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PROOFS AND REFUTATIONS IN LOWER SECONDARY SCHOOL GEOMETRY

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In this paper, we report on a teaching experiment in which we focused on students tackling 3D geometry problems in which, in general, they initially tended to produce 'primitive' conjectures by relying on visual images rather than geometrical reasoning. Following the work of Larsen and Zandieh (2008), we utilise the ideas of Lakatos (1976) on managing the refutation process and how the use of counter-examples can be important in promoting the growth of students' capability with geometrical reasoning and proof. We found that students' primitive conjectures can cause an unexpected result and that this can trigger further reviewing ('Monster-barring') and modifications of the conjecture ('Exception-barring') amongst students. Whole classroom discussion followed by small group discussion allowed students to exchange various ideas and opinions and this process was important for their construction of a proof of their new conjecture ('Proof-analysis').

Key-words: geometry, conjecturing, proof, refutation; 3D tasks

INTRODUCTION

The teaching of geometry provides not only a key vehicle for developing learners' spatial thinking and visualisation skills in mathematics, but also a major opportunity to develop their capability with deductive reasoning and proving (Battista, 2007; Royal Society, 2001). Through classroom-based research (for example, Kunimune, Fujita & Jones, 2010; Fujita, Jones & Kunimune, 2010), we are working on several themes in the teaching of geometrical reasoning and proof at the lower secondary school level, encompassing the design of problem-solving situations in geometry for students, the integration of geometrical constructions, ways of providing students with explicit opportunities to examine the differences between experimental verifications and deductive proof, and approaches to the teaching of deductive geometry based around a set of 'already-learnt' properties which are shared and discussed within the classroom.

In this paper we extend our previous research by focusing both on the design of problem-solving situations in geometry for students *and* on the teaching of deductive geometry based around a set of 'already-learnt' properties. While designing suitable classroom tasks is very important in mathematics education (e.g. Wittmann, 1995), using such tasks with students does not necessarily lead to 'good' results: something which Schoenfeld (1988) has illustrated in detail. Hence, additional factors need to be considered if the teaching of geometry is going to be more effective. In this paper, and following the work of Larsen and Zandieh (2008), we utilise the ideas of Lakatos (1976) to show how managing the refutation process and the use of counter-examples can be important in promoting the growth of students' capability with geometrical reasoning and proof. The tasks we use involve geometrical reasoning on simple 3D shapes - one of the topics considered by several papers from the CERME geometry working group (e.g. Mithalal, 2010; Pitallis et al, 2010).

REFUTATIONS IN THE PROCESS OF PROVING IN MATHEMATICS

Given that conjecturing processes are known to be important in the teaching and learning of mathematics in general, and geometry in particular, (Cañadas et al, 2007), our focus in this paper is on the relationship between conjecture, refutation, and proof. It is known that, on the one hand, treatment and understanding of refutation and counter-examples are not straightforward for learners (e.g. Balacheff, 1991; Stylianides and Al-Murani, 2010): indeed, Potari, Zachariades and Zaslavsky (2009) show that even trainee teachers find it difficult to identify correct counter-examples to refute false statements. Yet, on the other hand, counter-examples play an important role within the process of conjecture production and proof construction. Mathematical activity, it has to be said, is not straight-forward, but rather more like a zigzag path. Mathematicians typically make a conjecture, find counter-examples, refine the conjecture, find more counter-examples and so on, during their proving process. Lakatos (1976, p. 127), in his historical and epistemological study, considered that the proof and refutation process consists of the following:

- Primitive conjecture
- Proof (a rough thought-experiment or argument, decomposing the primitive conjecture into sub-conjectures)
- 'Global' counter-examples emerge (counter to the primitive conjecture)
- Proof is re-examined as a new theorem or improved conjecture emerges

