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GEOMETRICAL AND SPATIAL REASONING: CHALLENGES FOR RESEARCH IN MATHEMATICS EDUCATION

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Abstract

In this paper I examine evidence from research to argue that geometry education at the school level needs to attend to two closely-entwined aspects of geometry: the spatial aspects and the aspects that relate to reasoning with geometrical theory. Both of these aspects can be taught, but the challenge for research in mathematics education is to find way in which both geometric and spatial reasoning can be taught in a way that each supports the other.

Keywords: geometry, spatial, reasoning, research, mathematics education

Introduction

In the foreword to a book entitled *The Best Writing on Mathematics 2010*, the great mathematician Bill Thurston (1946 - 2012), in one of his last contributions to mathematics education, wrote:

> "We humans have a wide range of abilities that help us perceive and analyze mathematical content. We perceive abstract notions not just through seeing but also by hearing, by feeling, by our sense of body motion and position. Our geometric and spatial skills are highly trainable, just as in other high-performance activities. In mathematics we can use the modules of our minds in flexible ways - even metaphorically. A whole-mind approach to mathematical thinking is vastly more effective than the common approach that manipulates only symbols" (Thurston, 2011, p. xiii)

This quote captures, in a most elegant way, the themes of this paper: that geometric and spatial reasoning are essential to mathematics and that they can be taught in ways that enhance overall mathematical thinking. The challenge for research in mathematics education is how geometric and spatial reasoning can be taught in a way that supports what Thurston calls the "whole-mind approach to mathematical thinking" (ibid). In this paper I examine evidence from research to argue that geometry education at the school level needs to attend to two closely-entwined aspects of geometry: the spatial aspects and the aspects that relate to reasoning with geometrical theory. These twin aspects of geometry, the spatial and the deductive, I argue, are not separate; rather, they are interlocked. Just as the renowned mathematician Michael Atiyah refers to geometry as one of the two "pillars of mathematics" (Atiyah 2001, p. 657), alongside algebra, I argue in this paper that geometric and spatial reasoning are the *yin-yang* of geometry education in that they are interconnected and inter-dependent in such a way that each gives rise to the other.

In this paper, I first examine the nature of geometrical and spatial reasoning. Then, after a review of research with primary-school pupils, I review issues that impact on learners' progression in spatial and geometrical reasoning through the secondary school years. I conclude by suggesting issues that continue to present a challenge for research and where more evidence is needed. The overall thrust of what I say is adapted from the chapter on spatial and geometrical reasoning that I led for the book entitled *Key Ideas in Teaching Mathematics* due to be published in February 2013 (Watson, Jones & Pratt, in press). Where I can I use evidence from research that I have conducted, often in collaboration with colleagues internationally.

The nature of geometrical and spatial reasoning

A useful definition of geometry is one attributed to the mathematician, Christopher Zeeman: "geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight" (Royal Society, 2001, p. 12). This definition encapsulates what can be thought of as the dual nature of geometry in that it is both one of the most practical and reality-related components of mathematics, and it is an important area of mathematical theory. This means, on the one hand, that geometry can be seen all around us (and is widely utilised in art, design, architecture, engineering, and so on) while, on the other hand, it is simultaneously a theoretical field that allows geometers and other mathematicians, together with cosmologists and other scientists, to work with hypothetical objects in n-dimensional space using, amongst other things, mathematical visualisation techniques with high-powered computers.

The notion of 'figural concept' (Fischbein, 1993; Fischbein and Nachlieli, 1998) captures the combined role of the figural and the conceptual in geometry. This means that in 'seeing' a circle represented on paper, or on a computer screen, what we see is a textual representation of something which is an element of geometrical theory. One way to work with this dual nature of geometry is to distinguish between a ‘drawing’ and a ‘figure’ (Parzysz, 1988) in that, as Laborde (1993, p. 49) explains, ‘drawing refers to the material entity, while figure refers to a theoretical object’. Another way is to follow Phillips et al. (2010, p.3-4) and take a geometric diagram as “an unusual thing in that it is not an abstraction of an experienced object. Rather, it is an attempt to take an abstract concept and make it concrete”. In this sense, the term ‘geometric diagram’ is being used by Phillips (and by Laborde, 2004) to capture the idea that any geometric object that we see is both a material ‘drawing’ *and* a theoretical ‘figure’.

