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INTRODUCING THE STRUCTURE OF PROOF IN LOWER SECONDARY SCHOOL GEOMETRY: A LEARNING PROGRESSION BASED ON FLOW-CHART PROVING

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This paper reports on a learning progression based on flow-chart proving and aimed at providing a basis for introducing the structure of proof in lower secondary school geometry. The proposed learning progression has three phases: constructing flow-chart proofs in an open situation, constructing formal proofs by reference to flow-chart proofs in a closed situation, and refining formal proofs by placing them into flow-chart proof format in a closed problem situation. Through teaching this progression in three Grade 8 classrooms (students aged 14), and assessing the learning through a test administered several months later, our evidence indicates that students who studied proof and proving with this learning progression were better able to construct a proof than other students who did not follow this approach.

proof, flow-chart proving, learning progression, geometry, lower secondary school

INTRODUCTION

Even though the teaching and learning of proof is universally recognized as a key element of mathematics curricula, it remains the case that students at the lower secondary school level can experience difficulties in understanding proof (eg: Hanna & de Villiers, 2012; Mariotti, 2006). Improving instructional approaches is one strategy and in our research we focus on the introductory phases of the teaching of proofs and proving.

Based on the idea of learning progressions as “successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic” (National Research Council, 2007, p. 214), we designed a learning progression for the introductory phases of learning how to structure proofs in lower secondary school geometry. The purposes of this paper are twofold: a) to provide a theoretical outline of some of the design principles underpinning the learning progression on how to structure proofs in lower secondary school geometry, and b) to evaluate the effects of implementing these lessons in terms of knowledge and understanding which students gained from the lessons.

THEORETICAL UNDERPINNINGS

The theoretical underpinnings that we utilise for designing an introductory stage of learning to construct proofs in geometry in lower secondary school relate to several themes; the nature of proving activities, the design of learning progressions, and the use of flow-chart proofs as a format for geometrical proofs.

Proving as an explorative activity

In considering proving as an explorative activity, we refer to several theoretical positions: views on the relative nature of mathematical truths (Fawcett, 1938), the heuristic and fallibilistic nature of mathematical processes (Lakatos, 1976), and the nature of the proof construction process (McCrone & Martin, 2009). Based on these ideas, we consider that proving activities are flexible, dynamic and productive in nature, and various aspects of proving activities are interrelated and resonant with each other.

We can see that proving activities 'breathe life' into mathematics teaching and learning and are intellectually stimulating in numerous ways, for example: producing propositions inductively/deductively/analogically, planning and constructing proofs for these produced propositions, and reflecting on and looking back at producing propositions, including planning and constructing proofs to overcome local and global difficulties and counter-examples, and then refining propositions and proofs (Miyazaki & Fujita, in press).

Developing learning progressions

The idea of learning progressions, as Empson (2011, p. 574) explains, is "now virtually synonymous with learning trajectory" Given our goal of researching the introductory learning of how to structure proofs in lower secondary school geometry, we borrow from the notion of 'hypothetical learning trajectory' (HLT) that it includes "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon, 1995, p. 133). This set of components can underpin the design of a sequence of teaching; see, for example, Clements and Sarama (2004), Simon and Tzur (2004), Stylianides and Stylianides (2009). For our focus on learning to structure proofs in lower secondary school geometry, our learning progression comprises the following components: a) Learning goals: by the end of the teaching, students will be able to i) plan and construct a proof in geometry, and ii) understand the structure of proofs in geometry; this entails students beginning to grasp elements of the structure of proof, and then gradually being able to see the entire structure (Miyazaki & Fujita, 2010); b) The learning process and activities: proof construction based on the flow-chart proof format (McMurray, 1978), with both open and closed problem situations; c) The thinking and learning in which the students might engage: this encompasses thinking forward and backward, planning and constructing a proof, reflection of the structure of proofs, and so on.

Flow-chart proving in an open problem situation

A key feature of our learning progression is the use of flow-chart proofs that shows a 'story line' of the proof; beginning with the kinds of assumptions from which the conclusion is deduced, and including the kinds of theorems being used, how the assumptions and conclusion are connected, and so on. As McMurray (1978) and others have suggested, flow-chart proofs can be introduced to students before they learn the more formal 'two column proof' format.

We consider that the power of flow-chart proofs can be particularly enriched in 'open' situations where students can construct multiple solutions by deciding the assumptions and intermediate propositions necessary to deduce a given conclusion.

