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SUPPORTING STUDENTS TO OVERCOME CIRCULAR ARGUMENTS IN SECONDARY SCHOOL MATHEMATICS: THE USE OF THE FLOWCHART PROOF LEARNING PLATFORM

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The extent to which students are competent in identifying circular arguments in mathematical proofs remains an open question, as does how it might be possible to enhance their competency. In this paper we report on a study of learners encountering logical circularity while tackling geometry proof problems using a web-based proof learning support environment. The selected episodes presented in the paper illustrate how learners who have just started learning to construct mathematical proofs make various mistakes, including using circular arguments. Using the feedback supplied by the web-based proof learning support environment, and with suitable guidance from the teacher on the structural aspects of a proof, learners can start bridging the gap in their logic and thereby begin to overcome circular arguments in mathematical proofs.

INTRODUCTION

Bardelle (2010) provides an example of some undergraduate mathematics students in Italy being presented with the diagram in Figure 1 as a 'visual proof' of Pythagoras' theorem. The students were asked to use the figure to help them develop a more formal written proof of the theorem.

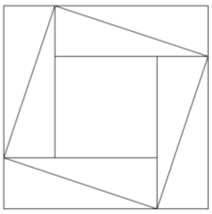


Figure 1: a 'visual proof' of Pythagoras' theorem

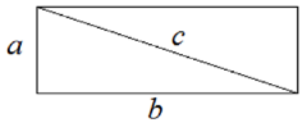


Figure 2: a rectangle from Figure 1

Bardelle relates how one student focused on the rectangles that surround the central square. By defining *a* as the short side and *b* the longer one (as in Figure 2), the student used Pythagoras' theorem to get $c=\sqrt{a^2+b^2}$ and thence, by squaring both sides, the student obtained Pythagoras theorem $c^2=a^2+b^2$. This is an example of a student using a circular argument or *circulus probandi* (arguing in a circle). It entails *assuming just what it is that one is trying to prove* (Weston, 2000, p75). In logic, circular reasoning is considered a fallacy as the proposition to be proved is assumed (either implicitly or explicitly) in one of the premises.

In a comprehensive consideration of the key questions for mathematics education research on the teaching and learning of proof and proving, Hanna and de Villiers

(2008, p333) raise the issue of the extent to which students are competent in identifying circular arguments in proofs. They also ask how it might be possible to enhance such competency in students. In this paper we report on a study of learners working with logical circularity while tackling proof problems. Our research questions encompass how it is that they create a proof which has a logical circularity, and how they modify their thinking through to constructing a correct proof. To answer these questions we analysed selected episodes collected as students work on geometry proof problems using a web-based proof learning support environment (for more details of this web-based system, see Miyazaki et al, 2011).

CIRCULAR ARGUMENTS IN DEDUTIVE REASONING

Rips (2002) has argued that the psychological study of reasoning should have a natural interest in patterns of thought like circular reasoning, since such reasoning may indicate fundamental difficulties that people may have in constructing and in interpreting even everyday discourse. However, Rips claims that up until his study in 2002 there appeared to be no prior empirical research on circular reasoning. While Rips reports on a study of young adults, Baum, Danovitch and Keil (2008) report findings with younger students - indicating that by 5 or 6 years of age, children show a preference for non-circular explanations and that this appears to have become robust by the time youngsters are about 10 years of age.

While learners' preference for non-circular explanations may be robust by the time they are ten years old, within mathematics education Kunimune, Fujita and Jones (2010) report on data on Grade 8 and 9 pupils showing that as many as a half of Grade 9 students and two-thirds of Grade 8 pupils are not able to determine why a particular geometric proof presented to them was invalid; that is they could not see the logical circularity in the proof. Likewise in Germany, Heinze and Reiss (2004) report that from Grade 8 to 13 an unchanging two-thirds of pupils fail to recognise circular arguments in mathematical proofs. Such evidence illustrates that pupils are in need of considerable support in order to identify and overcome circular arguments in mathematical proofs. As Freudenthal (1971, p427) observed "you have to educate your mathematical sensitivity to feel, on any level, what is a circular argument".

THEORETICAL FRAMEWORK

We take as our starting point that a mathematical proof generally consists of deductive reasoning starting from assumptions and leading to conclusions. Within this reasoning process, at least two types of deductive reasoning are employed: universal instantiation (which deduces a singular proposition from a universal proposition) and syllogism (where the conclusion necessarily results from the premises).

