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REPRESENTATIONS AND REASONING IN 3-D GEOMETRY IN LOWER SECONDARY SCHOOL

Keith Jones*, Taro Fujita** and Susumu Kunimune***

*School of Education, University of Southampton, UK

**School of Education, University of Plymouth, UK

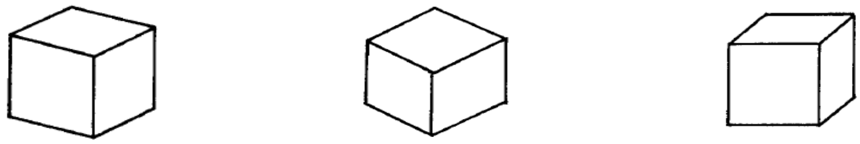
***Faculty of Education, Shizuoka University, Japan

A key question for research in geometry education is how learners' reasoning is influenced by the ways in which geometric objects are represented. When the geometric objects are three-dimensional, a particular issue is when the representation is two-dimensional (such as in a book or on the classroom board). This paper reports on data from lower secondary school pupils (aged 12-15) who tackled a 3-D geometry problem that used a particular representation of the cube. The analysis focuses on how the students used the representation in order to deduce information and solve the 3-D problem. This analysis shows how some students can take the cube as an abstract geometrical object and reason about it beyond reference to the representation, while others need to be offered alternative representations to help them 'see' the proof.

INTRODUCTION

The teaching of geometry provides both a means of developing learners' spatial visualisation skills *and* a vehicle for developing their capacity with deductive reasoning and proving (Battista, 2007; Royal Society, 2001). One long-standing issue for research is how learners' reasoning is influenced by the ways in which geometric objects are represented (see Hershkowitz, 1990; Mesquita, 1998). While the term 'representation' can refer to internal (mental) *and* to external (concrete) representations, in this paper the focus is on external representations such as the various representations of a cube in Figure 1. As this figure captures, a particular case of interest is when the geometric object being represented is three-dimensional while the medium of representation is two-dimensional, such as is necessary in this paper.

One phenomenon related to learners' understanding of geometric representations is the well-established 'prototype effect' by which a certain representation is judged more representative than another (Hershkowitz, 1990, p82). Due to this 'effect', it seems that learners are much better at recognizing isosceles triangles that are 'standing on their base' than ones that are presented in a different orientation. When representing 3-D geometric objects such as a cube on a two-dimensional medium such as paper (or the classroom board), Parzysz (1988; 1991) reports that not only do learners prefer the parallel perspective (in which parallels are drawn as parallels), but, in particular, they prefer the oblique parallel perspective in which the cube is drawn with one face as a square (in French this is the *perspective cavalière*). Figure 1 shows two orthogonal projections (*a* and *b*) and an example of an oblique parallel projection (*c*). It is the latter, according to Parzysz, that learners prefer. In many respects, this oblique parallel perspective is the 'classical' representation of a cube in two dimensions. A further convention is the use of dotted lines to show the 'hidden' edges of the cube.



(*a*) (*b*) (*c*)

Figure 1: orthogonal and oblique parallel projections of the cube

An important issue that this oblique parallel perspective representation raises for research in geometry education is the way in which learners' reasoning might be influenced by the form of the representation, given the difficulties pupils have with 3-D representations even when 3-D dynamic geometry software is available (Mithalal, 2009). This paper reports on data from lower secondary school pupils (aged 12-15) who tackled a problem involving a cube that was presented using the oblique parallel perspective representation. The research question we focus on is how the students use the representation in order to deduce information and solve the problem.

THEORETICAL VIEWS ON REASONING IN 3-D GEOMETRY

For our theoretical framework we integrate a number of ideas relating to students' reasoning processes in 3-D geometry. In particular, we utilise the ideas of 'productive reasoning processes' (Fischbein, 1987), 'capabilities in 3D geometry thinking' (Pittalis & Christou, 2010) and 'the characteristics of 2D representations of 3D shapes' (Mesquita, 1998).

Fischbein (1987, p. 41) argued that a "productive reasoning process" aims at solving a "genuine problem". In a later article he suggested that in productive reasoning "images and concepts interact intimately" (Fischbein, 1993, p. 144). By this we surmise that Fischbein is referring the notion of 'figural concept' as capturing the combined role of the figural and the conceptual in geometry. Within the context of 3-D geometry reasoning, Pittalis & Christou (2010, pp. 192-4) synthesise various capabilities in 3D geometrical thinking. While all the capabilities they identify are likely to be important, in this paper we refer to the capabilities 'to recognise the properties of 3D shapes and compare 3D objects' and 'to manipulate different representational models of 3D objects'. Both these capabilities, we would suggest, are likely to involve the figural and conceptual aspects of geometrical thinking.

