k=0}^\infty (1-k)\tau^{k-2}G_k(\tau) \sum_{\alpha \in S_{n-1}} r_{\vec{\eta}}(\epsilon_k)_\rho{}^\alpha B_{\vec{\eta}}^\tau(\gamma|\alpha) \,, \tag{3.28}$$

see [60] for the n-point derivations $r_{\vec{\eta}}(\epsilon_k)$ and (2.21) for their two-point examples.

In the closed-string case, the $\overline{V(\dots|\tau)}$ in (3.12) are not affected by holomorphic derivatives, and one can import a simplified version of the differential equations in [61] where contributions $\sim \bar{\eta}_j \partial_{\eta_j}$ are absent,

$$2\pi i \partial_\tau J_{\vec{\eta}}^\tau(\gamma|\rho) = \sum_{k=0}^\infty (1-k)(\tau-\bar{\tau})^{k-2}G_k(\tau) \sum_{\alpha \in S_{n-1}} \mathrm{sv}\, r_{\vec{\eta}}(\epsilon_k)_\rho{}^\alpha J_{\vec{\eta}}^\tau(\gamma|\alpha) \,. \tag{3.29}$$

By the differential equations (2.18) of the Green functions, also the term $\sim \zeta_2$ in $r_{\vec{\eta}}(\epsilon_0)$ is absent which we have indicated through the sv-notation,

$$\mathrm{sv}\, r_{\vec{\eta}}(\epsilon_k) = \begin{cases} r_{\vec{\eta}}(\epsilon_k) & : k \geq 4 \\ r_{\vec{\eta}}(\epsilon_0) - 2\zeta_2 s_{12\dots n} & : k = 0 \end{cases} \,, \tag{3.30}$$

where $r_{\vec{\eta}}(\epsilon_2) = 0$. The building blocks of the closed-string differential operator in (3.29) are related to those in the open-string analogue (3.28) through an extension SV of the single-valued map

$$\mathrm{SV}\left[\tau^{k-2}G_k(\tau)r_{\vec{\eta}}(\epsilon_k)\right] = \mathrm{SV}\left[\tau^{k-2}\right]\mathrm{SV}\left[G_k(\tau)\right]\mathrm{SV}\left[r_{\vec{\eta}}(\epsilon_k)\right]$$

$$= (\tau-\bar{\tau})^{k-2}G_k(\tau)\,\mathrm{sv}\, r_{\vec{\eta}}(\epsilon_k) \,, \tag{3.31}$$

which is taken to preserve the properties of sv,

$$\mathrm{SV}\, \zeta_{n_1,n_2,\dots,n_r} = \mathrm{sv}\, \zeta_{n_1,n_2,\dots,n_r} \,, \qquad \mathrm{SV}\, T = 2iy \tag{3.32}$$

and to furthermore preserve $G_k(\tau)$ and the η_j-variables, cf. (3.31). In other words, the differential operator $\mathcal{O}_{\vec{\eta}}^\tau = \sum_{k=0}^\infty (1-k)\tau^{k-2}G_k(\tau)r_{\vec{\eta}}(\epsilon_k)$ appearing in (3.28) and its closed-string analogue in (3.29) are related by

$$2\pi i \partial_\tau B_{\vec{\eta}}^\tau(\gamma|\rho) = \sum_{\alpha \in S_{n-1}} \mathcal{O}_{\vec{\eta}}^\tau(\rho|\alpha)B_{\vec{\eta}}^\tau(\gamma|\alpha) \quad \leftrightarrow \quad 2\pi i \partial_\tau J_{\vec{\eta}}^\tau(\gamma|\rho) = \sum_{\alpha \in S_{n-1}} \left[\mathrm{SV}\, \mathcal{O}_{\vec{\eta}}^\tau(\rho|\alpha)\right] J_{\vec{\eta}}^\tau(\gamma|\alpha) \,. \tag{3.33}$$

From the above discussion, both the $\tau \to i\infty$ asymptotics and the differential operators of the open- and closed-string integrals $B_{\vec{\eta}}^\tau$ and $J_{\vec{\eta}}^\tau$ are related by the SV map (3.32). Hence, we propose that the solutions of (3.33) yield an appropriate extension of the SV map

$$J_{\vec{\eta}}^\tau(\gamma|\rho) = \mathrm{SV}\, B_{\vec{\eta}}^\tau(\gamma|\rho) \,. \tag{3.34}$$

By construction, this SV map commutes with the holomorphic τ-derivative and, under the assumption (3.22), it is consistent at the level of the Laurent polynomials at the cusp. Compatibility of a single-valued map at genus one with $\tau \to i\infty$ generalizes the fact that the single-valued

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map of multiple polylogarithms commutes with evaluation [46]. Moreover, by the evidence to be discussed in section 4.3, the SV map is expected to be compatible with the shuffle product. As we will see in the next section, the α'-expansion of (3.34) induces an elliptic single-valued map for the eMZVs generated by $B_{\vec\eta}^\tau$ which yields the MGFs generated by $J_{\vec\eta}^\tau$.

Let us consider the scope of our definition (3.34). Firstly, not all the holomorphic iterated Eisenstein integrals appear in the α'-expansion of $B_{\vec\eta}^\tau$. As was discussed in [17, 18] and will become clearer when we discuss the α'-expansion of the solution of (3.33), relations among the $r_{\vec\eta}(\epsilon_k)$ such as (2.22) lead to dropouts of certain iterated Eisenstein integrals from eMZVs and $Y_{\vec\eta}^\tau$ and thereby from $B_{\vec\eta}^\tau$ and $J_{\vec\eta}^\tau$. Hence, (3.34) does not comprise the SV-map for the combinations of iterated Eisenstein integrals affected by these dropouts, starting with double integrals involving G_4 and G_{10}.

By contrast, the SV-map of arbitrary convergent eMZVs can be extracted from (3.34) at sufficiently high multiplicity: This follows from the fact that, for $\omega(n_1, \ldots, n_r|\tau)$ in (2.24) with given entries n_j (where $n_1, n_r \neq 1$), one can engineer a combination of genus-one open-string integrals, where the desired eMZV occurs at the zeroth order in s_{ij}.[13]

Finally, one could wonder whether holomorphic cusp forms lead to ambiguities in the definition of the $V(1, \ldots, n|\tau)$ in (3.7):[14] Starting from $n = 14$ points, their defining properties including simple-pole residues, the modularity of their constituents and their behavior (3.19) at the cusp are unchanged when adding combinations of holomorphic cusp forms and lower-weight $V_w(1, \ldots, n|\tau)$. However, adding a cusp form without any z_j-dependent coefficient to $V(1, \ldots, n|\tau)$ leads to a contradiction with the requirement that the τ-independent η^{1-n} order of $B_{\vec\eta}^\tau$ is mapped to the same term $\sim \eta^{1-n}$ in the corresponding $J_{\vec\eta}^\tau$ integral. Products of $V_w(1, \ldots, n|\tau)$ with cusp forms in turn would violate the pole structure (3.11) that reflects the boundary structure of the dual cycles. Hence, the above requirements do not leave any room to modify $V(1, \ldots, n|\tau)$ by holomorphic cusp forms.

3.6 Dual modular weights for cycles

Given that the antielliptic $\overline{V_w(\ldots|\tau)}$-functions (3.6) carry modular weight $(0, w)$, their combinations $\overline{V(\ldots|\tau)}$ (3.7) mix different modular weights. Hence, the α'-expansion of the generating function (3.12) with $\overline{V(\ldots|\tau)}$ in the integrand mixes modular forms of different weight, even at fixed order in η_j. One may wish to isolate the contributions at fixed modular weights and study

$$J_{w,\vec\eta}^\tau(\gamma|\rho) = \frac{(2i)^{n-1}}{(-2\pi i)^w} \int_{\mathfrak{T}^{n-1}} \Big(\prod_{j=2}^{n} d^2 z_j \Big) \prod_{1 \leq i < j}^{n} e^{s_{ij}\mathcal{G}_{\mathfrak{T}}(z_{ij}, \tau)} \overline{V_w(1, \gamma(2, \ldots, n)|\tau)} \varphi_{(\tau-\bar\tau)\vec\eta}^\tau(1, \rho(2, \ldots, n)),$$

(3.35)

with $0 \leq w \leq n-2$, where the terms at homogeneity degree m in the η_j are modular forms of weight $(0, 1-n-m+w)$. One can still identify combinations of integration cycles (3.2) to

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[13]By a similar argument, each MGF can be realized in the s_{ij}^0-order of Y^τ-integrals at sufficiently high multiplicity, see section 2.5 of [61].

[14]We are grateful to Nils Matthes for valuable discussions on this point.

write (3.35) at fixed modular weight w and ordering γ as the single-valued version of known open-string integrals: Each $V_w(1, 2, \ldots, n|\tau)$ with $w \leq n{-}2$ is expressible via permutation sums

$$V_w(1, 2, \ldots, n|\tau) = (2\pi i)^w \sum_{\gamma \in S_{n-1}} c_{w,\gamma} V(1, \gamma(2, \ldots, n)|\tau) \tag{3.36}$$

with coefficients $c_{w,\gamma} \in \mathbb{Q}$, e.g.

$$V_0(1, \ldots, n|\tau) = 1 = \sum_{\gamma \in S_{n-1}} V(1, \gamma(2, \ldots, n)|\tau) \tag{3.37}$$

$$V_1(1, 2, 3|\tau) = i\pi \big[V(1, 2, 3|\tau) - V(1, 3, 2|\tau) \big] \tag{3.38}$$

$$V_1(1, 2, 3, 4|\tau) = 2\pi i \big[V(1, 2, 3, 4|\tau) - V(1, 4, 3, 2|\tau) \big] \tag{3.39}$$

$$V_2(1, 2, 3, 4|\tau) = \frac{(2\pi i)^2}{6} \big[2V(1, 2, 3, 4|\tau) + 2V(1, 4, 3, 2|\tau) - V(1, 2, 4, 3|\tau)$$
$$- V(1, 3, 4, 2|\tau) - V(1, 3, 2, 4|\tau) - V(1, 4, 2, 3|\tau) \big] . \tag{3.40}$$

These relations and coefficients $c_{w,\gamma}$ can be traced back to the symmetries of the V_w-functions including the cyclicity (3.13), reflection property

$$V_w(1, 2, \ldots, n|\tau) = (-1)^w V_w(n, \ldots, 2, 1|\tau) \tag{3.41}$$

and corollaries of the Fay identity [98] which have been discussed in [78, 99]. An independent method based on the degeneration (3.19) to determine the $c_{w,\gamma}$ is described in appendix A. As a result, there are less than $(n{-}1)!$ independent permutations $V_w(1, \gamma(2, \ldots, n)|\tau)$ at fixed $0 \leq w \leq n{-}2$ and $n \geq 3$. Their counting is governed by the unsigned Stirling number $S_{n-1, n-w-1}$ of the first kind (where $S_{a,b}$ counts the number of permutations of a elements with b disjoint cycles) as exemplified in table 1. In particular, permutations of $V_{w=n-2}(1, \ldots, n|\tau)$ are related

n \ w	0	1	2	3	4	5
2	1	0	0	0	0	0
3	1	1	0	0	0	0
4	1	3	2	0	0	0
5	1	6	11	6	0	0
6	1	10	35	50	24	0
7	1	15	85	225	274	120

Table 1: *Examples of the unsigned Stirling numbers $S_{n-1, n-w-1}$ which count the number of independent permutations $\gamma \in S_{n-1}$ of $V_w(1, \gamma(2, \ldots, n)|\tau)$.*

by Kleiss–Kuijf relations [100, 99]

$$V_{n-2}(1, (a_2, \ldots, a_j) \shuffle (a_{j+1}, \ldots, a_n)|\tau) = 0, \qquad j = 2, 3, \ldots, n{-}1 \tag{3.42}$$

such as

$$V_1(1,2,3|\tau) = -V_1(1,3,2|\tau), \qquad V_2(1,2,3,4|\tau) + \text{cyc}(2,3,4) = 0, \qquad (3.43)$$

consistent with the counting $S_{n-1,1} = (n-2)!$ of independent permutations.

Given the decomposition (3.36) of a given V_w function with rational coefficients $c_{w,\gamma}$, one can by (3.34) write each $J^\tau_{w,\vec\eta}$ integral (3.35) as a combination of single-valued B-cycle integrals

$$J^\tau_{w,\vec\eta}(2,\ldots,n|\rho) = \text{SV} \sum_{\gamma \in S_{n-1}} c_{w,\gamma} B^\tau_{\vec\eta}(\gamma|\rho). \qquad (3.44)$$

For instance, the equivalent

$$J^\tau_{0,\eta_2,\eta_3}(2,3|\rho) = \text{SV}\big[B^\tau_{\eta_2,\eta_3}(2,3|\rho) + B^\tau_{\eta_2,\eta_3}(3,2|\rho)\big] \qquad (3.45)$$

$$J^\tau_{1,\eta_2,\eta_3}(2,3|\rho) = \frac{1}{2}\text{SV}\big[B^\tau_{\eta_2,\eta_3}(2,3|\rho) - B^\tau_{\eta_2,\eta_3}(3,2|\rho)\big]$$

of (3.34) together with (3.37) and (3.38) suggests to assign a formal "dual modular weight" 0 and 1 to the symmetric and antisymmetric three-point cycles, respectively,

$$\mathfrak{B}(2,3) + \mathfrak{B}(3,2) \quad \leftrightarrow \quad \text{dual modular weight } 0 \qquad (3.46)$$

$$\mathfrak{B}(2,3) - \mathfrak{B}(3,2) \quad \leftrightarrow \quad \text{dual modular weight } 1.$$

Similarly, combining (3.34) with (3.37), (3.39) and (3.40) leads to the following dual modular weights (d.m.w.) for four-point cycles

$$\mathfrak{B}(2,3,4) + \mathfrak{B}(4,3,2) + \mathfrak{B}(2,4,3) + \mathfrak{B}(3,4,2) + \mathfrak{B}(3,2,4) + \mathfrak{B}(4,2,3) \quad \leftrightarrow \quad \text{d.m.w. } 0$$

$$\mathfrak{B}(2,3,4) - \mathfrak{B}(4,3,2) \quad \leftrightarrow \quad \text{d.m.w. } 1 \qquad (3.47)$$

$$2\mathfrak{B}(2,3,4) + 2\mathfrak{B}(4,3,2) - \mathfrak{B}(2,4,3) - \mathfrak{B}(3,4,2) - \mathfrak{B}(3,2,4) - \mathfrak{B}(4,2,3) \quad \leftrightarrow \quad \text{d.m.w. } 2,$$

see section 5.5 for a more detailed discussion of the weight-two case. Finally, the all-multiplicity formula (3.36) translates into

$$\sum_{\gamma \in S_{n-1}} c_{w,\gamma} \mathfrak{B}(\gamma(2,3,\ldots,n)) \quad \leftrightarrow \quad \text{dual modular weight } w, \qquad (3.48)$$

see appendix A for the rational coefficients $c_{w,\gamma}$ and table 1 for the counting of independent n-point cycles with dual modular weight w.

4 Single-valued iterated Eisenstein integrals from α'-expansions

The goal of this section is to provide the explicit form of the single-valued map SV on eMZVs by reading (3.34) at the level of the α'- and η_j-expansions of $B^\tau_{\vec\eta}$ and $J^\tau_{\vec\eta}$. We will represent eMZVs through iterated Eisenstein integrals [17] in their formulation with integration kernels $\tau^j G_k$, $k \geq 4$ [101],

$$\mathcal{E}\begin{bmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{bmatrix}; \tau\Big] = (-1)^\ell \int\limits_{0<q_1<q_2<\ldots<q_\ell<q} \frac{dq_1}{q_1}\frac{dq_2}{q_2}\cdots\frac{dq_\ell}{q_\ell} \prod_{r=1}^\ell \frac{(2\pi i \tau_r)^{j_r} G_{k_r}(\tau_r)}{(2\pi i)^{k_r}}. \qquad (4.1)$$

The entries are taken to obey $k_r \geq 4$ and $0 \leq j_r \leq k_r - 2$, and we employ tangential-base-point regularization for the divergences as $q_r \to 0$ [101], which implies that the iterated Eisenstein integrals $\mathcal{E}[\ldots; \tau]$ vanish in the regularized limit $\tau \to i\infty$.

4.1 Improving the differential equations

We shall now derive the structure of the α'-expansion of $B_{\vec{\eta}}^\tau$ and $J_{\vec{\eta}}^\tau$ by repeating the key steps of [18] in solving the differential equation (2.19) of $Y_{\vec{\eta}}^\tau$. The first step is to introduce redefined generating series $\widehat{B}_{\vec{\eta}}^\tau$ and $\widehat{J}_{\vec{\eta}}^\tau$ by

$$\widehat{B}_{\vec{\eta}}^\tau = \exp\left(-\frac{r_{\vec{\eta}}(\epsilon_0)}{2\pi i \tau}\right) B_{\vec{\eta}}^\tau, \qquad \widehat{J}_{\vec{\eta}}^\tau = \exp\left(-\frac{\mathrm{sv}\, r_{\vec{\eta}}(\epsilon_0)}{2\pi i (\tau - \bar{\tau})}\right) J_{\vec{\eta}}^\tau. \tag{4.2}$$

After this redefinition, the $k = 0$ terms involving $\frac{\mathrm{G}_0}{\tau^2}$ and $\frac{\mathrm{G}_0}{(\tau - \bar{\tau})^2}$ with $\mathrm{G}_0 = -1$ are absent from the analogues of the differential equations (3.28) and (3.29), see (4.4) and (4.5) below. Throughout this section, we suppress the permutations γ, ρ labeling $B_{\vec{\eta}}^\tau(\gamma|\rho)$ and $J_{\vec{\eta}}^\tau(\gamma|\rho)$, and all matrix representations $r_{\vec{\eta}}(\epsilon_k)$ are understood to act matrix-multiplicatively on the ρ-entry.