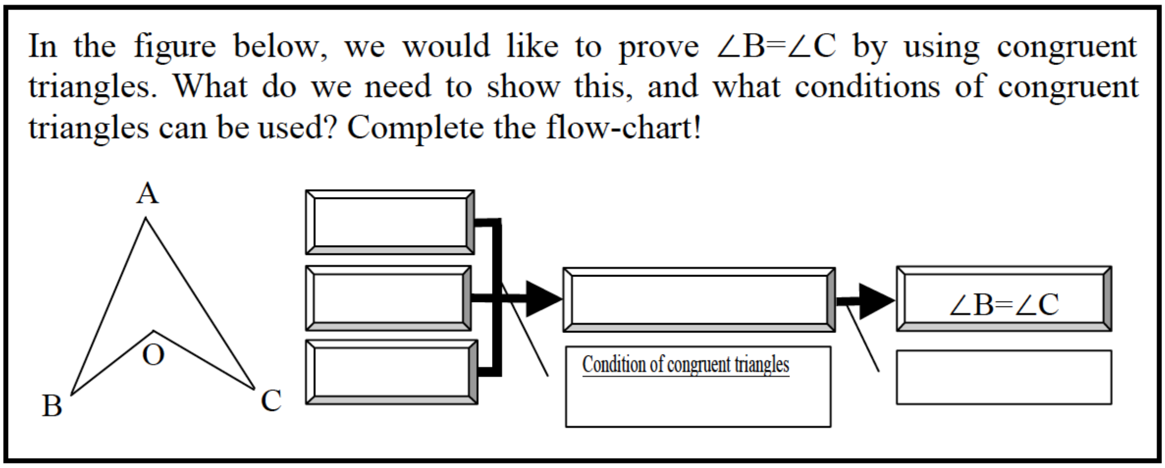


Figure 1: An example of flow-chart proving in an open situation

For example, the problem in Figure 1 is intentionally designed so that students can freely choose which assumptions they use to draw a conclusion ∠B=∠C. After drawing a line AO, for instance, students might think backwards from the conclusion to decide which triangles should be congruent to show ∠B=∠C, and what condition of congruent figures should be used. Then, they might show that ∠B=∠C by using the theorem "If two figures are congruent, then corresponding angles are equal." However, other solutions are also possible. One might be to use the fact that they have already found AO=AO as a same line and hence that ΔABO≡ΔACO can be shown by assuming AB=AC and BO=CO using the SSS condition. As students can construct more than one suitable proof, we refer to this type of problem situation as 'open'.

One purpose of flow-chart proving in an open problem situation is to encourage students to think backward from the problem conclusions. Over time, students are expected to connect two directions of thinking; thinking forwards from the assumptions, and thinking backwards from the conclusions. During this process it can happen that students produce a proof with logical circularity. Our research findings suggest that confronting logical circularity provides a chance to explore the reason why this form of argument is unacceptable and leads to students becoming more aware of their own understanding of the structure of proof (Miyazaki and Fujita, 2010; Fujita, Jones and Miyazaki, 2011).

LEARNING PROGRESSION FOR PROOF STRUCTURE AND CONSTRUCTION

Three phases of the teaching of proof construction

Our learning progression for the introductory learning of proof structure and construction by using flow-chart proof in Grade 8 (aged 14), informed by the theoretical underpinnings laid out above, has the following three phases: constructing flow-chart proofs in an open situation, constructing a formal proof by reference to a flow-chart proof in a closed situation, refining formal proofs by placing them into a flow-chart proof format in a closed situation. We explain the reasons for these phases below.

In the first phase, students construct flow-chart proofs in an open problem situation. Since students at this very early stage of learning about proofs might see a formal proof as a rather meaningless set of symbols about the properties of geometric shapes, students may not understand why they should engage in such mathematical arguments. In particular, they may have difficulty in connecting the problem assumptions to the problem conclusion in a deductive fashion. Through their activity in the first phase of our proposed learning progression, they are expected to learn how to think forwards/backwards between assumptions/conclusions as they construct their proof. They also are encouraged to organize their thinking in order to connect assumptions and conclusions. Thus this phase can support them to understand how to 'assemble' a proof as a structural entity.