As Mesquita (1998, p184) explains, an external representation of a geometrical problem does not, by itself, enable one to solve the problem, but it may contribute to the definition of the structure of the problem. One way this happens, according to Mesquita (*ibid*), is if the representation gives support to geometrical intuition, which in some situations can be very powerful, by helping individuals "to apprehend relationships among geometrical objects". Yet, Mesquita goes on to show, external representations can lead to some ambiguities with the result that particular geometrical relationships might appear as ‘evident’ to students in a way that can prevent geometrical reasoning from developing. What Mesquita (ibid, p186) calls the *double status* of a geometrical representation is that it can represent “either an abstract geometrical object, or a particular concretization”. It is this *double status* that impacts on student reasoning.

All this means that, with a particular geometry problem that makes use of a particular representation, students may, or may not, be able to recognise theorems or properties which their teacher might expect them to use to form, and then prove, a conjecture because the representations may, or may not, appear to the students as ‘typical’. This is what Mesquita is referring to when she shows that an external representation may become an “obstacle” to student understanding. In this paper, what we are interested in is how a representation which is given intentionally by a teacher might, or might not, lead students to engage in conceptual reasoning to make sense of what they ‘see’ and what they can deduce from the available information in the problem as presented to them; in other words, how the external representation supports, or not, their reasoning and how it might, or might not, provide an obstacle to their reasoning.

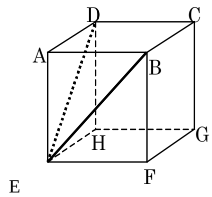
METHODOLOGY

The case of Japan illustrates how geometry teaching plays a role in developing students’ ideas about proof and proving, as illustrated by the learning progression generally used in primary and lower secondary schools in Japan (emphasis added, as explained below):

- In primary school (Grades 1-6), basic properties of plane and *solid figures are studied informally, mainly in relation to everyday life objects. Students also start developing their drawing skills to represent 3D shapes on a 2D plane*;
- In Grade 7, students (aged 12-13) study geometrical constructions, symmetry, and selected properties of *solid figures (names of 3D shapes, nets, sections of cube, surface areas and volume)* informally, but logically, to establish the basis of the learning of proof (*note that the measure of the angle between two lines in 3D space is not formally considered*);
- In Grade 8, students (aged 13-14) are introduced to formal proof through studying properties of angles, lines, congruent triangles, and parallelograms, during which they learn the structure of proofs, how to construct proofs, and how to explore and prove properties of triangles and quadrilaterals;
- In Grade 9, students (aged 14-15) study similar figures and properties of circles, drawing on their consolidated capacity to use proof in geometry and Pythagorean theorems with both 2D and *3D shapes*.

As evident in this progression, students in Japanese lower secondary school have relatively limited opportunities to study and explore 3-D geometry (shown in italics above). As a consequence, students in general have difficulties when they are faced with 3-D geometry problems, e.g. when, in Grade 9, they are finding the lengths of a diagonal of a cube by utilising the Pythagorean theorem. Furthermore, it is uncertain what understanding of 3-D representations the students gain during their lessons, and how they proceed with their reasoning when given such representations when tackling geometric problems. To investigate this issue, and address our research questions set out in the introduction, in this paper we present an analysis of quantitative and qualitative data which were collected through our classroom-based research.

The quantitative data come from a survey that was conducted in 2002. In the survey a total of 570 students in Grades 7-9 in two ordinary lower secondary schools were asked, at the end of their school year, to answer the problem in Figure 2. As can be seen, the representation in the Figure is the oblique parallel perspective the one typically used in the geometry classroom in Japan.



What is the size of the angle BED?

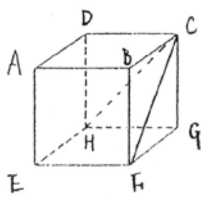
State your reason why.