Since the $r_{\vec{\eta}}(\epsilon_k)$ are expected (as tested for a wide range of k and n) to inherit the ad-nilpotency relations of the derivation algebra,

$$\mathrm{ad}_{r_{\vec{\eta}}(\epsilon_0)}^{k-1} r_{\vec{\eta}}(\epsilon_k) = 0, \qquad k \geq 4 \tag{4.3}$$

the combinations $\exp\left(-\frac{r_{\vec{\eta}}(\epsilon_0)}{2\pi i \tau}\right) r_{\vec{\eta}}(\epsilon_{k \geq 4}) \exp\left(\frac{r_{\vec{\eta}}(\epsilon_0)}{2\pi i \tau}\right)$ in the differential equations of the redefined integrals (4.2) truncate to a finite number of terms and we obtain

$$2\pi i \partial_\tau \widehat{B}_{\vec{\eta}}^\tau = \sum_{k=4}^\infty (1-k) \sum_{j=0}^{k-2} \frac{1}{j!} \left(\frac{-1}{2\pi i}\right)^j \tau^{k-2-j} \mathrm{G}_k(\tau) r_{\vec{\eta}}\left(\mathrm{ad}_{\epsilon_0}^j(\epsilon_k)\right) \widehat{B}_{\vec{\eta}}^\tau \tag{4.4}$$

$$2\pi i \partial_\tau \widehat{J}_{\vec{\eta}}^\tau = \sum_{k=4}^\infty (1-k) \sum_{j=0}^{k-2} \frac{1}{j!} \left(\frac{-1}{2\pi i}\right)^j (\tau - \bar{\tau})^{k-2-j} \mathrm{G}_k(\tau) r_{\vec{\eta}}\left(\mathrm{ad}_{\epsilon_0}^j(\epsilon_k)\right) \widehat{J}_{\vec{\eta}}^\tau. \tag{4.5}$$

We have used that $\mathrm{ad}_{r_{\vec{\eta}}(\epsilon_0)}(\cdot) = [r_{\vec{\eta}}(\epsilon_0), \cdot] = \mathrm{ad}_{\mathrm{sv}\, r_{\vec{\eta}}(\epsilon_0)}(\cdot)$ (since the term $\sim \zeta_2$ in (3.30) suppressed by sv is commutative) and employ the shorthands

$$r_{\vec{\eta}}\left(\mathrm{ad}_{\epsilon_0}^j(\epsilon_k)\right) = \mathrm{ad}_{r_{\vec{\eta}}(\epsilon_0)}^j r_{\vec{\eta}}(\epsilon_k), \qquad r_{\vec{\eta}}(\epsilon_{k_1} \epsilon_{k_2}) = r_{\vec{\eta}}(\epsilon_{k_1}) r_{\vec{\eta}}(\epsilon_{k_2}). \tag{4.6}$$

4.2 The α'-expansion of $B_{\vec{\eta}}^\tau$

By the differential equation

$$2\pi i \partial_\tau \mathcal{E}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix}; \tau\right] = -(2\pi i)^{2-k_\ell} (2\pi i \tau)^{j_\ell} \mathrm{G}_{k_\ell}(\tau) \mathcal{E}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_{\ell-1} \\ k_1 & k_2 & \cdots & k_{\ell-1} \end{smallmatrix}; \tau\right] \tag{4.7}$$

of the iterated Eisenstein integrals (4.1), one can solve the differential equation (4.4) of the generating series through the path-ordered exponential

$$\widehat{B}_{\vec{\eta}}^\tau = \sum_{\ell=0}^\infty \sum_{\substack{k_1, k_2, \ldots, k_\ell \\ =4,6,8,\ldots}} \sum_{j_1=0}^{k_1-2} \sum_{j_2=0}^{k_2-2} \cdots \sum_{j_\ell=0}^{k_\ell-2} \left(\prod_{i=1}^\ell \frac{(-1)^{j_i}(k_i-1)}{(k_i-j_i-2)!}\right) \mathcal{E}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix}; \tau\right] \tag{4.8}$$

$$\times r_{\vec{\eta}}\left(\mathrm{ad}_{\epsilon_0}^{k_\ell-j_\ell-2}(\epsilon_{k_\ell}) \ldots \mathrm{ad}_{\epsilon_0}^{k_2-j_2-2}(\epsilon_{k_2}) \mathrm{ad}_{\epsilon_0}^{k_1-j_1-2}(\epsilon_{k_1})\right) \widehat{B}_{\vec{\eta}}^{i\infty}$$

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for some initial value $\widehat{B}_{\vec{\eta}}^{i\infty}$ to be discussed below. By inverting the redefinition (4.2) and moving the exponential to act directly on the initial value, we obtain the open-string analogue

$$B_{\vec{\eta}}^{\tau} = \sum_{\ell=0}^{\infty} \sum_{\substack{k_1,k_2,\ldots,k_\ell \\ =4,6,8,\ldots}} \sum_{j_1=0}^{k_1-2} \sum_{j_2=0}^{k_2-2} \cdots \sum_{j_\ell=0}^{k_\ell-2} \left(\prod_{i=1}^{\ell} \frac{(-1)^{j_i}(k_i-1)}{(k_i-j_i-2)!} \right) \beta\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{smallmatrix} ; \tau \right] \tag{4.9}$$

$$\times r_{\vec{\eta}}\left(\mathrm{ad}_{\epsilon_0}^{k_\ell-j_\ell-2}(\epsilon_{k_\ell}) \ldots \mathrm{ad}_{\epsilon_0}^{k_2-j_2-2}(\epsilon_{k_2}) \mathrm{ad}_{\epsilon_0}^{k_1-j_1-2}(\epsilon_{k_1}) \right) \exp\left(\frac{r_{\vec{\eta}}(\epsilon_0)}{2\pi i \tau} \right) \widehat{B}_{\vec{\eta}}^{i\infty}$$

of the key result for the α'-expansion of $Y_{\vec{\eta}}^{\tau}$ in (3.11) of [18]. In commuting $\exp(\frac{r_{\vec{\eta}}(\epsilon_0)}{2\pi i \tau})$ past the ϵ_{k_j}, the iterated Eisenstein integrals are rearranged into the combinations

$$\beta\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix} ; \tau \right] = \sum_{p_1=0}^{k_1-j_1-2} \binom{k_1-j_1-2}{p_1} \left(\frac{i}{2T} \right)^{p_1} \mathcal{E}\left[\begin{smallmatrix} j_1+p_1 \\ k_1 \end{smallmatrix} ; \tau \right] \tag{4.10}$$

and more generally

$$\beta\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{smallmatrix} ; \tau \right] = \sum_{p_1=0}^{k_1-j_1-2} \sum_{p_2=0}^{k_2-j_2-2} \cdots \sum_{p_\ell=0}^{k_\ell-j_\ell-2} \binom{k_1-j_1-2}{p_1} \binom{k_2-j_2-2}{p_2} \cdots \binom{k_\ell-j_\ell-2}{p_\ell}$$

$$\times \left(\frac{i}{2T} \right)^{p_1+p_2+\ldots+p_\ell} \mathcal{E}\left[\begin{smallmatrix} j_1+p_1 & j_2+p_2 & \ldots & j_\ell+p_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{smallmatrix} ; \tau \right] . \tag{4.11}$$

Note that (4.9) is an alternative[15] organization of open-string α'-expansions at genus one as compared to [60, 59]. Non-planar B-cycle integrals obey the same differential equation (3.28) as the planar ones and therefore have an α'-expansion of the same form (4.9), only their initial values $\widehat{B}_{\vec{\eta}}^{i\infty}$ need to be adapted to the non-planar integration cycle.

The modified iterated Eisenstein integrals $\beta[\ldots]$ satisfy the differential equations

$$2\pi i \tau^2 \partial_\tau \beta\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_\ell \\ k_1 & k_2 & \ldots & k_\ell \end{smallmatrix} ; \tau \right] = \sum_{i=1}^{\ell} (k_i-j_i-2) \beta\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_{i-1} & j_i+1 & j_{i+1} & \ldots & j_\ell \\ k_1 & k_2 & \ldots & k_{i-1} & k_i & k_{i+1} & \ldots & k_\ell \end{smallmatrix} ; \tau \right] \tag{4.12}$$

$$- \delta_{j_\ell,k_\ell-2} \tau^{k_\ell} \mathrm{G}_{k_\ell}(\tau) \beta\left[\begin{smallmatrix} j_1 & j_2 & \ldots & j_{\ell-1} \\ k_1 & k_2 & \ldots & k_{\ell-1} \end{smallmatrix} ; \tau \right] ,$$

which allows us to directly check that (4.9) obeys (3.28). The integrals $\beta[\ldots]$ inherit the property that they vanish for $\tau \to i\infty$ from the $\mathcal{E}[\ldots]$. Note that the definition (4.11) is equivalent to integral representations such as

$$\beta\left[\begin{smallmatrix} j_1 \\ k_1 \end{smallmatrix} ; \tau \right] = -\frac{(2\pi i)^{1+j_1-k_1}}{\tau^{k_1-j_1-2}} \int_{i\infty}^{\tau} \mathrm{d}\tau_1 \, \mathrm{G}_{k_1}(\tau_1)(\tau-\tau_1)^{k_1-j_1-2} \tau_1^{j_1} \tag{4.13}$$

$$\beta\left[\begin{smallmatrix} j_1 & j_2 \\ k_1 & k_2 \end{smallmatrix} ; \tau \right] = \frac{(2\pi i)^{2+j_1+j_2-k_1-k_2}}{\tau^{k_1+k_2-j_1-j_2-4}} \int_{i\infty}^{\tau} \mathrm{d}\tau_2 \, \mathrm{G}_{k_2}(\tau_2)(\tau-\tau_2)^{k_2-j_2-2} \tau_2^{j_2} \int_{i\infty}^{\tau_2} \mathrm{d}\tau_1 \, \mathrm{G}_{k_1}(\tau_1)(\tau-\tau_1)^{k_1-j_1-2} \tau_1^{j_1} .$$

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[15]On top of the modular S transformation relating $Z_{\vec{\eta}}^{\tau}$ and $B_{\vec{\eta}}^{\tau}$, the $\mathcal{E}[\ldots]$ in (4.8) involve integration kernels $\tau^j \mathrm{G}_k$ with $0 \le j \le k-2$ instead of the $\mathrm{G}_0^0 = -1$ in [60, 59]. In other words, the relations (4.3) in the derivation algebra are built into (4.8), whereas the results in the references may require the use of shuffle relations to manifest the absence of $\mathcal{E}[\ldots \begin{smallmatrix} j \ge k-1 \\ k \end{smallmatrix} \ldots]$.

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The definition (4.11) of the $\beta[\ldots]$ preserves the shuffle relations of the iterated Eisenstein integrals (4.1), for instance

$$\mathcal{E}\begin{bmatrix} j_1 \\ k_1 \end{bmatrix};\tau\Big]\,\mathcal{E}\begin{bmatrix} j_2 \\ k_2 \end{bmatrix};\tau\Big] = \mathcal{E}\begin{bmatrix} j_1\ j_2 \\ k_1\ k_2 \end{bmatrix};\tau\Big] + \mathcal{E}\begin{bmatrix} j_2\ j_1 \\ k_2\ k_1 \end{bmatrix};\tau\Big] \quad \Rightarrow \quad \beta\begin{bmatrix} j_1 \\ k_1 \end{bmatrix};\tau\Big]\,\beta\begin{bmatrix} j_2 \\ k_2 \end{bmatrix};\tau\Big] = \beta\begin{bmatrix} j_1\ j_2 \\ k_1\ k_2 \end{bmatrix};\tau\Big] + \beta\begin{bmatrix} j_2\ j_1 \\ k_2\ k_1 \end{bmatrix};\tau\Big].$$

(4.14)

4.3 The α'-expansion of $J_{\vec{\eta}}^\tau$

One can extend the above strategy to expand $B_{\vec{\eta}}^\tau$ via (4.4) to the $J_{\vec{\eta}}^\tau$ integrals. The idea is to solve their differential equation (4.5) order by order in α' via

$$2\pi i \partial_\tau \mathcal{E}^{\mathrm{sv}}\begin{bmatrix} j_1\ j_2\ \cdots\ j_\ell \\ k_1\ k_2\ \ldots\ k_\ell \end{bmatrix};\tau\Big] = -(2\pi i)^{2-k_\ell+j_\ell}(\tau-\bar\tau)^{j_\ell}\mathrm{G}_{k_\ell}(\tau)\mathcal{E}^{\mathrm{sv}}\begin{bmatrix} j_1\ j_2\ \cdots\ j_{\ell-1} \\ k_1\ k_2\ \ldots\ k_{\ell-1} \end{bmatrix};\tau\Big],$$

(4.15)

using the combinations $\mathcal{E}^{\mathrm{sv}}$ of holomorphic iterated Eisenstein integrals (4.1) and their complex conjugates introduced in [18]. Their depth $\ell=1$ instances are completely known from the reference

$$\mathcal{E}^{\mathrm{sv}}\begin{bmatrix} j_1 \\ k_1 \end{bmatrix};\tau\Big] = \sum_{r_1=0}^{j_1}(-2\pi i\bar\tau)^{r_1}\binom{j_1}{r_1}\Big(\mathcal{E}\begin{bmatrix} j_1-r_1 \\ k_1 \end{bmatrix};\tau\Big] + (-1)^{j_1-r_1}\overline{\mathcal{E}\begin{bmatrix} j_1-r_1 \\ k_1 \end{bmatrix};\tau\Big]}\Big),$$

(4.16)

and their generalizations to depth $\ell \geq 2$ involve antiholomorphic integration constants $\overline{\alpha[\ldots]}$,

$$\mathcal{E}^{\mathrm{sv}}\begin{bmatrix} j_1\ j_2 \\ k_1\ k_2 \end{bmatrix};\tau\Big] = \sum_{r_1=0}^{j_1}\sum_{r_2=0}^{j_2}(-2\pi i\bar\tau)^{r_1+r_2}\binom{j_1}{r_1}\binom{j_2}{r_2}$$

(4.17)

$$\times \Big\{\mathcal{E}\begin{bmatrix} j_1-r_1\ j_2-r_2 \\ k_1\ k_2 \end{bmatrix};\tau\Big] + (-1)^{j_1-r_1}\overline{\mathcal{E}\begin{bmatrix} j_1-r_1 \\ k_1 \end{bmatrix};\tau\Big]}\,\mathcal{E}\begin{bmatrix} j_2-r_2 \\ k_2 \end{bmatrix};\tau\Big]$$

$$+ (-1)^{j_1+j_2-r_1-r_2}\overline{\mathcal{E}\begin{bmatrix} j_2-r_2\ j_1-r_1 \\ k_2\ k_1 \end{bmatrix};\tau\Big]}\Big\} + \overline{\alpha\begin{bmatrix} j_1\ j_2 \\ k_1\ k_2 \end{bmatrix};\tau\Big]}.$$

The integration constants $\overline{\alpha[\ldots]}$ are invariant under $\tau \to \tau + 1$ since the $\mathcal{E}^{\mathrm{sv}}[\ldots]$ and the contributions from the $\mathcal{E}[\ldots],\overline{\mathcal{E}[\ldots]}$ to (4.17) are. They are known on a case-by-case basis, for instance

$$\alpha\begin{bmatrix} 1\ 0 \\ 4\ 4 \end{bmatrix}] = \alpha\begin{bmatrix} 0\ 1 \\ 4\ 4 \end{bmatrix}] = 0\,,$$

$$\alpha\begin{bmatrix} 2\ 0 \\ 4\ 4 \end{bmatrix}] = \frac{2\zeta_3}{3}\Big(\mathcal{E}\begin{bmatrix} 0 \\ 4 \end{bmatrix}] + \frac{i\pi\tau}{360}\Big) = -\alpha\begin{bmatrix} 0\ 2 \\ 4\ 4 \end{bmatrix}]\,,$$

(4.18)

$$\alpha\begin{bmatrix} 2\ 1 \\ 4\ 4 \end{bmatrix}] = \frac{2\zeta_3}{3}\Big(2\pi i\tau\mathcal{E}\begin{bmatrix} 0 \\ 4 \end{bmatrix}] - \mathcal{E}\begin{bmatrix} 1 \\ 4 \end{bmatrix}] - \frac{\pi^2\tau^2}{360}\Big) = -\alpha\begin{bmatrix} 1\ 2 \\ 4\ 4 \end{bmatrix}]\,,$$

and the complete list of $\overline{\alpha\begin{bmatrix} j_1\ j_2 \\ k_1\ k_2 \end{bmatrix}]}$ at $k_1 + k_2 \leq 12$ can be found in an ancillary file within the arXiv submission of this work. The integration constants at arbitrary depth can be determined from the reality properties of the $Y_{\vec{\eta}}^\tau$ integrals [18]. The method in the reference to fix the $\overline{\alpha[\ldots]}$ hinges on the fact that the coefficients in the η_j- and $\bar\eta_j$-expansion of $Y_{\vec{\eta}}^\tau$ are closed under complex conjugation. For the $J_{\vec{\eta}}^\tau$-series in turn the antiholomorphic modular weights $\bar w$ of the integrands $\overline{V(\ldots)}$ in (3.12) are bounded by $\bar w \leq n-2$, so the complex conjugates of higher orders in the η_j expansion are not part of the series. Hence, in the present formulation, the expansion

of the $Y_{\vec{\eta}}^{\tau}$ in [18] is a necessary input to obtain well-defined $\mathcal{E}^{\mathrm{sv}}$. This expansion depends on the knowledge of the initial values of $Y_{\vec{\eta}}^{\tau}$ which is currently available from sphere integrals to arbitrary weight only for two points and is under investigation for higher multiplicity [95].

By repeating the steps towards (4.8) and (4.9), we arrive at the structure of the α'-expansion

$$J_{\vec{\eta}}^{\tau} = \sum_{\ell=0}^{\infty} \sum_{\substack{k_1,k_2,\dots,k_\ell \\ =4,6,8,\dots}} \sum_{j_1=0}^{k_1-2} \sum_{j_2=0}^{k_2-2} \cdots \sum_{j_\ell=0}^{k_\ell-2} \left(\prod_{i=1}^{\ell} \frac{(-1)^{j_i}(k_i-1)}{(k_i-j_i-2)!} \right) \beta^{\mathrm{sv}}\!\begin{bmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{bmatrix}\!;\tau \Big] \tag{4.19}$$
$$\times\, r_{\vec{\eta}}\!\left(\mathrm{ad}_{\epsilon_0}^{k_\ell-j_\ell-2}(\epsilon_{k_\ell})\dots \mathrm{ad}_{\epsilon_0}^{k_2-j_2-2}(\epsilon_{k_2})\mathrm{ad}_{\epsilon_0}^{k_1-j_1-2}(\epsilon_{k_1})\right) \exp\left(-\frac{\mathrm{sv}\, r_{\vec{\eta}}(\epsilon_0)}{4y}\right) \widehat{J}_{\vec{\eta}}^{i\infty}$$

with an initial value $\widehat{J}_{\vec{\eta}}^{i\infty}$ to be discussed below and the combinations analogous to (4.11) [18]

$$\beta^{\mathrm{sv}}\!\begin{bmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{bmatrix}\!;\tau\Big] = \sum_{p_1=0}^{k_1-j_1-2} \sum_{p_2=0}^{k_2-j_2-2} \cdots \sum_{p_\ell=0}^{k_\ell-j_\ell-2} \binom{k_1-j_1-2}{p_1}\binom{k_2-j_2-2}{p_2}\cdots\binom{k_\ell-j_\ell-2}{p_\ell}$$
$$\times \left(\frac{1}{4y}\right)^{p_1+p_2+\dots+p_\ell} \mathcal{E}^{\mathrm{sv}}\!\begin{bmatrix} j_1+p_1 & j_2+p_2 & \cdots & j_\ell+p_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{bmatrix}\!;\tau\Big]. \tag{4.20}$$

The expansion (4.19) solves (3.29) since the β^{sv} inherit their differential equation from (4.15),

$$2\pi i(\tau-\bar{\tau})^2 \partial_\tau \beta^{\mathrm{sv}}\!\begin{bmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{bmatrix}\!;\tau\Big] = \sum_{i=1}^{\ell} (k_i-j_i-2)\beta^{\mathrm{sv}}\!\begin{bmatrix} j_1 & j_2 & \cdots & j_{i-1} & j_i+1 & j_{i+1} & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_{i-1} & k_i & k_{i+1} & \cdots & k_\ell \end{bmatrix}\!;\tau\Big] \tag{4.21}$$
$$- \delta_{j_\ell,k_\ell-2}(\tau-\bar{\tau})^{k_\ell}\mathrm{G}_{k_\ell}(\tau)\beta^{\mathrm{sv}}\!\begin{bmatrix} j_1 & j_2 & \cdots & j_{\ell-1} \\ k_1 & k_2 & \cdots & k_{\ell-1} \end{bmatrix}\!;\tau\Big],$$

see (4.12) for the holomorphic counterpart for $\partial_\tau \beta[\dots]$. Both the $\mathcal{E}^{\mathrm{sv}}[\dots]$ and the $\beta^{\mathrm{sv}}[\dots]$ are expected to preserve the shuffle multiplication of their holomorphic counterparts (4.1) and (4.11): The differential equations (4.15) and (4.20) recursively imply that shuffle relations among $\mathcal{E}^{\mathrm{sv}}[\dots]$ and the $\beta^{\mathrm{sv}}[\dots]$ can at most be violated by antiholomorphic functions such as the integration constants $\overline{\alpha[\dots]}$ in (4.17).[16] All examples of $\alpha\begin{bmatrix} j_1 & j_2 \\ k_1 & k_2 \end{bmatrix}$ up to including $k_1+k_2=12$ were checked to preserve the shuffle relations, and their explicit form can also be found in an ancillary file to the arXiv submission of this work. Note that these checks cover the more intricate cases with $(k_1,k_2)=(4,6)$ and $(k_1,k_2)=(4,8)$ where imaginary cusp forms occur among the MGFs [18,87].

