University of Southampton

Faculty of Engineering and Physical Sciences
School of Physics and Astronomy

Quantum Gravity and the Renormalization Group

by

Matthew Peter Kellett

ORCiD: 0000-0001-9573-4127

A thesis for the degree of
Doctor of Philosophy

December 2020

University of Southampton

Abstract

Faculty of Engineering and Physical Sciences
School of Physics and Astronomy

Doctor of Philosophy

Quantum Gravity and the Renormalization Group

by Matthew Peter Kellett

This work seeks to address the problem of quantum gravity from the point of view of the renormalisation group. After the introduction of the needed concepts, it is seen that a problem of quantum gravity that stems from the "conformal factor instability", if the consequences are fully explored, can open the door to a rich phenomenology. In particular, it is seen that toric universes can be constrained to be highly symmetric when sufficiently small, which is potentially applicable to the initial conditions for inflation. The idea of regularising gravity via a supermanifold is covered, following similar treatments of gauge theory in order to preserve the symmetry (diffeomorphism invariance) at all points along the renormalization group flow. Furthermore, the machinery provided by the conformal factor instability provides us with a genuinely perturbative theory of quantum gravity, which can be calculated to have the same effective action as one might expect, but the understanding of the QFT is very different, with the Gaussian fixed point being defined "off space-time".

Contents

List of Figures

List of Tables

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:
    - Matthew P. Kellett and Tim R. Morris. Renormalization group properties of the conformal mode of a torus. *Class. Quant. Grav.*, 35(17):175002, 2018,
    - Matthew Kellett and Tim R. Morris. Parisi-Sourlas supergravity. *Class. Quant. Grav.*, 37(19):195018, 2020,
    - Matthew Kellett, Alex Mitchell, and Tim R. Morris. The continuum limit of quantum gravity at second order in perturbation theory. arXiv:2006.16682 2020

Signed:.. Date:..................

Acknowledgements

Firstly, I would like to thank my supervisor, Tim Morris for his assistance and advice both in the preparation of this thesis and more generally across the past 4 years.

This thesis has been written in the midst of the CoViD-19 pandemic, and as such I would like to thank my parents partially for their support, and also for allowing me to move back in during the preparation of this thesis.

Beyond this, there are many people who have helped me in various ways during the last 4 years. Special mentions go for the fellow denizens of 4007, who listened as I attempted to "rubber duck" my way through things I didn't understand, as well as the Southampton University Snooker & Pool Club for giving me something to do when I'd had enough of work and needed a break (although one may argue that this potentially harmed my productivity). In addition, I would like to thank Southampton FolkSoc for both being another way to relax, and for not judging me for my terrible ukulele playing.

Because of the aforementioned pandemic, it has been more important than ever to stay in contact with as many people as possible. In this spirit I would also like to thank everybody who took the time to socialise via video call (as indeed this is the only way possibly now) which includes (but is not limited to) those in the above societies, as well as many friends from Preston and Cambridge who have made the pandemic significantly less unpleasant than it otherwise would have been.

In memory of my Granddad, who passed away during the production of this thesis. He will be missed by many.

Chapter 1

Introduction, Background and Motivation

The two parts of any undertaking in physics are the problem and the method. The problem that this thesis seeks to address is an old (and popular) one: can we find a consistent theory of quantum gravity? It is worth motivating this. One way to motivate looking for such a theory is a purely aesthetic one: we would like to unify the forces. The Standard Model has been incredibly successful in describing electromagnetism and the strong and weak nuclear forces [8], and it would indeed be nice if we could unify all of the forces at some high energy scale, which studies of electroweak theory [9–11] and other more modern developments [12–14] seem to indicate.

However, this raises a question: if the gravitational forces between elementary particles are so weak, why do we care about quantum gravity? After all, we only really observe gravity on large scales, and so we should not care about the quantum effects. Indeed, for the most part, classical General Relativity (GR) will serve extremely well. However, there are interesting cases where the gravitational fields are strong *and* the distances involved are short. These are black holes, and the proposed singularity at the beginning of the Universe. Perhaps the most famous recent example of evidence for the existence of black holes is the observations of the Event Horizon Telescope [15], but other evidence exists. In particular, the LIGO observations of gravitational waves are thought to be sourced by a black hole merger [16]. An understanding of quantum gravity may give in indication as to what the breakdown of the classical equations (the "singularities") is telling us. Now, perhaps one might say that these are only conjectured to be singular regions of spacetime, however theorems of Hawking and Penrose [17, 18] show that, rather generically, GR suffers from singularities, so at least some modification seems to be in order.

Another reason to attempt to quantize gravity comes from the simple observation that matter is quantum mechanical, and gravity couples to matter. Indeed, we have the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

where the LHS describes the curvature (i.e., gravity), and the RHS describes the energy and momentum of matter. Standard practice at this point is to average the RHS to get a classical quantity. While this works for many purposes such as gravitational waves, it is philosophically unsatisfactory since this introduces non-locality into the theory and thus it is not clear whether this is entirely consistent. This "semi-classical" theory is most often regarded as a good approximation in a "weak field" approximation. There are several arguments and thought experiments which suggest that the semi-classical theory cannot be consistent [19, 20], but these are not without some controversy [21, 22].

We expect any theory of quantum gravity to be similar to GR at large distances - the "infra-red" (IR) regime - and something very different at small distances. The idea that physics changes at differing scales is fundamental to the renormalization group, which will be our aforementioned method.

In this chapter, some basics of the renormalization group will be reviewed, and some background material for the remainder of this thesis will be introduced. In Section 1.1, the ideas and origins of RG are recapitulated, starting with the work of Kadanoff [23] and showing how these are generalised from the field of statistical mechanics to that of QFT.

Section 1.2 describes how the RG is used to construct a continuum QFT. This is done by linearising the flow around the Gaussian fixed point. The renormalized trajectory is defined to give a path towards an interacting continuum limit.

Following this, Section 1.3 gives an explicit example of calculating eigenoperators used in constructing a continuum limit. Although the example given here is a well-understood, some of the aspects highlighted in this treatment serve as an introduction to some details in Chapter 2.

Section 1.4 discusses the relationship between quantum gravity and the RG, and outlines some ongoing work which attempts to construct a fully renormalizable and phenomenologically consistent theory of gravity and finally Section 1.5 outlines the goals for the remainder of this thesis.

1.1 The renormalization group

The idea behind the renormalization group is rather simple: the underlying physics of an object does not change depending on what scale we observe them. As an example, a cup of tea is made of atoms. The laws governing atoms are the same no matter whether we deal with a single atom or the many moles required to make a cup of tea. However, using the atomic forces to describe a cup of tea would be confusing, difficult and practically impossible. Fortunately, we can use fluid dynamics to describe the cup of tea to great effect. However, the underlying physics has not changed. We know that fluid dymanics is an “effective” description; it ignores many of the short distance - or ultra-violet (UV) - effects in order to capture the macroscopic physics.

The idea of effective theories is prevalent in modern physics. One of the first was the Fermi theory of interacting fermions [24]. More examples include the Standard Model and many theorize that GR is also an effective theory due to its apparent non-renormalizability [25]. The goal of quantum gravity is to find a theory that includes quantum effects and reduces to GR as an effective theory.

Statistical mechanics has the idea of an “effective theory” at its heart - ignoring (or simplifying) the underlying microscopic physics in order to concentrate on the important macroscopic physics. The quantities we care about: temperature, entropy, heat capacity etc., all come from dealing with the macroscopic properties of systems. Studies of many different systems show exactly the same “critical exponents” (numbers which govern the behaviour of various quantities near phase transitions) in different situations. This again points to the notion that the exact details of the microscopic physics are irrelevant to the macroscopic behaviour. In this context, we call it *universality*. And it is in statistical mechanics that the renormalization group was developed. It is worth reviewing this since many of the concepts are easier to have intuition for here than in the field theory counterpart.

1.1.1 Kadanoff blocking

The basis of modern renormalization group methods is based on Kadanoff blocking in statistical mechanics, where we move from one scale to another [23]. For concreteness, we consider the Ising model. Consider a 2-dimensional lattice (for ease of illustration) of spins $\{s_i\}$ which take values $s_i = \pm 1$. Suppose each spin can only interact with adjacent spins (‘nearest-neighbour” interactions). Kadanoff blocking consists of “coarse graining” these spins. That is, we consider a reduced set of spins with some way of getting from old spins to new spins. One example is shown in Figure 1.1. This is a “real space renormalization group” transformation.

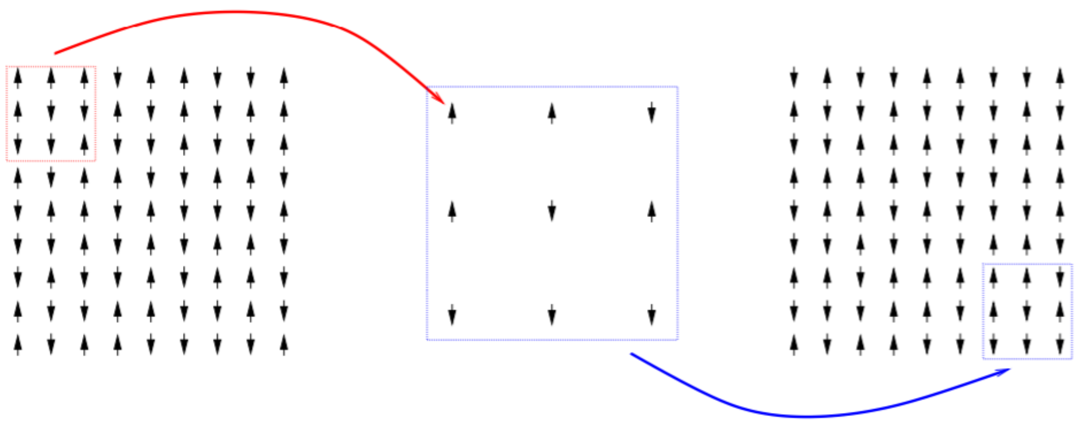

Figure 1.1: Visualisation of Kadanoff blocking. The original lattice of spins is broken into 3×3 blocks, each of which is replaced by a single "effective" spin which is simply whichever of up or down has the most representation in that block (red step). The length scale is then changed to allow comparison to the orginal problem (blue step). In this last step, the figure shows additional lattice sites from outside the original image. Figure taken from [1].

The two systems should have different parameters for the nearest-neighbour coupling. Furthermore, this coarse-graining also gives rise to additional interactions. More importantly, however, if we consider quantities that are calculated on distances much greater than the lattice spacing, then both ways of describing the system will give the same answer. Note that this points towards to universality concept once again: we have lost information about the short distance (UV) physics, but the long distance (IR) physics has not changed at all.

It is worth noting that although we don't explicitly state this, we expect the blocks used in this coarse-graining procedure are in some way sensible. There are many ways to draw them, however we expect them to be connected at the very least. This idea of locality (or quasi-locality) will be used in the field theory case.

1.1.2 Flow equations

When looking at field theory, it will prove to be much more expedient to consider momentum space renormalization, as opposed to the real space renormalization above. The concept is much the same if we translate "short distance" to "high energy" and "long distance" to "low energy". We will simply refer to these as the UV and IR regimes. From this point of view, the coarse-graining procedures above correspond to reductions in a momentum "cutoff scale". The crucial idea going forward is smoothly changing cutoff scales, rather than the blocking of Kadanoff. This is mostly due to Wilson [26]. More comprehensive reviews of what follows can be found, e.g. [1, 2].

First, we note that we should work in Euclidean space. There are a few reasons for this. One is due to the notion of locality of the blocking transformations - without this Wick rotation we could have "sites" blocked together with arbitrary spatial separation provided they were lightlike separated. Essentially it is easier to account for lightlike behaviour in general with this modification.

