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Discrete-time systems with slope restricted nonlinearities: Zames-Falb multiplier analysis using external positivity

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SUMMARY

This paper exploits positive systems theory in the search for Zames-Falb multipliers for the analysis of discrete-time Lurie systems, where the nonlinearity is assumed to be slope-restricted. Although a similar problem has been tackled in a continuous time context, the results in discrete-time take a different form and require a somewhat different approach to overcome certain technical problems. The work has two compelling features: (i) the arising analysis algorithms are completely convex; and (ii) numerical results compare well with the state-of-the-art in some cases, providing the least conservative result in one instance.

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KEY WORDS: robust control, absolute stability, integral quadratic constraints

1. INTRODUCTION

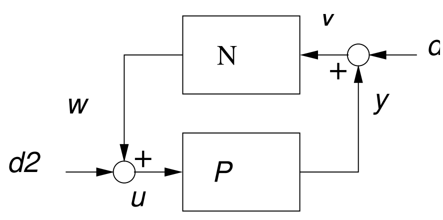

Figure 1. System under consideration

The study of Lurie systems, shown in Figure 1, is a classical problem in the theory of automatic control systems and has been studied since the 1950s: during this and the following decade, most of the fundamental results on the stability analysis of this type of system were obtained - see, for example, [32, 31, 16, 5] and references therein. Lurie systems are characterised by two distinct sub-systems: one being a linear time-invariant part; the other being static and nonlinear. The properties of the static nonlinear part essentially determine the strength of the results available.

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An important sub-class of Lurie systems is that when the nonlinearity $\phi(\cdot)$ is slope-restricted, its importance deriving from the fact that many practical nonlinearities fall into this class. The analysis of systems containing slope-restricted nonlinearities has been investigated by both Lyapunov (e.g. [21]) and input-output (e.g. [32]) approaches, but probably the most powerful approach was proposed by O'Shea [19] and refined by Zames and Falb shortly afterwards [32]. The "Zames-Falb" multiplier approach to the stability analysis of such systems is now reasonably common-place, with a rapid increase of papers on the subject in the last decade or so [10, 17, 4, 28, 9, 8, 12].

Most of the literature on Zames-Falb multipliers/slope restricted nonlinearities concentrates on continuous time systems with far less attention given to their discrete counterparts. There are some notable papers on the analysis of discrete-time Lurie systems with slope-restrictions. These include approaches relying on Lyapunov methods ([1, 20]), generic IQC-based approaches [14, 11], and various papers on discrete-time Zames-Falb approaches ([17, 30, 6]). The state-of-the-art in discrete-time Zames-Falb analysis is described cogently in the recent paper by Carrasco, Heath and co-authors in [6]. In that paper, the authors compare various techniques for analysing discrete-time slope-restricted Lurie systems, and also propose a new approach based on FIR[†] Zames-Falb multipliers. These FIR multipliers provide superior results over most other methods, including the Lyapunov approaches of [1, 20] and those corresponding to discrete-time versions of [29, 8]. An interesting aside is that the discrete-time approaches can also be used to provide irrational multipliers for continuous time systems and the approach appears to be computationally convenient.

One may note that the discrete-time results of [6] however, may still appear to be conservative and, for some of the examples given in the paper, the estimate of the maximum slope size is still some way short of that predicted by the Kalman conjecture. While it is not expected that the Kalman conjecture holds in all cases, one might expect better results in some of them. Thus, while undeniably [6] provides the least Conservative results for discrete-time Lurie systems with slope-restrictions, currently available, the work in this paper is motivated by conservatism which may yet exist in some instances.

The novelty in this paper is the application of the theory available for externally positive systems to discrete-time Zames-Falb multiplier searches. In some ways this paper is therefore a companion paper to [27] where the same was done for continuous-time systems, but the methods used and some of the obstacles overcome differ in the discrete-time case. The numerical results obtained are of variable competetiveness: sometimes rivalling the state-of-the-art [6], but sometimes yielding results some way from this.

The paper is structured as follows. Firstly, the problem considered is stated and discussed. Secondly, results from positive systems theory and in particular a class of systems which have a symmetric state-space representation are provided. The next section provides the main results; some enhancements are suggested next; numerical examples are then given; the paper then concludes.

1.1. Notation

Notation is fairly standard. For a square matrix M the notation $M > 0$ (≥ 0) means the matrix is symmetric and positive (semi-) definite. Negative definiteness is defined similarly. I denotes the identity matrix, of appropriate dimensions; I_m denotes the identity matrix of size m.

This paper mainly considers scalar-valued signals (sequences). For scalar signals, $x(k) \in \mathbb{R}$, the l_1 norm is defined as

$$\|x\|_1 := \sum_{k=-\infty}^{\infty} |x(k)| \tag{1}$$

With some abuse of notation, $\|H(z)\|_1$ also denotes the l_1 norm of the impulse response associated with the transfer function $H(z)$. A signal $x(k) \in l_1$ if $\|x\|_1$ is finite.

[†]Finite Impulse Response

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The l_2 norm of a signal $x(k) \in \mathbb{R}$ is defined as

$$\|x\|_2^2 := \sum_{k=-\infty}^{\infty} |x(k)|^2 \tag{2}$$

$x(k) \in l_2$ if $\|x\|_2$ is finite. Signals $x(k)$ which are zero for all $k < 0$ belong to the subspace $l_2[0, \infty)$. The extended space l_{2e} is the set of all signals whose truncations belongs to $l_2[, \infty)$.

The space of real rational transfer functions bounded on the unit circle is denoted by $\mathcal{RL}_\infty$, with norm defined as

$$\|G(z)\|_{\mathcal{L}_\infty} = \sup_{\omega \in (-\pi, \pi)} |G(e^{j\omega})| \tag{3}$$

The subspace $\mathcal{RH}_\infty \quad (\mathcal{RH}_\infty^\perp) \quad \subset \mathcal{RL}_\infty$ denotes those transfer functions analytic in (outside) the open unit disk.

A system $G(z)$ with impulse response $g(k)$ is causal (anti-causal) if $g(k) = 0$ for all $k < 0$ ($k > 0$). A system $G(z)$ is said to be bounded if its impulse response, $g(k) \in l_1$. If $G(z) \in \mathcal{RH}_\infty$ it is interpreted as a bounded causal operator; $G(z) \in \mathcal{RH}_\infty^\perp$ is a bounded anti-causal operator. $G^\sim(z) = G'(z^{-1})$ is the adjoint of the transfer function matrix $G(z)$; $G(z) \in \mathcal{RH}_\infty$ implies $G(z)^\sim \in \mathcal{RH}_\infty^\perp$.

2. LURIE SYSTEM STABILITY ANALYSIS

This paper considers the stability of the interconnection depicted in Figure 1 where $P(z) \in \mathcal{RH}_\infty$ is a single-input-single-output (SISO) linear time invariant system and $\phi(\cdot)$ is a static, time-invariant, slope-restricted nonlinearity. Following the "robust control" convention ([18]), positive feedback is used in Figure 1. The system in Figure 1 is said to be l_2 stable, or simply *stable*, if $y(k)$ and $w(k)$ belong to $l_2[0, \infty)$ for all exogenous signals $d_1(k), d_2(k) \in l_2[0, \infty)$. $P(z)$ is assumed to have the following state-space representation,

$$P(z) \sim \begin{cases} x(k+1) &= A_p x(k) + B_p u(k) \\ y(k) &= C_p x(k) + D_p u(k) \end{cases} \tag{4}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times 1}$, $C_p \in \mathbb{R}^{1 \times n_p}$, $D_p \in \mathbb{R}$.

The nonlinearity $\phi(\cdot)$ is assumed to be such that its generalised derivative (slope) is restricted to the interval $[0, \alpha]$, viz $\partial\phi \in [0, \alpha]$; that is, it satisfies the following inequality

$$0 \leq \frac{\phi(x) - \phi(y)}{x - y} \leq \alpha \quad \forall x, y \in \mathbb{R}, \quad \alpha > 0 \tag{5}$$

Moreover, it is assumed that $\phi(\cdot)$ belongs to either $\mathcal{N}_{[0,\alpha]}^S$ or $\mathcal{N}_{[0,\alpha]}^{S,odd}$ defined below.