While mathematicians, historians and philosophers remain engaged in on-going discussions into the validity of this process (see, for example, Hanna 2007, p. 10), there is some evidence in the mathematics education literature that Lakatos' framework can be a useful guide to promoting students' conjecture production and proof construction process. For example, Larsen and Zandieh (2008) utilised Lakatos' framework to analyse undergraduate students' proof construction processes in abstract algebra. They categorises the types of proof and refutation activities in terms of students' responses, described in their words as follows (p. 208):

- *Monster-barring*; any response in which the counter-example is rejected on the grounds that it is not a true instance of the relevant concept

- *Exception-barring*; any response that results in a modification of the conjecture to exclude a counter-example without reference to the proof
- *Proof-analysis*; the resulting modification to the conjecture is intended to make the proof work rather than simply exclude the counter-example from the domain of the conjecture

Larsen and Zandieh showed that Lakatos' framework "can serve as heuristics for designing instruction" (p. 215). In a similar vein, Komatsu (2010) revealed how a focus on counter-examples can encourage primary school pupils to refine their conjectures and extend their reasoning to reach a correct answer in a number task. We designed the teaching experiment below with a view to giving lower secondary school students valuable opportunities based on the 'proof and refutation' framework.

RESEARCH SETTING

The teaching experiment was undertaken in a Japanese lower secondary school where geometry has a major role in developing pupils' ideas about proof and proving. In Japan, the curriculum states that, in geometry, students must be taught to "understand the significance and methodology of proof" (JSME, 2000, p. 24. In terms of the 'paradigm of geometry' proposed by Houdement and Kuzniak (2003), Japanese geometry teaching may be characterized as within the *Geometry II* paradigm (in that axioms are not necessarily explicit and are as close as possible to natural intuition of space as experienced by students in their normal lives).

In our teaching experiment, by following the principles of the geometry curriculum, the following lessons were designed for Grades 7 and 8 students (aged 12-14);

- 21 lesson for Grade 7 (students aged 13 yrs old at the time): Introduction of 3D shapes and nets (2 lessons), Points, lines and planes (1 lesson), Positions and angles in 3D shapes (3 lessons, **our focus in this paper**), Distances of two points (2 lessons), Rotated shapes, circles and sectors (1 lessons), Surface areas volumes of 3D prisms and pyramids (2 lessons), line and rotational symmetry (1 lesson), Construction of parallel lines and tangents of circles (2 lessons), vertically opposite angles, alternate and corresponding angles in parallel lines (3 lessons), and Angles in polygons (4 lessons).
- 28 lessons for Grade 8 (students aged 13 yrs old at the time): Congruent triangles (3 lessons), Theorems and definitions in geometry (3 lessons), Constructions and properties of isosceles triangles (5 lessons), Constructions and properties of parallelogram (4 lessons), Construction of a cube (2 lessons, **our focus in this paper**), Congruent right-angled triangles (2 lessons), Relationship between triangles and quadrilaterals (2 lessons), Properties of circles (3 lessons), Parallel lines and areas (2 lessons) and Summary (2 lessons).

These lessons were implemented in one class of 40 students in a university-attached school where the teachers and researchers work together to undertake classroom-based research. The students' standard in mathematics is generally high. The regular teacher of the class, in line with Sekiguchi's (2002) account, generally considers a good lesson to be one in which the students are encouraged to share their ideas and solutions with each other.

In this paper, we focus on the lessons from *Positions and angles in 3D shapes* (taught in Grade 7) and *Construction of a cube* (taught in Grade 8). Our reason for focusing on these lessons is that, in trialling the lessons, students in general tended to produce their 'primitive' conjectures by relying on visual images rather than through geometrical reasoning. Thus, our concern is how to break this situation.

Recent studies (e.g. Christou et al, 2006; Mithalal, 2010) have shown how the use of technology and dynamic 3D geometry environments might help counter the difficulties that students have in studying the properties of 3D shapes. In this paper, we consider the method of 'proof and refutation' with practical activities and group discussions might also be effective and accessible way of teaching. In the analysis that follows, we consider this issue by using 'proof and refutation' framework of *Monster-barring*, *Exception-barring,* and *Proof-analysis*.

