

THE FIRST ℓ^2 -BETTI NUMBER AND GROUPS ACTING ON TREES

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ABSTRACT. We generalise results of Thomas, Allcock, Thom–Petersen, and Kar–Niblo to the first ℓ^2 -Betti number of quotients of certain groups acting on trees by subgroups with free actions on the edge sets of the graphs.

1. INTRODUCTION

The ℓ^2 -Betti numbers $b_i^{(2)}(G)$ of a group G are defined in [11]. The ℓ^2 -Euler characteristic $\chi^{(2)}$ of G is the alternating sum of these Betti numbers and is denoted $\chi^{(2)}(G)$. Let \mathfrak{C} denote the class of groups F such that

- $b_1^{(2)}(F) = b_2^{(2)}(F) = 0$, and
- either $\chi^{(2)}(F) = 0$ or F is finite.

Note that \mathfrak{C} contains all ℓ^2 -acyclic groups (i.e. the groups for which $b_i^{(2)} = 0$ for all $i > 0$) and in particular it contains all amenable groups. Relevant background on ℓ^2 -cohomology is included in Section 2. In this note we prove the following theorem.

Theorem 1.1. *Let F be a group acting cocompactly on a tree, with vertex and edge stabilisers in \mathfrak{C} , let N be a subgroup normally generated by m elements, intersecting the vertex stabilisers trivially. Let G denote F/N and set $k := \chi^{(2)}(F) + m$. Then the following conclusions hold:*

- If $k \leq 0$, then G is infinite.*
- If $k < 0$, then $b_1^{(2)}(G) \geq -k > 0$.*
- If G is finite, then $k > 0$ and $|G| \geq \frac{1}{k}$.*

Note that the hypotheses of this theorem guarantee that N acts freely on the specified tree and in particular N is necessarily a free group. Note also that, according to [2, Corollary 1.4], if $b_1^{(2)}(G) > 0$ then G has no commensurated infinite amenable subgroup and according to [4, Corollary 6] does not have property (T). If we also have $b_2^{(2)}(G) = 0$, then G is in the class \mathcal{D}_{reg} by [14, Lemma 2.8]. We refer the reader to [3] for background on property (T) and to [14, Definition 2.6] for the definition of the class \mathcal{D}_{reg} . By the main result of Osin’s paper [12] we have the following corollary.

Corollary 1.2. *Let G , F and N be as in Theorem 1.1. Assume that G is finitely presented, (virtually) indicable and that $\chi^{(2)}(F) + m < 0$. Then G is (virtually) acylindrically hyperbolic.*

The simplest way in which the indicability hypothesis may arise is through *stable letters*: Let T denote the F -tree of Theorem 1.1. Let K denote the (necessarily

normal) subgroup generated by the vertex stabilisers. Then there is a subgroup $E \leq F$ that complements K and all such subgroups are free of uniquely determined rank. Such a subgroup may be referred to as *a subgroup of stable letters of the action*. The group G has an infinite cyclic quotient when $N \cap E$ has infinite index in E , in other words when there is a stable letter that is faithfully represented in G , and in this case G is indicable.

Recall that a group G is C^* -simple if the reduced group C^* -algebra, denoted $C_r^*(G)$, has exactly two norm closed 2-sided ideals, 0, and the algebra $C_r^*(G)$ itself. By [5, Corollary 6.7] we obtain the following.

Corollary 1.3. *With G , F and N as before, G is C^* -simple if and only if it has trivial amenable radical.*

Theorem 1.1 has some historical pedigree. It originally began life as a result about quotients of free groups due to Thomas (see Theorem 1.4(i)) and was proved using combinatorial methods [15]. The result was generalised by Allcock to incorporate a bound on the rank of the abelianisation of the quotient group [1]. The introduction of ℓ^2 -cohomology came when Peterson–Thom [13, Theorem 3.6] and Kar–Niblo [10] independently linked the inequality of Thomas to the first ℓ^2 -Betti number. These discoveries are summarized in the following result.

Theorem 1.4 (Thomas [15], Allcock [1], Peterson–Thom [13], Kar–Niblo [10]). *Let G be a group with a presentation*

$$\langle x_1, \dots, x_n; r_1^{k_1}, \dots, r_m^{k_m} \rangle$$

in which the elements r_i have order k_i when interpreted in G .