Definition 1
$\phi(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is said to belong to $\mathcal{N}_{[0,\alpha]}^S$ if

i) It is bounded.
ii) It has slope restriction $\partial\phi \in [0, \alpha]$.

$\phi(\cdot)$ is said to belong to $\mathcal{N}_{[0,\alpha]}^{S,odd}$ if, in addition, it is odd.

As frequently pointed out, $\partial\phi \in [0, \alpha]$ implies $\phi(\cdot)$ belongs to the sector $[0, \alpha]$ as well, and so satisfies the following inequality

$$0 \leq \frac{\phi(v)}{v} \leq \alpha \quad \forall v \in \mathbb{R} \tag{6}$$

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It may be possible that the nonlinearity $\phi(\cdot)$ satisfies a tighter sector bound independently, but this is not pursued in this paper.

For convenience, the Zames-Falb analysis here is carried out in the IQC framework ([18]). Although this is not necessary, the IQC framework makes the analysis convenient for combining with other forms of "uncertainty", and also makes the treatment of non-causal multipliers more straightforward. It is known (see [14, 15] for example) that if $\phi \in \mathcal{N}^S_{[0,\alpha]}$, then it satisfies the IQC given by

$$\int_0^{2\pi} \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \widehat{\phi(v)}(e^{j\omega}) \end{bmatrix}^\star \Pi_{ZF}(e^{j\omega}) \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \widehat{\phi(v)}(e^{j\omega}) \end{bmatrix} d\omega \geq 0 \quad \forall\, \hat{v}(e^{j\omega}) \in l_2 \tag{7}$$

where $\Pi_{ZF}(e^{j\omega})$ is obtained by evaluating $\Pi_{ZF}(z)$ on the unit circle and where $\Pi_{ZF}(z)$ is given by

$$\Pi_{ZF}(z) = \begin{bmatrix} 0 & \alpha M^{\sim}(z) \\ \alpha M(z) & -M^{\sim}(z) - M(z) \end{bmatrix} \tag{8}$$

$M(z)$ is the so-called *Zames-Falb* multiplier which will be defined shortly. For discrete-time systems the class of Zames-Falb multipliers contains both FIR and IIR elements. For this paper, attention is confined to the following class of multipliers, which take a similar form to the continuous time ones used in [27].

Definition 2
$\mathcal{M}^{odd}_R$ is the set of transfer functions $M(z)$ having the following structure:

$$M(z) = H_0 - H_c(z) - H_a^{\sim}(z)$$

where $H_c(z), H_a(z) \in \mathcal{RH}_\infty$ and is such that

$$\|M(z)\|_1 \leq 2H_0$$

$\mathcal{M}_R$ is the sub-class for which the impulse response of $M(z)$, $m(k)$, is such that

$$m(k) \leq 0 \quad \forall k \neq 0 \tag{9}$$

The significance of $\mathcal{M}_R$ is that it can be used to treat the case where the nonlinearity $\phi(\cdot)$ is not necessarily odd. This will be the class of multiplier which is considered henceforth. Note that the condition on $m(k)$ is an external *negativity* condition.

The main stability result used throughout the paper is a specialisation of the main discrete-time IQC stability result, which has been re-stated from [15].

Theorem 1
Consider the interconnection in Figure 1 where $P(z) \in \mathcal{RH}_\infty$ and $\phi \in \mathcal{N}^S_{[0,\alpha]}$. Assume the system is well-posed. Then the system is l_2 stable if there exists an $M(z) \in \mathcal{M}_R$ such that the following inequality holds

$$\begin{bmatrix} P(e^{j\omega}) \\ I \end{bmatrix}^\star \Pi_{ZF}(e^{j\omega}) \begin{bmatrix} P(e^{j\omega}) \\ I \end{bmatrix} < -\epsilon I \quad \forall \omega \in [-\pi, \pi] \tag{10}$$

where $\Pi_{ZF}(z)$ is given in equation (8).

Unlike the original statement in [15], no *homotopy* condition is required. This is because (see Remark 15 in [15]) the multiplier $\Pi_{ZF}(e^{j\omega})$ is such that its (2,2) entry, see equation (8), is negative semi-definite (Seiler and co-authors call this a positive-negative multiplier [14]), and hence the homotopy condition is unconditionally satisfied. Much of the remainder of the paper describes how the above theorem can be converted to a set of LMI conditions.

3. POSITIVE SYSTEMS RESULTS

The main novelty in the paper is to exploit results on discrete-time positive systems for Zames-Falb multiplier analysis. Consider a SISO discrete-time system $G(z)$ with state-space realisation

$$G(z) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Definition 3
$G(z)$ is said to be *externally positive* if its impulse response $g(k)$ is such that $g(k) \geq 0 \quad \forall k \in (-\infty, \infty)$.

Fact 1
A causal linear system $G(z)$ is said to be externally positive if its impulse response

$$g(k) = \begin{cases} D & k = 0 \\ CA^{k-1}B & k = 1, 2, \ldots \end{cases}$$

is positive for all $k \geq 0$.

For Zames-Falb multipliers in discrete-time the concept of external *negativity* is pertinent.

Definition 4
$G(z)$ is said to be *externally negative* if its impulse response $g(k)$ is such that $g(k) \leq 0 \quad \forall k \in (-\infty, \infty)$.

Remark 1: The concept of *external negativity* is intrinsic to the definition of the class of of Zames-Falb multipliers for not-necessarily-odd nonlinearities, $\mathcal{M}_R$, as introduced in Definition 2. Thus, because this paper will focus on the class $\mathcal{M}_R$, it is clear that verifying external negativity of the multiplier is a central requirement for the stability results reported. It is noted that the requirement of external negativity can be dropped, but at the expense of only guaranteeing stability for *odd* nonlinearities.

Fact 2
$G(z)$ is externally negative if and only if $-G(z)$ is externally positive.

Fact 2 allows one to cast all problems of external negativity in the external positivity framework. One of the useful properties of externally positive SISO systems is that their l_1 norm is given by their DC gain ([22]).

Lemma 1
Consider $G(z) \sim (A, B, C, 0) \in \mathcal{RH}_\infty$. Assume that $G(z)$ is externally positive. Then $\|G\|_1$ is given by

$$\|G(z)\|_1 = G(1) = C(I - A)^{-1}B$$

By Fact 2, it follows that if $G(z) \sim (A, B, C, 0)$ is externally negative, then $\|G(z)\|_1 = -G(1)$.

In the results which follow, symmetric systems will play a central role and the result below is of importance.

Lemma 2
Consider the system $G(z) \sim (A, C', C, D) \in \mathcal{RH}_\infty$ where $A = A'$. Then the system is externally positive if $A \geq 0$ and $D \geq 0$.

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Proof: In order for $G(z)$ to be externally positive, its impulse response $g(k) \geq 0$ for all integers $k \geq 0$. At $k = 0$ it is clear that $g(0) \geq 0$ iff $D \geq 0$. For $k = 1$,

$$g(k) = CC' \geq 0$$

Then for $k > 1$, define $l = k - 1 > 0$ and note that

$$g(l) = CA^l C$$

For odd values of l it is clear that $A \geq 0$ ensures that $g(l) \geq 0$; this condition is not necessary. □□

Remark 2: Lemma 2 is subtly different from the continuous time equivalent in [27]: in the continuous time case, symmetry of A is sufficient for ensuring external positivity; in the discrete time case, A must in addition be positive semi-definite. That is, there exist symmetric discrete-time systems $G(z) \sim (A, C', C, D)$ where A is symmetric, which are not externally positive. □

Similar to [27], the following lemma is useful.

