Advances in Mathematics 437 (2024) 109412

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Cohomology of group theoretic Dehn fillings II

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ARTICLE INFO

Article history:
Received 20 January 2023
Received in revised form 25 September 2023
Accepted 5 November 2023
Available online xxxx
Communicated by D. Margalit

MSC:
20F67
20F10
20E06

Keywords:
Cohomology of groups
Dehn filling
Hyperbolically embedded subgroup
Cohen-Lyndon property
SQ-universality
Poincaré duality
Simplicial volume
Cohomological finiteness conditions

ABSTRACT

We study the cohomology of group theoretic Dehn fillings. Applying the Cohen-Lyndon property for sufficiently deep Dehn fillings of hyperbolically embedded subgroups $H \hookrightarrow_h G$, obtained by the second named author in [67], we derive a spectral sequence that computes the cohomology of the corresponding Dehn filling quotients $\overline{G}$. As an application, we establish an isomorphism between the relative cohomology of the group pair (G, H) and its sufficiently deep Dehn filling quotient pair $(\overline{G}, \overline{H})$. This allows us to generalize the results of Fujiwara and Manning on simplicial volume of Dehn fillings of hyperbolic manifolds to Dehn fillings of Poincaré duality pairs.

We also strengthen the results of Olshanskii [58], Dahmani-Guirardel-Osin [27] and Hull [42] on SQ-universality and common quotients of acylindrically hyperbolic groups by adding cohomological finiteness conditions. We apply these results to obtain hyperbolic and acylindrically hyperbolic quotients with special properties.

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https://doi.org/10.1016/j.aim.2023.109412

1. Introduction

1.1. Dehn surgery in 3-manifolds

In the late 1970's, Thurston dramatically changed the study of 3-manifolds by introducing his Geometrization Conjecture. As supporting evidence, Thurston proved that many non-Haken 3-manifolds satisfy the conjecture [69], using the notion of a *Dehn surgery*, which is a two-step procedure of modifying a 3-manifold by first cutting off a solid torus and then gluing the torus back in a different way. Another motivation of Dehn surgery comes from the Lickorish-Wallace theorem, which states that every closed connected orientable 3-manifold can be constructed from the 3-sphere by using finitely many Dehn surgeries.

The second step of the surgery, called *Dehn filling*, starts with a 3-manifold M with toral boundary and constructs a new manifold by gluing a solid torus to M by identifying their boundaries. Topologically distinct ways of gluing a solid torus are parametrized by free homotopy classes of essential simple closed curves on ∂M (the image of the meridian circle of the solid torus under the identification), called *slopes*. For a slope s, the new manifold constructed by the corresponding Dehn filling is denoted by M_s. A celebrated result of Thurston asserts that most Dehn fillings preserve hyperbolicity.

Theorem 1.1 *(Thurston [69]). Let M be a compact orientable 3-manifold with boundary a torus, and with interior admitting a complete finite volume hyperbolic structure. Then for all but finitely many slopes s on ∂M, M_s admits a hyperbolic structure.*

1.2. Group theoretic Dehn fillings

There is an analogous construction in group theory, called *(group theoretic) Dehn filling*, which can be formalized as follows. Given a group G with a subgroup H and a normal subgroup N of H, the Dehn filling associated with the triple (G, H, N) is the quotient $G/\langle\langle N\rangle\rangle$, where $\langle\langle N\rangle\rangle$ is the normal closure of N in G.

The relation between these two versions of Dehn fillings can be seen as follows: under the assumptions of Theorem 1.1, the natural homomorphism $\pi_1(\partial M) \to \pi_1(M)$ is injective and thus $\pi_1(\partial M)$ can be thought of as a subgroup of $\pi_1(M)$. Let $G = \pi_1(M)$ and $H = \pi_1(\partial M)$. Every slope s on ∂M generates a normal subgroup $N_s \lhd H$. As s is the image of the meridian circle of the solid torus, which bounds a disc, we have $\pi_1(M_s) = G/\langle\langle N_s\rangle\rangle$.

Dehn filling is a fundamental tool in group theory. It appears, for instance, in the solution of the Virtual Haken Conjecture [3], the study of the Farrell-Jones Conjecture and the isomorphism problem of relatively hyperbolic groups [1,26], and the construction of purely pseudo-Anosov normal subgroups of mapping class groups [27]. Other applications of Dehn fillings can be found for example in [2,35].

Algebraic analogs of Theorem 1.1 can be proved for groups satisfying certain negative curvature conditions. The first result of this kind was for relatively hyperbolic groups by Osin [60] and independently, by Groves-Manning [33]. Later, Dahmani-Guirardel-Osin [27] introduced a generalization of relative hyperbolicity based on the notion of a hyperbolically embedded subgroup and proved a generalization of the main results of [60,33]. We postpone the definition and motivation of hyperbolically embedded subgroups until Section 3.3 and only discuss several examples for the moment. The reader is referred to the survey [63] for other examples. We use $H \hookrightarrow_h G$ to indicate that H is a hyperbolically embedded subgroup of G.

Example 1.2. If a group G is hyperbolic relative to its subgroup H, then $H \hookrightarrow_h G$ [27, Proposition 2.4]. In particular, if M is a compact orientable manifold with one boundary component and $M \setminus \partial M$ admits a complete finite volume hyperbolic structure, then $\pi_1(\partial M) \hookrightarrow_h \pi_1(M)$ [14,28].

Example 1.3. Another typical example arises if a group G acts on a Gromov hyperbolic space S acylindrically by isometries and $g \in G$ is a loxodromic element. Then there exists a maximal virtually-cyclic subgroup $E(g) \leqslant G$ containing g such that $E(g) \hookrightarrow_h G$ [27, Corollary 2.9]. In particular, if G is a word-hyperbolic group (resp. mapping class group of a finite type surface [27, Theorem 2.19], outer automorphism group of a finite rank free group [27, Theorem 2.20]) and g is an infinite order (resp. a pseudo-Anosov, a fully irreducible) element, then $E(g) \hookrightarrow_h G$.

The following is a group theoretic analog of Thurston's Theorem 1.1 due to Dahmani-Guirardel-Osin.

Theorem 1.4 *(Dahmani-Guirardel-Osin [27]). Let G be a group with a subgroup $H \hookrightarrow_h G$. Then there exists a finite set $\mathcal{F} \subseteq H \setminus \{1\}$ such that if $N \lhd H$ and $N \cap \mathcal{F} = \varnothing$, then the natural homomorphism $H/N \to G/\langle\langle N\rangle\rangle$ maps H/N injectively onto a hyperbolically embedded subgroup of $G/\langle\langle N\rangle\rangle$.*

In the setting of Theorem 1.1, we have $\pi_1(\partial M) \hookrightarrow_h M$ by Example 1.2. Theorem 1.4 then implies that for all but finitely many slopes s on ∂M, we have $\pi_1(\partial M)/N_s \hookrightarrow_h \pi_1(M_s)$. A deeper investigation, using more precise versions of Theorem 1.4 (e.g., [60, Theorem 1.1], [33, Theorem 7.2] and [27, Theorem 2.27]), shows that $\pi_1(M_s)$ is word-hyperbolic [60, Corollary 1.2] and one-ended [34, Corollary 1.11]. The Geometrization Conjecture, proved by Perelman, then implies that M_s admits a hyperbolic structure.

1.3. Motivation: cohomology of Dehn fillings

Theorem 1.1 asserts that M_s is often hyperbolic and thus its universal cover is $\mathbb{H}^3$. It follows that the cohomology of $\pi_1(M_s)$ can be understood by studying the cohomology of

M_s and the action of $\pi_1(M_s)$ on $\mathbb{H}^3$. It is therefore natural to investigate the cohomology of $G/\langle\langle N\rangle\rangle$ in the more general setting of Theorem 1.4 where one may look for a similar geometric footing.

Question 1.5. *For a group G with a subgroup $H \hookrightarrow_h G$ and a normal subgroup $N \lhd H$, what can be said about the cohomology of $G/\langle\langle N\rangle\rangle$?*

The main goal of this series of two papers is to address this question and to illustrate the implications of the results in this direction.

We should point out that even though we consider the general case of hyperbolically embedded subgroups $H \hookrightarrow_h G$, all of our results are new in the special case when G is hyperbolic relative to H.

Acknowledgments. Most of this work was done when the second named author was a graduate student at Vanderbilt University. He would like to thank his supervisor, Denis Osin, for the valuable discussions. This paper would not have been written without the help of Osin. The second named author would also like to thank Anna Marie Bohmann for the helpful comments and thank Ian Leary for answering his question and the suggestion of references. The second named author received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Grant agreement No. 850930) and an AMS–Simons Travel Grant (Grant agreement No. IP00672308). The first named author would like to thank Clara Löh and Kevin Li for helpful comments and suggestions. Finally, the authors are thankful to the referees for the careful reading of the paper and many detailed comments which have improved both its content and exposition.

2. Statements of main results

2.1. Cohomological properties of Dehn fillings

To simplify the statement, we introduce the following terminology and notation.

Definition 2.1. Let G be a group and H a subgroup of G. We say that a property $\mathcal{P}$ *holds for sufficiently deep normal subgroups* if there is a finite set $\mathcal{F} \subseteq H \setminus \{1\}$ such that $\mathcal{P}$ holds whenever N is a normal subgroup of H and $N \cap \mathcal{F} = \varnothing$.

Given a normal subgroup N of H, let $\overline{G} = G/\langle\langle N\rangle\rangle$ and $\overline{H} = H/N$.

Theorem 1.4 can now be restated as: let G be a group with a subgroup $H \hookrightarrow_h G$. Then for sufficiently deep $N \lhd H$, the natural homomorphism $\overline{H} \to \overline{G}$ maps $\overline{H}$ injectively onto a hyperbolically embedded subgroup of $\overline{G}$.

The following is a summary of our main results on cohomological properties of Dehn fillings.

Theorem A. *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then the following hold for all sufficiently deep $N \lhd H$ and all $\overline{G}$-modules A.*

(i) *There is a spectral sequence*

$$E_2^{p,q}(A) = \begin{cases} H^p(\overline{H}; H^q(N;A)) & \text{for } q > 0 \\ H^p(\overline{G}; A) & \text{for } q = 0 \end{cases} \Rightarrow H^{p+q}(G;A),$$

where the action of G on A factors through $\overline{G}$. In particular, the action of N on A fixes A pointwise.

(ii) *(Algebraic Excision) For all $n \geqslant 0$, there is a natural isomorphism induced by the quotient maps $G \to \overline{G}$ and $H \to \overline{H}$,*

$$H^n(\overline{G}, \overline{H}; A) \cong H^n(G, H; A).$$

The spectral sequence together with the algebraic excision have the following application.

Corollary 2.2. *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then for all sufficiently deep $N \lhd H$ and all $\overline{G}$-modules A, we have*

(i) *For all $n \geqslant \mathrm{cd}(H) + 2$,*

$$H^n(\overline{G}; A) \cong H^n(G; A) \oplus H^n(\overline{H}; A).$$

(ii) $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(H) + 1, \mathrm{cd}(\overline{H})\}$.

(iii) *If G is of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$ (resp. FP), then $\overline{G}$ is of type FP_n (resp. FP) if and only if $\overline{H}$ is of type FP_n (resp. FP).*

Here, $\mathrm{cd}(G)$ stands for the *cohomological dimension* of a group G, and $\mathbb{N}^+$ stands for the set of positive integers. This notion, *property FP_n*, *property FP*, and *relative cohomology of group pairs* are reviewed in Section 3.2.

Analogous homological statements to Theorem A and Corollary 2.2 also hold. The spectral sequence in Theorem A (i) is a refinement of the classical Lyndon-Hochschild-Serre spectral sequence [41,51] in the setting of Dehn fillings.

Let M and M_s be as in Theorem 1.1 and let $G = \pi_1(M), H = \pi_1(\partial M), N = \langle s \rangle$. Then Theorem A (ii) is an immediate consequence of excision. Therefore, Theorem A (ii) can be thought of as an algebraic analog of excision and computes $H^n(\overline{G}, \overline{H}; A)$ from $H^n(G, H; A)$. In the special case where $A = \mathbb{Z}\overline{G}$, a spectral sequence to compute $H^*(\overline{G}, \overline{H}; \mathbb{Z}\overline{G})$ with $H^*(G, H; \mathbb{Z}G)$ has been developed by [71]. We remark that the spectral sequence in [71] is different from ours.

Instead of proving Theorem A, we will prove more general results (see Section 4), which cover the case of *a hyperbolically embedded family of subgroups* and will be useful in the proof of Theorems B, C, and E below, and also cover the case of *weakly hyperbolically embedded subgroups* and can be applied to graph of groups (see Example 3.8 (e)).

Corollary 2.2 was recently used by Arenas [5], who gave a variation of the Rips Construction that produces cubulated hyperbolic groups of cohomological dimension bounded above by the cohomological dimension of an associated compact special cube complex.

2.2. Poincaré duality and simplicial volume

Simplicial volume is a homotopy invariant of oriented closed connected manifolds [32,48]. It was introduced by Gromov in his seminal paper [38].

In [29], Fujiwara and Manning generalize Gromov-Thurston's 2π-theorem [10] on Dehn fillings of 3-manifolds to higher dimensional finite volume hyperbolic manifolds M^n with toral cusps. The resulting 2π-fillings are pseudomanifolds and are manifolds if and only if all the filling cores have dimension exactly $n-2$. They prove that every 2π-filling admits a complete locally CAT(0) metric. In [30], they show that the simplicial volume of every 2π-filling is positive and bounded above by the relative simplicial volume $||\overline{M}, \partial\overline{M}||$ of M.

In certain cases, simplicial volume can also be defined in more abstract setting of groups and topological spaces. The following two theorems can be seen as natural generalizations and group theoretic analogs of the results of Fujiwara and Manning.

Theorem B. *Let G be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of subgroups such that $\mathcal{H} \hookrightarrow_h G$. Suppose, for some integer $2 \leqslant n$, $(G, \mathcal{H})$ is a PD(n)-pair and that there are sufficiently deep $\{\mathbb{Z} \cong N_i \lhd H_i\}_{i=1}^m$, such that every $\overline{H}_i$ is a PD$(n-2)$-group. Then $\overline{G}$ is a PD(n)-group.*

The hypothesis on subgroups in Theorem B is for example satisfied when each H_i is a torsion-free nilpotent group with center of rank at least 2 (see e.g. Lemma 6.4).

Theorem C. *Let G be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of subgroups such that $\mathcal{H} \hookrightarrow_h G$. Suppose, for some integer $n \geqslant 2$, $(G, \mathcal{H})$ is a PD(n)-pair and for a sufficiently deep $\{N_i \lhd H_i\}_{i=1}^m$, $\operatorname{cd}(\overline{H}_i) \leqslant n-2$ for each $1 \leqslant i \leqslant m$. Then, $\operatorname{cd}(\overline{G}) = n$, $H_n(\overline{G}; \mathbb{Z}) = \mathbb{Z}$. In addition,*

(i) *if the group $\overline{H}_i$ is amenable for each $1 \leqslant i \leqslant m$, then $||\overline{G}|| \leqslant ||G, \mathcal{H}||$, where $||\cdot||$ denotes the simplicial volume (see Definition 6.2);*
(ii) *if G is hyperbolic relative to $\mathcal{H}$, then $||\overline{G}|| > 0$.*

We should point out that conjecturally by Kropholler, amenable groups of finite cohomological dimension such as $\overline{H}_i$ are virtually solvable [23, Question, p. 2]. Also, by Poincaré duality group or pair we will always mean orientable ones.

Theorem C can for example be applied when G is the fundamental group of a Riemannian manifold with a complete pinched negative sectional curvature and finite volume. We give one such application which generalizes Theorem 1.5 of [30].

Corollary 2.3 *(Corollary 6.6). Let $\overline{M}$ be a compact oriented n-manifold with nilmanifold boundary components such that the center of the fundamental group of each boundary component is of rank at least 2. Suppose the interior of $\overline{M}$ admits a Riemannian metric with a complete pinched negative sectional curvature and finite volume. If M_T is a sufficiently deep Dehn filling manifold of $\overline{M}$, then M_T is a closed oriented aspherical n-manifold with*

$$0 < ||M_T|| \leqslant ||\overline{M}, \partial\overline{M}||.$$

For the definition of M_T we refer to Definition 6.5. It is worth mentioning that the asphericity of M_T above is not immediate and follows from the Cohen-Lyndon property.

2.3. Quotients of acylindrically hyperbolic groups

The notion of an acylindrically hyperbolic group was introduced by Osin [62] as a generalization of non-elementary hyperbolic and non-elementary relatively hyperbolic groups. Examples of acylindrically hyperbolic groups can be found in many classes of groups that attracted group theorists for years, e.g., mapping class groups of surfaces [53,13], outer automorphism groups of free groups [8], small cancellation groups [40], convergence groups [66], the Cremona group (see [27] and references therein; the main contribution towards showing the acylindrical hyperbolicity of the Cremona group is due to [22]), and tame automorphism groups of 3-dimensional affine spaces [49]. We refer to [63] for details and other examples.

Every acylindrically hyperbolic group G contains hyperbolically embedded subgroups [27, Theorem 6.14] and Dehn fillings can often be applied to construct useful quotients of G. We use Theorem A to study homological properties of those quotients.

Recall that every acylindrically hyperbolic group G has a unique maximal finite normal subgroup denoted by $K(G)$ [27, Theorem 6.14].

Theorem D. *Let G be an acylindrically hyperbolic group, and let C be any countable group. Then C embeds into a quotient $\overline{G}$ of $G/K(G)$ (in particular, $\overline{G}$ is a quotient of G) such that*

(i) *$\overline{G}$ is acylindrically hyperbolic;*
(ii) *if C is finitely generated, then $C \hookrightarrow_h \overline{G}$;*

(iii) *if G and C are torsion-free, then so is $\overline{G}$;*
(iv) *for all $n \geqslant 3$ and every $\overline{G}$-module A, we have*

$$H^n(\overline{G}; A) \cong H^n(G/K(G); A) \oplus H^n(C; A),$$

where the action of $G/K(G)$ (resp. C) on A is induced by the quotient map $G/K(G) \to \overline{G}$ (resp. the embedding $C \hookrightarrow \overline{G}$);
(v) $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(C)\}$*;*
(vi) *if C is finitely generated and G is of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$ (resp. FP), then $\overline{G}$ is of type FP_n (resp. FP) if and only if C is of type FP_n (resp. FP).*

As an application, we strengthen SQ-universality of hyperbolic groups given by Olshanskii [58] and independently by Delzant [25] by adding cohomological conditions.

Corollary 2.4 *(Corollary 8.2). Let G be a non-elementary hyperbolic group and C any hyperbolic group. Then there is a hyperbolic quotient $\overline{G}$ of $G/K(G)$ (in particular, $\overline{G}$ is a quotient of G) such that C embeds into $\overline{G}$ and the following hold.*

(i) *For all $n \geqslant 3$ and every $\overline{G}$-module A, we have*

$$H^n(\overline{G}; A) \cong H^n(G/K(G); A) \oplus H^n(C; A),$$

where the action of $G/K(G)$ (resp. C) on A is induced by the quotient map $G/K(G) \to \overline{G}$ (resp. the embedding $C \hookrightarrow \overline{G}$).
(ii) $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(C)\}$.

Theorem E. *Let G_1 and G_2 be finitely generated acylindrically hyperbolic groups. Then there exists a common quotient G of $G_1/K(G_1)$ and $G_2/K(G_2)$ (in particular, G is a common quotient of G_1 and G_2) such that*

(i) *G is acylindrically hyperbolic;*
(ii) *for all $n \geqslant 3$ and every G-module A, we have*

$$H^n(G; A) \cong H^n(G_1/K(G_1); A) \oplus H^n(G_2/K(G_2); A),$$

where the actions of $G_1/K(G_1)$ and $G_2/K(G_2)$ on A factor through G;
(iii) $\mathrm{cd}(G) \leqslant \max\{\mathrm{cd}(G_1), \mathrm{cd}(G_2)\}$*;*
(iv) *if G_1 and G_2 are of type FP_n for some $n \in \{2, 3, ..., \infty\}$ (resp. FP), then so is G.*

Homological analogs of Theorem D (iv), (v) and Theorem E (ii), (iii) also hold (see Remarks 7.3 and 7.11). Theorem D (i), (ii) are proved in [27, Theorem 2.33] and Theorem E (i) is proved in [42, Corollary 7.4]. The benefit of Theorems D and E is that they allow one to control the cohomology of the resulting acylindrically hyperbolic quotients.

As we illustrate in Section 8, this facilitates the constructions of various acylindrically hyperbolic groups satisfying certain cohomological properties. We list some of them below.

Corollary 2.5 *(Corollary 8.1). Every torsion-free acylindrically hyperbolic group G of type FP_∞ has a torsion-free acylindrically hyperbolic quotient $\overline{G}$ of type FP_∞ which contains the Thompson group F.*

The existence of a pair $H < G$, where G is hyperbolic and H is of type F but not hyperbolic, was a well-known open problem, raised in particular by Bestvina [7, Question 2.1], Brady [15, Question 7.2], and Jankiewicz-Norin-Wise [46, Section 7]. The first example of such a pair was constructed recently by Italiano-Martelli-Migliorini [43, Corollary 2]. By Corollary 2.4, we obtain:

Corollary 2.6 *(Corollary 8.3). Let $n \geqslant 5$ be an integer. Every non-elementary hyperbolic group G with $\mathrm{cd}(G) \leqslant n$ has a hyperbolic quotient $\overline{G}$ with $\mathrm{cd}(\overline{G}) = n$ such that $\overline{G}$ contains the Italiano-Martelli-Migliorini group. In particular, there is a type F non-hyperbolic subgroup $H < \overline{G}$.*

The next two corollaries strengthen a result of [42, Corollary 1.7] stating that every acylindrically hyperbolic group has an acylindrically hyperbolic quotient with Kazhdan's Property (T).

Corollary 2.7 *(Corollary 8.4). Every acylindrically hyperbolic group G of type FP_n for some $n \in \{2, 3, ..., \infty\}$ (resp. FP) has an acylindrically hyperbolic quotient $\overline{G}$ of type FP_n (resp. FP) with Kazhdan's Property (T) such that $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), 2\}$.*

Corollary 2.8 *(Corollary 8.6). Let G be any acylindrically hyperbolic group of type FP_∞. Then G has a family of acylindrically hyperbolic quotients $\{G_k\}_{k=2}^{\infty}$ such that for each k, G_k has Kazhdan's Property (T), is of type FP_{k-1} but not of type FP_k.*

In particular, since mapping class groups of surfaces of finite type, outer automorphism groups of free groups of finite rank and most 3-manifold groups are acylindrically hyperbolic and of type FP, they all exhibit such quotients.