We will use the compact DeWitt notation, which treats functions as vectors and their arguments as indices. Thus we have, for example,

$$J \cdot \phi = J_x \phi_x = \int d^d x J(x)\phi(x) = \int_x J(x)\phi(x) \tag{1.2}$$

but also, we have

$$J \cdot \phi = \int \frac{d^d p}{(2\pi)^d} J(p)\phi(-p) = \int_p J(p)\phi(-p) \tag{1.3}$$

and so we do not have to specify whether we are working in real space or momentum space (note the factors of 2π absorbed into the definition in the final equality). Similarly, when dealing with propagators we treat these as matrices so that

$$\phi \cdot \Delta^{-1} \cdot \phi = \int_{x,y} \phi(x)\Delta^{-1}(x,y)\phi(y) = \int_p \phi(p)\Delta^{-1}(p^2)\phi(-p). \tag{1.4}$$

For simplicity, we will consider a single scalar field. The equivalent to a lattice spacing is a momentum cutoff, Λ_0. With this cutoff, the path integral is

$$Z[J] = \int^{\Lambda_0} \mathcal{D}\phi \, e^{-S^{tot}_{\Lambda_0}[\phi] + J\cdot\phi}. \tag{1.5}$$

We describe $S^{tot}_{\Lambda_0}$ as the "bare action". This bare action is usually chosen to be very simple. It turns out to be convenient to write

$$S^{tot}_{\Lambda_0}[\phi] = \frac{1}{2}\phi \cdot p^2 \cdot \phi + S_{\Lambda_0}[\phi] \tag{1.6}$$

so that S_{Λ_0} contains only the interactions (including mass terms).

Changing the cutoff, we expect the action to change (like the couplings change in the Kadanoff picture). But the physics is not expected to change. The Kadanoff blocking involves increasing distances (before the rescaling) and this corresponds to a lowering of the cutoff to a new scale, say Λ. Figure 1.2 schematically shows what we are doing. To this end, we introduce a UV cutoff function $C^\Lambda(p)$ and its associated IR cutoff function $C_\Lambda(p)$. These are related by

$$C^\Lambda(p) + C_\Lambda(p) = 1. \tag{1.7}$$

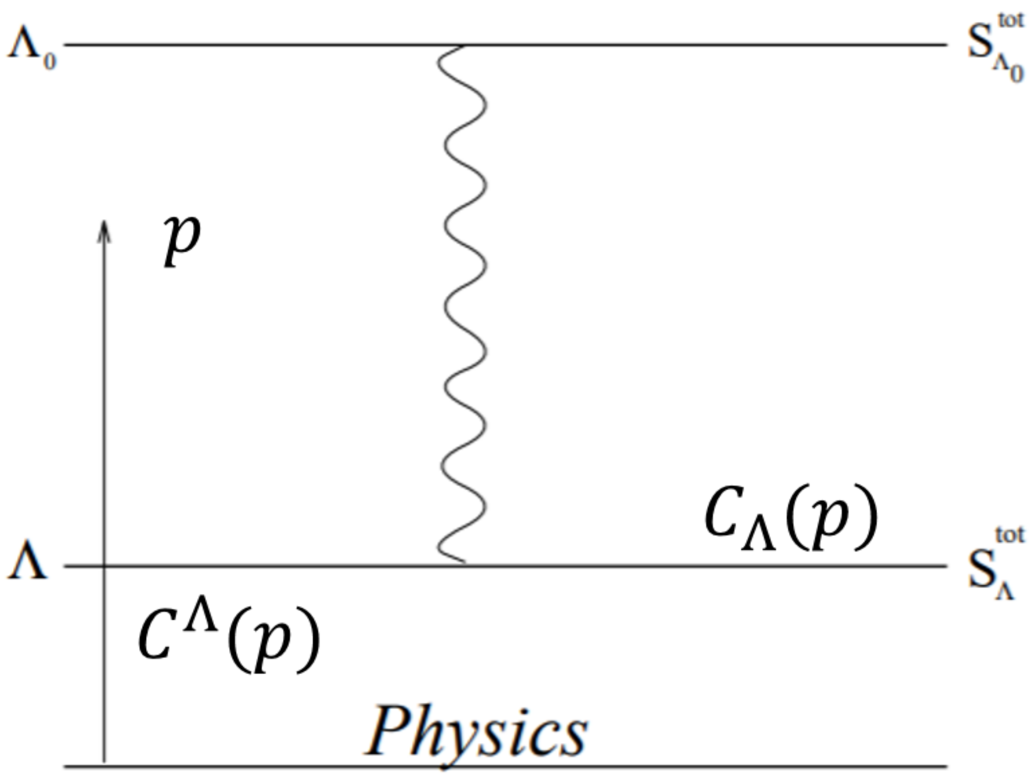

FIGURE 1.2: We see that we are integrating out the high momentum modes. Diagram adapted from [2].

For $C^\Lambda(p)$ to be a UV cutoff, we insist that

$$C^\Lambda(0) = 1, \qquad \lim_{\frac{p^2}{\Lambda^2}\to\infty} C^\Lambda(p) = 0. \tag{1.8}$$

For our analogy with Kadanoff blocking to hold, we require these functions to be smooth to preserve (quasi-)locality. We use these to modify the propagator by writing

$$\frac{1}{p^2} = \Delta^\Lambda(p) + \Delta_\Lambda(p) \tag{1.9}$$

where

$$\Delta^\Lambda(p) = \frac{C^\Lambda(p)}{p^2}, \qquad \Delta_\Lambda(p) = \frac{C_\Lambda(p)}{p^2}. \tag{1.10}$$

Finally, we write $\phi = \phi_< + \phi_>$ where $\phi_<$ is defined to have propagator Δ^Λ and $\phi_>$ is defined to have propagator Δ_Λ. Recall our generating functional is now (dropping the limit of integration)

$$Z[J] = \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\phi \cdot p^2 \cdot \phi - S_{\Lambda_0}[\phi] + J \cdot \phi\right) \tag{1.11}$$

which, after discarding a field independent constant of proportionality (more details in [2]) we can write as

$$Z[J] = \int \mathcal{D}\phi_< \mathcal{D}\phi_> \exp\Big(-\frac{1}{2}\phi_< \cdot (\Delta^\Lambda)^{-1} \cdot \phi_< - \frac{1}{2}\phi_> \cdot (\Delta_\Lambda)^{-1} \cdot \phi_> \\ - S_{\Lambda_0}[\phi_< + \phi_>] + J \cdot (\phi_< + \phi_>) \Big). \quad (1.12)$$

Restricting the support of J to "low energy" modes[1] (those with $|p| < \Lambda$) it can be shown that we can write

$$Z[J] = \int \mathcal{D}\phi_< \exp\left(-\frac{1}{2}\phi_< \cdot (\Delta^\Lambda)^{-1} \cdot \phi_< - S_\Lambda[\phi_<] + J \cdot \phi_< \right) \quad (1.13)$$

for some functional $S_\Lambda[\phi_<]$. Looking at the form of the exponent here, we can see that S_Λ is simply interacting part of the effective action at scale Λ. Note that both forms of $Z[J]$ represent the same physics, but are just a different description. Thus, we must have that

$$\frac{\partial}{\partial \Lambda} Z[J] = 0 \quad (1.14)$$

and from this one can deduce Polchinski's equation for the flow of the effective interactions [27]:

$$\frac{\partial}{\partial \Lambda} S_\Lambda[\phi] = \frac{1}{2} \frac{\delta S_\Lambda}{\delta \phi} \cdot \frac{\partial \Delta^\Lambda}{\partial \Lambda} \cdot \frac{\delta S_\Lambda}{\delta \phi} - \frac{1}{2} \mathrm{tr} \left(\frac{\partial \Delta^\Lambda}{\partial \Lambda} \cdot \frac{\delta^2 S_\Lambda}{\delta \phi \delta \phi} \right). \quad (1.15)$$

We can interpret the first term as being the tree-level ("classical") part, and the second term as being the one-loop ("quantum") part. To get more of an intuition for this, we can write this in terms of vertices to get the following representation shown in Figure 1.3.

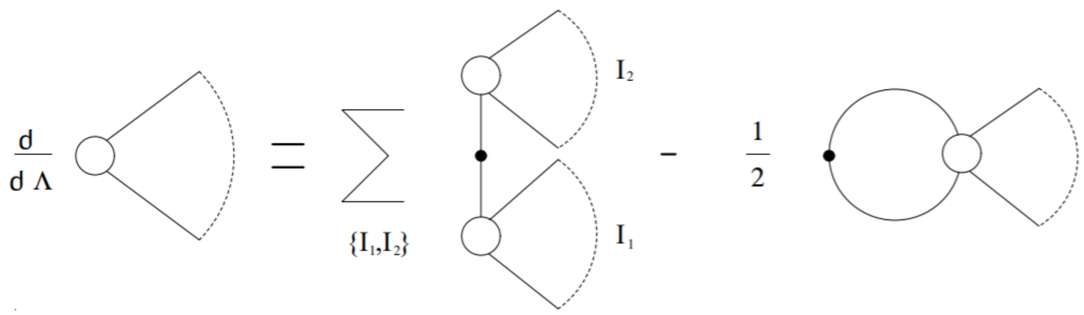

FIGURE 1.3: Visualisation of the flow equation for an n-point vertex. The black dot represents the differentiated cutoff propagator, and $\{I_1, I_2\}$ is some partition of n. Diagram taken from [3].

[1]This is not really necessary, and is only done here to simplify the analysis. The loose justification one can make for this step is the fact that when we lower the cutoff, we're only interested in low energy observables, as we wouldn't lower the cutoff otherwise.

For many purposes it is easier to deal with the average effective action Γ_Λ, which is the Legendre transform of the Wilsonian effective action:

$$S_\Lambda[\phi] = \Gamma_\Lambda[\phi_c] + \frac{1}{2}(\phi_c - \phi_<) \cdot (\Delta_\Lambda)^{-1} \cdot (\phi_c - \phi_<) \tag{1.16}$$

where

$$\phi_c = \frac{\delta}{\delta J} \ln Z[J]\bigg|_{J=0} \tag{1.17}$$

is the classical field. Which of these actions is more useful will depend on the precise situation (both are used in this thesis). The effective action also has a flow equation [3, 28]:

$$\frac{\partial}{\partial\Lambda}\Gamma_\Lambda[\phi_c] = -\frac{1}{2}\mathrm{tr}\left(\frac{1}{\Delta_\Lambda}\frac{\partial\Delta_\Lambda}{\partial\Lambda} \cdot \left(1 + \Delta_\Lambda \cdot \frac{\delta^2\Gamma_\Lambda}{\delta\phi_c\delta\phi_c}\right)^{-1}\right). \tag{1.18}$$

It is worth noting the following: both of these flow equations are non-perturbative. Indeed, they hold for very general actions. However, they are very complicated and usually some approximation needs to be made to make these equations in any way tractable. Where this approach has the advantage over standard perturbative renormalization, however, is the fact that we have the choice of what approximation we make - we are not restricted to insisting that the coupling is small (although this is often done). Examples include truncations, where only certain operators that can appear in the action are considered (often done in studies of asymptotic safety), or the Local Potential Approximation, which ignores any derivative operators other than the usual kinetic terms.

One final comment on this generalisation of Kadanoff blocking: so far, the analogy is incomplete since we have not performed the final step corresponding to a re-sizing of the lattice. In practice, the way we do this is to change all variables (couplings) for dimensionless versions and correcting the mass dimension with powers of Λ. This then rescales all couplings using the cutoff scale, and so the correspondence is complete. This will be assumed to be done from now on.

1.2 Flows and the critical surface

As far as we're aware, there is no overall scale at which physics itself fails. Therefore, we'd be inclined to take $\Lambda_0 \to \infty$ to get a "continuum limit". However, it is easier to do this directly, using a "fixed point" of the RG flow[2]

$$\partial_\Lambda S_* = 0. \tag{1.19}$$

[2] In this section, we will use the Wilsonian action S_Λ, but the same statements hold for the Legendre effective action Γ_Λ.