Lemma 3
Let $H(z) = H_c(z) + H_a^{\sim}(z)$ be a non-causal transfer function where $H_c(z), H_a(z) \in \mathcal{RH}_\infty$ and where $H_c(z)$ and $H_a^{\sim}(z)$ are both externally positive. Then $\bar{H}(z) := H_c(z) + H_a(z)$ is also negative and, furthermore,

$$\|H(z)\|_1 \leq \|\bar{H}(z)\|_1$$

Proof: If $H_a^{\sim}(z) \in \mathcal{RH}_\infty^{\perp}$ is externally positive, then so is $H_a(z) \in \mathcal{RH}_\infty$; because $H_c(z)$ is also externally positive then $\bar{H}(z)$ is also. The norm bound follows from elementary properties and the results given above.

$$\begin{aligned}
\|H(z)\|_1 &= \|H_c(z) + H_a^{\sim}(z)\|_1 && (11)\\
&\leq \|H_c(z)\|_1 + \|H_a^{\sim}(z)\|_1 && (12)\\
&= \|H_c(z)\|_1 + \|H_a(z)\|_1 && (13)\\
&= H_c(1) + H_a(1) \qquad \text{(by Lemma 1)} && (14)\\
&= \bar{H}(1) && (15)\\
&= \|\bar{H}(z)\|_1 && (16)
\end{aligned}$$

4. MAIN RESULTS

The main results consist of translating the frequency domain inequality (19) in Theorem 1 into an LMI and then providing a further LMI which ensures that $M(z) \in \mathcal{M}_R$. Simultaneous satisfaction of these LMI's then ensures that Theorem 1 holds and thus the interconnection in Figure 1 will be stable. As a preliminary step, first note ([27]) that $\Pi_{ZF}(z)$ admits the following factorisation:

$$\Pi_{ZF}(z) = \Psi_{ZF}^{\sim}(z) W_{ZF} \Psi_{ZF}(z) \quad W_{ZF} = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix} \tag{17}$$

where $\Psi_{ZF}(z) = [Y(z) \quad I_2]' \in \mathcal{RH}_\infty$ and $Y(z) \in \mathcal{RH}_\infty$

$$Y(z) := \begin{bmatrix} 0 & -\alpha H_a(z) \\ \alpha(H_0 - H_c(z)) & -H_0 + H_a(z) + H_c(z) \end{bmatrix} \tag{18}$$

Thus inequality (10) becomes

$$\begin{bmatrix} P(e^{j\omega}) \\ I_1 \end{bmatrix}^{\star} \Psi_{ZF}^{*}(e^{j\omega}) W_{ZF} \Psi_{ZF}(e^{j\omega}) \begin{bmatrix} P(e^{j\omega}) \\ I_1 \end{bmatrix} < -\epsilon I_1 \quad \forall \omega \in [-\pi, \pi] \tag{19}$$

or, more succinctly

$$L(e^{j\omega})^* W_{ZF} L(e^{j\omega}) \leq -\epsilon I_1 \quad \forall \omega \in [-\pi, \pi] \tag{20}$$

where

$$L(z) = [Y(z)' \quad I_2]' \begin{bmatrix} P(z) \\ I_1 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} 0 & -\alpha H_a(z) \\ \alpha(H_0 - H_c(z)) & -H_0 + H_a(z) + H_c(z) \\ I_1 & 0 \\ 0 & I_1 \end{bmatrix} \begin{bmatrix} P(z) \\ I_1 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} -\alpha H_a(z) \\ (H_0 - H_c(z))(\alpha P(z) - 1) + H_a(z) \\ P(z) \\ 1 \end{bmatrix} \tag{23}$$

Thus an intermediate result is the following corollary.

Corollary 1
Consider the interconnection in Figure 1 where $P(z) \in \mathcal{RH}_\infty$ and $\phi \in \mathcal{N}^S_{[0,\alpha]}$. Let $M(z) \in \mathcal{M}_R$ and assume that the closed loop is well posed. Then the system is l_2 stable if inequality (20) is satisfied.

A further intermediate result can be arrived at by applying the KYP Lemma to inequality (20), which ensures Theorem 1 is satisfied.

Proposition 1
Let $P(z) \in \mathcal{RH}_\infty$ and $\phi \in \mathcal{N}^S_{[0,\alpha]}$ and assume that the system in Figure 1 is well posed. Let $M(z) \in \mathcal{M}_R$, where $H_c(z), H_a(z) \in \mathcal{RH}_\infty$, and let $H_c(z)$ and $H_a(z)$ have the following state-space realisations:

$$H_c(z) \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right] \qquad H_a(z) \sim \left[\begin{array}{c|c} A_a & B_a \\ \hline C_a & 0 \end{array}\right] \tag{24}$$

If there exists a real symmetric matrix P such that the following inequality is satisfied

$$\begin{bmatrix} \tilde{A}'P\tilde{A} - P + \tilde{C}'W_{ZF}\tilde{C} & \tilde{A}'P\tilde{B} + \tilde{C}'W_{ZF}\tilde{D} \\ \star & \tilde{B}'P\tilde{B} + \tilde{D}'W_{ZF}\tilde{D} \end{bmatrix} < 0 \tag{25}$$

where

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{ccc|c} A_p & 0 & 0 & B_p \\ \alpha B_c C_p & A_c & 0 & B_c(\alpha D_p - 1) \\ 0 & 0 & A_a & B_a \\ \hline 0 & 0 & -\alpha C_a & 0 \\ \alpha H_0 C_p & -C_c & C_a & H_0(\alpha D_p - 1) \\ C_p & 0 & 0 & D_p \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{26}$$

then the system in Figure 1 is l_2 stable.

Proof: The proof is an application of the discrete-time KYP Lemma ([23]) to inequality (20), noting the state-space realisation of $L(z)$ (26). It is essentially the same as the continuous time case [27] and is thus omitted. □□

Although Proposition 1 contains a matrix inequality, it is bilinear and thus not tractable; in addition no detail about how to enforce the l_1 constraint has been given. These issues are addressed in the main result of the paper below.

Proposition 2
Consider Figure 1 and assume that it is well-posed, that $P(z) \in \mathcal{RH}_\infty$, and that $\phi(\cdot) \in \mathcal{N}^S_{[0,\alpha]}$. Define the matrix

$\mathcal{M}(A,B,C,D,X,Y,Z) :=$

$$\begin{bmatrix} -\mathbf{X_{11}} & \mathbf{X_{11}} & 0 & -\mathbf{C_c}' & -\mathbf{A_c}' & \mathbf{A_c}' & 0 \\ \star & -\mathbf{P_{11}}+Z & -\mathbf{N}-\alpha C'\mathbf{C_a} & \alpha C'\mathbf{H_0}+Y & -A'\mathbf{X_{11}}+\alpha C'\mathbf{C_c} & A'\mathbf{P_{11}}-\alpha C'\mathbf{C_c} & A'\mathbf{N} \\ \star & \star & -\mathbf{N} & -\mathbf{C_a}'(\alpha D-I_1) & 0 & -\mathbf{A_a}' & -\mathbf{A_a}' \\ \star & \star & \star & 2\mathbf{H_0}(\alpha D-I_1)+X & -B'\mathbf{X_{11}}+(\alpha D-I_1)\mathbf{C_c} & B'\mathbf{P_{11}}-(\alpha D-I_1)\mathbf{C_c}+\mathbf{C_a} & B'\mathbf{N}+\mathbf{C_a} \\ \star & \star & \star & \star & -\mathbf{X_{11}} & \mathbf{X_{11}} & 0 \\ \star & \star & \star & \star & \star & -\mathbf{P_{11}} & -\mathbf{N} \\ \star & \star & \star & \star & \star & \star & -\mathbf{N} \end{bmatrix} \tag{27}$$

Then the system in Figure 1 is l_2 stable if there exist positive definite matrices $\mathbf{P}_{11}$, $\mathbf{X}_{11}$, $\mathbf{N}$, negative definite matrices $\mathbf{A}_c$ and $\mathbf{A_a}$, matrices $\mathbf{C}_c$, $\mathbf{C_a}$, and a scalar $\mathbf{H_0} \geq 0$ such that the following inequalities hold:

$$\mathcal{M}(A_p,B_p,C_p,D_p,X,Y,Z) < 0 \tag{28}$$

$$\begin{bmatrix} \mathbf{H_0} & \mathbf{C_c} & \mathbf{C_a} & 0 & 0 \\ \star & \mathbf{X}_{11} & 0 & \mathbf{A_c} & 0 \\ \star & \star & \mathbf{N} & 0 & \mathbf{A_a} \\ \star & \star & \star & -\mathbf{A_c} & 0 \\ \star & \star & \star & \star & -\mathbf{A_a} \end{bmatrix} \geq 0 \tag{29}$$

where $X = 0$, $Y = 0$ and $Z = 0$.