Ever since the time of Euclid’s Elements (the third century BCE, or thereabouts), geometrical reasoning has been synonymous with the deductive method. As such, for the purposes of this paper, I take geometrical reasoning to align with deductive reasoning. In terms of spatial reasoning, this is defined by Clements and Battista (1992, p. 420) as “the set of cognitive processes by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated”. As such, spatial reasoning is a form of mental activity which makes possible the creation of spatial images and enables them to be manipulated in the course of solving practical and theoretical problems in mathematics. This links to visualisation, something which is generally taken as “the ability to represent, transform, generate, communicate, document, and reflect on visual information” (Hershkowitz, 1989, p. 75). Both spatial reasoning and visualisation play vital roles not only in geometry itself and in geometry education, but also more widely in mathematics and in mathematics education (Giaquinto, 2007; Jones, 2001).

In addition to Fischbein’s ‘figural concept’ noted above, influential researchers on the nature of spatial and geometrical reasoning, and its development in learners, include (in chronological order) Piaget, van Hiele, and Duval, amongst others. Here I have no space even to give a brief outline of each; for such detail, see Battista (2007, pp. 846-65). What such research suggests about the nature of spatial and geometrical reasoning is that various types of geometric ideas, both spatial and theoretical, appear to develop over time, becoming increasingly integrated and synthesised. Geometrical ideas symmetry, invariance, transformation, similarity and congruence relate to the more global mathematical ideas of proof and proving. Importantly, an ever-growing strand of research is examining the influence of the use of various classroom artefacts on the development of geometrical and spatial reasoning, especially the impact of computer-based tools (for teacher-oriented reviews, see Jones, 2005; 2012).

Geometrical and spatial reasoning across the primary school years

Research on geometrical and spatial reasoning during the pre-school and primary school years has examined classroom activities that engage learners in visualising, drawing, making, and communicating about two- and three-dimensional shapes (Levenson, et al., 2011; Roth; 2011). During these years, it seems that much geometry teaching focuses on the development of language for shape (for example, the names of polygons) and for location (for instance, left and right). Of course, knowledge of mathematical terminology is essential for modelling, visualising and communicating in all areas of mathematics. Even so, the problem can be that a heavy emphasis on descriptive language and definitions, even if relatively informal, at the expense of geometrical problem solving, might mean that children's progression in geometry during their primary school years is somewhat limited (Clements, 2003, pp.151-2; Jones and Mooney, 2003).

Through primary school, while young children may learn the names of simple shapes (though see below for some cautions regarding the influence of prototypical representations), it can be more difficult for them to recognise the relation between transformed shapes through rotation, reflection and enlargement. For example, primary school children are likely to need a lot of experience with transforming shapes before they are able to complete rotation or reflection patterns. This may be because children's earlier experiences of mathematical shapes focus primarily on enabling them to recognise the same shape whatever its location or size (for example, that a shape is a square no matter what size it is) rather than also helping them to be aware of relevant transformations of the shape. Research also indicates that children experience particular problems with measuring lengths and areas, even though they may understand the underlying logic of measurement. Similarly, learning how to represent angle mathematically is not straightforward for younger children, even though angles occur everywhere in their everyday life. For a very useful summary of such research, see Bryant (2009).