The $\mathcal{E}^{\mathrm{sv}}$ and β^{sv} are expected to occur in Brown's generating series of single-valued iterated Eisenstein integrals [101, 13, 14]. The construction of non-holomorphic modular forms in the references – so-called equivariant iterated Eisenstein integrals – are obtained by augmenting their single-valued counterparts by combinations of MZVs and objects of lower depth. At depth one, the equivariant iterated Eisenstein integrals are non-holomorphic Eisenstein series along

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[16]Moreover, any such violation of shuffle relations would need to be a combination of antiholomorphic iterated Eisenstein integrals (by the differential equation [18, (2.37)] for $\partial_{\bar{\tau}} Y_{\vec{\eta}}^{\tau}$) but at the same time line up with the modular weights in the η_j-expansion of $J_{\vec{\eta}}^{\tau}$, $Y_{\vec{\eta}}^{\tau}$ and the reality properties of the latter. It would be interesting to find a rigorous argument to rule out the existence of such antiholomorphic functions.

with their Cauchy–Riemann derivatives [101, 13, 14]. From their representation [18]

$$(\pi\nabla)^m \mathrm{E}_k = \left(-\frac{1}{4}\right)^m \frac{(2k-1)!}{(k-1)!(k-1-m)!}\left\{-\beta^{\mathrm{sv}}\left[\begin{smallmatrix} k-1+m \\ 2k \end{smallmatrix}\right] + \frac{2\zeta_{2k-1}}{(2k-1)(4y)^{k-1-m}}\right\}$$

$$\mathrm{E}_k = \frac{(2k-1)!}{[(k-1)!]^2}\left\{-\beta^{\mathrm{sv}}\left[\begin{smallmatrix} k-1 \\ 2k \end{smallmatrix}\right] + \frac{2\zeta_{2k-1}}{(2k-1)(4y)^{k-1}}\right\} \tag{4.22}$$

$$\frac{(\pi\overline{\nabla})^m \mathrm{E}_k}{y^{2m}} = \frac{(-4)^m(2k-1)!}{(k-1)!(k-1-m)!}\left\{-\beta^{\mathrm{sv}}\left[\begin{smallmatrix} k-1-m \\ 2k \end{smallmatrix}\right] + \frac{2\zeta_{2k-1}}{(2k-1)(4y)^{k-1+m}}\right\},$$

the β^{sv} are seen to take the role of the single-valued rather than equivariant iterated Eisenstein integrals at depth one. At higher depth, the precise relation of the β^{sv} to Brown's construction is an open question.

4.4 Initial values

It remains to specify the initial values $\widehat{B}_{\vec{\eta}}^{i\infty}$ and $\widehat{J}_{\vec{\eta}}^{i\infty}$ in the α'-expansions (4.9) and (4.19). The Laurent-polynomial contributions from the asymptotics (3.23) and (3.24) of the Green functions are still functions of τ and need to be translated into a series that solely depends on η_j and s_{ij}. Following the construction of a similar initial value for $Y_{\vec{\eta}}^{\tau}$ in section 3.4 of [18], we import the constant parts $\sim \tau^0$ and $\sim (\mathrm{Im}\,\tau)^0$ of the respective Laurent polynomials

$$\widehat{B}_{\vec{\eta}}^{i\infty} = \exp\left(\frac{ir_{\vec{\eta}}(\epsilon_0)}{2T}\right)B_{\vec{\eta}}^{\tau}\big|_{\mathrm{LP}}\big|_{\tau^0} \tag{4.23}$$

$$\widehat{J}_{\vec{\eta}}^{i\infty} = \exp\left(\frac{\mathrm{sv}\,r_{\vec{\eta}}(\epsilon_0)}{4y}\right)J_{\vec{\eta}}^{\tau}\big|_{\mathrm{LP}}\big|_{(\mathrm{Im}\,\tau)^0}. \tag{4.24}$$

In both cases, the exponentials ensure that the negative powers of T in $B_{\vec{\eta}}^{\tau}\big|_{\mathrm{LP}}$ and y in $J_{\vec{\eta}}^{\tau}\big|_{\mathrm{LP}}$ disappear order by order in α'. Hence, (4.23) and (4.24) pick up the lowest powers of T, y present in $\exp(\frac{ir_{\vec{\eta}}(\epsilon_0)}{2T})B_{\vec{\eta}}^{\tau}\big|_{\mathrm{LP}}$ and $\exp(\frac{\mathrm{sv}\,r_{\vec{\eta}}(\epsilon_0)}{4y})J_{\vec{\eta}}^{\tau}\big|_{\mathrm{LP}}$. The leading α'- and η_2-orders of the two-point initial values following from the expressions in (3.25) and (3.26) are

$$\begin{aligned}
\widehat{B}_{\eta_2}^{i\infty} = \frac{1}{\eta_2}&\left[1 + \frac{1}{6}s_{12}^2\zeta_2 + \frac{1}{12}s_{12}^3\zeta_3 + \frac{131}{720}s_{12}^4\zeta_4 + s_{12}^5\left(\frac{17}{360}\zeta_2\zeta_3 + \frac{43\zeta_5}{720}\right) + \mathcal{O}(s_{12}^6)\right] \\
&+ \eta_2\left[-2\zeta_2 - s_{12}\zeta_3 - \frac{29}{12}s_{12}^2\zeta_4 - s_{12}^3\left(\frac{1}{3}\zeta_2\zeta_3 + \frac{5\zeta_5}{6}\right) - s_{12}^4\left(\frac{1}{12}\zeta_3^2 + \frac{87\zeta_6}{40}\right) + \mathcal{O}(s_{12}^5)\right] \\
&+ \eta_2^3\left[-2\zeta_4 + s_{12}(2\zeta_2\zeta_3 - \zeta_5) + s_{12}^2\left(\frac{\zeta_3^2}{2} - \frac{33\zeta_6}{8}\right) + s_{12}^3\left(\frac{9}{4}\zeta_3\zeta_4 + \frac{3}{2}\zeta_2\zeta_5 - \frac{7\zeta_7}{4}\right) + \mathcal{O}(s_{12}^4)\right] \\
&+ \eta_2^5\left[-2\zeta_6 + s_{12}(2\zeta_4\zeta_3 + 2\zeta_2\zeta_5 - \zeta_7) + s_{12}^2\left(-\zeta_2\zeta_3^2 + \zeta_3\zeta_5 - \frac{43\zeta_8}{6}\right) + \mathcal{O}(s_{12}^3)\right] \quad (4.25) \\
&+ \eta_2^7\left[-2\zeta_8 + s_{12}(2\zeta_3\zeta_6 + 2\zeta_4\zeta_5 + 2\zeta_2\zeta_7 - \zeta_9) + \mathcal{O}(s_{12}^2)\right] + \eta_2^9\left[-2\zeta_{10} + \mathcal{O}(s_{12})\right] + \mathcal{O}(\eta_2^{11})
\end{aligned}$$

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as well as

$$\widehat{J}^{i\infty}_{\eta_2} = \frac{1}{\eta_2}\Big[1 + \frac{1}{6}s_{12}^3\zeta_3 + \frac{43}{360}s_{12}^5\zeta_5 + \mathcal{O}(s_{12}^6)\Big]$$
$$+ \eta_2\Big[-2s_{12}\zeta_3 - \frac{5}{3}s_{12}^3\zeta_5 - \frac{1}{3}s_{12}^4\zeta_3^2 + \mathcal{O}(s_{12}^5)\Big]$$
$$+ \eta_2^3\Big[-2s_{12}\zeta_5 + 2s_{12}^2\zeta_3^2 - \frac{7}{2}s_{12}^3\zeta_7 + \mathcal{O}(s_{12}^4)\Big] \qquad (4.26)$$
$$+ \eta_2^5\big[-2s_{12}\zeta_7 + 4s_{12}^2\zeta_3\zeta_5 + \mathcal{O}(s_{12}^3)\big]$$
$$+ \eta_2^7\big[-2s_{12}\zeta_9 + \mathcal{O}(s_{12}^2)\big] + \eta_2^9\mathcal{O}(s_{12}) + \mathcal{O}(\eta_2^{11})\,.$$

Higher orders in s_{12} and η_2 are readily available through the straightforward expansion of exponentials and Γ-functions in (3.25) and (3.26). In particular, these two-point expressions imply that all the coefficients in the s_{12}- and η_2- expansions are combinations of Riemann zeta values for $\widehat{B}^{i\infty}_{\eta_2}$ and odd Riemann zeta values for $\widehat{J}^{i\infty}_{\eta_2}$.

Starting from $n = 3$ points, the initial values $\widehat{B}^{i\infty}_{\vec{\eta}}$ will also feature irreducible MZVs of higher depth. Based on Enriquez' method to generate the Laurent polynomial of B-cycle eMZVs [3] (also see appendix B of [51]) we have determined the three-point initial values to certain orders, and the results are included in an ancillary file to the arXiv submission of this article. To the orders under consideration, we find the following coefficients of $\zeta_{3,5}$

$$\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|2,3)\big|_{\zeta_{3,5}} = \frac{1}{10}(\eta_{23} - 2\eta_3)(2\eta_{23} - \eta_3)(\eta_{23} + \eta_3)$$
$$\times \Big[2\eta_{23}^2 s_{13} - 2\eta_{23}\eta_3 s_{12} - 4\eta_{23}\eta_3 s_{13} + 2\eta_3^2 s_{13} - 2\eta_{23}\eta_3 s_{23} + \eta_{23}s_{12}s_{13} + \eta_3 s_{13}s_{23} + \mathcal{O}(s_{ij}^3)\Big]$$

$$\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|3,2)\big|_{\zeta_{3,5}} = -\frac{1}{10}(\eta_2 - 2\eta_{23})(2\eta_2 - \eta_{23})(\eta_2 + \eta_{23}) \qquad (4.27)$$
$$\times \Big[2\eta_2^2 s_{12} - 4\eta_2\eta_{23}s_{12} + 2\eta_{23}^2 s_{12} - 2\eta_2\eta_{23}s_{13} - 2\eta_2\eta_{23}s_{23} - \eta_{23}s_{12}s_{13} - \eta_2 s_{12}s_{23} + \mathcal{O}(s_{ij}^3)\Big]\,,$$

which by the single-valued maps sv $\zeta_{3,5} = -10\zeta_3\zeta_5$ and sv $\zeta_3\zeta_5 = 4\zeta_3\zeta_5$ enter the closed-string initial values via

$$\widehat{J}^{i\infty}_{\eta_2,\eta_3}(2,3|\rho(2,3))\big|_{\zeta_3\zeta_5} = -10\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|\rho(2,3))\big|_{\zeta_{3,5}} + 4\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|\rho(2,3))\big|_{\zeta_3\zeta_5}\,. \qquad (4.28)$$

Similarly, the MZV $\zeta_{3,5,3}$ seen in Laurent polynomials of both B-cycle integrals [51] and modular graph functions [12] will occur in both $\widehat{B}^{i\infty}_{\eta_2,\eta_3}$ and $\widehat{J}^{i\infty}_{\eta_2,\eta_3}$. The contributions to $\widehat{B}^{i\infty}_{\eta_2,\eta_3}$ involving MZVs of weight up to and including four can be found in appendix B.

Since the initial values are obtained from the Laurent polynomials and the exponents in (4.23) and (4.24) are related by the single-valued map, the conjecture (3.22) supported by tree-level results and extensive genus-one tests is equivalent to

$$\widehat{J}^{i\infty}_{\vec{\eta}}(\gamma|\rho) = \text{sv}\,\widehat{B}^{i\infty}_{\vec{\eta}}(\gamma|\rho)\,, \qquad (4.29)$$

in agreement with (4.25) and (4.26).

The $\tau \to i\infty$ asymptotics of n-point A-cycle integrals (2.9) has been expressed in terms of $(n+2)$-point disk integrals (2.1) in suitable kinematic limits [60]. Similarly, the Laurent

polynomials of n-point genus-one integrals $B_{\vec{\eta}}^\tau, J_{\vec{\eta}}^\tau$ are determined by genus-zero integrals at multiplicity $n+2$ and below, see (3.25) and (3.26) for the explicit two-point result. As will be further investigated in [95], the main challenge is to determine the admixture of lower-point genus-zero integrals that generalize the subtraction of $e^{iT(\frac{s_{12}}{2}\pm\eta_2)}$ and $e^{-y(s_{12}\pm2\eta_2)}$ from the Γ-functions in (3.25) and (3.26).

4.5 The single-valued map on iterated Eisenstein integrals

The proposed single-valued map (3.34) can now also be studied at the level of the α'-expansions. Using (4.29) and (3.30), we find that one obtains $J_{\vec{\eta}}^\tau$ as the single-valued version of $B_{\vec{\eta}}^\tau$ if the coefficients obey

$$\beta^{\mathrm{sv}}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};\tau\right] = \mathrm{SV}\,\beta\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};\tau\right].\tag{4.30}$$

This follows from the relation (4.29) among the initial values and the form of the $r_{\vec{\eta}}$ operators in the respective α'-expansions, recalling that $r_{\vec{\eta}}(\mathrm{ad}_{\epsilon_0}^j(\epsilon_k)) = \mathrm{sv}\,r_{\vec{\eta}}(\mathrm{ad}_{\epsilon_0}^j(\epsilon_k))$.

On the one hand, (4.30) fixes the single-valued map of the eMZVs in the expansion of $B_{\vec{\eta}}^\tau$ that enter through the iterated Eisenstein integrals $\beta[\ldots]$. On the other hand, (4.30) only applies to the combinations $\beta[\ldots]$ and $\beta^{\mathrm{sv}}[\ldots]$ that occur in the path-ordered exponentials (4.9) and (4.19). The SV map of individual $\beta\left[\begin{smallmatrix} j_1 & \cdots & j_\ell \\ k_1 & \cdots & k_\ell \end{smallmatrix};\tau\right]$ remains undetermined whenever the relations in the derivation algebra such as (2.22) lead to dropouts of certain $\beta[\ldots]$ and $\beta^{\mathrm{sv}}[\ldots]$ from $B_{\vec{\eta}}^\tau$ and $J_{\vec{\eta}}^\tau$ (starting with cases at $(k_1, k_2) = (10, 4)$ at depth $\ell = 2$).

Since the factors of $\frac{i}{2T}$ and $\frac{1}{4y}$ in (4.11) and (4.20) are furthermore related by SV, (4.30) is equivalent to

$$\mathcal{E}^{\mathrm{sv}}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};\tau\right] = \mathrm{SV}\,\mathcal{E}\left[\begin{smallmatrix} j_1 & j_2 & \cdots & j_\ell \\ k_1 & k_2 & \cdots & k_\ell \end{smallmatrix};\tau\right],\tag{4.31}$$

again up to cases where the relations in the derivation algebra cause dropouts. For instance, (4.16) implies that the single-valued version of holomorphic Eisenstein integrals (4.1) at depth one is given by

$$\mathrm{SV}\,\mathcal{E}\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\right] = \sum_{r=0}^{j}(-2\pi i\bar{\tau})^r\binom{j}{r}\left(\mathcal{E}\left[\begin{smallmatrix} j-r \\ k \end{smallmatrix};\tau\right] + (-1)^{j-r}\overline{\mathcal{E}\left[\begin{smallmatrix} j-r \\ k \end{smallmatrix};\tau\right]}\right),\tag{4.32}$$

where the contributions on the right-hand side can be recognized as

$$\sum_{r=0}^{j}(-2\pi i\bar{\tau})^r\binom{j}{r}\mathcal{E}\left[\begin{smallmatrix} j-r \\ k \end{smallmatrix};\tau\right] = (2\pi i)^{1-k+j}\int_\tau^{i\infty}\mathrm{d}\tau_1\,(\tau_1-\bar{\tau})^j\,\mathrm{G}_k(\tau_1),\tag{4.33}$$

$$(-1)^j\sum_{r=0}^{j}(2\pi i\bar{\tau})^r\binom{j}{r}\overline{\mathcal{E}\left[\begin{smallmatrix} j-r \\ k \end{smallmatrix};\tau\right]} = -(2\pi i)^{1-k+j}\int_{\bar{\tau}}^{i\infty}\mathrm{d}\bar{\tau}_1\,(\bar{\tau}_1-\bar{\tau})^j\,\overline{\mathrm{G}_k(\tau_1)},$$

respectively. Note that (4.30) fixes the SV-map of all the $\mathcal{E}\left[\begin{smallmatrix} j \\ k \end{smallmatrix};\tau\right]$ at depth one with $k \geq 4$ and $0 \leq j \leq k-2$ since the caveats related to relations in the derivation algebra only affect iterated Eisenstein integrals of depth $\ell \geq 2$.

The iterated Eisenstein integrals defined in (4.1) may be reorganized in terms of [17]

$$\mathcal{E}_0(k_1, k_2, \ldots, k_\ell; \tau) = 2\pi i \int_\tau^{i\infty} \mathrm{d}\tau_\ell \, \frac{\mathrm{G}^0_{k_\ell}(\tau_\ell)}{(2\pi i)^{k_\ell}} \, \mathcal{E}_0(k_1, k_2, \ldots, k_{\ell-1}; \tau) \tag{4.34}$$

with $\mathcal{E}_0(;\tau) = 1$. By subtracting the zero mode of the holomorphic Eisenstein series

$$\begin{aligned}
\mathrm{G}^0_k(\tau) &= \mathrm{G}_k(\tau) - 2\zeta_k \quad \text{for} \quad k \neq 0, \\
\mathrm{G}^0_0(\tau) &= \mathrm{G}_0(\tau) = -1,
\end{aligned} \tag{4.35}$$

the integrals (4.34) are made to converge if $k_1 > 0$.

At depth one they are related to the holomorphic iterated Eisenstein integrals via [51]

$$\mathcal{E}\left[\begin{smallmatrix} j \\ k \end{smallmatrix}; \tau\right] = j! \, \mathcal{E}_0(0^j, k\,; \tau) + \frac{B_k(2\pi i\tau)^{j+1}}{k!(j+1)} \tag{4.36}$$

with Bernoulli numbers B_k. From this, we see that the implicit action of SV on these functions is given at depth one by

$$\mathrm{SV}\,\mathcal{E}_0(0^j, k\,; \tau) = \frac{1}{j!} \mathcal{E}^{\mathrm{sv}}\left[\begin{smallmatrix} j \\ k \end{smallmatrix}; \tau\right] - \frac{B_k(-4y)^{j+1}}{k!(j+1)!} \,. \tag{4.37}$$

By (4.32), (4.36) and the shuffle relation

$$\mathcal{E}_0(0^j, k; \tau) = \sum_{r=0}^{j} \frac{(-1)^{j-r}}{r!} (2\pi i\tau)^r \mathcal{E}_0(k, 0^{j-r}; \tau), \tag{4.38}$$

two equivalent formulations of (4.37) are

$$\mathrm{SV}\,\mathcal{E}_0(0^j, k\,; \tau) = \mathcal{E}_0(0^j, k\,; \tau) + \sum_{r=1}^{j} \overline{\mathcal{E}_0(0^r\,; \tau)} \mathcal{E}_0(0^{j-r}, k\,; \tau) + \overline{\mathcal{E}_0(k, 0^j\,; \tau)} \tag{4.39}$$

$$\mathrm{SV}\,\mathcal{E}_0(k, 0^j\,; \tau) = \mathcal{E}_0(k, 0^j\,; \tau) + \sum_{r=1}^{j} \overline{\mathcal{E}_0(0^{j-r}, k\,; \tau)} \mathcal{E}_0(0^r\,; \tau) + \overline{\mathcal{E}_0(0^j, k\,; \tau)},$$

which match the expectation from [102].