Proof: Similar to the continuous-time case in [27], the proof can be partitioned for clarity. The first part of the proof is the derivation of the main LMI (27)-(28), which essentially translates the nonlinear matrix inequality in Proposition 1 into the LMI. The second part is the conversion of the requirement of that $M(z) \in \mathcal{M}_R$ into a LMI.

Part 1: Main LMI derivation. This part of the proof is essentially a combination of the procedure described in [27] and [6] and is thus deferred to the appendix.

Part 2: Ensuring $M(z) \in \mathcal{M}_R$. This part requires the proof that $\|M(z)\|_1 \leq 2H_0$ and that the impulse response of $M(z)$ is negative for $k \neq 0$. Recall that

$$M(z) = H_0 - H_c(z) - H_a^\sim(z)$$

where $H_c(z)$ and $H_a(z)$ are both strictly proper. *Temporarily* assume that $H_c(z)$ and $H_a(z)$ are externally positive, then this implies that the impulse response condition is negative for $k \neq 0$: this will be useful shortly. Next note that $\|M(z)\|_1 \leq 2H_0$ can be written as

$$\|H_0 - H_c(z) - H_a^\sim(z)\|_1 \leq 2H_0 \tag{30}$$

which holds if

$$H_0 + \|H_c(z) + H_a^\sim(z)\|_1 \leq 2H_0 \tag{31}$$

$$\Leftrightarrow \|H_c(z) + H_a^\sim(z)\|_1 \leq H_0 \tag{32}$$

Using the fact that $H_c(z)$ and $H_a(z)$ are externally positive, by Lemma 3 we thus have that the above inequality holds if

$$\|\bar{H}(z)\|_1 = \|H_c(z) + H_a(z)\|_1 \leq H_0 \tag{33}$$

Noting, $\bar{H}(z)$ has the state-space realisation

$$\bar{H}(z) \sim \left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_c & 0 & B_c \\ 0 & A_a & B_a \\ \hline C_c & C_a & 0 \end{array}\right]$$

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and that $\bar{H}(z)$ is externally negative, we then have, by Lemma 1, that

$$\|\bar{H}(z)\|_1 = C_H(I - A_H)^{-1}B_H$$

From the identities (93)-(98) in the appendix, it transpires (as noted in [7, 27]) that A_H, B_H and C_H can be expressed as

$$\left[\begin{array}{c|c} A_H & B_H \\ \hline C_H & 0 \end{array}\right] = \left[\begin{array}{c|c} -X\mathbf{A_H} & -X\mathbf{B_H} \\ \hline \mathbf{C_H} & 0 \end{array}\right] \tag{34}$$

where

$$\left[\begin{array}{c|c} \mathbf{A_H} & \mathbf{B_H} \\ \hline \mathbf{C_H} & 0 \end{array}\right] = \left[\begin{array}{cc|c} \mathbf{A_c} & 0 & \mathbf{B_c} \\ 0 & \mathbf{A_a} & \mathbf{B_a} \\ \hline \mathbf{C_c} & \mathbf{C_a} & 0 \end{array}\right] \tag{35}$$

and

$$X = \begin{bmatrix} \mathbf{P_{11}} - \mathbf{S_{11}} - \mathbf{N} & 0 \\ 0 & \mathbf{N} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{X_{11}} & 0 \\ 0 & \mathbf{N} \end{bmatrix}^{-1} \tag{36}$$

With this in mind, the l_1 bound (33) to be enforced becomes:

$$\|\bar{H}(z)\|_1 = -\mathbf{C_H}(I + X\mathbf{A_H})^{-1}X\mathbf{B_H} \leq H_0 \tag{37}$$

$$= -\mathbf{C_H}(X^{-1} + \mathbf{A_H})^{-1}\mathbf{B_H} \leq H_0 \tag{38}$$

Next, imposing symmetry on $\mathbf{A_H}$ and letting $\mathbf{B_H} = -\mathbf{C_H}'$, gives

$$\mathbf{C_H}(X^{-1} + \mathbf{A_H})^{-1}\mathbf{C_H}' \leq H_0 \tag{39}$$

$$\Leftrightarrow H_0 - \mathbf{C_H}(X^{-1} + \mathbf{A_H})^{-1}\mathbf{C_H}' \geq 0 \tag{40}$$

By the Schur complement,

$$\begin{bmatrix} H_0 & \mathbf{C_H} \\ \star & X^{-1} + \mathbf{A_H} \end{bmatrix} \geq 0 \tag{41}$$

which, after a further Schur complement and congruence transformation, becomes

$$\begin{bmatrix} H_0 & \mathbf{C_H} & 0 \\ \star & X^{-1} & \mathbf{A_H} \\ \star & \star & -\mathbf{A_H} \end{bmatrix} \geq 0 \tag{42}$$

Noting the definition of X in (36), this is then exactly inequality (29) in the proposition.

The temporary assumption that $\bar{H}(z)$ is externally positive is now removed. Note that a state-space realisation of $-\bar{H}(z)$ is, after appropriate similarity transformation and recalling that $\mathbf{B_H} = -\mathbf{C_H}$:

$$\bar{H}(z) \sim \left[\begin{array}{c|c} -X^{1/2}\mathbf{A_H}X^{1/2} & -X^{1/2}\mathbf{C_H}' \\ \hline -X^{1/2}\mathbf{C_H} & 0 \end{array}\right] \tag{43}$$

Lemma 2 reveals that $\bar{H}(z)$ will indeed be externally positive, if the matrix $-X^{1/2}\mathbf{A_H}X^{1/2}$ is positive semi-definite, which is then equivalent to $\mathbf{A_H}$ being negative semi-definite - but this is ensured by inequality (42): thus $\bar{H}(z)$, and therefore $H(z)$ is externally positive as required.

Remark 3: Apart from the differing algebra in the continuous time ([27]) and discrete-time case considered here, there are several significant differences which are relevant to the l_1 bound. Firstly, in the continuous time case, the $\mathcal{L}_1$ bound only involves the transformed multiplier state-space matrices $\mathbf{A_c}, \mathbf{C_c}$ and so on, whereas in the discrete-time case, the l_1 bound also features other decision variables due to the presence of the X^{-1} matrix in inequality (42). Also, in the continuous time case, external positivity of the multiplier is guaranteed by simply assuming $\mathbf{A_H}$ is symmetric;

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in the discrete time case, the condition requires negative definiteness of $\mathbf{A_H}$ (see final part of above proof and Lemma 2). □

Remark 4: A notable feature of the results, which are shared with the continuous time ones [27], is that the conditions in Proposition 2 are completely convex. No choice of pole location, order of multiplier or structure are required. Also, no line search is required due to the use of the tight l_1 bound associated with externally positive/negative systems. This contrasts with many other approaches where one either has to accept a line search or make certain choices about the multiplier in order to obtain computable results - see [6]. □

5. ENHANCING THE RESULTS

The results above contain one key source of conservatism: the state-space realisation of the dynamic portion of the *multiplier* $M(z)$ is stipulated to be symmetric. This may then lead Proposition 2 to predict stability only for smaller values of the upper bound of the slope, α, than is possible. Although it has been shown ([11]) that Zames-Falb multipliers actually incorporate various other classes of multiplier, because the search proposed in the foregoing sections is only performed over a subset of Zames-Falb multipliers, the addition of other multipliers in the search can be useful in practice. Moreover, with the IQC formulation, it is straightforward to combine different classes of multiplier.