2.4. A few words on the proofs of the main results

The first step of the proof of Theorem A is to establish, under the assumptions of the theorem, the isomorphism

$$H^n(\langle\langle N \rangle\rangle; A) \cong \mathrm{CoInd}_{\overline{H}}^{\overline{G}} H^n(N; A) \tag{1}$$

for all $n > 0$. Here, $\mathrm{CoInd}_{\overline{H}}^{\overline{G}}$ stands for the co-induction from $\mathbb{Z}\overline{H}$ to $\mathbb{Z}\overline{G}$. The Lyndon-Hochschild-Serre spectral sequence associated to the triple $(G, \langle\langle N \rangle\rangle, A)$ takes the form

$$E_2^{p,q}(A) = H^p(\overline{G}; H^q(\langle\langle N\rangle\rangle; A)) \Rightarrow H^{p+q}(G; A).$$

Shapiro's lemma together with (1) yields Theorem A (i). In fact, we will establish a more precise result which involves a morphism between the Lyndon-Hochschild-Serre spectral sequences associated to $(G, \langle\langle N\rangle\rangle, A)$ and (H, N, A). Part (ii) of Theorem A will be proved by an inspection of this morphism.

To prove Theorem B, we analyze the spectral sequence of Theorem A (i) and apply part (ii) and Corollary 2.2 (iii). To show that $\overline{G}$ is a Poincaré duality group, we use Johnson-Wall characterization [47]; namely, a group Γ is a Poincaré duality group of dimension n if and only if Γ is of type FP, $H^i(\Gamma, \mathbb{Z}\Gamma) = 0$ for $i \neq n$ and $H^n(\Gamma, \mathbb{Z}\Gamma) = \mathbb{Z}$.

The algebraic excision Theorem A (ii) is again key in proving Theorem C. Since all $\overline{H}_i$ are amenable, the natural map in bounded cohomology $H^n_b(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \to H^n_b(\overline{G}; \mathbb{R})$ is an isometric isomorphism. The duality pairing between bounded cohomology and ordinary homology leads to the inequality $||\overline{G}|| \leqslant ||G, \mathcal{H}||$. When G is hyperbolic relative to $\mathcal{H}$, then a sufficiently deep Dehn filling quotient $\overline{G}$ is hyperbolic relative to $\overline{\mathcal{H}}$ [60, Theorem 1.1]. This allows us to show that $||\overline{G}|| > 0$.

The proof of Theorem D is a modification of the proof of [27, Theorem 2.33]. Given any acylindrically hyperbolic group G, one can find a non-cyclic free group $F \hookrightarrow_h G_0 = G/K(G)$. For any countable group C and any finite subset $\mathcal{F} \subseteq F \smallsetminus \{1\}$, we will use small cancellation theory to construct a normal subgroup $N \lhd F$ such that $N \cap \mathcal{F} = \varnothing$, C embeds into F/N, and F/N has the desired cohomological properties. Theorem 1.4 then implies that C embeds into $G_0/\langle\langle N\rangle\rangle$ and Theorem A and Corollary 2.2 applied to $N \lhd F \hookrightarrow_h G_0$ yields the desired cohomological results.

The proof of [42, Corollary 7.4] uses small cancellation theory instead of Dehn filling. In order to apply our main result, we carry out an alternative approach. Given finitely generated acylindrically hyperbolic groups G_1 and G_2, we construct subgroups $H_1, H_2 < \widetilde{G} = G_1' * G_2'$, where $G_1' = G_1/K(G_1)$ and $G_2' = G_2/K(G_2)$, such that the family $\{H_1, H_2\}$ *hyperbolically embeds* into $\widetilde{G}$. For any finite sets $\{\mathcal{F}_i \subseteq H_i \smallsetminus \{1\}\}_{i=1,2}$, we will use small cancellation theory to construct normal subgroups $\{N_i \lhd H_i\}_{i=1,2}$ such that $N_i \cap \mathcal{F}_i = \varnothing$ and N_1 (resp. N_2) identifies a finite set of generators of G_2' (resp. G_1') with certain elements of G_1' (resp. G_2'). The quotient $G = \widetilde{G}/\langle\langle N_1 \cup N_2\rangle\rangle$ is thus a common quotient of G_1' and G_2'. Theorem E is then proved by applying general versions of Theorems 1.4 and A. The main difficulty of this argument is the construction of H_1 and H_2, which is presented in Section 7.2, using a technical tool provided by [27] (see also [60]) called *isolated components*.

2.5. Organization of the paper

We will start with preliminaries in Section 3, recalling basic definitions of group cohomology, the notion of (weakly) hyperbolically embedded subgroups, isolated components, acylindrically hyperbolic groups, and the structural result of [67] called the *Cohen-Lyndon property*. The proof of (the general version of) Theorem A and Corollary 2.2

is given in Section 4. Theorems B and C are proved in Sections 5 and 6, respectively. Theorems D and E are proved in Section 7 with applications given in Section 8.

3. Preliminaries

3.1. Direct sum and product of spectral sequences

We briefly recall a property concerning convergence of the direct sum and direct product of a (possibly infinite) family of spectral sequences of ring modules. For detailed introduction to spectral sequences we refer to [72], [19] and [20].

Let R be a unital ring, and let $E^{2,\lambda}_{p,q} \Rightarrow H^{\lambda}_{p+q}, \lambda \in \Lambda$ be a family of convergent homological spectral sequences of R-modules such that $E^{2,\lambda}_{p,q} = 0$ if either p or q is less than 0.

Lemma 3.1. *We have*

$$\bigoplus_{\lambda\in\Lambda} E^{2,\lambda}_{p,q} \Rightarrow \bigoplus_{\lambda\in\Lambda} H^{\lambda}_{p+q}, \quad \prod_{\lambda\in\Lambda} E^{2,\lambda}_{p,q} \Rightarrow \prod_{\lambda\in\Lambda} H^{\lambda}_{p+q}.$$

Moreover, the same statement holds for cohomological spectral sequences as well.

Proof. For simplicity, denote $\bigoplus_{\lambda\in\Lambda} E^{2,\lambda}_{p,q}$ as $E^2_{p,q}$. First note that the E^∞-terms satisfy

$$E^{\infty}_{p,q} = \bigoplus_{\lambda\in\Lambda} E^{\infty,\lambda}_{p,q}.$$

Indeed, fix p, q for the moment. We have

$$E^{\infty}_{p,q} = E^{p+q+1}_{p,q} = \bigoplus_{\lambda\in\Lambda} E^{p+q+1,\lambda}_{p,q} = \bigoplus_{\lambda\in\Lambda} E^{\infty,\lambda}_{p,q}.$$

Next, let us recall that the convergence $E^{2,\lambda}_{p,q} \Rightarrow H^{\lambda}_{p+q}$ means that for each λ and each H^{λ}_{p+q}, there is a filtration

$$0 = F_{-1}H^{\lambda}_{p+q} < F_0 H^{\lambda}_{p+q} < \dots < F_{p+q}H^{\lambda}_{p+q} = H^{\lambda}_{p+q}$$

such that for all p, q we have a short exact sequence

$$0 \to F_{p-1}H^{\lambda}_{p+q} \to F_p H^{\lambda}_{p+q} \to E^{\infty,\lambda}_{p,q} \to 0.$$

Consider the following filtration of $\bigoplus_{\lambda\in\Lambda} H^{\lambda}_{p+q}$:

$$0 = \bigoplus_{\lambda\in\Lambda} F_{-1}H^{\lambda}_{p+q} < \bigoplus_{\lambda\in\Lambda} F_0 H^{\lambda}_{p+q} < \dots < \bigoplus_{\lambda\in\Lambda} F_{p+q}H^{\lambda}_{p+q} = \bigoplus_{\lambda\in\Lambda} H^{\lambda}_{p+q}.$$

Then we have short exact sequences

$$0 \to \bigoplus_{\lambda\in\Lambda} F_{p-1}H^{\lambda}_{p+q} \to \bigoplus_{\lambda\in\Lambda} F_p H^{\lambda}_{p+q} \to \bigoplus_{\lambda\in\Lambda} E^{\infty,\lambda}_{p,q} \to 0$$

by taking the direct sum of the above short exact sequences, as the direct sum functor is exact [72, Theorem 2.6.15].

The proof for direct product is similar. We only point out that the direct product functor is exact for R-modules, as the category of R-modules satisfies Axiom (AB4*) of [36, Section 1.5]. Also, the proof for cohomological spectral sequences is almost identical. $\square$

3.2. Cohomology of groups

Let G be a group. Recall that the *homological* and *cohomological dimension* of G can be defined by

$$\mathrm{hd}(G) = \sup\{n \in \mathbb{N} \mid H_n(G, A) \neq 0 \text{ for some } \mathbb{Z}G\text{-module } A\},$$
$$\mathrm{cd}(G) = \sup\{n \in \mathbb{N} \mid H^n(G, A) \neq 0 \text{ for some } \mathbb{Z}G\text{-module } A\},$$

respectively.

G is of *type FP_n* for some $n \in \mathbb{N}^+ \cup \{\infty\}$ if there is a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z}$$

over $\mathbb{Z}G$ such that P_k are finitely generated G-modules for all $k \leqslant n$. G is of type FP if $\mathrm{cd}(G) < \infty$ and G is of type FP_∞.

Property FP_n can be characterized by the cohomology functor. The following will be useful in the proof of Theorem 4.8.

Theorem 3.2 *(Bieri [12, Theorem 1.3], Brown [17, Theorem 2]). For a group G, the following are equivalent.*

(a) *G is of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$.*
(b) *For every $k \leqslant n$ and every direct system $\{A_i\}_{i\in I}$ of G-modules such that $\varinjlim A_i = 0$, we have $\varinjlim H^k(G; A_i) = 0$.*

Given a family $\{H_\lambda\}_{\lambda\in\Lambda}$ of subgroups of G, [6] defined the relative (co)homology of the group pair $(G, \{H_\lambda\}_{\lambda\in\Lambda})$. We briefly recall the definition. Let Δ be the kernel of the augmentation $\bigoplus_{\lambda\in\Lambda} \mathbb{Z}[G/H_\lambda] \twoheadrightarrow \mathbb{Z}$ which sends every left H_λ-coset to 1. By definition,

$$H_n(G, \{H_\lambda\}_{\lambda\in\Lambda}; A) = \mathrm{Tor}^G_{n-1}(\Delta, A), \quad H^n(G, \{H_\lambda\}_{\lambda\in\Lambda}; A) = \mathrm{Ext}^{n-1}_G(\Delta, A) \tag{2}$$

for any G-module A. The dimension shift in the above definition ensures a long exact sequence between the absolute and relative (co)homology (see [6, Proposition 1.1]).

3.3. (Weakly) hyperbolically embedded subgroups

The notion of (weakly) hyperbolically embedded subgroups was introduced by [27], which is our main reference for Sections 3.3, 3.4, and 3.5. We first recall the definition and present some examples. The motivation will be discussed afterwards.

Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a family of subgroups of G, X a subset of G such that G is generated by X together with the union of all H_λ (in which case X is called a *relative generating set* of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$), and $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$. Consider the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$. Note that, for $\lambda \in \Lambda$ there is a natural embedding $\Gamma(H_\lambda, H_\lambda) \hookrightarrow \Gamma(G, X \sqcup \mathcal{H})$ whose image is the subgraph of $\Gamma(G, X \sqcup \mathcal{H})$ with vertices and edges labeled by elements of H_λ.

Remark 3.3. We do allow $X \cap H_\lambda \neq \varnothing$ and $H_\lambda \cap H_\mu \neq \{1\}$ for distinct $\lambda, \mu \in \Lambda$, in which case there will be multiple edges between some pairs of vertices of $\Gamma(G, X \sqcup \mathcal{H})$.

For $\lambda \in \Lambda$, an edge path in $\Gamma(G, X \sqcup \mathcal{H})$ between vertices of $\Gamma(H_\lambda, H_\lambda)$ is called H_λ-*admissible* if it does not contain any edge of $\Gamma(H_\lambda, H_\lambda)$. Note that an H_λ-admissible path is allowed to pass through vertices of $\Gamma(H_\lambda, H_\lambda)$.

For example, consider the simple case where $\{H_\lambda\}_{\lambda\in\Lambda} = \{H\}$ consists of only one subgroup $H \leqslant G$. The Cayley graph $\Gamma(G, X \sqcup H)$ is displayed in Fig. 1. The blue path is admissible. The red path is an edge from 1 to h labeled by $h \in H$, and thus is inadmissible. If h happens to be an element of X, i.e., there exists $x \in X$ with $x = h$, and the red path were labeled by x instead of h, then the red path would be admissible.

Definition 3.4. For every pair of elements $h, k \in H_\lambda$, let $\widehat{d_\lambda}(h, k) \in [0, \infty]$ be the length of a shortest H_λ-admissible path connecting the vertices labeled by h and k. If no such path exists, set $\widehat{d_\lambda}(h, k) = \infty$. The laws of summation on $[0, \infty)$ extend naturally to $[0, \infty]$ and it is easy to verify that $\widehat{d_\lambda} : H_\lambda \times H_\lambda \to [0, +\infty]$ defines a metric on H_λ, which is called the *relative metric on H_λ with respect to X*.

Definition 3.5. We say that the family $\{H_\lambda\}_{\lambda\in\Lambda}$ *weakly hyperbolically embeds into* (G, X) (denoted by $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$) if G is generated by the set X together with union of all $H_\lambda, \lambda \in \Lambda$, and the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ is a Gromov hyperbolic space.

If $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ and for each $\lambda \in \Lambda$, the metric space $(H_\lambda, \widehat{d_\lambda})$ is proper, i.e., every ball of finite radius contains only finitely many elements, then $\{H_\lambda\}_{\lambda\in\Lambda}$ *hyperbolically embeds into* (G, X) (denoted by $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$). If in addition, X and Λ are finite, then we say that G is *hyperbolic relative* to $\{H_\lambda\}_{\lambda\in\Lambda}$.

Further, we say that the family $\{H_\lambda\}_{\lambda\in\Lambda}$ *hyperbolically embeds into* G, denoted by $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$, if there exists some subset $X \subseteq G$ such that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$.

Notation 3.6. In case $\{H_\lambda\}_{\lambda\in\Lambda} = \{H\}$ is a singleton, we will drop the braces and write $H \hookrightarrow_{wh} (G, X)$ and $H \hookrightarrow_h G$.

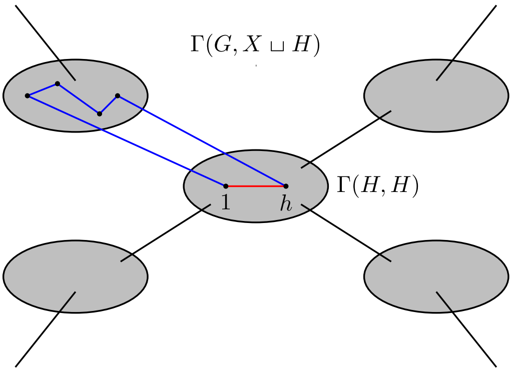

Fig. 1. Illustration of $H \hookrightarrow_h G$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

We refer to Fig. 1 for an illustration of the situation $H \hookrightarrow_h G$. The grey discs represent cosets of H in G. The black edges are labeled by elements of X. The edges and discs appear in a tree-like pattern as $\Gamma(G, X \sqcup H)$ is Gromov hyperbolic.

Remark 3.7. Notice that the above definition of relative hyperbolicity is not commonly used in literature. One of the most commonly used definitions for relative hyperbolicity which we will use later is the following: a group G is *hyperbolic relative* to a family $\{H_\lambda\}_{\lambda\in\Lambda}$ of its subgroups if G has a finite relative presentation with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$ with a linear relative isoperimetric function (see [27, Definition 3.6]). The equivalence of these two definitions is proved in [27, Remark 4.41 and Theorem 4.42].

Example 3.8.

(a) $H \hookrightarrow_{wh} (G, G)$ for every subgroup $H \leqslant G$.
(b) $H \hookrightarrow_h (G, G)$ for every finite subgroup $H \leqslant G$.
(c) $G \hookrightarrow_h (G, \varnothing)$.
(d) If G can be decomposed as a free product of its subgroups $\{G_\lambda\}_{\lambda\in\Lambda}$ (denoted by $G = *_{\lambda\in\Lambda} G_\lambda$), then $\{G_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, \varnothing)$ [27, Example 4.12].
(e) More generally, suppose that $G = \pi_1(\mathcal{G})$, where $\mathcal{G}$ is a graph of groups. Let $\{G_v\}_{v\in V\mathcal{G}}$ be the collection of vertex subgroups and $\{G_e\}_{e\in E\mathcal{G}}$ the collection of edge subgroups. By [27, Example 4.12], $\{G_v\}_{v\in V\mathcal{G}} \hookrightarrow_{wh} (\pi_1(\mathcal{G}), X)$, where the subset $X \subseteq G$ consists of stable letters (i.e., there is a spanning tree $T\mathcal{G}$ of $\mathcal{G}$ such that X consists of generators corresponding to the edges of $\mathcal{G} \smallsetminus T\mathcal{G}$).

Recall that a group is *word-hyperbolic* if it is finitely generated and for some (equivalently, any) finite generating set the corresponding Cayley graph is Gromov hyperbolic. The notion of weak hyperbolic embedding is thus an attempt to study possibly non-word-hyperbolic groups via Gromov hyperbolic spaces. Example 3.8 (a) illustrates a triviality of this notion, in the sense that the weak hyperbolic embedding does not provide any information about the group G. Notice that in that example, the corresponding relative metric is bounded. One might therefore refine the notion by imposing additional conditions on the relative metric, for example, requiring local finiteness of the relative metric and obtaining the notion of hyperbolic embedding. We note that Examples 3.8 (b) and (c) exhibit two kinds of trivialities of hyperbolic embedding, and a further refinement is given by the notion of an *acylindrically hyperbolic group* (see Section 3.5).

The next lemma tells us how to find hyperbolically embedded subgroups.

Lemma 3.9 *(Dahmani-Guirardel-Osin [27, Lemma 4.21]). Suppose that* $\mathrm{card}(\Lambda) < \infty$, *$G$ acts on a Gromov hyperbolic space (S, d) by isometries, and the following three conditions are satisfied, then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$.*

(C$_1$) *Every H_λ acts on S properly.*
(C$_2$) *There exists $s \in S$ such that for every $\lambda \in \Lambda$, the H_λ-orbit $H_\lambda(s)$ of s is quasi-convex in S.*
(C$_3$) *For every $\epsilon > 0$ and some $s \in S$, there exists $R > 0$ such that the following holds. Suppose that for some $g \in G$ and $\lambda, \mu \in \Lambda$, we have*

$$\mathrm{diam}\left(H_\mu(s) \cap (gH_\lambda(s))^{+\epsilon}\right) \geqslant R,$$

then $\lambda = \mu$ and $g \in H_\lambda$, where $(gH_\lambda(s))^{+\epsilon}$ denotes the ϵ-neighborhood of $gH_\lambda(s)$ in S.

The following proposition, which roughly says that being a hyperbolically embedded subgroup is a transitive property, will be used later.

Proposition 3.10 *(Dahmani-Guirardel-Osin [27, Proposition 4.35]). Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a finite family of subgroups of G, let $X \subseteq G$, and let $Y_\lambda \subseteq H_\lambda$ for every $\lambda \in \Lambda$. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$ and for every $\lambda \in \Lambda$, there is a family of subgroups $\{K_{\lambda,\mu}\}_{\mu\in M_\lambda} \hookrightarrow_h (H_\lambda, Y_\lambda)$. Then*

$$\bigcup_{\lambda\in\Lambda}\{K_{\lambda,\mu}\}_{\mu\in M_\lambda} \hookrightarrow_h \left(G, X \cup \left(\bigcup_{\lambda\in\Lambda} Y_\lambda\right)\right).$$

The cohomological finiteness properties of a group G are inherited by its hyperbolically embedded subgroups:

Lemma 3.11 *(Dahmani-Guirardel-Osin [27, Remark 4.26 and Corollary 4.32]). Let G be a group with a hyperbolically embedded family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Suppose that G is of type FP_n for some $n \in \{2, 3, ..., \infty\}$. Then for every λ, the group H_λ is of type FP_n.*

3.4. Isolated components

In the proof of Theorem E, we need to construct specific hyperbolically embedded subgroups. A tool to do this is the notion of an isolated component, which was introduced by [60] for relatively hyperbolic groups and generalized to hyperbolically embedded subgroups by [27]. In this section, we recall the definition and collect several results.

We start with conventions. Let G be a group and X a generating set of G. Consider the Cayley graph $\Gamma(G, X)$. Let p be a path in $\Gamma(G, X)$. The *label* of p, denoted $\mathbf{Lab}(p)$, is obtained by concatenating all labels of the edges in p and is a word over X. The length p is denoted by $\ell_X(p)$, and the initial (resp. terminal) vertex of p is denoted by p^- (resp. p^+).

Now suppose that $\{H_\lambda\}_{\lambda\in\Lambda}$ is a family of subgroups of G. Let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$, and let X be a relative generating set of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$. For $\lambda \in \Lambda$, let $\widehat{d}_\lambda$ be the relative metric on H_λ with respect to X. The following terminology goes back to [59].

Definition 3.12. Let p be a path in $\Gamma(G, X \sqcup \mathcal{H})$. For every $\lambda \in \Lambda$, an H_λ-*subpath* q of p is a nontrivial subpath of p such that $\mathbf{Lab}(q)$ is a word over the alphabet H_λ (if p is a cycle, we allow q to be a subpath of some cyclic shift of p). An H_λ-subpath q of p is an H_λ-*component* if q is not properly contained in any other H_λ-subpath. Two H_λ-components q_1 and q_2 of p are *connected* if there exists a path t in $\Gamma(G, X \sqcup \mathcal{H})$ such that t connects a vertex of q_1 to a vertex of q_2, and that $\mathbf{Lab}(t)$ is a letter from H_λ. An H_λ-component q of p is *isolated* if it is not connected to any other H_λ-component of p.

Remark 3.13. The definition of connectedness in [27] is seemingly different from the version above: instead of requiring $\mathbf{Lab}(t)$ to be a letter from H_λ, [27] only requires $\mathbf{Lab}(t)$ to be a word over H_λ for some $\lambda \in \Lambda$. However, as every element of H_λ belongs to the generating set $X \sqcup \mathcal{H}$, if there is a path t connecting a vertex of q_1 to a vertex of q_2 with $\mathbf{Lab}(t)$ being a word over H_λ, then there is another path t' connecting a vertex of q_1 to a vertex of q_2 with $\mathbf{Lab}(t')$ being a single letter of H_λ ($\mathbf{Lab}(t')$ is the element of H_λ represented by $\mathbf{Lab}(t)$).