In the second phase, students first construct a flow-chart proof in a closed situation (similar to the typical form of proof problem that appear in textbooks). Next they construct a formal proof through transposing a flow-chart proof into a paragraph proof. Through their learning in the previous phase (where they constructed flow-chart proofs) they have a better understanding of the structure of proof and develop the capability to think forwards/backwards between assumptions/conclusions. In the second phase, students are expected to transform their flow-chart proof into a paragraph proof. Here, students need to learn how to use assumptions, and the 'arrows' of flow-chart proof are replaced by mathematical language.

Finally, in the third phase students first construct paragraph proofs in closed problem situations, and then refine their proofs by placing them into flow-chart proof format. The reason for this third phase is that although students get familiar with constructing paragraph proofs gradually by the end of second phase, they usually make some mistakes in their paragraph proofs. By translating paragraph proofs into flow-chart proof format in this phase, the intention is that students spot their mistakes related to the structure of proof, and they learn to look back over proofs, correct any mistakes, and make proofs better by themselves.

Nine lessons as an introductory stage of the teaching of the structure of proof

By using our proposed learning progression of flow-chart proving, we designed nine lessons taking into account open/closed situations, varying steps of deductive reasoning, and different problems and contexts. Our design, developed in cooperation with expert mathematics teachers, included detailed teaching guidelines, together with worksheets for students' activity. The nine lessons are summarised in Table 1.

Table 1: Learning progression across nine lessons

Phase	No.	Open/closed situation	Steps in reasoning	Problems and contexts
Constructing flow-chart proofs	1	Open	1	Constructing flow-chart proofs by using congruency of two triangles connected by a vertex
	2	Open	1	Constructing flow-chart proofs by using congruency of two triangles connected by a side
	3	Open	1	Constructing flow-chart proofs using congruency of two triangles connected by a vertex, while clarifying figural properties as assumptions
	4	Open	2	Constructing flow-chart proofs using congruency of two triangles connected by a side, while clarifying figural properties as assumptions
Constructing a formal proof by reference to a flow-chart proof	5	Closed	1	Constructing a formal proof with a flow-chart proof using the problem from Lesson 3
	6	Closed	2	Constructing a formal proof with a flow-chart proof using the problem from Lesson 4
Refining formal proofs by placing them into flow-chart proof format	7	Closed	2	Constructing a formal proof and refining it in the flow-chart format using the problem of Lesson No.6
	8	Closed	2	Constructing a formal proofs by using congruency of two triangles overlapping each other and refining it with the flow-chart format
	9	Closed	2	Constructing a formal proofs by using properties of parallel lines and congruency of two triangles connected by a vertex and refining it with the flow-chart format

EVALUATING THE LEARNING PROGRESSION

Methods

In a widely-used Japanese 8th Grade textbook (for 14 years old), authorized by the Ministry of Education, there are three main sections of geometry: 1) properties of parallel lines and angles, properties of congruent figures, and conditions of congruent triangles through informal proofs; 2) what is a formal proof and how to construct it; 3) properties of triangles and quadrangles by using formal proofs. The suggested plan is that the 1st, 2nd and 3rd sections need 11, 4, and 15 lessons, respectively. Our learning progression described above is designed to replace the second section. As a result, the second section needed five additional lessons. The first section and the third section remained the same.

From October 2010 to January 2011, when most 8th Graders in Japan learn geometry, three mathematics teachers in a state-funded lower secondary school implemented our sequence of nine geometry lessons in accordance with our learning progression. Each teacher taught one class. In total, 94 students were taught. Afterwards they implemented the usual lessons on the properties of triangles and quadrangles using the textbook that is widely used in Japan.

Our assessment was conducted in May 2011 (when the students were learning algebra), some four months after the teaching of the nine lessons. We used test items from the Japanese National Survey that was conducted for all students in Japan in April 2009. The National Survey consisted of two sets of problems: Math A and Math B. The first set of problems, Math A, checked basic knowledge and skills. The second section, Math B, checked advanced mathematical thinking in the real world and the mathematical world (for an example of a Math B problem, see Figure 2). In order to compare our result with the National Survey, we used both sets of questions and allocated the same time for our students to answer. Furthermore, to ensure the quality of the assessment, the marking of our survey was conducted by the same organization that marked the National Survey.

Results and discussion

As shown in Table 2, in terms of the basic geometry problems in Math A (questions especially related to the content for 8th Graders) there was no difference between our sample and National survey. This indicates that the nine lessons based on our learning progression did not impact on the development of the students' basic knowledge and skills.

