Cite as: Jones, K., Pratt, D. & Watson, A. (2014). Key ideas in teaching mathematics: Reasoning with decimals in Key Stage 3. *NCETM Secondary Mathematics Magazine*, 108, 3-5.

Secondary Magazine 108

National Centre for Excellence in the Teaching of Mathematics

Key Ideas in Teaching Mathematics – Reasoning with decimals in Key Stage 3

In this and other issues, the Secondary Magazine will feature a set of six articles, written by Anne Watson, Keith Jones and Dave Pratt, the authors of the recent publication Key Ideas in Teaching Mathematics. While not replicating the text of this publication, the articles will follow the themes of the chapters and are intended to stimulate thought and discussion, as mathematics teachers begin to consider the implications of the changes to the National Curriculum. This article is the fourth in the series and focusses on Reasoning with decimals in Key Stage 3. Future articles will feature Place Value, Algebra and Probabilistic Reasoning. Previous articles focussed on similarity, ratio and trigonometry in Key Stage 3, Geometric and spatial reasoning in Key Stage 3, and statistical reasoning in Key Stage 3.

A recent discussion on the TES mathematics teaching forum concerned the following question that a teacher had set for homework: how many tenths in 1.5? A response that might immediately be expected is 15 tenths. Another suggested response was 5 tenths, with the explanation that 1.5 is one unit and 5 tenths. A further response was that there are 10 tenths, in that there are always 10 "tenths" in whatever is the "whole". One suggested way forward was that perhaps such variation in response could be avoided if the original question was worded how many tenths is 1.5; but then perhaps the variation in response provoked by the original question can provide a valuable teaching opportunity.

The debate illustrates that there is more to reasoning with decimals than might be expected, given that children are exposed to ideas associated with place value from an early age (from early counting through, for example, to everyday experience of measures). In the mathematics curriculum, the notion of place value, and of decimal place value in particular, begins in the early primary school years and continues through secondary school – with a progression from decimal notation and equivalents towards terminating and recurring decimals, while taking in ideas of rounding and significant figures. To be successful, students need to coordinate place value ideas with aspects of whole number and fraction knowledge. Making the transition to being able to reason with decimals relies on pupils having a thorough understanding of ideas previously met and these earlier ideas becoming fully integrated with new information. If pupils persist in trying to treat decimals as if they were whole numbers, then pupil reasoning with decimals can remain uncertain - as exemplified by the following:

- "more digits means bigger" (e.g. 0.21345 is larger than 0.3); this can occur when pupils use a method that is successful for whole numbers (ie 21345 is larger than 3).
- "more digits means smaller" (e.g. 0.31345 is smaller than 0.25); this can be a reaction to learning that "more digits means bigger" is incorrect, or it can result from incorrectly generalising that because one hundredth is smaller than one tenth (and so on) then the more digits the smaller the number.
- "zeros to the right of a decimal number increases the size of that number" (e.g., 0.400 is larger than 0.40, which is larger than 0.4); this is related to the case above of treating the decimal part as a whole number.
- "zeros on the left can be ignored" (e.g. 0.4 is the same as 0.04, and both are the same as 0.004); this occurs from incorrectly generalising that just as the number 8 is not changed by placing a zero in front of it (ie 08), the zeros after the decimal point can also be disregarded.
- "the decimal point can be ignored" (e.g. $0.2+4 = 0.6$, and variants: $0.07+0.4 = 0.11$, $6\times0.4 = 24$, and $42\div 0.6 = 7$); this also occurs when pupils use arithmetic methods that are successful for whole numbers.

In all these cases, what is needed is for pupils to build their knowledge in such a way that they come to recognise the features of whole numbers that are similar to decimal fractions and those that are unique to whole numbers. Although it is clear that pupils struggle with understanding rational numbers, in general it is decimals that present one of the greatest challenges. TIMSS data, for example, indicate that pupils do not always perform as well on questions involving decimals compared to those involving ordinary fractions - perhaps because of the difficulty some students have in shifting from understanding fractions as 'parts of something' to understanding them as single numbers.