Suppose that q is an H_λ-component of a path $p \subseteq \Gamma(G, X \sqcup \mathcal{H})$. Then q^- (resp. q^+) is labeled by an element $g \in G$ (resp $h \in G$) and we have $g^{-1}h \in H_\lambda$. In this case, let

$$\widehat{\ell}_\lambda(q) = \widehat{d}_\lambda(1, g^{-1}h).$$

A nice property of isolated components is that in a geodesic polygon p, the total $\widehat{\ell}$-length of isolated components of p is bounded linearly by the number of sides of p. More precisely:

Proposition 3.14 *(Dahmani-Guirardel-Osin [27, Proposition 4.14] (see also [60, Proposition 3.2])). If $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$, then there exists a number $D > 0$ satisfying the following property: Let p be an n-gon in $\Gamma(G, X \sqcup \mathcal{H})$ with geodesic sides $p_1, ..., p_n$*

and let I be a subset of the set of sides of p such that every side $p_i \in I$ is an isolated H_{λ_i}-component of p for some $\lambda_i \in \Lambda$. Then

$$\sum_{p_i \in I} \widehat{\ell}_{\lambda_i}(p_i) \leqslant Dn.$$

In fact, the above proposition is the reason why certain properties (for example, Theorems 3.23 and 3.25 below) hold for *sufficiently deep* Dehn fillings (see Definition 3.19).

The technical lemma below will be used in Section 7.2 along with Lemma 3.9 to construct hyperbolically embedded subgroups.

Lemma 3.15 *(Dahmani-Guirardel-Osin [27, Lemma 4.21]). Suppose $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$. Let W be the set consisting of all words w over $X \sqcup \mathcal{H}$ such that*

(W1) *w contains no subwords of type xy, where $x, y \in X$;*
(W2) *if w contains a letter $h \in H_\lambda$ for some $\lambda \in \Lambda$, then $\widehat{d}_\lambda(1, h) > 50D$, where D is given by Proposition 3.14;*
(W3) *if $h_1 x h_2$ (resp. $h_1 h_2$) is a subword of w, where $x \in X, h_1 \in H_\lambda, h_2 \in H_\mu$, then either $\lambda \neq \mu$ or the element represented by x in G does not belong to H_λ (resp. $\lambda \neq \mu$).*

Then the following hold.

(a) *Every path in the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ labeled by a word from W is a $(4, 1)$-quasi-geodesic.*
(b) *If p is a path in $\Gamma(G, X \sqcup \mathcal{H})$ labeled by a word from W, then for every $\lambda \in \Lambda$, every H_λ-component of p is isolated.*
(c) *For every $\epsilon > 0$, there exists $R > 0$ satisfying the following condition. Let p, q be two paths in $\Gamma(G, X \sqcup \mathcal{H})$ such that*

$$\ell_{X\sqcup\mathcal{H}}(p) \geqslant R, \qquad \boldsymbol{Lab}(p), \boldsymbol{Lab}(q) \in W,$$

and p, q are oriented ϵ-close, i.e.,

$$\max\{d_{X\sqcup\mathcal{H}}(p^-, q^-), d_{X\sqcup\mathcal{H}}(p^+, q^+)\} \leqslant \epsilon,$$

where $d_{X\sqcup\mathcal{H}}$ is the combinatorial metric of $\Gamma(G, X \sqcup \mathcal{H})$. Then there exist five consecutive components of p which are respectively connected to five consecutive components of q. In other words,

$$p = ra_1x_1a_2x_2a_3x_3a_4x_4a_5s, \qquad q = tb_1y_1b_2y_2b_3y_3b_4y_4b_5u,$$

such that the following hold.

(i) *r (resp. t) is a subpath of p (resp. q) whose label does not end with a letter from $\mathcal{H}$.*
(ii) *s (resp. u) is a subpath of p (resp. q) whose label does not start with a letter from $\mathcal{H}$.*
(iii) *For $i = 1, ..., 4$, x_i and y_i are either trivial subpaths or subpaths labeled by a letter over X;*
(iv) *For $i = 1, ..., 5$, a_i and b_i are connected H_{λ_i}-components.*

Remark 3.16. Conclusion (b) of Lemma 3.15 is not stated in [27, Lemma 4.21], but it is proved in the second paragraph of the proof of [27, Lemma 4.21].

3.5. Acylindrical hyperbolicity

We notice that Examples 3.8 (b) and (c) are two occasions where having a hyperbolically embedded subgroup does not provide any information about the group G. We also notice that in those two cases, the hyperbolically embedded subgroup is either finite or improper. It is therefore natural to look at the groups G with a proper infinite hyperbolically embedded subgroup. By [27, Theorem 7.19] and [62, Theorem 1.2], this is equivalent to saying that G is acylindrically hyperbolic.

Definition 3.17. A group G is *acylindrically hyperbolic* if G admits a non-elementary acylindrical action on some Gromov hyperbolic space by isometries.

For the definition an acylindrical action, the reader is referred to [63]. Intuitively, one can think of acylindricity as an analog of properness. An acylindrical action of a group G is *non-elementary* if its orbits are unbounded and G is not virtually-cyclic [62, Theorem 1.1].

Techniques of hyperbolically embedded subgroups are mainly applied to acylindrically hyperbolic groups, because every acylindrically hyperbolic group contains infinitely many hyperbolically embedded virtually free subgroups.

Theorem 3.18 *(Dahmani-Guirardel-Osin [27, Theorem 6.14]). Let G be an acylindrically hyperbolic group. Then G has a unique maximal finite normal subgroup, denoted by $K(G)$. Moreover, for every $n \in \mathbb{N}^+$, there exists a free group F of rank n such that $F \times K(G) \hookrightarrow_h G$.*

3.6. Sufficient deepness and Cohen-Lyndon triples

One consequence of Proposition 3.14 is that acylindrical hyperbolicity is preserved by Dehn fillings, provided that the Dehn fillings are *sufficiently deep* and done on hyperbolically embedded subgroups (see Theorem 3.23).

Definition 3.19. Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ for some subset $X \subseteq G$. For every $\lambda \in \Lambda$, let $\widehat{d}_\lambda$ be the relative metric on H_λ with respect to X. A property $\mathcal{P}$ *holds for all sufficiently deep normal subgroups* if there exists a constant $C > 0$ such that $\mathcal{P}$ holds for every family of normal subgroups $\{N_\lambda \lhd H_\lambda\}_{\lambda\in\Lambda}$ with $\widehat{d}_\lambda(1, n) > C$ for all $n \in N_\lambda \smallsetminus \{1\}$.

Example 3.20. If $G = H_1 *_A H_2$ is a free product with amalgamation, then $\{H_1, H_2\} \hookrightarrow_{wh} (G, \varnothing)$ by Example 3.8 (e). For normal subgroups $\{N_i \lhd H_i\}_{i=1,2}$, consider the property

$\mathcal{P}$: The quotient $G/\langle\langle N_1 \cup N_2\rangle\rangle$ splits as an amalgamated free product, where $\langle\langle N_1 \cup N_2\rangle\rangle$ denotes the normal closure of $N_1 \cup N_2$ in G.

Then $\mathcal{P}$ holds for all sufficiently deep normal subgroup $\{N_i \lhd H_i\}_{i=1,2}$, because $\mathcal{P}$ holds whenever $N_i \cap A = \{1\}$, which amounts to saying that $\widehat{d}_i(n) > 1$ for all $n \in N_i \smallsetminus \{1\}$, where $\widehat{d}_i$ is the relative metric on H_i corresponding to the weak hyperbolic embedding $\{H_1, H_2\} \hookrightarrow_{wh} (G, \varnothing)$.

Remark 3.21. In Definition 3.19, if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G, X)$, then the relative metrics $\widehat{d}_\lambda$ are locally finite. Thus,

$$\operatorname{card}\Big(\{h \in H_\lambda \mid \widehat{d}_\lambda(1, h) \leqslant C\}\Big) < \infty$$

for all $C > 0$. Therefore:

Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$. Suppose that a property $\mathcal{P}$ holds for all sufficiently deep normal subgroups $\{N_\lambda \lhd H_\lambda\}_{\lambda\in\Lambda}$. Then there exist finite sets $\{\mathcal{F}_\lambda \subseteq H_\lambda \smallsetminus \{1\}\}_{\lambda\in\Lambda}$ such that $\mathcal{P}$ holds whenever $N_\lambda \cap \mathcal{F}_\lambda = \varnothing$ for all $\lambda \in \Lambda$.

If in addition, $|\Lambda| < \infty$, then a property $\mathcal{P}$ holds for all sufficiently deep normal subgroups $\{N_\lambda \lhd H_\lambda\}_{\lambda\in\Lambda}$ if and only if there exists a finite set $\mathcal{F} \subseteq \big(\bigcup_{\lambda\in\Lambda} H_\lambda\big) \smallsetminus \{1\}$ such that $\mathcal{P}$ holds whenever $N_\lambda \cap \mathcal{F} = \varnothing$. In particular, if G is a group with a hyperbolically embedded subgroup $H \hookrightarrow_h G$, then Definition 2.1 is a special case of Definition 3.19.

To simplify statements, we use the following notation.

Notation 3.22. Let G be a group and S a subset of G. Then $\langle\langle S\rangle\rangle$ denotes the normal closure of S in G. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda}$ is a family of subgroups of G and $\{N_\lambda \lhd H_\lambda\}_{\lambda\in\Lambda}$ is a family of normal subgroups. We call $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ a *group triple*. We also let

$$\mathcal{N} = \bigcup_{\lambda\in\Lambda} N_\lambda, \quad \overline{G} = G/\langle\langle \mathcal{N}\rangle\rangle, \quad \overline{H}_\lambda = H_\lambda/N_\lambda.$$

Theorem 3.23 *(Dahmani-Guirardel-Osin [27, Theorem 7.19], Osin [62, Theorem 1.2]). Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$. Then for all sufficiently* *deep normal subgroups* $\{N_\lambda \lhd H_\lambda\}_{\lambda\in\Lambda}$, *the natural homomorphism* $\overline{H}_\lambda \to \overline{G}$ *is injective for* $\lambda \in \Lambda$ *and we have* $\{\overline{H}_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h \overline{G}$. *Moreover, if for some* $\lambda \in \Lambda$, $\mathrm{card}(\overline{H}_\lambda) = \infty$ *and* $\overline{H}_\lambda$ *is a proper subgroup of* $\overline{G}$, *then* $\overline{G}$ *is acylindrically hyperbolic.*

The normal subgroup $\langle\langle \mathcal{N} \rangle\rangle$ in the above theorem can be described more precisely: it has a particular free product structure.

Definition 3.24. A group triple $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ is called a *Cohen-Lyndon triple* if there exist left transversals T_λ of $H_\lambda\langle\langle \mathcal{N} \rangle\rangle$ in G such that

$$\langle\langle \mathcal{N} \rangle\rangle = \ast_{\lambda\in\Lambda, t\in T_\lambda} tN_\lambda t^{-1}.$$

The above free product structure was first proved by Cohen-Lyndon [21] for free groups, hence the name. The following theorem was partially proved by [27, Theorem 7.19], which was later improved by [67].

Theorem 3.25 *(Sun [67, Theorem 5.1]). Let* G *be a group with a family of subgroups* $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ *for some* $X \subseteq G$. *Then for all sufficiently deep normal subgroups* $\{N_\lambda \lhd H_\lambda\}_{\lambda\in\Lambda}$, $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ *is a Cohen-Lyndon triple.*

4. Cohomology of Dehn fillings

In this section, we prove Theorem A. To simplify the notation, we use 3.22. Recall, given a group G, a subgroup $H \leqslant G$ and a $\mathbb{Z}H$-module M, the *induced module* is

$$\mathrm{Ind}_H^G M := \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

and the *co-induced module* is

$$\mathrm{CoInd}_H^G M := \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

The left action of G on $\mathbb{Z}G$ induces a $\mathbb{Z}G$-module structure on both the induced and the co-induced modules [19, §III.5].

Throughout, we adopt the standard convention that given $\mathbb{Z}G$-modules M and N, then $M \otimes N := M \otimes_{\mathbb{Z}} N$ and $M \otimes_G N := M \otimes_{\mathbb{Z}G} N$. We repeatedly make use of the following tenor identities which are a consequence of the associativity of the tensor products [19, §III.3 and III.5].

Lemma 4.1 *(Tensor product identities). Let* G *be a group, let* H *be a subgroup of* G, *and let* K *be a normal subgroup of* G. *For any* G*-modules* M, N *and* H*-module* L, *we have*

(i) $M \otimes \mathbb{Z}[G/H] \cong \mathrm{Ind}_H^G \mathrm{Res}_H^G M$ *as* G*-modules,*
(ii) $M \otimes_G N \cong (M \otimes_K N) \otimes_{G/K} \mathbb{Z}$,

(iii) $(L \otimes_H \mathbb{Z}G) \otimes_G N \cong L \otimes_H N$.

Proposition 4.2. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple. Then for any G-module A and $q > 1$ (also for any $\overline{G}$-module A and $q > 0$) there are isomorphisms*

$$H_q(\langle\langle\mathcal{N}\rangle\rangle; A) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Ind}_{\overline{H}_\lambda}^{\overline{G}} H_q(N_\lambda; A),$$

$$H^q(\langle\langle\mathcal{N}\rangle\rangle; A) \cong \prod_{\lambda\in\Lambda} \mathrm{CoInd}_{\overline{H}_\lambda}^{\overline{G}} H^q(N_\lambda; A)$$

induced by the inclusions $N_\lambda \hookrightarrow \langle\langle\mathcal{N}\rangle\rangle$.

We remark that an easy consequence of $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ being a Cohen-Lyndon triple is that the natural maps $\overline{H}_\lambda \to \overline{G}$ are injective [67, Lemma 6.4], and thus it makes sense to consider the (co)inductions $\mathrm{Ind}_{\overline{H}_\lambda}^{\overline{G}}$ and $\mathrm{CoInd}_{\overline{H}_\lambda}^{\overline{G}}$.

Proof. We will prove the homological version. The proof of the cohomological version will be analogous.

Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be the family of left transversals associated to the Cohen-Lyndon triple $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$, given by Definition 3.24. Then $\langle\langle\mathcal{N}\rangle\rangle = \ast_{\lambda\in\Lambda, t\in T_\lambda} tN_\lambda t^{-1}$. Consider the Bass-Serre tree X associated to this free product decomposition with vertex set V and edge set E. Viewing X as a 1-dimensional CW-complex, we get a short exact sequence of $\langle\langle\mathcal{N}\rangle\rangle$-modules

$$0 \to \mathbb{Z}[E] \xrightarrow{\partial_1} \mathbb{Z}[V] \xrightarrow{\partial_0} \mathbb{Z} \to 0,$$

where $C_1(X) = \mathbb{Z}[E]$, $C_0(X) = \mathbb{Z}[V]$, and ∂_* is the usual boundary map.

Since the terms in the above exact sequence are $\mathbb{Z}$-free modules, tensoring it with A preserves exactness [16, §V.6, p. 278-279] and we obtain the short exact sequence

$$0 \to \mathbb{Z}[E] \otimes A \to \mathbb{Z}[V] \otimes A \to A \to 0. \tag{3}$$

Let $F_i \twoheadrightarrow \mathbb{Z}$ be a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Tensoring the above sequence by $F_i \otimes_{\langle\langle\mathcal{N}\rangle\rangle} -$, we obtain a short exact sequence

$$0 \to F_i \otimes_{\langle\langle\mathcal{N}\rangle\rangle} (\mathbb{Z}[E] \otimes A) \to F_i \otimes_{\langle\langle\mathcal{N}\rangle\rangle} (\mathbb{Z}[V] \otimes A) \to F_i \otimes_{\langle\langle\mathcal{N}\rangle\rangle} A \to 0. \tag{4}$$

Note that there is an isomorphism of $\langle\langle\mathcal{N}\rangle\rangle$-modules

$$\mathbb{Z}[V] \cong \bigoplus_{\lambda\in\Lambda, t\in T_\lambda} \mathbb{Z}[\langle\langle\mathcal{N}\rangle\rangle / tN_\lambda t^{-1}].$$

The short exact sequence (4) then transforms to a short exact sequence of chain complexes

$$0 \to F_i \otimes_{\langle\langle \mathcal{N} \rangle\rangle} (\mathbb{Z}[E] \otimes A) \to \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} F_i \otimes_{tN_\lambda t^{-1}} A \xrightarrow{\theta} F_i \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A \to 0. \tag{5}$$

The long exact sequence corresponding to (5) yields

$$\cdots \to H_q(\langle\langle \mathcal{N} \rangle\rangle; \mathbb{Z}[E] \otimes A) \to \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} H_q(tN_\lambda t^{-1}; A) \xrightarrow{\theta_*} H_q(\langle\langle \mathcal{N} \rangle\rangle; A) \to$$

$$\cdots \to \mathbb{Z} \otimes_{\langle\langle \mathcal{N} \rangle\rangle} (\mathbb{Z}[E] \otimes A) \xrightarrow{f} \mathbb{Z} \otimes_{\langle\langle \mathcal{N} \rangle\rangle} (\mathbb{Z}[V] \otimes A) \to \mathbb{Z} \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A \to 0.$$

Since $\mathbb{Z}[E]$ is a free $\langle\langle \mathcal{N} \rangle\rangle$-module, $H_q(\langle\langle \mathcal{N} \rangle\rangle; \mathbb{Z}[E] \otimes A) = 0$ for $q > 0$ and f is injective if $\langle\langle \mathcal{N} \rangle\rangle$ acts trivially on A. This yields an isomorphism

$$\theta_* : \bigoplus_{\lambda \in \Lambda, t \in T_\lambda} H_q(tN_\lambda t^{-1}; A) \to H_q(\langle\langle \mathcal{N} \rangle\rangle; A) \tag{6}$$

for $q > 1$ and also for $q = 1$ if $\langle\langle \mathcal{N} \rangle\rangle$ acts trivially on A. The action of $\overline{G}$ on $H_q(\langle\langle \mathcal{N} \rangle\rangle; A)$ and the isomorphism θ_* endow $\bigoplus_{\lambda \in \Lambda, t \in T_\lambda} H_q(tN_\lambda t^{-1}; A)$ with a $\overline{G}$-action. Next, we will show that this action is a direct sum of permutation actions.

The $\overline{G}$-action on $H_q(\langle\langle \mathcal{N} \rangle\rangle; A)$ comes from the G-action on A and conjugation of G on $\langle\langle \mathcal{N} \rangle\rangle$ which is induced by the diagonal G-action on $F_* \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A$. More explicitly, each $g \in G$ acts as

$$\tau_g : F_i \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A \to F_i \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A, \ [x, a] \mapsto [gx, ga].$$

For each $\lambda \in \Lambda$ and $t \in T_\lambda$, the group $tH_\lambda t^{-1}$ acts on $F_i \otimes_{tN_\lambda t^{-1}} A$ by

$$[x, a] \mapsto [gx, ga], \ \forall g \in tH_\lambda t^{-1}.$$

This gives us a $tH_\lambda t^{-1}$-equivariant restriction of θ

$$\theta : F_i \otimes_{tN_\lambda t^{-1}} A \to F_i \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A,$$

which in turn induces the $tH_\lambda t^{-1}$-equivariant inclusion

$$\theta_* : H_q(tN_\lambda t^{-1}; A) \hookrightarrow H_q(\langle\langle \mathcal{N} \rangle\rangle; A).$$

Now, for each $\lambda \in \Lambda$ and $t \in T_\lambda$, the action of t on $H_q(\langle\langle \mathcal{N} \rangle\rangle; A)$ induces a map between two summands on the left-hand side of the isomorphism (6). To see this, note that the map defined by

$$\sigma_t : F_i \otimes_{N_\lambda} A \to F_i \otimes_{tN_\lambda t^{-1}} A,$$
$$[x, a] \mapsto [tx, ta],$$

satisfies $\theta \circ \sigma_t = \tau_t \circ \theta$. In homology, we then have $(\sigma_t)_* = \theta_*^{-1} \circ (\tau_t)_* \circ \theta_*$ as claimed.

We have shown that for each $\lambda \in \Lambda$, the G-action on $\bigoplus_{t \in T_\lambda} H_q(tN_\lambda t^{-1}; A)$ restricts to the $tH_\lambda t^{-1}$-action on the summand $H_q(tN_\lambda t^{-1}; A)$ and that it permutes $H_q(N_\lambda; A)$ and $H_q(tN_\lambda t^{-1}; A)$. It is not difficult to show now, see for example [19, Proposition III.5.3], that

$$\bigoplus_{t \in T_\lambda} H_q(tN_\lambda t^{-1}; A) \cong \mathrm{Ind}_{\overline{H}_\lambda}^{\overline{G}} H_q(N_\lambda; A).$$

This finishes the proof of the homology isomorphism of the lemma.

Next, we outline the proof of the cohomology isomorphism which is similar.