Note that since we have replaced all dimensionful couplings with powers of Λ and dimensionless couplings, this implies that there is no dependence on any dimensional couplings. That is, we have a fixed point when our action describes a massless and free field. This is known as the "Gaussian fixed point", and always exists for any action. However, physics at the Gaussian fixed point is extremely boring. The interesting case is what happens away from the Gaussian fixed point. The first thing to note is that renormalizability implies [2]

$$S_\Lambda[\phi] = S[\phi](g_i(\Lambda)) \tag{1.20}$$

that is, that under the RG flow, the form of the action does not change, only the couplings. This "self-similar" evolution is what allows us to get a non-trivial continuum limit in terms of "renormalized variables". The process is illustrated in Figure 1.4.

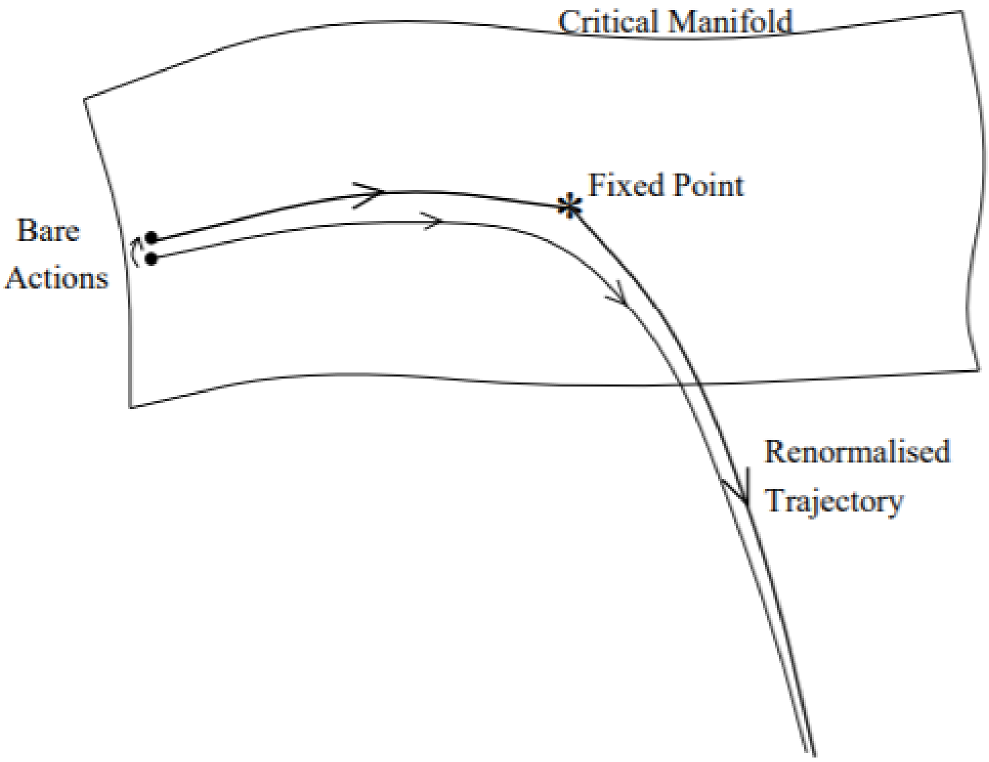

FIGURE 1.4: The critical manifold and renormalized trajectory. Diagram taken from [2].

First we note that we can write

$$S_\Lambda[\phi] = S_*[\phi] + \sum_i \hat{g}_i(\mu/\Lambda)^{\lambda_i} O_i[\phi] \tag{1.21}$$

to parametrize solutions in the vicinity of the Gaussian fixed point. Here, the $\hat{g}_i$ are dimensionless couplings, the λ_i are the RG eigenvalues of the operators O_i and μ is an arbitrary mass scale. Note that in general we have an infinite number of these. However, if we move away from the fixed point along those operators with $\lambda_i < 0$, the RG flow will send our action straight back to the fixed point. These "irrelevant" perturbations span the (infinite dimensional) critical manifold, on which any bare action will flow to the Gaussian fixed point. If we move along one of the other, "relevant" directions, the flow will move away from the critical manifold, typically not giving a well-defined limit.

However, if we tune our bare action to be slightly off the critical manifold, then the RG flow will head in the direction of the fixed point, before shooting away near the "renormalized trajectory" (RT), which is achieved in the limit of tuning the bare action back towards the critical surface. Unfortunately, in this limit the relevant couplings diverge under RG flow. To remedy this, while tuning our bare action back to the critical manifold, we also modify the couplings as $\Lambda \to \infty$ to be (roughly)

$$g_i = \hat{g}_i(\mu/\Lambda)^{\lambda_i} \tag{1.22}$$

such that these renormalized couplings remain finite on the RT, and so the far end of the RT corresponds to a non-trivial (i.e., interacting) continuum limit.

Since these couplings parametrize a solution to a first-order differential equation, they each require one boundary value to specify them. For these reason, we generally only consider theories with a finite number of relevant operators for the sake of predictivity.

1.3 Eigenoperators for the scalar field

It instructive to see how using the RG can give us the operators to construct a scalar field theory. In this section, we follow the procedure in, e.g., [29]. This construction will be modified in Chapter 2. Starting from the Polchinski equation (1.15), we linearise around the Gaussian fixed point to get

$$\partial_\Lambda S_\Lambda = -\frac{1}{2}\mathrm{tr}\left(\partial_\Lambda \Delta^\Lambda \cdot \frac{\delta^2 S_\Lambda}{\delta\phi\delta\phi}\right). \tag{1.23}$$

Although we are doing this near the Gaussian fixed point $S_\Lambda = 0$, this is sufficient to get the form of the eigenoperators at all scales due to the self-similarity of the RG transformation. We will consider non-derivative interactions (local potential approximation) so we write

$$S_\Lambda[\phi] = \epsilon \int d^4x V(\phi(x), \Lambda), \tag{1.24}$$

where ϵ is a small parameter. This gives us

$$\partial_\Lambda V = -\frac{1}{2}\partial^2_{\phi\phi} V \partial_\Lambda \int \frac{d^4p}{(2\pi)^4}\frac{C^\Lambda(p)}{p^2}. \tag{1.25}$$

To make the final step of the RG transformation, we change to dimensionless variables using powers of Λ:

$$\tilde{x} = x\Lambda, \qquad \tilde{\phi} = \frac{\phi}{\Lambda}, \qquad \tilde{V} = \frac{V}{\Lambda^4} \tag{1.26}$$

and, in addition we define

$$t = \ln\left(\frac{\mu}{\Lambda}\right), \tag{1.27}$$

the "RG time" (increasing towards the IR) where μ is some arbitrary energy scale. Expressed in these dimensionless variables, the only dependence on Λ that $\tilde{V}$ can have is due to the scaling of the couplings:

$$\tilde{V}(\tilde{\phi}, t) = \left(\frac{\mu}{\Lambda}\right)^{\lambda} \tilde{V}(\tilde{\phi}) \tag{1.28}$$

(note that we can treat each coupling separately since we are in the linear regime). We also take the time to define the one-loop tadpole integral[3]

$$\Omega_\Lambda = |\langle\phi(x)\phi(x)\rangle| = \int \frac{d^4p}{(2\pi)^4} \frac{C^\Lambda(p)}{p^2} \tag{1.29}$$

and the dimensionless version

$$\frac{1}{2a^2} = \frac{1}{2\Lambda}\partial_\Lambda \Omega_\Lambda = \int \frac{d^4\tilde{p}}{(2\pi)^4} \frac{C(\tilde{p})}{\tilde{p}^2} \tag{1.30}$$

where $p = \tilde{p}\Lambda$ and $C^\Lambda(p) = C(\tilde{p})$. Note that although a is a dimensionless constant, it is clearly not universal due to its dependence on the cutoff. Making all of these substitutions into (1.25) leads to the eigenoperator equation:

$$-\lambda\tilde{V} - \tilde{\phi}\tilde{V}' + 4\tilde{V} = -\frac{\tilde{V}''}{2a^2}. \tag{1.31}$$

This equation is of Sturm-Liouville form and thus we know that its solutions form a discrete set. The solutions in this case are (almost) the Hermite polynomials:

$$\mathcal{O}_n(\tilde{\phi}) = \frac{H_n(a\tilde{\phi})}{(2a)^n} = \tilde{\phi}^n - \frac{n(n-1)}{4a^2}\tilde{\phi}^{n-2} + \ldots \tag{1.32}$$

where $\lambda = 4 - n$ and n is a non-negative integer. Since equation (1.31) is of Sturm-Liouville form, we know that the solutions are orthogonal with respect to an appropriate weight function:

$$\int_{-\infty}^{\infty} d\tilde{\phi}\, e^{-a^2\tilde{\phi}^2} \mathcal{O}_n(\tilde{\phi})\mathcal{O}_m(\tilde{\phi}) = \frac{1}{a}\frac{1}{(2a^2)^n} n!\,\sqrt{\pi}\delta_{nm} \tag{1.33}$$

and we know that this set of functions is complete in the Hilbert space of solutions to equation (1.31). This space, which we call $\mathcal{L}_+$, is the set of functions which are square-integrable under the Sturm-Liouville measure $e^{-a^2\tilde{\phi}^2}$. In particular, this means

[3]The modulus sign here is extraneous, but is included for consistency with Chapter 2.

we can define couplings to be

$$\tilde{g}_n = \frac{(2a^2)^n a}{\sqrt{\pi} n!} \int_{-\infty}^{\infty} d\tilde{\phi}\, e^{-a^2\tilde{\phi}^2} \tilde{V}(\tilde{\phi}) \mathcal{O}_n(\tilde{\phi}) \tag{1.34}$$

and then the partial sums of the potential and the linear combination of the operator defined by these couplings converges to the potential in the sense of this measure, that is

$$\int_{-\infty}^{\infty} d\tilde{\phi}\, e^{-a^2\tilde{\phi}^2} \left(\tilde{V}(\tilde{\phi}) - \sum_{n=0}^{N} \tilde{g}_n \mathcal{O}_n \right) \to 0 \qquad \text{as } N \to \infty \tag{1.35}$$

and thus we can parametrize perturbations around the fixed point by the countable infinity of couplings $\tilde{g}_n$.

For this specific case, much of this is unnecessary. Note that for a well-defined continuum limit, we only care about relevant operators. That is, those with $\lambda > 0$.[4] Thus, $\mathcal{O}_0$ (vacuum energy - meaningless in the absence of gravity) and $\mathcal{O}_2$ (adding a mass term) are the only operators that survive this process. Thus, for a single scalar field, the only well-defined continuum field theories are those that are free and (possibly) massive. While this is inconvenient for this case, the machinery of the above can be applied for any fields, and a modification of the above is presented in Chapter 2.

1.4 Gravity and the RG

It is often stated in the literature that gravity is not renormalizable [25] (at least perturbatively). It is instructive to see why this view is taken. The most simple explanation given is that the only coupling, $\kappa = \sqrt{32\pi G}$, in the theory is irrelevant with $[\kappa] = -1$, and thus there can be no interacting fixed point.

There have been many attempts to reconcile gravity and the RG, and one of the most notable is the idea of asymptotic safety, in which the view is taken that there is a non-Gaussian UV fixed point. One review is [30], but there are many others. This has been supported with evidence from polynomial truncations in the Ricci scalar R, up to R^{70} [31]. Another example that has come up in recent years is causal dynamical triangulations (CDT) [32] which seeks to build an inherently quantum picture of spacetime itself, with a renormalization scale built in (the size of the triangulations used).

There are many attempts to solve this problem currently, and as yet we have very little other than aesthetic and mathematical principles to guide us. In some sense, we will only truly be doing physics with quantum gravity once predictions are made that can realistically be falsified. In this thesis, the view is taken that the Einstein-Hilbert

[4] $\mathcal{O}_4$ is marginal to this order, as $\lambda = 0$. One must go to higher orders to determine the behaviour of $\tilde{g}_4$, but suffice to say that it is marginally irrelevant, so can be ignored in the continuum limit.

action is, in fact perturbatively renormalizable with an appropriate shift in how one defines the process of quantization.

