Cite as: Ding, L., Jones, K., & Zheng, Y. (2009). *Teaching geometrical proof problem solving in China: a case analysis from the perspective of the dynamic approach of the teacher*. Paper presented at 3rd International Symposium on the History and Pedagogy of Mathematics. Beijing (China), 22-25 May 2009.

Teaching geometrical proof problem solving in China: A case analysis from the perspective of the dynamic approach of the teacher

*DING Liping**, *Keith JONES*** and *ZHENG Yuxin* ***
*Massey University, New Zealand; ** University of Southampton, U.K.; ***Nanjing University, China.

The purpose of the research reported in this paper is to analyse the teaching of proof problem solving from the perspective of the dynamic approach of the teacher. The paper utilises the case of a Chinese expert teacher teaching proof problem solving in geometry at Grade 8 in Shanghai, China. Data come from the learning responses of a sample of students, together with an account of each lesson taught be the teacher. The analysis of this data reveals the efficacy of this teacher's instructional approach, focusing in particular on the development of two main forms of mathematical thinking, namely deductive and inductive reasoning.

Introduction

Mathematical proof is taken the world over as a line of reasoning that shows convincingly that some chosen mathematical statement is necessarily true. For some researchers, mathematical proof is seen as distinct from argumentation (the latter being "a reasoned discourse that is not necessarily deductive but uses arguments of plausibility", see Hanna & de Villiers, 2008, p. 331). For other researchers, proof is regarded as part of a continuum with argumentation, rather than as something distinct. Each of these views can profoundly influence didactical practice in the classroom (*ibid*).

In China, proof receives considerable emphasis in school mathematics. For instance, Zhang, Li & Tang (2004, p. 198) state that Euclidean geometry "is useful in teaching rigorous deductive reasoning and formal proof" and that to prove Pythagoras Theorem "one needs to use rigorous algebraic or geometric methods; just using a cut-and-paste method is not acceptable".

To date, little research has been published on how school teachers in China approach the relationship between argumentation and mathematical proof in their classroom teaching. To develop insight into this issue, this paper re-analyses the case-study of a Chinese expert teacher's instructional practice in geometrical proof problem solving that was reported to ICMI Study 19 (Ding and Jones, 2009). The approach in this paper is to focus on the dynamic approach of the teacher, in line with what Zheng (2006, p. 387) refers to as the “heuristic nature of teaching”.

The dynamic approach of the teacher in the Chinese classroom

The central role of the teacher in Chinese pedagogy has been investigated by a range of studies (examples include Dai, 1998; Ma, 1999; Paine, 1990; Tu, 2006; Zhang, 2005). Zheng (2006) argues that, influenced by the Chinese philosophical thought and in particular the *Yin-Yang* theory of *Taoism*, a fundamental feature of Chinese mathematics education is likely to be the seeking of a balance between a range of what can sometimes appear to be opposing ideas - such as the relationship between the central role of the teacher and the active role of students (for more details see *ibid*, p.385). Zheng goes on to explain that teaching in China is not regarded as “a process of conveying well-developed knowledge” (*ibid*, p.387), rather teachers in China do their best to make the content of their lessons fully understandable to their students by, amongst other things, “paying more attention to the process of creation or discovery”. In this dynamic approach, the teacher does not merely provide the students with direct instruction or explanation of, for example, a proof; rather the teacher leads the students to experience the *process* of solving the proof problem, from exploring the problem to making a conjecture and then proving it.

As pointed out by Zheng, to improve mathematics education, it is essential to develop a deeper understanding of the strengths and the weaknesses of all the features of Chinese pedagogy in mathematics. In this paper, the aim is to illuminate the dynamic approach utilised by an expert teacher and, in doing so, deepen the analysis to reveal the potentials and limitations of the selected example of classroom practice.

Relating the dynamic approach to teaching to other theories

In this section of the paper, the dynamic approach to teaching is related to relevant research studies conducted in other countries. In particular, attention is drawn to studies of students’ dynamic exploration of open problem-solving situations reported by Boero, Garuti and Mariotti (1996), and the theoretical hypothesis of “transformational reasoning” proposed by Simon (1996).