First, the analog of (3) is the exact sequence

$$0 \to A \to \mathrm{Hom}(\mathbb{Z}[V], A) \to \mathrm{Hom}(\mathbb{Z}[E], A) \to 0,$$

which can be identified with the cochain complex $C^*(X, A)$ of the tree X with the coefficients in A. Applying $\mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(F_i, -)$ to the above sequence, we obtain a short exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(F_i, A) \to \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(F_i, \mathrm{Hom}(\mathbb{Z}[V], A)) \\ &\to \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(F_i, \mathrm{Hom}(\mathbb{Z}[E], A)) \to 0. \end{aligned}$$

Using the tensor-hom adjunction, we obtain

$$\begin{aligned} \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(F_i, \mathrm{Hom}(\mathbb{Z}[V], A)) &\cong \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(F_i \otimes \mathbb{Z}[V], A) \\ &\cong \prod_{\lambda \in \Lambda} \mathrm{Hom}_{tN_\lambda t^{-1}}(F_i, A), \end{aligned}$$

where second isomorphism follows from Lemma 4.1 and the adjointness of restriction and extension of scalars functors [19, §III.3, eq. (3.3)]. Substituting in the long exact sequence above and taking the cohomology, gives the long exact sequence

$$\begin{gathered} 0 \to \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(\mathbb{Z}, A) \to \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(\mathbb{Z}[V], A) \xrightarrow{f} \mathrm{Hom}_{\langle\langle \mathcal{N} \rangle\rangle}(\mathbb{Z}[E], A) \to \\ \cdots \to H^q(\langle\langle \mathcal{N} \rangle\rangle; A) \xrightarrow{\theta^*} \prod_{\lambda \in \Lambda, t \in T_\lambda} H^q(tN_\lambda t^{-1}; A) \to H^q(\langle\langle \mathcal{N} \rangle\rangle; \mathrm{Hom}(\mathbb{Z}[E], A)) \to \cdots \end{gathered}$$

If $\langle\langle \mathcal{N} \rangle\rangle$ acts trivially on A, then the first three terms in the above sequence compute the cohomology of the quotient graph $X/\langle\langle \mathcal{N} \rangle\rangle$ which is contractible. Therefore f is surjective in this case. It follows that θ^* is an isomorphism for $q > 1$ and also for $q = 1$ if $\langle\langle \mathcal{N} \rangle\rangle$ acts trivially on A. By an analogous argument to the homological case but now using [19, Proposition III.5.8], it follows that

$$\prod_{t \in T_\lambda} H^q(tN_\lambda t^{-1}; A) \cong \mathrm{CoInd}_{\overline{H}_\lambda}^{\overline{G}} H^q(N_\lambda; A). \quad \square$$

Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple and A any G-module. There are Lyndon-Hoschild-Serre spectral sequences

$$\begin{aligned} E^{\lambda,2}_{p,q} &= H_p(\overline{H}_\lambda; H_q(N_\lambda; A)) \Rightarrow H_{p+q}(H_\lambda; A), \\ F^2_{p,q} &= H_p(\overline{G}; H_q(\langle\langle \mathcal{N} \rangle\rangle; A)) \Rightarrow H_{p+q}(G; A) \end{aligned}$$

associated with the triples $(H_\lambda, N_\lambda, A)$ and $(G, \langle\langle \mathcal{N} \rangle\rangle, A)$, respectively (see for example [19, Chapter VII]). Let $E_{p,q} = \bigoplus_{\lambda\in\Lambda} E^\lambda_{p,q}$. Then

$$E^2_{p,q} = \bigoplus_{\lambda\in\Lambda} H_p(\overline{H}_\lambda; H_q(N_\lambda; A)) \Rightarrow \bigoplus_{\lambda\in\Lambda} H_{p+q}(H_\lambda; A)$$

by Lemma 3.1.

The inclusions $\overline{H}_\lambda \hookrightarrow \overline{G}$ and $N_\lambda \hookrightarrow \langle\langle \mathcal{N} \rangle\rangle$ induce a morphism $\phi : E_{p,q} \to F_{p,q}$. By Proposition 4.2 and Shapiro's lemma, the restriction $\phi : E^2_{p,q} \to F^2_{p,q}$ is an isomorphism for all $q > 1$ and also for $q = 1$ if $\langle\langle \mathcal{N} \rangle\rangle$ acts trivially on A. In short:

Theorem 4.3. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple. Then for every G-module A, there is a natural morphism of homological spectral sequences $\phi : E_{p,q} \to F_{p,q}$ such that*

$$\begin{aligned} E^2_{p,q} &= \bigoplus_{\lambda\in\Lambda} H_p(\overline{H}_\lambda; H_q(N_\lambda; A)) \Rightarrow \bigoplus_{\lambda\in\Lambda} H_{p+q}(H_\lambda; A), \\ F^2_{p,q} &= H_p(\overline{G}; H_q(\langle\langle \mathcal{N} \rangle\rangle; A)) \Rightarrow H_{p+q}(G; A), \end{aligned}$$

ϕ is induced by the inclusions $\overline{H}_\lambda \hookrightarrow \overline{G}, N_\lambda \hookrightarrow \langle\langle \mathcal{N} \rangle\rangle$, and ϕ restricts to an isomorphism $\phi : E^2_{p,q} \xrightarrow{\cong} F^2_{p,q}$ for all $q > 1$ and also for $q = 1$ if $\langle\langle \mathcal{N} \rangle\rangle$ acts trivially on A.

Moreover, the analogous statement holds for cohomology as well.

We further investigate the morphism ϕ of the above theorem. Let $P_i \twoheadrightarrow \mathbb{Z}$ be a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\overline{G}$ and $S_i \twoheadrightarrow \mathbb{Z}$ a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The spectral sequence $E_{p,q}$ is induced by the double complex $C_{p,q} = \bigoplus_{\lambda\in\Lambda} (P_p \otimes S_q) \otimes_{H_\lambda} A$ and $F_{p,q}$ is induced by $D_{p,q} = (P_p \otimes S_q) \otimes_G A$. The surjections

$$(P_p \otimes S_q) \otimes_{H_\lambda} A \twoheadrightarrow (P_p \otimes S_q) \otimes_G A$$

induce a surjection $C_{p,q} \twoheadrightarrow D_{p,q}$, which in turn induces the morphism ϕ. Let $R_{p,q}$ be the kernel of the surjection $C_{p,q} \twoheadrightarrow D_{p,q}$ and $E_{p,q}(R)$ the spectral sequence associated with $R_{p,q}$. It turns out that $E_{p,q}(R) \Rightarrow H_{p+q+1}(G, \{H_\lambda\}_{\lambda\in\Lambda}; A)$ and we have the following commutative diagram of long exact sequences.

Proposition 4.4. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple and A a $\overline{G}$-module. Then there is a commutative diagram of exact sequences*

$$
\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{\lambda\in\Lambda} H_i(H_\lambda;A) \arrow[r, "\iota_*"] \arrow[d] & H_i(G;A) \arrow[r] \arrow[d] & H_i(G,\{H_\lambda\}_{\lambda\in\Lambda};A) \arrow[r] \arrow[d, "id"] & \bigoplus_{\lambda\in\Lambda} H_{i-1}(H_\lambda;A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bigoplus_{\lambda\in\Lambda} H_i(\overline{H}_\lambda;A) \arrow[r, "\overline{\iota}_*"] & H_i(\overline{G};A) \arrow[r] & H_i(G,\{H_\lambda\}_{\lambda\in\Lambda};A) \arrow[r] & \bigoplus_{\lambda\in\Lambda} H_{i-1}(\overline{H}_\lambda;A) \arrow[r] & \cdots
\end{tikzcd}
$$

where ι_* and $\overline{\iota}_*$ are induced by the inclusions $H_\lambda \hookrightarrow G$ and $\overline{H}_\lambda \hookrightarrow \overline{G}$, respectively. Moreover, the cohomological analog of the above statement holds as well.

Proof. We will only prove the homological version since the cohomological version is similar. First, we compute the limit of the spectral sequence $E_{p,q}(R)$.

Claim 4.4.1. $E_{p,q}(R) \Rightarrow H_{p+q+1}(G,\{H_\lambda\}_{\lambda\in\Lambda};A)$.

Proof of the claim. Let Δ be the kernel of the augmentation $\bigoplus_{\lambda\in\Lambda}\mathbb{Z}[G/H_\lambda] \to \mathbb{Z}$. Then we have a short exact sequence

$$0 \to P_p \otimes S_q \otimes \Delta \to P_p \otimes S_q \otimes \left(\bigoplus_{\lambda\in\Lambda}\mathbb{Z}[G/H_\lambda]\right) \to P_p \otimes S_q \to 0. \tag{7}$$

Note that

$$(P_p \otimes S_q \otimes \mathbb{Z}[G/H_\lambda]) \otimes_G A \cong (P_p \otimes S_q) \otimes_{H_\lambda} A.$$

Thus, tensoring (7) by $-\otimes_G A$ yields

$$0 \to (P_p \otimes S_q \otimes \Delta) \otimes_G A \to C_{p,q} \to D_{p,q} \to 0. \tag{8}$$

Let $T_n = \bigoplus_{p+q=n} P_p \otimes S_q$. Notice that $T_n \otimes \Delta \twoheadrightarrow \Delta$ is a free resolution of Δ over $\mathbb{Z}G$. Therefore, it can be used to compute the relative homology of G, which by definition given in (2), is

$$H_n(G,\{H_\lambda\}_{\lambda\in\Lambda};A) = \mathrm{Tor}^G_{n-1}(\Delta,A) = H_{n-1}((T_* \otimes \Delta)\otimes_G A).$$

Hence, to prove the claim, it suffices to show that

$$(P_p \otimes S_q \otimes \Delta) \otimes_G A \cong R_{p,q},$$

which will be established once we show that (8) is exact. It is easy to check that

$$0 \to A \otimes \Delta \to A \otimes \left(\bigoplus_{\lambda\in\Lambda}\mathbb{Z}[G/H_\lambda]\right) \to A \to 0$$

is exact (by e.g. [72, Corollary 3.1.5]). Since P_p is free abelian and S_q is a free G-module, we have an exact sequence

$$0 \to (P_p \otimes \Delta \otimes A) \otimes_G S_q \to \left[P_p \otimes A \otimes \left(\bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda] \right) \right] \otimes_G S_q \to (P_p \otimes A) \otimes_G S_q \to 0,$$

which, by basic tensor identities, transforms to the desired one. $\square$

Let us return to the proof of Proposition 4.4. We have a short exact sequence

$$0 \to R_{p,q} \to C_{p,q} \to D_{p,q} \to 0,$$

whose vertical homology gives the following long exact sequence of the E^1-terms of the associated spectral sequences

$$\cdots \to H_q(R_{p,*}) \to H_q(C_{p,*}) \to H_q(D_{p,*}) \to H_{q-1}(R_{p,*}) \to \cdots$$

We claim that $H_q(D_{p,*}) = H_q(C_{p,*})$ for all $q > 0$. To see this, first note that for all p and all $q > 0$, applying Lemma 4.1, we have

$$\begin{aligned}
D_{p,q} &= (P_p \otimes S_q) \otimes_G A \\
&= (S_q \otimes P_p \otimes A) \otimes_G \mathbb{Z} && \text{by Lemma 4.1 (ii)} \\
&= S_q \otimes_G (P_p \otimes A) && \text{by Lemma 4.1 (ii)} \\
&= (S_q \otimes_{\langle\langle \mathcal{N} \rangle\rangle} (P_p \otimes A)) \otimes_{\overline{G}} \mathbb{Z} && \text{by Lemma 4.1 (ii)} \\
&= ((S_q \otimes_{\langle\langle \mathcal{N} \rangle\rangle} \mathbb{Z}) \otimes P_p \otimes A) \otimes_{\overline{G}} \mathbb{Z} && \text{as } \langle\langle \mathcal{N} \rangle\rangle \text{ acts trivially on } P_p \text{ and } A \\
&= P_p \otimes_{\overline{G}} ((S_q \otimes_{\langle\langle \mathcal{N} \rangle\rangle} \mathbb{Z}) \otimes A) && \text{by Lemma 4.1 (ii)} \\
&= P_p \otimes_{\overline{G}} (S_q \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A) && \text{as } \langle\langle \mathcal{N} \rangle\rangle \text{ acts trivially on } A.
\end{aligned}$$

As P_p is a free $\mathbb{Z}\overline{G}$-module, we have

$$\begin{aligned}
H_q(D_{p,*}) &= P_p \otimes_{\overline{G}} H_q(S_* \otimes_{\langle\langle \mathcal{N} \rangle\rangle} A) \\
&= P_p \otimes_{\overline{G}} H_q(\langle\langle \mathcal{N} \rangle\rangle; A) \\
&= P_p \otimes_{\overline{G}} \left(\bigoplus_{\lambda \in \Lambda} \operatorname{Ind}_{\overline{H}_\lambda}^{\overline{G}} H_q(N_\lambda; A) \right) \\
&= \bigoplus_{\lambda \in \Lambda} \left(P_p \otimes_{\overline{G}} \operatorname{Ind}_{\overline{H}_\lambda}^{\overline{G}} H_q(N_\lambda; A) \right) \\
&= \bigoplus_{\lambda \in \Lambda} (P_p \otimes_{\overline{H}_\lambda} H_q(N_\lambda; A))
\end{aligned}$$

by Proposition 4.2 and Shapiro's lemma.

Similarly, we have

$$H_q(C_{p,*}) = \bigoplus_{\lambda \in \Lambda} \left(P_p \otimes_{\overline{H}_\lambda} H_q(N_\lambda; A) \right),$$

as desired.

The equality $H_q(D_{p,*}) = H_q(C_{p,*})$ for $q > 0$ shows that $E^1_{p,q}(R) = H_q(R_{p,*}) = 0$ for all $q > 0$ and hence $E^2_{p,0}(R) = H_{p+1}(G, \{H_\lambda\}_{\lambda\in\Lambda}; A)$ for all p. Also, since $\mathbb{Z} \otimes_{\langle\!\langle \mathcal{N} \rangle\!\rangle} A = A$, we have

$$E^1_{p,0}(C) = H_0(C_{p,*}) = \bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A,$$
$$E^1_{p,0}(D) = H_0(D_{p,*}) = P_p \otimes_{\overline{G}} A.$$

So, we get a short exact sequence

$$0 \to E^1_{p,0}(R) \to \bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A \to P_p \otimes_{\overline{G}} A \to 0, \tag{9}$$

whose long exact sequence is the bottom row of the desired diagram.

Similar to the proof of the claim, one can show that there is a short exact sequence

$$0 \to (S_q \otimes \Delta) \otimes_G A \to \bigoplus_{\lambda\in\Lambda} S_q \otimes_{H_\lambda} A \to S_q \otimes_G A \to 0. \tag{10}$$

As S_q is a free $\mathbb{Z}G$-module and we can view P_p as a $\mathbb{Z}G$-module (through the $\overline{G}$-action), there is a $\mathbb{Z}G$-module homomorphism from the resolution $S_q \twoheadrightarrow \mathbb{Z}$ to $P_p \twoheadrightarrow \mathbb{Z}$. This induces a map $\bigoplus_{\lambda\in\Lambda} S_q \otimes_{H_\lambda} A \to \bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A$ and a map $S_q \otimes_G A \to P_p \otimes_{\overline{G}} A$, yielding a commutative diagram

$$\begin{array}{ccc} \bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A & \longrightarrow & P_p \otimes_{\overline{G}} A \\ \uparrow & & \uparrow \\ \bigoplus_{\lambda\in\Lambda} S_q \otimes_{H_\lambda} A & \longrightarrow & S_q \otimes_G A \end{array} \tag{11}$$

By composing $\bigoplus_{\lambda\in\Lambda} S_q \otimes_{H_\lambda} A \to \bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A$ with $(S_q \otimes \Delta) \otimes_G A \to \bigoplus_{\lambda\in\Lambda} S_q \otimes_{H_\lambda} A$, we get a map $(S_q \otimes \Delta) \otimes_G A \to \bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A$, whose image lies in

$$\ker(\bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A \to P_p \otimes_{\overline{G}} A) = E^1_{p,0}(R)$$

as the diagram (11) commutes. We therefore have a map from (10) to (9), whose associated long exact sequence is the commutative diagram in the statement. □

Remark 4.5. Similar to the proof of Proposition 4.4, one can show that there is a short exact sequence

$$0 \to (P_p \otimes \overline{\Delta}) \otimes_{\overline{G}} A \to \bigoplus_{\lambda\in\Lambda} P_p \otimes_{\overline{H}_\lambda} A \to P_p \otimes_{\overline{G}} A \to 0, \tag{12}$$

where P_p and A are as in the proof of Proposition 4.4, and $\overline{\Delta}$ is the kernel of the augmentation $\bigoplus_{\lambda\in\Lambda}\mathbb{Z}[\overline{G}/\overline{H}_\lambda]\twoheadrightarrow\mathbb{Z}$. The natural map from (9) to (12) gives rise to a commutative diagram of the corresponding long exact sequences, which together with the five lemma yields an isomorphism

$$H_*(\overline{G},\{\overline{H}_\lambda\}_{\lambda\in\Lambda};A)\cong H_*(G,\{H_\lambda\}_{\lambda\in\Lambda};A) \tag{13}$$

(of course, under the assumption that $(G,\{H_\lambda\}_{\lambda\in\Lambda},\{N_\lambda\}_{\lambda\in\Lambda})$ is a Cohen-Lyndon triple and A is a $\overline{G}$-module). Moreover, the analogous isomorphism for cohomology holds as well.

An easy consequence of Proposition 4.4 is a direct sum decomposition of (co)homology.

Corollary 4.6. *Let $(G,\{H_\lambda\}_{\lambda\in\Lambda},\{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple and A a $\overline{G}$-module.*

(a) *If for some $p\in\mathbb{N}$, $\bigoplus_{\lambda\in\Lambda}H_p(H_\lambda;A)=0$ and the natural map $\bigoplus_{\lambda\in\Lambda}H_{p-1}(H_\lambda;A)\to H_{p-1}(G;A)$ is injective, then*

$$H_p(\overline{G};A)\cong H_p(G;A)\oplus\left(\bigoplus_{\lambda\in\Lambda}H_p(\overline{H}_\lambda;A)\right).$$

(b) *If for some $p\in\mathbb{N}$, $\prod_{\lambda\in\Lambda}H^p(H_\lambda;A)=0$ and the natural map $H^{p-1}(G;A)\to\prod_{\lambda\in\Lambda}H^{p-1}(H_\lambda;A)$ is surjective, then*

$$H^p(\overline{G};A)\cong H^p(G;A)\oplus\left(\prod_{\lambda\in\Lambda}H^p(\overline{H}_\lambda;A)\right).$$

Proof. We only prove the homological version (a) and point out that the proof of (b) is analogous. To shorten the notation, we denote $H_*(-;A)$ by $H_*(-)$, $\bigoplus_{\lambda\in\Lambda}H_*(H_\lambda;A)$ by $H_*(\mathcal{H})$, $\bigoplus_{\lambda\in\Lambda}H_*(\overline{H}_\lambda;A)$ by $H_*(\overline{\mathcal{H}})$, and $H_*(G,\{H_\lambda\}_{\lambda\in\Lambda};A)$ by $H_*(G,\mathcal{H})$. Use Proposition 4.4 and consider the commutative diagrams of exact sequences

$$\begin{array}{ccccccc}
H_{p+1}(\overline{\mathcal{H}}) & \xrightarrow{\phi_{p+1}} & H_{p+1}(\overline{G}) & \xrightarrow{\lambda_{p+1}} & H_{p+1}(G,\mathcal{H}) & \xrightarrow{\xi_{p+1}} & H_p(\overline{\mathcal{H}})\\
{\scriptstyle\psi_{p+1}}\big\uparrow & & {\scriptstyle\theta_{p+1}}\big\uparrow & & {\scriptstyle id_{p+1}}\big\uparrow & & {\scriptstyle\psi_p}\big\uparrow\\
H_{p+1}(\mathcal{H}) & \xrightarrow{r_{p+1}} & H_{p+1}(G) & \xrightarrow{\rho_{p+1}} & H_{p+1}(G,\mathcal{H}) & \xrightarrow{\eta_{p+1}} & H_p(\mathcal{H})
\end{array}$$

$$\begin{array}{ccccccc}
H_p(\overline{\mathcal{H}}) & \xrightarrow{\phi_p} & H_p(\overline{G}) & \xrightarrow{\lambda_p} & H_p(G,\mathcal{H}) & \xrightarrow{\xi_p} & H_{p-1}(\overline{\mathcal{H}})\\
{\scriptstyle\psi_p}\big\uparrow & & {\scriptstyle\theta_p}\big\uparrow & & {\scriptstyle id_p}\big\uparrow & & {\scriptstyle\psi_{p-1}}\big\uparrow\\
H_p(\mathcal{H}) & \xrightarrow{r_p} & H_p(G) & \xrightarrow{\rho_p} & H_p(G,\mathcal{H}) & \xrightarrow{\eta_p} & H_{p-1}(\mathcal{H})
\end{array}$$

where the horizontal maps come from the exact sequences for the pairs $(G, \mathcal{H})$, $(\overline{G}, \overline{\mathcal{H}})$ and ψ_*, θ_* are natural maps induced by the surjections $G \twoheadrightarrow \overline{G}, H_\lambda \twoheadrightarrow \overline{H}_\lambda$, respectively.

The hypothesis implies that $\rho_p : H_p(G) \to H_p(G, \mathcal{H})$ is an isomorphism which in turn shows that the map λ_p is surjective. Since $H_p(\mathcal{H}) = 0$, ρ_{p+1} is surjective. So λ_{p+1} is also surjective. This shows $\xi_{p+1} = 0$ and hence ϕ_p is injective. Since $\theta_p \circ (\rho_p)^{-1}$ is a section for λ_p, the result follows. □

An immediate corollary to the above is an estimate of the (co)homological dimension. To shorten the notation, if $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ is a Cohen-Lyndon triple, let

$$\begin{aligned} \mathrm{hd}(\mathcal{H}) &= \sup_{\lambda\in\Lambda}\{\mathrm{hd}(H_\lambda)\}, \ \ \mathrm{hd}(\overline{\mathcal{H}}) = \sup_{\lambda\in\Lambda}\{\mathrm{hd}(\overline{H}_\lambda)\}, \\ \mathrm{cd}(\mathcal{H}) &= \sup_{\lambda\in\Lambda}\{\mathrm{cd}(H_\lambda)\}, \ \ \mathrm{cd}(\overline{\mathcal{H}}) = \sup_{\lambda\in\Lambda}\{\mathrm{cd}(\overline{H}_\lambda)\}. \end{aligned}$$

Corollary 4.7. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a Cohen-Lyndon triple. Then*

$$\mathrm{hd}(\overline{G}) \leqslant \max\{\mathrm{hd}(G), \mathrm{hd}(\mathcal{H}) + 1, \mathrm{hd}(\overline{\mathcal{H}})\}, \quad \mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(\mathcal{H}) + 1, \mathrm{cd}(\overline{\mathcal{H}})\}.$$

By Theorem 3.2, the commutativity of colimits of coefficients with the cohomology functor can be used to characterize property FP_n, which is our next goal.

Theorem 4.8. *Let G be a group with a finite family of subgroups $\{H_i\}_{i=1}^m \hookrightarrow_h G$. Suppose that G is of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$. Then for sufficiently deep $\{N_i \lhd H_i\}_{i=1}^m$, $\overline{G}$ is of type FP_n if and only if all $\overline{H}_i$ are of type FP_n.*

Proof. Suppose that $\overline{G}$ is of type FP_n. By Theorem 3.23, we may assume that $\{\overline{H}_i\}_{i=1}^m \hookrightarrow_h \overline{G}$. Then by Lemma 3.11, $\overline{H}_i$ are of type FP_n as $\overline{G}$ is.