5.1. Popov Multipliers

In the continuous times case [28], Popov multipliers were combined with Zames-Falb multipliers to provide less conservative results. Although strictly speaking the class of Popov multipliers are "phase contained" within the Zames-Falb multipliers, experience has shown ([28, 12]) it is sometimes useful to include them in the search algorithms to obtain better results.

Despite the results of [13] and [3], the Popov Criteria does not quite have an analogue in the discrete-time domain, the nearest being the Jury-Lee and Tsypkin Criteria - see discussion in [2]. However, the results of [3] can be expressed in IQC form and thus combined with the Zames-Falb multipliers to provide, possibly, a less conservative search algorithm. The computational cost of incorporating the Popov-type results is low since only two extra scalar decision variables are required in the LMI's. The discrete-time Popov multiplier is best developed in the time domain, along the lines of [14], and then it can be translated into the frequency domain to be combined with the Zames-Falb condition. The first lemma needed is essentially obtained by reverse-engineering results from [13] and [3].

Lemma 4
Assume the nonlinearity $\phi(\cdot) \in \mathcal{N}^S_{[0,\alpha]}$, then the following inequalities hold for all $\eta \geq 0$ and all $v(k+1)$ and $v(k)$:

$$(i)\; M_1 = 2\eta\phi(v(k))\Big(v(k+1) - v(k)\Big) + \eta\alpha\Big(v(k+1) - v(k)\Big)^2 \geq 0 \tag{44}$$

$$(ii)\; M_2 = -2\eta\phi(v(k))\Big(v(k+1) - v(k)\Big) + \eta\alpha\Big(v(k+1)^2 - v(k)^2\Big) \geq 0 \tag{45}$$

Proof: First note that $\phi \in \mathcal{N}^S_{[0,\alpha]}$ implies that $\partial\phi \in [0, \alpha]$ and hence inequality (5) holds. Furthermore, $\partial\phi \in [0, \alpha]$ implies that $\phi \in \mathrm{Sector}[0, \alpha]$ and hence inequality (6) holds too.

i) To prove the first inequality note that

$$M_1 = 2\eta\phi(v(k))\Big(v(k+1) - v(k)\Big) + 2\alpha\eta \int_{v(k)}^{v(k+1)} (\sigma - v(k))d\sigma \tag{46}$$

The slope restriction (5) then implies that $\phi(\sigma) - \phi(v(k)) \leq \alpha(\sigma - v(k))$ so

$$M_1 \geq 2\eta\phi(v(k))\Big(v(k+1) - v(k)\Big) + 2\eta \int_{v(k)}^{v(k+1)} \Big(\phi(\sigma) - \phi(v(k)\Big) d\sigma \tag{47}$$

$$= 2\eta \int_{v(k)}^{v(k+1)} \phi(\sigma) d\sigma \geq 0 \tag{48}$$

where the final inequality follows from the sector bound (6).

ii) To prove the second inequality note

$$M_2 = -2\eta\phi(v(k))\Big(v(k+1) - v(k)\Big) + 2\alpha\eta \int_{v(k)}^{v(k+1)} \sigma d\sigma \tag{49}$$

Next, as noted in [3], from monotonicity bestowed by the slope restriction (5) we have

$$\int_{v(k)}^{v(k+1)} \phi(\sigma) d\sigma \geq \int_{v(k)}^{v(k+1)} \phi(v(k)) d\sigma \tag{50}$$

$$= \phi(v(k))\Big(v(k+1) - v(k)\Big) \tag{51}$$

Using this in (49)

$$M_2 \geq -2\eta \int_{v(k)}^{v(k+1)} \phi(\sigma) d\sigma + 2\alpha\eta \int_{v(k)}^{v(k+1)} \sigma d\sigma \tag{52}$$

$$= 2\eta \int_{v(k)}^{v(k+1)} (\alpha\sigma - \phi(\sigma))\, d\sigma \geq 0 \tag{53}$$

where the final inequality follows from the sector condition (6).

From the above lemma, it is easy to see that the following (hard) time domain IQC holds (see [14])

$$\sum_{k=0}^{\infty} \begin{bmatrix} v(k) \\ \phi(v(k)) \end{bmatrix}' \tilde{\Pi}_P(z) \begin{bmatrix} v(k) \\ \phi(v(k)) \end{bmatrix} \geq 0 \quad \forall v(k) \in l_{2e} \tag{54}$$

where, with some abuse of notation,

$$\tilde{\Pi}_P(z) = \begin{bmatrix} \beta\eta_1(z-1)(z-1) + \beta\eta_2(z^2-1) & (\eta_1 - \eta_2)(z-1) \\ \star & 0 \end{bmatrix} \quad \eta_1, \eta_2 \geq 0 \tag{55}$$

On the unit Circle $|z| = 1$, the following frequency domain IQC then holds

$$\int_0^{2\pi} \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \widehat{\phi(v)(e^{j\omega})} \end{bmatrix}^* \Pi_P(e^{j\omega}) \begin{bmatrix} \hat{v}(e^{j\omega}) \\ \widehat{\phi(v)(e^{j\omega})} \end{bmatrix} d\omega \geq 0 \tag{56}$$

where $\Pi_P(z)$ can be written as

$$\Pi_P(z) = \Psi_P(z)^{\sim} W_P \Psi_P(z)$$

and

$$\Psi_P(z) = \begin{bmatrix} (z-1) & 0 \\ 0 & I_1 \end{bmatrix} \quad W_P = \begin{bmatrix} \alpha\eta_1 & \eta_1 - \eta_2 \\ \star & 0 \end{bmatrix} \tag{57}$$

This IQC can then be combined with the Zames-Falb IQC in order to get the following result:

Proposition 3
Consider Figure 1 and assume that it is well-posed, that $P(z) \in \mathcal{RH}_\infty$ and strictly proper, and that $\phi(\cdot) \in \mathcal{N}^S_{[0,\alpha]}$. Then the interconnection is l_2 stable if there exist positive definite matrices $\mathbf{P}_{11}$, $\mathbf{X}_{11}$, $\mathbf{N}$, negative definite matrices $\mathbf{A}_c$ and $\mathbf{A_a}$, matrices $\mathbf{C}_c$, $\mathbf{C_a}$, and scalars $\mathbf{H_0} \geq 0$, $\eta_1 \geq 0$, $\eta_2 \geq 0$ such that the following inequality

$$\mathcal{M}(A_p, B_p, C_p, D_p, X, Y, Z) < 0 \tag{58}$$

where

$$X = \alpha\eta_1(A_p - I)'C_p'C_p(A_p - I) \tag{59}$$

$$Y = \alpha\eta_1(A_p - I)'C_p'C_pB_p + (\eta_1 - \eta_2)(A_p - I)'C_p' \tag{60}$$

$$Z = \alpha\eta_1 C_pB_pB_p'C_p' + (\eta_1 - \eta_2)(C_pB_p + B_p'C_p) \tag{61}$$

and inequality (29) hold.