1.5 Outlook and scope

The following chapters each describe aspects of quantum gravity and the renormalization group. Chapter 2, based on the work in [5] describes the conformal factor instability and a resolution which, on top of solving this problem, may yet open the door to quantum gravity. In addition, it explores the topological effects of using RG techniques on a non-trivial manifold, which may yet find application in lattice field theory.

Chapter 3, describes the work of [6] in constructing a supermanifold in order to introduce Pauli-Villars regulator fields, along the lines of what has already been done in gauge theory [33]. The result is mathematically interesting yet somewhat incomplete as the final step of introducing a symmetry breaking superscalar field has not yet been done, having been left to future publications.

Chapter 4 draws from the work of [4, 7, 34, 35] and looks at how the RG equation and a generalisation of the BRST quantization procedure, the Batalin-Vilkoviski (BV) formalism, are compatible. This produces a way to construct the action of quantum gravity order by order, with the only input being the fields and their symmetries at the free level. In Chapter 5, we also look at the effect of adding a scalar field, and show that the additional terms that this generates are precisely what one would expect when coupling a scalar to gravity if one were to do so in a "naive" way.

Finally, we look at what avenues are left to explore in this formulation of quantum gravity, and look at ways one might include Standard Model particles into this formalism.

Chapter 2

The Conformal Mode and The Torus

2.1 Introduction

Much of this chapter is adapted from [5], with some additional material from [29].

In [29], it was shown that the conformal factor field φ has profoundly different RG properties to a standard scalar field in QFT, and that these properties may be what are required to construct a perturbatively renormalizable theory of quantum gravity. These properties were shown to lead to a novel effect which links the size of a universe to its inhomogeneity, at least for $\mathbb{T}^4$. Then, in [5], this was taken further to include $\mathbb{T}^3 \times \mathbb{R}$ (with the idea of identifying $\mathbb{R}$ with Euclidean time) and twisted tori. This work is described in what follows, including some analytic results regarding global extrema.

2.1.1 The problem

The path integral for the Einstein-Hilbert action (in Euclidean space) has a number of problems. Chief of which is the fact that the Euclidean action is unbounded from below (in fact it is unbounded from above also, but this doesn't cause any problems). Note that

$$S_{EH} = \int d^4x \mathcal{L}_{EH}, \qquad \mathcal{L}_{EH} = -\frac{2\sqrt{g}R}{\kappa^2} \tag{2.1}$$

and so the Euclidean path integral

$$Z = \int \mathcal{D}[g_{\mu\nu}] e^{-S_{EH}} \tag{2.2}$$

diverges for metric configurations with large positive Ricci curvature.

To see how this is related to the conformal factor, we write

$$g_{\mu\nu} = \left(1 + \frac{\kappa}{2}\varphi\right)\delta_{\mu\nu} + \kappa h_{\mu\nu} \tag{2.3}$$

where $h_{\mu\nu}$ is traceless (so we identify this as the graviton), φ is the conformal factor and $\kappa = \sqrt{32\pi G}$, where G is Newton's constant. Upon substituting this into S_{EH}, and fixing to a Feynman - de Donder gauge we see that the Lagrangian is of the form

$$\mathcal{L}_{EH} = -\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(\partial_\rho h_{\mu\nu})^2 + \dots \tag{2.4}$$

and therefore we see that the divergence of the path integral comes about for spacetime configurations with a conformal factor that varies sufficiently quickly. This is known as the "conformal factor instability" and there have been several attempts to address this. One, by Hawking et. al. [36] suggests analytically continuing the conformal factor $\varphi \mapsto i\varphi$ to change this minus sign to a plus sign, but we will take the view that this sign is to be kept and taken seriously.

It is worth taking time to justify not using the treatment in [36] or other alternatives. Indeed, the authors showed that the procedure of continuing the conformal factor to the imaginary axis does not in fact affect perturbative results. However, it is not clear whether this procedure makes sense non-perturbatively [37]. One approach considered within the asymptotic safety scenario involves a truncation to a finite set of operators, leading to a "$-R + R^2$" action [38]. The opposing sign of the R^2 term serves to stabilise the conformal sector, but also results in an unsuppressed non-perturbative Planckian scale modulated phase which breaks Lorentz symmetry, which would clearly pose phenomenological problems if physical [39–41].

2.1.2 A way out: Wilsonian RG

Now, it seems that this theory is doomed from the start, given that the Euclidean path integral is (worse than usually) ill-defined. However, if one treats Z as a formal object, then we can derive flow equations for this theory in exactly the same way as we would otherwise. Furthermore, in both this and the usual case we can reverse the derivation to get the path integral from the flow equation. Thus we take the view that the RG equation is the way that we define the theory, as opposed to the path integral, which we merely view as a formal object.

For the remainder of this chapter, the traceless part of the metric will be discarded. At this point we are not really doing gravity, but rather simply QFT with a scalar that has a negative kinetic term. The treatment here should be compared with that of Section 1.3. What follows is also covered in [29, 42] Looking at the Wilsonian effective action, we can split the kinetic terms and the interactions as

$$S^{\text{tot}}_\Lambda[\varphi] = S_\Lambda[\varphi] - \frac{1}{2}\varphi \cdot (\Delta^\Lambda)^{-1} \cdot \varphi \tag{2.5}$$

where the propagator

$$\Delta^\Lambda(p) = \frac{C^\Lambda(p)}{p^2} \tag{2.6}$$

is regulated by the UV cutoff function $C^\Lambda(p) = C(p^2/\Lambda^2)$, which satisfies

$$\lim_{u\to 0} C(u) = 1, \qquad \lim_{u\to\infty} C(u) = 0 \tag{2.7}$$

where convergence to the UV limit is sufficiently fast to regulate all momentum integrals. Following the steps of Section 1.1.2 leads to a modification of the Polchinski equation

$$\frac{\partial}{\partial\Lambda} S_\Lambda[\varphi] = -\frac{1}{2}\frac{\delta S_\Lambda}{\delta\varphi} \cdot \frac{\partial \Delta^\Lambda}{\partial\Lambda} \cdot \frac{\delta S_\Lambda}{\delta\varphi} + \frac{1}{2}\text{tr}\left(\frac{\partial\Delta^\Lambda}{\partial\Lambda} \cdot \frac{\delta^2 S_\Lambda}{\delta\varphi\delta\varphi}\right). \tag{2.8}$$

Comparing with equation (1.15), we see that the only difference is an overall minus sign on the right hand side. This looks innocent at first glance, but as we will see this overall sign has important properties for the RG behaviour of the theory.

Since we are only interested in the form of the eigenoperators, we can linearise about the Gaussian fixed point ($S_\Lambda = 0$) to yield

$$\partial_\Lambda S_\Lambda = \frac{1}{2}\text{tr}\left(\partial_\Lambda \Delta^\Lambda \cdot \frac{\delta^2 S_\Lambda}{\delta\varphi\delta\varphi}\right) \tag{2.9}$$

and as before, we use the LPA to get the flow equation for the potential

$$\partial_t V = -\Omega_\Lambda \partial^2_{\varphi\varphi} V \tag{2.10}$$

where we have defined

$$\Omega_\Lambda = |\langle\varphi(x)\varphi(x)\rangle| \tag{2.11}$$

as before[1]. Note that equation (1.25) can be written like this only with a change in sign on the RHS. This form shows how the change in sign has a drastic effect on the behaviour of the RG flow. In the case of the positive sign, we have a heat diffusion equation[2] with time increasing as we flow to the IR. This means that, given a solution at Λ_0, we can always flow to a solution at $\Lambda < \Lambda_0$ as we wish. However, with the change in sign, then as we flow towards the IR then generically, at some point, the flow will fail. That is, we are only guaranteed that a flow to the UV exists [43]. We will see later that this can have profound consequences for the theory.

[1] Note that now the modulus sign is required for consistency with Section 1.3.
[2] After a change of variables.

To find the eigenoperators, we follow the steps of Section 1.3 to get the eigenoperator equation:

$$-\lambda\tilde{V} - \tilde{\varphi}\tilde{V}' + 4\tilde{V} = \frac{V''}{2a^2} \tag{2.12}$$

which again, is of Sturm-Liouville form, but now with measure $e^{+a^2\tilde{\varphi}^2}$. As seen in [29], there are multiple sets of solutions, including a continuous family. This causes issues with notions of completeness that we're used to in QFT. However, we can restrict the set of solutions (for V) to be spanned by

$$\delta^{(n)}_\Lambda(\varphi) = \frac{\partial^n}{\partial\varphi^n}\delta^{(0)}_\Lambda(\varphi), \quad \text{where} \quad \delta^{(0)}_\Lambda(\varphi) = \frac{1}{\sqrt{2\pi\Omega_\Lambda}}\exp\left(-\frac{\varphi^2}{2\Omega_\Lambda}\right) \tag{2.13}$$

by insisting that the potential is square-integrable under the Sturm-Liouville measure

$$\int_{-\infty}^{\infty} d\varphi V^2(\varphi,\Lambda)\exp\left(\frac{\varphi^2}{2\Omega_\Lambda}\right) < \infty \tag{2.14}$$

at the bare level (where $\Lambda = \Lambda_0$). Note that since the potential "smooths out" as RG time goes backwards (c.f. time-reversed diffusion equation), then imposing this at the bare level also ensures that it holds for $\Lambda > \Lambda_0$.

These eigenoperators satisfy $[\delta^{(n)}_\Lambda(\varphi)] = -1 - n$ and span a Hilbert space of potentials satisfying equation (2.14), which we call $\mathcal{L}_-$. Therefore, we can write the general solution to the flow equation as

$$V(\varphi,\Lambda) = \sum_{n=0}^{\infty} g_n \delta^{(n)}_\Lambda(\varphi) \tag{2.15}$$

where $[g_n] = 5 + n$. We thus have an infinite tower of relevant operators. Normally this would be disastrous from the point of view of predictivity, but this is resolved in Section 2.1.3. If the couplings are chosen such that the flow exists to the IR, then we can extract the physical potential

$$V_p(\varphi) = \lim_{\Lambda\to 0} V(\varphi,\Lambda). \tag{2.16}$$

Due to the backward-parabolic nature of the RG equation, we can extract the potential at any Λ if we have the physical potential. Indeed, we have

$$V(\varphi,\Lambda) = \int_{-\infty}^{\infty}\frac{d\pi}{2\pi}\,\mathcal{V}_p(\pi)e^{-\Omega_\Lambda\frac{\pi^2}{2}+i\pi\varphi} \tag{2.17}$$

where $\mathcal{V}_p$ is the Fourier transform of the physical potential.

Now, solutions of (2.14) are characterized [29] by an *amplitude suppression scale* at large field

$$V_p(\varphi) \sim e^{-\frac{\varphi^2}{\Lambda_p^2}} \tag{2.18}$$

which is (up to a non-universal constant) the point at which the IR evolved potential leaves $\mathcal{L}_-$.

2.1.3 Connection to quantum gravity

As stated earlier, we are not really doing gravity at this stage, just QFT with a scalar that has a negative kinetic term. However, as seen in [34], the eigenoperators for the conformal factor have a key role to play in constructing a theory of quantum gravity. Indeed, the general interaction term (ignoring ghosts etc.) can be written as

$$f^\sigma_\Lambda(\varphi)\sigma(\partial, h, \partial\varphi) + \dots \tag{2.19}$$

with σ some Lorentz invariant monomial in the fields (the dots indicate tadpole corrections) and

$$f^\sigma_\Lambda(\varphi) = \sum_{n=n_\sigma}^{\infty} g^\sigma_n \delta^{(n)}_\Lambda(\varphi) \tag{2.20}$$

is the "coefficient function". Thus these conformal factor eigenoperators are indeed worth studying for the construction of a theory of quantum gravity, as well as being sufficiently of interest and novel to be worth studying on their own. Note that since $\delta^{(n)}_\Lambda$ has arbitrarily negative mass dimension, we can use it to render any monomial σ renormalizable.