- (i) *If $n - \sum_{i=1}^m \frac{1}{k_i} \geq 1$ then G is infinite.*
- (ii) *If G is finite then $|G| \geq \frac{1}{1 - n + \sum_{i=1}^m \frac{1}{k_i}}$.*
- (iii) *If $n - \sum_{i=1}^m \frac{1}{k_i} > 1$ then G is non-amenable.*

Deduction of Theorem 1.4 from Theorem 1.1. Let G be a group with a presentation

$$G = \langle x_1, \dots, x_n \mid r_1^{k_1}, \dots, r_m^{k_m} \rangle.$$

Adding m fresh generators y_1, \dots, y_m , we can give the following alternative presentation of the same group:

$$G = \langle x_1, \dots, x_n, y_1, \dots, y_m \mid y_1^{k_1}, \dots, y_m^{k_m}, r_1 y_1^{-1}, \dots, r_m y_m^{-1} \rangle.$$

Let F be the group with presentation

$$F = \langle x_1, \dots, x_n, y_1, \dots, y_m \mid y_1^{k_1}, \dots, y_m^{k_m} \rangle$$

and let N be the subgroup of F normally generated by $r_1 y_1^{-1}, \dots, r_m y_m^{-1}$. Then F is a free product of cyclic groups: in particular it is virtually free and has Euler characteristic $\chi(F) = \sum_{i=1}^m \frac{1}{k_i} - n - m + 1$. The condition that the r_i have order k_i in the original presentation ensures that N does not meet any of the finite subgroups of F and so is torsion-free. Applying Theorem 1.1 with these choices of F , N , G yields Theorem 1.4. \square

Finally, we also provide a computation of the first ℓ^2 -Betti number for certain groups acting on trees. This generalises a result of Lück [7], which covers the case of an amalgamated free product, and a result of Tsouvalas [16, Corollary 3.7]. Tsouvalas assumes the vertex stabilisers are either residually finite or virtually

torsion-free and the edge stabilisers are finite. Here we replace both of these assumptions with Lück's less restrictive assumption that the first ℓ^2 -Betti numbers of the edge stabilisers vanish. So, for example, the theorem applies to fundamental groups of graphs of \mathfrak{C} -groups.

Theorem 1.5. *Let F be a group acting on a tree and let V and E denote sets of representatives of F -orbits of vertices and edges. Assume for each $e \in E$ that $b_1^{(2)}(F_e) = 0$, then*

$$b_1^{(2)}(F) = \sum_{v \in V} \left(b_1^{(2)}(F_v) - \frac{1}{|F_v|} \right) + \sum_{e \in E} \frac{1}{|F_e|},$$

where for a group G , $\frac{1}{|G|}$ is interpreted as 0 if G is infinite.

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2. BACKGROUND ON ℓ^2 -HOMOLOGY

Let G be a group. Then both G and the complex group algebra $\mathbb{C}G$ act by left multiplication on the Hilbert space $\ell^2 G$ of square-summable sequences. The group von Neumann algebra $\mathcal{N}G$ is the ring of G -equivariant bounded operators on $\ell^2 G$. The regular elements of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}G$ can be identified with the *ring of affiliated operators*, and is denoted by $\mathcal{U}G$. One has the inclusions $\mathbb{C}G \subseteq \mathcal{N}G \subseteq \ell^2 G \subseteq \mathcal{U}G$ and it is also known that $\mathcal{U}G$ is a self-injective ring which is flat over $\mathcal{N}G$. For more details concerning these constructions we refer the reader to [11] and especially to Theorem 8.22 of §8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [11, §8.3]. Now let Y be a G -CW complex as defined in [11, Definition 1.25 of §1.2]. The ℓ^2 -homology groups of Y are then defined to be the equivariant homology groups $H_i^G(Y; \mathcal{U}G)$, and we have

$$b_i^{(2)}(Y) = \dim_{\mathcal{U}G} H_i^G(Y; \mathcal{U}G).$$

The ℓ^2 -Betti numbers of a group G are then defined to be the ℓ^2 -Betti numbers of EG . By [11, Theorem 6.54(8)], the zeroth ℓ^2 -Betti number of G is equal to $1/|G|$ where $1/|G|$ is defined to be zero if G is infinite. Moreover, if G is finite then $b_n^{(2)}(G) = 0$ for $n \geq 1$.