Proof: The proof is similar to that of Proposition 2 and will only be briefly sketched. The main differences are that the augmented IQC multiplier is given by

$$\Pi(z) = \Pi_{ZF}(z) + \Pi_P(z)$$

which, using the partitions (17) and (57), can be written as

$$\Pi(z) = \begin{bmatrix} \Psi_{ZF}(z) \\ \Psi_P(z) \end{bmatrix}^\sim \begin{bmatrix} W_{ZF} & 0 \\ 0 & W_p \end{bmatrix} \begin{bmatrix} \Psi_{ZF}(z) \\ \Psi_P(z) \end{bmatrix}$$

Stability then follows by evaluating Theorem 1 with the expression of $\Pi(z)$ given above instead of $\Pi_{ZF}(z)$ alone. The procedure and details are almost identical to those used in the proof of Proposition 2. □□

Remark 5: Inequality (ii) of Lemma 4 actually defines a so-called soft IQC (see [14]) for all $v \in l_2$. To see this note that summing inequality (ii) from $k = 0$ to $k = \infty$ gives:

$$\sum_{k=0}^{\infty} 2\eta\phi(v(k))\big(v(k+1) - v(k)\big) + \eta\alpha\big(v(k+1)^2 - v(k)^2\big)$$

$$= \sum_{k=0}^{\infty} 2\eta\phi(v(k))\big(v(k+1) - v(k)\big) + \eta\alpha\left\{\big(v(1)^2 - v(0)^2\big) + \big(v(2)^2 - v(1)^2\big) + \ldots\right.$$

$$= \sum_{k=0}^{\infty} 2\eta\phi(v(k))\big(v(k+1) - v(k)\big) - \eta\alpha\left\{v(0)^2 + v(\infty)^2\right\}$$

where $\lim_{k\to\infty} v(k) = 0$ since $v \in l_2$. Therefore we have the soft time domain IQC:

$$\sum_{k=0}^{\infty} 2\eta\phi(v(k))\Big(v(k+1) - v(k)\Big) \geq \alpha\eta v(0)^2 \geq 0 \tag{62}$$

Note the term involving α is not present - as it is not when evaluating the frequency domain version on the unit Circle. □

5.2. *Yakubovic Multipliers*

As noted in [11], Yakubovic multipliers have a straightforward form in discrete-time and it is easy to prove that a nonlinearity $\phi \in \mathcal{N}^S_{[0,\alpha]}$ satisfies the following inequality, for all $\tau \geq 0$.

$$2\tau\Big(\alpha(v(k+1) - v(k)) - (\phi(v(k+1)) - \phi(v(k)))\Big)\Big(\phi(v(k+1)) - \phi(v(k))\Big) \geq 0 \tag{63}$$

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In fact, it is convenient to write this in recursive form, viz

$$2\tau\Big(\alpha(v(k)-v(k-1))-(\phi(v(k))-\phi(v(k-1)))\Big)\Big(\phi(v(k))-\phi(v(k-1))\Big)\geq 0 \tag{64}$$

which then leads to the time domain (hard) IQC:

$$\sum_{k=0}^{\infty}\begin{bmatrix} v(k) \\ \phi(v(k)) \end{bmatrix}' \Pi_Y(z)\begin{bmatrix} v(k) \\ \phi(v(k)) \end{bmatrix}\geq 0 \quad \forall v_k\in l_{2e} \tag{65}$$

where, similar to before, $\Pi_Y(z)$ can be written as

$$\Pi_Y(z)=\Psi_Y(z)^{\sim}W_Y\Psi_Y(z) \tag{66}$$

where

$$\Psi_Y(z)=\begin{bmatrix} (1-z^{-1}) & 0 \\ 0 & (1-z^{-1}) \end{bmatrix} \tag{67}$$

and

$$W_Y=\begin{bmatrix} 0 & \alpha\tau \\ \alpha\tau & -2\tau \end{bmatrix} \tag{68}$$

Inequality (65) defines a dynamic IQC and in general will require an extra two states to implement. With the above IQC in mind, the Popov IQC can also be written in recursive form with $\Psi_P(z)$ instead defined as

$$\Psi_P(z)=\begin{bmatrix} (1-z^{-1}) & 0 \\ 0 & z^{-1} \end{bmatrix} \tag{69}$$

The nonlinearity $\phi\in\mathcal{N}_{[0,\alpha]}$ then satisfies the IQC defined by the multiplier $\Pi(z)=\Pi_{ZF}(z)+\Pi_P(z)+\Pi_Y(z)$ which can be written as

$$\Pi(z)=\begin{bmatrix} \Psi_{ZF}(z) \\ \Psi_P(z) \\ \Psi_Y(z) \end{bmatrix}^{\sim}\begin{bmatrix} W_{ZF} & 0 & 0 \\ 0 & W_P & 0 \\ 0 & 0 & W_Y \end{bmatrix}\begin{bmatrix} \Psi_{ZF}(z) \\ \Psi_P(z) \\ \Psi_Y(z) \end{bmatrix} \tag{70}$$

It is useful to note that a state-space realisation of

$$\begin{bmatrix} P(z) \\ (1-z^{-1})P(z) \\ (1-z^{-1})I \\ z^{-1}I \end{bmatrix} \tag{71}$$

is given by

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C}_1 & D_p \\ \bar{C}_2 & D_p \\ \bar{C}_3 & I \\ \bar{C}_4 & 0 \end{array}\right]=\left[\begin{array}{ccc|c} A_p & 0 & 0 & B_p \\ C_p & 0 & 0 & D_p \\ 0 & 0 & 0 & I \\ \hline C_p & 0 & 0 & D_p \\ C_p & -I & 0 & D_p \\ 0 & 0 & -I & I \\ 0 & 0 & I & 0 \end{array}\right] \tag{72}$$

With this in mind a similar result to Proposition 2 can be stated, combining Zames-Falb, Popov and Yakubovic multipliers.

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Proposition 4
Consider Figure 1 and assumed that it is well-posed, that $P(z) \in \mathcal{RH}_\infty$, and that $\phi(\cdot) \in \mathcal{N}^S_{[0,\alpha]}$. Then the feedback interconnection is l_2 stable if there exist positive definite matrices $\mathbf{P}_{11}$, $\mathbf{X}_{11}$, $\mathbf{N}$, negative definite matrices $\mathbf{A}_c$ and $\mathbf{A_a}$, matrices $\mathbf{C}_c$, $\mathbf{C_a}$, and scalars $\mathbf{H_0} \geq 0$, $\eta_1 \geq 0$, $\eta_2 \geq 0$ and $\tau > 0$ such that the following inequality

$$\mathcal{M}(\bar{A}, \bar{B}, \bar{C}_1, D_p, X, Y, Z) < 0 \tag{73}$$

where

$$X = \alpha\eta_1 \bar{C}_2'\bar{C}_2 + (\eta_1 - \eta_2)(\bar{C}_2'\bar{C}_4 + \bar{C}_4'\bar{C}_2) + \alpha\tau(\bar{C}_2'\bar{C}_3 + \bar{C}_3'\bar{C}_2) - 2\tau\bar{C}_3'\bar{C}_3 \tag{74}$$

$$Y = \alpha\eta_1 \bar{C}_2' D_p + (\eta_1 - \eta_2)\bar{C}_4' D_p + \alpha\tau\bar{C}_2' + \tau\bar{C}_3'(\alpha D_p - 2) \tag{75}$$

$$Z = \alpha\eta_1 D_p' D_p \tag{76}$$

and inequality (29) hold.

Proof: The proof follows in a similar manner, again, to Proposition 2, except that the multiplier given in equation (70) is used, and the state-space realisation (72) is exploited. □□

Remark 6: The extra decision variable τ, associated with the Yakubovic multiplier will increase the computational burden of solving the LMI's associated with Proposition 4. However, a further increase in computational burden results from the two extra states required to implement the Yakubovic multiplier, which are captured by the state-space realisation (72): implying the Zames-Falb multiplier will be of order $n_p + 4$, since the causal and anti-causal multipliers each have an extra 2 states. □

6. NUMERICAL RESULTS

The main advantages of the results herein are summarised in Remark 2: Propositions 2-4 are completely convex algorithms, with the only input data required in order to compute the maximum slope size α being the state-space data of the plant $P(z)$. It is interesting to compare Propositions 2-4 to other schemes available in the literature. The numerical examples used below are exactly those used in [6]: some of these have been re-used from elsewhere in the literature, but some are new and there is a good spectrum of plants, albeit of fairly low order. Here, comparisons are made with the "best" algorithm in [6], which is an LMI-based scheme derived using FIR representations of Zames-Falb multipliers. This is arguably the "state-of-the-art" search for discrete-time slope-restricted systems. Comparisons are also made against the recent results of [20] which appears to be the best Lyapunov-based approach and has the advantage of rapid computation. The Nyquist value is also shown, giving an idea of potential conservatism, although recently more accurate results have become available in [25]. The numerical results were obtained using a 2.6 Ghz Intel i5-3320M laptop equipped with a hard-disk, 4 GB RAM and running Linux Mint 17.1 and Matlab R2015a - not state-of-the-art hardware. The LMI's were implemented using the algorithms in the Matlab Robust Control Toolbox.