ANALYSIS OF EPISODES FROM OUR CLASSROOM EXPERIMENT

Episode 1 – what size is angle PQR in a cube?

In Grade 7 in Japanese schools, the main purpose of geometry teaching is to introduce students to geometrical reasoning through the study of 3D shapes and the angle properties of 2D shapes. In this episode (during the third of three lessons on *Positions and angles in 3D shapes*), and after learning some basic concepts of cubes and cuboids during the previous five lessons, the students were asked to investigate the size of the angle PQR in a cube ABCDEFG (see figure 1).

Of the forty students in the class, 25 of them considered that 'the angle is 90 degrees', 11 thought that 'angles will be changed' and 4 said 'I don't know'. As such, the dominant 'primitive' conjecture can be taken to be 'the angle is 90 degrees'. One student (referred to as student 1) stated his reasoning as follows:

Student 1: I think wherever P, Q, and R are, the size is 90 degree. Because angle PQR looks like 90 degrees if you look at it from the face BFGC.

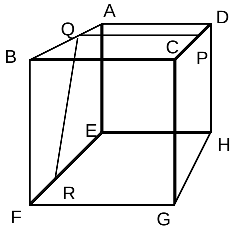


Figure 1: angle PQR in a cube

Students exchanged their ideas and opinions in groups and subsequently in whole classroom discussions which led them to modify their conjecture. The following presentations were made by students during the whole classroom discussion:

Student 2: I investigated by cutting a model of a cube. If we cut AC and AF, then we have an angle, and I think it won't be 90 degrees as the angles are formed by AC and AF.

Student 3: I also used a model, and I used protractor as well. I have got about 60 degrees, and not 90 degrees.

We consider these as *Exception-barring* responses, as their focus is not rejection of the 'primitive' conjecture, but the production of a new conjecture that 'angles will be changed'. After these presentations, the following idea was proposed by a student:

Student 4: I consider why Students 2's and 3's angles are 60 degrees. If we connect C and F, then there will be a triangle. It is a bit difficult to see the figure on the blackboard [as this is a 2D representation of a cube], but these lines should be the same and since all the angles are the same, this triangle should be an equilateral triangle. Therefore, angle CAF is 60 degrees.

We consider this as a *Proof-analysis* response wherein the new conjecture 'angles will be changed' is now justified by a simple proof.

Episode 2 – What shape is face DPFQ in a cube?

In Grade 8 in Japan, students continue to study geometry and are gradually introduced to more formal ways of geometrical reasoning. In the two lessons on *Construction of a cube* (16th and 17th lessons of their geometry work), the students undertook the following problem: 'Consider the net of a cube [see Figure 2]. Construct a net including the face DPFQ [where P and Q are the mid-points of AE and CG respectively].'

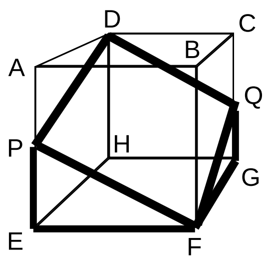


Figure 2: a half-cube for Grade 8 students to construct

In this task, the students were not only expected to identify the face DPFQ, but also to construct an actual net and make the model. This additional practical requirement is particularly important in the teaching experiment as we consider this is more likely to create 'unexpected situations' (such as the square DPFQ does not fit) for many students more easily than a question that solely asks students to determine the shape of the face DPFQ. In the latter case, students might say that the face DPFQ is a square, but it might be more difficult for them to recognise that it is not.