2.1.4 Outlook for this chapter

Having established the eigenoperators for the scalar field with a negative kinetic term, Section 2.2 shows how these are modified when we are on a manifold other than $\mathbb{R}^4$. Although the effects of this in generality necessarily must wait for a full theory of quantum gravity (since they explicitly relate to the geometry of the manifold), we see than in general, the failure of the RG flow to the IR implies a bound on a certain function $\mathcal{S}$ of the geometry. Finally, we use this function to define a measure of inhomogeneity, largely inspired by the work that follows.

In Section 2.3, deal with the the simplest manifolds we have the machinery to deal with, $\mathbb{T}^4$ and $\mathbb{T}^3 \times \mathbb{R}$, and see how $\mathcal{S}$ changes as we vary the lengths of the fundamental loops. As hoped, it is found that $\mathcal{S}$ decreases, and indeed becomes negative for sufficiently inhomogeneous geometries. The numerical work in this section is supplemented by analytic results in Section 2.4, both of which point to maximally symmetric manifolds maximising $\mathcal{S}$.

Finally, in Section 2.5, we generalise these results to twisted tori. We see that, generally inhomogeneity reduces $\mathcal{S}$, but with the addition of a twist parameter, a much more intricate set of phenomena is revealed. A small section of moduli space is explored and numerical results presented.

2.2 RG evolution on manifolds

We wish to see how these $\delta_\Lambda^{(n)}$ evolve on a general manifold $\mathcal{M}$. We expect the main alterations compared to $\mathbb{R}^4$ will be seen in the IR, since the UV is regulated by Λ_0, which we expect to send to infinity and thus, by the definition of the local properties of a manifold, $\mathcal{M}$ will be indistinguishable from $\mathbb{R}^4$.

To see how the difference arises, it will be useful to write, from equation (2.17)

$$\delta_{\Lambda_0}^{(n)}(\varphi) = \exp\left(\frac{1}{2}\Omega_{\Lambda_0}\frac{\partial^2}{\partial\varphi^2}\right)\delta^{(n)}(\varphi) \tag{2.21}$$

where $\delta^{(n)}$ is the nth derivative of the δ-function, which is the physical limit of the above eigenoperators. If one starts from these bare operators and solves equation (2.9) down to some scale $\Lambda = k$ then one obtains

$$\int_x \delta_k^{(n)}(\varphi) = \exp\left(-\frac{1}{2}\mathrm{tr}\left[\Delta_k^{\Lambda_0}\cdot\frac{\delta^2}{\delta\varphi\delta\varphi}\right]\right)\int_x \delta_{\Lambda_0}^{(n)}(\varphi) \tag{2.22}$$

where now, integrals over spacetime (including those implied in DeWitt notation) are to be read as

$$\int_x = \int d^4x\sqrt{g}. \tag{2.23}$$

We have also defined a propagator that is regulated in both the UV and IR, where

$$\Delta_k^{\Lambda_0} = \frac{C_k^{\Lambda_0}(p)}{p^2} = \frac{C^{\Lambda_0}(p) - C^k(p)}{p^2}. \tag{2.24}$$

Combining the above, we see that we can write the evolved operators as

$$\delta_{k,\Lambda_0}^{(n)}(\varphi) = \exp\left(\frac{1}{2}\Omega_{k,\Lambda_0}\frac{\partial^2}{\partial\varphi^2}\right)\delta^{(n)}(\varphi) \tag{2.25}$$

where

$$\Omega_{k,\Lambda_0}(x) = |\langle\varphi(x)\varphi(x)\rangle|_{\mathbb{R}^4} - |\langle\varphi(x)\varphi(x)\rangle|_{\mathcal{M}}. \tag{2.26}$$

The first term on the RHS here is the one-loop tadpole at $\Lambda = \Lambda_0$, whereas the second term is evaluated on the manifold and regulated by $C_k^{\Lambda_0}$. On $\mathbb{R}^4$, this term is $\Omega_{\Lambda_0} - \Omega_k$, and so $\Omega_{k,\Lambda_0} = \Omega_k$. By contrast, on $\mathcal{M}$, we expect changes to this when the IR evolution reaches $k \sim 1/L$, where L is some characteristic length scale for the manifold.

Getting the physical Ω for a manifold is a simple matter of removing the regulators:

$$\Omega_p(x) = \lim_{\substack{\Lambda_0 \to \infty \\ k \to 0}} \Omega_{k,\Lambda_0}(x). \tag{2.27}$$

One can then use the evolution equation for the $\delta_\Lambda^{(n)}$ to get the physical eigenoperators:

$$\delta_p^{(n)}(\varphi) = \frac{\partial^n}{\partial \varphi^n} \delta_\Lambda^{(0)}(\varphi), \quad \text{where} \quad \delta_p^{(0)}(\varphi) = \frac{1}{\sqrt{2\pi\Omega_p}} \exp\left(-\frac{\varphi^2}{2\Omega_p}\right). \tag{2.28}$$

In the case $\mathcal{M} = \mathbb{R}^4$, we have $\Omega_p = 0$, and thus the eigenoperators return to being the δ-function and its derivatives. However, we can, on dimensional grounds write

$$\Omega_p(x) = \frac{\mathcal{S}(x)}{4\pi L^2} \tag{2.29}$$

with L a characteristic length scale of $\mathcal{M}$, and $\mathcal{S}$ we expect to be a finite and universal (independent of regularization details) function of the geometry[3].

As seen in [29], Ω_p can be negative, and in this case, the RG can fail at some point in the IR. In particular, if the flow exists for $k \to 0$, we have

$$V_p(\varphi(x), x) \sim \exp\left(-\frac{\varphi^2}{\Lambda_p^2 + 2\Omega_p(x)}\right) \tag{2.30}$$

for large φ, and we have the constraint

$$\mathcal{S}(x) > -2\pi L^2 \Lambda_p^2 \quad \forall x \in \mathcal{M}. \tag{2.31}$$

Now we define

$$\mathcal{S}_{\min} = \inf_{x \in \mathcal{M}} \mathcal{S}(x). \tag{2.32}$$

If the manifold has $\mathcal{S}_{\min} < 0$, then by our constraint the manifold must have a minimum size given by

$$L_{\min} = \sqrt{\frac{-\mathcal{S}_{\min}}{2\pi\Lambda_p^2}} \tag{2.33}$$

and thus the amplitude suppression scale has a role in controlling the size of such manifolds. Now we define $\mathcal{S}_{\max}$ to be the maximum of $\mathcal{S}_{\min}$ over all manifolds with a given topology. We will see in the remainder of this chapter that a larger $\mathcal{S}$ is associated with more symmetric manifolds, and therefore we interpret

$$\mathcal{I}_{\mathcal{M}} = \mathcal{S}_{\max} - \mathcal{S}_{\min} \tag{2.34}$$

as a measure of inhomogeneity in $\mathcal{M}$[4]. We find that generally, $\mathcal{S}_{\max}$ is $\mathcal{O}(1)$.

[3] The factor of 4π is merely there for convenience - it could just as well be subsumed into $\mathcal{S}$.

[4] It is worth remembering that $\mathcal{S}_{\min}$ depends only on $\mathcal{M}$, whereas $\mathcal{S}_{\max}$ depends on its topological class.

With this definition, clearly a smaller $\mathcal{S}_{\min}$ corresponds to a more inhomogeneous manifold, and this will be borne out in the examples that follow. To summarize, however, we can now say that homogeneous universes are constrained to be small:

$$\mathcal{I}_{\mathcal{M}} < \mathcal{S}_{\max} + 2\pi L^2 \Lambda_p^2 \tag{2.35}$$

and this would have important and far-reaching implications in cosmology.

We would now like to use some examples to test out these ideas. However, it is unclear exactly how to interpret the integrals in equation (2.25) without a fully developed theory of quantum gravity. Thus we are restricted to those manifolds with $g_{\mu\nu} = \delta_{\mu\nu}$. This greatly restricts our options, but we can us tori, which have a rich moduli space and are simple enough that calculations can be fruitfully performed.

2.3 Flat tori

2.3.1 Four-torus

In [29], Ω_p was calculated for the four-torus $\mathcal{M} = \mathbb{T}^4$. An account of the derivation follows. Suppose that the lengths of the non-contractible loops are L_μ $(\mu = 1, \ldots, 4)$. Then we write

$$|\langle\varphi(x)\varphi(x)\rangle|_{\mathcal{M}} = \frac{1}{V} \sum_{n \neq 0} \frac{C_k^{\Lambda_0}(p_n)}{p_n^2} \tag{2.36}$$

where $p_n^\mu = 2\pi n_\mu / L_\mu$ (no sum), $n \in \mathbb{Z}^4 \backslash \{0\}$ and $V = \prod_{\mu=1}^4 L_\mu$ is the volume of $\mathbb{T}^4$. Note that in this case, $\mathcal{S}(x) = \mathcal{S}_{\min}$ since the manifold is translation invariant. Note that we have removed the zero mode since in this configuration, $\varphi = \varphi_0$ and thus near the Gaussian fixed point, the integrand of the path integral does not depend on the field value, and so these configurations must be divided out. This step also makes the sum IR finite as $k \to 0$. It turns out it will be convenient to add the zero mode back in as an intermediary step:

$$\lim_{p \to 0} \frac{C_k^{\Lambda_0}(p)}{p^2} = C'(0) \left(\frac{1}{\Lambda_0^2} - \frac{1}{k^2} \right). \tag{2.37}$$

With the zero mode included in the sum, we can use the Poisson summation formula to write the sum as an integral, then we must subtract this zero mode contribution:

$$|\langle\varphi(x)\varphi(x)\rangle|_{\mathcal{M}} = \int_p \frac{C_k^{\Lambda_0}(p)}{p^2} \sum_n e^{i l_n \cdot p} - \frac{C'(0)}{V} \left(\frac{1}{\Lambda_0^2} - \frac{1}{k^2} \right) \tag{2.38}$$

where $l_n^\mu = n_\mu L_\mu$ (no sum) and $n \in \mathbb{Z}^4 \backslash \{0\}$ is now the winding number. Clearly, when $n = 0$, the first term is clearly the flat space propagator with the regulators: $\Omega_{\Lambda_0} - \Omega_k$.

With this, we have

$$\Omega_{k,\Lambda_0} = \Omega_k + \frac{C'(0)}{V}\left(\frac{1}{\Lambda_0^2} - \frac{1}{k^2}\right) - \int_p \frac{C_k^{\Lambda_0}(p)}{p^2}\sum_{n\neq 0} e^{il_n\cdot p} \tag{2.39}$$

which, since the third term is a sum of propagators between distinct points, is finite as $\Lambda_0 \to \infty$. Thus, since we have already noted that Ω_{k,Λ_0} is IR finite, the final result is finite in the abscence of regulators and, in particular, we are free to choose our cutoff profile. For what follows, we will take $C(p^2/\Lambda^2) = e^{-\frac{p^2}{\Lambda^2}}$, and the take the $\Lambda_0 \to \infty$ and $k \to 0$ limits where it is safe to do so. Thus, we have

$$\Omega_p = \frac{1}{k^2 V} - \int_p \int_0^{\frac{1}{k^2}} d\alpha\, e^{-\alpha p^2} \sum_{n\neq 0} e^{il_n\cdot p}. \tag{2.40}$$

where we have expressed the IR cutoff in terms of a Schwinger parameter. Performing the (Gaussian) integral over momentum and changing variables to $\alpha = L^2 t/4\pi$ where $L = V^{\frac{1}{4}}$, we have

$$\Omega_p = \frac{1}{k^2 V} - \frac{1}{4\pi L^2}\int_0^{\frac{4\pi}{k^2 L^2}} \frac{dt}{t^2}\left(\prod_{\mu=1}^4 \Theta\left(\frac{l_\mu^2}{t}\right) - 1\right) \tag{2.41}$$

where $l_\mu = L_\mu/L$ and we've defined the third Jacobi theta function:

$$\Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}. \tag{2.42}$$

To deal with the remaining integral, we define

$$s_4(l_\mu) = \int_0^1 \frac{dt}{t^2}\left(\prod_{\mu=1}^4 \Theta\left(\frac{l_\mu^2}{t}\right) - 1\right) \tag{2.43}$$

and to deal with the remainder of the integral, we make the substitution $t \mapsto 1/t$. We also make use of the formula $\Theta(x) = \frac{1}{\sqrt{x}}\Theta\left(\frac{1}{x}\right)$. What remains is $s_4(1/l_\mu)$, plus a constant and an IR divergent part which cancels the divergence from the first term in Ω_p. Thus, we have

$$\Omega_p = \frac{\mathcal{S}_4(l_\mu)}{4\pi L^2} \tag{2.44}$$

where

$$\mathcal{S}_4(l_\mu) = 2 - s_4(l_\mu) - s_4(1/l_\mu). \tag{2.45}$$

A few things to note about this. Firstly, as expected, $\mathcal{S}_4$ does not depend on the overall sign of the manifold, just the relation of the lengths of the fundamental loops to each other. In addition, it is invariant if we permute the lengths between each other. Perhaps more surprisingly, there is an invariance when we send $l_\mu \mapsto 1/l_\mu$, which is somewhat similar to T-duality in string theory. The significance of this is yet to be determined, if indeed there is any.

