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STUDENTS' GEOMETRICAL CONSTRUCTIONS AND PROVING ACTIVITIES: A CASE OF COGNITIVE UNITY?

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Geometrical constructions (whether with paper and pencil or with appropriate software) are widely considered to be a suitable vehicle for secondary school students to gain experience of proof and proving. It is also recognised, however, that there can be a tension between the practical aspect of physically carrying out a geometrical construction and the theoretical aspect of constructing the related proof. This paper reports on data from a teaching experiment in which Grade 9 students worked on a series of challenging geometrical construction problems. Analysing the data through the lens of cognitive unity, we report on whether, or to what extent, geometrical constructions in particular encourage the uniting of student conjecture production and proof construction. Our analysis suggests that this is not automatic.

INTRODUCTION

A characteristic feature of geometry is its dual nature as simultaneously a theoretical domain and perhaps the most reality-linked part of mathematics. This dual nature has consequences. When students are examining the properties of geometrical objects practically (say with ruler and compasses, or with suitable software), they can ascertain the properties of shapes and generate various conjectures such as 'these triangles look congruent', 'these angles must be the same', 'the sum of these angles must be 360 degrees', etc. For mathematics learning it is desirable that students verify such conjectures through suitable ways of proving. It is well established, however, that not all students naturally understand why more formal proof is necessary, and they can feel that experimental verification is sufficient (see, for example, Kunimune, 1987; Mariotti, 2007; Kunimune, Fujita and Jones, 2009).

A key challenge in mathematics education is to devise ways of enabling students successfully to move between the practical and theoretical domains in geometry. The purpose of this paper is to analyse in what ways the use of challenging geometrical constructions might be a suitable vehicle for lower secondary school students to gain valuable experience of proof and proving of geometrical statements.

GEOMETRICAL CONSTRUCTIONS AND PROVING ACTIVITIES

Undertaking geometrical constructions with straightedge (or marked ruler) and compass – and being challenged with difficult, if not impossible constructions – is a very ancient human activity going back beyond Euclid and the ancient Greeks (for a comprehensive exposition, see, for example, Martin, 1998). The significance for mathematics learning is the choice of these tools. As Mariotti (2002) explains, the meaning of *circle* - the geometric figure - is incorporated in the use of the compass.

Nevertheless, while, on the one hand, use of the compass realises the graphic representation of circles - on the other hand, for learners the passage *from* the use of the compass to trace circular shapes *to* the conception of the circle as 'the locus of the points equidistant for the centre' is not immediate.

Jore and Parzysz (2005) detail how lower secondary school students can find themselves in a 'twilight zone' between the practical aspect of physically carrying out a geometrical construction and the theoretical aspect of proof. They conclude (p121) that "giving too much precision in the description of the gestures to be performed to make a 'construction' may conceal the metaphorical role of language, with the consequence that some students will remain quite a long time in a 'geometry of drawings'…" and that ways to enable students "to move towards a 'theoretical geometry' is indeed an important question, and we think it worthy of investigation by researchers".

THEORETICAL FRAMEWORK

In earlier research we proposed that developing students' '*geometrical eye*' (defined as '*the power of seeing geometrical properties detach themselves from a figure*' Godfrey, 1910; Fujita and Jones, 2003) be encouraged at each stage of geometry education. We have also examined the importance of providing students with explicit opportunities to examine differences between experimental verification and deductive proof in geometry lessons (see, for example, Kunimune, Fujita and Jones, 2009)

In this paper, we focus on the notion of *cognitive unity*, defined as the continuity between the processes of conjecture production and proof construction (Boero, 1996). Boero *et al* (2007) provide some evidence of cognitive unity in a series of teaching experiments. Pedemonte (2007) has explored the notion of cognitive unity further, and proposed *structural continuity* to give a more detailed account of the relationship between argumentation and mathematical proof.

In our research we are interested in how we might design teaching that helps students prove suitable geometrical statements. In terms of the notion of cognitive unity, our approach in this paper is to analyse the circumstances when students *unite*, or not, their conjecture production and proof construction. As existing research suggests, students are more likely to have a richer understanding of proof if they could engage in argumentation processes which lead to the forming of conjectures, rather than merely reading and following pre-prepared proofs (Marriotti, 2001). It is also commonly thought that geometrical constructions provide students with opportunities in which they can form conjectures and consider why their constructions work.

To identify whether we could observe cognitive unity in students' arguments, we follow the model devised Toulmin (1958), see Figure 1. This model has been used successfully to analyse students' mathematical argumentations and proving activities (for example, Hoyles and Küchemann, 2002) and, in particular, as Pedemonte (2007) has demonstrated, this model is useful in analysing the structure of students' argumentations and the generation of proof in mathematics.