The results of the comparison are summarised in Table I. Several features are apparent:

- The results reports in [6] are the best (least conservative) available.
- All three algorithms developed in this paper provide the same slope bounds, but computational times are dramatically slower when Proposition 4 is used. One would expect Propositions 3 and 4 to provide less conservative results if the sector condition is tighter than the slope condition, but this is not the case here.
- The results developed in this paper provide similar slope estimates to those in [20], except in two cases: Example 1, where [20] is clearly superior; and Example 4 where Propositions 2-4 are clearly superior.

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It is interesting to compare the continuous and discrete-time Zames-Falb searches, presented in [27] and here, respectively. Both searches use the concepts of external positivity, and in particular symmetry of the multiplier state-space realisation. In the continuous time case, the poles of the Zames-Falb multiplier are all real, but may be both positive and negative, leading to the multiplier transfer function being of the form

$$M(s) = H_0 - \sum_{i=1}^{2n_p} \frac{k_i}{s+\rho_i} \quad \rho_i \in \mathbb{R}$$

This effectively results from the A-matrix of the multiplier being symmetric and means that all poles have unity damping. In the discrete-time case, the A-matrix of the multiplier (see equation (43)) is symmetric and positive definite, meaning that the multiplier has the form

$$M(z) = H_0 - \sum_{i=1}^{2n_p} \frac{k_i}{z+\rho_i} \quad \rho_i \in \mathbb{R}_{\geq 0}$$

Again, this means that all poles have unity damping. Thus although, in a state-space setting, the discrete-time results may appear to be more restrictive, in fact the continuous and discrete-time results have a similar nature.

Remark: 7 The results here have been developed on the basis of the multiplier having a *symmetric* state-space realisation in order to guarantee external negativity. This symmetry condition is accompanied by conservatism as it is clearly not a necessary condition for external negativity. Therefore, if one could search over a wider class of externally negative multipliers, improved results would be expected. A note of caution is in order however: the advantage of the symmetric multiplier formulation is that it easily allows an l_1 bound for the multiplier to be expressed as an LMI and is "robust" to the various similarity transformations required (see Part 2 in the proof of Prop 2). It is possible that the results of [26] could be exploited, but this requires more work.

Example	Maximum slope size						Nyquist
	Prop. 2	Prop. 3	Prop. 4	Best of [6] (odd)	Best of [6] (non-odd)	Park	value
1	12.431 (3.25)	12.431 (4.75)	12.431 (74.95)	13.5113 (FIR)	13.028	12.996	36.1
2	0.7262 (3.35)	0.7262 (4.77)	0.7262 (47.52)	1.1056 (FIR)	0.8027	0.7397	2.7455
3	0.3027 (2.19)	0.3027 (3.02)	0.3027 (53.71)	0.3121 (FIR)	0.3120	0.3054	0.31259
4	2.6827 (2.91)	2.6827 (3.96)	2.6827 (109.43)	3.8240 (FIR)	3.8240	2.5904	7.907
5	2.4475 (0.12)	2.4475 (0.14)	2.4475 (0.46)	2.4475 (various)	2.4475	2.4475	2.4475
6	0.9067 (0.47)	0.9067 (0.71)	0.9067 (6.42)	1.0870 (FIR)	0.9115	0.9108	1.087
7	0.1695 (3.74)	0.1695 (5.08)	0.1695 (26.71)	0.5280 (FIR)	0.4922	0.1695	1.1766

Table I. Comparison of Props. 2-4 and the best results of [6]. The bracketed figure in the second column shows the computational time (in seconds) required for the algorithms in Propositions 2-4 to run, using the hardware described above.

7. CONCLUSIONS

A stability analysis for discrete-time Lurie systems has been presented. The results rely on some concepts from positive systems theory and, more specifically, systems with symmetric state-space realisations. These concepts allow the Zames-Falb search in discrete-time to be cast as a purely convex optimisation problem, with few choices required from the user. The main results of the paper (Propositions 2 - 4) do not necessarily provide the least conservative results in all cases, yet they are competitive with the state-of-the-art, are computationally efficient, and do not require any user intervention.

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A. DERIVATION OF INEQUALITY (27)

This appendix sketches the derivation of inequality (27) in Proposition 1. The algebra is straightforward but a little intricate, and essentially involves converting the (nonlinear) matrix inequality in Proposition 1 into the *linear* matrix inequality (27). First note that inequality (25) in Proposition (1) is equivalent to the inequality

$$\begin{bmatrix} -P+\tilde{C}'W_{ZF}\tilde{C} & \tilde{C}'W_{ZF}\tilde{D} & \tilde{A}' \\ \star & \tilde{D}'W_{ZF}\tilde{D} & \tilde{B}' \\ \star & \star & -P^{-1} \end{bmatrix} < 0 \qquad (77)$$

Next, following [27], P is assigned the following structure.

$$P = \begin{bmatrix} \mathbf{P_{11}} & P_{12} & P_{13} \\ \star & P_{22} & 0 \\ \star & \star & P_{33} \end{bmatrix} \qquad (78)$$

Note the structure of P potentially introduces conservatism, but allows linear matrix inequalities to be obtained. *Temporarily assume that* $P > 0$ and define $Q := P^{-1}$ and note that

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q'_{12} & Q_{22} & Q_{23} \\ Q'_{13} & Q'_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} \mathbf{P_{11}} & P_{12} & P_{13} \\ P'_{12} & P_{22} & 0 \\ P'_{13} & 0 & P_{33} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (79)$$

where, P_{12}, P_{13}, Q_{12} and Q_{13} are full rank matrices. Shortly, it will be verified that the inequalities in the theorem do indeed allow the assumption that P is nonsingular to be made. Next, consider the (full rank) matrices:

$$\Pi_1 := \begin{bmatrix} Q_{11} & I & 0 \\ Q'_{12} & 0 & 0 \\ Q'_{13} & 0 & I \end{bmatrix} \quad \Pi_2 := \begin{bmatrix} I & 0 & 0 \\ \mathbf{P_{11}} & P_{12} & P_{13} \\ P'_{13} & 0 & P_{33} \end{bmatrix} \qquad (80)$$

Note that the following identities arise: $\Pi'_1 P = \Pi_2$, $\Pi'_1 = Q\Pi_2$, $\Pi_2 Q \Pi'_2 = \Pi'_1\Pi'_2$ and $\Pi'_1 P \Pi_1 = \Pi_2\Pi_1$. Note also that

$$\Pi'_1 P \Pi_1 = \Pi_2 Q \Pi'_2 = \begin{bmatrix} Q_{11} & I & 0 \\ I & P_{11} & P_{13} \\ 0 & P'_{13} & P_{33} \end{bmatrix} \qquad (81)$$

