Cite as: Jones, K., Fujita, T., & Kunimune, S. (2012). Promoting productive reasoning in the teaching of geometry in lower secondary school: Towards a future research agenda. *Pre-proceedings of the 12th International Congress on Mathematical Education* (ICME-12) (pp. 2387-2396). Seoul, Korea: ICMI.

12th International Congress on Mathematical Education

Topic Study Group 10

8 July – 15 July, 2012, COEX, Seoul, Korea

PROMOTING PRODUCTIVE REASONING IN THE TEACHING OF GEOMETRY IN LOWER SECONDARY SCHOOL: TOWARDS A FUTURE RESEARCH AGENDA

Keith Jones — University of Southampton, UK — d.k.jones@soton.ac.uk

Taro Fujita — University of Plymouth, UK — taro.fujita@plymouth.ac.uk

Susumu Kunimune — Shizuoka University, Japan — ecskuni@ipc.shizuoka.ac.jp

In the teaching of geometry an important aim is for students to develop ways of reasoning and proving which means that they can tackle complex problems productively. In this paper we synthesise selected recent studies in the teaching and learning of geometry, focusing on the student age range of 12-15. We do this in order to suggest questions for future research. The issues that we identify focus on issues of mathematical definitions, mathematical representations, and the form of instruction used by teachers. No doubt there are other relevant issues that could be identified; plus there is the likelihood that some or all such issues overlap and interact. Undertaking suitably-designed empirical work is a next step, in conjunction with the development and refinement of suitable theoretical ideas. Building on opportunities for further international research networking would be helpful in taking forward such research on the teaching and learning of geometry.

geometry; reasoning; definitions; representations; pedagogy; theory

INTRODUCTION

The teaching of geometry provides not only a key vehicle for developing learners' spatial thinking and visualisation skills in mathematics, but also a major opportunity to develop their capability with deductive reasoning and proving (Battista, 2007; Royal Society, 2001). Even so, it remains uncertain what classroom factors might trigger productive mathematical reasoning and proof in school geometry lessons. In this paper, we select several themes relating to the teaching and learning of proof in school geometry which we identify from existing research as issues that influence the development of students' geometrical reasoning in the mathematics classroom. Our purpose is to look towards a future research agenda by identifying research questions that it would be useful to investigate in future research.

We concentrate on three issues: mathematical definitions, mathematical representations, and the form of teacher's instructions. We focus on these issues because a) definitions are essential to mathematical reasoning, b) in geometry lessons, geometrical shapes are represented in various ways and by various means (such as representing 3-D objects by 2-D representations), and c) the form of instruction used by teachers during their lessons is known to impact in various ways on students' reasoning.

In the next section of the paper we introduce the idea of productive reasoning processes in geometry. We do this in order to focus on what can help to promote such reasoning.

Following this we identify the issues of mathematical definitions, mathematical representations, and the form of teacher's instructions as one worthy of further research.

PRODUCTIVE REASONING PROCESSES IN GEOMETRY

Developing students' ideas about proof and proving in geometry

The role that geometry teaching can play in developing students' ideas about proof and proving can be illustrated by the learning progression in general use in lower secondary schools in Japan:

- In Grade 7, students (aged 12-13) study selected properties of plane and solid figures informally, but logically, to establish the basis of the learning of proof
- In Grade 8, students (aged 13-14) are introduced to formal proof through studying properties of angles, parallel lines, and congruent triangles, during which they learn the structure of proofs, how to construct proofs, and how to explore and prove properties of triangles and quadrilaterals
- In Grade 9, students (aged 14-15) study similar figures, properties of circles, and the Pythagorean theorem, drawing on their consolidated capacity to use proof in geometry

In the case of Japan, this learning progression applies to *all* students, not just the more capable students. As such, while many students in Japan are successful in learning about proof, many have difficulties. The 2008 Japanese national survey, for example, reported that about 44% of Japanese Grade 9 students could analytically/synthetically plan and construct proofs (NIER, 2008); this means that over half of students could not.