5 Examples

We shall now spell out a variety of examples that illustrate both the pairing of cycles with dual antielliptic integrands $\overline{V(\ldots|\tau)}$ and the action of the single-valued map on eMZVs. Both eMZVs and MGFs occur in the simultaneous expansion of the generating series $B^\tau_{\vec{\eta}}, J^\tau_{\vec{\eta}}$ in s_{ij} and η_j. The coefficients in their η_j-expansion will be referred to as *component integrals*, and we will use the shorthand

$$f^{(a)}_{ij} = f^{(a)}(z_i - z_j, \tau), \qquad \overline{f^{(b)}_{ij}} = \overline{f^{(b)}(z_i - z_j, \tau)} \tag{5.1}$$

for the Kronecker–Eisenstein coefficients defined by (2.14) that occur in the integrands. More precisely, the building blocks (2.13) and (3.6) of the integrands of $B_{\vec{\eta}}^\tau, J_{\vec{\eta}}^\tau$ involve the following combinations of (5.1) with $\eta_{23...n} = \eta_2 + \eta_3 + \ldots + \eta_n$,

$$\varphi_{\vec{\eta}}^\tau(1, 2, \ldots, n) = \sum_{a_2,\ldots,a_n \geq 0} \eta_{23...n}^{a_2-1} \eta_{3...n}^{a_3-1} \cdots \eta_n^{a_n-1} f_{12}^{(a_2)} f_{23}^{(a_3)} \cdots f_{n-1,n}^{(a_n)}$$

$$V_w(1, 2, \ldots, n|\tau) = \sum_{\substack{a_1,a_2,\ldots,a_n \geq 0 \\ a_1+a_2+\ldots+a_n=w}} f_{12}^{(a_1)} f_{23}^{(a_2)} \cdots f_{n-1,n}^{(a_{n-1})} f_{n,1}^{(a_n)} . \tag{5.2}$$

5.1 Two-point α'-expansions

At two points, the general definitions (3.1) and (3.12) only admit a single permutation in $B_{\eta_2}^\tau = B_{\eta_2}^\tau(2|2)$ and $J_{\eta_2}^\tau = J_{\eta_2}^\tau(2|2)$,

$$B_{\eta_2}^\tau = \int_{-\tau/2}^{\tau/2} \frac{dz_2}{\tau \eta_2} \sum_{a=0}^\infty \tau^a \eta_2^a f_{12}^{(a)} e^{s_{12}\mathcal{G}_{\mathfrak{B}}(z_{12},\tau)} \tag{5.3}$$

$$J_{\eta_2}^\tau = \int_{\mathfrak{I}} \frac{d^2 z_2}{\text{Im}\,\tau} \frac{1}{\eta_2} \sum_{a=0}^\infty (\tau-\bar{\tau})^a \eta_2^a f_{12}^{(a)} e^{s_{12}\mathcal{G}_{\mathfrak{I}}(z_{12},\tau)} ,$$

and we introduce the following notation for component integrals

$$B_{(a)}^\tau = B_{\eta_2}^\tau \big|_{\eta_2^{a-1}} = \tau^a \int_{-\tau/2}^{\tau/2} \frac{dz_2}{\tau} f_{12}^{(a)} e^{s_{12}\mathcal{G}_{\mathfrak{B}}(z_{12},\tau)} \tag{5.4}$$

$$J_{(a)}^\tau = J_{\eta_2}^\tau \big|_{\eta_2^{a-1}} = (\tau-\bar{\tau})^a \int_{\mathfrak{I}} \frac{d^2 z_2}{\text{Im}\,\tau} f_{12}^{(a)} e^{s_{12}\mathcal{G}_{\mathfrak{I}}(z_{12},\tau)} .$$

Then, combining the initial values (4.25) and (4.26) with the α'-expansions (4.9) and (4.19) yields expressions like

$$B_{(0)}^\tau = 1 + s_{12}^2\Big(-3\beta[\tfrac{1}{4};\tau] + \frac{\zeta_2}{6} + \frac{i\zeta_3}{2T} + \frac{3\zeta_4}{4T^2}\Big)$$
$$+ s_{12}^3\Big(-5\beta[\tfrac{2}{6};\tau] + 12\zeta_2\beta[\tfrac{0}{4};\tau] + \frac{\zeta_3}{12} + \frac{19i\zeta_4}{24T} - \frac{\zeta_5}{4T^2} + \frac{\zeta_2\zeta_3}{T^2} - \frac{4i\zeta_6}{3T^3}\Big) + \mathcal{O}(s_{12}^4)$$
$$B_{(2)}^\tau = -2\zeta_2 + s_{12}\Big(3\beta[\tfrac{2}{4};\tau] - \zeta_3 + \frac{3i\zeta_4}{T}\Big) \tag{5.5}$$
$$+ s_{12}^2\Big(10\beta[\tfrac{3}{6};\tau] - 18\zeta_2\beta[\tfrac{1}{4};\tau] - \frac{29\zeta_4}{12} - \frac{i\zeta_5}{T} + \frac{3i\zeta_2\zeta_3}{T} + \frac{43\zeta_6}{8T^2}\Big) + \mathcal{O}(s_{12}^3)$$
$$B_{(4)}^\tau = -2\zeta_4 + s_{12}\Big(5\beta[\tfrac{4}{6};\tau] - 6\zeta_2\beta[\tfrac{2}{4};\tau] + 2\zeta_2\zeta_3 - \zeta_5 - \frac{11i\zeta_6}{2T}\Big) + \mathcal{O}(s_{12}^2)$$

as well as

$$J_{(0)}^\tau = 1 + s_{12}^2\Big(-3\beta^{\text{sv}}[\tfrac{1}{4};\tau] + \frac{\zeta_3}{2y}\Big) + s_{12}^3\Big(-5\beta^{\text{sv}}[\tfrac{2}{6};\tau] + \frac{\zeta_3}{6} + \frac{\zeta_5}{8y^2}\Big) + \mathcal{O}(s_{12}^4)$$

$$J_{(2)}^\tau = s_{12}(3\beta^{\text{sv}}[\tfrac{2}{4};\tau] - 2\zeta_3) + s_{12}^2\Big(10\beta^{\text{sv}}[\tfrac{3}{6};\tau] - \frac{\zeta_5}{y}\Big) + \mathcal{O}(s_{12}^3) \tag{5.6}$$

$$J_{(4)}^\tau = s_{12}(5\beta^{\text{sv}}[\tfrac{4}{6};\tau] - 2\zeta_5) + \mathcal{O}(s_{12}^2)$$

upon extracting suitable powers of η_2. The action of SV on the $\zeta_n, T, \beta[\ldots]$ as in (3.32) and (4.30) relates $J_{(a)}^\tau = \mathrm{SV}\, B_{(a)}^\tau$ as expected from (3.34). Examples of $\beta[\ldots]$ beyond depth one occur at the next orders in s_{ij}, e.g.

$$
\begin{aligned}
B_{(0)}^\tau \big|_{s_{12}^4} =\ & -21\beta[\tfrac{3}{8};\tau] + 9\beta[\tfrac{1}{4}\tfrac{1}{4};\tau] - 18\beta[\tfrac{2}{4}\tfrac{0}{4};\tau] - \frac{\zeta_2}{2}\beta[\tfrac{1}{4};\tau] + 40\zeta_2\beta[\tfrac{1}{6};\tau] \\
& + 6\zeta_3\beta[\tfrac{0}{4};\tau] - \frac{3i\zeta_3}{2T}\beta[\tfrac{1}{4};\tau] - \frac{18i\zeta_4}{T}\beta[\tfrac{0}{4};\tau] - \frac{9\zeta_4}{4T^2}\beta[\tfrac{1}{4};\tau] \\
& + \frac{131\zeta_4}{720} + \frac{5i\zeta_5}{12T} + \frac{i\zeta_2\zeta_3}{12T} + \frac{23\zeta_6}{32T^2} + \frac{\zeta_3^2}{8T^2} - \frac{9i\zeta_3\zeta_4}{8T^3} + \frac{i\zeta_2\zeta_5}{T^3} - \frac{3i\zeta_7}{8T^3} + \frac{85\zeta_8}{192T^4}
\end{aligned}
\tag{5.7}
$$

as well as

$$
\begin{aligned}
J_{(0)}^\tau \big|_{s_{12}^4} =\ & -21\beta^{\mathrm{sv}}[\tfrac{3}{8};\tau] + 9\beta^{\mathrm{sv}}[\tfrac{1}{4}\tfrac{1}{4};\tau] - 18\beta^{\mathrm{sv}}[\tfrac{2}{4}\tfrac{0}{4};\tau] \\
& + 12\zeta_3\beta^{\mathrm{sv}}[\tfrac{0}{4};\tau] - \frac{3\zeta_3}{2y}\beta^{\mathrm{sv}}[\tfrac{1}{4};\tau] + \frac{5\zeta_5}{12y} - \frac{\zeta_3^2}{8y^2} + \frac{3\zeta_7}{32y^3}
\end{aligned}
$$

$$
\begin{aligned}
J_{(0)}^\tau \big|_{s_{12}^5} =\ & -135\beta^{\mathrm{sv}}[\tfrac{4}{10};\tau] - 60\beta^{\mathrm{sv}}[\tfrac{3}{6}\tfrac{0}{4};\tau] + 15\beta^{\mathrm{sv}}[\tfrac{1}{4}\tfrac{2}{6};\tau] + 15\beta^{\mathrm{sv}}[\tfrac{2}{6}\tfrac{1}{4};\tau] - 60\beta^{\mathrm{sv}}[\tfrac{2}{4}\tfrac{1}{6};\tau] \\
& - \frac{1}{2}\zeta_3\beta^{\mathrm{sv}}[\tfrac{1}{4};\tau] + \frac{6\zeta_5}{y}\beta^{\mathrm{sv}}[\tfrac{0}{4};\tau] - \frac{3\zeta_5}{8y^2}\beta^{\mathrm{sv}}[\tfrac{1}{4};\tau] + 40\zeta_3\beta^{\mathrm{sv}}[\tfrac{1}{6};\tau] - \frac{5\zeta_3}{2y}\beta^{\mathrm{sv}}[\tfrac{2}{6};\tau] \\
& + \frac{43\zeta_5}{360} + \frac{\zeta_3^2}{12y} + \frac{7\zeta_7}{32y^2} - \frac{3\zeta_3\zeta_5}{16y^3} + \frac{15\zeta_9}{128y^4}\,.
\end{aligned}
\tag{5.8}
$$

5.2 Extracting single-valued eMZVs

The above α'-expansions at two points have been generated in earlier work in terms of eMZVs [51, 59] and MGFs [10, 11, 61], respectively. The results in the references include

$$
B_{(0)}^\tau = 1 + s_{12}^2\Big(\tfrac{1}{2}\omega(0,0,2|-\tfrac{1}{\tau}) + \tfrac{5\zeta_2}{12}\Big) + s_{12}^3\Big(\tfrac{1}{18}\omega(0,0,3,0|-\tfrac{1}{\tau}) - \tfrac{4}{3}\zeta_2\omega(0,0,1,0|-\tfrac{1}{\tau}) + \tfrac{\zeta_3}{12}\Big) + \mathcal{O}(s_{12}^4)
$$

$$
J_{(0)}^\tau = 1 + \tfrac{1}{2}s_{12}^2 \mathrm{E}_2(\tau) + \tfrac{1}{6}s_{12}^3\big(\mathrm{E}_3(\tau) + \zeta_3\big) + \mathcal{O}(s_{12}^4)\,,
\tag{5.9}
$$

where $\mathrm{E}_2 = -6\beta^{\mathrm{sv}}[\tfrac{1}{4};\tau] + \tfrac{\zeta_3}{y}$, and the modular transformation may be evaluated to yield [51, 93]

$$
\omega(0,0,2|-\tfrac{1}{\tau}) = -\frac{T^2}{180} - \frac{\zeta_2}{6} + \frac{3\zeta_4}{2T^2} + \omega(0,0,2|\tau) + \frac{8i}{T}\zeta_2\omega(0,1,0,0|\tau)\,.
\tag{5.10}
$$

For the examples in (5.9), the component version $J_{(a)}^\tau = \mathrm{SV}\, B_{(a)}^\tau$ of $J_{\eta_2}^\tau = \mathrm{SV}\, B_{\eta_2}^\tau$ implies that

$$
\mathrm{SV}\,\omega(0,0,2|-\tfrac{1}{\tau}) = \mathrm{E}_2(\tau)
\tag{5.11}
$$

and a similar analysis for higher orders in s_{12} and at $a \neq 0$ yields for instance

$$
\mathrm{SV}\,\omega(0,3|-\tfrac{1}{\tau}) = 2\pi\nabla\mathrm{E}_2(\tau)\,, \qquad\qquad \mathrm{SV}\,\omega(0,0,4|-\tfrac{1}{\tau}) = -\frac{4}{3}\pi\nabla\mathrm{E}_3(\tau)
\tag{5.12}
$$

$$
\mathrm{SV}\,\omega(0,0,3,0|-\tfrac{1}{\tau}) = 3\mathrm{E}_3(\tau)\,, \qquad\qquad \mathrm{SV}\,\omega(0,5|-\tfrac{1}{\tau}) = -\frac{4}{3}(\pi\nabla)^2\mathrm{E}_3(\tau)\,.
$$

At depth two, relating $B^\tau_{(0)}\big|_{s^4_{12}} \leftrightarrow J^\tau_{(0)}\big|_{s^4_{12}}$ (see (5.7) and (5.8)) or $B^\tau_{(2)}\big|_{s^3_{12}} \leftrightarrow J^\tau_{(2)}\big|_{s^3_{12}}$ yields

$$\mathrm{E}_{2,2}(\tau) = \mathrm{SV}\left(-\frac{7}{5}\omega(0,0,0,0,4|-\tfrac{1}{\tau}) - \omega(0,0,0,2,2|-\tfrac{1}{\tau}) + \frac{1}{2}\omega(0,0,2|-\tfrac{1}{\tau})^2 + \frac{3}{20}\omega(0,0,4|-\tfrac{1}{\tau})\right)$$

$$\pi\nabla\mathrm{E}_{2,2}(\tau) = \mathrm{SV}\left(-\frac{1}{60}\omega(0,5|-\tfrac{1}{\tau}) + \frac{3}{5}\omega(0,0,0,5|-\tfrac{1}{\tau}) - \frac{1}{2}\omega(0,0,2,3|-\tfrac{1}{\tau})\right), \tag{5.13}$$

where the combinations

$$\mathrm{E}_{2,2} = \left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^4 \left(\mathcal{C}\big[\begin{smallmatrix}2&1&1\\2&1&1\end{smallmatrix}\big] - \frac{9}{10}\mathcal{C}\big[\begin{smallmatrix}4&0\\4&0\end{smallmatrix}\big]\right)$$

$$= -18\beta^{\mathrm{sv}}\big[\begin{smallmatrix}2&0\\4&4\end{smallmatrix}\big] + 12\zeta_3\beta^{\mathrm{sv}}\big[\begin{smallmatrix}0\\4\end{smallmatrix}\big] + \frac{5\zeta_5}{12y} - \frac{\zeta_3^2}{4y^2} \tag{5.14}$$

$$\pi\nabla\mathrm{E}_{2,2} = \frac{(\mathrm{Im}\,\tau)^5}{\pi^3}\left(\mathcal{C}\big[\begin{smallmatrix}3&1&1\\1&1&1\end{smallmatrix}\big] - \frac{8}{5}\mathcal{C}\big[\begin{smallmatrix}5&0\\3&0\end{smallmatrix}\big]\right)$$

$$= 9\beta^{\mathrm{sv}}\big[\begin{smallmatrix}2&1\\4&4\end{smallmatrix}\big] - 6\zeta_3\beta^{\mathrm{sv}}\big[\begin{smallmatrix}1\\4\end{smallmatrix}\big] - \frac{5\zeta_5}{12} + \frac{\zeta_3^2}{2y}$$

of MGFs (2.26) are engineered to remove G_8 from the differential equations [51]. The systematics of depth-one relations between eMZVs and non-holomorphic Eisenstein series including higher-weight generalizations of (5.12) is detailed in appendix C.1 and leads to the closed-form results

$$\mathrm{SV}\sum_{j=0}^{\ell-1}\frac{B_j}{j!}\omega(0^{\ell-j}, 2k+\ell|-\tfrac{1}{\tau}) = (-1)^\ell\frac{(k+\ell-1)!}{(2k+\ell-1)!}(-4\pi\nabla)^k\mathrm{E}_{k+\ell}, \quad k\geq0,\ \ell\geq1,\ k+\ell\geq2 \tag{5.15}$$

$$\mathrm{SV}\sum_{j=0}^{\ell-1}\frac{B_j}{j!}\omega(0^{\ell-j}, \ell-2k|-\tfrac{1}{\tau}) = (-1)^{\ell+k}\frac{(\ell-k-1)!}{(\ell-1)!}\frac{(\pi\overline{\nabla})^k\mathrm{E}_{\ell-k}}{(2y)^{2k}}, \quad k\geq0,\ \ell-2k\geq1,\ \ell-k\geq2$$

for combinations of eMZVs of different length that are weighted by Bernoulli numbers B_j [103]. Similarly, the analogue of (5.13) for the MGF $\mathrm{E}_{2,3} = (\frac{\mathrm{Im}\,\tau}{\pi})^5\big(\mathcal{C}\big[\begin{smallmatrix}3&1&1\\3&1&1\end{smallmatrix}\big] - \frac{43}{35}\mathcal{C}\big[\begin{smallmatrix}5&0\\5&0\end{smallmatrix}\big]\big)$ and its holomorphic derivatives is spelled out in appendix C.2.

Note that the single-valued map of A-cycle eMZVs at argument τ rather than $-\frac{1}{\tau}$ generically leads to combinations of MGFs of different modular weights. For instance, changing the argument $-\frac{1}{\tau}$ to τ in (5.11) gives rise to

$$\mathrm{SV}\,\omega(0,0,2|\tau) = -\frac{y^2}{15} + \mathrm{E}_2(\tau) + \frac{\pi\overline{\nabla}\mathrm{E}_2(\tau)}{y} \tag{5.16}$$

instead of a single modular invariant $\mathrm{E}_2(\tau)$. This can be seen by expressing all of $\omega(0,0,2|\tau)$, $\mathrm{E}_2(\tau)$ and $\overline{\nabla}\mathrm{E}_2(\tau)$ in terms of convergent iterated Eisenstein integrals (4.34) and applying their single-valued map (4.39)[17]. Alternatively, (5.16) can be deduced by setting $\tau \to -\frac{1}{\tau}$ in (5.10)

---

[17]The representations in terms of convergent iterated Eisenstein integrals needed to verify (5.16) are

$$\omega(0,0,2|\tau) = -6\mathcal{E}_0(4,0;\tau) - \frac{1}{3}\zeta_2$$

$$\mathrm{E}_2(\tau) = \frac{y^2}{45} + \frac{\zeta_3}{y} - 12\mathrm{Re}[\mathcal{E}_0(4,0;\tau)] - \frac{6}{y}\mathrm{Re}[\mathcal{E}_0(4,0,0;\tau)]$$

$$\pi\overline{\nabla}\mathrm{E}_2(\tau) = \frac{2y^3}{45} - \zeta_3 + 24y^2\overline{\mathcal{E}_0(4;\tau)} + 12y\overline{\mathcal{E}_0(4,0;\tau)} + 6\mathrm{Re}[\mathcal{E}_0(4,0,0;\tau)]$$

and exploiting the result $\mathrm{SV}\,\omega(0,1,0,0|-\frac{1}{\tau}) = \frac{3\pi\overline{\nabla}\mathrm{E}_2(\tau)}{8y^2}$ that will be extracted from a four-point example in section 5.5. The much cleaner result (5.11) for $\mathrm{SV}\,\omega(0,0,2|-\frac{1}{\tau})$ as compared to $\mathrm{SV}\,\omega(0,0,2|\tau)$ is another manifestation of the fact that the differential equations (3.28) of B-cycle integrals are more closely related to the closed-string counterparts (3.29) than the A-cycle differential equations in (2.19).