Table 2: Test results for basic geometrical problems in Math A

<table>
<tr><th colspan="2" rowspan="2">Question number and topic</th><th colspan="2">Proportion of correct answers (%)</th></tr>
<tr><th>Our sample</th><th>National Survey</th></tr>
<tr><td>6(3)</td><td>Choose a congruent triangle with a given triangle</td><td>66.0</td><td>64.7</td></tr>
<tr><td>6(4)</td><td>Answer the degree of a circular angle</td><td>56.4</td><td>59.6</td></tr>
<tr><td>6(5) 1</td><td rowspan="2">Choose appropriate assumptions to complete a proof that shows the sum of inner angles of triangle is 180 degrees.</td><td>79.8</td><td>78.6</td></tr>
<tr><td>6(5) 2</td><td>86.2</td><td>82.8</td></tr>
<tr><td>7</td><td>Write a condition of parallelogram with using symbols according to a figure.</td><td>60.6</td><td>57.3</td></tr>
<tr><td>8</td><td>Understand a diagram used in a proof</td><td>56.4</td><td>57.6</td></tr>
</table>

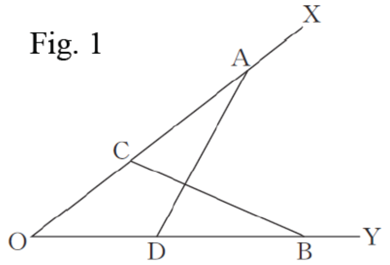
In contrast, the impact of our learning progression using flow-chart proofs is shown by the results the students obtained on the advanced problems in Math B (the part of the test that aimed at checking if students could construct a formal proof based on a suggested plan of proving); see Figure 2 for the test questions, and Table 3 for the results.

Takuya is trying to solve the following problem.

Problem

In fig. 1, let us take points A, B, C and D on OX and OY of ΔXOY so that OA = OB and OC = OD. When A and D, and B and C are connected, prove AD = BC.

Fig. 1

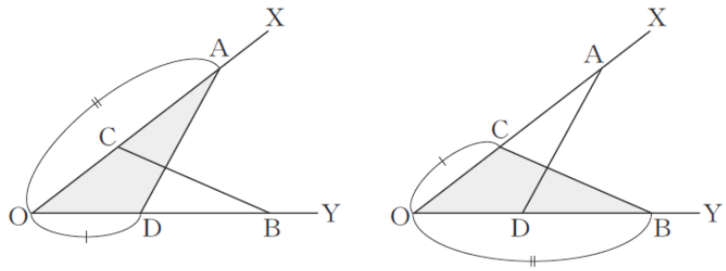


Takuya described his plan to prove it as follows.

Takuya's memo

#1 To prove AD = BC, it is enough to show ΔAOD ≡ΔBOC.

#2 When I see ΔAOD and ΔBOC of Fig.1 more clearly, I can divide it into two parts and show what are assumed as follows.



#3 Based on #2, I think I can prove ΔAOD ≡ΔBOC.

(1) Which property should be used to say 'in order to prove AD=BC, it is enough to show ΔAOD ≡ΔBOC' as seen in #1 of Takuya's memo? Choose from a)-d).

a) In congruent figures, corresponding sides are equal.
b) In congruent figures, corresponding angles are equal.
c) In congruent figures, perimeters are equal.
d) In congruent figures, areas are equal.

(2) Prove AD＝BC of **Problem**.

Figure 2. Advanced geometry problems for Grade 8 students

Table 3: Results of the advanced geometry problems for Grade 8 students

	Question 1	Question 2			
	Correct (%)	Correct (%)			No answer (%)
		Complete proof	Incomplete proof	Total	
Our sample	73.3	44.7	4.3	48.9	21.3
National survey	63.3	34.2	9.1	43.3	28.6

In the Math B test items (see Figure 2), Question 1 checks if students can reason backwards from the conclusion AD=BC or not. As the data in Table 3 show, 73.3% of the students in our sample answered correctly. This is 10% higher than the national average of 63.3%. This result shows that more students in our sample can identify what would be necessary to deduce the conclusion as seen in section 1 of 'Takuya's memo'. We consider that this positive result is due to our students' experience with flow-chart proofs in open situations in the first phase of our learning progression. In this phase, students learn to complete a flow-chart proof and experience thinking forwards/backwards between assumptions/conclusions. In this way, the students in our teaching experiment have more experience in planning a proof by finding which properties can be used as assumptions in open problems.