In using Toulmin's model, we consider that cognitive unity is identified when students specify at least the 'data', 'claim' and 'warrant' in their mathematical argumentation as this suggests that the 'distance' between the argumentation and the proof would be quite small and this would likely lead students to being more successful in proof production. Note, however, that we also take it that the 'proof' being worked on by the students does not necessarily have to be correct at the stage at which we might identify cognitive unity. We take this position because rebuttal is a recognised component of argumentation (see Figure 1).

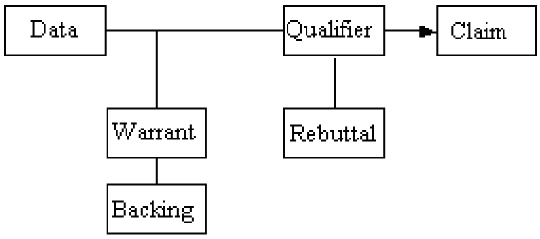

Fig. 1: Toulmin's model

METHODOLOGY

Our research was undertaken in Japanese lower secondary schools. Here, proof is taught mainly in geometry, with the curriculum stating that, in geometry, students must be taught to "understand the significance and methodology of proof" (JSME, 2000, p. 24). In general, the research evidence indicates that, while students can successfully construct simple proof in geometry, they may not necessarily know why such proof is necessary, and they may not distinguish between experimental verification and more formal proof (see, for example, Kunimune, 1987; Kunimune, Fujita and Jones, 2009). In terms of geometrical constructions (with ruler and compasses), while these can be taught in Grade 7, these constructions are often not proved until Grade 8 (after students have learnt how to prove simple geometrical statements), and not studied in Grade 9 any further.

In a series of teaching experiments, we investigated the use of more complex geometrical constructions - and their associated proofs - in Grade 9. In the analysis that follows, we report on one of our teaching experiments in which Grade 9 students learnt theorems and properties of similar triangles. In this teaching experiment, 13 one-hour lessons were designed and taught (in 2002). Each lesson was recorded using video cameras and the research team kept detailed field notes. The 13 lessons were split into three units of work: in unit 1 (3 lessons) the students worked on the problem of how to enlarge a given triangle; in unit 2 (4 lessons) students studied the conditions of similar triangles, constructing the largest square in a triangle; in unit 3 (6 lessons) the students tackled the problem of trisecting a given line. The teacher generally began lessons by identifying a set of already learnt properties which are used as a form of axioms (a similar idea to that of the 'germ theorems' of Bartolini Bussi, 1996). This provided students with known starting points for their proofs.

The structure of each unit was designed as follows: first, students undertake constructions individually, students then share their solutions in small groups, they then present their construction and, if possible, their proof, and then the teacher summarises the classroom discussion.

DATA ANALYSIS AND DISCUSSION

In this teaching experiment, for the first unit (of three lessons) the students worked on the problem of *how to enlarge a given triangle ABC*. In the experimental class, the students found various ways of tackling the problem, with nine different methods being proposed and examined. The unit concluded with the conditions for similar triangles being introduced by the teacher. In the second unit (of four lessons), the idea of similarity was extended to a square, and the students collaboratively presented three methods of solving the problem of *how to construct the largest square within a given triangle ABC.*

In this paper, because of limited space, we report on selected episodes from the third unit (of six lessons) in which the students worked on the problem of *how to trisect a given line*.

The students first undertook this problem individually. Our data shows that, at the beginning, the students were, on the whole, rather uncertain of how to proceed, saying 'I think I know how to bisect a line', 'Trisecting a line? No way!' and so on. However, they gradually made progress. Then, the students discussed their ideas in small groups. While some groups just presented their methods without proof, other groups started to produce proof for their methods. For example, Group D students exchanged the following arguments:

D1: I considered if we could trisect angles, then we could trisect a line. What I did was, like this figure [see Figure 2], first I constructed a right-angled isosceles triangle ABC, and then an equilateral triangle based on CB. I then constructed an angle bisector of the angle DCB. Now, angle ACF=angle FCG=angle GCB=30 degrees, and hence the line AB is trisected, AF=FG=GB.

D2: But, I don't think AF=FG=GB

D1: Is it just an error in my construction? [field notes suggest that the student was assuming that his reasoning should be correct]

D2: Really?

D1: Because if we can trisect the angle, then the line must be trisected?

D2: I don't think so. It seems AF=BG, but I really think FG looks shorter than the other two.