5.3 Symmetrized cycles and graph functions

At $n \geq 3$ points, most of the antielliptic functions in (3.8) introduce non-constant $\overline{f_{ij}^{(a)}}$ into the closed-string integrands, except for the simplest case $V_0(1,2,\ldots,n|\tau) = 1$ dual to a permutation sum over B-cycles, see (3.37),

$$J_{0,\vec{\eta}}^\tau(*|\rho) = \int_{\mathfrak{T}^{n-1}} \left(\prod_{j=2}^n \frac{\mathrm{d}^2 z_j}{\mathrm{Im}\,\tau}\right) \prod_{1\leq i<j}^n e^{s_{ij}\mathcal{G}_{\mathfrak{T}}(z_{ij},\tau)} \rho\left\{ \prod_{k=2}^n \sum_{a_k=0}^\infty (\tau-\bar{\tau})^{a_k} \eta_{k,k+1\ldots n}^{a_k-1} f_{k-1,k}^{(a_k)} \right\} \qquad (5.17)$$

$$= \mathrm{SV} \sum_{\gamma\in S_{n-1}} B_{\vec{\eta}}^\tau(\gamma|\rho)\,.$$

As indicated by the $*$-notation, the $J_{0,\vec{\eta}}^\tau(*|\rho)$ integral (3.35) on the left-hand side is independent of the ordering $*$ since its integrand V_0 is. The symmetrized open-string integrals on the right-hand side were studied in [51, 76, 16] as the generating series of holomorphic graph functions,

$$\sum_{\gamma\in S_{n-1}} B_{\vec{\eta}}^\tau(\gamma|\rho) = \left(\int_{-\tau/2}^{\tau/2} \prod_{j=2}^n \mathrm{d}z_j\right) \prod_{1\leq i<j}^n e^{s_{ij}\mathcal{G}_{\mathfrak{B}}(z_{ij},\tau)} \rho\left\{ \prod_{k=2}^n \sum_{a_k=0}^\infty (\tau\eta_{k,k+1\ldots n})^{a_k-1} f_{k-1,k}^{(a_k)} \right\}, \quad (5.18)$$

where each puncture is integrated independently over the entire B-cycle. More specifically, the references considered the components $f_{k-1,k}^{(a_k)} \to f_{k-1,k}^{(0)} = 1$ at the most singular order in the η_j,

$$M_n^{\mathrm{open}} = \sum_{\gamma\in S_{n-1}} B_{\vec{\eta}}^\tau(\gamma|\rho)\Big|_{\eta_{23\ldots n}^{-1}\eta_{3\ldots n}^{-1}\ldots\eta_n^{-1}} = \frac{1}{\tau^{n-1}} \left(\prod_{j=2}^n \int_{-\tau/2}^{\tau/2} \mathrm{d}z_j\right) \prod_{1\leq i<j}^n e^{s_{ij}\mathcal{G}_{\mathfrak{B}}(z_{ij},\tau)} \qquad (5.19)$$

$$= 1 + \frac{1}{2} \sum_{1\leq i<j}^n s_{ij}^2\, \mathbf{B}\!\left[\,\begin{smallmatrix}\bullet\!-\!\bullet\end{smallmatrix}\,\right] + \frac{1}{6} \sum_{1\leq i<j}^n s_{ij}^3\, \mathbf{B}\!\left[\,\begin{smallmatrix}\bullet\!=\!\bullet\end{smallmatrix}\,\right] + \sum_{1\leq i<j<k}^n s_{ij}s_{ik}s_{jk}\, \mathbf{B}\!\left[\,\begin{smallmatrix}\triangle\end{smallmatrix}\,\right] + \mathcal{O}(s_{ij}^4)\,,$$

where the dependence on the permutation ρ drops out, and the integrands at fixed order in s_{ij} are polynomials in B-cycle Green functions. In passing to the second line, each monomial in $\mathcal{G}_{\mathfrak{B}}(z_{ij},\tau)$ is mapped to a graph Γ that labels the B-cycle graph functions $\mathbf{B}[\Gamma]$, where a factor of $\mathcal{G}_{\mathfrak{B}}(z_{ij},\tau)$ is represented by an edge connecting vertices z_i and z_j. One-particle reducible graphs Γ_{1PR} lead to vanishing $\mathbf{B}[\Gamma_{\mathrm{1PR}}]$ since $\int_{-\tau/2}^{\tau/2} \mathrm{d}z\, \mathcal{G}_{\mathfrak{B}}(z,\tau) = 0$, i.e. higher orders of (5.19) stem from all combinations of one-particle irreducible graphs with four and more edges in total. Any $\mathbf{B}[\Gamma]$ is expressible in terms of B-cycle eMZVs [51] since the α'-expansion of each component integral of the series $B_{\vec{\eta}}^\tau(\gamma|\rho)$ is.

Similarly, modular graph functions $\mathbf{D}[\Gamma]$ (as opposed to modular graph forms) were defined [10, 11] by n-point torus integrals over monomials in $\mathcal{G}_{\mathfrak{T}}(z_{ij},\tau)$, where each torus Green functions

is again visualized through an edge between vertices z_i and z_j. The $\mathbf{D}[\Gamma]$ associated with dihedral graphs Γ are proportional to the lattice sums (2.26) with $a_j = b_j$, and also more complicated graph topologies can be straightforwardly translated into lattice sums.

The generating series of n-point modular graph functions resides at the most singular order of (5.17) w.r.t. η_j where the insertions of $f_{ij}^{(a)}$ are absent,

$$M_n^{\text{closed}} = J_{0,\vec{\eta}}^{\tau}(*|\rho)\big|_{\eta_{23...n}^{-1}\eta_{3...n}^{-1}\cdots\eta_n^{-1}} = \int_{\mathfrak{T}^{n-1}} \left(\prod_{j=2}^{n} \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \prod_{1\leq i<j}^{n} e^{s_{ij}\mathcal{G}_{\mathfrak{T}}(z_{ij},\tau)} \tag{5.20}$$

$$= 1 + \frac{1}{2} \sum_{1\leq i<j}^{n} s_{ij}^2 \, \mathbf{D}\!\left[\vcenter{\hbox{\includegraphics{a}}}\right] + \frac{1}{6} \sum_{1\leq i<j}^{n} s_{ij}^3 \, \mathbf{D}\!\left[\vcenter{\hbox{\includegraphics{b}}}\right] + \sum_{1\leq i<j<k}^{n} s_{ij}s_{ik}s_{jk} \, \mathbf{D}\!\left[\vcenter{\hbox{\includegraphics{c}}}\right] + \mathcal{O}(s_{ij}^4),$$

and we have $\mathbf{D}[\Gamma_{\text{1PR}}] = 0$ by $\int_{\mathfrak{T}} \mathrm{d}^2 z \, \mathcal{G}_{\mathfrak{T}}(z,\tau) = 0$. As a consequence of (5.17) at the most singular order in the η_j, modular graph functions are single-valued B-cycle graph functions,

$$M_n^{\text{closed}} = \mathrm{SV}\, M_n^{\text{open}} \quad \Leftrightarrow \quad \mathbf{D}[\Gamma] = \mathrm{SV}\, \mathbf{B}[\Gamma], \tag{5.21}$$

which ultimately follows from the Betti–deRham duality between $V_0 = 1$ and the symmetrized cycles $\sum_{\gamma\in S_{n-1}} \mathfrak{B}(\gamma(2,\ldots,n))$.

The relations in (5.21) have firstly appeared in [51] with a proposal "esv" for an elliptic single-valued map in the place of SV. The esv-map of [51] has the same action (3.32) on MZVs and Laurent polynomials in τ as the SV-map in this work. In particular, all pairs of B-cycle eMZVs and modular graph functions related via $\text{esv}\,\omega(\ldots|-\frac{1}{\tau}) \sim \mathbf{D}[\ldots]$ in the reference are also related via $\mathrm{SV}\,\omega(\ldots|-\frac{1}{\tau}) \sim \mathbf{D}[\ldots]$ as a consequence of (5.21). For suitable representations of the q-series of eMZVs via $\mathcal{E}_0$ defined by (4.34), the Fourier expansions of all modular graph functions up to weight six could be reproduced from the replacement $\mathcal{E}_0 \to 2\operatorname{Re}(\mathcal{E}_0)$ prescribed by esv [51]. However, it was an open problem in the reference to reconcile esv with the shuffle property of iterated Eisenstein integrals. The SV action (4.30) in turn is expected to be compatible with the shuffle multiplication of the $\beta[\ldots]$ and $\beta^{\text{sv}}[\ldots]$ by the discussion in section 4.3: Shuffle-compatibility can be verified on a case-by-case basis from the antiholomorphic integration constants $\overline{\alpha[\ldots]}$ in (4.17) and higher-depth generalizations.

Note that subleading orders in the η_j-expansion of (5.17) generate infinite families of additional relation between MGFs and single-valued eMZVs beyond (5.21). The comparison of open- and closed-string integrals with additional insertions of $f_{12}^{(a_2)} f_{23}^{(a_3)} \ldots f_{n-1,n}^{(a_n)}$ identifies MGFs of various modular weights as single-valued B-cycle eMZVs.

5.4 Three-point cycles and $\overline{V_1(1,2,3|\tau)}$

The simplest instance of $J_{\vec{\eta}}^{\tau}(\gamma|\rho) = \mathrm{SV}\, B_{\vec{\eta}}^{\tau}(\gamma|\rho)$ with non-constant antielliptic integrands $\overline{V(\ldots|\tau)}$ occurs at three points. The single-valued map relates an antisymmetric integration cycle on the

open-string side in (3.45) to the closed-string integral

$$J^\tau_{1,\eta_2,\eta_3}(2,3|2,3) = -\frac{1}{2\pi i}\int_{\mathfrak{T}^2}\frac{\mathrm{d}^2 z_2 \mathrm{d}^2 z_3}{(\mathrm{Im}\,\tau)^2}\prod_{1\leq i<j}^{3}e^{s_{ij}\mathcal{G}_{\mathfrak{T}}(z_{ij},\tau)}$$

$$\times \sum_{a,b=0}^{\infty}(\tau-\bar\tau)^{a+b}\eta_{23}^{a-1}\eta_3^{b-1}f_{12}^{(a)}f_{23}^{(b)}\left(\overline{f_{12}^{(1)}}+\overline{f_{23}^{(1)}}+\overline{f_{31}^{(1)}}\right). \tag{5.22}$$

Since contributions with even $a+b$ integrate to zero, the simplest component integrals involve permutations of $f_{12}^{(1)}\overline{f_{12}^{(1)}}$ or $f_{23}^{(1)}\overline{f_{12}^{(1)}}$ at the orders of η_3^{-1} or η_{23}^{-1},

$$J^\tau_{1,\eta_2,\eta_3}(2,3|2,3)\big|_{\eta_{23}^0\eta_3^{-1}} = -\frac{\mathrm{Im}\,\tau}{\pi}\int_{\mathfrak{T}^2}\frac{\mathrm{d}^2 z_2 \mathrm{d}^2 z_3}{(\mathrm{Im}\,\tau)^2}\prod_{1\leq i<j}^{3}e^{s_{ij}\mathcal{G}_{\mathfrak{T}}(z_{ij},\tau)}f_{12}^{(1)}\left(\overline{f_{12}^{(1)}}+\overline{f_{23}^{(1)}}+\overline{f_{31}^{(1)}}\right)$$

$$= \frac{1}{s_{12}}+\frac{s_{123}^2}{2s_{12}}\mathrm{E}_2+\frac{s_{123}^3}{6s_{12}}(\mathrm{E}_3+\zeta_3)+\frac{s_{13}s_{23}}{2}(3\mathrm{E}_3+\zeta_3) \tag{5.23}$$

$$+\frac{s_{123}^4}{s_{12}}\left(\mathrm{E}_{2,2}+\frac{3}{20}\mathrm{E}_4+\frac{1}{8}\mathrm{E}_2^2\right)+s_{13}s_{23}s_{123}\left(\frac{9}{2}\mathrm{E}_{2,2}+\frac{21}{20}\mathrm{E}_4\right)+\mathcal{O}(s_{ij}^4),$$

which furnish the simplest examples of kinematic poles $\sim s_{ij}^{-1}$ in a $J^\tau_{\vec\eta}$-series. The corresponding antisymmetrized B-cycle integral features the same types of kinematic poles in component integrals involving $f_{ij}^{(1)}$, e.g.[18]

$$\frac{1}{2}\left[B^\tau_{\eta_2,\eta_3}(2,3|2,3)-B^\tau_{\eta_2,\eta_3}(3,2|2,3)\right]\big|_{\eta_{23}^0\eta_3^{-1}} = \frac{1}{\tau}\int_{-\frac{\tau}{2}<z_2<z_3<\frac{\tau}{2}}\mathrm{d}z_2\,\mathrm{d}z_3\,f_{12}^{(1)}\prod_{1\leq i<j}^{3}e^{s_{ij}\mathcal{G}_{\mathfrak{B}}(z_{ij},\tau)}$$

$$= \frac{1}{s_{12}}+\frac{s_{123}^2}{s_{12}}\left(\frac{1}{2}\omega(0,0,2|-\tfrac{1}{\tau})+\frac{5\zeta_2}{12}\right)+\frac{s_{123}^3}{s_{12}}\left(\frac{1}{18}\omega(0,0,3,0|-\tfrac{1}{\tau})-\frac{4}{3}\zeta_2\omega(0,0,1,0|-\tfrac{1}{\tau})+\frac{\zeta_3}{12}\right)$$

$$+s_{13}s_{23}\left(\frac{1}{2}\omega(0,0,3,0|-\tfrac{1}{\tau})+\frac{\zeta_3}{4}\right)+\frac{s_{123}^4}{s_{12}}\left(-\omega(0,0,0,2,2|-\tfrac{1}{\tau})-\frac{5}{4}\omega(0,0,0,0,4|-\tfrac{1}{\tau})\right. \tag{5.24}$$

$$+\frac{1}{8}\omega(0,0,4|-\tfrac{1}{\tau})+\frac{5}{8}\omega(0,0,2|-\tfrac{1}{\tau})^2+\frac{13}{24}\zeta_2\omega(0,0,2|-\tfrac{1}{\tau})-2\zeta_2\omega(0,0,0,0,2|-\tfrac{1}{\tau})+\frac{343\zeta_4}{576}\right)$$

$$-s_{13}s_{23}s_{123}\left(\frac{9}{2}\omega(0,0,0,2,2|-\tfrac{1}{\tau})+\frac{21}{4}\omega(0,0,0,0,4|-\tfrac{1}{\tau})-\frac{1}{2}\omega(0,0,4|-\tfrac{1}{\tau})\right.$$

$$\left.-\frac{9}{4}\omega(0,0,2|-\tfrac{1}{\tau})^2+\frac{1}{2}\zeta_2\omega(0,0,2|-\tfrac{1}{\tau})-3\zeta_2\omega(0,0,0,0,2|-\tfrac{1}{\tau})+\frac{11\zeta_4}{40}\right)+\mathcal{O}(s_{ij}^4).$$

We have used that, by the antisymmetry $f_{12}^{(1)}=-f_{21}^{(1)}$ of the integrand, the contribution from the ordering $-\frac{\tau}{2}<z_3<z_2<\frac{\tau}{2}$ is minus that of the ordering $-\frac{\tau}{2}<z_2<z_3<\frac{\tau}{2}$. Comparison of (5.24) with (5.23) confirms the relation (3.45) under the SV-map at the respective orders in s_{ij} and η_j. Up to the restriction of the Koba–Nielsen factor to three instead of five punctures, (5.23) and (5.24) are the type of integrals over $f_{ij}^{(1)}\overline{f_{pq}^{(1)}}$ seen in genus-one five-point amplitudes of type II superstrings [104, 105].

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[18]By slight abuse of notation, we denote the ordering of punctures z_i, z_j on the imaginary axis by $-\frac{\tau}{2}<z_i<z_j<\frac{\tau}{2}$.

5.5 Four-point cycles and $\overline{V_2(1,2,3,4|\tau)}$

The esv map [51] has also been applied to the four-gluon amplitude of the heterotic string [52], where the torus integral[19]

$$J_{\text{het}}^\tau = \frac{1}{(2\pi i)^2} \int_{\mathfrak{T}^3} \left(\prod_{j=2}^{4} \frac{\mathrm{d}^2 z_j}{\text{Im}\,\tau} \right) \overline{V_2(1,2,3,4|\tau)} \prod_{1 \leq i < j}^{4} e^{s_{ij} \mathcal{G}_{\mathfrak{T}}(z_{ij},\tau)}$$
$$= J_{2,\eta_2,\eta_3,\eta_4}^\tau (2,3,4|2,3,4) \big|_{\eta_{234}^{-1} \eta_{34}^{-1} \eta_4^{-1}} \tag{5.25}$$

was related to the open-string integration cycle dual to (3.38). More specifically, the MGFs in [52]

$$J_{\text{het}}^\tau \big|_{k_j^2=0} = -\frac{3 s_{13} \pi \overline{\nabla} \mathrm{E}_2}{4y^2} - (s_{13}^2 + 2 s_{12} s_{23}) \frac{\pi \overline{\nabla} \mathrm{E}_3}{6y^2} \tag{5.26}$$
$$+ s_{13}(s_{12} s_{23} - s_{13}^2) \left(\frac{\pi \overline{\nabla} \mathrm{E}_4}{5y^2} + \frac{3\mathrm{E}_2 \overline{\nabla} \mathrm{E}_2}{2y^2} + \frac{3 \overline{\nabla} \mathrm{E}_{2,2}}{y^2} \right) + \mathcal{O}(s_{ij}^4)$$

were proposed to be the single-valued versions of the eMZVs in the α'-expansion of

$$B_{\text{het}}^\tau = \frac{1}{6} \left[2B_{234}^\tau + 2B_{432}^\tau - B_{243}^\tau - B_{342}^\tau - B_{324}^\tau - B_{423}^\tau \right]$$
$$B_{ijk}^\tau = B_{\eta_2,\eta_3,\eta_4}^\tau (i,j,k|2,3,4) \big|_{\eta_{234}^{-1} \eta_{34}^{-1} \eta_4^{-1}}, \tag{5.27}$$

namely [4]

$$B_{\text{het}}^\tau \big|_{k_j^2=0} = -2 s_{13} \omega(0,1,0,0|-\tfrac{1}{\tau}) - \frac{2}{3}(s_{13}^2 + 2 s_{12} s_{23}) \left[\omega(0,1,0,1,0|-\tfrac{1}{\tau}) + \omega(0,1,1,0,0|-\tfrac{1}{\tau}) \right]$$
$$+ \frac{4}{3} s_{13}(s_{13}^2 - s_{12} s_{23}) \left[\omega(0,0,1,0,0,2|-\tfrac{1}{\tau}) + \omega(0,0,0,1,0,2|-\tfrac{1}{\tau}) \right. \tag{5.28}$$
$$\left. - \omega(0,1,0,1,1,0|-\tfrac{1}{\tau}) - \zeta_2 \omega(0,1,0,0|-\tfrac{1}{\tau}) \right] + \mathcal{O}(s_{ij}^4).$$

As indicated by $\big|_{k_j^2=0}$, the α'-expansions (5.26) and (5.28) have been obtained in the limit of four-point on-shell kinematics with two independent Mandelstam invariants instead of six. However, the relation (3.34) between n-point closed-string and single-valued open-string integrals is conjectured to be valid for the $\frac{1}{2}n(n-1)$ independent Mandelstam variables $\{s_{ij}, \ 1 \leq i < j \leq n\}$ with $s_{ij} = s_{ji}$. At four points, the corollary

$$J_{2,\eta_2,\eta_3,\eta_4}^\tau (2,3,4|\rho) = \frac{1}{6} \text{SV} \left[2B_{\eta_2,\eta_3,\eta_4}^\tau (2,3,4|\rho) + 2B_{\eta_2,\eta_3,\eta_4}^\tau (4,3,2|\rho) \right. \tag{5.29}$$
$$\left. - B_{\eta_2,\eta_3,\eta_4}^\tau (2,4,3|\rho) - B_{\eta_2,\eta_3,\eta_4}^\tau (3,4,2|\rho) - B_{\eta_2,\eta_3,\eta_4}^\tau (3,2,4|\rho) - B_{\eta_2,\eta_3,\eta_4}^\tau (4,2,3|\rho) \right]$$

of the relation (3.40) between $V_2(1,2,3,4|\tau)$ and permutations of the $V(1,2,3,4|\tau)$ functions is claimed to hold for all of $\{s_{12}, s_{13}, s_{23}, s_{14}, s_{24}, s_{34}\}$ independent. The coefficient of $\eta_{234}^{-1} \eta_{34}^{-1} \eta_4^{-1}$ in (5.29) with $\rho = 2,3,4$ then implies

$$J_{\text{het}}^\tau = \text{SV}\, B_{\text{het}}^\tau \tag{5.30}$$

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[19]The quantity J_{het}^τ in (5.25) is defined to be $(2\pi i)^{-2}$ times the complex conjugate of the integral $\mathcal{I}_{1234}^{(2,0)}$ in (2.44) and (4.35) of [52]. Similarly, B_{het}^τ in (5.27) is obtained from the integral $Z_{1234}^{(2)}$ in section 5.2 of [52] through modular S transformation.

and explains the relations between the α'-expansions (5.26) and (5.28) observed in [52] in the on-shell limit $k_j^2 = 0$. In particular, the prescription (4.30) for the single-valued map of the iterated-Eisenstein-integral representation of B_{het}^τ produces the complete $q, \bar{q}$-expansion of the MGFs in (5.26), whereas certain antiholomorphic contributions could not be reproduced by esv in [52].

