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THE INFLUENCE OF 3D REPRESENTATIONS ON STUDENTS' LEVEL OF 3D GEOMETRICAL THINKING

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While representations of 3D shapes are used in the teaching of geometry in lower secondary school, it is known that such representations can provide difficulties for students. In this paper, we report findings from a classroom experiment in which Grade 7 students (aged 12-13) tackled a problem in 3D geometry that was, for them, quite challenging. To analyse students' reasoning about 3D shapes, we constructed a framework of levels of 3D geometrical thinking. We found that students at a lower level of 3D thinking could not manipulate representations effectively, while students operating at a higher level of 3D thinking controlled representations well and could reason correctly.

INTRODUCTION

In geometry teaching, despite the study of 2D figures often taking precedence over the study of 3D figures, most school curricula aim to develop learners' understanding of 3D figures. As such, an issue for research is to seek ways to develop learners' spatial thinking and reasoning in 3D geometry (Gutiérrez et al., 2004). In reporting an earlier study of students' reasoning in 3D geometry (see Jones, Fujita, and Kunimune, 2012), we focused on how particular types of 3D representation can influence lower secondary school students' reasoning about 3D shapes. The purpose of this paper is to propose a theoretical framework to capture students' levels of thinking with 3D shapes and their representations. In particular, we address the following research questions:

- What framework can be constructed to capture students' spatial thinking in 3D geometry?
- What characteristics of thinking can be identified when students tackle challenging problems in 3D geometry?

In what follows, we take as our starting point the levels of 3D geometrical thinking proposed by Gutiérrez (1992) and our previous study (Jones, Fujita and Kunimune, 2012). We then construct a framework to capture students' levels of thinking with 3D shapes. To construct our framework, we take a bottom up approach, i.e. our framework is mainly derived from data from 570 G7-9 students. We then evaluate our framework further by analysing classroom episodes taken from a sequence of two lessons from a teaching experiment with Grade 7 students. In considering how our framework is useful to capture students' levels of thinking effectively, insights from our findings are discussed in terms of how we might improve students' thinking with 3D shapes – a form of mathematical thinking which is challenging for many students.

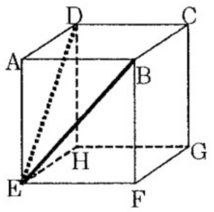
STUDENTS' REASONING IN 3D GEOMETRY: SURVEY RESULTS

By considering existing studies, and in order to obtain detailed analyses of students' thinking and reasoning with 3D shapes in relation to their interpretation of graphical information (Bishop, 1983) and decoding 3D figures (Pittalis & Christou, 2013), the following three theoretical components 'Reasoning with 3D shapes', 'Manipulation of 3D representations', and 'Levels of thinking of 3D shapes' are important.

Research on students' reasoning and representations of 3D shapes

While physical models of 3D shapes can be used in the teaching of geometry, representations of 3D shapes (on the 2D board or in textbooks or other materials) are the main mediational means. Existing research evidence indicates that representations of 3D shapes can have various impacts on learners' reasoning processes. Parzysz (1988; 1991), for example, reported that not only do learners prefer the parallel perspective (in which parallels are drawn as parallels), but, in particular, they prefer the oblique parallel perspective in which the cube is drawn with one face as a square. Such external representations can lead to some ambiguities for students with the result that particular geometrical relationships might appear as 'evident' in a way that can prevent geometrical reasoning from developing in the most appropriate way. In line with this, Ryu et al. (2007) reported that while some of the mathematically-gifted students they studied could, for example, imagine the rotation of a represented 3D object, other such students had difficulty in imagining a 3D object from its 2D representation.

Informed by research by Hershkowitz (1990) and Mesquita (1998), in research reported in Jones, Fujita and Kunimune (2012) we found that some students can take the cube as an abstract geometrical object and reason about it beyond reference to the representation, while others were influenced by the visual appearances of 3D representations and could not reason correctly. The 570 G7-9 students' answers for the question illustrated in Figure 1 (which was one of the survey questions) were classified into the following five categories: (A) global judgment; e.g. 90^o, no reason (19.3%); (B1) incorrect answer influenced by visual information; e.g. half of $\angle$AEF = $90^o/2$ = 45^o (44%); (B2) incorrect answer with some manipulations of a cube but influenced by visual information; e.g. drawing a net, and then $45^o + 45^o = 90^o$ (10.3%); (C) incorrect answer by using sections of cube but influenced by visual information; e.g. in triangle BDE, $\angle$B = $\angle$D = 45^o, therefore $\angle$AEF = 90^o (5%) ;(D) correct answer with correct reasoning; e.g. in triangle BDE, EB=BD=DE and therefore $\angle$BED = 60^o (6.3%) ;(E) no answer (15.3%). The result implies that it is difficult for many students to reason correctly with given representation.