Question 2 asks students to construct a formal proof by reference to the plan of 'Takuya's memo'. This plan shows that $\Delta AOD \equiv \Delta BOC$ is adequate to deduce the conclusion AD=BC (#1), that each of two pairs of sides (as the given conditions) are equal (#2), and that it might be possible to deduce $\Delta AOD \equiv \Delta BOC$ from the given conditions (#3). As the data in Table 3 shows, 48.9% of the students in our sample answered correctly; this is 5.6% higher than the national average, 43.3%. Furthermore, 21.3% of our sample gave no answer, which is 7.3% lower than the national average of 28.6%.

When we examined the quality of answer to Question 2 more closely, the correct answers are divided into two categories. Category 1 includes complete answers that provided correct reasons (e.g. OA=OB because this is an assumption) and the appropriate theorems to be used (e.g. congruent conditions of triangles). Category 2 includes the correct answers without these details. The data in Table 3 shows that the proportion of students who answered Question 2 completely was greater in our sample (at 44.7%) than in the National survey (at 34.2%).

These results suggests that our learning progression has an effect on increasing the quality of students' proof construction as they can express more precisely what reasons and theorems would be necessary to complete a proof. We consider this improvement is due to the use of flow-chart formats through our learning progression. The flow-chart format influenced students to pay more attention to the structural elements of proofs, and their relationships, both when they construct flow-chart proofs and when they construct formal proofs by reference to flow-chart proofs.

CONCLUDING REMARKS

Based on the evidence presented in this paper, and as a tentative conclusion that needs further replication, we consider that the teaching of proof and proving with flow-chart proof based on our learning progression shows encouraging results as an introductory form of instruction. In particular, our survey results indicate that students who learnt proofs and proving in geometry by following our learning progression are more likely to plan a proof better and construct a proof in accordance with their plan. This is due to a) their experience with open problems that encourage them to think backwards/forwards to seek assumptions and conclusions in proofs, and b) they could grasp structure of proofs better through using flow-chart proofs.

In taking our research forwards, we are focusing on two things. One focus is on why our learning progression works well in helping to develop students' proof planning skills. To address this issue we have qualitative data (in addition to the quantitative survey data) on how our students learn flow-chart and formal proofs within our learning progression - this learning being akin to Simon's (1995) use of the term 'Actual Learning Trajectory' (see also Simon & Tzur, 2004).

A second focus is providing students with more opportunities to construct flow-chart proofs in open situations (as used in the first phase of our learning progression). To do this are developing a web-based learning platform, www.schoolmath.jp/flowchart_en/home.htm (available in both Japanese and English) that provides flow-chart proof tasks for use in teaching our learning progression (see Miyazaki, Fujita, Murakami, Baba & Jones, 2011). Figure 3 provides an example of an open problem situation from our web-based learning platform. Initial trialling is suggesting that using the web-based learning platform, and the feedback it provides, can promote students' productive and flexible thinking as they learn to prove.