First, the teacher introduced the problem by referring to the students' experiences in Grade 7:

Teacher: Do you remember we made the solid ABCDEGH [illustrated as Figure 3]

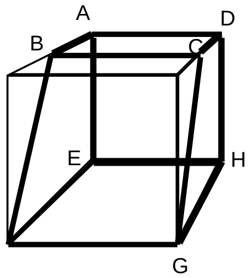


Figure 3: a solid ABCDEGH

Students: Yes, I remember. I think we managed to make it.

Teacher: Yes, and today, we try the task 'Let us consider a net of this 3D shape (where P and Q are the mid-points of AE and CG respectively). Construct a net including the face DPFQ'.

In this problem, a challenging point, on the one hand, is that the quadrilateral DPFQ is not a square, but a rhombus. On the other hand, this can lead the students to making a conjecture, refuting their conjecture, modifying the conjecture and so on, until their final decisions. After investigating this task individually, the students found that their 'primitive' conjecture 'the DPFQ is a square' might not be true as a square did not fit their models. The students then started exchanging their ideas within each group. For example, students in Group A (with students referred to as A1, A2, etc) had the following discussion (relating to models represented by Figures 4 and 5):

Student A1: I think DPFQ is a square. First the original shape was a cube, and all faces are squares, and therefore ΔAPD≡ΔEPF≡ΔGQF≡ΔCQD and all the sides are the same [note that this student's model was incomplete as the quadrilateral DPFQ did not fit perfectly].

Student A2: I thought, like you, that DPFQ is a square, but it did not fit… I drew a square first, and cut and pasted in my model.

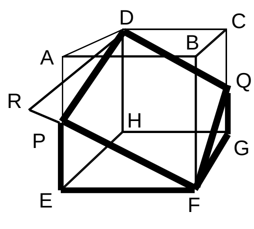


Figure 4: the model by student A2

Student A3: But [see Figure 5] if we follow A2's method, then I wonder if we would have a rhombus? I think, if the first shape we make is a square, then all sides should be the same, DQ=DR, and we cut ΔDRP, and this is a right-angled triangle. Therefore, DP is longer than DR, and DP≠DQ, and this is not a rhombus?

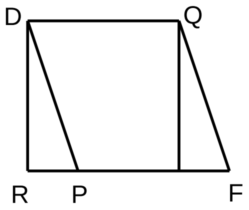


Figure 5: student A3's reasoning about shape DPFQ

We consider the above responses as *Monster-barring* and (incorrect) *Proof-Analysis*. The students tried to reject the counter-example and keep their original conjecture by using (incorrect) reasoning. It is interesting that their *Monster-barring* led to a proof which they tried to use to justify their original conjecture.

In another group (group B), however, two students (B1 and B2) first made their models without drawing DPFQ, and then student B3 showed his answers as follows (see Figure 6):

Student B1: My method is probably cheating, but I drew a net without DPFQ, and then made a model without a lid. Then, I put my half-completed model on a piece of paper, traced DPFQ and then made the lid (DPFQ).

Student B2: My method is similar to B1, but I did it a bit differently. I also made a model without a lid, and then I measured the angle PDQ, and it was 79 degrees. I made a quadrilateral with the angle PDQ 79 degrees, and then put the lid.

Student B3: I tried the method which is similar to B1 and B2, started from a net without DPFQ, and made a model. But I noticed that the length of PQ, the diagonal of DPFQ is the same as EG, the diagonal of HEFG. If we use this fact, we can construct ΔDQP by using ruler and compass. If we can construct ΔDQP, then we can also construct ΔPQF, then we can complete the net [see Figure 6]

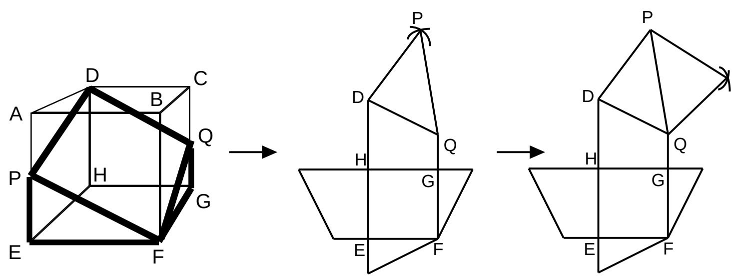


Figure 6: the net made by student B3

The above process can be considered as *Exception-barring*. This is because the students' original conjecture was abandoned and new ideas were searched for to make the situation consistent. Neither arguments by student B1 nor B2 were proofs.