What is the size of the angle BED?

State your reason why.

Figure 1: Angle in a cube problem (survey problem version)

3D geometry thinking levels

Based on the van Hiele model of thinking in geometry, something widely used to describe and analyse learners' thinking in 2D geometry, Gutiérrez (1992) proposed levels of 3D thinking. This was used by Gutiérrez et al. (2004) to investigate students' levels of thinking and their proof capabilities with problems involving prisms. The result was a characterization of students' levels of 3D spatial thinking, with the lower levels characterised as relying on simple descriptions based on drawings, while at higher level students begin using reasoning more analytically (pp. 512-513). Extending this, we note that Pittalis and Christou (2013) argue that interpreting representations of 3D figures utilises two capabilities: a) recognising the properties of 3D shapes and comparing 3D objects, and b) manipulating different representational models of 3D objects. From these points of view, we first undertook an initial analysis of the survey data mentioned above of 570 G7-9 students' answers. We found that students' incorrect responses were the result either of inappropriate reasoning with 3D shapes' properties or inappropriate manipulating of shapes, or both. In this paper, we utilise this information to refine Gutiérrez 's work, and propose the framework set out in Table 1 to capture students' geometrical thinking with 3D shapes.

Level	Reasoning with 3D properties	Manipulating	Features of students' 3D thinking
1	No	No	Students' thinking is influenced by 2D representation.
2a	Yes(not appropriate)	No	Students start utilising 3D properties of shapes but without effective manipulations.
2b	Yes(not appropriate)	Yes(not appropriate)	Students utilise 3D properties & manipulate the figure but it is not appropriate.
2c	Yes(not appropriate)	Yes	Students utilise 3D properties & manipulate the figure appropriately but with an incorrect answer.
3	Yes	Yes	Students utilise 3D properties & manipulate the figure appropriately, and obtain the correct answer.

Table 1: Levels of thinking in 3D geometry

STUDY CONTEXT AND METHODOLOGY

Building on our earlier classroom-based research in 3D geometry (e.g. Jones, Fujita, & Kunimune, 2012), in this paper, to address our research questions, we analyse episodes taken from two lessons in which students tackled the problem in Figure 1 by using our framework presented in Table 1 (note in these lessons the size of angle BGD is asked instead of the angle BED). We refer to the data from classroom episodes because the data from the survey is rather superficial to reveal students' thinking and further qualitative analysis of students' reasoning with this problem is worthwhile. In particular, the cube in the problem uses the oblique parallel projection and the angle to be found is changed from the original survey problem. All this means that it is not straightforward to know the size of the angle BGD because of this representation.

The main data are taken from a class of 28 Grade 7 students (aged 12-13) from a public school in Japan. The class teacher, Mrs M, has more than 20 years teaching experience, and is particularly interested in students' geometrical reasoning processes. Given that teachers' interactions with students are crucial to encourage students' reasoning (e.g. Jones & Herbst, 2012), in general her roles in the lessons were to facilitate students' discussions by suggesting where to direct their attention in the problem, which properties might be used, and so on. Through following the Japanese geometry curriculum, the students had already studied selected properties of solid figures such as nets, sections of a cube, surface areas and volume (note that the measure of the angle between two lines in 3D space is not formally studied within the prescribed curriculum). We video-recorded two lessons (each 50 minutes) in which the students worked with the problem. Field notes were kept and the audio was transcribed. In addition, student worksheets from both lessons were collected to obtain information on how the students' reasoning changed across the two lessons. All data were analysed qualitatively in terms of the theoretical framework presented above. We particularly analysed students' interactions with the teacher during the lessons and their answers and explanations in their worksheets. Through this we determined the levels of students' thinking in terms of the characteristics of their reasoning and manipulations of representations.

FINDINGS AND ANALYSIS

Lesson progression

A key to correctly answering the problem is to deduce that triangle BGD is an equilateral triangle. During the two lessons, the 25 students (3 students were absent during the first lesson) attempted enthusiastically to solve the problem and Mrs M led the class well. In the first lesson, after the problem was posed, the students began by tacking the problem individually. Their initial answers from their worksheets are shown in Table 2.