Conversely, suppose that all $\overline{H}_i$ are of type FP_n. To shorten notations, we write $H^*(G, \mathcal{H}; -)$ for $H^*(G, \{H_i\}_{i=1}^m; -)$, $H^*(\mathcal{H}; -)$ for $\prod_{i=1}^m H^*(H_i; -)$, and $H^*(\overline{\mathcal{H}}; -)$ for $\prod_{i=1}^m H^*(\overline{H}_i; -)$. Let $\{A_j\}_{j\in J}$ be a direct system of $\overline{G}$-modules such that $\varinjlim A_j = 0$. By Proposition 4.4, we have a commutative diagram of exact sequences for each $j \in J$:

$$\begin{tikzcd} \cdots \arrow[r] & H^k(G, \mathcal{H}; A_j) \arrow[r] \arrow[d, "id"] & H^k(\overline{G}; A_j) \arrow[r] \arrow[d] & H^k(\overline{\mathcal{H}}; A_i) \arrow[r] \arrow[d] & H^{k+1}(G, \mathcal{H}; A_j) \arrow[r] \arrow[d, "id"] & \cdots \\ \cdots \arrow[r] & H^k(G, \mathcal{H}; A_j) \arrow[r] & H^k(G; A_j) \arrow[r] & H^k(\mathcal{H}; A_j) \arrow[r] & H^{k+1}(G, \mathcal{H}; A_j) \arrow[r] & \cdots \end{tikzcd}$$

The above remains a commutative diagram of exact sequences after taking direct limit, by [72, Theorem 2.6.15]. For $k \leqslant n$, we have $\varinjlim H^k(G; A_j) = 0$ by Theorem 3.2. By Lemma 3.11, we also have $\varinjlim H^k(\mathcal{H}; A_j) = 0$, and thus $\varinjlim H^k(G, \mathcal{H}; A_j) = 0$, which implies that $\varinjlim H^k(\overline{G}; A_j) = \varinjlim H^k(\overline{\mathcal{H}}; A_j) = 0$ except for $k = n$. Since $\varinjlim H^n(\overline{\mathcal{H}}; A_j) = 0$, we have $\varinjlim H^n(\overline{G}; A_j) = 0$ as well, and thus $\overline{G}$ is of type FP_n, again by Theorem 3.2. □

We emphasize that the finiteness of Λ is needed in the above proof to guarantee that $\varinjlim H^k(\mathcal{H}; A_j) = \varinjlim H^k(\overline{\mathcal{H}}; A_j) = 0$ for $k \leqslant n$.

Corollary 4.9. *Let G be a group with a finite family of subgroups $\{H_i\}_{i=1}^m \hookrightarrow_h G$. Suppose that G is of type FP. Then for sufficiently deep $\{N_i \lhd H_i\}_{i=1}^m$, $\overline{G}$ is of type FP if and only if all $\overline{H}_i$ are of type FP.*

Proof. Theorem 4.8 implies that $\overline{G}$ is of type FP_∞ if and only if all $\overline{H}_i$ are of type FP_∞. So we only need to prove that $\mathrm{cd}(\overline{G}) < \infty$ if and only if $\mathrm{cd}(\overline{H}_i) < \infty$ for all i. Suppose that $\mathrm{cd}(\overline{G}) < \infty$. Then since $\overline{H}_i$ are subgroups of $\overline{G}$, we have $\mathrm{cd}(\overline{H}_i) \leqslant \mathrm{cd}(\overline{G}) < \infty$. Conversely, suppose that $\mathrm{cd}(\overline{H}_i) < \infty$ for all i. Since G is of type FP, we have $\mathrm{cd}(G) < \infty$. As H_i are subgroups of G, we also have $\mathrm{cd}(H_i) < \infty$ for all i. Then Corollary 4.7 gives

$$\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \max_{1\leqslant i\leqslant m}\{\mathrm{cd}(H_i)+1\}, \max_{1\leqslant i\leqslant m}\{\mathrm{cd}(\overline{H}_i)\}\} < \infty. \quad \square$$

Proof of Theorem A. By Theorem 3.25, (G, H, N) is a Cohen-Lyndon triple for sufficiently deep $N \lhd H$. Item (ii) follows directly from Propositions 4.4 and Remark 4.5.

Theorem 4.3 provides us a spectral sequence

$$E_2^{p,q} \Rightarrow H^{p+q}(G; A)$$

and isomorphisms

$$E_2^{p,q} \cong H^p(\overline{H}; H^q(N; A))$$

for $q > 0$. Theorem A (i) then follows by observing that $H^p(\overline{G}; H^0(\langle\langle N\rangle\rangle; A)) \cong H^p(\overline{G}; A)$. Similarly, one can prove the homological version. $\square$

Corollary 2.2 follows directly from Proposition 4.6, Corollary 4.7, Theorem 4.8, and Corollary 4.9.

We collect the results of this section and state the full version of Theorem A.

Theorem 4.10. *Let G be a group with a hyperbolically embedded family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$. Then the following hold for all sufficiently deep $N_\lambda \lhd H_\lambda, \lambda \in \Lambda$ and all $\overline{G}$-modules A, where $\overline{G} = G/\langle\langle \cup_{\lambda\in\Lambda} N_\lambda\rangle\rangle$.*

(i) *There is a spectral sequence*

$$E_2^{p,q}(A) = \begin{cases} \prod_{\lambda\in\Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A)) & \text{for } q > 0 \\ H^p(\overline{G}; A) & \text{for } q = 0 \end{cases} \Rightarrow H^{p+q}(G; A),$$

where $\overline{H}_\lambda = H_\lambda/N_\lambda$ for all λ and the action of G on A factors through $\overline{G}$. In particular, the action of $\langle\langle \cup_{\lambda\in\Lambda} N_\lambda\rangle\rangle$ on A fixes A pointwise.

(ii) *(Algebraic Excision) For all $n \geqslant 0$ and $\lambda \in \Lambda$, there is a natural isomorphism induced by the quotient maps $G \to \overline{G}$ and $H_\lambda \to \overline{H}_\lambda$,*

$$H^n(\overline{G}, \overline{H}_\lambda; A) \cong H^n(G, H_\lambda; A).$$

(iii) *For all $n \geqslant \sup_{\lambda\in\Lambda} \operatorname{cd}(H_\lambda) + 2$,*

$$H^n(\overline{G}; A) \cong H^n(G; A) \oplus \left(\prod_{\lambda\in\Lambda} H^n(\overline{H}_\lambda; A)\right).$$

(iv) $\operatorname{cd}(\overline{G}) \leqslant \max\{\operatorname{cd}(G), \sup_{\lambda\in\Lambda} \operatorname{cd}(H_\lambda) + 1, \sup_{\lambda\in\Lambda} \operatorname{cd}(\overline{H}_\lambda)\}$.

(v) *If $|\Lambda| < \infty$ and G is of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then $\overline{G}$ is of type FP_n if and only if every $\overline{H}_\lambda$ is of type FP_n.*

(vi) *If $|\Lambda| < \infty$ and G is of type FP, then $\overline{G}$ is of type FP if and only if every $\overline{H}_\lambda$ is of type FP.*

We end this section by showing that the assumption $n \geqslant \operatorname{cd}(H) + 2$ in Corollary 2.2 (i) cannot be dropped.

Example 4.11. Let G be a free group with basis $\{x, y\}$ and let $H = \langle h\rangle \leqslant G$ where $h = xyx^{-1}y^{-1}$. Then $H \hookrightarrow_h G$ by Example 1.3 and $\operatorname{cd}(H) + 1 = 2$. Let $N = \langle h^k\rangle \lhd H$. Note that we can pick k large enough so that N avoids any given finite subset of $H \setminus \{1\}$. By [52, Theorem 11.1], $H^2(\overline{G}; \mathbb{Z}) \cong \mathbb{Z}$, and it is well-known that $H^2(G; \mathbb{Z}) = 0$ and $H^2(\overline{H}; \mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$. Thus, $H^2(\overline{G}; \mathbb{Z}) \ncong H^2(G; \mathbb{Z}) \oplus H^2(\overline{H}; \mathbb{Z})$. Similarly, $\operatorname{hd}(H) + 1 = 2$ and one can show that $H_2(\overline{G}; \mathbb{Z}) \ncong H_2(G; \mathbb{Z}) \oplus H_2(\overline{H}; \mathbb{Z})$.

5. Dehn fillings and duality

Let G a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a finite collection of subgroups. Following Bieri-Eckmann [6], we say that $(G, \mathcal{H})$ is a *duality pair* of dimension n, with dualizing module C, if for all $k \in \mathbb{Z}$ and all G-modules A, one has

$$H^k(G; A) \cong H_{n-k}(G, \mathcal{H}; C \otimes A)$$
$$H^k(G, \mathcal{H}; A) \cong H_{n-k}(G; C \otimes A)$$

given by the cap product by the fundamental class $e \in H_n(G, \mathcal{H}; C)$. Here, we adapt the convention that $H^k(G; A) = H^k(G, \mathcal{H}; A) = H_k(G, \mathcal{H}; C \otimes A) = H_k(G; C \otimes A) = 0$ for all $k < 0$. In which case, it follows that $C \cong H^n(G, \mathcal{H}; \mathbb{Z}G)$. If $C = \mathbb{Z}$ with trivial G-action, the pair is called an (orientable) *Poincaré duality* pair, in short, a PD(n)-pair.

Let $(G, \mathcal{H})$ be a duality pair of dimension n with dualizing module C and let Δ be the kernel of the augmentation $\bigoplus_{i=1}^m \mathbb{Z}[G/H_i] \to \mathbb{Z}$. By [6, Theorem 4.2], G is a duality group of dimension $n-1$ with dualizing module $\Delta \otimes C$ and each H_i is a duality group of dimension $n-1$ with dualizing module C (thinking as an H_i-module).

The following result generalizes [71, Corollary 1.5] which deals with the case where $(G, \mathcal{H})$ is a type F_∞ relatively hyperbolic group pair.

Corollary 5.1. *Let G be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of subgroups such that $\mathcal{H} \hookrightarrow_h G$. Suppose $(G, \mathcal{H})$ is a duality pair of dimension n, with dualizing module C. Then for all sufficiently deep $\{N_i \lhd H_i\}_{i=1}^m$ and $k \in \mathbb{Z}$*

$$H^k(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) \cong \begin{cases} \bigoplus_{i=1}^m \operatorname{Ind}_{\overline{H}_i}^{\overline{G}} H_{n-k}(N_i; C) & \text{for } k \neq n, \\ C \otimes_{\langle\langle \mathcal{N} \rangle\rangle} \mathbb{Z} & \text{for } k = n. \end{cases}$$

Proof. By Theorem 4.10 (ii) (see also Remark 4.5) and duality, for $k < n$ one has

$$H^k(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) \cong H_{n-k}(G; C \otimes \mathbb{Z}\overline{G}) \cong H_{n-k}(G; \operatorname{Ind}_{\langle\langle \mathcal{N} \rangle\rangle}^G \operatorname{Res}_{\langle\langle \mathcal{N} \rangle\rangle}^G C) \cong H_{n-k}(\langle\langle \mathcal{N} \rangle\rangle; C),$$

where the second isomorphism follows from Lemma 4.1. The claim now follows from Theorem 3.25 and Proposition 4.2. Also,

$$\begin{aligned} H^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) &\cong H_0(G; C \otimes \mathbb{Z}\overline{G}) \cong (C \otimes \mathbb{Z}\overline{G}) \otimes_{\mathbb{Z}G} \mathbb{Z} \cong (C \otimes_{\langle\langle \mathcal{N} \rangle\rangle} \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z} \\ &\cong C \otimes_{\langle\langle \mathcal{N} \rangle\rangle} \mathbb{Z}. \quad \square \end{aligned}$$

It is worth noting that the assumption that the collection $\mathcal{H}$ is finite in Corollary 5.1 cannot be dropped, since if $(G, \mathcal{H})$ is a duality pair, then $\mathcal{H}$ must be finite [6, Theorem 4.2].

Theorem 5.2. *Let G be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of subgroups such that $\mathcal{H} \hookrightarrow_h G$. Suppose, for some integer $0 < n$, $(G, \mathcal{H})$ is a PD(n)-pair and that there are sufficiently deep $\{\mathbb{Z} \cong N_i \lhd H_i\}_{i=1}^m$ such that each member of $\overline{\mathcal{H}} = \{\overline{H}_i\}_{i=1}^m$ is a PD($n-2$)-group. Then $\overline{G}$ is a PD(n)-group.*

Proof. For each $1 \leqslant i \leqslant m$, $H^k(\overline{H}_i; \mathbb{Z}\overline{G}) = 0$ if $k \neq n-2$ and by Corollary 5.1, $H^k(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) = 0$ if $k \neq n-1, n$. The long exact sequence in cohomology for the pair $(\overline{G}, \overline{\mathcal{H}})$ shows that $H^k(\overline{G}; \mathbb{Z}\overline{G}) = 0$ if $k \neq n-2, n-1, n$ and for $k = n$ it gives $H^n(\overline{G}; \mathbb{Z}\overline{G}) \cong H^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z}\overline{G}) \cong \mathbb{Z}$.

Next, we consider the spectral sequence of Theorem 4.10

$$E_2^{p,q} = \begin{cases} \prod_{i=1}^m H^p(\overline{H}_i; H^q(N_i; \mathbb{Z}\overline{G})) & \text{for } q > 0 \\ H^p(\overline{G}; \mathbb{Z}\overline{G}) & \text{for } q = 0 \end{cases} \Rightarrow H^{p+q}(G; \mathbb{Z}\overline{G}).$$

Since each $N_i \cong \mathbb{Z}$, $E_2^{p,q} = 0$ for $q > 1$ and $E_2^{p,1} \cong \prod_{i=1}^m H^p(\overline{H}_i; \mathbb{Z}\overline{G})$. Since there are only two nontrivial rows on the $E_2^{p,q}$-page of the spectral sequence, by the cohomological analog of [72, Ex. 5.2.2], see also [20, Theorem XV.5.11], we obtain a long exact sequence

$$\begin{aligned}\ldots \to \prod_{i=1}^{m} H^{k-2}(\overline{H}_i; H^1(N_i;\mathbb{Z}\overline{G})) \xrightarrow{d_2} H^k(\overline{G};\mathbb{Z}\overline{G}) \to H^k(G;\mathbb{Z}\overline{G}) \\ \to \prod_{i=1}^{m} H^{k-1}(\overline{H}_i; H^1(N_i;\mathbb{Z}\overline{G})) \xrightarrow{d_2} \ldots\end{aligned} \tag{14}$$

For $k = n-2$, (14) yields

$$\begin{aligned} H^{n-2}(\overline{G};\mathbb{Z}\overline{G}) &\cong H^{n-2}(G;\mathbb{Z}\overline{G}) \\ &\cong H_2(G,\mathcal{H};\mathbb{Z}\overline{G}) \\ &\cong H_2(\overline{G},\overline{\mathcal{H}};\mathbb{Z}\overline{G}) = 0. \end{aligned}$$

For $k = n-1$, (14) and the morphism between the Lyndon-Hochschild-Serre spectral sequences associated to the inclusions $(H_\lambda, N_\lambda) \hookrightarrow (G, \langle\langle N\rangle\rangle)$ give us

$$\begin{tikzcd} 0 \arrow[r] & H^{n-1}(\overline{G};\mathbb{Z}\overline{G}) \arrow[r] & H^{n-1}(G;\mathbb{Z}\overline{G}) \arrow[r] \arrow[d, "r^{n-1}"] & \prod_{i=1}^m H^{n-2}(\overline{H}_i; H^1(N_i;\mathbb{Z}\overline{G})) \arrow[d, "="] \\ & & \prod_{i=1}^m H^{n-1}(H_i;\mathbb{Z}\overline{G}) \arrow[r, "\cong"] & \prod_{i=1}^m H^{n-2}(\overline{H}_i; H^1(N_i;\mathbb{Z}\overline{G})) \end{tikzcd}$$

where the bottom isomorphism follows from the Lyndon-Hochschild-Serre spectral sequence applied to the extensions $1 \to N_i \to H_i \to \overline{H}_i \to 1$. To show that $H^{n-1}(\overline{G};\mathbb{Z}\overline{G}) = 0$ amounts to showing that r^{n-1} is injective. By [6, Theorem 2.1], there is a commutative diagram

$$\begin{tikzcd} H^{n-1}(G;\mathbb{Z}\overline{G}) \arrow[r, "r^{n-1}"] \arrow[d, "\cong"] & \prod_{i=1}^m H^{n-1}(H_i;\mathbb{Z}\overline{G}) \arrow[d, "\cong"] \\ H_1(G,\mathcal{H};\mathbb{Z}\overline{G}) \arrow[r, "\partial"] & \bigoplus_{i=1}^m H_0(H_i;\mathbb{Z}\overline{G}) \end{tikzcd}$$

where by duality the vertical maps are isomorphisms. Thus, it suffices to show that the connecting homomorphism ∂ is injective. This follows from the commutativity of the diagram

$$\begin{tikzcd} H_1(G,\mathcal{H};\mathbb{Z}\overline{G}) \arrow[r, "\partial"] \arrow[d, "\cong"] & \bigoplus_{i=1}^m H_0(H_i;\mathbb{Z}\overline{G}) \arrow[d, "\cong"] \\ H_1(\overline{G},\overline{\mathcal{H}};\mathbb{Z}\overline{G}) \arrow[r, "\overline{\partial}"] & \bigoplus_{i=1}^m H_0(\overline{H}_i;\mathbb{Z}\overline{G}) \end{tikzcd}$$

and the fact the kernel of $\overline{\partial}$ is $H_1(\overline{G},\mathbb{Z}\overline{G}) = 0$. This finishes the proof of $H^{n-1}(\overline{G};\mathbb{Z}\overline{G}) = 0$.

By [6, Theorem 6.2], G and each $\overline{H}_i$ are of type FP. So, by Corollary 4.9, $\overline{G}$ is of type FP. We have also established that $H^k(\overline{G};\mathbb{Z}\overline{G}) = 0$ if $k \neq n$ and $H^n(\overline{G};\mathbb{Z}\overline{G}) \cong \mathbb{Z}$. By [6, Theorem 6.2(i)], $\overline{G}$ is a PD(n)-group. □

6. Simplicial volume of Dehn fillings

For detailed background on bounded cohomology and simplicial volume, we refer to [38], [44] and [32].

Let G be a group. Consider the singular chain complex $C_*(BG;\mathbb{R})$ endowed with the ℓ^1-norm

$$|c|_1 = \sum_{i=1}^{k} |a_i|, \quad \forall c = \sum_{i=1}^{k} a_i\sigma_i \in C_*(BG;\mathbb{R}).$$

The cochain complex of bounded cochains $C_b^*(BG;\mathbb{R})$ coincides with the normed dual complex of $C_*(BG;\mathbb{R})$. The norm on chains induces a ℓ^1-semi-norm $||\cdot||_1$ on $H_*(G;\mathbb{R}) = H_*(C_*(BG;\mathbb{R}))$ and ℓ^∞-semi-norm $||\cdot||_\infty$ on $H_b^*(G;\mathbb{R}) = H^*(C_b^*(BG;\mathbb{R}))$. When $\mathcal{H} = \{H_i\}_{i=1}^m$ is a collection of subgroups of G, $H_*(G,\mathcal{H};\mathbb{R})$ and $H_b^*(G,\mathcal{H};\mathbb{R})$ and their semi-norms are defined analogously [55, §9.2].

As before, when $N_i \lhd H_i$, we let $\mathcal{N} = \bigcup_{i=1}^m N_i$, $\overline{\mathcal{H}} = \{\overline{H}_i\}_{i=1}^m$ and $\overline{G} = G/\langle\langle\mathcal{N}\rangle\rangle$.

Lemma 6.1. *Let G be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of subgroups such that $\mathcal{H} \hookrightarrow_h G$. Suppose, for some integer $n \geqslant 2$, $(G,\mathcal{H})$ is a PD(n)-pair and for a sufficiently deep $\{N_i \lhd H_i\}_{i=1}^m$, $\mathrm{cd}(\overline{H}_i) \leqslant n-2$ for each $1 \leqslant i \leqslant m$. Then $H_n(\overline{G};\mathbb{Z}) = \mathbb{Z}$.*

Proof. The result follows from the long exact sequence in homology of the pair $(\overline{G},\overline{\mathcal{H}})$ and, by Theorem 4.10, the isomorphism $H_n(\overline{G},\overline{\mathcal{H}};\mathbb{Z}) \cong H_n(G,\mathcal{H};\mathbb{Z}) = \mathbb{Z}$. □

Definition 6.2. Under the hypothesis of Lemma 6.1, we define the *simplicial volume* of $\overline{G}$ by

$$||\overline{G}|| = \inf\{|c|_1 \mid c \in C_n(B\overline{G};\mathbb{R}),\ [c] = [\overline{G}] \in H_n(\overline{G};\mathbb{R})\},$$

where $[G]$ is the image of the fundamental class under the change of coefficients map $H_n(\overline{G};\mathbb{Z}) \to H_n(\overline{G};\mathbb{R})$. Similarly, we define

$$||G,\mathcal{H}|| = \inf\{|f|_1 \mid f \in C_n(BG,\bigsqcup_{i=1}^m BH_i);\mathbb{R}),\ [f] = [G,\mathcal{H}] \in H_n(G,\mathcal{H};\mathbb{R})\}.$$

The simplicial volume of $(\overline{G},\overline{\mathcal{H}})$ is defined analogously.

The following result is a generalization of [30, Theorem 1.5].

Theorem 6.3. *Let G be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of subgroups such that $\mathcal{H} \hookrightarrow_h G$. Suppose, for some integer $n \geqslant 2$, $(G, \mathcal{H})$ is a PD(n)-pair and for a sufficiently deep $\{N_i \lhd H_i\}_{i=1}^m$, $\operatorname{cd}(\overline{H}_i) \leqslant n-2$ for each $1 \leqslant i \leqslant m$. Then, $\operatorname{cd}(\overline{G}) = n$, $H_n(\overline{G}; \mathbb{Z}) = \mathbb{Z}$. In addition,*

(i) *if the group $\overline{H}_i$ is amenable for each $1 \leqslant i \leqslant m$, then $||\overline{G}|| \leqslant ||G, \mathcal{H}||$;*
(ii) *if G is hyperbolic relative to $\mathcal{H}$, then $||\overline{G}|| > 0$.*

Proof. Recall that since $(G, \mathcal{H})$ is a PD(n)-pair, each H_i is a PD($n-1$)-group [6, Theorem 4.2] and hence $\operatorname{cd}(H_i) = n - 1$. The first two claims now follow from Theorem 4.10 (iv) and Lemma 6.1.