In relation to this, we have reported that while most Grade 9 students (aged 14-15) can write down a proof, around 70% do not understand why proofs are needed (see Kunimune, 1987; Kunimune, Fujita & Jones, 2009; 2010). In this research, we capture students' understanding of proof in terms of two components: these are 'Construction of proof' and 'Generality of proof'. The first of these, 'Construction of proof in geometry', recognises that, on the one hand, students have to learn how to 'construct' deductive arguments in geometry by knowing sufficient about definitions, assumptions, proofs, theorems, logical circularity, and so on. The second of these two components, 'Generality of proof', recognises that, on the other hand, students also have to understand the generality of proof in geometry, including the universality and generality of geometrical theorems (proved statements), the roles of figures, the difference between formal proof and experimental verification, and so on.

All told, our research reveals that an important aspect of the teaching and learning of proof is that for students there is a difference between being able to construct a proof and understanding why proofs are needed.

Developing productive reasoning process in geometry

Fischbein (1987, p. 41) argued that a "productive reasoning process" aims at solving a "genuine problem". In a later article he argued that in productive reasoning "images and concepts interact intimately" (Fischbein, 1993, p. 144). In geometry lessons we consider the following process (informed by the ideas in Becker & Shimada, 1997 and with some similarities to what Stein *et al.*, 2008, call reform-oriented lessons) is a 'productive reasoning process':

- A problem is first introduced, and then students share various ideas to prove their conjecture
- Students prove their conjecture (this includes cases where, if necessary, the conjecture is modified and then proved)
- Students share their reasoning and proofs and undertake further proving activities

In our research, we have reported on how students can undertake challenging geometrical problems and, through this, generate proofs (e.g. Fujita, Jones & Kunimune, 2010; Fujita, Kunimune, Kumakura & Matsumoto, 2011; Jones, Fujita & Kunimune, 2012).). In this paper our aim is to review a number of the issues that have arisen during our classroom-based studies in order to develop research questions that can be addressed in future research. The issues we highlight are concerned with mathematical definitions, mathematical representations, and the form of teacher instruction during classroom lessons. While none of these issues is entirely new, our suggestion is that much more is now known about each of these issues so that this is an opportune time to consolidate what is known and look to devising a research agenda to take forward research on these matters.

DEFINITIONS, REPRESENTATIONS, AND TEACHER INSTRUCTION

Definitions in geometry

There is no doubt that mathematical definitions play a central role when exploring, developing and teaching mathematics (Vinner, 1991). In school mathematics, various definitions are introduced to students and it is expected that they utilise such definitions as they develop their mathematical reasoning. For example, a parallelogram might be defined as 'a quadrilateral which has two pairs of parallel lines'. By starting from this definition, various properties can be deduced, as can relationships between other quadrilaterals (for example, a square can be regarded as a parallelogram as it also has two sets of parallel lines). Yet, as Freudenthal (1971, pp. 424) pointed out "Though the teacher can impose definitions…, this means degrading mathematics to something like spelling, ruled by arbitrary prescriptions" (see, also, Freudenthal, 1973, chapter XVI).

The issue of whether to teach definitions in geometry, or to teach students how to define, was taken up by de Villiers (1998). From his study of secondary school students, de Villiers concluded that students should be actively involved in formulating and evaluating definitions. This might mean, according to de Villiers, working with less formal definitions (what he called visual, uneconomic and economic definitions) in the earlier stages of education as these would be more meaningful to students as they progress from elementary through secondary education. Informed by the work of Lakatos (1961, p. 54) that "a definitional procedure is a procedure of concept formation", Ouvrier-Buffet (2006) offers a theoretical perspective for what she calls "definition construction processes". In a follow-up paper (Ouvrier-Buffet, 2011, p. 179), she concludes that "it remains difficult to design didactical situations involving defining processes" because "an important part of learning mathematics is actually to become aware of the importance of defining just like the importance of proving”.