By applying (5.30) at the level of the α'-expansions (5.26) and (5.28), one can infer

$$\frac{\pi\overline{\nabla}E_2}{y^2} = \frac{8}{3}\,SV\,\omega(0,1,0,0|-\tfrac{1}{\tau})$$

$$\frac{\pi\overline{\nabla}E_3}{y^2} = 4\,SV\left[\omega(0,1,0,1,0|-\tfrac{1}{\tau}) + \omega(0,1,1,0,0|-\tfrac{1}{\tau})\right] \tag{5.31}$$

$$\frac{\pi\overline{\nabla}E_{2,2}}{y^2} = SV\left[\frac{8}{5}\omega(0,0,0,0,0,3|-\tfrac{1}{\tau}) + \frac{2}{5}\omega(0,0,0,3|-\tfrac{1}{\tau})\right.$$
$$\left. - \frac{11}{75}\omega(0,3|-\tfrac{1}{\tau}) - 8\omega(0,0,0,0,1,2|-\tfrac{1}{\tau}) + \frac{\zeta_3}{3}\right].$$

Moreover, higher orders in the η_j-expansion of (5.29) yield infinite families of relations between the α'-expansions of open- and closed-string integrals over additional factors $f_{1i}^{(a)} f_{ij}^{(b)} f_{jk}^{(c)}$.

5.6 Imaginary cusp forms and double zetas

We shall finally exemplify the appearance of cuspidal MGFs from single-valued open-string integrals whose Laurent polynomial at the order of $q^0\bar{q}^0$ vanishes. A systematic study of imaginary cusp forms among the two-loop MGFs can be found in [54], also see [87] for examples of real cusp forms. The simplest imaginary cusp forms occur among the lattice sums (2.26) at modular weights $(5, 5)$ whose basis can be chosen[20] to include [18]

$$B_{2,3} = \left(\frac{\text{Im}\,\tau}{\pi}\right)^5 \left(\mathcal{C}\left[\begin{smallmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{smallmatrix}\right] - \overline{\mathcal{C}\left[\begin{smallmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{smallmatrix}\right]}\right) + \frac{(\nabla E_2)\overline{\nabla}E_3 - (\overline{\nabla}E_2)\nabla E_3}{6(\text{Im}\,\tau)^2}$$

$$B_{2,3}' = B_{2,3} + \frac{1}{2}\left(\frac{\text{Im}\,\tau}{\pi}\right)^5 \left(\mathcal{C}\left[\begin{smallmatrix} 0 & 2 & 3 \\ 3 & 0 & 2 \end{smallmatrix}\right] - \overline{\mathcal{C}\left[\begin{smallmatrix} 0 & 2 & 3 \\ 3 & 0 & 2 \end{smallmatrix}\right]}\right) - \frac{21}{4}E_{2,3} - \frac{1}{2}\zeta_3 E_2. \tag{5.32}$$

The β^{sv}-representations involve double-integrals over $G_4 G_6$ [18],

$$B_{2,3} = 450\beta^{\text{sv}}\left[\begin{smallmatrix} 2 & 1 \\ 4 & 6 \end{smallmatrix}\right] - 450\beta^{\text{sv}}\left[\begin{smallmatrix} 3 & 0 \\ 6 & 4 \end{smallmatrix}\right] + 270\beta^{\text{sv}}\left[\begin{smallmatrix} 2 & 1 \\ 6 & 4 \end{smallmatrix}\right] - 270\beta^{\text{sv}}\left[\begin{smallmatrix} 1 & 2 \\ 4 & 6 \end{smallmatrix}\right]$$
$$- 3\zeta_3\beta^{\text{sv}}\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right] - 300\zeta_3\beta^{\text{sv}}\left[\begin{smallmatrix} 1 \\ 6 \end{smallmatrix}\right] + \frac{45\zeta_3\beta^{\text{sv}}\left[\begin{smallmatrix} 2 \\ 6 \end{smallmatrix}\right]}{y} + \frac{45\zeta_5\beta^{\text{sv}}\left[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right]}{y} - \frac{27\zeta_5\beta^{\text{sv}}\left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]}{4y^2} - \frac{13\zeta_5}{120},$$

$$B_{2,3}' = 1260\beta^{\text{sv}}\left[\begin{smallmatrix} 2 & 1 \\ 4 & 6 \end{smallmatrix}\right] - 840\zeta_3\beta^{\text{sv}}\left[\begin{smallmatrix} 1 \\ 6 \end{smallmatrix}\right] + \frac{7\zeta_5}{240} - \frac{\zeta_3^2}{2y} - \frac{147\zeta_7}{64y^2} + \frac{21\zeta_3\zeta_5}{8y^3}, \tag{5.33}$$

and the associated integration constants $\overline{\alpha[\ldots]}$ can be found in the reference and in an ancillary file within the arXiv submission of this article. Both $B_{2,3}, B_{2,3}'$ and their Cauchy–Riemann

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[20]The choice of basis in [18] is tailored to delay the appearance of holomorphic Eisenstein to higher Cauchy-Riemann derivatives as far as possible. That is why the real MGFs $-\frac{21}{4}E_{2,3} - \frac{1}{2}\zeta_3 E_2$ have been added to the imaginary cusp forms $B_{2,3} + \frac{1}{2}\left(\frac{\text{Im}\,\tau}{\pi}\right)^5 (\mathcal{C}\left[\begin{smallmatrix} 0 & 2 & 3 \\ 3 & 0 & 2 \end{smallmatrix}\right] - \overline{\mathcal{C}\left[\begin{smallmatrix} 0 & 2 & 3 \\ 3 & 0 & 2 \end{smallmatrix}\right]})$ in (5.32).

derivatives drop out from the two-point J- and Y-integrals. At three points, one can identify their derivatives as single-valued eMZVs,

$$\pi\nabla\mathrm{B}_{2,3} = \mathrm{SV}\left[-\tfrac{1}{2}\omega(0,0,2,2,2|-\tfrac{1}{\tau}) - 2\omega(0,0,0,1,5|-\tfrac{1}{\tau}) - \tfrac{3}{8}\omega(0,0,0,2,4|-\tfrac{1}{\tau})\right.$$
$$+ \tfrac{\zeta_3}{8}\omega(0,3|-\tfrac{1}{\tau}) + \tfrac{17}{48}\omega(0,3|-\tfrac{1}{\tau})^2 + \tfrac{7}{8}\omega(0,0,4|-\tfrac{1}{\tau})\omega(0,0,2|-\tfrac{1}{\tau})$$
$$\left.- \tfrac{7}{2}\omega(0,0,0,3|-\tfrac{1}{\tau})\omega(0,3|-\tfrac{1}{\tau}) + \tfrac{137}{16}\omega(0,0,0,0,6|-\tfrac{1}{\tau}) - \tfrac{15}{32}\omega(0,0,6|-\tfrac{1}{\tau})\right]$$

$$\pi\nabla\mathrm{B}'_{2,3} = \mathrm{SV}\left[-\tfrac{1}{2}\omega(0,0,2,2,2|-\tfrac{1}{\tau}) - 11\omega(0,0,0,1,5|-\tfrac{1}{\tau}) + \tfrac{295}{8}\omega(0,0,0,0,6|-\tfrac{1}{\tau})\right.$$
$$- \tfrac{25}{16}\omega(0,0,6|-\tfrac{1}{\tau}) - 11\omega(0,0,0,3|-\tfrac{1}{\tau})\omega(0,3|-\tfrac{1}{\tau}) + \tfrac{11}{12}\omega(0,3|-\tfrac{1}{\tau})^2 \qquad (5.34)$$
$$\left.- \tfrac{1}{4}\omega(0,0,4|-\tfrac{1}{\tau})\omega(0,0,2|-\tfrac{1}{\tau})\right]$$

$$(\pi\nabla)^2\mathrm{B}'_{2,3} = \mathrm{SV}\left[-\tfrac{189}{80}\omega(0,0,2,5|-\tfrac{1}{\tau}) + \tfrac{63}{160}\omega(0,0,4,3|-\tfrac{1}{\tau}) + \tfrac{603}{40}\omega(0,0,0,7|-\tfrac{1}{\tau})\right.$$
$$\left.- \tfrac{699}{320}\omega(0,7|-\tfrac{1}{\tau}) - \tfrac{1323}{160}\omega(0,3|-\tfrac{1}{\tau})\omega(0,0,4|-\tfrac{1}{\tau})\right]$$

$$(\pi\nabla)^3\mathrm{B}'_{2,3} = \mathrm{SV}\left[-\tfrac{63}{40}\omega(0,3,5|-\tfrac{1}{\tau}) + \tfrac{567}{64}\omega(0,0,8|-\tfrac{1}{\tau}) - \tfrac{63}{8}\omega(0,3|-\tfrac{1}{\tau})\omega(0,5|-\tfrac{1}{\tau})\right]$$

by inspecting the contributions of $f_{12}^{(3)}f_{23}^{(3)}$ or $f_{12}^{(4)}$ to J_{0,η_2,η_3}^τ and $f_{12}^{(3)}$ to J_{1,η_2,η_3}^τ. The appearance of the undifferentiated $\mathrm{B}_{2,3}$ and $\mathrm{B}'_{2,3}$ is relegated to the J-integrals at four points (or the Y-integrals at three points [18]), and comparison with the B-cycle integrals yields

$$\mathrm{B}_{2,3} = \mathrm{SV}\left[\tfrac{143}{20}\omega(0,0,0,0,0,5|-\tfrac{1}{\tau}) - \tfrac{11}{2}\omega(0,0,0,0,1,4|-\tfrac{1}{\tau}) + 2\omega(0,0,0,0,2,3|-\tfrac{1}{\tau})\right.$$
$$- 2\omega(0,0,0,1,2,2|-\tfrac{1}{\tau}) - \tfrac{91}{40}\omega(0,0,0,5|-\tfrac{1}{\tau}) - \tfrac{449}{7200}\omega(0,5|-\tfrac{1}{\tau}) - \tfrac{5}{12}\omega(0,3|-\tfrac{1}{\tau})\omega(0,0,2|-\tfrac{1}{\tau})$$
$$+ \tfrac{5}{2}\omega(0,0,2|-\tfrac{1}{\tau})\omega(0,0,0,3|-\tfrac{1}{\tau}) + \tfrac{23}{12}\omega(0,0,2,3|-\tfrac{1}{\tau}) - \tfrac{15}{2}\omega(0,3|-\tfrac{1}{\tau})\omega(0,0,0,0,2|-\tfrac{1}{\tau})$$
$$\left.+ \tfrac{\zeta_3}{4}\omega(0,0,2|-\tfrac{1}{\tau}) - \tfrac{43\zeta_5}{480}\right] \qquad (5.35)$$

$$\mathrm{B}'_{2,3} = \mathrm{SV}\left[\tfrac{463}{10}\omega(0,0,0,0,0,5|-\tfrac{1}{\tau}) - 22\omega(0,0,0,0,1,4|-\tfrac{1}{\tau}) + 5\omega(0,0,0,0,2,3|-\tfrac{1}{\tau})\right.$$
$$- 2\omega(0,0,0,1,2,2|-\tfrac{1}{\tau}) - \tfrac{121}{20}\omega(0,0,0,5|-\tfrac{1}{\tau}) - \tfrac{1069}{3600}\omega(0,5|-\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,3|-\tfrac{1}{\tau})\omega(0,0,2|-\tfrac{1}{\tau})$$
$$\left.+ \omega(0,0,2|-\tfrac{1}{\tau})\omega(0,0,0,3|-\tfrac{1}{\tau}) + \tfrac{25}{6}\omega(0,0,2,3|-\tfrac{1}{\tau}) - 24\omega(0,3|-\tfrac{1}{\tau})\omega(0,0,0,0,2|-\tfrac{1}{\tau}) - \tfrac{11\zeta_5}{96}\right].$$

The open-string counterparts of $\mathrm{B}_{2,3}, \mathrm{B}'_{2,3}$ and their Cauchy–Riemann derivatives involve the simplest combinations of B-cycle eMZVs with an irreducible $\zeta_{3,5}$ in their Laurent polynomials: The methods of [3] (also see appendix B of [51]) yield the following examples of $\tau \to i\infty$ degenerations in (5.34) and (5.35),

$$\mathrm{B}_{2,3}\big|_{\mathrm{LP}} = \mathrm{SV}\left[\frac{iT^3\zeta_2}{5040} + \frac{29iT\zeta_4}{630} + \frac{\zeta_3\zeta_2}{8} - \frac{8153i\zeta_6}{2880T} - \frac{9\zeta_5\zeta_2}{8T^2} - \frac{73\zeta_3\zeta_4}{16T^2} + \frac{1837i\zeta_8}{240T^3}\right.$$
$$\left.+ \frac{3i}{10T^3}(2\zeta_{3,5} + 5\zeta_3\zeta_5) - \frac{39\zeta_5\zeta_4}{8T^4} + \frac{45\zeta_3\zeta_6}{8T^4} + \frac{33i\zeta_{10}}{20T^5}\right]$$
$$= 0 \qquad (5.36)$$

$$(\pi\nabla)^3\mathrm{B}'_{2,3}\big|_{\mathrm{LP}} = \mathrm{SV}\left[\frac{T^8}{17280} + \frac{iT^5\zeta_3}{120} + \frac{T^4\zeta_4}{40} - \frac{7T^2\zeta_6}{80} - \frac{8211\zeta_8}{640}\right.$$
$$\left.+ \frac{63\zeta_{3,5}}{40} + \frac{189i\zeta_5\zeta_4}{8T} + \frac{63i\zeta_3\zeta_6}{4T} + \frac{14553\zeta_{10}}{160T^2}\right]$$
$$= \frac{2y^8}{135} - \frac{8y^5\zeta_3}{15} - \frac{63\zeta_3\zeta_5}{4}.$$

One can see from the order of T^{-3} or y^{-3} that the cuspidal nature of $B_{2,3}$ hinges on the depth-two result sv $\zeta_{3,5} = -10\zeta_3\zeta_5$. The non-vanishing Laurent polynomial of $(\pi\nabla)^3 B_{2,3}'$ is due to the real MGFs $-\frac{21}{4}E_{2,3} - \frac{1}{2}\zeta_3 E_2$ in (5.32).

Note that the simplest instances of $\zeta_{3,7}$ and $\zeta_{3,5,3}$ arise in the Laurent polynomials of B-cycle eMZVs with MGFs (2.26) of weights $\sum_{j=1}^r (a_j + b_j) = 12$ and 14 in their SV-image. The appearance of $\zeta_{3,5,3}$ in modular graph functions and eMZVs can be found in [12] and [51], respectively. While $\zeta_{3,7}$ drops out from MGFs under the single-valued map, it enters for instance the T^0-order of the Laurent polynomial of $\omega(0,3,7|-\frac{1}{\tau})$ whose SV-image contributes to the quantity $(\pi\nabla)^4 B_{2,4}'$ in section 9.2 of [87]

$$\omega(0,3,7|-\tfrac{1}{\tau})\big|_{\mathrm{LP}} = -\frac{T^{10}}{1261260} + \frac{2iT^5\zeta_5}{315} + \frac{2T^4\zeta_6}{63} + \frac{7iT^3\zeta_7}{45} + \frac{7T^2\zeta_8}{6} \tag{5.37}$$
$$- \zeta_{3,7} - 14\zeta_3\zeta_7 - 6\zeta_5^2 + \frac{27\zeta_{10}}{2} + \frac{84i\zeta_{11}}{T} + \frac{30i\zeta_5\zeta_6}{T} + \frac{84i\zeta_3\zeta_8}{T} + \frac{1353\zeta_{12}}{2T^2}\,.$$

One can eventually find all $\mathbb{Q}$-independent MZVs[21] in the Laurent polynomials of B-cycle eMZVs. This follows from both the degeneration limits of the elliptic KZB associator [3] and from the fact that any MZV is expressible via $\mathbb{Q}[2\pi i]$-linear combinations of multiple modular values [107].

Note that a large number of Laurent polynomials of B-cycle eMZVs obtained from a FORM implementation [96] of the methods of [3,51] are available for download from [97].

6 Conclusions and outlook

In this work, we have studied generating series of configuration-space integrals that arise in open- and closed-string amplitudes at genus one. The differential equations and $\tau \to i\infty$ degenerations of these generating series served as a framework to propose the explicit form of an elliptic single-valued map. Our construction is based on a tentative genus-one uplift of the Betti–deRham duality between integration cycles on a disk boundary and antiholomorphic Parke–Taylor integrands which drives the relation between closed-string and single-valued open-string tree amplitudes [40–45]. These considerations lead us to construct closed-string genus-one integrals over specific antielliptic functions which are thought of as Betti–deRham dual to open-string integration cycles in view of their singularities at $z_i \to z_j$ and their degeneration at $\tau \to i\infty$.

Most importantly, the differential equations of the open- and closed-string integrals under investigation only differ by $\tau^j G_k(\tau)$ vs. $(\tau-\bar\tau)^j G_k(\tau)$ in the respective differential operators with holomorphic Eisenstein series G_k. Accordingly, we generate the elliptic multiple zeta values and modular graph forms in their α'-expansions via path ordered exponentials with the same polynomial structures in kinematic invariants and formal expansion variables. The τ-dependent building blocks are iterated Eisenstein integrals in both cases – holomorphic ones with kernels $\tau^j G_k(\tau)$ for the open-string integrals and their single-valued versions involving kernels $(\tau-\bar\tau)^j G_k(\tau)$ for closed strings.

Our proposal for an elliptic single-valued map is defined through the relation between the generating series of open- and closed-string integrals. By their respective α'-expansion, we

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[21]See [106] for a computer implementation of $\mathbb{Q}$-relations among MZVs.

obtain the single-valued map for all iterated Eisenstein integrals occurring in the open-string series. This in turn determines the single-valued map of any convergent elliptic multiple zeta value in terms of modular graph forms.