For part (i), by Theorem 4.10 (ii), $p_n : H_n(G, \mathcal{H}; \mathbb{Z}) \to H_n(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z})$ induced by $(G, \mathcal{H}) \twoheadrightarrow (\overline{G}, \overline{\mathcal{H}})$ is an isomorphism. Since each $\overline{H}_i$ is amenable, $j^n : H_b^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \to H_b^n(\overline{G}; \mathbb{R})$ is an isometric isomorphism [32, Theorem 5.14]. By Duality Principle [32, Lemma 6.1], we obtain

$$
\begin{aligned}
||\overline{G}|| &= \sup\{\langle j^n(\psi), [\overline{G}]\rangle \mid \psi \in H_b^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}),\ ||\psi||_\infty \leqslant 1\}, \\
&= \sup\{\langle \psi, j_n([\overline{G}])\rangle \mid \psi \in H_b^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}),\ ||\psi||_\infty \leqslant 1\}, \\
&= ||j_n([\overline{G}])||_1, \\
&= ||\overline{G}, \overline{\mathcal{H}}||, \\
&\leqslant ||G, \mathcal{H}||,
\end{aligned}
$$

where the last inequality follows from the functoriality of the ℓ^1-semi-norm [48, Proposition 2.1].

For part (ii), since G is hyperbolic relative to $\mathcal{H}$, then $\overline{G}$ is hyperbolic relative $\overline{\mathcal{H}}$ [60, Theorem 1.1]. Consider the commutative diagram of long exact sequences

$$
\begin{tikzcd}
\prod_{i=1}^m H_b^{n-1}(\overline{H}_i; \mathbb{R}) \arrow[r] \arrow[d] & H_b^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \arrow[r] \arrow[d] & H_b^n(\overline{G}; \mathbb{R}) \arrow[r] \arrow[d, "c"] & \prod_{i=1}^m H_b^n(\overline{H}_i; \mathbb{R}) \arrow[d] \\
\prod_{i=1}^m H^{n-1}(\overline{H}_i; \mathbb{R}) \arrow[r] & H^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \arrow[r] & H^n(\overline{G}; \mathbb{R}) \arrow[r] & \prod_{i=1}^m H^n(\overline{H}_i; \mathbb{R})
\end{tikzcd}
$$

Since $\operatorname{cd}(\overline{H}_i) \leqslant n - 2$, the map $H^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \to H^n(\overline{G}; \mathbb{R})$ is onto. Since $\overline{G}$ is hyperbolic relative to $\overline{\mathcal{H}}$, the comparison map $H_b^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R}) \to H^n(\overline{G}, \overline{\mathcal{H}}; \mathbb{R})$ is onto [31, Theorem 1.1], see also [55]. It follows that the comparison map $c : H_b^n(\overline{G}; \mathbb{R}) \to H^n(\overline{G}; \mathbb{R})$ is also onto which shows that $||\overline{G}|| > 0$ [32, Lemma 6.1], see also [32, Proposition 7.10]. □

6.1. Dehn fillings of pinched negatively curved manifolds

In this section, we illustrate how the results of the previous sections apply in a geometric setting. First, we need a lemma.

Lemma 6.4. *Let G be a group and $\mathcal{H} = \{H_i\}_{i=1}^m$ a collection of finitely generated torsion-free nilpotent subgroups with $rk(Z(H_i)) \geqslant 2$, for all i. Then there are infinitely many collections of subgroups $\{N_i\}_{i=1}^m$ such that $\mathbb{Z} \cong N_i \lhd Z(H_i)$, $\overline{H}_i$ is torsion-free and $j_k : H_k(\overline{G}; \mathbb{Z}) \to H_k(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z})$ is an isomorphism for all $k > \max\{\mathrm{hd}(H_i) \mid 1 \leqslant i \leqslant m\}$.*

Proof. For each i, the center $Z(H_i)$ contains a factor $\langle x_i, y_i\rangle \cong \mathbb{Z}^2$. Let $N_i^{s,t} = \langle x_i^s y_i^t\rangle \lhd H_i$ where $s, t \in \mathbb{Z}$ are co-prime. Since s, t are co-prime, $Z(H_i)/N_i^{s,t}$ is torsion-free. Hence the quotient group $\overline{H}_i = H_i/N_i^{s,t}$ is torsion-free, since it has a composition series with torsion-free factor groups. So, $\overline{H}_i$ is a torsion-free nilpotent group. The long exact sequence in homology for the pair $(\overline{G}, \overline{\mathcal{H}})$ establishes the isomorphism $j_k : H_k(\overline{G}; \mathbb{Z}) \to H_k(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z})$ for all

$$\begin{aligned} k &> \max\{\mathrm{hd}(\overline{H}_i) \mid 1 \leqslant i \leqslant m\} + 1 \\ &= \max\{\mathrm{hd}(H_i) \mid 1 \leqslant i \leqslant m\}. \end{aligned}$$

The above equality holds since homological dimension is additive for group extensions of finitely generated torsion-free nilpotent groups [12, Theorem 5.5], see also [12, Theorem 7.10(a)]. The collections $\{N_i^{s,t}\}_{i=1}^m$ enumerated by the co-prime pairs (s, t) have the required properties. □

Definition 6.5. Let M be a Riemannian n-manifold with a complete pinched negative sectional curvature and finite volume. All the cuspidal ends of M are almost-flat manifolds and hence by the results of Gromov and Ruh are infra-nil [37,65]. In particular, they are finitely covered by nilmanifolds. Let us assume here that all the cuspidal ends $\{L_i\}_{i=1}^m$ of M are nilmanifolds such that $rk(Z(\pi_1(L_i)) \geqslant 2$. Let $\overline{M}$ be the natural compactification of M with boundary components $\{L_i\}_{i=1}^m$. Let $G = \pi_1(\overline{M})$ and $H_i = \pi_1(L_i)$. Then G is hyperbolic relative to the fundamental groups of the cuspidal ends and in particular $\{H_i\}_{i=1}^m \hookrightarrow_h G$ [14,28]. By Theorem 4.10 (ii) and Lemma 6.4, there are infinitely many collections of sufficiently deep normal subgroups $\{N_i\}_{i=1}^m$ satisfying the conclusions of both such that $j_n : H_n(\overline{G}; \mathbb{Z}) \xrightarrow{\cong} H_n(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z})$. Let $\{N_i\}_{i=1}^m$ be one such collection. By [60, Theorem 1.1], we can also assume that $\{N_i\}_{i=1}^m$ are sufficiently deep so that $\overline{G}$ is hyperbolic relative to $\overline{\mathcal{H}}$. Let $T_i \hookrightarrow L_i \xrightarrow{\pi_i} B_i$ be the circle bundle with a nilmanifold base which arises as a quotient of the Malcev completion (see e.g. [24, §1.2, p. 9]) $\mathbb{R} \hookrightarrow \mathbf{L}_i \twoheadrightarrow \mathbf{B}_i$ of the short exact sequence $N_i \hookrightarrow H_i \twoheadrightarrow \overline{H}_i$. Denote by $C(L_i, T_i)$ the mapping cylinder of π_i for each i (see Fig. 2 for an illustration). Note that $C(L_i, T_i)$ is manifold with boundary L_i. We define

$$M_T = \overline{M} \bigcup_{\phi_1 \sqcup \cdots \sqcup \phi_m} (C(L_1, T_1) \sqcup \cdots \sqcup C(L_m, T_m)),$$

where each ϕ_i is the canonical identification of $\partial C(L_i, T_i)$ with L_i and call it a *sufficiently deep Dehn filling* of $\overline{M}$.

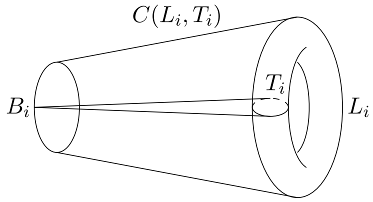

Fig. 2. Mapping cylinder $C(L_i,T_i)$.

Given a closed oriented n-manifold M possibly with boundary ∂M, the *simplicial volume of* $(M,\partial M)$ is defined as

$$||M,\partial M|| = ||[M,\partial M]||_1$$

where $[M,\partial M]\in H_n(M,\partial M;\mathbb{R})$ is the image of the fundamental class under the change of coefficients map $H_n(M,\partial M;\mathbb{Z})\to H_n(M,\partial M;\mathbb{R})$. The following result is again a generalization of [30, Theorem 1.5].

Corollary 6.6. *Let $\overline{M}$ be a compact oriented n-manifold with nilmanifold boundary components $\{L_i\}_{i=1}^m$ as above. Suppose M_T is a sufficiently deep Dehn filling of $\overline{M}$. Then, M_T is a closed oriented aspherical n-manifold with fundamental group $\overline{G}$ and*

$$0<||M_T||\leqslant ||\overline{M},\partial\overline{M}||.$$

Proof. We first show that M_T is aspherical with fundamental group $\overline{G}$.

Let $\pi:\widetilde{M}\to M$ be the universal covering map and for each $1\leqslant i\leqslant m$, denote by $\mathbf{L}_i$ the universal cover of L_i which is the Malcev completion [24, §1.2] of H_i. It is a simply connected nilpotent Lie group containing H_i as a uniform lattice. Observe that $\overline{M}$ is homeomorphic to a submanifold $R\subseteq M$ where R is obtained by removing the interiors of the cusps of M [28]. The universal cover Y of R is a subspace of $\widetilde{M}$ with boundary a G-orbit of almost-flat totally geodesic submanifolds homeomorphic to $\mathbf{L}_i$ for each $1\leqslant i\leqslant m$. Let $K=Y/\langle\langle\mathcal{N}\rangle\rangle$, which is a cover of R with fundamental group $\langle\langle\mathcal{N}\rangle\rangle$. The boundary of K is a disjoint union of $\overline{G}$-orbits of submanifolds homeomorphic to $L_i'=\mathbf{L}_i/N_i$ for each $1\leqslant i\leqslant m$. The circle bundle $T_i\hookrightarrow L_i\xrightarrow{\pi_i}B_i$ is the quotient of the short exact sequence $\mathbb{R}\hookrightarrow\mathbf{L}_i\twoheadrightarrow\mathbf{B}_i$ of simply connected nilpotent Lie groups by the action of H_i. Hence, it lifts to the circle bundle $T_i\hookrightarrow L_i'\xrightarrow{\pi_i'}\mathbf{B}_i$. Denote by $C(L_i',T_i)$ the mapping cylinder of π_i'. Define

$$M'=K\bigcup_{\psi_1\sqcup\cdots\sqcup\psi_m}(\overline{G}\times_{\overline{H}_1}C(L_1',T_1)\sqcup\cdots\sqcup\overline{G}\times_{\overline{H}_m}C(L_m',T_m)),$$

where each ψ_i is the canonical identification of $\overline{G}\times_{\overline{H}_i}\partial C(L_i',T_i)$ with $\overline{G}L_i'$. The manifold M' is simply connected by construction. By Mayer-Vietoris homology sequence and the Cohen-Lyndon property of Theorem 3.25, it follows M' has also trivial homology and is therefore contractible. The group $\overline{G}$ acts freely on M' and the quotient $M'/\overline{G}$ is homeomorphic to M_T.

Since $(\overline{M}, \partial\overline{M}) \simeq (BG, \bigsqcup_{i=1}^{m} BH_i)$, we have $[\overline{M}, \partial\overline{M}] \in H_n(G, \mathcal{H}; \mathbb{R})$. Similarly, since $M_T \simeq B\overline{G}$, we have $[M_T] \in H_n(\overline{G}; \mathbb{R})$. Applying Theorem 6.3, we obtain

$$0 < ||M_T|| \leqslant ||\overline{M}, \partial\overline{M}||. \quad \square$$

Remark 6.7. In the geometric setting of Corollary 6.6, the isomorphism

$$p_n : H_n(G, \mathcal{H}; \mathbb{Z}) \xrightarrow{\cong} H_n(\overline{G}, \overline{\mathcal{H}}; \mathbb{Z})$$

used in the proof of Theorem 6.3 also follows from excising the interiors of the attached submanifolds $\{C(L_i, T_i)\}_{i=1}^{m}$ from M_T which gives an isomorphism between the (co)homologies of $(\overline{M}, \partial M)$ and $(M_T, \{C(L_i, T_i)\}_{i=1}^{m})$. So, one can think of the isomorphism in Theorem 4.10 (ii) as a group theoretic analog of topological excision.

We should also remark that the right-hand-side inequality of Corollary 6.6 can be deduced from Gromov's Additivity Theorem [32, Theorem 7.6].

7. Quotients of acylindrically hyperbolic groups

7.1. Cohomology and embedding theorems

We prove Theorem D in this subsection. The reader is referred to Section 2.4 for an outline of the proof. In the sequel, we employ the convention that if X is a set of alphabets and w is a word over X, then $\|w\|$ denotes the length of w. In certain cases, it might be possible to view w as a word over another alphabet Y. In such a case, we will use $\|w\|_X$ (resp. $\|w\|_Y$) to denote the number of letters of X (resp. Y) in w. For two words w and v, we write $w \equiv v$ to indicate that there is a letter-by-letter equality between w and v.

Lemma 7.1. *Let F_4 be a free group of rank 4, $\mathcal{F} \subseteq F_4$ a finite set, and C a countable group with $\mathrm{cd}(C) \geqslant 2$. Then there exists a quotient R of F_4 such that the following hold.*

1. *R can be decomposed as a free product $R = \mathbb{Z} * R_0$ with $\mathrm{card}(R_0) = \infty$. Moreover, C embeds into R_0.*
2. *The quotient map $F_4 \to R$ is injective on $\mathcal{F}$.*
3. *If C is torsion-free, then so is R.*
4. $\mathrm{hd}(R) \leqslant \max\{\mathrm{hd}(C), 2\}$, $\mathrm{cd}(R) \leqslant \mathrm{cd}(C)$.
5. *For every $n \geqslant 3$ and every R-module A, we have*

$$H_n(R; A) \cong H_n(C; A),\ \ H^n(R; A) \cong H^n(C; A),$$

where the action of C on A is induced by the embedding $C \hookrightarrow R$.

(6) *If C is finitely generated, then R_0 is hyperbolic relative to C.*
(7) *If C is of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then so is R.*
(8) *If C is of type FP, then so is R.*

Remark 7.2. Lemma 7.1 (1) (2) (6) are proved in [27, Lemma 8.4]. We refine the method therein so as to impose (co)homological conditions.

The proof of Lemma 7.1 relies on small cancellation theory, the reader is referred to [50, Chapter V] for a treatment.

Proof. Let $\{x, y, z, t\}$ be a free basis of F_4, let $F_3 < F_4$ be the subgroup generated by x, y, t, and let $\{c_i\}_{i \in I}$ be a generating set of C. There exist freely reduced words $\{w_i\}_{i\in I}, \{v_i\}_{i\in I}$ over the alphabet $\{x, y\}$ such that

(a) the words $\{c_i w_i\}_{i\in I}$ satisfy the $C'(1/2)$ small cancellation condition over the free product $\langle x\rangle * \langle y\rangle * C$;
(b) the words $\{v_i\}_{i\in I}$ satisfy the $C'(1/2)$ small cancellation condition over the alphabet $\{x, y\}$;
(c) the words $\{tc_i w_i t^{-1} v_i\}_{i\in I}$ satisfy the $C'(1/6)$ small cancellation condition over the free product $\langle x\rangle * \langle y\rangle * \langle t\rangle * C$.

Indeed, we can first construct words w_i satisfying condition (a), and then pick sufficiently long words v_i to ensure conditions (b) and (c).

Let N (resp. N_0) be the normal subgroup of $F_4 * C$ (resp. $F_3 * C$) generated by $\{tc_i w_i t^{-1} v_i\}_{i\in I}$, and let

$$R_0 = (F_3 * C)/N_0, \qquad R = (F_4 * C)/N.$$

For $i \in I$, let $\overline{t}$ (resp. $\overline{c}_i, \overline{w}_i, \overline{v}_i, \overline{z}$) be the image of t (resp. c_i, w_i, v_i, z) under the quotient map $F_4 * C \to R$. Then we have

$$R = \langle \overline{z}\rangle * R_0 = \mathbb{Z} * R_0.$$

Note that $\overline{t}\overline{c}_i\overline{w}_i\overline{t}^{-1}\overline{v}_i = 1$ and we can rewrite this equation as $\overline{c}_i = \overline{t}^{-1}\overline{v}_i^{-1}\overline{t}\overline{w}_i^{-1}$. Thus, R is generated by $\overline{t}, \overline{z}, \overline{w}_i, \overline{v}_i, i \in I$, and hence is a quotient of F_4.

Let

$$\alpha : F_4 \to R$$

be the corresponding quotient map, which is the restriction of the quotient map $F_4 * C \to R$ to F_4. It follows from the Greendlinger's lemma for free products [50, Chapter V Theorem 9.3] that if $\|w_i\|, \|v_i\|, i \in I$, are sufficiently large, then α is injective on $\mathcal{F}$ and thus statement (2) is guaranteed.

Let $L = \langle x\rangle * \langle y\rangle * C$, let $U \leqslant L$ be the subgroup generated by $\{c_i w_i\}_{i\in I}$, and let $V \leqslant L$ be the subgroup generated by $\{v_i\}_{i\in I}$.

Claim 7.2.1. *U (resp. V) is a free group with basis $\{c_i w_i\}_{i\in I}$, (resp. $\{v_i\}_{i\in I}$). In particular, U and V are both of rank* $\mathrm{card}(I)$*.*

Proof of Claim 7.2.1. We prove the claim for U. The proof for V is similar. Let

$$u \equiv \prod_{k=1}^{\ell}(c_{i_k}w_{i_k})^{\epsilon_k}$$

be a nonempty freely reduced word over the alphabet $\{c_i w_i\}_{i\in I}$, where $i_k \in I$ and $\epsilon_k = \pm 1$ for $k = 1, ..., \ell$. Think of u as a word over the alphabet $\langle x\rangle \cup \langle y\rangle \cup C$ and then reduce u to its normal form $\overline{u}$ corresponding to the free product $\langle x\rangle * \langle y\rangle * C$ (see [50, Chapter IV] for the definition of normal forms). By condition (a) and that the words w_i do not involve inverses of the generators x, y, for each factor $(c_{i_k}w_{i_k})^{\epsilon_k}$ of u, a non-empty subword of $(c_{i_k}w_{i_k})^{\epsilon_k}$ survives in $\overline{u}$. In particular, $\overline{u}$ is nonempty and thus u does not represent 1 in L. □

Note that the relations $\overline{t}\overline{c}_i\overline{w}_i\overline{t}^{-1}\overline{v}_i = 1$ can be rewritten as $\overline{t}\overline{c}_i\overline{w}_i\overline{t}^{-1} = \overline{v}_i^{-1}$. Thus, R_0 is the HNN-extension of L with associated subgroups U and V. In particular, L embeds into R_0. As $\mathrm{card}(L) = \infty$, we have $\mathrm{card}(R_0) = \infty$. Since C embeds into L, C embeds into R_0. Thus, statement (1) holds.

If C is torsion-free, then so is L. Being an HNN-extension of L, R_0 is also torsion-free, and thus so is $R = \mathbb{Z} * R_0$, which is statement (3).

By [11, Theorem 3.1], there is a long exact sequence for any R_0-module A,

$$\cdots \to H^{n-1}(U;A) \to H^n(R_0;A) \to H^n(L;A) \to H^n(U;A) \to \cdots \tag{15}$$

As U is free, exact sequence (15) implies for $n \geqslant 3$,

$$H^n(R_0;A) \cong H^n(L;A) \cong H^n(C;A).$$

As $R = \mathbb{Z} * R_0$, the cohomology part of statement (5) holds. Similarly, one can prove the homology part of statement (5). Statement (4) follows from statement (5) and $\mathrm{cd}(C) \geqslant 2$.

If C is finitely generated, then we can construct R using a finite generating set of C. Then R_0 is the quotient of $F_3 * C$ by adding finitely many relations $tc_iw_it^{-1}v_i$, and thus has a finite relative presentation over C. The Greendlinger's lemma for free products implies that the relative isoperimetric function of R_0 with respect to C is linear. Thus, R_0 is hyperbolic relative to C (see Remark 3.7), which is statement (6).

If C is of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then since W is of type FP, we have R_0 is of type FP_n by [12, Proposition I.2.13(b)]. As $R = \mathbb{Z} * R_0$, R is of type FP_n, which is statement (7). Finally, statement (8) follows from (4) and (7). □

Proof of Theorem D. By Theorem 3.18, G has a unique maximal finite normal subgroup $K(G)$. By [42, Lemma 5.10], $G_0 = G/K(G)$ is acylindrically hyperbolic.

If $\mathrm{cd}(C) = 0$, then $C = \{1\}$. Let $\overline{G} = G_0$. By Theorem 3.18, $C \hookrightarrow_h \overline{G}$. First consider statement (vi). As $\overline{G}$ and G are quasi-isometric (note that the assumption of (vi) implies that G and $\overline{G}$ are finitely generated), [4, Corollary 9] implies (vi). Other conclusions of Theorem D hold trivially.

If $\mathrm{cd}(C) = 1$, then by the Stallings-Swan theorem [68, corollary to Theorem 1], C is free. By Theorem 3.18, there exists a finitely generated non-cyclic free group $F \hookrightarrow_h G_0$. Let $\overline{G} = G_0$. It is well-known that the free group C embeds into F. Thus, C also embeds into $\overline{G}$. All conclusions except for (ii) hold trivially. If in addition, C is finitely generated, then C is a finite rank free group and we can let $F = C$. Thus, (ii) also holds.

Let us assume $\mathrm{cd}(C) \geqslant 2$. By Theorem 3.18, there exists $X \subseteq G_0$ and a free subgroup $F_4 \leqslant G_0$ of rank 4 such that $F_4 \hookrightarrow_h (G_0, X)$. There exists a finite set $\mathcal{F} \subseteq F_4 \smallsetminus \{1\}$ such that if $N \lhd F_4$ satisfies $N \cap \mathcal{F} = \varnothing$, then the conclusions of Theorems A and 3.23 and [27, Theorem 7.15] hold.

By Lemma 7.1, C embeds into an infinite quotient $R = \mathbb{Z} * R_0$ of F_4 such that the conclusions of Lemma 7.1 hold and the quotient map $F_4 \to R$ is injective on $\mathcal{F}$. Let N be the kernel of $F_4 \to R$. Then $N \cap \mathcal{F} = \varnothing$. Let $\overline{G} = G/\langle\langle N \rangle\rangle$.

As $R = \mathbb{Z} * R_0$, R_0 is a proper subgroup of R and in particular, R_0 is a proper subgroup of G. By Example 3.8 (d), $R_0 \hookrightarrow_h R$. Proposition 3.10 and $R \hookrightarrow_h G$ then imply that $R_0 \hookrightarrow_h G$. As $\mathrm{card}(R_0) = \infty$, Theorem 3.23 implies that $\overline{G}$ is acylindrically hyperbolic, that is, statement (i) holds. As C embeds into R_0, C also embeds into $\overline{G}$.

