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"WHY DO WE HAVE TO PROVE THIS?" FOSTERING STUDENTS' UNDERSTANDING OF 'PROOF' IN GEOMETRY IN LOWER SECONDARY SCHOOL

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This paper reports findings that indicate that as many as 80% of lower secondary age students can continue to consider that experimental verifications are enough to demonstrate that geometrical statements are true - even while, at the same time, understanding that proof is required to demonstrate that geometrical statements are true. Further data show that attending more closely to the matter of the 'Generality of proof' can disturb students' beliefs about experimental verification and make deductive proof meaningful for them.

INTRODUCTION

While it is recognized internationally that it is very important to teach 'proof' to all students, it is the case that this is not an easy task. A range of research and professional experience has found that it is difficult for many students to behave like professional mathematicians when they study deductive geometry, and they often find difficult to know how to advance logical arguments (Mariotti, 2007). One key issue is that students may not understand *why* they have to prove statements as experimental verification can seem enough. Students can consider more formal forms of mathematical argument, such as proof, as not necessary.

In this paper we report findings from a series of research projects carried out in Japan on the learning and teaching of proof, specifically in geometry. We address the issue of students' natural cognitive needs for conviction and verification and how these needs might be changed and developed through instructional activity. In what follows, we first present how students in lower secondary schools perceive 'proof' in geometry by using data from 418 Japanese lower secondary school students (206 from Grade 8, and 212 from Grade 9) collected in 2005. We then offer some suggestions developed from classroom-based research (undertaken since the 1980s) about how we might encourage students' understanding of deductive proof in geometry.

THE TEACHING OF PROOF IN GEOMETRY IN JAPAN

The specification of the mathematics curriculum for Japan, the 'Course of Study', can be found in the *Mathematics Programme in Japan* (English edition published by the Japanese Society of Mathematics Education, 2000). It should be noted that no differentiation is required in the 'Course of Study', and mixed-attainment classes are common in Japan. 'Geometry' is one of the important areas in lower secondary schools (the other areas are 'Number and Algebra' and 'Quantitative Relations'), and the curriculum states that in geometry students must be taught to "understand the significance and methodology of proof" (JSME, 2000, p. 24).

However, research repeatedly shows that many students demonstrate poor or sometimes no understanding of proof in geometry (for example, Kunimune, 2000).

STUDENTS' UNDERSTANDING OF PROOF IN GEOMETRY

In the research summarized in this paper, we capture students' understanding of proof in terms of the following two aspects: 'Generality of proof' and 'Construction of proof'. On the one hand, students have to understand the generality of proof in geometry; universality and generality of geometrical theorems (proved statements), roles of figures, difference between formal proof and experimental verification, and so on, and we call this aspect 'Generality of proof in geometry'. On the other hand, they also have to learn how to 'construct' deductive arguments in geometry; definitions, axioms, assumptions, proof, theorems, logical circularity, axiomatic systems and so on, i.e. 'Construction of proof in geometry'.

Considering these two aspects, the following levels of understanding are proposed (we do not, in this paper, relate these levels to the van Hiele model):

- Level I: at this level, students consider experimental verifications are enough to demonstrate that geometrical statements are true (Level Ia: Do not achieve both 'Generality of proof' and 'Construction of proof' and Level Ib: Achieved 'Construction of proof' but not 'Generality of proof')
- Level II: at this level, students understand that proof is required to demonstrate geometrical statements are true (Level IIa: Achieved 'Generality of proof', but not understand logical circularity and Level IIb: Understood logical circularity)
- Level III: at this level, students can understand simple logical chains between theorems

The following questions are used to measure students' levels of understanding:

Q1 Read the following explanations by three students who demonstrate why the sum of inner angles of triangle is 180 degree.