In order to understand the structure of proof, students need to pay attention to elements of proof such as its premises and conclusions and their inter-relationships. Both Heinze and Reiss (2004) and McCrone and Martin (2009) identify appreciation of proof structure as an important component of learner competence with proof. In this paper we use the following levels of learner understanding of proof structure elaborated by Miyazaki and Fujita (2010):

- *Pre-structural*: this is the most basic status in terms of understanding of the proof structure, where learners regard proof as a kind of 'cluster' of possibly meaningless symbolic objects and they cannot see that, within the structure of proof, *singular propositions* are those which are universally instantiated from universal *propositions*, that *syllogism* is necessary to connect *singular propositions*, and so on.
- *Partial-structural*: given that a proof consists of elements of proof such as singular and universal propositions, deductive reasoning, and their relational network, if learners have started paying attention to each element, then we consider they are at the Partial-structural elemental sub-level. To reach the next level, learners need to recognise some relationships between these elements (such as universal instantiations and syllogism). If learners have started paying attention to each relationship, then we consider them to be at the Partial-structural relational sub-level.
- *Holistic-structural*: at this level, learners understand the relationships between singular and universal propositions, and see a proof as 'whole' in which assumptions and conclusions are logically connected through universal instantiations and syllogism (much like the 'warp' and the 'weft' when weaving textiles). Once learners have 'Holistic-structural' understanding, they should be able to start refining proofs, become aware of the hierarchical relationships between theorems, be able to construct their own proofs, and so on.

The Pre-, Partial-, and Holistic-structural levels of understanding of proof structure is summarised in Figure 3.

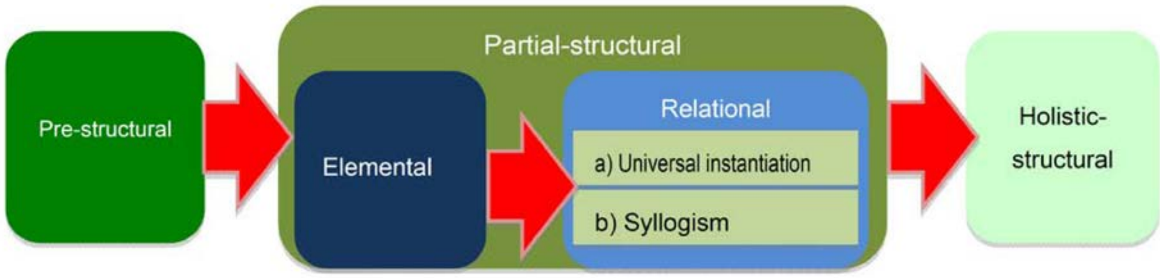


Figure 3: Pre-, Partial-, and Holistic-structural levels of understanding of proof

According to this framework of Pre-, Partial-, and Holistic-structural levels of understanding of proof, most learners who are just starting to learn proofs would be at either the Pre or Partial-structural level. In particular, if learners do not fully understand the role of syllogism, then they would be likely to accept or construct a proof which includes logical circularity.

METHODOLOGY

To investigate students' understanding of logical circularity in mathematic proofs, a web-based learning platform (hereinafter the system) was utilised (for details of this, see Miyazaki et al, 2011). The current version is online at:
http:// www.schoolmath.jp/flowchart_en/home.html

For this learning platform, flow-chart proofs (see Ness, 1962) are adopted and both open and closed problems in geometry are available to learners, including ones that involve the properties of parallel lines and congruent triangles. Learners tackle proof problems by dragging sides, angles and triangles to cells of the flow-chart proof and the system automatically transfers figural to symbolic elements so that learners can concentrate on logical and structural aspects of proofs. The geometry problems that student tackle when using the learning platform include both ordinary proof problems such as 'prove the base angles of an isosceles triangles are equal' (we call these 'closed' problems) and problems by which students construct different proofs by changing premises under certain given limitations (we call these 'open' problems). Each time the learners selects a next step in their flow-chart proof, the web-based system checks for any error via a database of possible next steps. If there is an error, the learners receive orderly feedback in accordance with the type of error (such as error in the deductive chain, error in selecting the appropriate theorem, error in the antecedent and the consequent of a singular proposition, and so on).