If C is finitely generated, then Lemma 7.1 implies that R_0 is hyperbolic relative to C, in particular, $C \hookrightarrow_h R_0$. As $R_0 \hookrightarrow_h \overline{G}$, we have $C \hookrightarrow_h \overline{G}$ by Proposition 3.10. Thus, statement (ii) holds.

Suppose that G and C are torsion-free and there is a non-trivial finite-order element $\overline{g} \in \overline{G}$. Then $G_0 = G$. Denote the image of X under the quotient map $G \twoheadrightarrow \overline{G}$ by $\overline{X}$. Then $R \hookrightarrow_h (\overline{G}, \overline{X})$ [27, Theorem 7.15 (b)]. As $\overline{g}$ has finite order, it acts elliptically on the Cayley graph $\Gamma(\overline{G}, \overline{X} \sqcup R)$. By [27, Theorem 7.15 (f)], there is an element $g \in G$ such that g is mapped to $\overline{g}$ under the quotient map $G \twoheadrightarrow \overline{G}$ and g acts elliptically on $\Gamma(G, X \sqcup F_4)$. As $\overline{g}$ has finite order, $g^n \in \langle\langle N \rangle\rangle$ for some $n > 0$. Since g^n is elliptic as g is, there is some $h \in G$ such that $hg^nh^{-1} \in N \leqslant F_4$ [27, Theorem 7.15 (d)]. By Lemma 7.1 (3), R is torsion free and thus $hgh^{-1} \notin F_4$. But

$$|(hgh^{-1})F_4(hgh^{-1})^{-1} \cap F_4| \geqslant |\langle hg^nh^{-1} \rangle| = \infty,$$

which is in contradiction with the almost malnormality of F_4 in G [27, Proposition 2.10]. We have proved statement (iii).

Statement (iv) follows from Corollary 2.2 (i).

Consider statement (v). Corollary 2.2 (ii) implies that

$$\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G_0), \mathrm{cd}(F_4) + 1, \mathrm{cd}(R)\}.$$

If $K(G) \neq \{1\}$, then G has torsion and thus $\mathrm{cd}(G) = \infty$ [19, Chapter VIII Corollary 2.5], in which case (v) is a void statement. Thus, let us assume $K(G) = \{1\}$ and thus $G_0 = G$. As $\mathrm{cd}(R) \leqslant \mathrm{cd}(C)$ and $\mathrm{cd}(C) \geqslant 2$, we have

$$\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(F_4)+1, \mathrm{cd}(R)\} \leqslant \max\{\mathrm{cd}(G), 2, \mathrm{cd}(C)\} = \max\{\mathrm{cd}(G), \mathrm{cd}(C)\}.$$

If G and C are of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then Lemma 7.1 implies that so is R. As G and G_0 are quasi-isometric, G_0 is of type FP_n [4, Corollary 9]. As F_4 is of type FP_∞, Corollary 2.2 (iii) implies that $\overline{G}$ is of type FP_n. Conversely, if C is finitely generated and G and $\overline{G}$ are of type FP_n for some $n \in \mathbb{N}^+ \cup \{\infty\}$, then $C \hookrightarrow_h \overline{G}$ by the previously proved statement (ii). Thus, C is of type FP_n [27, Theorem 2.11]. The previously proved statement (v) implies that $\mathrm{cd}(\overline{G}) < \infty$ if and only if $\max\{\mathrm{cd}(G), \mathrm{cd}(C)\} < \infty$, which further implies the statement about type FP. This finishes the proof of statement (vi). □

Remark 7.3. Analogous to the above proof, in the setting of Theorem D, we have $\mathrm{hd}(\overline{G}) \leqslant \max\{\mathrm{hd}(G), \mathrm{hd}(C), 2\}$.

7.2. Constructing hyperbolically embedded subgroups

We will prove Theorem E in the next subsection, where the following technical result will be needed. The reader is referred to Section 2.4 for an outline of the proof.

Proposition 7.4. *Suppose that G is a group, Λ is a finite set and $\{k_\lambda\}_{\lambda\in\Lambda}$ is a set of positive integers. Let $a_{\lambda,i} \in G$ for $\lambda \in \Lambda$ and $i = 1, 2, \cdots, k_\lambda$. Also suppose that there is a family of subgroups $\{F_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$ such that*

(a) *each F_λ is free of rank $2k_\lambda$ with basis $\{f_{\lambda,i}, g_{\lambda,i}\}_{i=1}^{k_\lambda}$; and*
(b) *$a_{\lambda,i} \notin F_\lambda \smallsetminus \{1\}$ for all $\lambda \in \Lambda$ and $i = 1, 2, \cdots, k_\lambda$.*

Then for sufficiently large $n \in \mathbb{N}^+$, the set

$$\{f^n_{\lambda,i} a_{\lambda,i} g^n_{\lambda,i}\}_{i=1}^{k_\lambda} \subseteq G$$

freely generates a subgroup $H_\lambda \leqslant G$ and

$$\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G. \tag{16}$$

The rest of this subsection is devoted to the proof of the above proposition. We first find a "sufficiently large" $n \in \mathbb{N}^+$. As $F_\lambda = *_{i=1}^{k_\lambda}(\langle f_{\lambda,i}\rangle * \langle g_{\lambda,i}\rangle)$, we have $\{\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle\}_{i=1}^{k_\lambda} \hookrightarrow_h F_\lambda$ by Example 3.8. It follows from Proposition 3.10 that there exists a set $X \subseteq G$ such that

$$\{\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle\}_{\lambda\in\Lambda, i\in\{1,\ldots,k_\lambda\}} \hookrightarrow_h (G, X). \tag{17}$$

By [27, Corollary 4.27], we may assume that $a_{\lambda,i} \in X$ for all $\lambda \in \Lambda$ and $i = 1, 2, \cdots, k_\lambda$, as Λ and k_λ are finite.

For $\lambda \in \Lambda$ and $i \in \{1, ..., k_\lambda\}$, let

$$\widehat{d}_{\lambda,i,f} : \langle f_{\lambda,i}\rangle \times \langle f_{\lambda,i}\rangle \to [0, +\infty], \qquad \widehat{d}_{\lambda,i,g} : \langle g_{\lambda,i}\rangle \times \langle g_{\lambda,i}\rangle \to [0, +\infty]$$

be the relative metrics corresponding to (17). The metrics $\widehat{d}_{\lambda,i,f}$ and $\widehat{d}_{\lambda,i,g}$ are locally finite. As $\mathrm{card}(\Lambda), k_\lambda < \infty$, for sufficiently large n, we will have

$$\widehat{d}_{\lambda,i,f}(1, f^n_{\lambda,i}), \widehat{d}_{\lambda,i,g}(1, g^n_{\lambda,i}) > 50D$$

for all λ and i, where $D > 0$ is given by Lemma 3.14. We fix one such n and let $H_\lambda \leqslant G$ be as in Proposition 7.4. For simplicity, denote the set $\{f^n_{\lambda,i} a_{\lambda,i} g^n_{\lambda,i}\}_{i=1}^{k_\lambda}$ by U_λ.

Notice that $F_\lambda \cap F_\mu = \{1\}$ whenever $\lambda \neq \mu$ [27, Proposition 4.33]. The following are easy consequences of this fact.

$$\langle f_{\lambda,i}\rangle \cap \langle g_{\mu,j}\rangle = \{1\} \qquad \text{for all } \lambda, \mu \in \Lambda, \tag{18}$$

$$\langle f_{\lambda,i}\rangle \cap \langle f_{\mu,j}\rangle = \langle g_{\lambda,i}\rangle \cap \langle g_{\mu,j}\rangle = \{1\} \qquad \text{for all } \lambda, \mu \in \Lambda \text{ with } \lambda \neq \mu, \tag{19}$$

$$\langle f_{\lambda,i}\rangle \cap \langle f_{\lambda,j}\rangle = \langle g_{\lambda,i}\rangle \cap \langle g_{\lambda,j}\rangle = \{1\} \qquad \text{for all } \lambda \in \Lambda \text{ and } i, j \in \{1, ..., k_\lambda\} \text{ with } i \neq j. \tag{20}$$

In particular, $\{\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle\}_{\lambda\in\Lambda, i\in\{1,\ldots,k_\lambda\}}$ is a distinct family of hyperbolically embedded subgroups of G.

For simplicity, let

$$\mathcal{K}_\lambda = \left(\bigsqcup_{i=1}^{k_\lambda} \langle f_{\lambda,i}\rangle\right) \sqcup \left(\bigsqcup_{i=1}^{k_\lambda} \langle g_{\lambda,i}\rangle\right), \qquad \mathcal{K} = \bigsqcup_{\lambda\in\Lambda} \mathcal{K}_\lambda.$$

Remark 7.5. We will apply Lemma 3.15 for the group G, the hyperbolically embedded family of subgroups $\{\langle f_{\lambda,i}\rangle, \langle g_{\lambda,i}\rangle\}_{\lambda\in\Lambda, i\in\{1,\ldots,k_\lambda\}} \hookrightarrow_h (G, X)$, and reduced words w over U_λ for every $\lambda \in \Lambda$ and every freely reduced word w over U_λ. We can think of w as a word over the alphabet $X \sqcup \mathcal{K}$, i.e., regard every $f^n_{\lambda,i}$ (resp. $g^n_{\lambda,i}$) as a letter from $\langle f_{\lambda,i}\rangle$ (resp. $\langle g_{\lambda,i}\rangle$) and regard every $a_{\lambda,i}$ as a letter from X. In this sense, w satisfies the conditions (W1), (W2), and (W3) of Lemma 3.15.

Lemma 7.6. *For all $\lambda \in \Lambda$, H_λ is free with basis U_λ.*

Proof. We need to show that for every $\lambda \in \Lambda$, every nonempty freely reduced word over U_λ does not represent 1 in G. Suppose that

$$w \equiv \prod_{k=1}^{\ell} (f^n_{\lambda,i_k} a_{\lambda,i_k} g^n_{\lambda,i_k})^{\epsilon_k}$$

is a freely reduced word over U_λ for some $\lambda \in \Lambda$ such that w represents 1 in G, where $\epsilon_k = \pm 1$ for $k = 1, ..., \ell$. As Remark 7.5, we think of w as a word over $X \sqcup \mathcal{K}$. Then w labels a 3ℓ-gon p in $\Gamma(G, X \sqcup \mathcal{K})$ with geodesic sides. Notice that p has 2ℓ components. By Lemma 3.15 (b), each of these components is isolated. Proposition 3.14 then implies

$$2n \cdot 50D \leqslant 3nD,$$

which is absurd. Therefore, such a word w does not exist. □

Consider the action $G \curvearrowright \Gamma(G, X \sqcup \mathcal{K})$. Note that $\Gamma(G, X \sqcup \mathcal{K})$ is hyperbolic, by (17). Let $d_{X \sqcup \mathcal{K}}$ be the combinatorial metric of $\Gamma(G, X \sqcup \mathcal{K})$. We verify that, with respect to this action, the family $\{H_\lambda\}_{\lambda \in \Lambda}$ satisfies the conditions (C_1), (C_2), and (C_3) of Lemma 3.9, which then implies $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h G$. The verification is divided into the following Lemmas 7.7, 7.8, and 7.9.

Lemma 7.7. *For every $\lambda \in \Lambda$, the action $H_\lambda \curvearrowright \Gamma(G, X \sqcup \mathcal{K})$ is proper.*

Proof. Fix $\lambda \in \Lambda$. It suffices to prove that for every $R > 0$, there are only finitely many $h \in H_\lambda$ such that $d_{X \sqcup \mathcal{K}}(1, h) \leqslant R$. Let $h \in H_\lambda$ such that $d_{X \sqcup \mathcal{K}}(1, h) \leqslant R$ and let w be a freely reduced word over U_λ representing h in G. As in Remark 7.5, think of w as a word over $X \sqcup \mathcal{K}$. By Lemma 3.15 (a), w labels a $(4, 1)$-quasi-geodesic in $\Gamma(G, X \sqcup \mathcal{K})$. Thus,

$$\|w\|_{U_\lambda} = \frac{\|w\|_{X \sqcup \mathcal{K}}}{3} \leqslant \frac{4R + 1}{3}.$$

There are only finitely many words w satisfying the above inequality. It follows that the number of $h \in H_\lambda$ such that $d_{X \sqcup \mathcal{K}}(1, h) \leqslant R$ is finite. □

For every $\lambda \in \Lambda$, we identify H_λ with the subset of $\Gamma(G, X \sqcup \mathcal{K})$ labeled by elements of H_λ. Equivalently, we identify H_λ with the H_λ-orbit of the identity vertex of $\Gamma(G, X \sqcup \mathcal{K})$.

Lemma 7.8. *For every $\lambda \in \Lambda$, the orbit H_λ is quasi-convex in $\Gamma(G, X \sqcup \mathcal{K})$.*

Proof. Fix $\lambda \in \Lambda$. Let $h \in H_\lambda$ and let γ be a geodesic in $\Gamma(G, X \sqcup \mathcal{K})$ from the vertex 1 to the vertex h. As $\Gamma(G, X \sqcup \mathcal{K})$ is a Gromov hyperbolic space, there exists $R > 0$ such that if α and β are $(4, 1)$-quasi-geodesics with the same endpoint, then $d_{Hau}(\alpha, \beta) \leqslant R$, where d_{Hau} is the Hausdorff metric with respect to $d_{X \sqcup \mathcal{K}}$.

Let w be a freely reduced word over U_λ representing h in G. As in Remark 7.5, think of w as a word over $X \sqcup \mathcal{K}$. By Lemma 3.15 (a), w labels a $(4, 1)$-quasi-geodesic α in

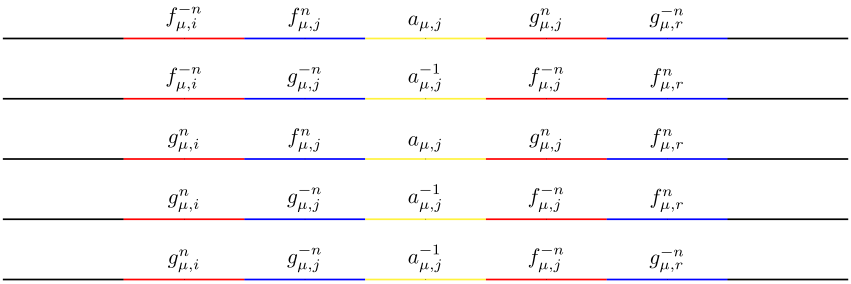

Fig. 3. Some of the possible configurations of the two pairs of adjacent components of p. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$\Gamma(G, X \sqcup \mathcal{K})$. Note that α lies in the 2-neighborhood of the orbit H_λ, and γ lies in the R-neighborhood of α. Thus, γ lies in the $(R+2)$-neighborhood of H_λ. $\square$

For $\lambda, \mu \in \Lambda$, the orbits H_λ and H_μ are subsets of $\Gamma(G, X \sqcup \mathcal{K})$. Thus, it makes sense to talk about the diameter of $H_\mu \cap (gH_\lambda)^{+\epsilon}$ in $\Gamma(X \sqcup \mathcal{K})$, which is denoted by $\operatorname{diam}\left(H_\mu \cap (gH_\lambda)^{+\epsilon}\right)$.

Lemma 7.9. *For every $\epsilon > 0$, there exists $R > 0$ such that the following holds. Suppose that for some $g \in G$ and $\lambda, \mu \in \Lambda$, we have*

$$\operatorname{diam}\left(H_\mu \cap (gH_\lambda)^{+\epsilon}\right) \geqslant R.$$

Then $\lambda = \mu$ and $g \in H_\lambda$.

Proof. Fix $\epsilon > 0$ and let R be the constant given by Lemma 3.15 (c). Suppose that $\operatorname{diam}\left(H_\mu \cap (gH_\lambda)^{+\epsilon}\right) \geqslant R$ for some $g \in G$ and $\lambda, \mu \in \Lambda$. Then there exist vertices $v_1, v_2 \in H_\mu$ and $v_3, v_4 \in gH_\lambda$ such that

$$d(v_1, v_2) \geqslant R, \quad d(v_1, v_3), d(v_2, v_4) \leqslant \epsilon.$$

Let p (resp. q) be a path from v_1 (resp. v_3) to v_2 (resp. v_4) such that $\mathbf{Lab}(p)$ (resp. $\mathbf{Lab}(q)$) is a freely reduced word over U_μ (resp. U_λ). Then $\ell(p) \geqslant R$ and p, q are oriented ϵ-close. By Lemma 3.15 (b), there exist five consecutive components of p which are connected to five consecutive components of q. In particular, there exist two pairs of adjacent components of p which are connected to four consecutive components of q. Some of the possible configurations of these two pairs of adjacent components are shown by Fig. 3, where each horizontal line represents one possible configuration, the red and blue segments represent the two pairs of adjacent components of p, and the corresponding labels are written on top of the subpaths.

Below, we assume, without loss of generality, that these two pairs of adjacent components are of the form

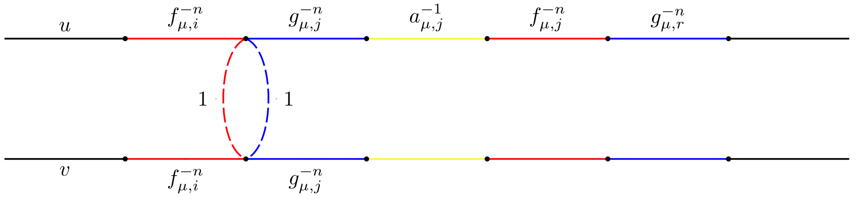

Fig. 4. Case 1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

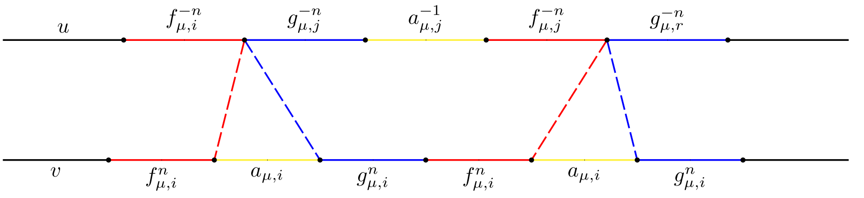

Fig. 5. Case 2. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$f_{\mu,i}^{-n} g_{\mu,j}^{-n}, f_{\mu,j}^{-n} g_{\mu,r}^{-n}.$$

Other possible configurations can be analyzed similarly. We distinguish two cases.

Case 1. The first pair of adjacent components of p are respectively connected to a pair of adjacent components of q.

Case 1 is displayed by Fig. 4, where the red (resp. blue) dashed line represents a path with label in $\langle f_{\mu,i}^n \rangle$ (resp. $\langle g_{\mu,j}^n \rangle$) connecting the corresponding red (resp. blue) components. Equations (18), (19), and (20) imply that $\lambda = \mu$ and the red (resp. blue) component of q is labeled by $f_{\mu,i}^{-n}$ (resp. $g_{\mu,j}^{-n}$). As the red and blue dashed lines form a loop, another consequence of (18) is that both of these dashed lines are labeled by 1.

Let p_1 (resp. q_1) be the subpath of p (resp. q) labeled by $uf_{\mu,i}^{-n}$ (resp. $vf_{\mu,i}^{-n}$). Then $p_1^+ = q_1^+$. By the structure of U_μ, $\mathbf{Lab}(p_1)$ and $\mathbf{Lab}(q_1)$ are words over U_μ and thus represent elements in H_μ. Therefore, $p_1^+ \in H_\mu \cap gH_\mu$. It follows that $g \in H_\mu$.

Case 2. The first pair of adjacent components of p are respectively connected to two consecutive, but not adjacent, components of q.

Case 2 is displayed by Fig. 5. Once again, Equations (18), (19), and (20) imply $\lambda = \mu$. The structures of U_μ imply $i = j = r$. The red (resp. blue) dashed line on the left is labeled by an element in $\langle f_{\mu,i}^n \rangle$ (resp. $\langle g_{\mu,i}^n \rangle$). As these dashed lines and the yellow segment labeled by $a_{\mu,i}$ form a loop, assumption (b) of Proposition 7.4 implies that both of these dashed lines are labeled by 1. Similarly, the red and blue dashed lines on the right are both labeled by 1.

Therefore, the word $g_{\mu,i}^{2n} f_{\mu,i}^{2n} a_{\mu,i}$ labels a loop in $\Gamma(G, X \sqcup \mathcal{K})$ and thus represents 1 in G, which is in contradiction with assumption (b) of Proposition 7.4. Hence, Case 2 is in fact impossible. $\square$

Proof of Proposition 7.4. The first assertion follows from Lemma 7.6 and formula (16) follows Lemmas 3.9, 7.7, 7.8, and 7.9. □

7.3. Common quotients of acylindrically hyperbolic groups

In this subsection, we prove Theorem E. Given finitely generated acylindrically hyperbolic groups G_1 and G_2, we construct a common quotient G of $G_1' = G_1/K(G_1)$ and $G_2' = G_2/K(G_2)$ satisfying the conclusions of that theorem, where $K(G_1)$ (resp. $K(G_2)$) is the maximal finite normal subgroup of G_1 (resp. G_2).

The idea is to consider $\widetilde{G} = G_1' * G_2'$ and pick a finite generating set A (resp. B) of G_1' (resp. G_2'). The quotient G is constructed by adding particular relations (which will be done by Dehn filling) to $\widetilde{G}$ which identify elements of A (resp. B) with certain elements of G_2' (resp. G_1').

There exists a finite generating set $A = \{a_1, ..., a_k\}$ (resp. $B = \{b_1, ..., b_k\}$) of G_1' (resp. G_2') for some $k \in \mathbb{N}^+$. To simplify the argument, let

$$a_{k+1} = a_{k+2} = b_{k+1} = b_{k+2} = 1. \tag{21}$$

To perform Dehn filling on $\widetilde{G}$, the first step is to find hyperbolically embedded subgroups. By [42, Lemma 5.10], G_1' and G_2' are acylindrically hyperbolic with $K(G_1') = K(G_2') = \{1\}$. Thus, Theorem 3.18 implies that there exist free groups

$$F_1 \hookrightarrow_h G_1', \qquad F_2 \hookrightarrow_h G_2',$$

each of which has rank $2k + 4$. By Example 3.8, we have $\{G_1', G_2'\} \hookrightarrow_h \widetilde{G}$. Thus, Proposition 3.10 implies

$$\{F_1, F_2\} \hookrightarrow_h \widetilde{G}.$$

In fact, F_1, F_2 are not quite the subgroups that we want, and we will apply Proposition 7.4 to construct other hyperbolically embedded subgroups from A, B, F_1, F_2. Note that for $i = 1, 2, \cdots, k + 2$,

$$a_i \notin F_2 \setminus \{1\}, \quad b_i \notin F_1 \setminus \{1\}.$$

Let $\{f_{1,i}, g_{1,i}\}_{i=1}^{k+2}$ (resp. $\{f_{2,i}, g_{2,i}\}_{i=1}^{k+2}$) be a basis of the free group F_1 (resp. F_2). Then Proposition 7.4 implies the following.

