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Intuition and Geometrical Problem Solving

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This paper describes a small-scale study of the role of geometrical intuition in the solving of a geometrical problem. The use of the dynamic geometry package Cabri-Géomètre *allowed students to explore the problem in a way that suggested an intuitive solution to them. Subsequently, the students justified their solution. This illustrates how a deductive and an intuitive approach can prove to be mutually reinforcing when solving geometrical problems.*

The availability of new tools, such as dynamic geometry packages, allows familiar problems to be approached in what may be new ways. In this paper I consider *why* people make the decisions that they do when solving geometrical problems. In doing so, I explore the role of geometrical intuition, something that has been emphasised for some time (van Hiele 1986, Fischbein 1987). Of specific interest are the claims that appropriate use of the computer may make intuition more accessible for study. Papert, for instance, claims that, in the right circumstances, the computer can "help bridge the gap between formal knowledge and intuitive understanding" (Papert 1980 p 145). The computer does this, Turkle and Papert (1991 p 162) suggest, by standing "betwixt and between the world of formal systems and physical things: it has the ability to make the abstract concrete".

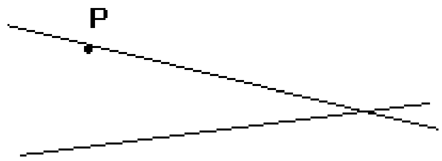
Schoenfeld has written extensively about his work with students solving (and not solving) the geometrical problem given below (see, for instance, Schoenfeld 1985 and 1986). In each of the four elements of Schoenfeld's framework for analysing mathematical problem solving it is possible to find a role for intuition. The problem-solvers' *resources* include intuitive knowledge, *heuristics* involve knowing when to use which strategy, *control* focuses on major decisions about what to do in a problem and *belief systems* shape cognition, even when the problem-solver is not consciously aware of holding those beliefs. As a result, Schoenfeld suggests (1986), rather than being disjoint, "a deductive approach to mathematical discovery .... and an empirical intuitive approach .. are, in fact, mutually reinforcing".

It is useful, at this point, to use Fischbein's definition of intuition as a special type of cognition, characterised by self-evidence and immediacy, and with the following properties (Fischbein 1987 p 43-56): *intrinsic certainty, perseverance, coerciveness, theory status, extrapolativeness, globality*, and *implicitness*. In Fischbein's view, intuitions are theories or coherent systems of beliefs. This conception has similarities to Cooney's (1991) idea that the representation of an intuition is likened to a mini-theory, a model that supports reaching a conclusion, with certainty, on the basis of incomplete information.

I now report some of the results of a study of pairs of recent mathematics graduates tackling a series of geometrical construction problems using the dynamic geometry package *Cabri-Géomètre.* The problem solving sessions using *Cabri* were videotaped. Later, the researcher and the students reviewed the videotape and the students were encouraged to talk about the reasons they made the decisions they did during their problem solving session. These review sessions were audio-recorded.

Problem

You are given two intersecting straight lines and a point P on one of them. Show how to construct a circle that is tangent to both lines and has P as its point of tangency to one of the lines.



Critical decisions in the solution of this problem are:

1. Constructing a perpendicular line through P
2. Constructing the angle bisector of the angle between the two intersecting lines
   or constructing a circle centred at the intersection and passing through P giving an intersection with the second line; a perpendicular line through this point intersects the perpendicular through P at the centre of the required circle.

One subject pair, both male and both with some experience of geometrical constructions, began by reproducing the problem diagram on the computer screen. Their first approach was to construct a circle with a centre chosen somewhere between the two intersecting lines, and with point P as the radius point. They then used the facility available with *Cabri* to drag the centre of the circle so that it appeared also to be tangential to the lower of the two intersecting lines. Though this gave them a solution, they were not happy with this and searched for a way of being “absolutely sure”.

Subject CR says “Well, the tangent is perpendicular to the line of radius, isn’t it?” so they constructed a perpendicular line through P and constrained the centre of the circle to lie on this perpendicular. Then subject CR suggested that they construct a perpendicular line to the lower of the two intersecting lines and move it into the correct place. At this point, TC wonders if the centre of the circle lies on the bisector of the angle between the two intersecting lines. With that the problem was solved.

Another subject pair, one female (KH) and one male (KJ), both with some experience of geometrical constructions, used a similar approach. They began by creating the diagram for problem 1 and then proceeded to construct two perpendiculars, one through P and a second perpendicular to the lower of the two intersecting lines. As with the first pair, this second perpendicular line was then dragged into place. At this point, KH says “I tell you the other thing we could do and that's bisect that angle to find out where they should cross”. With that they too had solved the problem.

For both pairs, once they had solved the problem, they discussed their solution method. This resulted in them drawing up an argument that would properly serve as a proof. In this way, the solution of the problem suggested the structure of a deductive proof. None of the pairs studied used the alternative method suggested above.