In much school geometry teaching it remains that case that Euclidean-type definitions continue to be used. For example, a square might be defined as ‘a quadrilateral with equal sides and with right-angles’. Yet a square can be defined in other ways; say as ‘a quadrilateral whose diagonals are the same length and that bisect each other at 90 degrees’, or ‘a quadrilateral with four lines of symmetry’, and so on. In line with other researchers (see, for example, Koseki, 1987) we have found in our research (e.g. Fujita & Jones, 2007; Fujita, 2012; Kunimune, 2000) that learners experience difficulties with definitions in geometry. For example, we have found that even those students who can provide correct definitions of some simple geometrical shapes may have difficulties in accepting, for instance, that a square is also a parallelogram because they may argue that ‘parallelograms do not have right angles’.

All such research indicates that students’ reasoning and proving processes can depend on what definitions are provided for them during classroom activities. This means that questions remain about the mathematical definitions that can be used when formulating geometrical problems for lower secondary school students and how such different definitions might be introduced by the teacher in various problem solving contexts.

Representations in geometry

While many physical objects have a geometrical shape which can be examined by the human senses (i.e. they can be touched, seen, constructed, and so on), geometrical shapes are also abstract mathematical objects. For example, a cube can be represented by a physical object, a 2D sketch on the blackboard, or a virtual dynamic object on a computer screen, and so on. According to Mesquita (1998, pp183-187):

- Representations can be *external* (“embodied materially on paper or other support”) and *iconical* (“centered on visual image”)
- An *external representation* can have a double status of “finiteness” (in the sense of “finite and diversified forms”) and “ideal objectiveness” (a detachment from the material constrains linked to the external representation)
- Some *external representations* are more typical than the others (because the typicality of a representation “results from the fact that individuals more easily associate some external representations than others to a given problem or situation”)

In geometry lessons, external representations can be used when introducing theorems, posing problems, explaining proofs, and so on. Yet such representations might be perceived by students in ways that are different to what a teacher expects (note that miss-matches between teacher and students ideas in geometry are recognised as problematic by the van Hiele model; see, van Hiele, 1986). In an example provided by Mesquita, even though a teacher drawing a triangle on a board might intend it to show general properties of triangles, students might consider it a ‘particular’ triangle with, say, specific side lengths such as 3, 4 and 5. As Mesquita (1998) notes, some representations are more ‘typical’ than others and this might result in various cognitive differences amongst learners.

In our classroom-based research we have examples of how some representations seem more ‘typical’ to students than other representations. For example, as we report in Fujita et al. (2011) and Jones et al. (2012), we conducted classroom-based studies in which Grade 9 students were asked to determine the shape of quadrilateral DPFQ in a cube (P and Q are the mid-points of AE and CG respectively) given the representation drawn on the blackboard shown on the left-hand side of Figure 1.

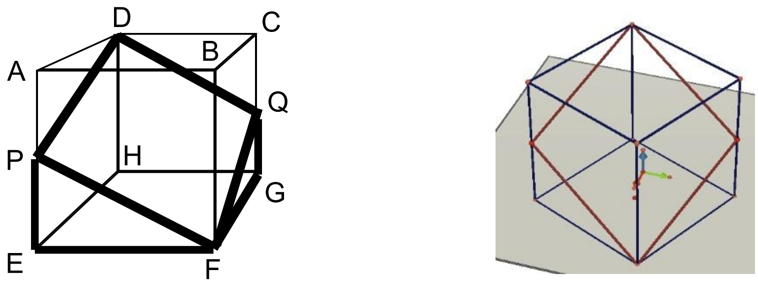

Figure 1. Different orientations of the representation of a cube

In our research we found that some students initially said that DPFQ is a square (DPFQ is, in fact, a rhombus). In our study, the orientation of the cube in the representation shown on the left-hand Figure 1 was deliberate. This raises the question, of course, of how a differently-orientated representation of the cube, or perhaps a representation constructed using 3D software such as Cabri 3D (as in the right-hand section of Figure 1), would impact on students reasoning. Would some still initially say that DPFQ is a square?