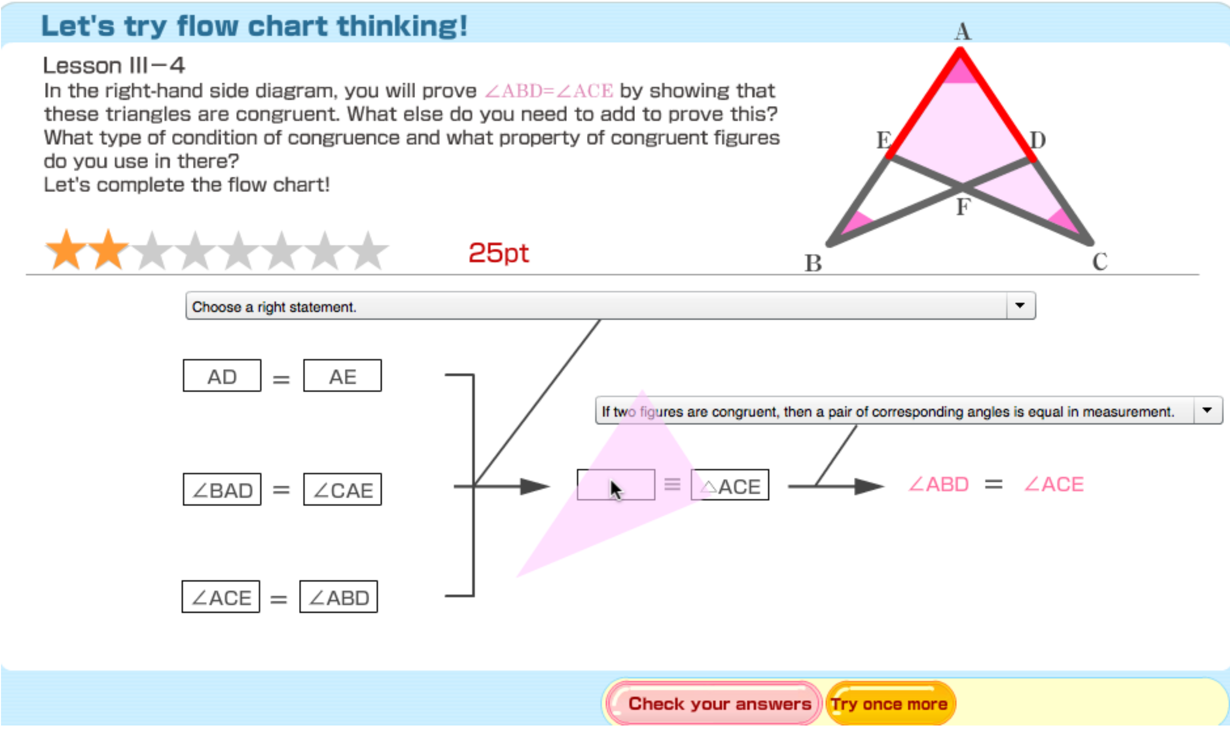


Figure 3. Example task from a web-based learning platform for flow-chart proving

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REFERENCES

Clements, D.H., & Sarama, J. (2004). Learning trajectories in mathematics education, *Mathematical Thinking and Learning*, *6*(2), 81-89

Empsom, S.B. (2011). On the idea of learning trajectories: promises and pitfalls, *The Mathematics Enthusiast*, *8*(3), 571-596.

Fawcett, H.P. (1938). *The Nature of Proof*. (13th NCTM Yearbook), New York: AMS Press.

Fujita, T., Jones, K., & Miyazaki, M. (2011). Supporting students to overcome circular arguments in secondary school mathematics. In Ubuz, B. (Ed.), *Proc. 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 353-360). Ankara, Turkey: PME.

Hanna, G., & de Villiers, M. (Eds.) (2012). *Proof and Proving in Mathematics Education: the ICMI Study*. New York: Springer.

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge: Cambridge University Press.

Mariotti, M.A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, present and future* (pp.173–204). Rotterdam: Sense.

McCrone, S.M.S., & Martin, T.S. (2009). Formal proof in high school geometry: student perceptions of structure, validity and purpose. In M. Blanton, D. Stylianou, & E. Knuth (Eds.) *Teaching and Learning Proof across the Grades* (pp. 204–221). London: Routledge.

McMurray, R. (1978). Flow proofs in geometry. *Mathematics Teacher*, *71*, 592–595.

Miyazaki, M., & Fujita, T. (2010). Students' understanding of the structure of proof: Why do students accept a proof with logical circularity? In Y. Shimizu, Y. Sekiguchi, and K. Hino (Eds), *Proceedings of EARCOME5* (pp. 172-179). Tokyo, Japan: EARCOME.

Miyazaki, M., Fujita, T., Murakami, Y., Baba, N. & Jones, K. (2011). Secondary school mathematics learners constructing geometric flow-chart proofs with a web-based learning support system. In M. Joubert, A. Clark-Wilson, & M. McCabe (Eds.), *Proc.10th International Conference on Technology in Mathematics Teaching* (pp. 45-46). Portsmouth: ICTMT.

Miyazaki, M., & Fujita, T. (in press). Proving as an explorative activity in mathematics education. In B. Sriraman, & et al. (Eds.), *The first sourcebook on Asian research in mathematics education*. Charlotte, NC: Information Age Publishing.

National Research Council (2007). *Taking Science to School: learning and teaching science in grades K-8.* [Duschl, R.A., Schweingruber, H.A. & Shouse, A.W. (Eds.)]. Washington, DC: National Academy Press.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 114–145.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Stylianides, G.J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314-352.