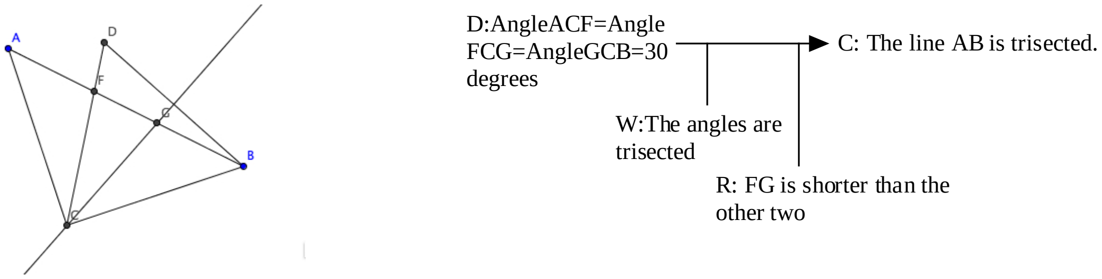

Fig. 2: Construction by Group D, together with analysis of student argument

Our analysis puts this as an example of cognitive unity, even though not a correct proof. While D1 tried to prove his method, in his argument the warrant was 'angles are trisected' but this was challenged by D2, and D1 could not defend his argument.

A similar argument was observed in Group E:

E1: I tried to solve this by constructing an equilateral triangle ABC and square ADEB (Figure 3). If we join the midpoints of CA and BC, and the midpoint of DE, then I think we can trisect AB, but I don't know why.

E2: Is it true?

E1: I checked it with compass and it seems this method works.

E2: Then how do we prove your method?

E1: That is what I am asking you!

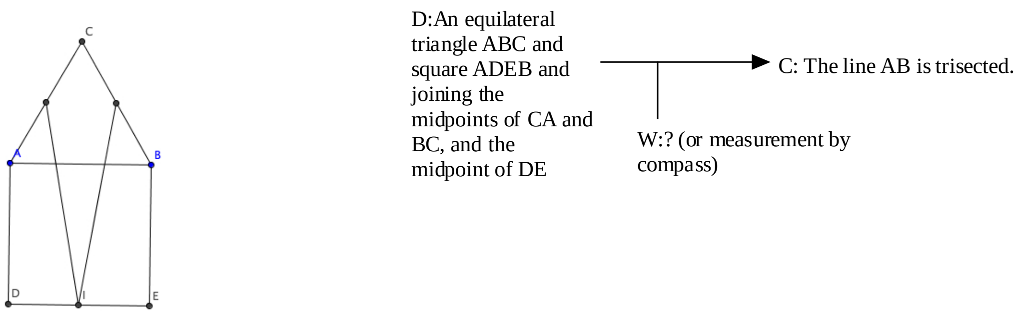

Fig. 3: Construction by student E1, together with analysis of student argument

Our analysis puts this as *not* the example of cognitive unity. Student E1 relied on verification by measurement (through use of the compass) and could not find his 'warrant'. The argument made by E1 was challenged by E2, and E1 could not construct a proof at this point.

Student E3 provided the following:

E3: AB [see Figure 4] is trisected by constructing a square whose diagonal is AB, and joining a vertex and midpoints. A proof for why this works is that because ΔADF and ΔBEF are similar, and ratio is 2:1, so AF:FB=2:1, and this proves that AF':F'F:FB=1:1:1.

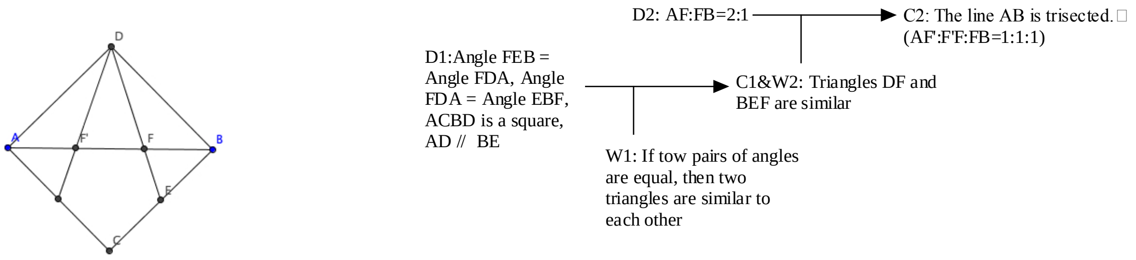

Fig. 4: Construction by student E3, together with analysis of student argument

Our analysis puts this as an example of cognitive unity. In this solution, the conditions and properties of similar triangles, which the student had learnt in unit 1, were used to prove the conjecture. At this point, students had already spent more than 10 hours with geometrical construction and proof, and many students appreciated the necessity and importance of mathematical proof in geometry. By using the approach of E3, E1 managed to modify his original idea, proved a modified solution (Figure 5), and then finally proved that his original idea (Figure 3) was not a correct method:

E1: I thought that I could trisect AB when I constructed this [Figure 3], but I think I found this is not true. So I proved that we cannot trisect the line AB. We just saw the construction [in Figure 4] is true, so I use this approach in my proof, to prove ΔADH and ΔBFH are similar. Now, I draw an equilateral triangle on AB [Figure 5], and by doing this, we *can* trisect the AB, and proof is similar [to Figure 3].