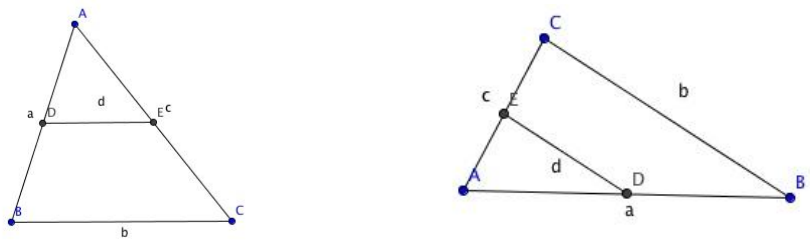

Figure 2. Different orientations of triangles illustrating the mid-point theorem

Another example that arose in another of our classroom-based studies (yet to be published) involves how the mid-point theorem might consistently be introduced to students with a particular orientation (see the left-hand side of Figure 2). As such, students who are used to the orientation used in the left-hand side of Figure 2 might not be able to recognise the theorem in the right-hand section of Figure 2. In the sense of the ‘geometrical eye’ (Godfrey 1910; Fujita and Jones, 2003), the students cannot ‘see’ the theorem in the right-hand figure, and hence they are not able to utilise it in their proving and problem solving.

The issue this raises for us is how the reasoning that would be explored by the students during their classroom work would depend on the representation used in the problem presentation. This is not a straightforward issue. As we note above in connection with our studies reported in Fujita et al. (2011) and Jones et al. (2012), we know from our research that it can happen that a somewhat limited representation, given intentionally, can lead students to engage in productive reasoning. For example, we found that the specific students who made an initial conjecture that 'DPFQ is a square' (based on the left-hand representation in Figure 1) were able to modify their conjectures (following small-group discussions and whole classroom sharing) and go on to prove that DPFQ is a rhombus. Such research shows how an intentionally-limited choice of problem representation can entail a task for learners from which, with a suitable instructional strategy by the teacher, productive learning can result. Yet questions remain about how different mathematical representations influence students' decision-making, conjecture-production, and proof construction processes in the classroom, and how can such representations can be utilised by teachers to develop students' productive reasoning process.

Teachers' instructions during geometry lessons

Research in mathematics education is continuing to document how mathematics teachers' instructional interventions impact on children's learning. Yet, as Hiebert and Grouws (2007) explain in their comprehensive review, there remains much to learn about the inter-relationships between teachers' practices and students' learning. Indeed, according to Hiebert and Grouws, robust theories of teachers' instructional practices do not, as yet, exist.

In the teaching of geometry, to promote geometrical reasoning teachers use various instructional techniques and strategies (Herbst, 2006; Jones, Fujita, & Ding, 2004; Jones & Herbst, 2012). In a geometry lesson when students are trying to solve a proof problem, the teacher might suggest to the students where to direct their attention in the problem, which theorems might be used, and so on. In our classroom research we have observed some of the ways in such instructional techniques influence the students' proving process. An example of this taken from Fujita, Jones and Kunimune (2010) was when students were set the challenge of trisecting a line AB (see Figure 3). Successful students did this by the following construction: two equilateral triangles are constructed, the mid-point (D) of AC is taken, and D and E are linked. When the students tried to prove AF:FB=1:2, a teacher might say 'Let's focus on lines AC and EB being parallel' as a way of helping the students to deduce that triangles AFD and BFE are similar.

While, in this case, the teacher's instructional strategy of focusing students' attention on the fact that 'line AC and EB are parallel' might help students prove that they have trisected line AB, this focussing their attention on that fact that lines AC and EB are parallel might restrict other methods which might have been developed by the students.

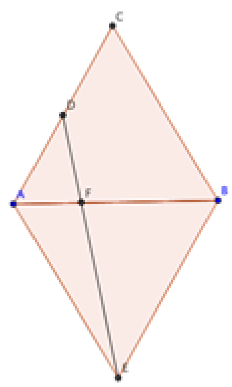

Figure 3. Trisecting a line AB

In some ways this conundrum for the teacher is related to discussions of the efficacy, or otherwise, of teaching by 'telling' (Smith, 1996) and what need Chazan and Ball (1999, p. 10) identify as the need for classroom-based research to identify and understand "what kind of "telling" it was, what motivated this "telling," and what the teacher thought the telling would do", together with ways of "probing the sense that different students make of different teacher moves". Perhaps the dilemma that mathematics teachers face when considering strategies for focusing students' attention on specific aspects of a classroom task is captured by the following paraphrasing of the opening section of the famous soliloquy in Shakespeare's play *Hamlet*:

To say a lot, or not very much, that is the question for mathematics teachers.
Whether 'tis nobler to say too much
and risk overly influencing students so that they cannot think freely
Or say not very much
and risk students failing to learn to prove

With this in mind it is clear that much remains to be researched about how various instructional strategies and interventions by teachers influence and shape students' proving products during the teaching of proof in geometry.

TOWARDS A FUTURE RESEARCH AGENDA

In contemplating future research it is sobering to reflect on the words of Freudenthal (1971, pp. 417-418) "The deductive structure of traditional geometry has never been a convincing didactical success.... It failed because its deductivity could not be reinvented by the learner but only imposed". As this observation by Freudenthal confirms, students' learning processes, and teachers' teaching approaches, in geometry are neither simple nor straightforward. In our research aimed at improving the situation in geometry teaching, we have developed several principles of curriculum design and teaching. While we have obtained positive outcomes from our ideas for developing productive geometrical reasoning among students, new questions have arisen during our research. In this paper we have focused on three selected themes and explored issues that could usefully be studied in future research. We conclude by listing a question relating to each of the three themes:

- What are the mathematical definitions that can be used when formulating geometrical problems for lower secondary school students and how might such different definitions be introduced in various problem solving contexts?
- How do different mathematical representations influence students' decision-making, conjecture-production, and proof construction processes, and how can such representations be utilised by teachers to encourage productive reasoning process?
- What is the influence of different teaching instructions by teachers on students' decision-making, conjecture-production, and proof construction processes?

We are aware that the research of ours that we have reported in this paper relates to studies conducted primarily in classrooms in Japan. Our view is that research studies on the three themes that we have reviewed could contribute to identifying issues in the teaching of geometry in international arenas because the three themes discussed in this paper are essential in the teaching of geometry. No doubt there are other issues, plus there is the likelihood that some or all such issues overlap and interact during the complexity that is the teaching and learning of geometry. Undertaking suitably-designed empirical research is a clear next task; as is the development and refinement of suitable theoretical ideas. One way to take these ideas forward might be to build on opportunities for international research networking in order to promote further research on the issues that we identify in this paper and others that are relevant to improving the teaching and learning of geometry.

References

Battista, M.T. (2007). The development of geometric and spatial thinking. In: F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 843-908). Charlotte, NC: NCTM/Information Age Publishing.

Becker, J. P., & Shimada, S. (1997). The Open-ended Approach: A new proposal for teaching mathematics. Reston, VA: NCTM.

Chazan, D., & Ball, D.L. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2-10.

de Villiers, M. (1998). To teach definitions in geometry or to teach to define? In A. Olivier, & K. Newstead (Eds.), *Proc. 22nd Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 248–255). Stellenbosch, RSA: PME.

Fischbein, E. (1987). Intuition in Science and Mathematics: An educational approach. Dordrecht: Reidel.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139–62.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics, 3*(3&4), 413–435.

Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht: Kluwer Academic Publishers.

Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *Journal of Mathematical Behavior, 31*(1), 60-72.

Fujita, T., & Jones, K. (2003). The place of experimental tasks in geometry teaching: Learning from the textbooks design of the early 20th Century. *Research in Mathematics Education, 5*(1-2), 47-62.

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals, *Research in Mathematics Education, 9*(1-2), 3-20.

Fujita, T. Jones, K., Kunimune, S., Kumakura, H., & Matsumoto, S. (2011). Refutations in lower secondary school geometry. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proc. 7th Congress of the European Society for Research in Mathematics Education* (pp. 660-669). Rzeszów, Poland: ERME.

Fujita, T., Jones, K., & Kunimune, S. (2010). Students' geometrical constructions and proving activities : a case of cognitive unity? In M.M. Pinto & T.F. Kawasaki (Eds.), *Proc. 34th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol 3, pp. 9-16). Belo Horizonte, Brazil: PME.