Student A 'I measured each angle, and they are 50, 53 and 77. 50+53+77=180. Therefore, the sum is 180 degree.' Accept/Not accept

Student B 'I drew a triangle and cut each angle and put them together. They formed a straight line. Therefore, the sum is 180 degree.' Accept/Not accept

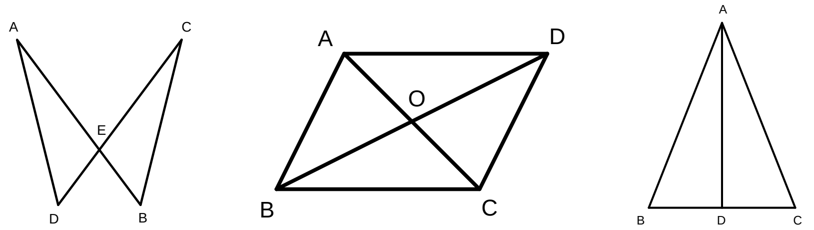
Student C Demonstration by using properties of parallel line (an acceptable proof) Accept/Not accept

Q2 Prove AD=CB when $\angle A = \angle C$, and AE=CE.

Q3The following argument demonstrates that 'the diagonals of a parallelogram intersect at their middle points' carefully. 'In a parallelogram ABCD, let O be the intersection of its diagonals. In ΔABO and ΔCDO, AB // DC. Therefore, $\angle BAO = \angle DCO$ and $\angle ABO = \angle CDO$. Also, AB = CD. Therefore $\Delta ABO \equiv \Delta CDO$. Therefore, AO = CO and BO = DO, i.e. the diagonals of a parallelogram intersect at their middle points'

Now, why can we say a) AB // DC, b) AB = CD, and c) ΔABO ≡ ΔCDO?

Q4 Do you accept the following argument which demonstrates that 'in an isosceles triangle ABC, the base angles are equal'? If you don't accept, then write down your reason. 'Draw an angle bisector AD from ∠A. In ΔABD and ΔACD, AB = AC, ∠BAD = ∠CAD and ∠B = ∠C. Therefore, ΔABD ≡ ΔACD and hence ∠B = ∠C'



Q2 Q3 Q4

Q1 checks whether learners can understand difference between experimental verification and formal proof in geometry. Q2 checks whether learners can understand a simple proof. Q3 checks whether learners can identify assumptions, conclusions and so on in formal proof. Finally, Q4 checks whether learners can identify logical circularity within a formal proof (proof is invalid as '∠B = ∠C' is used to prove '∠B = ∠C'). To achieve Level II, students have to answer correctly for Q1. Students who perform well in Q2 and 3 can be considered at least at Level Ib as they achieve good understanding in 'Construction of proof'. Figure 1 summarizes the criteria and levels.

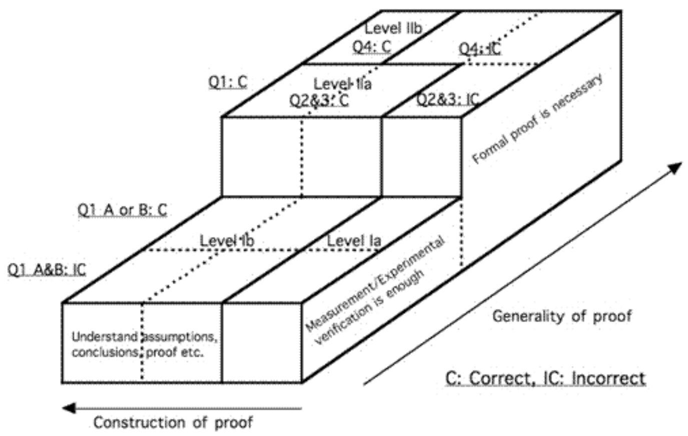


Figure 1: criteria and levels of generality and proof construction

This framework has been used in several research studies in Japan. For example, the results of surveys carried out in 1987, 2000 and 2005 indicate that over 60% students consider that experimental verification is enough to say it is true that the sum of the inner angles of triangle is 180 degree. Here, we briefly report on data collected in 2005 (with 206 students from Grade 8, and 212 students from Grade 9), see Tables 1 and 2.

The results presented in Table 1 indicate that whereas students can accept (or understand) that a formal proof (Student C explanation) is a valid way of verification, many also consider experimental verification (student B's explanation) as acceptable. There are, however, changes from Grade 8 to Grade 9, as, by the later grade, more students reject empirical arguments or demonstrations. This is because Grade 9 students have more experience with formal proof, whereas in Grade 8 the students are only just started studying proof (for more on this, see Fujita and Jones, 2003).