When young children are learning about 2D and 3D shapes, research has documented the ways in which they are likely to do some or all of the following: under-generalise by including irrelevant characteristics that inhibit generalisation, over-generalise by omitting key properties with a result that their generalisation is too wide, and incur language-related misconceptions (for example, that 'diagonal' means 'slanting'). In a summary of such research, Hershkowitz (1990, p.82) shows how, for learners, each geometric object has "one or more prototypical examples that are attained first" that are "usually the subset of examples that had the 'longest' list of attributes of all the critical attributes of the concept and those specific (non-critical) attributes that had strong visual characteristics". For example, learners are much better at recognising isosceles triangles that are 'standing on their base' compared to those presented in a different orientation.

Other issues that learners encounter related to naming shapes (and lines) are linked to matters of definition, and to learners' embryonic understanding of necessary and sufficient conditions, and of inclusivity in defining (see below for more on issues of definition and defining). Examples of such issues include use of terms such as 'oblong' (for a rectangle that is not a square) and 'diamond' (for a specific orientation of a rhombus that is almost certainly a square), and the confusion between 'regular' and 'symmetrical'.

One further thing that research suggests is not always fully taken into account in primary mathematics education is that children come to school with a good deal of knowledge about spatial relations, primarily because we inhabit a spatial world surrounded by spatial objects. This means, as Bryant (2009, p.3) puts it, that "one of the most important challenges in mathematical education is how best to harness this implicit knowledge in lessons". For some examples of how this can be achieved at the primary school level, see Lehrer, et al. (1999). More such research is needed.

Progression in geometrical and spatial reasoning during secondary school

As is clear from what has already been said in this chapter, during their later school years it seems that students continuously move back and forth between what Laborde (1998) calls 'spatio-graphic geometry' (ie spatial reasoning) and 'theoretical geometry' (ie deductive reasoning). This means that when attempting a geometric proof, for example, a secondary school student might move from making conjectures using measures taken from a geometrical drawing, to using definitions and theorems, then go back to the drawing, and so on. This moving between 'spatio-graphic geometry' and ‘theoretical geometry’ relates to the issue of the sometimes uneasy relationship between measuring and proving in geometry.

This uneasy relationship exists even though in area measurement, for example, precise solutions can be obtained by considering the theoretical relationships between geometric shapes. For example, the ‘base x height’ rule for the area of rectangles applies in the same way to parallelograms and this can be proved by transforming a rectangle into a parallelogram with the same height and base (knowing that the transformation does not change the area). Similarly, the rule for finding the area of a triangle [Area = ½ (base × height)] can be justified by the fact that every triangle can be transformed into a parallelogram with the same base and height by doubling that triangle. Thus, rules for precise area measurement via formulae are built from the theoretical relations between geometric shapes. Here it is worth noting, as Bryant (2009, p. 22) confirms, that more research is needed on learners’ understanding of this centrally-important aspect of geometry and measurement.

At secondary school level, and even beyond to undergraduate level, learners can experience difficulties in using definitions appropriately and may not fully appreciate the role of definitions in geometry (Edwards and Ward, 2004; Vinner, 1991). Yet, as Freudenthal (1971, pp. 424) pointed out “Though the teacher can impose definitions…, this means degrading mathematics to something like spelling, ruled by arbitrary prescriptions”. As such, one way to overcome such issues is for students to be actively engaged in the defining of geometric objects, as exemplified by de Villiers (1998) in the case of quadrilaterals.

In terms of the idea of geometric similarity, Friedlander and Lappan (1987, p.36) list a range of mathematics that is related, including enlargement, scale factor, projection, area growth, and indirect measurement. These, say Friedlander and Lappan, are “frequently encountered by children in their immediate environment and in their studies of natural and social sciences” (ibid.). What is more, similar geometric shapes provide helpful mental images of ratios and equivalent fractions, and provide a model for some rational number concepts. Ideas of similarity extend to trigonometry and to the notion of self-similarity that is characteristic of fractal geometry.