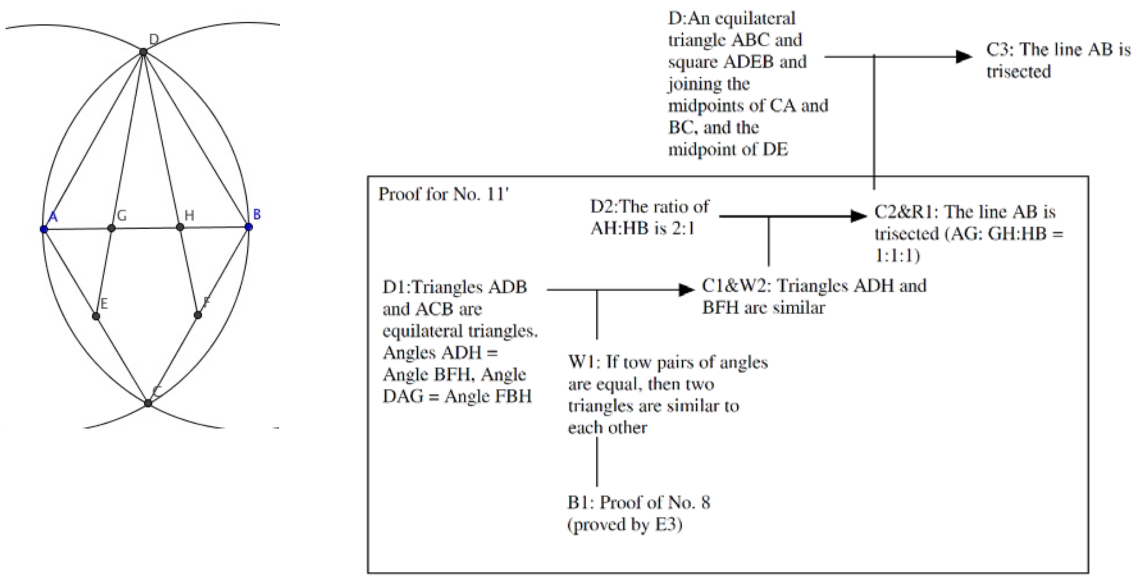

Fig. 5: Revised construction by student E1, together with analysis of argument

E1: Now, compare this [Figure 5] to my construction [in Figure 3], and C and D are not in the same place, as the height of the triangle ACB is shorter than the height of the square. We know we can trisect the AB by using this approach [Figure 5], and therefore, my method [in Figure 3] does not work.

Our analysis puts this as an example of cognitive unity, as E1, who at one point was failing to construct a proof, was able to use another student's proof most effectively as 'Backing (B1)' in his own argument and proof. As Figure 5 shows, the logical chain which this student reached is quite sophisticated. Not only that, but the student was able to use this proof as 'rebuttal' to disprove the earlier idea [shown in Figure 3].

By the end of the last lesson of the third (and final) unit of work, the groups of students had examined 13 different methods of tackling the problem of *how to trisect a given line*. Within these13 different methods, 11 were correctly proved by students themselves. As the above episodes show, the students in this class actively attempted not only to solve challenging geometrical constructions, but also to prove their constructions.

In many cases, cognitive unity was observed, and proofs were shared and used to prove other conjectures. Our findings suggest that it is important, not only to encourage the uniting of students' conjecture production and proof construction, but also to give them opportunities to share their mathematical argument and reasoning within the classroom.

CONCLUDING REMARKS

In this paper, we report on results from one of our teaching experiments that focused on how to bridge conjecture production and proof construction. We focused in particular on geometrical construction and examined this through the theoretical lens of cognitive unity. As we demonstrate above, challenging geometrical construction problems can encourage students' mathematical arguments, reasoning and proof. The key issue for us is whether, or to what extent, geometrical constructions in particular encouraged the uniting of student conjecture production and proof construction. Our analysis suggests that this is not automatic. The teaching experiment suggests that careful design can enable students to develop a good 'manner' to study geometry as they could engage argumentation processes which can support the forming of conjectures (so that students are not just reading and following pre-prepared proofs). In addition, it is the case that, in these teaching episodes, geometrical statements the students already know and have proved were often shared and recognised as starting points of further proofs. Furthermore, by the end of the third unit, it was clear that the students had a much greater appreciation of how to use already known facts to proceed with further investigations in mathematics. We are aware, however, that further research is necessary to give a fuller answer to the matter of how, and to what extent, geometrical constructions encourage the uniting of student conjecture production and proof construction.

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