Boero, Garuti & Mariotti (1996) summarise the findings of various teaching experiments dealing with Italian students' construction and proof of conjectures in the dynamic learning environment of sun shadows. These researchers hypothesized that the dynamic exploration of the problem situation plays a crucial role, both at the stage of conjecture production and during the proof. The researchers noted that when students made conjectures of the problem they were trying to solve, they performed the dynamic exploration of the problem situation in different ways: "indicating with their hands the imagined movement of the sun, or moving themselves, or moving the oblique stick, or moving the platform supporting the sticks, etc." (*ibid*, p. 124). Moreover, most students continued the dynamic exploration of the problem situation during the construction of proof. However, there was a functional difference in the thinking process of the dynamic exploration implemented during the construction of conjectures and the proof, moving "from a support to the selection and the specification of the conjecture" to "a support for the implementation of a logical connection" (*ibid*, p. 126).

Simon (1996) argues that the quest of mathematics learners to understand mathematics and to determine mathematical validity leads not only to inductive and deductive reasoning, but also to a third type of reasoning— *transformational reasoning* (TR). He defined TR as follows:

> "Transformational reasoning is the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated." (*ibid*, p. 201)

This statement shows that transformational reasoning can be a powerful way of understanding mathematics, for it involves not only the ability to carry out a particular mental or physical enactment, but also the realization of the appropriateness of that process to a particular mathematical situation. Moreover, TR may not only produce a different way of thinking about mathematical situations, it may also involve a different set of questions. Noticeably, it is not yet clear how transformational reasoning approach is generated by students, for such reasoning "requires both the inquisitiveness with respect to the workings of a mathematical system and the developed ability to translate the system into a mental or physical representation that can be 'run'" (*ibid*, p. 203).

To date, many of the existing studies in the research field dealing with the relationship of argumentation and proof concentrate largely on examining the nature of tasks from a learner's cognitive perspective rather than from a didactical perspective (for a recent review, see Mariotti, 2007). A key issue that remains unclear concerns the necessary interventions of the teacher in helping students to understand the necessity of mathematical proof rather than relying on argumentation, no matter how ordinarily plausible the argument. For educational purpose, it becomes essential to examine the relationship between teachers' instructional practices and the development of students' reasoning. This is an important motivation to re-analysis the case reported in ICMI Study 19 (Ding & Jones, 2009).

A case analysis of the dynamic approach of the teacher

This section of the paper presents an analysis of an expert Chinese teacher's instructional practice in geometrical proof problem solving at Grade 8 in Shanghai, China. The analysis reveals an emphasis on both the development of transformational reasoning *and* the development of deductive reasoning. The key significance of the analysis is the way this development is managed through the dynamic approach of the teacher, utilising aspects of the "heuristic nature of teaching" (Zheng, 2006, p. 387). In what follows, the analysis focuses first on the development of transformational reasoning. After this, the analysis focuses on the development of deductive reasoning. The name of the teacher is a pseudonym; line numbers against extracts of dialogue come from transcripts of the lessons, with some lines omitted to aid clarity (extended transcripts are available in Ding, 2008).

An emphasis on the development of transformational reasoning

Proof problem: *Given - Triangle ABC and AED are equilateral triangles; CD=BF. Prove: 1) triangles ADC and CFB are congruent; 2) Quadrilateral CDEF is a parallelogram* (see figure 1.1)

Teacher Lily, in the second of a sequence of lessons on developing her students' understanding of this multi-step proof problem, first guided her students to explore a sub-problem that they had encountered before. Lily began her instruction by redrawing part of the whole figure, equilateral triangle ABC, on the blackboard (in other words, figure 1.1 was redrawn as figure 1.2). Then she drew students' attention to the first proof problem, namely to prove the congruent triangles ADC and CFB as follows:

418 Lily: We learned equilateral triangle at grade 7. In the problem, there are two problems. If the first problem is taken, it will be very difficult to directly think about the second question. But if you start to think from the first problem, it will be much easier. So the first problem is actually a 'stepping stone' towards solving the second problem.