Trying some numerical examples, we find that $\mathcal{S}_4 = 1.765$ at the symmetric point ($l_\mu = 1$), and can be made negative by deviating sufficiently far from this. In fact, $\mathcal{S} = 0$ at the following points:

- $l_1 = 2.709$ and the other $l_\mu = 0.7173$;
- $l_1 = 0.3691$ and the other $l_\mu = 1.394$;
- $l_1 = l_2 = 2.457$ and $l_3 = l_4 = 0.4069$;
- $l_\mu = 1.487 l_{\mu+1}$ ($\mu = 1, 2, 3$).

Note that the first two are dual to each other, whereas the other two are self-dual (combined with permutation symmetry). When the lengths deviate more than this, then $\mathcal{S}_4$ is negative, and thus the bound (2.33) applies, and in this case, this means that, in terms of the spacetime volume

$$V > \frac{\mathcal{S}_4(l_\mu)^2}{4\pi^2\Lambda_p^4} \tag{2.46}$$

and so we see in this case that small universes are constrained to be highly symmetric[5].

2.3.2 Spatial three-torus

Now we look at a slightly more realistic model, $\mathbb{T}^3 \times \mathbb{R}$, with the idea that the real line corresponds to the time direction after undoing the Wick rotation. Just as the $\mathbb{T}^4$ case [44], we can relate this computation to those in the literature discussing finite size effects in lattice quantum field theory [45]. We subtract the zero mode, which again would render our expression IR divergent. The details of why this is valid are more complicated, and would presumably go along the lines of [45], but a precise justification would have to wait for a full theory of quantum gravity.

Suppose the lengths of the fundamental loops of $\mathbb{T}^3$ are L_i, and $V_3 = \prod_{i=1}^3 L_i$. We also define $L = V_3^{\frac{1}{3}}$, and the dimensionless lengthths as $l_i = L_i/L$. We then have

$$|\langle\varphi(x)\varphi(x)\rangle|_{\mathcal{M}} = \frac{1}{V_3}\int_{p_4}\sum_{n\neq 0}\frac{C_k^{\Lambda_0}(p)}{p^2} \tag{2.47}$$

where $p = (p_i^n, p_4)$, $p_i^n = 2\pi n_i/L_i$ (no sum) and $n \in \mathbb{Z}^3\backslash\{0\}$.

[5] Although it seems this formula can hold for $\mathcal{S}$ positive, the bound for $L_{\min}$ was derived with the assumption of negative $\mathcal{S}$, so this has no meaning when $\mathcal{S}_4 > 0$.

As with the $\mathbb{T}^4$ case, this expression is UV and IR finite, and so we have our choice in regularisation. Adding and subtracting the zero mode and applying the Poisson summation formula as before, we can see that

$$\Omega_{k,\Lambda_0} = \Omega_k + \frac{1}{V_3}\int_{p_4}\frac{C_k^{\Lambda_0}(p_4)}{p_4^2} - \int_p \frac{C_k^{\Lambda_0}(p)}{p^2}\sum_{n\neq 0} e^{i\vec{l}_n\cdot\vec{p}}. \tag{2.48}$$

Using the same regulator as before $C^\Lambda(p) = C(p^2/\Lambda^2)$, using the Schwinger parameter trick and taking limits where it is safe to do so, we see that

$$\Omega_p = \frac{1}{k\sqrt{\pi}V_3} - \frac{1}{4\pi L^2}\int_0^{\frac{4\pi}{k^2L^2}}\frac{dt}{t^2}\left(\prod_{i=1}^3 \Theta\left(\frac{l_i}{t}\right) - 1\right). \tag{2.49}$$

Analogously to above, we define

$$s_3(l_i) = \int_0^1 \frac{dt}{t^2}\left(\prod_{i=1}^3 \Theta\left(\frac{l_i}{t}\right) - 1\right) \tag{2.50}$$

and we split the integral about $t = 1$, take $t \mapsto 1/t$ and use $\Theta(x) = \frac{1}{\sqrt{x}}\Theta\left(\frac{1}{x}\right)$. This results in the IR divergences cancelling, and we are left with

$$\Omega_p = \frac{\mathcal{S}_3(l_i)}{4\pi L^2} \tag{2.51}$$

where $\mathcal{S}_3(l_i) = 3 - s_3(l_i) - \tilde{s}_3(1/l_i)$ and

$$\tilde{s}_3(l_i) = \int_0^1 \frac{dt}{t^{\frac{3}{2}}}\left(\prod_{i=1}^3 \Theta\left(\frac{l_i}{t}\right) - 1\right). \tag{2.52}$$

Once again, we see that the result is symmetric under interchange of the lengths of the fundamental loops, as would be expected from the symmetries of the torus. We also note that since $s_3 \neq \tilde{s}_3$, the inversion symmetry $l_i \mapsto 1/l_i$ is seen to be a quirk of the fact that previously we had four compact dimensions. Also, we see that $\mathcal{S}_3$ does not depend on the overall size of the manifold, as with $\mathcal{S}_4$.

To get some intuition, we input some values:

- $l_1 = l_2 = l_3 = 1$: $\mathcal{S}_3 = 2.8373$;
- $l_1 = 1$, $l_2 = 2$ and $l_3 = \frac{1}{2}$: $\mathcal{S}_3 = 0.8538$;
- $l_1 = 1$, $l_2 = 3$ and $l_3 = \frac{1}{3}$: $\mathcal{S}_3 = -4.2936$;
- $l_1 = l_2 = 2$ and $l_3 = \frac{1}{4}$: $\mathcal{S}_3 = -8.95463$;
- $l_1 = l_2 = 3$ and $l_3 = \frac{1}{9}$: $\mathcal{S}_3 = -73.1222$;

- $l_1 = 2$, $l_2 = 3$ and $l_3 = \frac{1}{6}$: $\mathcal{S}_3 = -28.4098$;
- $l_1 = \frac{1}{2}$, $l_2 = \frac{1}{3}$ and $l_3 = 6$: $\mathcal{S}_3 = -15.7999$.

Note that the largest value was seen in the maximally symmetric case, and decreases as anisotropy increases, becoming negative at some points. That last two provide an explicit example that the inversion symmetry from before fails in this case. In the case that $\mathcal{S}_3 < 0$, the bound (2.33) applies, and thus we have

$$V_3 > \left(\frac{-\mathcal{S}_3}{2\pi\Lambda_p^2}\right)^{\frac{3}{2}} \tag{2.53}$$

indicating that once again, small universes are constrained to be highly symmetric.

2.4 Some analytic results

We wish to see how much our hypothesis regarding anisotropies are supported by analytic results. First, we wish to show that the symmetric point is an extremum of $\mathcal{S}_d$ ($d = 3, 4$). We write $l_\mu = e^{z_\mu}$. Any first-order perturbation $\delta z_\mu = \epsilon\alpha_\mu$ gives a change in $\mathcal{S}_d$ proportional to $\sum_\mu \alpha_\mu$. However, since $\prod_\mu l_\mu = 1$, we have the constraint that $\sum_\mu \alpha_\mu = 0$, and so the first order change vanishes.

We have shown that the symmetric point is a local extremum, but we wish to see whether it is a global maximum. For simplicity (and at the cost of some generality), we set only two of the $l \neq 1$. Without loss of generality, we can then set $l_1 = \sqrt{\chi}$ and $l_2 = \frac{1}{\sqrt{\chi}}$. Then in both of the above cases, the only dependence on χ comes from the combination

$$\theta(\chi) = \Theta\left(\frac{\chi}{t}\right)\Theta\left(\frac{1}{t\chi}\right). \tag{2.54}$$

Now we note that this is invariant under $\chi \mapsto 1/\chi$, and so we can restrict our attention to $\chi \in (0, 1]$. Now we use a result of Ramanujan (given in, e.g., Berndt [46]):

$$\ln\Theta(x) = 2\sum_{k=1}^{\infty}\frac{q^{2k-1}}{(2k-1)(1+q^{2k+1})} \tag{2.55}$$

where $q = e^{-\pi x}$. From this we can see that

$$\partial_x\Theta(x) = -2\pi\Theta(x)\sum_{k=1}^{\infty}\frac{1}{\left(1+e^{\pi(2k+1)x}\right)^2} \tag{2.56}$$

and thus we have

$$\partial_\chi\theta(\chi) = \frac{2\pi\theta(\chi)}{t\chi^2}\sum_{k=1}^{\infty}\frac{1+\exp\left(\frac{\pi(2k-1)\chi}{t}\right)+\chi\left(1+\exp\frac{\pi(2k-1)}{\chi t}\right)}{\left(1+\exp\frac{\pi(2k-1)}{\chi t}\right)^2\left(1+\exp\frac{\pi(2k-1)\chi}{t}\right)^2}f_k(\chi) \tag{2.57}$$

where

$$f_k(\chi) = 1 + \exp\left(\frac{\pi(2k-1)\chi}{t}\right) - \chi\left(1 + \exp\frac{\pi(2k-1)}{\chi t}\right) \tag{2.58}$$

will determine the sign of each term, since every the other piece is always positive. Now, we have that $f_k(1) = 0$, and we wish to have $f_k'(\chi) > 0$ for $\chi \in (0,1]$ so that $\partial_\chi \theta < 0$ in this range. Since $2k-1 > 1$, we have

$$f_k'(\chi) = \pi \exp\left(\frac{\pi(2k-1)\chi}{t}\right) - 1 + \exp\left(\frac{\pi(2k-1)}{\chi t}\right)\left(\frac{\pi}{\chi} - 1\right) \tag{2.59}$$

which is indeed positive in the required range. Thus, $\partial_\chi \theta \leq 0$ for $\chi \in (0,1]$. Therefore, $\theta(\chi)$ has a global minimum at $\chi = 1$, and so the symmetric point is a global maximum for $\mathcal{S}_d$, since extrema of θ govern extrema of s_4, s_3 and $\tilde{s}_3$.