For data collection, a range of individual or grouped learners (up to 4) tackled one or more mathematical activities with the web-based system and their conversations were recorded by video camera and then transcribed. In the next section we report selected cases involving five learners: two high-attaining secondary school students aged 14 years old (WS1 and WS2) and three undergraduate primary trainee teachers (an individual, R, and a pair, J1 and J2). None of these learners had prior experience of mathematical proof in geometry.

DATA ANALYSIS AND DISCUSSION

In the problem in Figure 4 (lesson 2-b00), the learners are asked to prove 'AB=CD', with reasoning in both universal instantiation and syllogism being required to deduce a proper conclusion. This is an example of an open problem in that while learners have to use 'AO=CO' for their proof, they can decide for themselves which other properties to use. In this problem, they could either consider AO=CO, BO=DO and $\angle$AOB=$\angle$COD (the SAS condition) or use AO=CO, $\angle$AOB=$\angle$COD and $\angle$OAB=$\angle$OCD (the ASA condition).

Case 1: after practicing with an introductory problem, and understanding that there are three conditions that can be used to say that two triangles are congruent, two 14-year-old students, WS1 and WS2, undertook the problem in Figure 4.

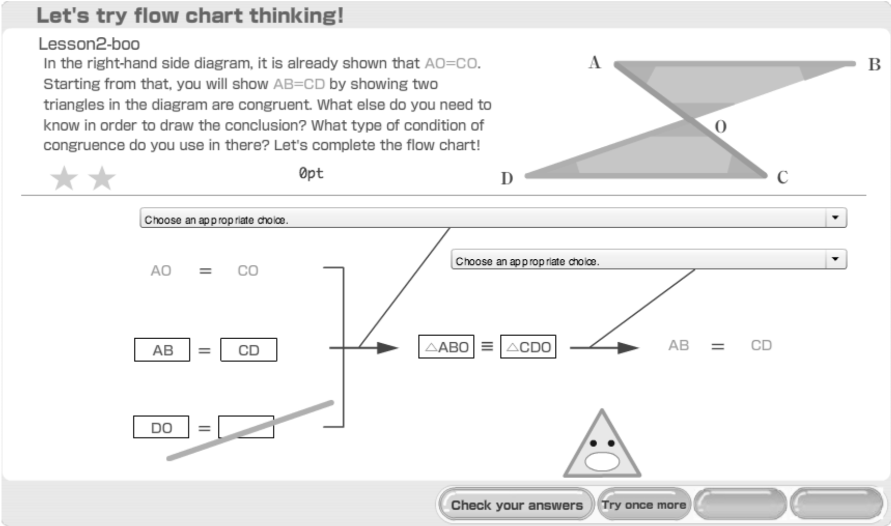
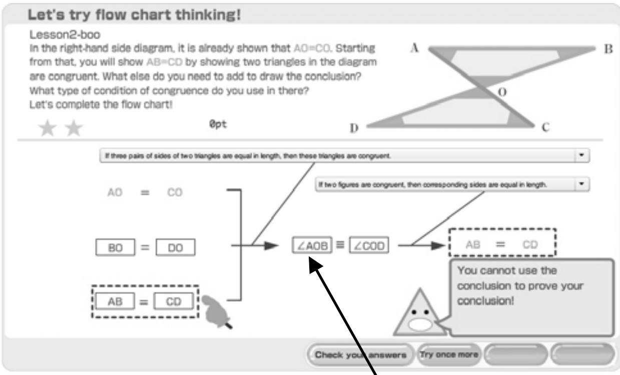


Figure 4: System interface and Lesson 2-b00

Without any hesitation, their first attempt involved using the SSS condition as follows (I: Interviewer)

50	WS2	That one and that one (BO and DO)? That one looks bigger than that one. Is it that one (∠AOB=∠COD)? [student chooses SSS condition, and checks answer] No.



51	WS1	I don't think that angle is … [indicating]
52	I	What does it say?
53	WS2	[Reading the hint] You cannot use the conclusion to prove your conclusion.
54	I	What do you want to prove?
55	WS1	We want to prove that the three pairs of sides …..I don't know, I am really confused.

They made a mistake (line 50) as they put ∠AOB=∠COD are congruent, rather than ΔOAB and ΔOCD. More importantly, they failed to notice that they should not use 'AB=CD' in their proof. This is evidence that they did not have good understanding of universal instantiation (line 55) or of logical circularity (line 50). The system highlighted the use of logical circularity by showing a box saying "you cannot use the conclusion to prove your conclusion". After receiving this hint from the system, and with additional support from the interviewer, the students started considering that 'AB=CD' should not be used in their proof. With this they began to understand, as shown below in the dialogue immediately below, why AB=CD should not be used.