As can be seen from Table 2, only five of the students considered that the angle was 60°. This indicates that their reasoning is likely to be influenced by the external representation of the problem. After this initial stage, the teacher asked the students to share their ideas and answers, and the six answers in Figure 2 were presented. Mrs M then asked the students to comment on these, but no students stated their opinions. This was the end of the first lesson.

Answer	90°	60°	22.5°	30°	45°	35°	90° or 60°
Number of students	8	5	4	2	2	1	2

In total 24 answers; one student did not write any answer.

Table 2: Students' initial answers to the problem

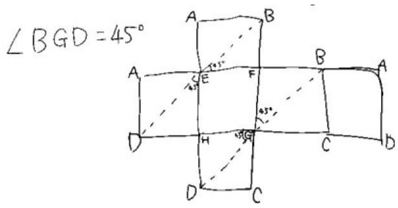
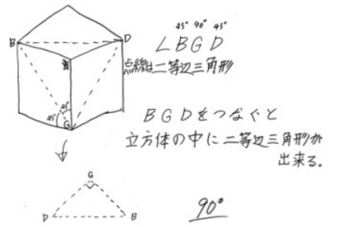
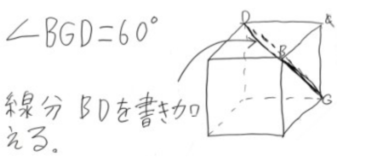
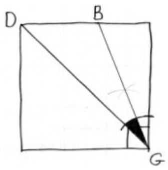
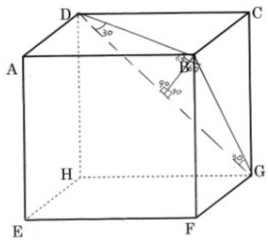
(1) 35° (Student F1) I thought it is 35° because I used a set square (to measure the angle in the representation), and it seems 10° smaller.	(2) 45° (student Y) I used a net and if I cut it from B to D, then it is 45°. So ∠ BGD＝45°  	(3) 90° (Student F2) I rotated the cube and I can make an isosceles triangle BGD. And ∠ BGD＝90° 
(4) 60° (Student IM) In a cube all diagonals of each face are the same. I added a line BD and we have a triangle BGD which is an equilateral triangle. And ∠ BGD＝60°  	(5) 22.5° (Student K) A line DG halves a square, and another line BG further halves it. So 90×1/2＝45, and 45×1/2＝22.5.  	(6) 30° (Student H)  No explanation and angles are measured as 2D angles.

Figure 2: Presented students' answers

At the start of the second lesson the 24 students (four were absent during the second lesson) continued to exchange their ideas and reasoning. First, Mrs M asked the students whether they changed their answers or not, based on the presentations at the end of the first lesson. The students' revised answers are shown in Table 3. While the number of students giving the answer 60° had risen to 16, during the second lesson some students still argued why they could not see the angle as 60°.Mrs M asked the students to consider an explanation which would help everyone in the class to consider whether the answer was 60°. One student explained BGD is an equilateral triangle. After another student (student IM) presented his proof to refute 90°, Mrs M then used a physical model of a cube to demonstrate the reasoning. That completed the second lesson.

Answer	90°	60°	22.5°	30°	45°	35°	22.5° or 60°
Number of students	6	16	0	1	0	0	1

Table 3: Students' revised answers to the problem

Students' level of thinking

As might be expected, students in the class illustrated various levels of 3D thinking. In the first lesson, some students determined the size of the angle neither by referring to the properties of 3D shapes nor by manipulating the presented figure. In our case, students F1 (the Figure 2-(1)) or H (in Figure 2-(6)) for example, used measurement from the given representation, and did not have any idea why this would not be correct. These students can be considered as Level 1 in our framework.

Meanwhile, some students started using the properties of a cube and simple (but ineffective) manipulations. For example, like student K in Figure 2-(5), student J (Figure 3 left) did not add anything on the given figure but used properties of angles of a square (90°) and concluded 22.5°. This is Level 2a. Student C utilised a net to consider the size of the angle BGD to deduce the angle is 90°, as illustrated in Figure 3 (right). As evident, this approach does not work because the angles in the net and the angle required in the problem are different. Yet this student cannot see this by using only the net representation, i.e. they cannot utilise properties of cubes independently from the used representations (a similar case is student Y, Figure 2-(2)). Such thinking can be considered as level 2b, i.e. utilised properties of shapes and started to manipulate the given representation, but neither of them were effective and appropriate.