This construction hinges on the compatibility of the initial values at $\tau \to i\infty$ under the single-valued map of multiple zeta values [48, 46]. We have given evidence for their compatibility by identifying the key building blocks of genus-zero integrals at the cusp – appropriate pairs of disk orderings and Parke–Taylor integrands. However, the detailed expressions for the asymptotic expansions beyond two points in terms of genus-zero integrals is left for future work. At present, the procedure also relies on the reality properties of a generating series $Y_{\vec{\eta}}^{\tau}$ of a more general class of closed-string integrals. Our method does not yet provide a direct construction of single-valued iterated Eisenstein integrals solely from open-string data.

6.1 Genus-one integrals versus string amplitudes

The results of this work concern infinite families of configuration-space integrals at genus one, and their application to genus-one string amplitudes requires the following leftover steps:

For both open and closed strings, it remains to integrate over the modular parameter τ of the respective surface. In the closed-string case, τ-integrals over modular graph forms are typically performed on the basis of their Laplace equations [9, 108–112] and Poincaré-series representations [113–115,54,116–118]. The τ-integration of open-string integrals has for instance been discussed in [119–122], and a general method applicable to arbitrary depth may be based on the representation of elliptic multiple zeta values in terms of iterated Eisenstein integrals (including their "over-integrated" instances with kernels $\tau^{j}G_{k}$ at $j > k-2$ [118]) and properties of multiple modular values [101]. It would be particularly interesting to relate closed-string and single-valued open-string integrals at genus one *after* integration over τ.

For open strings, the $Z_{\vec{\eta}}^{\tau}$- or $B_{\vec{\eta}}^{\tau}$-series are claimed to exhaust all the configuration-space integrands built from $f^{(k)}(z_i - z_j, \tau)$ that are inequivalent under Fay identities and integration by parts. Similarly, the $Y_{\vec{\eta}}^{\tau}$-series built from double copies of the open-string integrands is expected to contain all torus integrals of this type. Hence, by the arguments of [78, 4, 52], $B_{\vec{\eta}}^{\tau}$ and $Y_{\vec{\eta}}^{\tau}$ should[22] capture the conformal-field-theory correlators in the integrands of n-point genus-one amplitudes of massless states (and possibly also of massive states) in bosonic, heterotic and type-II string theories. In all cases, the component integrals in the η_j-expansions of the $Z_{\vec{\eta}}^{\tau}$-, $B_{\vec{\eta}}^{\tau}$ or $Y_{\vec{\eta}}^{\tau}$ series need to be dressed with kinematic factors that are determined by the correlators and carry the polarization dependence of the respective string amplitude.

The integrands of $J_{\vec{\eta}}^{\tau}$ only involve antielliptic combinations of $\overline{f^{(k)}(z_i - z_j, \tau)}$ and omit infinite classes of component integrals of $Y_{\vec{\eta}}^{\tau}$. For a given genus-one closed-string amplitude, it is therefore not a priori clear if its correlator is generated by the integrand of $J_{\vec{\eta}}^{\tau}$. Still, the correlators for the four- and five-point type-II amplitudes can be recovered from the subsectors $J_{w,\vec{\eta}}^{\tau}$ of the

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[22]It has been shown in [52] that the integrands of massless genus-one amplitudes in bosonic, heterotic and type-II theories are expressible in terms of products of $f^{(k)}(z_i - z_j, \tau)$ and their z_i-derivatives. The conjectural part is that arbitrary products of $f^{(k)}(z_i - z_j, \tau)$ (possibly including derivatives) are expressible in terms of the $\varphi_{\vec{\eta}}^{\tau}$ in (2.13) with their specific chain structure via repeated use of Fay identities and integration by parts [59, 60].

$J_{\vec{\eta}}^{\tau}$-series at fixed modular weights: The four-point correlator of [79] resides at the η^{-3} order of $J_{0,\eta_2,\eta_3,\eta_4}^{\tau}$, and the five-point correlators of [104, 105] can be assembled from the most singular η-orders of $J_{w,\eta_2,\eta_3,\eta_4,\eta_5}^{\tau}$ at $w = 0, 1$. Similarly, the four- and five-point amplitudes of gluons and gravitons in heterotic string theories can in principle be extracted from the same $J_{w,\vec{\eta}}^{\tau}$ which also appear in type II, where higher orders in η_j are needed to capture the bosonic sectors. It would be interesting to see if this pattern persists at higher points in supersymmetric amplitudes, and whether the $J_{\vec{\eta}}^{\tau}$ are sufficient to generate bosonic-string amplitudes at low multiplicity.

6.2 Further directions

This work spawns a variety of further directions and open questions of relevance to both physicists and mathematicians:

The single-valued image of elliptic multiple zeta values is proposed to contain combinations of holomorphic iterated Eisenstein integrals and their complex conjugates denoted by β^{sv} and constructed from the α'-expansion of closed-string integrals in [18]. It would be important to work out their detailed relation to Brown's earlier construction of single-valued iterated Eisenstein integrals [13, 14]. In particular, it remains to relate the MZVs in the antiholomorphic contributions to β^{sv} (fixed from reality properties of Koba–Nielsen integrals in [18]) to the combinations of multiple modular values entering Brown's construction. This will hopefully bypass the need to use these reality properties as independent input for the construction of β^{sv} as done so far.

Several aspects of our construction are based on conjectures with strong support from a variety of non-trivial examples. As pointed out in the relevant passages in earlier parts of this work, it would be desirable to find mathematically rigorous proofs that

- any Koba–Nielsen integral at genus one involving products and derivatives of Kronecker–Eisenstein coefficients $f^{(k)}(z, \tau)$ can be expanded in the coefficients of the series $Z_{\vec{\eta}}^{\tau}$ and $Y_{\vec{\eta}}^{\tau}$

- the matrices $r_{\vec{\eta}}(\epsilon_k)$ and $R_{\vec{\eta}}(\epsilon_k)$ in open- and closed-string differential equations (2.19) preserve the commutation relations of Tsunogai's derivations ϵ_k

- the single-valued images β^{sv} of iterated Eisenstein integrals satisfy shuffle relations, i.e. that the antiholomorphic integration constants α do not introduce any obstructions

- the coefficients of the s_{ij}- and η_j- expansion of the initial values $\widehat{J}_{\vec{\eta}}^{i\infty}$ and $\widehat{Y}_{\vec{\eta}}^{i\infty}$ are single-valued multiple zeta values

The proposal of the present work concerns single-valued integration [48, 49] in the modular parameter τ. An alternative approach is to recover modular graph forms from single-valued functions of torus punctures [10, 15]. In this context, it would be rewarding to find an explicit realization of single-valued integration in z for elliptic polylogarithms and their complex conjugates, for instance by building upon the ideas of [15] and the depth-one results in [123].

At genus zero, the identification of sphere integrals as single-valued disk integrals is equivalent to the Kawai–Lewellen–Tye (KLT) relations between closed-string and squares of open-string

tree-level amplitudes [124]. Accordingly, one could wonder if the combinations of holomorphic and antiholomorphic iterated Eisenstein integrals in the β^{sv} or modular graph forms can arise from products of open-string type generating functions and their complex conjugates. If such a genus-one echo of KLT relations exists, then one can expect a close relation to the monodromy relations among open-string integrals [125, 122, 126, 127] and in particular their study in the light of twisted deRham theory [128]. And it could open up a new perspective on the quest for loop-level KLT relations to revisit the generating functions of closed-string integrals in the framework of chiral splitting [129, 130], by performing the α'-expansion at the level of the loop integrand.

A particularly burning question concerns a higher-genus realization of single-valued integration and the associated relations between open- and closed-string amplitudes. A promising first step could be to identify suitable holomorphic open-string analogues of the modular graph forms [131, 132] and modular graph tensors [133] at higher genus. More generally, the simplified correlators of maximally supersymmetric genus-two amplitudes at four points [134, 135] and five points [136, 137] provide valuable showcases of Koba–Nielsen integrals relevant to open- and closed-string scattering. Furthermore, the construction of the generating series in this work was inspired by extended families of genus-one Koba–Nielsen integrals that arise from heterotic or bosonic strings [52]. Hence, the genus-two correlators of the heterotic string and the combinations of theta functions studied in [138, 139] could give important clues on higher-genus versions of the elliptic functions and generating series in this work.

Acknowledgments

A   Relations among the elliptic V and V_w functions

In this appendix, we spell out a method to determine the rational coefficients $c_{w,\gamma}$ in the expansion (3.36) of elliptic functions $V_w(\ldots)$ of fixed modular weights in terms of the $V(\ldots)$ functions in (3.7). This will be done by exploiting the $\tau \to i\infty$ degeneration (3.19) of the $V(\ldots)$ which

fixes the $c_{w,\gamma}$ in the ansatz (3.36) via

$$\lim_{\tau \to i\infty} \frac{V_w(1, 2, \ldots, n|\tau)}{(2\pi i)^w \sigma_1 \sigma_2 \ldots \sigma_n} = \sum_{\gamma \in S_{n-1}} c_{w,\gamma} \, \mathrm{PT}^{(1)}(1, \gamma(2, 3, \ldots, n)) \tag{A.1}$$

$$\mathrm{PT}^{(1)}(1, 2, \ldots, n) = (-1)^{n-1} \lim_{\sigma_- \to \infty} |\sigma_-|^2 \Big(\mathrm{PT}(+, n, n-1, \ldots, 2, 1, -) + \mathrm{cyc}(1, 2, \ldots, n) \Big)$$

$$= \frac{1}{\sigma_{12}\sigma_{23} \ldots \sigma_{n-1,n}\sigma_n} + \mathrm{cyc}(1, 2, \ldots, n). \tag{A.2}$$

The combinations $\mathrm{PT}^{(1)}$ are known as one-loop Parke–Taylor factors from an ambitwistor-string context [94], and we have used $\sigma_+ = 0$ in passing to the last line. In order to determine the degeneration of the left-hand side of (A.1), we expand the elliptic functions

$$V_w(1, 2, \ldots, n|\tau) = \sum_{\substack{a_1, a_2, \ldots, a_n \geq 0 \\ a_1 + a_2 + \ldots + a_n = w}} g_{12}^{(a_1)} g_{23}^{(a_2)} \ldots g_{n-1,n}^{(a_{n-1})} g_{n,1}^{(a_n)}$$

$$\frac{\theta_1'(0, \tau)\theta_1(z_{ij}+\eta, \tau)}{\theta_1(z_{ij}, \tau)\theta_1(\eta, \tau)} = \sum_{a=0}^{\infty} \eta^{a-1} g_{ij}^{(a)} \tag{A.3}$$

in terms of the meromorphic Kronecker–Eisenstein coefficients $g_{ij}^{(a)}$ instead of the $f_{ij}^{(a)}$ in (5.2), starting with $g_{ij}^{(0)} = 1$ and $g_{ij}^{(1)} = \partial_{z_i} \log \theta_1(z_{ij}, \tau)$. Their $\tau \to i\infty$ limits [3, 4] (also see (3.18))

$$\lim_{\tau \to i\infty} g_{jk}^{(a)} = \begin{cases} 1 & : \ a = 0 \\ i\pi \frac{\sigma_j + \sigma_k}{\sigma_j - \sigma_k} & : \ a = 1 \\ -2\zeta_a & : \ a \in 2\mathbb{N} \\ 0 & : \ a \in 2\mathbb{N}+1 \end{cases} \tag{A.4}$$

ensure that the combination $(2\pi i)^{-w} V_w(1, 2, \ldots, n|\tau)$ in (A.1) degenerates to a rational function of the σ_j, where all factors of $i\pi$ cancel. Hence, the only σ_j dependence of $V_w(\ldots|\tau \to i\infty)$ occurs via $\lim_{\tau \to i\infty} g_{jk}^{(1)} = i\pi \frac{\sigma_j + \sigma_k}{\sigma_j - \sigma_k}$.

By applying the degeneration (A.4) to the elliptic function V_w in (A.3), the leftover challenge in determining the $c_{w,\gamma}$ in (A.1) is to expand the terms of the form $(\sigma_1 \sigma_2 \ldots \sigma_n)^{-1} \prod_{i=1}^r \frac{\sigma_{j_i} + \sigma_{k_i}}{\sigma_{j_i} - \sigma_{k_i}}$ on the left-hand side in terms of Parke–Taylor factors. For the choices of $\sigma_{j_i}, \sigma_{k_i}$ that arise from the degeneration of $V_{w \leq n-2}$, these Parke–Taylor decompositions can be performed by the methods of [140]: As explained in section 3 of the reference, the net effect of the rational factor $\frac{\sigma_{j_i} + \sigma_{k_i}}{\sigma_{j_i} - \sigma_{k_i}}$ is to modify the signs of the Parke–Taylor factors on the right-hand side of

$$\frac{1}{\sigma_1 \sigma_2 \ldots \sigma_n} = (-1)^{n-1} \lim_{\sigma_- \to \infty} |\sigma_-|^2 \sum_{\rho \in S_n} \mathrm{PT}(+, \rho(1, 2, \ldots, n), -) \tag{A.5}$$

More specifically, with the notation

$$\mathrm{sgn}_{jk}^{\rho} = \begin{cases} +1 & : \ j \text{ is on the right of } k \text{ in } \rho(1, 2, \ldots, n) \\ -1 & : \ j \text{ is on the left of } k \text{ in } \rho(1, 2, \ldots, n) \end{cases} \tag{A.6}$$

the modification of (A.5) by degenerations of $g^{(1)}_{j_i k_i}$ can be written as [140]

$$\lim_{\tau \to i\infty} \frac{g^{(1)}_{j_1 k_1} g^{(1)}_{j_2 k_2} \cdots g^{(1)}_{j_r k_r}}{(2\pi i)^r \sigma_1 \sigma_2 \ldots \sigma_n} = \frac{(-1)^{n-1}}{2^r} \lim_{\sigma_- \to \infty} |\sigma_-|^2 \qquad \text{(A.7)}$$
$$\times \sum_{\rho \in S_n} \text{sgn}^\rho_{j_1 k_1} \text{sgn}^\rho_{j_2 k_2} \ldots \text{sgn}^\rho_{j_r k_r} \text{PT}(+, \rho(1, 2, \ldots, n), -) \, .$$

The contributions of $(2\pi i)^{-2k} g^{(2k)}_{jk}$ in turn degenerate to rational constants by (A.4) which multiply the overall sum over permutations ρ. Hence, (A.7) allows to straightforwardly expand the left-hand side of (A.1) in terms of Parke–Taylor factors in an $n!$-element basis of $\text{PT}(+, \ldots, -)$. Matching the Parke–Taylor coefficients with those on the right-hand side determines the $c_{w,\gamma}$ in (3.36). It is a special property of the elliptic functions V_w that their degeneration conspires to combinations of $\text{PT}^{(1)}$ in (A.2), i.e. that the $(n-1)!$ independent $c_{w,\gamma}$ are sufficient to accommodate the $n!$ permutations of $\text{PT}(+, 1, \ldots, n, -)$.

For instance, the decompositions in (3.38) to (3.40) follow from the special cases of (A.7)

$$\lim_{\tau \to i\infty} \frac{V_1(1, 2, 3|\tau)}{(2\pi i)\sigma_1 \sigma_2 \sigma_3} = \frac{1}{2} \lim_{\sigma_- \to \infty} |\sigma_-|^2 \sum_{\rho \in S_3} \text{PT}(+, \rho(1, 2, 3), -)(\text{sgn}^\rho_{12} + \text{sgn}^\rho_{23} + \text{sgn}^\rho_{31})$$

$$\lim_{\tau \to i\infty} \frac{V_1(1, 2, 3, 4|\tau)}{(2\pi i)\sigma_1 \sigma_2 \sigma_3 \sigma_4} = -\frac{1}{2} \lim_{\sigma_- \to \infty} |\sigma_-|^2 \sum_{\rho \in S_4} \text{PT}(+, \rho(1, 2, 3, 4), -)(\text{sgn}^\rho_{12} + \text{sgn}^\rho_{23} + \text{sgn}^\rho_{34} + \text{sgn}^\rho_{41})$$

$$\lim_{\tau \to i\infty} \frac{V_2(1, 2, 3, 4|\tau)}{(2\pi i)^2 \sigma_1 \sigma_2 \sigma_3 \sigma_4} = -\lim_{\sigma_- \to \infty} |\sigma_-|^2 \sum_{\rho \in S_4} \text{PT}(+, \rho(1, 2, 3, 4), -)\left\{ \frac{1}{3} + \frac{1}{4}(\text{sgn}^\rho_{12}\text{sgn}^\rho_{34} + \text{sgn}^\rho_{23}\text{sgn}^\rho_{41}) \right.$$
$$\left. + \frac{1}{4}(\text{sgn}^\rho_{12}\text{sgn}^\rho_{23} + \text{sgn}^\rho_{23}\text{sgn}^\rho_{34} + \text{sgn}^\rho_{34}\text{sgn}^\rho_{41} + \text{sgn}^\rho_{41}\text{sgn}^\rho_{12}) \right\} \qquad \text{(A.8)}$$

once the right-hand sides are matched with the combinations of one-loop Parke–Taylor factors $\text{PT}^{(1)}$ in (A.1) and (A.2).

B The initial value $\widehat{B}^{i\infty}_{\eta 2, \eta 3}$ at three points

This appendix gathers the three-point initial values $\widehat{B}^{i\infty}_{\eta 2, \eta 3}(2, 3|\rho(2, 3))$ for the α'-expansion (4.9) of B-cycle integrals up to including weight four. The corresponding orders of $\widehat{J}^{i\infty}_{\eta 2, \eta 3}(2, 3|\rho(2, 3))$ relevant to the α'-expansion (4.19) of J-integrals are obtained from the single-valued map $(\zeta_2, \zeta_3, \zeta_4) \to (0, 2\zeta_3, 0)$. Since even (odd) orders in the η_j-expansion integrate to zero on the odd (even) integration cycles $\mathfrak{B}(2, 3) \pm \mathfrak{B}(3, 2)$, we will separate the two types of contributions in order to infer $\widehat{B}^{i\infty}_{\eta 2, \eta 3}(2, 3|3, 2)$ from a relabeling of $\widehat{B}^{i\infty}_{\eta 2, \eta 3}(2, 3|2, 3)$.

The expressions in this appendix along with various higher-order terms in the s_{ij}- and η_j-expansions can be found in an ancillary file within the arXiv submission of this article.

B.1 Even orders in η_j

The terms of even orders in η_j in the three-point initial values are given by

$$
\begin{aligned}
\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|2,3)\big|_{\text{even}} &= \frac{1}{\eta_{23}\eta_3}\Big(\frac{1}{2} + \frac{\zeta_2}{12}(s_{12}^2 + s_{13}^2 + s_{23}^2) + \frac{\zeta_3}{24}(s_{12}^3 + s_{13}^3 + s_{23}^3) \\
&\qquad + \zeta_4\Big[\frac{131}{1440}(s_{12}^4 + s_{13}^4 + s_{23}^4) + \frac{5}{144}(s_{12}^2 s_{13}^2 + s_{12}^2 s_{23}^2 + s_{23}^2 s_{13}^2) + \frac{1}{18}s_{12}s_{13}s_{23}s_{123}\Big] + \dots\Big) \\
&\quad + \frac{\eta_{23}}{\eta_3}\Big(-\zeta_2 - \frac{\zeta_3}{2}s_{12} - \zeta_4\Big[\frac{29s_{12}^2}{24} + \frac{5s_{13}^2}{12} + \frac{s_{13}s_{23}}{3} + \frac{5s_{23}^2}{12}\Big] + \dots\Big) \\
&\quad + \frac{\eta_3}{\eta_{23}}\Big(-\zeta_2 - \frac{\zeta_3}{2}s_{23} - \zeta_4\Big[\frac{29s_{23}^2}{24} + \frac{5s_{12}^2}{12} + \frac{s_{12}s_{13}}{3} + \frac{5s_{13}^2}{12}\Big] + \dots\Big) \\
&\quad + \Big(\frac{3\zeta_2 s_{13}}{s_{123}} + \frac{\zeta_3}{2}s_{13} + \frac{15\zeta_4 s_{13}}{4s_{123}}(s_{12}^2 + s_{12}s_{23} + s_{23}^2) - \frac{41}{24}\zeta_4(s_{12} + s_{23})s_{13} + \frac{49\zeta_4 s_{13}^2}{24} + \dots\Big) \\
&\quad + \eta_{23}\eta_3(5\zeta_4) - \frac{\eta_{23}^3}{\eta_3}\zeta_4 - \frac{\eta_3^3}{\eta_{23}}\zeta_4 + \dots
\end{aligned}
\tag{B.1}
$$

and

$$
\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|3,2)\big|_{\text{even}} = \widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|2,3)\big|_{\text{even}}\big|^{\eta_2 \leftrightarrow \eta_3}_{s_{12} \leftrightarrow s_{13}}
\tag{B.2}
$$

with MZVs of weight ≥ 5 in the ellipsis.