Figure 2: angle in a cube (survey problem version)

As qualitative data, we analyse an episode taken from a lesson for a class of 46 Grade 8 students in a selective lower secondary school for girls. As such, it should be noted that the students' standards are relatively high. In designing the geometry lessons from which the episode is taken we employed the following process as a 'productive reasoning process' (informed by ideas in Becker & Shimada, 1997 and with some similarities to what Stein et al, 2008, call "reform-oriented lessons"; note that this process is more likely to occur within a classroom in which students can freely share their ideas in geometry, see Fujita, Jones & Kunimune, 2010):

- A problem is introduced, and the students generate conjectures and share ideas that could be used to prove their conjectures.
- Students attempt to prove their conjectures; incorrect proofs might be generated and, if necessary, the conjecture is modified and then proved.
- Students share their reasoning and proofs; incorrect proofs are revisited, and students undertake further proving activities.

In the classroom episode that we analyse, the students are tackling the same 'angle in a cube' problem, but this time the orientation of the triangle inside the cube is different; see Figure 3 (and compare to Figure 2).



What is the size of the angle FCH?

State your reason why.

Figure 3: angle in a cube (classroom problem version)

In our analysis of the classroom episode, we focus on the interplay between the figural and conceptual aspects of reasoning; in particular on how the students tried to interpret the representations and undertake their reasoning to prove their conjectures.

FINDINGS AND ANALYSIS

Survey results

As mentioned above, in the survey a total of 570 students across Grades 7-9 tackled the problem in Figure 2. We categorised the students' answers as follows: (A) global judgment; e.g. 90^o, no reason; (B1) incorrect answer influenced by visual information; e.g. half of angle AEF = 90/2=45^o; (B2) incorrect answer with some manipulations of a cube but influenced by visual information; e.g. drawing a net, and then 45^o + 45^o =90^o; (C2) incorrect answer by using sections of cube but influenced by visual information; e.g. in triangle BDE, angle B = angle D = 45^o, therefore AEF = 90^o; (D) correct answer with correct reasoning; e.g. in triangle BDE, EB=BD=DE and therefore AEF = 60^o; (E) no answer. Table 1 gives the percentages of student answers in each category.

	A	B1	B2	C	D	E
Grade 7 (N=146)	13	58	8	4	2	15
Grade 8 (N=204)	26	44	7	4	3	16
Grade 9 (N=220)	19	29	16	7	14	15

Table 1: Categorisation of student responses to the survey (in percentages)

As evident from the results in Table 1, the response of around two-thirds of the students are in A or B categories; that is, they made a global response with no reasoning, or their response was incorrect but clearly influenced in some way by the visual information in the representation of the cube. Even by Grade 9 only 14% of students gave a correct answer with correct reasoning. These results suggest that students in general are not able to manipulate 3-D representations well and that their reasoning, if apparent in their responses, is likely to have been influenced by visual information from the specific representation.

As the students were asked to solve this problem under the restricted conditions of a survey, they did not have an opportunity to engage in the form of productive reasoning processes that can occur in the classroom. To investigate this point, we designed, in conjunction with a teacher, a classroom experiment in which students could exchange their conjectures and reasoning about the problem.

Classroom teaching results

This section reports on an episode taken from a lesson with 46 Grade 8 students. In the first phase of the lesson, the problem in Figure 3 was introduced, and the students generated conjectures. At this point, 28 students considered that angle FCH would be 60^o, three said 90^o, and 15 said 'I am not sure'. During the lesson, when ideas were shared amongst the class, three of the unsure students opted for the answer of 60^o, making a total of 31 students (ie 67%) conjecturing that FCH was 60^o.

In the next phase of the lesson, the students engaged in a productive reasoning process by discussing their ideas to deduce the size of the angle FCH. One student (S1) who considered the size would be 90 ^{o} explained her reasoning by using a net; see Figure 4 (in the dialogue, T is the teacher):

22	S1	I used a net, but it might be wrong? Maybe it is 60^{o}? I don’t know.
23	T	Don’t worry; please explain your idea to everyone?
24	S1	Because FCG and GCG are 45^{o}, then add them together and get 90^{o}?

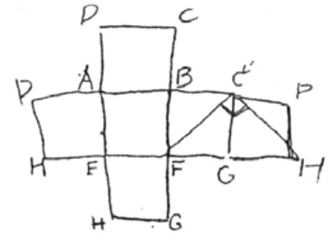


Figure 4: Student use of a net to solve the ‘angle in a cube’ problem

This type of response was seen in the survey data.

A student (S13) then challenged this reasoning, and the teacher started asking for opinions as to why the angle was 60^{o}.