86	WS1	It is the same as that. [reviewing WS2's answer] You have done AB=CD again!
87	WS2	Why can't we do that?
88	WS1	Because it is the same conclusion.

After realising that AB=CD should not be used, they finally constructed a correct proof. Nevertheless, the above example illustrates that understanding the meanings and roles of premises and conclusions are difficult for learners who have just started learning mathematical proof. Moreover, from the structure of proof point of view, our evidence shows that learners who cannot see the whole structural relationships between premises and conclusion (namely that they are not at the Holistic structural level) cannot identify the logical circularity. In order to identify logical circularity as a serious error, learners need to understand at least the role of syllogism which connects premises with conclusions. It means learners need to understand the aspect of syllogism included in the relational Partial-structural relational sub-level (see Fig 3).

Case 2: in the episode below, student R, a first year student on a primary teacher training course, first considered that it would be possible to use SSS condition as a way to tackle the open problem to prove 'AB=CD'. This indicates that R is lacking understanding of logical circularity. After making several mistakes, including logical circularity, student R finally reasoned why it was not possible to use SSS (see lines 34 - 40 below). This shows that student R was in the upper level of the Partial-structural relational level involving the understanding the aspect of syllogism at least.

34	R	I don't think anymore answers.
35	I	Are you confident to say so?
36	R	Yes.
37	I	If you choose ∠AOB&∠COD, and ∠ABO&∠CDO, then…
38	R	We need to use BO and DO but …
39	I	No, we can't use them as AO=CO is already assumed. Also we can't use AB=CD, because this is…
40	R	What you are trying to find! [laughs]

Case 3: J1 and J2, two first year students on a primary teacher training course, are towards the end of their work on the proof problem. In the extract below, they are not only considering why they cannot use the SSS condition for the problem (lines 149-151 below), but also eliminating other possibilities for answers (lines 152-157). This illustrates their capacity to identify logical circularity in proofs, and that their understanding of structure of proof is almost at the Holistic structural level as there is

evidence that they have started grasping the relationship between premises and conclusion.

147	J1	Um, try again?
148	J2	You could do all the …
149	J1	All the sides?
150	J2	Yes… actually no, because..
151	J1&J2	You are trying to prove [AB=CD] …
152	J1	And if you can't use this line [AB] then we can't use the other angle… because it is not included…
153	J2	You mean those [∠ABO&∠CDO]?
154	J1	Yes, it is not included [as AB cannot be used]… and we've already got others…
155	J1	How about AO-∠OAB-AB?
156	J2	You cannot use these, because…
157	J1	Because these ones [AB&CD] which we are trying to prove…

This example shows that students J1 and J2 could overcome the logical circularity gradually by considering possible combinations of premises and conclusion and checking whether their proof fell into logical circularity or not. This might mean that the kinds of activity available with the web-based flow-chart proof system are useful to understand the whole structural relationship between premises and conclusions more deeply, to encourage learners to shift the level of the understanding of proof structure, and that this may lead to them, in the end, overcoming the error of logical circularity.

CONCLUSIONS

The selected episodes presented in this paper illustrate how learners who have just started learning to construct mathematical proofs make various mistakes, including using a conclusion to prove the same conclusion. Our conjecture is that the cause of this is their incomplete understanding of whole structure of proof, especially their lack of understanding of the role of syllogism. The web-based learning environment with its open problem situations using flow-chart-type proof, as we show in this paper, can reveal learners' naive status of understanding, in particular their lack of understanding of syllogism (for example, cases WS1 and WS2, and R).

While it is appears difficult for learners to consider why logical circularity cannot be used in a proof (see the example of WS1 and WS2), to overcome such difficulties it is important for teachers to encourage learners to attend to the structural relationships between premises and conclusion and how they could be bridged (via syllogism). As support, the feedback supplied by the web-based proof system provides guidance on what help might be given learners to help develop their understanding. By focusing on the structural aspects of a proof, the learners start bridging the gap in their logic in syllogism (see example R, lines 34 - 40). For some learners (for example, J1 and J2), by using the open problem situation, logical circularity is eliminated by considering possible combinations of premises and conclusion (see case J1 and J2, lines 141-157). This suggests that both considering possible combinations of premises and conclusion, and checking whether the proof falls into logical circularity or not, are useful for overcoming errors of logical circularity.

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