Godfrey, C. (1910). The Board of Education circular on the teaching of geometry. *Mathematical Gazette, V*, 195-200.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education, 37*(4), 313-347.

Hershkowitz, R. (1990), Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds) *Mathematics and Cognition* (pp. 70-95). Cambridge: Cambridge University Press.

Hiebert, J., & Grouws, D.A. (2007). The effects of classroom mathematics teaching on students' learning. In F.K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 371- 404). Charlotte, NC: Information Age Publishing.

Jones, K., Fujita, T., & Kunimune, S. (2012). Representations and reasoning in 3-D geometry in lower secondary school. In T-Y Tso (Ed.), *Proc. 36th Conf. of the Int. Group for the Psychology of Mathematics Education*. Taipei, Taiwan.

Jones, K., Fujita, T., & Ding, L. (2004). Structuring mathematics lessons to develop geometrical reasoning: comparing lower secondary school practices in China, Japan and the UK. Paper presented at the *British Educational Research Association Annual Conference 2004* (BERA 2004), Manchester.

Jones, K., & Herbst, P. (2012). Proof, proving, and teacher-student interaction: Theories and contexts. In Gila Hanna & Michael de Villiers (Eds.), *Proof and Proving in Mathematics Education* (the 19th ICMI Study) (chapter 11). New York: Springer.

Koseki, K. (Ed.) (1987). *The Teaching of Geometrical Proof*. Tokyo, Meiji Tosho Publishers. [in Japanese].

Kunimune, S. (1987). The study of understanding about 'The significant of demonstration in learning geometrical figure', *Journal of the Japanese Society of Mathematical Education, 69* (supplement 47/48), 3-23. [in Japanese].

Kunimune, S. (2000). A change in understanding with demonstration in geometry, *Journal of the Japanese Society of Mathematical Education. 82*(3). 2-12. [in Japanese]

Kunimune, S., Fujita, T., & Jones, K. (2009). "Why do we have to prove this?" Fostering students' understanding of 'proof' in geometry in lower secondary school. In F-L. Lin, F-J. Hsieh, G. Hanna, & M. de Villiers (Eds), *Proc. ICMI study 19 conference: Proof and Proving in Mathematics Education* (Vol 1, 256-261). Taipei, Taiwan: National Taiwan Normal University.

Kunimune, S., Fujita, T., & Jones, K. (2010). Strengthening students' understanding of 'proof' in geometry in lower secondary school. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds), *Proc. 6th Congress of the European Society for Research in Mathematics Education* (pp756-765). Lyon, France: ERME.

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge: Cambridge University Press.

Mesquita, A.L. (1998). On conceptual obstacles linked with external representations in geometry. *Journal of Mathematical Behavior*, *17*(2), 183–196.

National Institute for Education Policy Research (NIER). (2008). *Reports of National Assessment of Academic Ability over the Whole of Japan*. Retrieved from http://www.nier.go.jp/08chousakekkahoukoku/08chuu_data/houkokusho/05_chuu_bunseki_suugaku.pdf [in Japanese]

Ouvrier-Buffet, C. (2006). Exploring mathematical definition construction processes. *Educational Studies in Mathematics, 63*(3), 259–282.

Ouvrier-Buffet, C. (2011). A mathematical experience involving defining processes: in-action definitions and zero-definitions. *Educational Studies in Mathematics*, *76*(2), 165-182.

Royal Society (2001). *Teaching and Learning Geometry 11-19*. London, Royal Society/Joint Mathematical Council.

Smith, J. P., III. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education, 27*(4), 387–402.

Stein, M.K., Engle, R.A., Smith, M.S., & Hughes, E.K. (2008). Orchestrating productive mathematical discussions. *Mathematical Thinking and Learning*, *10*(4), 313–340

van Hiele, P.M. (1986). *Structure and Insight: A theory of mathematics education*. Orlando, Florida: Academic Press.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65–81). Dordrecht: Kluwer Academic Publishers.