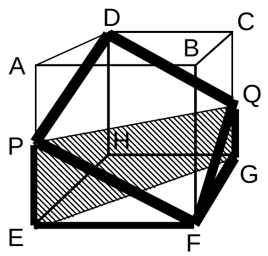
In addition, it is difficult to consider B3's argument as a proof as his method still does not explain what DPFQ is.

After the group discussion, all the group arguments were shared with the whole class. After listening to the presentation of student B3, a student G1 (from group G) added his reasoning as follows:

> Student G1: I did like B3's way, but if you looked at the shape without the lid from above, we can see PQ is equal to EG, and as the four sides of DPFQ are the same, so I think it is a rhombus. I then measured PQ and then used compass to complete the face DPFQ.

Student G1's response is again *Exception-barring*, and now a new conjecture 'the face DPFQ is a rhombus' is shared in the classroom. Finally, student H1 (from group H) presented his idea and the new conjecture was proved as follows (see Figure 7):

> Student H1: My idea is that I dissected the solid first. If we cut it vertically from PQ to EG, then it will be a rectangle. Therefore, PQ=EG. Also, if we cut it by connecting DH and F, then it will be a right-angled triangle, and DF is its hypotenuse and the other line is HF [and therefore, DF is longer than PQ].



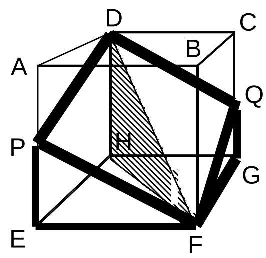


Figure 7: illustration of student H1's proof of why the face DPFQ is a rhombus

This reasoning, triggered by group discussions and whole classroom discussion, is considered as *Proof-analysis*. It is also interesting to see that the properties of quadrilaterals and triangles are used effectively by the student to justify the reasoning. Before this lesson, in addition to the 21 lessons in Grade 7, students have already completed 15 geometry lessons in which they practiced their geometrical reasoning in using a set of already-learnt properties which are shared and discussed within the classroom. The properties of quadrilaterals and right-angled triangles were already studied, and this student (H1) used them effectively to advance his reasoning.

DISCUSSION AND CONCLUDING COMMENTS

These episodes show that the first conjecture 'DPFQ is a square' caused an unexpected situation, and then this triggered further reviews (*Monster-barring*) and modifications of the conjecture (*Exception-barring*) amongst students. Whole classroom discussion followed by small group discussion allowed students to exchange various ideas and opinions and this process was important for their construction of a proof of their new conjecture (*Proof-analysis*).

In focusing, in this paper, on students' conjecture production and proof construction within the proof and refutation framework, we can conclude that the framework is useful not only for describing students' proving processes but also in indicating some helpful instructional approaches in geometry lessons. Through our analysis of data from our classroom-based research, we illustrate how managing students' discussions of counter-examples, both in group and whole classroom work, can act as a vehicle for promoting the development of their geometrical reasoning. We found that *Monster-barring* can sometimes lead to an incorrect proof from students (for example, students A1, A2 and A3 in the second episode). As such, *Exception-barring* and classroom discussions are important to construct legitimate proofs (*Proof-analysis*) (see student 4 in the first episode, and student H1 in the second). In future research, in addition to continuing to design suitable tasks for students, we aim to investigate other factors which could facilitate students' conjecture production and proof construction in geometry.

NOTE

The lessons in this teaching experiment were based on the Japanese 'Course of Study' first published in 2000 (JSME, 2000).

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