Let $C_*(Y; \mathcal{U}G)$ denote the standard cellular chain complex of Y with coefficients in $\mathcal{U}G$. We have the formula

$$\dim_{\mathcal{U}G} C_i(Y; \mathcal{U}G) = \sum_{\sigma} \frac{1}{|G_{\sigma}|}$$

where σ runs through a set of orbit representatives of i -dimensional cells in Y and if G_{σ} is infinite then $1/|G_{\sigma}|$ is taken to be equal to 0. Standard arguments of homological algebra give the connection between two Euler characteristic computations:

$$(1) \quad \sum_i (-1)^i b_i^{(2)}(Y) = \sum_i (-1)^i \dim_{\mathcal{U}G} C_i(Y; \mathcal{U}G).$$

We record two other consequences, both of which can be found in [11, Section 6.6].

Lemma 2.1. *Let G be a group and let Y be a cocompact G -complex with finite stabilizers. Then we have*

- (i) $\chi^{(2)}(G) = \sum_{i=0}^{\infty} \left((-1)^i \sum_{v \in \mathcal{S}_i} \frac{1}{|G_{v_i}|} \right) = \sum_{i=0}^{\infty} (-1)^i b_i^{(2)}(G);$
(ii) if G has a finite index subgroup H with a finite $K(H, 1)$ then

$$\chi^{(2)}(G) = \chi(G).$$

We will need the following lemma for the proofs in the next section. One should think of it as a mild generalisation of Theorem 6.54(2) in [11]

Lemma 2.2 (Comparison with the Borel construction up to rank). *Let X be a G -CW complex. Suppose for all $x \in X$ the isotropy group G_x is finite or $b_p^{(2)}(G_x) = 0$ for all $0 \leq p \leq n$, then*

$$b_p^{(2)}(X) = b_p^{(2)}(EG \times X) \quad \text{for } 0 \leq p \leq n.$$

Proof. It suffices to prove that the dimension of the kernel and cokernel of the map

$$\text{pr}_p : H_p^G(EG \times X; \mathcal{N}G) \rightarrow H_p^G(X; \mathcal{N}G)$$

induced by the projection $EG \times X \rightarrow X$ are trivial for $0 \leq p \leq n$. By an identical argument to [11, Theorem 6.54(2)] it suffices to prove for each isotropy subgroup $H \leq G$ and $0 \leq p \leq n$ the kernel and cokernel of the map $\text{pr}_p : H_p^H(EH; \mathcal{N}H) \rightarrow H_p^H(*; \mathcal{N}H)$ have trivial dimension. If H is finite this follows from [11, Theorem 6.54(8a)], and is immediate if $b_p^{(2)}(H) = 0$ for all $0 \leq p \leq n$. \square

3. THE MAIN THEOREM

To prove Theorem 1.1, one needs the following method of computing the ℓ^2 -Euler characteristic of a group acting on a tree analogous to Chiswell's result [9] for rational Euler characteristic.

Proposition 3.1 (Chatterji–Mislin [8]). *Let F be a group acting on a tree and let V and E denote sets of representatives of F -orbits of vertices and edges. If the ℓ^2 -Euler characteristic of each vertex and edge group is finite, then*

$$\chi^{(2)}(F) = \sum_{v \in V} \chi^{(2)}(F_v) - \sum_{e \in E} \chi^{(2)}(F_e).$$

Proof of Theorem 1.1. There is a cocompact action of F on a tree T with vertex and edge stabilizers in the class \mathcal{C} . Let V and E denote the vertex and edge sets. Let \bar{T} denote the quotient graph T/N and write \bar{V} and \bar{E} for its vertex and edge sets. Now $G = F/N$ acts cocompactly on \bar{T} with vertex and edge stabilizers in \mathcal{C} . The augmented chain complex of T is the short exact sequence

$$0 \rightarrow \mathbb{Z}E \rightarrow \mathbb{Z}V \rightarrow \mathbb{Z} \rightarrow 0.$$

Upon factoring out the action of N we have the exact sequence

$$(2) \quad H_1(N, \mathbb{Z}) \rightarrow \mathbb{Z}\bar{E} \rightarrow \mathbb{Z}\bar{V} \rightarrow \mathbb{Z} \rightarrow 0,$$

this being the tail end of the long exact sequence of homology of N . Let $\{r_i : i = 1, \dots, m\}$ denote a normal generating set for N . Choose a vertex v_0 in T to be a fixed basepoint. For $1 \leq i \leq m$ consider the geodesic from v_0 to $v_0 r_i$. In the quotient graph \bar{T} this geodesic descends to a loop because v_0 and $v_0 r_i$ become identified in \bar{T} . Now 2-discs can be glued to each loop. By adjoining free G -orbits of 2-discs

equivariantly we can build a 2-complex Y with an action of G , whose 1-skeleton is \overline{T} , and which has augmented cellular chain complex

$$(3) \quad \mathbb{Z}G^m \rightarrow \mathbb{Z}\overline{E} \rightarrow \mathbb{Z}\overline{V} \rightarrow \mathbb{Z} \rightarrow 0.$$

The exactness of (2) ensures the exactness of 3.