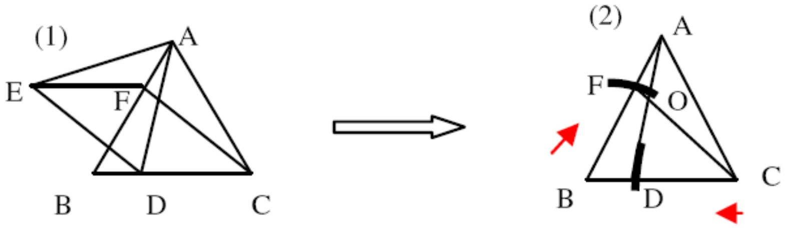

Figure 1.1 and 1.2

Lily then turned to explaining the importance of the given (CD=BF) of the problem. She explained this as follows:

425. Lily: CD=BF. What does this mean?
426. Students do not respond
427 Lily: It means that D and F are dynamic points, aren't they? (The teacher repeated the question a couple of times, dialogue lines 427-429 omitted).
430 Lily: OK. CD=BF. This means that D and F are dynamic points. D could be here, could be here, could be here, right? (The teacher recreated the figure by using compasses to draw D and F, making CD=BF, and then used a ruler to link C and F, A and D. see the result of her drawing in figure 1.2.)
434 Lily: D and F are dynamic points. Now they move such that CD=BF. So if D goes this way. F goes that way. … The different dynamic points go in different directions at the same speed, right? … So the length they (D and F) moved should be the same, shouldn't they? (The teacher put red arrows in the figure on the blackboard, see figure 1.2)
435 Lily: If you are told like this statement, you might understand that this means CD=BF. We could describe a problem in different way, yet the meaning could be same. In this problem, it means that CD=BF.
436 Lily: Well. Now, are you familiar with this figure? [see figure 1.2]

The teacher encouraged students in the whole class to observe and compare between figures 1.1 and 1.2 on the blackboard (lines 437-439 omitted).

440 Lily: You could think about this figure during the lesson break [see figure 1.2]. You learnt about the equilateral triangle at Grade 7. In the process of the movement of D and F, D and F move regularly. Could you find what is never changed in the movement?

In this instructional process, teacher Lily dynamically presented the static figure (see figure 1.2) on the blackboard. Here, transformational reasoning (Simon, 1996) is nurtured and developed, for the teacher chiefly emphasizes the result of 'running' the system, not as a way of accumulating outputs as in an inductive approach, but rather to develop a feel for the system. Noticeably, the mental enactment of the dynamic movement of the figure was addressed ('mental enactment' refers to operations carried out in mental images; see Simon, 1996, p. 201). The teacher's instructional intention is to develop students' understanding of the relationship between the given (CD=BF) of the problem with other parts of the equilateral triangle ABC. Through asking students the question "*Could you find what is never changed in the movement?*" (see line 440), students are led to be more conscious of regularities in the dynamic movement of the figure.

Having illustrated the emphasis on the development of transformational reasoning, the next section considers the development of deductive reasoning.

An emphasis on the development of deductive reasoning

Students continued to have difficulty in perceiving the hidden geometrical objects and properties of the figure (AD=CF and angle AOF=60^0, see figure 1.2). To surmount this difficulty, at the beginning of the following lesson, Lily instigated a whole-class discussion of the problem:

37. Lily: In this figure, could you find what is not changed, when D and F are moving? [see figure 1.2] (Students discussed this, line 38 omitted)
40. Some students: DC=BF.
41. Lily: DC=BF? This is already given. Except this, what else is not changed?
42. Some students: Oh, AF=BD. Because AB=BC.
43. Lily: AB=BC? This is given, as it is an equilateral triangle (ABC).
More students discussed CF=AD in the class. Lily encouraged a boy student to stand up and to present his finding to the class (lines 44-49, omitted).
50. Wang WY (boy): Two triangles are congruent (probably ADC and CFB). AD and CF are always equal.