Using the congruence transformation

$$\begin{bmatrix} \Pi'_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \Pi_2 \end{bmatrix} \qquad (82)$$

on inequality (77) yields

$$M_{KYP} = \begin{bmatrix} -\Pi'_1 P\Pi_1 + \Pi'_1\tilde{C}'W_{ZF}\tilde{C}\Pi_1 & \Pi'_1\tilde{C}'W_{ZF}\tilde{D} & \Pi'_1\tilde{A}'\Pi'_2 \\ \star & \tilde{D}'W_{ZF}\tilde{D} & \tilde{B}'\Pi'_2 \\ \star & \star & -\Pi_2 P^{-1}\Pi'_2 \end{bmatrix} < 0 \qquad (83)$$

where after some algebra, the individual terms can be calculated as

$$\Pi_1'\tilde{C}'W_{ZF}\tilde{C}\Pi_1 = \begin{bmatrix} -\alpha(Q_{11}C_p'C_aQ_{13}' + Q_{13}C_a'C_pQ_{11}) & -\alpha Q_{13}C_a'C_p & -\alpha Q_{11}C_p'C_a \\ -\alpha C_p'C_aQ_{13}' & 0 & -\alpha C_p'C_a \\ -\alpha C_a'C_pQ_{11} & -\alpha C_a'C_p & 0 \end{bmatrix} \quad (84)$$

$$\Pi_1'\tilde{C}'W_{ZF}\tilde{D} = \begin{bmatrix} \alpha Q_{11}C_p'H_0 - Q_{12}C_c' - Q_{13}C_a'(\alpha D_p - I) \\ \alpha C_p'H_0 \\ -C_a'(\alpha D_p - I) \end{bmatrix} \quad (85)$$

$$\Pi_1'\tilde{A}'\Pi_2' = \begin{bmatrix} Q_{11}A_p' & Q_{11}A_p'\mathbf{P_{11}}+\alpha Q_{11}C_p'B_c'P_{12}' + Q_{12}A_c'P_{12}' + Q_{13}A_a'P_{13}' & Q_{11}A_p'P_{13} + Q_{13}A_a'P_{33} \\ A_p' & A_p'\mathbf{P_{11}}+\alpha C_p'B_c'P_{12}' & A_p'P_{13} \\ 0 & A_a'P_{13}' & A_a'P_{33} \end{bmatrix} \quad (86)$$

$$\tilde{B}'\Pi_2' = \begin{bmatrix} B_p' & B_p'\mathbf{P_{11}} + (\alpha D_p - I)'B_c'P_{12}' + B_a'P_{13}' & B_p'P_{13} + B_a'P_{33} \end{bmatrix} \quad (87)$$

Note that the inequality is still nonlinear. Therefore a series of successive congruence transformations are applied to inequality (83) in the following manner:

$$M_T = T_3T_2T_1M_{KYP}T_1'T_2'T_3' < 0 \quad (88)$$

where

$$T_1 = \text{blockdiag}(\mathbf{S_{11}}, I_n, I_n, I_1, \mathbf{S_{11}}, I_n, I_n) \quad (89)$$
$$T_2 = \text{blockdiag}(I_n, I_n, P_{33}^{-1}, I_1, I_n, I_n, P_{33}^{-1}) \quad (90)$$
$$T_3 = \text{blockdiag}(I_n, I_n, P_{13}, I_1, I_n, I_n, P_{13}) \quad (91)$$

and $\mathbf{S_{11}} = Q_{11}^{-1}$. Noting that

$$Q_{13} = -Q_{11}P_{13}P_{33}^{-1} \quad \Leftrightarrow \mathbf{S_{11}}Q_{13} = -P_{13}P_{33}^{-1} \quad (92)$$

and making substitutions based on the following identities (inspired by [24] and used in [27]):

$$\begin{aligned} \mathbf{A_c} &:= P_{12}A_cQ_{12}'\mathbf{S_{11}} && (93) \\ \mathbf{B_c} &:= P_{12}B_c && (94) \\ \mathbf{C_c} &:= C_cQ_{12}'\mathbf{S_{11}} && (95) \\ \mathbf{A_a} &:= -P_{13}A_aP_{33}^{-1}P_{13}' && (96) \\ \mathbf{B_a} &:= -P_{13}B_a && (97) \\ \mathbf{C_a} &:= C_aP_{33}^{-1}P_{13}' && (98) \\ \mathbf{N} &:= P_{13}P_{33}^{-1}P_{13}' && (99) \end{aligned}$$

then allows inequality (88) to be written as

$$\begin{bmatrix} -S_{11}+\alpha(C_p'\mathbf{C_a}+\mathbf{C_a}'C_p) & -S_{11}+\alpha\mathbf{C_a}'C_p & -\alpha C_p'\mathbf{C_a} & \alpha C_p'H_0-\mathbf{C_c}'+\mathbf{C_a}'(\alpha D_p-I) & A_p'S_{11} & A_p'P_{11}+\alpha C_p'\mathbf{B_c}'+\mathbf{A_c}'+\mathbf{A_a}' & A_p'\mathbf{N}+\mathbf{A_a}' \\ \star & -P_{11} & -\mathbf{N}-\alpha C_p'\mathbf{C_a} & \alpha C_p'H_0 & A_p'S_{11} & A_p'P_{11}+\alpha C_p'\mathbf{B_c}' & A_p'\mathbf{N} \\ \star & \star & -\mathbf{N} & -\mathbf{C_a}'(\alpha D_p-I) & 0 & -\mathbf{A_a}' & -\mathbf{A_a}' \\ \star & \star & \star & (\alpha D_p-I)'H_0'+H_0(\alpha D_p-I) & B_p'S_{11} & B_p'P_{11}+(\alpha D_p-I)'\mathbf{B_c}'-\mathbf{B_a}' & B_p'\mathbf{N}-\mathbf{B_a}' \\ \star & \star & \star & \star & -S_{11} & -S_{11} & 0 \\ \star & \star & \star & \star & \star & -P_{11} & -\mathbf{N} \\ \star & \star & \star & \star & \star & \star & -\mathbf{N} \end{bmatrix} < 0 \quad (100)$$

which is an LMI in the decision variables. However, the result can be expressed in a tidier form with another use of a congruence transformation. Indeed, pre-multiplying the left-hand side of inequality

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by the matrix

$$T_4 = \begin{bmatrix} I & -I & I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & -I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and post-multiplying by T_4', gives, after some algebra, the LMI

$$\begin{bmatrix} -\mathbf{X_{11}} & \mathbf{X_{11}} & 0 & -\mathbf{C_c}' & -\mathbf{A_c}' & \mathbf{A_c}' & 0 \\ \star & -\mathbf{P_{11}} & -\mathbf{N}-\alpha\mathbf{C_a}'C_p & \alpha C_p'\mathbf{H_0} & -A_p'\mathbf{X_{11}}-\alpha C_p'\mathbf{B_c}' & A_p'\mathbf{P_{11}}+\alpha C_p'\mathbf{B_c}' & A_p'\mathbf{N} \\ \star & \star & -\mathbf{N} & -\mathbf{C_a}'(\alpha D_p - I) & 0 & -\mathbf{A_a}' & -\mathbf{A_a}' \\ \star & \star & \star & (\alpha D_p-I)'\mathbf{H_0}'+\mathbf{H_0}(\alpha D_p-I) & -B_p'\mathbf{X_{11}}-(\alpha D_p-I)\mathbf{B_c}' & B_p'\mathbf{P_{11}}+(\alpha D_p-I)\mathbf{B_c}'-\mathbf{B_a}' & B_p'\mathbf{N}-\mathbf{B_a}' \\ \star & \star & \star & \star & -\mathbf{X_{11}} & \mathbf{X_{11}} & 0 \\ \star & \star & \star & \star & \star & -\mathbf{P_{11}} & -\mathbf{N} \\ \star & \star & \star & \star & \star & \star & -\mathbf{N} \end{bmatrix} < 0 \quad (101)$$

where $\mathbf{X_{11}} := \mathbf{P_{11}} - \mathbf{S_{11}} - \mathbf{N}$. Imposing the symmetry conditions blue$\mathbf{B_c} = -\mathbf{C_c'}$, $\mathbf{B_a} = -\mathbf{C_a}'$, $\mathbf{A_c} = \mathbf{A_c}'$ and $\mathbf{A_a} = \mathbf{A_a}'$, this then becomes (27). The reason for the symmetry stipulations are clear in Part 2 of the proof in the main body of the paper.

The final stage of this part of the proof is to remove the temporary assumption that the matrix $P > 0$: note that $P > 0$ is equivalent to $\Pi_1' P \Pi_1 > 0$. Applying a Schur complement to equation (81), one finds that this is equivalent to the matrix $X > 0$; this is then guaranteed by inequality (42) and, equivalently, inequality (29) in the proposition.