With time and direct effort, pupils can learn to distinguish whole number from rational number concepts (at times where a distinction needs to be made) and develop a meaningful understanding of how fractions and decimals are represented symbolically. In secondary mathematics, a key idea is the transition from additive to multiplicative reasoning and the move pupils need to make from seeing fractions and decimals as two numbers to seeing them as either as numbers in their own right or as the result of dividing where the divisor does not divide exactly into the dividend.

There can be two approaches to teaching decimals. One approach tends to emphasise integrating decimals with other proportion concepts such ratio and percentages. Another approach focuses on building meaningful understanding of decimal numeration based on place value, perhaps through a link to measurement. The two approaches are not opposing camps; rather both approaches seek to construct meaningful links between related ideas such as between measurement, fractions, decimals and percentages. The teaching of topics such as probability can support the linking between decimals and fractions. As with much of the mathematics we teach, there is no unique path in sequencing different ideas to arrive at the understanding that decimals and fractions can be pure numbers or proportional operators, but percentages can only ever be proportional operators.

It can be that over-reliance on one approach ends up exacerbating the problem. For example, over-reliance on the context of money can lead to difficulties in comparing pairs of decimals such as 4.4502 and 4.45. Children over-relying on halves and quarters can struggle with other fractional proportions. For example, they can struggle with tenths because tenths cannot be achieved by halving and more halving. Here money does not help because a penny does not look like a tenth of a 10p piece. Even with length measures, which can be a useful context for work on decimals, pupils have been known to identify a 1-place decimal with cm and 2-place decimals with mm. While in some countries it can be acceptable for learners to say 0.85 as what would translate as 'nought point eighty five', because they are thinking about hundredths or about mm, focussing on the size of the parts, in this case comparing cm and mm in general, can lead such pupils to conclude that 8.1 is larger than 8.15 because centimetres are larger than millimetres. Nevertheless, refining measurement units to measure quantities more precisely can be a useful approach in teaching. Use of the “double number line”, integrating two units of metric measures such as metre and centimetre, can also be beneficial. Pupils who tackle contextualized problems can build up their understanding of decimals by utilising their ‘everyday’ knowledge as it relates to the meaning of decimal numbers and the results of decimal calculations

Even so, the use of the decimal system to express large numbers (such as a population) or in commonplace metric measures can mask the continuous nature of decimals, including the notion of the density of the real numbers (these being the field of all rational and irrational numbers). As a result, pupils can have difficulty in imagining what happens with all the number that they can think of between, say, 12.849 and 12.850. A range of other issues relating to real numbers tends to be glossed over in school mathematics. For example, converting an ordinary fraction into a decimal fraction frequently gives rise to infinite decimals, yet little curriculum time is generally devoted to infinite decimals or clarifying their representation by never-ending decimals. The result is that well-known questions like the following are usually left unaddressed: “is 0.999 . . . exactly equal to 1, or only approximately?”. What is masked is the jump from finite to infinite processes, and how infinite processes can be safely treated mathematically. Similarly, while pupils encounter numbers like $\sqrt{2}$, π, and e (the latter at A-level), there is little time at secondary school for a consideration of irrational numbers.

While it is beyond school mathematics to resolve the contradiction between the continuous nature of the number line continuum and the discrete nature of the numbers themselves (the apparent contradiction can be resolved by defining Real numbers using either Dedekind cuts or Cauchy sequences), infinite processes (the basis of the transition from Calculus to Analysis) do occur in school mathematics; and one topic in which such infinite processes occur is within a consideration of decimals. In touching on notions of the density, and completeness, of the real numbers, reasoning with decimals is revealed as a key idea in school mathematics; one that continues beyond school into higher level mathematics.

Keith Jones, Dave Pratt and Anne Watson

In keeping this series of articles brief, there is no space for full references; these can be found in the book Key Ideas in Teaching Mathematics

Image credit
Page header by Cordey (adapted), some rights reserved