After CF=AD was made explicit in the class, Lily moved to draw students' attention to another hidden property of the figure – the location relationship of AD and CF.

58. Lily: Obviously, they (AD, CF) are not parallel. They are intersected, aren't they? How is the angle they formed? Will it change? You could use a protractor to measure the figure on your book. You could measure the angle before and after the movement [see figure 1.2]

59.1. Some students: It will be the same. (One students responded 60^{o} . (#57))

59.2. Liuliu (boy): [Noticing his classmate's response] 60^{o} , 60^{o} . Only need to prove two parallel lines [probably CF//ED in figure 1.1]

60. Lily: How do you explain that they are equal? No change? How much is the angle then?

More students like Liuliu suggested 60^{o} of angles AOF and COD (#61-62).

64. Lily: If this angle (AOF) is 60^{o} . How to prove? (The teacher used number 1 to represent angle AOF, see figure 1.1).

Some students like Beibei (girl) wondered why angle AOF is 60^{o} , while Lily encouraged an explanation of the finding (lines 65-67).

68. Beibei: (asked Liuliu) Why is it 60^{o}? Parallel?

69. Liuliu: If both of them are 60^{o} , then they are always parallel. (Probably if angle AOF=angle ADE=60^{o}, then FC//ED.)

70. Linlin (boy): Oh, in the middle, there is a pair of vertically opposite angles! (Probably angle AOF=angle COD)

The teacher invited a boy student to present his ideas to the whole class (line 71).

72 Zheng YQ (boy): Because angle 1= angle DAC + angle ACF. (The teacher then used number 2 to represent angle DAC.)

75.1 Some students, Linlin and Liuliu: Ah? It is angle ACF? [surprised tone)

76. Zheng YQ: Because of the congruent triangles (ADC and FBC), angle 2=angle FCB.

76.1 Some students: Oh, the bottom angle! [Probably angle ACD[[Surprised tone]

In this instructional process, the teacher continued the dynamical presentation of the static figure (line 37) to enhance students' ability to carry out a particular mental enactment of the operation, while the physical enactment of the operation such as an experiment of measurement of angles to verify their guesses was just suggested as an after lesson activity (see line 58). Noticeably, many students in the class did not appear to have trouble perceiving the images when they heard the teacher's dynamical description of it. Note that the teacher's intention is to lead students to think *how it works* and *why it works* (see lines 60, 64, 68). These questions are both for understanding and validation. As illustrated by Simon (1996, p. 204), "insight into the workings of the system leads to knowledge of what results to expect and why. Validity is inherent in such understanding, not because it is deductively established, but because the learner has 'seen' the relationship between the initial state and the result".

Moreover, two variations were applied by the teacher to lead students to deductive reasoning. First, the teacher varied the mathematical problems from a "*problem to find*" at the start of the exploration in the previous lesson (as illustrated by the teaching episode in the previous section) to a "*problem to prove*" at the development of the exploration (see the teaching episode cited in this section). Secondly, the teacher varied her questioning from involving students to make their own guess (see lines 37, 58) to leading them to make a deductive reasoning for their conjectures (see line 60). For more discussion of the two variations see Ding and Jones (2009); for more on the theory of variation, see Gu (1994) and Gu, Huang, & Marton (2004).

Discussion

The dynamical approach applied in the case encompassed:

- dynamical presentation of the static proof figure;
- the variation from a "*problem to find*" to a "*problem to prove*";
- the variation of questioning, from involving students in making their own guess to them making deductive reasoning for their conjectures.

The significant role of the teacher in helping students to prove, and to understand proofs, was in terms of the following:

- the teacher's emphasis on the nature of the reasoning produced by student, with transformational reasoning as a way of understanding this proof problem;
- the teacher's emphasis on the ability to carry out a particular mental enactment of the operation to the nature of deductive arguments taken into account by students as reliable arguments for validation;
- the teacher's questions on *how it works* and *why it works* for understanding and validation.