Bearing in mind the geometrical ideas of symmetry, invariance, transformation, similarity and congruence, there are many reasons, as Freudenthal (1971, p.434) explains, why a focus on symmetries is a good idea. This is not to say that symmetry is simple or uncomplicated to teach. Research tracing the development of students' knowledge of symmetry in school geometry, such as that by Leikin et al. (2000), has revealed a range of difficulties that learners encounter with ideas of symmetry. These range from straightforward errors such as identifying an incorrect symmetry axis, or failing to recognise a correct symmetry axis, to difficulties with reflecting in oblique lines. There are matching difficulties when secondary school students work with symmetry in three-dimensions (Cooper, 1992). Research with suitable digital technologies is providing examples of how students might gain a more multi-faceted appreciation of symmetry (e.g., Hoyles and Healy, 1997; Clements et al., 2001).

Invariance, like symmetry, while a central idea of mathematics in general, is especially relevant and important in geometry. Most theorems in geometry can be seen as resulting from the study of what change is permitted that leaves some relationships or property invariant. There is research indicating that the use of transformations can be a means by which ideas of invariance can be studied most easily and by which the formal definitions of congruence and similarity can be related to learners' previous intuitive ideas. Here is a place where research indicates that digital technologies such as DGS can play a valuable role (see Hollebrands et al., 2008; Laborde et al., 2006).

While an important aim of geometry teaching is for students to develop their geometric and spatial reasoning in order that they can tackle relatively complex problems productively, research with which I have been involved indicates that even though many Grade 9 students in Japan can write down a proof, *around 70% do not understand why proofs are needed* (Kunimune, Fujita & Jones, 2009; 2010). Such research raises a number of challenges for research. These include: what geometrical definitions might be used when formulating geometrical problems for classroom use and how might students be involved in constructing definitions, and with what consequences? How do different (or even differently-orientated) representation of geometric objects, including representations constructed using software, impact on students reasoning in geometry? What is the impact of teacher's instructions on students' reasoning in geometry?

Concluding comments

This paper argues that school geometry is not solely about naming shape or proving circle theorems; rather, as Malkevitch (2009, p.14) illustrates, geometry is more akin to "the branch of mathematics that studies visual phenomena" in all their glories and richness. This is why geometry is such an important part of the school mathematics curriculum, and why the teaching of geometry across the school years needs to ensure a sustained focus on the twinned aspects of geometry: the spatial aspects, and the aspects that relate to reasoning with geometrical theory. In forming the *yin-yang* of geometry education, each gives rise to the other and each only exists in relation to the other.

Del Grande (1990, p.19) argued some time ago that "geometry has been difficult for pupils due to an emphasis on the deductive aspects of the subject and a neglect of the underlying spatial abilities acquired by hands-on activities that are necessary prerequisites for understanding and mastery of geometrical concepts". Bill Thurston put it this way:

> "We have an inexorable instinct that prompts us to convey through speech content that is not easily spoken. Because of this tendency, mathematics takes a highly symbolic, algebraic, and technical form. Few people listening to a technical discourse are hearing a story. Most readers of mathematics (if they happen not to be totally baffled) register only technical details - which are essentially different from the original thoughts we put into mathematical discourse. The meaning, the poetry, the music, and the beauty of mathematics are generally lost....
>
> Another source of the cloud of illusions that often obscures meaning in mathematics arises from the contrast between our amazingly rich abilities to absorb geometric information and the weakness of our innate abilities to convey spatial ideas - except for things we can point to or act out.... Since our minds all have much in common, we can indeed describe mental images in words, than surmise and reconstruct them through suggestive powers. This is a process of developing mental reflexes and, like other similar tasks, it is time-consuming. We just need to be aware that this is the task and that it is important, so that we won't instinctively revert to a symbolic and denatured encoding….
>
> A whole-mind approach to mathematical thinking is vastly more effective than the common approach that manipulates only symbols" (Thurston, 2011, p. xi-xiii)

The great challenge for research in mathematics education is how geometric and spatial reasoning can be taught in a way that supports what Thurston calls the "whole-mind approach to mathematical thinking" (ibid).

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