Lemma 7.10. *For sufficiently large $\ell \in \mathbb{N}^+$, $\{f_{1,i}^{\ell} b_i g_{1,i}^{\ell}\}_{i=1}^{k+2}$ (resp. $\{f_{2,i}^{\ell} a_i g_{2,i}^{\ell}\}_{i=1}^{k+2}$) freely generates a subgroup $H_1 \leqslant \widetilde{G}$ (resp. $H_2 \leqslant \widetilde{G}$) and*

$$\{H_1, H_2\} \hookrightarrow_h \widetilde{G}.$$

Proof of Theorem E. As G_1' and G_2' are acylindrically hyperbolic, we have $|G_1'| = |G_2'| = \infty$ and thus $\mathrm{cd}(G_1'), \mathrm{cd}(G_2') \geqslant 1$. Suppose $\mathrm{cd}(G_1') = \mathrm{cd}(G_2') = 1$. Then G_1' and G_2' are free by the Stallings-Swan theorem [68, corollary to Theorem 1]. Without loss of generality, we may assume that the rank of G_1' is greater than or equal to the rank of G_2'. It follows that G_2' is a quotient of G_1'. Let $G = G_2'$. Statements (i), (ii), and (iii) follow trivially. Statement (iv) also holds because if G_2 is of type FP_n for some $n \in \{2, 3, ..., \infty\}$, then G_2' is a finite rank free group and thus of type FP_∞.

Thus, let us assume $\max\{\mathrm{cd}(G_1'), \mathrm{cd}(G_2')\} \geqslant 2$. Fix a sufficiently large $\ell \in \mathbb{N}^+$ and let H_1 and H_2 be the subgroups given by Lemma 7.10. By Remark 3.21, Lemma 7.10, and Theorems 3.23, 3.25, there exist finite sets $\mathcal{F}_1 \subseteq H_1 \smallsetminus \{1\}$ and $\mathcal{F}_2 \subseteq H_2 \smallsetminus \{1\}$ such that if $N_1 \lhd H_1, N_2 \lhd H_2$ and $N_1 \cap \mathcal{F}_1 = N_2 \cap \mathcal{F}_2 = \varnothing$, then the following hold.

(a) $\{H_1/N_1, H_2/N_2\} \hookrightarrow_h \widetilde{G}/\langle\langle N_1 \cup N_2 \rangle\rangle$.
(b) $(\widetilde{G}, \{H_1, H_2\}, \{N_1, N_2\})$ is a Cohen-Lyndon triple and thus Theorems 4.6 and 4.8 and Corollary 4.7 can be applied to it.

Let $\{u_i\}_{i=1}^k$ (resp. $\{v_i\}_{i=1}^k$) be freely reduced words over the alphabet $\{f_{1,k+1}^\ell b_{k+1} g_{1,k+1}^\ell, f_{1,k+2}^\ell b_{k+2} g_{1,k+2}^\ell\}$ (resp. $\{f_{2,k+1}^\ell a_{k+1} g_{2,k+1}^\ell, f_{2,k+2}^\ell a_{k+2} g_{2,k+2}^\ell\}$) satisfying the $C'(1/6)$ small cancellation condition. Note that $u_i \in G_1'$ and $v_i \in G_2'$ for $i = 1, ..., k$ by (21). Let N_1 (resp. N_2) be the normal subgroup of H_1 (resp. H_2) generated by $\{f_{1,i}^\ell b_i g_{1,i}^\ell u_i\}_{i=1}^k$ (resp. $\{f_{2,i}^\ell a_i g_{2,i}^\ell v_i\}_{i=1}^k$).

By Lemma 7.10, H_1 and H_2 are freely generated by $\{f_{1,i}^\ell b_i g_{1,i}^\ell\}_{i=1}^{k+2}$ and $\{f_{2,i}^\ell a_i g_{2,i}^\ell\}_{i=1}^{k+2}$, respectively. Thus, H_1/N_1 and H_2/N_2 can be presented as

$$\begin{aligned} H_1/N_1 &= \langle f_{1,i}^\ell b_i g_{1,i}^\ell \quad (i = 1, ..., k+2) \mid f_{1,j}^\ell b_j g_{1,j}^\ell u_j \quad (j = 1, ..., k) \rangle \\ &= \langle f_{1,k+1}^\ell b_{k+1} g_{1,k+1}^\ell, f_{1,k+2}^\ell b_{k+2} g_{1,k+2}^\ell \rangle, \end{aligned} \tag{22}$$

$$\begin{aligned} H_2/N_2 &= \langle f_{2,i}^\ell a_i g_{2,i}^\ell \quad (i = 1, ..., k+2) \mid f_{2,j}^\ell a_j g_{2,j}^\ell v_j \quad (j = 1, ..., k) \rangle \\ &= \langle f_{2,k+1}^\ell a_{k+1} g_{2,k+1}^\ell, f_{2,k+2}^\ell a_{k+2} g_{2,k+2}^\ell \rangle, \end{aligned} \tag{23}$$

where the last equality of (22) (resp. (23)) follows from eliminating $f_{1,i}^\ell b_i g_{1,i}^\ell$ (resp. $f_{2,i}^\ell a_i g_{2,i}^\ell$) for $i = 1, ..., k$ using Tietze transformations (see [50, Chapter II]).

Thus, H_1/N_1 and H_2/N_2 are free groups of rank 2. In particular,

$$\mathrm{card}(H_1/N_1) = \infty. \tag{24}$$

By the Greendlinger's lemma for free groups [50, Chapter V Theorem 4.5], if $\|u_i\|, \|v_i\|, 1 \leqslant i \leqslant k$, are sufficiently large, then

$$N_1 \cap \mathcal{F}_1 = N_2 \cap \mathcal{F}_2 = \varnothing.$$

Let

$$G = \widetilde{G}/\langle\langle N_1 \cup N_2 \rangle\rangle.$$

As $a_{k+1} = a_{k+2} = b_{k+1} = b_{k+2} = 1$, G is a common quotient of G_1' and G_2'. In particular, G is a common quotient of G_1 and G_2.

If $H_1/N_1 = G$, then G is a non-cyclic free group and thus is acylindrically hyperbolic. If H_1/N_1 is a proper subgroup of G, then equation (24), item (a), and Theorem 3.23 imply that G is acylindrically hyperbolic. Statement (i) is proved.

Consider statement (ii). For every $n \geqslant 3$ and every G-module A, we have

$$\begin{aligned}
&H^n(G; A) \\
\cong &H^n(\widetilde{G}; A) \oplus H^n(H_1/N_1; A) \\
&\oplus H^n(H_2/N_2; A) && \text{by Corollary 4.6 and that } H_1, H_2 \text{ are free} \\
\cong &H^n(\widetilde{G}; A) && \text{as } H_1/N_1 \text{ and } H_2/N_2 \text{ are free groups} \\
\cong &H^n(G_1'; A) \oplus H^n(G_2'; A) && \text{as } \widetilde{G} = G_1' * G_2',
\end{aligned}$$

which is (ii).

Consider statement (iii). If either $K(G_1)$ or $K(G_2)$ is not the trivial group $\{1\}$, then (iii) is trivial. So let us assume that $K(G_1) = K(G_2) = \{1\}$. Then Corollary 4.7 implies

$$\begin{aligned}
\operatorname{cd}(G) &\leqslant \max\{\operatorname{cd}(\widetilde{G}), \operatorname{cd}(H_1/N_1), \operatorname{cd}(H_2/N_2), \operatorname{cd}(H_1)+1, \operatorname{cd}(H_2)+1\} \\
&= \max\{\operatorname{cd}(\widetilde{G}), 1, 2\} && \text{as } H_1, H_2, H_1/N_1, \text{ and } H_2/N_2 \text{ are free groups} \\
&= \max\{\operatorname{cd}(G_1'), \operatorname{cd}(G_2')\} && \text{as } \operatorname{cd}(G_1'), \operatorname{cd}(G_2') \geqslant 2 \\
&= \max\{\operatorname{cd}(G_1), \operatorname{cd}(G_2)\}.
\end{aligned}$$

Finally, suppose that G_1 and G_2 are of type FP_n for some $n \in \{2, 3, ..., \infty\}$. Then so are G_1' and G_2' [4, Corollary 9]. As H_1/N_1 and H_2/N_2 are free groups of finite rank, they are of type FP_∞. Therefore, Theorem 4.8 implies that G is of type FP_n and thus statement (iv) holds. □

Remark 7.11. Analogous to the above proof, in the setting of Theorem E, we have $\operatorname{hd}(G) \leqslant \max\{\operatorname{hd}(G_1), \operatorname{hd}(G_2), 2\}$.

8. Some applications

In this section, we give some applications of Theorems D and E.

8.1. Infinite dimension torsion-free FP_∞ groups

If a group G has $\mathrm{cd}(G) < \infty$, then G is necessarily torsion-free. Of course, the converse is not true, e.g., $\mathrm{cd}(\mathbb{Z}^\infty) = \infty$. Observe however that $\mathbb{Z}^\infty$ is not of type FP_∞. In fact, Bieri asked if every torsion-free group of type FP_∞ is also of type FP or if there is an FP_∞ group with a non-finitely generated free abelian subgroup [70, Problem F11]. In [9], Brown and Geoghegan settled both questions, by showing that Thompson's group F is of type FP_∞.

For every torsion-free acylindrically hyperbolic group G, Theorem D embeds F into an acylindrically hyperbolic quotient $\overline{G}$ of G, which is a torsion-free FP_∞ group of infinite cohomological dimension. We thus have the following.

Corollary 8.1. *Every torsion-free acylindrically hyperbolic group G of type FP_∞ has a torsion-free acylindrically hyperbolic quotient $\overline{G}$ of type FP_∞ which contains the Thompson group F. In particular,* $\mathrm{cd}(\overline{G}) = \infty$.

8.2. Quotients of hyperbolic groups

SQ-universality of non-elementary hyperbolic groups was proved by Olshanskii [58] and independently by Delzant [25]. In Theorem D, if G and C are word-hyperbolic, a hyperbolic quotient $\overline{G}$ of G can be constructed so that the conclusions hold. The next corollary is thus a strengthening of the aforementioned result.

Corollary 8.2. *Let G be a non-elementary hyperbolic group and C any hyperbolic group. Then there is a hyperbolic quotient $\overline{G}$ of $G/K(G)$ (in particular, $\overline{G}$ is a quotient of G), where $K(G)$ is the maximal finite normal subgroup of G, such that C embeds into $\overline{G}$ and the following hold.*

(i) *For all $n \geqslant 3$ and every $\overline{G}$-module A, we have*

$$H^n(\overline{G}; A) \cong H^n(G/K(G); A) \oplus H^n(C; A),$$

where the action of $G/K(G)$ (resp. C) on A is induced by the quotient map $G/K(G) \to \overline{G}$ (resp. the embedding $C \hookrightarrow \overline{G}$).

(ii) $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(C)\}$.

Proof. By passing to $G/K(G)$, we may assume that $K(G) = \{1\}$. By [64, Corollary 4.21], there is a free subgroup $F_2 < G$ of rank 2 such that G is hyperbolic relative to F_2. By [45], four random elements of F_2 freely generate a free subgroup $F_4 \leqslant F_2$ that is malnormal in F_2. By Marshall Hall's theorem, there exists a finite-index subgroup H of F_2 such that F_4 is a free factor of H. In particular, F_4 is quasi-convex in F_2, and thus [14, Theorem 7.11] implies that F_2 is hyperbolic relative to F_4. Thus, G is hyperbolic relative to F_4 [59]. By Lemma 7.1, C embeds into a quotient R of F_4 such that R is hyperbolic relative to C and the quotient map $F_4 \twoheadrightarrow R$ is injective on any given finite set $\mathcal{F} \subseteq F_4 \smallsetminus \{1\}$. Since C is hyperbolic, so is R [59, Corollary 2.41]. Let $\overline{G}$ be the quotient of G by the Dehn filling $F_4 \twoheadrightarrow R$. By avoiding a suitable finite set $\mathcal{F}$, we may assume that $\overline{G}$ is hyperbolic relative to R [60, Theorem 1.1]. Therefore, $\overline{G}$ is hyperbolic [59, Corollary 2.41]. Items (i) and (ii) are immediate consequences of Theorem D. □

Recently in [43, Corollary 2], Italiano-Martelli-Migliorini settled a long-standing open problem by constructing the first example of hyperbolic group which contains a subgroup of type F that is not hyperbolic. Their group is the fundamental group of a 5-dimensional hyperbolic pseudo-manifold that fibers over the circle. The fundamental group of the fiber is of type F but not hyperbolic.

Corollary 8.3. *Let $n \geqslant 5$ be an integer. Every non-elementary hyperbolic group G with $\mathrm{cd}(G) \leqslant n$ has a hyperbolic quotient $\overline{G}$ with $\mathrm{cd}(\overline{G}) = n$ such that $\overline{G}$ contains the Italiano-Martelli-Migliorini group. In particular, there is a type F non-hyperbolic subgroup $H < \overline{G}$.*

Proof. Let C_1 be the group constructed by Italiano-Martelli-Migliorini in [43, Corollary 2]. Then $\mathrm{cd}(C_1) = 5$. Let C_2 be a hyperbolic group with $\mathrm{cd}(C_2) = n$, and let $C = C_1 * C_2$. Since $\mathrm{cd}(G) < \infty$, we have $K(G) = \{1\}$. Corollary 8.2 thus yields a hyperbolic quotient $\overline{G}$ such that C embeds into $\overline{G}$ and $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), \mathrm{cd}(C)\} = \max\{\mathrm{cd}(G), \mathrm{cd}(C_1), \mathrm{cd}(C_2)\} = n$. Since C embeds into $\overline{G}$, we also have $\mathrm{cd}(\overline{G}) \geqslant \mathrm{cd}(C) \geqslant \mathrm{cd}(C_2) = n$, and thus $\mathrm{cd}(\overline{G}) = n$. □

8.3. Property (T) quotients

A group G has *Kazhdan's property (T)* if every affine isometric action of G on a Hilbert space has a global fixed point. The next result strengthens [42, Corollary 1.7].

Corollary 8.4. *Every acylindrically hyperbolic group G of type FP_n for some $n \in \{2, 3, ..., \infty\}$ (resp. FP) has an acylindrically hyperbolic quotient $\overline{G}$ of type FP_n (resp. FP) with Kazhdan's Property (T) such that $\mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), 2\}$.*

Proof. Let Γ be an type FP acylindrically hyperbolic property (T) group with $\mathrm{cd}(\Gamma) = 2$. For example, one can let Γ be a random group in the Gromov density model with density $1/3 < d < 1/2$. By [39, Section 9.B] (see also [56, Theorem 1]), with overwhelming probability Γ is hyperbolic and has an aspherical presentation, and thus is of type FP and satisfies $\mathrm{cd}(\Gamma) \leqslant 2$. By [73, Theorem 4] (see also [57, Section I.3.g]), with overwhelming probability Γ has property (T). The result now follows from Theorem E applied to G and Γ. □

The above result can for example be applied to mapping class groups of surfaces of finite type, outer automorphism groups of free groups of finite rank and (non-virtually polycyclic) fundamental groups of compact orientable 3-manifolds which all exhibit nice cohomological finiteness conditions.

8.4. Acylindrically hyperbolic quotients distinguishable by cohomology

Let G be any acylindrically hyperbolic group. For each $k \geqslant 3$, Theorem D implies that one can embed $\mathbb{Z}^k$ into a quotient G_k of G such that $H^n(G_k;\mathbb{Z}) \cong H^n(G;\mathbb{Z}) \oplus H^n(\mathbb{Z}^k;\mathbb{Z})$ for all $n \geqslant 3$. Suppose $\mathrm{cd}(G) < \infty$. Then $H^*(G_k;\mathbb{Z}) \neq H^*(G_\ell;\mathbb{Z})$ for $k, \ell > \mathrm{cd}(G)$. Suppose that G is of type FP_∞ instead. Then $H^k(G;\mathbb{Z})$ is finitely generated, and thus $H^k(G_k;\mathbb{Z}) \ncong H^k(G_\ell;\mathbb{Z})$ for $k \neq \ell$. This establishes the following.

Corollary 8.5. *Let G be any acylindrically hyperbolic group. Then there exists an infinite family $\{G_k\}_{k=3}^\infty$ of acylindrically hyperbolic quotients of G such that $H^n(G_k;\mathbb{Z}) \cong H^n(G;\mathbb{Z}) \oplus H^n(\mathbb{Z}^k;\mathbb{Z})$ for all $n, k \geqslant 3$. In particular, if $\mathrm{cd}(G) < \infty$ or G is of type FP_∞, then each pair of elements of $\{G_k\}_{k=l}^\infty$ have non-isomorphic integral cohomology for $l = \mathrm{cd}(G) + 1$ and $l = 3$, respectively.*

If instead of embedding $\mathbb{Z}^k$, we embed the family of Houghton's groups or Abel's groups [18], then we will obtain quotients that are separated by homological finiteness properties.

Corollary 8.6. *Let G be any acylindrically hyperbolic group of type FP_∞. Then G has a family of acylindrically hyperbolic quotients $\{G_k\}_{k=2}^\infty$ such that for each k, G_k has Kazhdan's Property (T), is of type FP_{k-1} but not of type FP_k.*

We remark that being of type FP_n is a quasi-isometric invariant [4, Corollary 9]. Therefore, the quotients $\{G_k\}_{k\geqslant 1}$ in Corollary 8.6 are pairwise non-quasi-isometric.

Proof. By Corollary 8.4, we can pass to a quotient $\overline{G}$ of G that is acylindrically hyperbolic and of type FP_∞ and has property (T). There exists a family of groups $\{H_k\}_{k=2}^\infty$ such that for all k, H_k is of type FP_{k-1} but not of type FP_k. For example, one can let $\{H_k\}_{k=2}^\infty$ be the family of Houghton's groups or Abel's groups [18, Theorems 5.1 and 6.1]. Theorem D then embeds each H_k to a quotient G_k of $\overline{G}$ with the desired properties. □

8.5. First Betti number and a question of Osin

Motivated by the Virtual Positive First Betti Number Conjecture which has now been settled by Agol [3], Osin posed the following question.

Question 8.7 *(Osin, [61, Problem 2.3]). Let G be a Poincaré duality group of dimension 3 such that G is hyperbolic relative to its subgroup H. If G has positive virtual first*

Betti number, is it true that for most $N \lhd H$, $G/\langle\langle N\rangle\rangle$ also has positive virtual first Betti number?

Question 8.7 is only about virtual Betti numbers. For actual Betti numbers there are counterexamples. For example, let S be an orientable surface of genus two. Take a suitable pseudo-Anosov homeomorphism ϕ on S such that the mapping torus T_ϕ has $b_1(T_\phi) = 1$. Then $\pi_1(T_\phi)$ is hyperbolic and is a semi-direct product $\pi_1(T_\phi) = \pi_1(S) \rtimes \langle t\rangle$. $\pi_1(T_\phi)$ is hyperbolic relative to $\langle t\rangle$ [14], and for every $n \geqslant 1$, $b_1(\pi_1(T_\phi)/\langle\langle t^n\rangle\rangle) = 0$ since the image of t generates $H_1(\pi_1(T_\phi); \mathbb{Q})$. Using this example, we prove the following.

Corollary 8.8. *Let G be any acylindrically hyperbolic group. Then G has an acylindrically hyperbolic quotient $\overline{G}$ such that $b_1(\overline{G}) = 0, \mathrm{cd}(\overline{G}) \leqslant \max\{\mathrm{cd}(G), 3\}$, and $H^n(\overline{G}; A) \cong H^n(G; A)$ for all $n \geqslant 4$ and any $\overline{G}$-module A.*

Proof. The idea is to construct an acylindrically hyperbolic group G' such that $b_1(G') = 0$ and $\mathrm{cd}(G') \leqslant 3$. Once such a G' has been constructed, Theorem E will produce a common quotient $\overline{G}$ of G and G' with the desired properties.

So it remains to construct G'. Let T_ϕ, t be as above. Take eight elements $a_1, a_2, \cdots, a_8$ that generate $\pi_1(S)$ as a group. Then $a_1, a_2, \cdots, a_8, t$ generate $\pi_1(T_\phi)$ as a group. Let X be the set consisting of $a_1, a_2, \cdots, a_8, t$ and their inverses. Randomly generate two words w_1, w_2 over X of length n and identify w_1, w_2 with the elements of $\pi_1(T_\phi)$ they represent.

By [54, Theorem 1], as $n \to \infty$, with probability 1 we have that $\langle w_1, w_2\rangle \leqslant \pi_1(T_\phi)$ is free of rank 2 and $\langle w_1, w_2\rangle \hookrightarrow_h \pi_1(T_\phi)$. Moreover, as $n \to \infty$, with positive probability we have that both w_1 and w_2 have non-zero t-exponents. Thus, there exist $w_1, w_2 \in \pi_1(T_\phi)$ such that

(a) $\langle w_1, w_2\rangle \leqslant \pi_1(T_\phi)$ is free of rank 2,
(b) $\langle w_1, w_2\rangle \hookrightarrow_h \pi_1(T_\phi)$, and
(c) w_1 and w_2 have non-zero t-exponents.

Let w be a word over w_1, w_2 such that w is not a proper power, satisfies the $C'(1/6)$ small cancellation condition, and w has positive t-exponent. Let N be the normal subgroup of $\langle w_1, w_2\rangle$ generated by w. By [52, Theorem 11.1], $\langle w_1, w_2\rangle/N$ has $\mathrm{cd}(\langle w_1, w_2\rangle/N) \leqslant 2$. Let $\langle\langle N\rangle\rangle$ be the normal closure of N in $\pi_1(T_\phi)$. By making w sufficiently long, we can guarantee that N avoids any given finite subset of $\langle w_1, w_2\rangle \smallsetminus \{1\}$, and therefore guarantee that $\pi_1(T_\phi)/\langle\langle N\rangle\rangle$ is acylindrically hyperbolic and $\mathrm{cd}(\pi_1(T_\phi)/\langle\langle N\rangle\rangle) \leqslant 3$ by Theorems 3.23 and A. Since w has positive t-exponent, we have $b_1(\pi_1(T_\phi)/\langle\langle N\rangle\rangle) = 0$. So it suffices to let $G' = \pi_1(T_\phi)/\langle\langle N\rangle\rangle$. □

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