	Student A		**Student B**		**Student C**	
	Accept	Not accept	Accept	Not accept	Accept	Not accept
Grade 8	62%	32%	70%	21%	74%	15%
Grade 9	36%	58%	52%	38%	80%	6%

Table 1: Results of Q1

	Q2	Q3 a)	Q3 b)	Q3 c)	Q4
Grade 8	57%	82%	80%	53%	34%
Grade 9	63%	85%	81%	59%	49%

Table 2: Result of Q2-4

The results in Table 2 indicate the following in terms of students' understanding of 'Generality of proof' and 'Construction of proof'.

- Q2: More than half of students can construct a simple proof.
- Q3: Students show relatively good performance for a) and b), and these indicate that students have good understanding about deductive arguments of simple properties. Q3 c) is more difficult as students are required to have knowledge about the conditions of congruent triangles.
- Q4: The results suggest that more than half of students cannot 'see' why the proof in Q4 is invalid; that is, they cannot understand the logical circularity in this proof.

In summary, as shown in Table 3, 90% of Grade 8 and 77% of Grade 9 students were found to be at level I. Data from 1987 and 2000 show similar results (see Kunimune, 1987, 2000).

Level	**Ia**	**Ib**	**IIa or above**
Grade 8	33%	57%	9%
Grade 9	28%	49%	22%

Table 3: levels of understanding

MOVING STUDENTS TO DEDUCTIVE THINKING

As evident in a recent review of research on proof and proving by Mariotti (2007, p181), the 'discrepancy' between experimental verifications and deductive reasoning is now a recognized problem; Japan is not an exception. The findings given above indicate that Japanese Grade 8 and 9 students are achieving in terms of 'Construction of proof', but not necessarily in terms of 'Generality of proof'. There is a gap between the two aspects. This means that students might be able to 'construct' formal proof, yet they may not appreciate the significance of such formal proof in geometry. They may believe that formal proof is a valid argument, while, at the same time, they also believe experimental verification is equally acceptable to 'ensure' universality and generality of geometrical theorems. The data for Grade 9 students can be considered as quite concerning as almost 80% of students remain at level I even though they have studied formal proof at Grade 8 (with 90% of relevant intended lessons in a Japanese textbook were devoted for 'justifying and proving' geometrical facts' in G8 (Fujita and Jones, 2003)).

Now we turn to the question of working with students on why formal proof is needed. Kunimune (2000) reports on a series of lessons for Grade 8 students which were designed and implemented to disturb students' beliefs about experimental verification. In these lessons, students were asked to compare and discuss various ways of verifying the geometrical statement that the sum of the inner angles of triangles is 180 degrees. This statement was chosen as way of trying to bridge the gap between empirical and deductive approaches as students often encounter the angle sum statement in primary schools and they study this again with deductive proof in lower secondary schools. While we do not have space to provide the data we can provide a summary from studies by Kunimune (1987; 2000) of ways which can be useful in encouraging students to develop an appreciation of why formal proof is necessary in geometry.

- Students first exchange their ideas on various ways of verification; they comment on accuracy or generality of experimental verification; they discuss the advantages/disadvantages of experimental verifications. Students' comments such as 'A protractor is not always accurate ...', 'The triangle is not general', and so on, often cause a state of disequilibrium in students (*viz* Piaget), and make students doubt the universality and generality of experimental verification.
- Advice from teachers is necessary to encourage students to reflect critically on different ways of verifications (*viz* establishment of 'social norm' in classrooms, Yackel and Cobb, 1996).

Kunimune (1987; 2000) found that, after such lessons, around 40% of students previously at Level Ib have moved to Level II. They no longer accept experimental verification and start considering that deductive proof is the only acceptable argument in geometry. A later post-test carried out one month after the lessons found that about 60% of students are at Level IIa.

In summary, we conclude that the matter of the 'Generality of proof' could usefully be explicitly addressed in geometry lessons in lower secondary schools.

CONCLUDING COMMENTS

This paper outlines research findings from Japan suggesting that, in terms of 'Generality of proof' and 'Construction of proof', many students in lower secondary school remain at Level I (where they hold the view that experimental verifications are enough to demonstrate that geometrical statements are true), even after intensive instruction in how to proceed with proofs in geometry. Classroom studies have tested ways of challenging such views about empirical ways of verification which indicate that it is necessary to establish classroom discussions to disturb students' beliefs about experimental verification and to make deductive proof meaningful for them.

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