Findings of our study substantiate the statement by Zhang (2005, p. 3) that "the leading role of a teacher is mainly to ensure the direction of learning, to be hands-on in teaching as an example for students to follow, and to help students to make a progress". Similarly, Stigler & Hiebert (1999, p. 48) say that the effective teacher "allows the students to participate more directly in the development of the procedures ….the teacher is still in control, carefully constraining the task to ensure certain outcomes”.

Nevertheless, while the teacher in the case in this paper dynamically sought a way of creating an effective instructional situation to involve the whole class in learning, the students were led by the teacher to make certain explorations and discoveries. In this way, students’ freedom to explore and think through the problem remained under the control of the teacher’s instruction. This is a case of what Professor Zheng Yuxin calls the ‘indoor flying’ approach – it is like flying, but it is like flying indoors as it is constrained in certain ways. This ‘indoor flying’ approach can be effective in the development of certain types of mathematical thinking, such as deductive reasoning, as demonstrated in this paper, yet perhaps it should not be considered as a truly heuristic approach because some aspects of the teacher’s considerable instructional practice in this case remain somewhat unanswered. For instance, why did the teacher consider investigating the problem in a dynamic way? Why, in particular, did the teacher ask particular questions such as what remained invariant and what varied during the motion. Why should angle AOF receive particular attention by the teacher? Was the aim that specific pieces of knowledge come naturally together? That is, which came first during the problem solving, inductive facts or deductive knowledge?

Concluding comments

It is fitting to give the teacher in this case the last word. In an interview conducted as part of the larger study (see Ding, 2008), the teacher’s thoughts in dealing with the relationship between argumentation and mathematical proof in the classroom teaching is evident as follows:

> “Deductive geometry is a difficult subject. It is different from other school subjects; for in teaching other subjects, the teacher’s questioning may not necessarily demand student thinking at a high level. Or perhaps it is more easily to involve students in plausible reasoning in other subjects as students more easily gain access to information or data in such a subject. But in respect of deductive geometry, I think that it is most important to develop students’ independent thinking. Thus, when a geometrical proof problem is posed, students should give time to think about it on their own. After that, students may be led to explore the analytic path of the problem and then in turn to use the synthetic method to make the proof. For instance, what is the given in the problem? What may the given be turned into? In this teaching process, some students will be able to answer such forms of questions, while others may not

> be able to do so. Thus, it is better for the teacher to encourage some able students to lead the rest of thee students in the class into the problem solving activity. Surely when I ask a question to the whole class, I purposefully select individual students to answer it. For me, it is essential to find the right student in the class to answer a certain question. When teaching a new theorem, it may be reasonable to provide students time freely to explore the problem situation and to encourage plausible reasoning. But now, in these lessons, in starting to teach students how to apply theorems to solving proof problems, I need to emphasize the logical nature of such a problem-solving process. Without any guidance, what can we expect of students' free arguments in such a process? If I distribute questions to different students, I can improve the efficiency of the whole class teaching as well. So the challenge facing teachers in teaching proof problem solving is about how to distribute effectively the sequence of questions to different individual students in the class." (Lily, interviewed on 27th December, 2006, translated by DING Liping)

This statement shows that the teacher knows very well their central role in teaching deductive reasoning to students. It also shows how the teacher sees the relationship between argumentation and mathematical proof - that it is important to provide students with time freely to explore a problem situation and, with this, to encourage plausible reasoning. After that, students may be led to explore the analytic path of the problem and then, in turn, to use the synthetic method to make the proof. This provides a beginning to an examining of how mathematics teachers in China approach the relationship between argumentation and mathematical proof in their classroom teaching.