26	S13	But…, Mr T, is S1’s answer only on a plane and we need to fold [to make a cube], so it is different?
27	T	What do you mean? CG will be folded?
28	S13	Yes.
29	T	OK, do you understand what S13 said? [many students nod]
37	T	OK, I would like to listen to opinions about why the angle is 60^{o}
47	S22	Well, in a cube all faces are the same square, and DHGC, BCGF and HEFG are all the same, and HG, CF and HF are all diagonals of the same size squares. So the lengths [of the sides of triangle FCH] are the same, and it is an equilateral.

At this point, the three students who initially said 90^{o} changed their idea.

While students S13 and S22 were using the properties of the cube to construct a reasonable argument, two students showed hesitation in accepting this reasoning.

58	S9	I do not understand how to join H and F.
59	S28	I can accept the explanation (by S22) but in the figure (Figure 4), I cannot see any equilateral triangle.

At this, student S31 suggested an alternative idea.

64	S31	I have another idea. We can see the cube from A to C.
65	T	We can see from A to C? Can you draw a picture?

Student S32 had a different suggestion, so that teacher asked S31 and S32 to draw their ideas, as shown in Figure 5 (S31 on the left and S32 on the right). The teacher asked the class what they thought of these representations.

67 Ss	It is very clear. It is like an equilateral triangle.

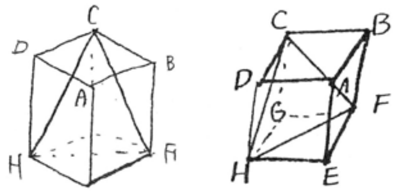


Figure 5: Student's alternative ideas for representing the 'angle in a cube' problem

Hence, by shifting the view of the oblique parallel projection (right-hand of Figure 5) or by shifting to an orthogonal projection (left hand of Figure 5) all the students agreed that triangle CFH is equilateral and hence that angle FCH is 60º. The students could 'see' that this is the case from the representations in Figure 5.

DISCUSSION

In the survey results, even with Grade 9 students, only 14% could give a fully correct response to the 'angle in a cube' problem (ie correct answer with correct reasoning). In the classroom situation, as many as 67% of Grade 8 students could do this (following some sharing of ideas. Such a difference is not altogether surprising and not the point of this paper, though what this comparison does point to is the impact of the "productive reasoning process" during which "images and concepts interact intimately" (Fischbein, 1993, p. 144).

The successful reasoning of many of the Grade 8 students in the classroom teaching experiment shows evidence of the capabilities identified by Pittalis & Christou (2010, pp. 192-4), particularly the capabilities 'to recognise the properties of 3D shapes and compare 3D objects' and 'to manipulate different representational models of 3D objects'. There is also evidence of what Mesquita (1988, p186) calls the *double status* of a geometrical representation in that it can represent "either an abstract geometrical object, or a particular concretization". Students such as S22 were able to take the cube as an abstract geometrical object and could reason about triangle CFH beyond reference to the representation (shown in Figure 3) provided by the teacher. Yet other students in the same class, such as S9 and S28, needed to 'see' that triangle CFH was equilateral. At this, student S31 provided a way to do this by using the orthogonal projection of the cube (left-hand part of Figure 5), while student S32 provided a different viewpoint of the oblique parallel projection (right-hand of Figure 5). Using these representations, then triangle CFH 'appears' to the students to be equilateral. This combination of reasoning and representation convinced doubting students and may help to make proof seem more meaningful (Kunimune, Fujita & Jones, 2010).

CONCLUDING COMMENTS

Some of the student responses to the survey showed them trying to use the net of a cube as a representation to help them solve the 'angle in a cube' problem. Some of the students in the classroom teaching experiment tried the same representation. On the whole, students did not find success when using the net representation. Some of the students who were successful could take the cube as an abstract geometrical object and reason about it beyond reference to the representation provided by the teacher. Others, who could not 'see' the more abstract reasoning benefitted from being offered alternative representations to help them 'see' the solution. This illustrates how students' reasoning with 3-D geometry problems is influenced by the use of various representations. Of course the situation would be different if 3-D dynamic geometry software (such as *Cabri 3D*) was being used since students could utilise various viewpoints as if the computer representation were a 'concrete' model. Nevertheless, as Mithalal (2009) shows, even with 3-D dynamic geometry software, students need to go beyond visual information in order to solve geometry problems.

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