B.2 Odd orders in η_j

The terms of odd orders in η_j in the three-point initial values are given by

$$
\begin{aligned}
\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|2,3)\big|_{\text{odd}} &= \frac{1}{\eta_3}\Big(\frac{1}{s_{12}} + \frac{\zeta_2 s_{123}^2}{6s_{12}} + \zeta_3\Big[\frac{s_{13}s_{23}}{4} + \frac{s_{123}^3}{12s_{12}}\Big] + \zeta_4\Big[\frac{131s_{123}^4}{720s_{12}} - \frac{s_{23}s_{123}s_{13}}{20}\Big] + \dots\Big) \\
&\quad + \frac{1}{\eta_{23}}\Big(\frac{1}{s_{23}} + \frac{\zeta_2 s_{123}^2}{6s_{23}} + \zeta_3\Big[\frac{s_{12}s_{13}}{4} + \frac{s_{123}^3}{12s_{23}}\Big] + \zeta_4\Big[\frac{131s_{123}^4}{720s_{23}} - \frac{s_{12}s_{123}s_{13}}{20}\Big] + \dots\Big) \\
&\quad + \eta_3\Big(-\frac{2\zeta_2}{s_{12}} - \frac{\zeta_3 s_{123}}{s_{12}} + \zeta_4\Big[-\frac{29s_{123}^2}{12s_{12}} + \frac{s_{13}}{4}\Big] + \dots\Big) \\
&\quad + \eta_{23}\Big(-\frac{2\zeta_2}{s_{23}} - \frac{\zeta_3 s_{123}}{s_{23}} + \zeta_4\Big[-\frac{29s_{123}^2}{12s_{23}} + \frac{s_{13}}{4}\Big] + \dots\Big) \\
&\quad + \frac{\eta_{23}^2}{\eta_3}\Big(-\zeta_3 + \zeta_4\Big[\frac{2}{3}s_{123} + \frac{s_{12}}{4}\Big] + \dots\Big) + \frac{\eta_3^2}{\eta_{23}}\Big(-\zeta_3 + \zeta_4\Big[\frac{2}{3}s_{123} + \frac{s_{23}}{4}\Big] + \dots\Big) \\
&\quad + \eta_{23}^3\Big(-\frac{2\zeta_4}{s_{23}} + \dots\Big) + \eta_3^3\Big(-\frac{2\zeta_4}{s_{12}} + \dots\Big) + \dots
\end{aligned}
\tag{B.3}
$$

and

$$
\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|3,2)\big|_{\text{odd}} = -\widehat{B}^{i\infty}_{\eta_2,\eta_3}(2,3|2,3)\big|_{\text{odd}}\big|^{\eta_2 \leftrightarrow \eta_3}_{s_{12} \leftrightarrow s_{13}},
\tag{B.4}
$$

again with MZVs of weight ≥ 5 in the ellipsis.

C Examples of single-valued eMZVs

C.1 Systematics at depth one

The simplest examples (5.11) and (5.12) of single-valued eMZVs extracted from the two-point integrals (5.4) are special cases of the SV map (4.30) on holomorphic iterated Eisenstein integrals. For their depth-one combinations $\beta\begin{bmatrix} j \\ k \end{bmatrix}$ in (4.10), the SV image $\beta^{\mathrm{sv}}\begin{bmatrix} j \\ k \end{bmatrix}$ yields the following Cauchy–Riemann derivatives $\nabla = 2i(\operatorname{Im}\tau)^2\partial_\tau$ of non-holomorphic Eisenstein series [18]

$$\beta^{\mathrm{sv}}\begin{bmatrix} k-1 \\ 2k \end{bmatrix} = -\frac{[(k-1)!]^2}{(2k-1)!}\mathrm{E}_k + \frac{2\zeta_{2k-1}}{(2k-1)(4y)^{k-1}}$$

$$\beta^{\mathrm{sv}}\begin{bmatrix} k-1+m \\ 2k \end{bmatrix} = -\frac{(-4)^m(k-1)!\,(k-1-m)!\,(\pi\nabla)^m\mathrm{E}_k}{(2k-1)!} + \frac{2\zeta_{2k-1}}{(2k-1)(4y)^{k-1-m}} \qquad \text{(C.1)}$$

$$\beta^{\mathrm{sv}}\begin{bmatrix} k-1-m \\ 2k \end{bmatrix} = -\frac{(k-1)!\,(k-1-m)!\,(\pi\overline{\nabla})^m\mathrm{E}_k}{(-4)^m(2k-1)!y^{2m}} + \frac{2\zeta_{2k-1}}{(2k-1)(4y)^{k-1+m}}\,,$$

also see (4.22). While the objects on the right-hand side are expressible in terms of the lattice sums $\mathcal{C}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ in (2.27) via (2.28), the $\beta\begin{bmatrix} j \\ k \end{bmatrix}$ are simple combinations of B-cycle eMZVs $\omega(0^p, k|-\frac{1}{\tau})$, where 0^p stands for a sequence $0,0,\ldots,0$ of p successive zeros. On these grounds, $\beta^{\mathrm{sv}}\begin{bmatrix} j \\ k \end{bmatrix} = \mathrm{SV}\,\beta\begin{bmatrix} j \\ k \end{bmatrix}$ translates to simple relations such as

$$\mathrm{SV}\,\omega(0, 2k+1|-\tfrac{1}{\tau}) = -\frac{(\tau-\bar\tau)^{2k+1}}{2\pi i}\mathcal{C}\begin{bmatrix} 2k+1 & 0 \\ 1 & 0 \end{bmatrix}\,, \quad k \geq 1$$

$$\mathrm{SV}\,\omega(0, 0, 2k+2|-\tfrac{1}{\tau}) = \frac{(\tau-\bar\tau)^{2k+2}}{(2\pi i)^2}\mathcal{C}\begin{bmatrix} 2k+2 & 0 \\ 2 & 0 \end{bmatrix}\,, \qquad k \geq 0 \quad \text{(C.2)}$$

$$\mathrm{SV}\left(\omega(0,0,0,2k+3|-\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,2k+3|-\tfrac{1}{\tau})\right) = -\frac{(\tau-\bar\tau)^{2k+3}}{(2\pi i)^3}\mathcal{C}\begin{bmatrix} 2k+3 & 0 \\ 3 & 0 \end{bmatrix}\,, \quad k \geq -1$$

$$\mathrm{SV}\left(\omega(0,0,0,0,2k+4|-\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,0,2k+4|-\tfrac{1}{\tau})\right) = \frac{(\tau-\bar\tau)^{2k+4}}{(2\pi i)^4}\mathcal{C}\begin{bmatrix} 2k+4 & 0 \\ 4 & 0 \end{bmatrix}\,, \qquad k \geq -1$$

as well as

$$\mathrm{SV}\left(\omega(0,0,0,0,0,2k+5|-\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,0,0,2k+5|-\tfrac{1}{\tau}) + \tfrac{7}{360}\omega(0,2k+5|-\tfrac{1}{\tau})\right)$$

$$= -\frac{(\tau-\bar\tau)^{2k+5}}{(2\pi i)^5}\mathcal{C}\begin{bmatrix} 2k+5 & 0 \\ 5 & 0 \end{bmatrix}\,, \qquad k \geq -2\,. \qquad \text{(C.3)}$$

The relative factors of $-\frac{1}{6}$ and $\frac{7}{360}$ among the eMZVs of different lengths are engineered to streamline the iterated-Eisenstein-integral representation [17] and generalize as follows [103]

$$\mathrm{SV}\sum_{j=0}^{\ell-1}\frac{B_j}{j!}\omega(0^{\ell-j}, 2k+\ell|-\tfrac{1}{\tau}) = (-1)^\ell\frac{(\tau-\bar\tau)^{2k+\ell}}{(2\pi i)^\ell}\mathcal{C}\begin{bmatrix} 2k+\ell & 0 \\ \ell & 0 \end{bmatrix}\,, \qquad k \geq 1-\left\lceil\frac{\ell}{2}\right\rceil,\ \ell \geq 1\,. \quad \text{(C.4)}$$

In obtaining (C.2) and (C.3) from (C.4), we have used $\mathrm{SV}\,\omega(m) = 0 \,\forall\, m \geq 1$ and the following simplifications of the only eMZV $\omega(0^{\ell-1}, 2k+\ell)$ whose length and weight adds up to an even

number [17],

$$\omega(0,0,2k{+}1) = \frac{1}{2}\omega(0,2k{+}1)$$

$$\omega(0,0,0,2k) = \frac{1}{2}\omega(0,0,2k) - \frac{1}{24}\omega(2k)$$ (C.5)

$$\omega(0,0,0,2k{+}1) = \frac{1}{2}\omega(0,0,0,2k{+}1) - \frac{1}{24}\omega(0,2k{+}1)\,.$$

Based on the dictionary (2.28) between lattice sums $\mathcal{C}\left[\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right]$ and non-holomorphic Eisenstein series, one can reformulate (C.4) as

$$\mathrm{SV}\sum_{j=0}^{\ell-1}\frac{B_j}{j!}\omega(0^{\ell-j},2k{+}\ell|{-}\tfrac{1}{\tau}) = (-1)^\ell \frac{(k{+}\ell{-}1)!}{(2k{+}\ell{-}1)!}(-4\pi\nabla)^k \mathrm{E}_{k+\ell}\,, \qquad k \geq 0\,,\ \ \ell \geq 1\,, \quad \text{(C.6)}$$

where $k = 0$ needs to be excluded if $\ell = 1$, for instance

$$\mathrm{SV}\,\omega(0,2k{+}1|{-}\tfrac{1}{\tau}) = -\frac{k!}{(2k)!}(-4\pi\nabla)^k \mathrm{E}_{k+1}\,, \qquad\qquad k \geq 1 \qquad \text{(C.7)}$$

$$\mathrm{SV}\,\omega(0,0,2k{+}2|{-}\tfrac{1}{\tau}) = \frac{(k{+}1)!}{(2k{+}1)!}(-4\pi\nabla)^k \mathrm{E}_{k+2}\,, \qquad\qquad k \geq 0\,.$$

Moreover, by extending (C.4) to $k \to -k$ and applying the complex conjugate of (2.28), we also obtain antiholomorphic Cauchy–Riemann derivatives as single-valued eMZVs (with $\ell - 2k > 0$),

$$\mathrm{SV}\sum_{j=0}^{\ell-1}\frac{B_j}{j!}\omega(0^{\ell-j},\ell{-}2k|{-}\tfrac{1}{\tau}) = (-1)^\ell \frac{(\tau{-}\bar\tau)^{\ell-2k}}{(2\pi i)^\ell}\,\mathcal{C}\left[\begin{smallmatrix} \ell-2k & 0 \\ \ell & 0 \end{smallmatrix}\right] \qquad\qquad \text{(C.8)}$$

$$= (-1)^{\ell+k}\frac{(\ell{-}k{-}1)!}{(\ell{-}1)!}\frac{(\pi\overline{\nabla})^k \mathrm{E}_{\ell-k}}{(2y)^{2k}}\,.$$

The simplest examples include

$$\mathrm{SV}\left(\omega(0,0,0,1|{-}\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,1|{-}\tfrac{1}{\tau})\right) = \frac{\pi\overline{\nabla}\mathrm{E}_2}{8y^2}$$

$$\mathrm{SV}\left(\omega(0,0,0,0,2|{-}\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,0,2|{-}\tfrac{1}{\tau})\right) = -\frac{\pi\overline{\nabla}\mathrm{E}_3}{12y^2} \qquad \text{(C.9)}$$

$$\mathrm{SV}\left(\omega(0,0,0,0,0,1|{-}\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,0,0,1|{-}\tfrac{1}{\tau}) + \tfrac{7}{360}\omega(0,1|{-}\tfrac{1}{\tau})\right) = -\frac{(\pi\overline{\nabla})^2\mathrm{E}_3}{192y^4}$$

$$\mathrm{SV}\left(\omega(0,0,0,0,0,3|{-}\tfrac{1}{\tau}) - \tfrac{1}{6}\omega(0,0,0,3|{-}\tfrac{1}{\tau}) + \tfrac{7}{360}\omega(0,3|{-}\tfrac{1}{\tau})\right) = \frac{\pi\overline{\nabla}\mathrm{E}_4}{16y^2}\,,$$

and the first two lines are equivalent to those in (5.31).

C.2   Examples with real MGFs at depth two

By inspecting the s_{ij}^4 order of the two-point integrals $B_{(0)}^\tau, J_{(0)}^\tau$ and the s_{ij}^3 order of $B_{(2)}^\tau, J_{(2)}^\tau$, we have obtained the representations (5.13) of $\mathrm{E}_{2,2}$ and $\pi\nabla\mathrm{E}_{2,2}$ as single-valued eMZVs. One can

extract similar representations for $\mathrm{E}_{2,3}$, $\pi\nabla\mathrm{E}_{2,3}$ and $(\pi\nabla)^2\mathrm{E}_{2,3}$ from the s_{ij}^5 order of $B_{(0)}^\tau, J_{(0)}^\tau$, the s_{ij}^4 order of $B_{(2)}^\tau, J_{(2)}^\tau$ and the s_{ij}^3 order of $B_{(4)}^\tau, J_{(4)}^\tau$, respectively:

$$\mathrm{E}_{2,3} = \mathrm{SV}\left(-\frac{167}{35}\omega(0,0,0,0,0,5|-\tfrac{1}{\tau}) + 2\omega(0,0,0,0,1,4|-\tfrac{1}{\tau}) + \frac{97}{210}\omega(0,0,0,5|-\tfrac{1}{\tau})\right.$$
$$\left. -\frac{1}{3}\omega(0,0,2,3|-\tfrac{1}{\tau}) + 2\omega(0,0,0,0,2|-\tfrac{1}{\tau})\omega(0,3|-\tfrac{1}{\tau}) + \frac{7}{200}\omega(0,5|-\tfrac{1}{\tau})\right)$$

$$\pi\nabla\mathrm{E}_{2,3} = \mathrm{SV}\left(-\frac{1}{12}\omega(0,3|-\tfrac{1}{\tau})^2 + \frac{13}{168}\omega(0,0,6|-\tfrac{1}{\tau}) + \omega(0,3|-\tfrac{1}{\tau})\omega(0,0,0,3|-\tfrac{1}{\tau})\right. \quad\text{(C.10)}$$
$$\left. -\frac{41}{28}\omega(0,0,0,0,6|-\tfrac{1}{\tau}) + \frac{1}{2}\omega(0,0,0,2,4|-\tfrac{1}{\tau})\right)$$

$$(\pi\nabla)^2\mathrm{E}_{2,3} = \mathrm{SV}\left(\frac{25}{336}\omega(0,7|-\tfrac{1}{\tau}) + \frac{5}{8}\omega(0,3|-\tfrac{1}{\tau})\omega(0,0,4|-\tfrac{1}{\tau}) - \frac{23}{28}\omega(0,0,0,7|-\tfrac{1}{\tau})\right.$$
$$\left. +\frac{1}{4}\omega(0,0,2,5|-\tfrac{1}{\tau}) + \frac{1}{8}\omega(0,0,4,3|-\tfrac{1}{\tau})\right)$$

The corresponding lattice-sum representations [51,87] and β^{sv} representations [18] are given by

$$\mathrm{E}_{2,3} = \left(\frac{\mathrm{Im}\,\tau}{\pi}\right)^5\left(\mathcal{C}[\begin{smallmatrix}3&1&1\\3&1&1\end{smallmatrix}] - \frac{43}{35}\mathcal{C}[\begin{smallmatrix}5&0\\5&0\end{smallmatrix}]\right)$$
$$= -120\beta^{\mathrm{sv}}[\begin{smallmatrix}2&1\\4&6\end{smallmatrix}] - 120\beta^{\mathrm{sv}}[\begin{smallmatrix}3&0\\6&4\end{smallmatrix}] + \frac{12\zeta_5}{y}\beta^{\mathrm{sv}}[\begin{smallmatrix}0\\4\end{smallmatrix}] + 80\zeta_3\beta^{\mathrm{sv}}[\begin{smallmatrix}1\\6\end{smallmatrix}] - \frac{\zeta_5}{36} + \frac{7\zeta_7}{16y^2} - \frac{\zeta_3\zeta_5}{2y^3}$$

$$\pi\nabla\mathrm{E}_{2,3} = \frac{(\mathrm{Im}\,\tau)^6}{\pi^4}\left(3\mathcal{C}[\begin{smallmatrix}1&1&4\\1&1&2\end{smallmatrix}] + 2\mathcal{C}[\begin{smallmatrix}1&2&3\\1&0&3\end{smallmatrix}] - \frac{43}{7}\mathcal{C}[\begin{smallmatrix}6&0\\4&0\end{smallmatrix}]\right)$$
$$= 90\beta^{\mathrm{sv}}[\begin{smallmatrix}2&2\\4&6\end{smallmatrix}] + 60\beta^{\mathrm{sv}}[\begin{smallmatrix}3&1\\6&4\end{smallmatrix}] + 30\beta^{\mathrm{sv}}[\begin{smallmatrix}4&0\\6&4\end{smallmatrix}] \quad\text{(C.11)}$$
$$- 60\zeta_3\beta^{\mathrm{sv}}[\begin{smallmatrix}2\\6\end{smallmatrix}] - 12\zeta_5\beta^{\mathrm{sv}}[\begin{smallmatrix}0\\4\end{smallmatrix}] - \frac{6\zeta_5}{y}\beta^{\mathrm{sv}}[\begin{smallmatrix}1\\4\end{smallmatrix}] - \frac{7\zeta_7}{8y} + \frac{3\zeta_3\zeta_5}{2y^2}$$

$$(\pi\nabla)^2\mathrm{E}_{2,3} = \frac{(\mathrm{Im}\,\tau)^7}{\pi^3}\left(4\mathcal{C}[\begin{smallmatrix}0&2&5\\1&0&2\end{smallmatrix}] - 4\mathcal{C}[\begin{smallmatrix}3&0\\1&0\end{smallmatrix}]\mathcal{C}[\begin{smallmatrix}4&0\\2&0\end{smallmatrix}] - \frac{62}{7}\mathcal{C}[\begin{smallmatrix}7&0\\3&0\end{smallmatrix}]\right)$$
$$= -45\beta^{\mathrm{sv}}[\begin{smallmatrix}2&3\\4&6\end{smallmatrix}] - 15\beta^{\mathrm{sv}}[\begin{smallmatrix}3&2\\6&4\end{smallmatrix}] - 30\beta^{\mathrm{sv}}[\begin{smallmatrix}4&1\\6&4\end{smallmatrix}]$$
$$+ 30\zeta_3\beta^{\mathrm{sv}}[\begin{smallmatrix}3\\6\end{smallmatrix}] + 12\zeta_5\beta^{\mathrm{sv}}[\begin{smallmatrix}1\\4\end{smallmatrix}] + \frac{3\zeta_5}{2y}\beta^{\mathrm{sv}}[\begin{smallmatrix}2\\4\end{smallmatrix}] + \frac{7\zeta_7}{8} - \frac{3\zeta_3\zeta_5}{y}\,.$$

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