Let V_0 and E_0 be sets of orbit representatives of vertices and edges in Y . Now, applying [11, Theorem 6.80(1)], then(1) and finally Proposition 3.1, we have that

$$\begin{aligned} \chi^{(2)}(Y) &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y) \\ \sum_{v \in V_0} \frac{1}{|G_v|} - \sum_{e \in E_0} \frac{1}{|G_e|} + m &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y) \\ \chi^{(2)}(F) + m &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y). \end{aligned}$$

Applying Lemma 2.2, we have

$$\begin{aligned} \chi^{(2)}(F) + m &= b_0^{(2)}(EG \times Y) - b_1^{(2)}(EG \times Y) + b_2^{(2)}(EG \times Y). \\ &\geq b_0^{(2)}(EG \times Y) - b_1^{(2)}(EG \times Y). \end{aligned}$$

Applying [11, Theorem 6.54(1a)] to the diagonal map $EG \rightarrow EG \times Y$ we see that

$$\chi^{(2)}(F) + m \geq b_0^{(2)}(G) - b_1^{(2)}(G).$$

Let $k = \chi^{(2)}(F) + m$. If $k \leq 0$, then $b_0^{(2)}(G) - b_1^{(2)}(G) \leq 0$ and so G is infinite, this proves (i). Now, assume $k < 0$. In this case G is infinite and therefore $b_0^{(2)}(G) = 0$. It follows that $b_1^{(2)}(G) \geq -k > 0$, this proves (ii).

If G is finite, then $b_0^{(2)}(G) = \frac{1}{|G|}$, $b_1^{(2)}(G) = 0$, and $k > 0$. In particular, $k \geq \frac{1}{|G|} > 0$ and (iii) follows. \square

4. ON THE ℓ^2 -INVARIANTS FOR CERTAIN GROUPS ACTING ON TREES

Proof of Theorem 1.5. Let V and E denote sets of representatives of F -orbits of vertices and edges for the action of F on the tree. We consider the relevant part of the E^1 -page for the F -equivariant spectral-sequence (see Chapter VII.9 of [6]) applied to the tree:

$$\begin{array}{ccc} & \uparrow & \\ 1 & \oplus_{v \in V} H_1^F(F \times_{F_v} EF_v; \mathcal{U}F) & 0 \\ & \downarrow & \\ 0 & \oplus_{v \in V} H_0^F(F \times_{F_v} EF_v; \mathcal{U}F) & \xleftarrow{d^1} \oplus_{e \in E} H_0^F(F \times_{F_e} Fe; \mathcal{U}F) \\ & \downarrow & \\ & 0 & 1 \end{array}$$

If F is finite then $b_1^{(2)}(F) = 0$, so d^1 is injective and $E_{1,0}^2 = 0$. The result follows from the fact $E_{0,1}^1 = 0$.

Now, assume F is infinite, then d^1 is surjective since $b_0^{(2)}(F) = 0$. Thus,

$$\dim_{\mathcal{U}F}(\text{Ker}(d^1)) = \sum_{e \in E} b_0^{(2)}(F_e) - \sum_{v \in V} b_0^{(2)}(F_v).$$

Now, the spectral sequence obviously collapses on the E^2 -page and $E_{0,1}^1 = E_{0,1}^2$. Since von Neumann dimension is additive over short exact sequences, we have

$$\begin{aligned} b_1^{(2)}(F) &= \dim_{\mathcal{U}F}(\text{Ker}(d^1)) + \dim_{\mathcal{U}F}(E_{0,1}^2) \\ &= \left(\sum_{e \in E} b_0^{(2)}(F_e) - \sum_{v \in V} b_0^{(2)}(F_v) \right) + \sum_{v \in V} b_1^{(2)}(F_v), \end{aligned}$$

and the result follows. \square

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