2.5 Twisted tori

We are still restricted to using manifolds with $g_{\mu\nu} = \delta_{\mu\nu}$, but we can still consider manifolds with the same topology, that is, twisted tori. The twisted four-torus is defined by the equivalence relation $x_\mu \sim x_\mu + L v_\mu$, where the v_μ defined by the lattice $\underline{\Lambda}_4$:

$$v_\mu \in \underline{\Lambda}_4 = \left\{\sum_i n^i l_i^\mu ; n \in \mathbb{Z}^4\right\} \tag{2.60}$$

where the primitive vectors l_i are not all orthogonal. Note that we have factored out the length scale $L = V_4^{\frac{1}{4}}$, where V_4 is the volume of the 4-torus. With this normalization, we have $\det(l_1, l_2, l_3, l_4) = 1$. If we wish to follow the derivation above, we need to define the dual lattice $\underline{\Lambda}_4^*$ with which we define the momentum modes:

$$\underline{\Lambda}_4^* = \{u \in \mathbb{R}^4 : u \cdot v \in 2\pi\mathbb{Z}, \forall v \in \underline{\Lambda}_4\}. \tag{2.61}$$

We then have $p \in \underline{\Lambda}_4^*/L$. Removing the zero mode as before, we have

$$|\langle\varphi(x)\varphi(x)\rangle|_{\mathcal{M}} = \frac{1}{V_4} \sum_{Lp \in \underline{\Lambda}_4^* \setminus \{0\}} \frac{C_k^{\Lambda_0}(p)}{p^2}. \tag{2.62}$$

Now it is a simple case of following the steps from before. To do so, we need the lattice theta function, which is a well known subject in modular forms (for example, see [47]):

$$\Theta_{\underline{\Lambda}}(t) = \sum_{v \in \underline{\Lambda}} e^{-\pi t v^2}. \tag{2.63}$$

Performing the same steps as for the untwisted case, and using another application of the Poisson summation formula[6]:

$$\Theta_{\underline{\Lambda}}(t) = t^{-\frac{d}{2}}\Theta_{\underline{\Lambda}^*}(1/t) \tag{2.64}$$

we arrive at

$$\Omega_p = \frac{\mathcal{S}_4(\underline{\Lambda}_4)}{4\pi L^2} \tag{2.65}$$

where $\mathcal{S}_4(\underline{\Lambda}_4) = 2 - s(\underline{\Lambda}_4) - s(\underline{\Lambda}_4^*)$, where we've defined

$$s(\underline{\Lambda}) = \int_0^1 \frac{dt}{t^2}\left(\Theta_{\underline{\Lambda}}\left(\frac{1}{t}\right) - 1\right). \tag{2.66}$$

Similarly, in the twisted $\mathbb{T}^3 \times \mathbb{R}$ case, we can similarly generalise to

$$\Omega_p = \frac{\mathcal{S}_3(\underline{\Lambda}_3)}{4\pi L^2} \tag{2.67}$$

where $\mathcal{S}_3(\underline{\Lambda}_3) = 3 - s(\underline{\Lambda}_3) - \tilde{s}(\underline{\Lambda}_3^*)$, with s as above, and

$$\tilde{s}(\underline{\Lambda}) = \int_0^1 \frac{dt}{t^{\frac{3}{2}}}\left(\Theta_{\underline{\Lambda}}\left(\frac{1}{t}\right) - 1\right). \tag{2.68}$$

It is easy to confirm that these results reduce to the previous results in the case of orthogonal primitive lattice vectors. Note also that the inversion symmetry in the $\mathbb{T}^4$ case is a special case of the $\underline{\Lambda} \mapsto \underline{\Lambda}^*$ symmetry. In fact, this is somewhat similar to symmetries in one-loop calculations in String Theory [48, 49], but the significance of this is yet to be determined. Clearly, we also still have the symmetry of interchanging the primitive vectors. In addition, we are free to redefine the primitive lattice vectors by

$$l_i \mapsto l_i + \sum_{j\neq i} n_j l_j, \quad n_j \in \mathbb{Z} \tag{2.69}$$

since the new primitive vectors will have the same span as the old ones, and hence define the same lattice. This is a $PSL(d,\mathbb{Z})$ symmetry, which we will have to take account of when finding the analytic properties of $\mathcal{S}$ since, for example in $d = 2$, if $l_1 = (1,0)$, $l_2 = (1,1)$, $l_1' = (1,0)$ and $l_2' = (0,1)$, then $\{l_1, l_2\}$ and $\{l_1', l_2'\}$ define the same lattices, and hence, despite appearances, $\{l_1', l_2'\}$ does *not* correspond to a twisted torus.

2.5.1 Analytic properties

The easiest realisation of the twisted torus is to start with orthogonal primitive vectors l_μ, and to twist $l_1 \mapsto l_1 + a l_2$ for $a \in \mathbb{R}$. In fact, due to the $PSL(d,\mathbb{Z})$ symmetry, we

[6] If we hadn't normalised the lattice (and its dual) there would be an inverse factor of volume on the RHS.

can restrict attention to $a \in [0,1)$, since $a \in \mathbb{Z}$ corresponds to the same lattice. For $d \geq 2$, we define $\underline{\Lambda}_2$ to be the sublattice generated by $\{l_1 + al_2, l_2\}$. In this case, the dependence of $\mathcal{S}$ on a is only due to $\Theta_{\underline{\Lambda}_2}$ and $\Theta_{\underline{\Lambda}_2^*}$. In these theta functions, we have sums over n_1, n_2, and for $v \in \underline{\Lambda}_2$ we have

$$v^2 = l_1^2 n_1^2 + l_2^2 (n_1 a + n_2)^2 \tag{2.70}$$

and when $v \in \underline{\Lambda}_2^*$, we have

$$v^2 = (n_1 a - n_2)^2 / l_1^2 + n_2^2 / l_2^2. \tag{2.71}$$

Note that in both cases, if $l_1 \gg l_2$, then the dependence on a is exponentially suppressed, whereas it is exponentially enhanced for $l_1 \ll l_2$. This may well be expected as this means that adding a small vector to a large vector makes less difference than adding a large vector to a small vector. We will see this explicitly in some numerical examples.

It is also worth noting that the above are symmetric under $a \mapsto -a$ (upon summation over n_1, n_2) and therefore, $a = 0$ is a local extremum of $\mathcal{S}$. Combining this with the modular symmetry $a \mapsto a + 1$ shows that $\mathcal{S}$ is symmetric about $a = \frac{1}{2}$, and so this is also a local extremum of $\mathcal{S}$.

When computing the numerical examples below, it is worth noting that the lattice theta function is not in most algebraic computing packages. In these cases is easier to note that

$$s(\underline{\Lambda}) = \sum_{v \in \underline{\Lambda} \setminus \{0\}} \exp(-\pi v^2) \tag{2.72}$$

and

$$\tilde{s}(\underline{\Lambda}) = \sum_{v \in \underline{\Lambda} \setminus \{0\}} \frac{1}{|v|} \operatorname{erfc}\left(\sqrt{\pi v^2}\right) \tag{2.73}$$

thus, in both cases, if we organise the sum in order or increasing $|v|$, we can compute these with exponentially fast convergence.

2.5.2 Four-torus

To display examples of twisted tori, we define $\underline{\Lambda}$ by a matrix M where the rows are the primitive vectors. Figure 2.1 shows the effect of starting from a lattice with orthonormal primitive vectors, and increasing a from 0 to 1. We see that $a = 0$ (equivalently $a = 1$) is the minimum of $\mathcal{S}$, at 1.765. We also see that the maximum is at $a = \frac{1}{2}$, with $\mathcal{S} = 1.784$. Figure 2.2 shows the effect of twisting a smaller vector towards a larger one. We see that the minimum and maximum are in the same places ($a = 0$ and $a = \frac{1}{2}$, respectively), but the difference in $\mathcal{S}$ between the two is much greater than the previous case. When the difference between these vectors is greater,

$$M = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad :$$

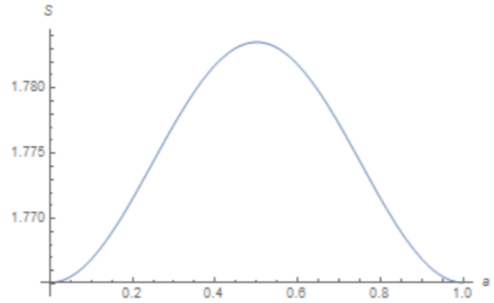

FIGURE 2.1: Variation of $\mathcal{S}$ as we twist the torus away from orthonormal primitive vectors.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & a & \frac{1}{2} \end{pmatrix} \quad :$$

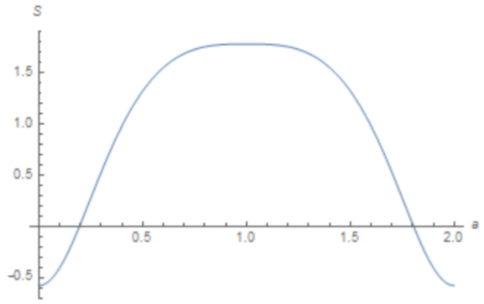

FIGURE 2.2: Variation of $\mathcal{S}$ as we twist a smaller vector towards a larger vector.

we see larger plateaux around $a = \frac{1}{2}$. This is largely due to the fact that the changing a makes less difference since the length of the vector is already large compared to the change from varying a.

In Table 2.1, we see the effect of twisting on this lattice in different directions. In all of these we see that twisting a smaller vector towards a larger one causes the largest change in $\mathcal{S}$.

Now, at increased inhomogeneities, we see fascinating new effects. From this point on, we use the parametrization of Section 2.5.1, and our general matrix will be

$$M_4(a, x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & ax & \frac{1}{x} \end{pmatrix} \tag{2.74}$$

In addition, due to the modular symmetry of the lattice we may consider only $a \in [0, 0.5]$ without losing any generality. Figure 2.3 shows how $\mathcal{S}$ changes with a for various values of x.

We see that when we reach $x = 3$, the stationary point at $a = \frac{1}{2}$ has become a local minimum, as opposed to a maximum when $x = 2$. In addition, when x increases

Matrix	$\mathcal{S}_{4\,\max}$	Range of $\mathcal{S}_4$
$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$	-0.547798	0.0293644
$\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ a & 0 & 0 & \frac{1}{2} \end{pmatrix}$	0.128869	0.706032
$\begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 2 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$	-0.577106	5.6×10^{-5}
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & a \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$	-0.577163	$\sim 10^{-10}$
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & a & \frac{1}{2} \end{pmatrix}$	1.78352	2.36068

TABLE 2.1: All possible ways to twist $M = \mathrm{diag}(1, 1, 2, \frac{1}{2})$, for which we have $\mathcal{S}_4 = -0.577163$. For those rows with two matrices, they are related by the inversion symmetry $\underline{\Lambda} \leftrightarrow \underline{\Lambda}^*$ (after relabelling), and those with one matrix are self-dual (up to relabelling). Note that we have the greatest change when a small vector is twisted towards a larger vector. In each case, the value $\mathcal{S}_{4\,\max}$ comes from maximal twist, that is when a is half the length of the vector we are twisting towards.

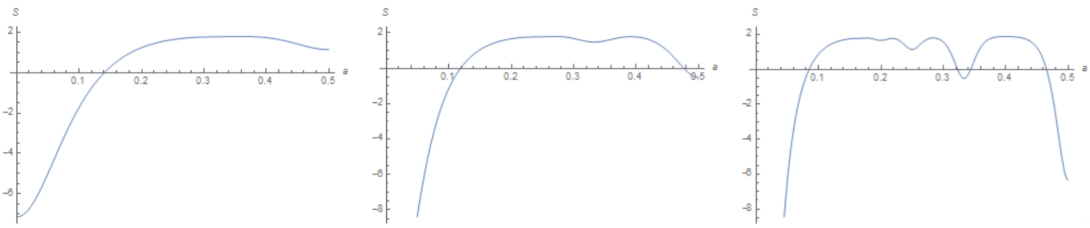

FIGURE 2.3: Variation of $\mathcal{S}$ as we vary $a \in [0, 0.5]$ for $x = 3, 4$ and 6 respectively.

further, new minima begin to appear, and the minimum at $a = \frac{1}{2}$ becomes negative. Increasing even further, not only do more minima appear, but also additional disconnected regions of $\mathcal{S} < 0$. It is worth noting that in the case $x = 4$, the minima appear to be at $a = \frac{1}{2}, \frac{1}{3}$, and for $x = 6$, we appear to have these, and additionally $a = \frac{1}{4}, \frac{1}{5}$. Thus, we are led to conjecture (with supporting evidence from non-integer x) that for a given x, we have minima at $a = \frac{1}{n}$ with $n < x$. It is not clear whether there is any physical significance to this pattern or whether this is simply a mathematical curiosity.