For pair 1 the suggestion to draw the angle bisector was made quite tentatively towards the end of their problem-solving attempt:

TC: Yes ... Ah! Now would the centre of the circle lie .. I'm just thinking something slightly different now, because I'm just trying to think, there must be a way of securing the centre accurately .. and I'm thinking .. does the centre of the circle ..sit on the bisector of the angle that’s made by those two lines ..

For pair 2 the student was more certain

KH: I tell you the other thing we could do and that’s to bisect that angle to find out where they should cross.

This is how the students accounted for their actions as they watched a video-recording of their problem-solving attempt later the same day. In the case of pair 1:

TC: .. [long pause] .. well, partly previous knowledge. I wasn’t .. completely sure. I wasn’t saying “Oh, yes. This is what does happen”. I just had a sneaky feeling that we were missing something and I couldn’t work out what it was, but I thought, well I’m sure the angle .. there must be some connection between the angle between the two lines and the centre [of the circle]. So, let’s put the line in and see what happens.

It turned out to be right, but it was just a sort of stab .. well, it wasn’t a stab in the dark completely ...

I can’t think why, but I was sure we should be bisecting the angle.

In the case of pair 2:

KH: Ohhh! .. [laughs] .. That's quite interesting because, maybe, .. the fact that there's a cross there [where the two perpendicular lines intersect 'opposite' where the original two lines intersect] actually encouraged me to think, well, we need to know where the cross is going to be. Perhaps if we hadn't drawn the other perpendicular, it would not have come so quickly.

Looking at that picture now I think .. it's ..er .. er .. I mean just having that sort of cross there on the screen opposite the angle there. I mean, that just spells it out. I think perhaps that's why it just came so quickly.

In both cases the students had some difficulty explaining their actions (a methodological issue that warrants further attention). Nevertheless, both previous experience and the visual image appeared to play a part in determining the course of action they were suggesting. In this context, Fischbein says, "Experience is a fundamental factor in shaping intuitions" (Fischbein 1987 p 85). However, Fischbein then goes on to say that "There is little systematic evidence available supporting that view, i.e. evidence demonstrating that new intuitions can be shaped by practice" (*ibid* p 85). In terms of the visual image, Fischbein claims that visualisation is "the main factor contributing to the production of the effect of immediacy" (*ibid* p 103). Fischbein then goes on to relate visualisation to the domain of mental models. The evidence available from this study seems to support Fischbein's views in the domain of solving geometrical problems.

Further analysis of the data from this study suggests that geometrical intuition has a role in the planning-implementation, and transition episodes of a problem-solving attempt (see Schoenfeld 1985 p 292 for details of these episodes). In addition, it is possible to tentatively identify the following mechanisms as participating in the formation of the subject's geometrical intuitions: premature closure, primacy effect, factors of immediacy (particularly visualisation and anchoring), and factors of globality (see Fischbein 1987 p 204-205 for an explanation of these mechanisms). However, because the analysis examined points of critical decision for the *successful* solution of the problem, instances of geometrical intuition may, inevitably, tend to form points of transition in the problem-solving process or occur during planning and implementing episodes. The analysis presented here does not consider how intuition may have led the subjects astray.

The framework Fischbein (1987) has proposed proved reasonably robust in terms of this study. His problem-solving categories of intuition have been identified, and a way suggested to differentiate between anticipatory and conclusive intuitions, such that subjects' awareness of the critical nature of the decision they are making seems to be associated with conclusive intuitions. Secondly, it has been possible to tentatively discern the mechanisms that participated in the generation of these geometrical intuitions. The explanations provided by the subjects in this study provide some supporting evidence.

A further aim of this study has been to examine the potential of a geometry package such as *Cabri* to provide a ‘window’ on geometrical intuitions. In discussing the use of *Cabri* with the subjects in the present study, and comparing it with using paper and pencil, they thought that *Cabri* allowed more exploration on their part, and more experimentation. This is particularly evident from the work of pairs described above, who, once they had ‘dragged’ the second perpendicular into approximately the correct position, knew that by constructing the angle bisector they could solve the problem. These examples, and there are others in the project, suggest that using *Cabri* may make geometrical intuitions more visible for study. Further work is necessary to verify this.

This study was designed to provide evidence of particular aspects of the nature and role of geometrical intuition in the process of solving geometrical problems, and of the possible mechanisms that participated in the generation of these geometrical intuitions. The students observed here used a mixture of a deductive approach in, for instance, drawing a perpendicular through point *P*, and an empirical intuitive approach provided for by *Cabri* by being able to ‘drag’ a second perpendicular into place. Once they had a solution, the ensuing discussion effectively provided a proof. This illustrates how a deductive and an intuitive approach can prove to be mutually reinforcing when solving geometrical problems.

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