To gain fuller insight into the potentials and limitations of Chinese classroom practice, an important focus for further analysis is the variation of the teachers' questioning – perhaps utilising the Vygotskian perspective of the zone of proximal development (Vygotsky, 1934/1986), taken as the difference between what a learner can do without help and what they can do with the support and assistance of the teacher (in the widest sense). For instance, analysis could usefully focus on the extent to which teacher practice may be considered as a good instruction which marches ahead of student development and leads it. Another focus might be how different children in the class may have differently-sized zones of proximal development, meaning that the teacher may be in the situation of determining the lowest threshold and the upper threshold in the instruction. Such analyses are motivated towards developing profound understanding of Chinese pedagogy in school mathematics.

References

Boero, P., Garuti, R. & Mariotti, M.A. (1996). Some dynamic mental processes underlying producing and proving conjectures. *Proceedings of PME-XX*, Valencia, vol.2, pp. 121-128.

Dai, Z.P. (1998). Didactic reforms of school mathematics of the country. In D.Zh. Zhang, Z.P. Dai, R.F. Tang, Sh.Q. Li, and M. Ni (eds.) *Guide to the research in mathematics education.* Nanjing: Jiangsu Educational Press. [戴再平, 我国数学教学方法的改革。张奠宙主编，戴再平， 唐瑞芬， 李士錡，倪明副主编，数学教育研究导引。江苏教育出版社]

Ding, L. and Jones, K. (2009). Instructional strategies in explicating the discovery function of proof for lower secondary school students. In Fou-Lai Lin, Feng-Jui Hsieh, Gila Hanna & Michael de Villiers (Eds), *Proceedings of the ICMI Study 19 conference: Proof and Proving in Mathematics Education* (Vol 1, pp.136-141). Taipei, Taiwan: National Taiwan Normal University.

Ding, L. (2008). *Developing insight into teachers' didactical practice in geometric proof problem solving*. Unpublished PhD thesis. Southampton, UK: University of Southampton.

Gu, L. (1994). *Methodological and teaching principles of teaching experiments in Qingpu*. Beijing, China: Educational Science. [in Chinese]

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In: L. Fan, N-Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: perspectives from insiders* (pp 309-347). Singapore: World Scientific.

Hanna, G. & de Villiers, M. (2008). ICMI study 19: proof and proving in mathematics education. *ZDM-The International Journal of Mathematics Education*. 40(2), 329–336.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mariotti, M. A. (2007). Proof and proving in mathematics education. In A. Guitierrez & P. Boero (Eds.) *Handbook of research on the psychology of mathematics education* (pp 173-204). Rotterdam: Sense Publishers.

Paine, L.W. (1990). The teacher as virtuoso: A Chinese model for teaching. *Teachers College Record*, 92 (1), 49-81.

Simon, M. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics, 30*, 197-210.

Stigler, J.W. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Tu, R.B. (2006). The teaching and learning environment of "knowledge orientation". In D.Zh. Zhang (ed.) *The "Two Basics": Mathematics Teaching in Mainland China.* Shanghai: Shanghai Educational Press. [涂荣豹，"知识本位"的教学环境。张奠宙编，中国数学双基教学。上海教育出版社]

Vygotsky/, L. [alternative spelling Vygotskii] (1934/1986). *Thought and language* [alternative title *Thinking and speaking*]. Boston: MIT Press. Revised edition.

Zhang, D.Zh. (2005). Some views of the theoretical ideas in "*The full time compulsory education mathematics curriculum standards*". *General Report of Mathematics*, Vol.44, *5*, pp.1-4. [张奠宙, 对《全日制义务教育数学课程标准》理念部分的意见. 数学通报, 2005 年, 第 44 卷，第 12 期, 1-4 页.]

Zhang, D., Li, Sh. & Tang, R. (2004). The “two basics”: Mathematics teaching and learning in mainland China. In L. Fan, N. Wong, J. Cai, and S. Li, (eds.) *How Chinese learn mathematics: perspectives from insiders* (pp 189-207). Singapore: World Scientific.

Zheng, Y. (2006). Mathematics education in China: From a cultural perspective. In F.K.S. Leung et al. (Eds.), *Mathematics education in different cultural traditions* (pp381-390). New York: Springer.