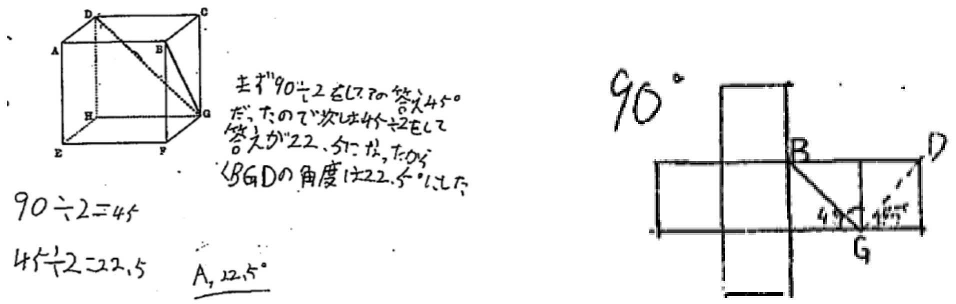


Figure 3: Answers by student J (left) and C (right)

Other students used more manipulations of shapes and also started using properties of shapes to construct some simple deductions. In our study an example of this was when student F2 joined B and D to form triangle BDG and started examining what the triangle BDG would be to deduce the size of the angle. However, the following exchange shows that this student could not recognise the triangle BGD as an equilateral because of how the representation of the cube looked:

Student F2	I joined B and D.
Mrs M	Join B and D, and then?
Student F2	Then I see a right-angled isosceles triangle (Figure 2-(3)).
Mrs M	OK, you thought the triangle is a right-angled isosceles triangle…
Student F2	So, G should be 90°?

It is notable that student F2 changed the oblique parallel projection to an orthogonal projection (see Figure 2-(3)) but still deduced an incorrect answer. This student's manipulation was appropriate (i.e. can lead to the correct answer), but their reasoning was influenced by the visual appearance of the triangle in the figure, and they were not able to utilise properties of shapes effectively.

Nevertheless, it was the case that even in the first lesson some students started manipulating the representations effectively to solve the problem. In such cases, their reasoning was not overly influenced by the external representation of the cube but was more controlled by logical thinking, which can be considered as level 3 thinking. For example, student IM explained his reasoning very clearly as follows:

Student IM	Because in a cube all diagonals should be the same length, this triangle is an equilateral.
Mrs M	OK, you thought it will be an equilateral because of the length of the diagonals.
Student IM	[nods] Each angle of an equilateral triangle is 60, ° so ∠BDG is 60°.

Student IM also showed his clear and advanced thinking and explained how he could refute 90 °as an answer as follows:

Student IM	(writing the following answer) *The sum of the inner angles of a triangle is 180°so 90 does not work. D=90, B=90, G=90. D+B+G=90+90+90=270.*
Mrs M	Can you explain this?
Student IM	If the line BD, DG and BG are all the same, then ··· no, sorry. If you add the angles of a triangle, then 180°, and if the ∠BGD is 90°, then it is an equilateral, so all angles should be the same and the other two angles are also 90°, and add them together it will be 270°. This does not work.

DISCUSSION

For our first research question 'What framework can be constructed to capture students' spatial thinking in 3D geometry?, we developed the framework of the levels of geometrical thinking with the two aspects 'reasoning with 3D properties' and 'manipulating 3D shapes representations' derived from our large empirical data set. In order to evaluate our framework, in this paper we used data from the classroom episodes in the form of students' explanations and tested how our framework can capture students' characteristics of thinking. As we have seen, the students presented a wide variety of their answers and reasoning, and our framework can provide a comprehensive classification of these answers and reasoning.

For our second research question 'What characteristics of thinking can be identified when students tackle challenging problems in 3D geometry?', from what we have observed in the two lessons and students' worksheets, our framework can successfully characterise students' thinking. At the first level, there are students who are strongly influenced by visual appearances of external representations. These students should be encouraged to explore their reasoning without relying on their naïve visual thinking.

Students at the second level cannot utilise manipulations of representations or properties of shapes. For example, although some students used effective manipulations such as drawing a line BD, some of these students could still not reach the correct answer. For these students, it is necessary to make them reflect on their reasoning or manipulations. Indeed, in our classroom episodes, it was useful when students in the class shared their ideas of various manipulations of shapes, reasoning and so on. In particular, student IM's refutation which was appropriate use of reasoning with properties worked well in leading many students to the correct answer. In future research, we plan to examine these modified level descriptions further using a larger data set. While we only have space in this paper to present one problem, we have additional analyses of other problems that reveal students' levels of thinking with 3D shapes in general.

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