Also of interest is fixing the twist and allowing the inhomogeneity to vary. Figure 2.4

shows what happens when we fix $a = \frac{1}{2}$ and allow x to vary. Here we note that,

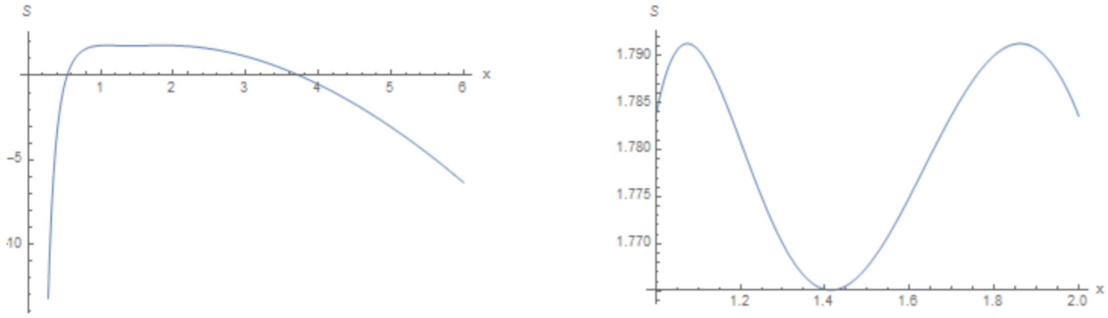

FIGURE 2.4: Plot of $\mathcal{S}(\frac{1}{2}, x)$. The second panel is a close-up for $x \in [1, 2]$. We see a local minimum at approximately $x = \sqrt{2}$.

broadly speaking, as inhomogeneity increases, $\mathcal{S}$ decreases and becomes negative, as we'd expect from the untwisted case. However, as we see in the second panel in Figure 2.4, the maximum does not occur at $x = 1$. Going by our definition of $\mathcal{I}_{\mathcal{M}}$ in Section 2.2, the theory appears to see the local maxima as the "most symmetric" points in this space of manifolds. Also of note is the fact that the local minimum appears to be around $x = \sqrt{2}$, and in fact if we fix $a = \frac{1}{3}$, the minimum appears around $x = \sqrt{3}$. Again, it is unclear whether this has any physical significance.

2.5.3 Spatial three-torus

Numerically, the effect of twisting on $\mathbb{T}^3 \times \mathbb{R}$ seems to be much the same as that for $\mathbb{T}^4$, except the changes seem to be damped somewhat. We see again that for mild inhomogeneities, the manifold prefers a more twisted configuration, for example,

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \text{gives } \mathcal{S}_3 = 1.327 \tag{2.75}$$

which should be compared to the untwisted $\mathcal{S}_3 = 0.8538$. Also, more dramatically, we see

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \quad \text{gives } \mathcal{S}_3 = 2.8538, \tag{2.76}$$

an even larger increase, as one would expect for a smaller vector twisting into a larger vector. Taking our lead from the $\mathbb{T}^4$ case, we define

$$M_3(a, x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & ax & \frac{1}{x} \end{pmatrix} \tag{2.77}$$

and find very similar behaviour to $\mathbb{T}^4$, but the effects seem more mild. Examples are given in Figure 2.5. Most important to note is the once again, the minima seem to

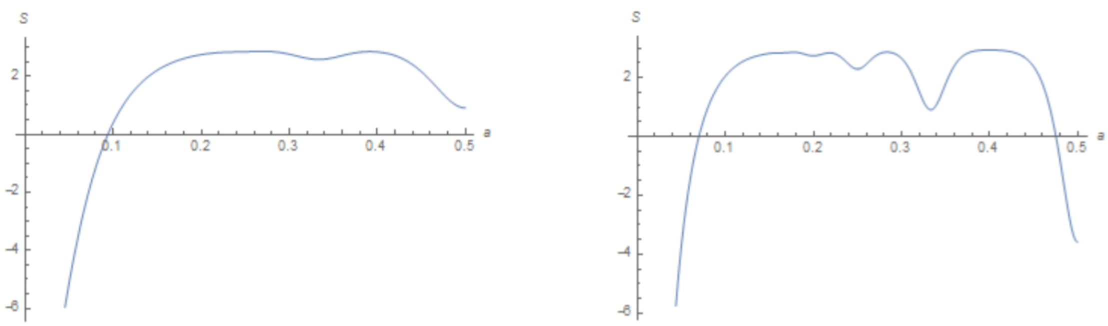

FIGURE 2.5: Variation of $\mathcal{S}_3$ as we vary $a \in [0, 0.5]$ for $x = 4$ and $x = 6$ respectively.

appear with the same arithmetic regularity as previously, however more inhomogeneity is needed to achieve $\mathcal{S}_3 < 0$. Also interesting is the fact that the plot for $\mathcal{S}_3(\frac{1}{2}, x)$ has a very similar shape to that in Figure 2.4. These seem to indicate that in both cases, the extrema that appear are due to extrema in $\Theta_{\underline{\Lambda}_2}$, as defined in Section 2.5.1.

2.6 Discussion

This chapter has focused on the renormalization group properties of the conformal mode of the metric. Although we treat it in isolation, we also see that knowledge of how the eigenoperators behave will be crucial in constructing an interacting theory of quantum gravity. We see that the wrong sign of the kinetic term has a profound impact on the theory, giving motivation for a modification of the definition of quantization which in turn gives rise to eigenoperators with novel properties.

The key aspect developed here is that the wrong sign of the kinetic term alters the direction in which the RG flow equation is well-posed. This means that, generically, the flow to the IR fails at some scale. Specifically, this happens when a universal function of the geometry $\mathcal{S}$ is negative and falls below some value set by the characteristic length of the manifold L and the amplitude suppression scale Λ_p. That is, if $\mathcal{S} < 0$, manifolds cannot be arbitrarily small (for finite Λ_p). We have seen that $\mathcal{S} < 0$ is possible in explicit examples, even when restricted to manifolds with $g_{\mu\nu} = \delta_{\mu\nu}$. Indeed, for $\mathbb{T}^4$ and $\mathbb{T}^3 \times \mathbb{R}$, we see that this occurs when the lengths of the fundamental loops are sufficiently different, i.e., when there is sufficient inhomogeneity. Put another way, small universes are constrained to be highly symmetric.

In addition, we see that toric universes "prefer" to twist, in that this increases $\mathcal{S}$, thus avoiding the constraint of the size of the manifold. We see that varying each of these leads to a rich variety of behaviours. This moduli space is vast, and here hardly any of it is explored. For example, relatively little has been investigated about in what happens when more than 2 lengths are varied from the symmetric point, and nothing has been made of multiple directions for twisting. It seems that there is a fascinating interplay between varying the lengths of the fundamental loops and varying the twisting between them. Thus, despite $\mathbb{T}^4$ and $\mathbb{T}^3 \times \mathbb{R}$ being the simplest examples available to us, it is clear that there is much we don't know about their behaviour.

Assuming that the inhomogeneity effects described here survive a full theory of quantum gravity, we would have the beginnings of answers to questions about the history of the universe. Examples are the initial conditions required for inflation (small universes constrained to be symmetric) and the "Why now?" problem (why the energy densities for dark energy and matter are of the same order of magnitude now). Clearly these effects alone qualify this for further study.

Unfortunately, in [34], it is seen that to achieve diffeomorphism invariance, Λ_p must be sent to infinity. This removes the restriction on the overall size of the manifold. Nonetheless, the effects shown may yet find application in lattice field theory, see e.g., [45]. To the current author's knowledge, effects of twisting have not been examined in this sphere, so the work above may yield fruit in this direction as well.

Chapter 3

Parisi-Sourlas Supergravity

3.1 Introduction

This chapter is largely based on [6] and surrounding work.

We wish to construct an RG flow for gravity which is diffeomorphism invariant at all stages of the flow. The first stage (at the classical level) was developed in [50]. This chapter looks to take the first steps to extending this into a fully diffeomorphism RG flow. In order to do this, we want to introduce UV degrees of freedom so that when integrations are regulated at the effective cutoff scale Λ, diffeomorphism invariance is preserved. This was done in gauge theory over a period of years [33, 51–75] by extending the $SU(N)$ symmetry group to the supergroup $SU(N|N)$. This introduces extra fermionic gauge degrees of freedom which are spontaneously broken at Λ and then act as Pauli-Villars fields, and they regulate the gauge theory for all scales Λ. At large scales, the bosonic and fermionic degrees of freedom cancel each other, as with Parisi-Sourlas supersymmetry [76].

The natural way to implement this in the gravity case is therefore to extend the diffeomorphism symmetry to a superdiffeomorphism symmetry, which is done by extending the manifold to a supermanifold. Fortunately, supermanifolds have been extensively developed in the mathematical literature, e.g. [77]. The key idea is the the D bosonic coordinates of normal spacetime are extended to

$$x^A = (x^\mu, \theta^a) \tag{3.1}$$

where we've introduced D fermionic coordinates θ^a. These fermionic coordinates are different to the standard supergravity case [78, 79] since the θ^a are not treated as spinors under the bosonic Lorentz group, but rather vectors in their own space (and scalars under changes in bosonic coordinates). Crucially, as we have added new directions to the space, it makes sense that the metric is extended to a supermetric

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} & g_{\mu a} \\ g_{b\nu} & g_{ab} \end{pmatrix} \tag{3.2}$$

where $g_{\mu a} = g_{a\mu}$ and $g_{ab} = -g_{ba}$ and all indices run from 1 to D. Now, there are D^2 bosonic degrees of freedom ($D(D+1)/2$ from $g_{\mu\nu}$ and $D(D-1)/2$ from g_{ab}) which at UV scales we hope are cancelled by the D^2 wrong statistics[1] fermionic fields $g_{\mu a}$, which we thus expect to act as the Pauli-Villars fields in this case. We thus describe this as *Parisi-Sourlas supergravity* in order to distinguish it from other types of supergravity.

Having extended the manifold to a supermanifold, we will see how the linearised Einstein-Hilbert action is modified, and in particular we want to see what new degrees of freedom have been introduced. To account for superdiffeomorphism invariance, we will have to fix a gauge in order to calculate propagators, which will allow us to see what physical degrees of freedom we have, and which are gauge artefacts. In particular, we expect the "wrong statistics" terms to be spontaneously broken and given a mass of scale Λ. This would require adding a symmetry breaking scalar, as in the case of $SU(N|N)$ [33].

There are, however, hints that symmetry breaking may occur already, since the kinetic terms in our action are only diagonalisable with a mass scale M. However, this does not behave as a mass in the usual way. In particular, it does not result in changes to the position of poles in the propagators, and thus does not change the physical mass of any physical field, but is instead more subtle.

3.2 A review of supermanifolds

In this section, we introduce notation and nomenclature largely taken from [77] and define what we will need to construct a theory of gravity on a supermanifold. In theory, we can define a supermanifold with any number of bosonic and fermionic coordinates, but with the view to treating the new degrees of freedom as Pauli-Villars fields, we will have D of each. Therefore, we will work on the superspace $\mathbb{R}_c^D \times \mathbb{R}_a^D$. We use greek indices to label the coordinates in $\mathbb{R}_c^D$ (the commuting/bosonic coordinates) and lower case latin indices to label those in $\mathbb{R}_a^D$ (the anticommuting/fermionic coordinates). It will also be convenient to work with general indices $A = (\mu, a)$ that run over the whole superspace.

[1] Indeed, from the point of view of the "base manifold" (essentially the bosonic manifold), $g_{\mu a}$ are seen as fermionic vectors, in violation of the spin-statistics relation.

3.2.1 Vectors, matrices and indices

Just as a vector is a map from functions to functions in normal geometry, a supervector $\mathbf{X}$ does the same over a superspace. We define $\mathbf{X}$ to be "c-type" if it maps c-functions to c-functions and a-functions to a-functions. Similarly, we define $\mathbf{X}$ to be "a-type" if it maps c-functions to a-functions and vice versa. Using a "standard basis", a c-type supervector has c-numbers in the first D entries and a-numbers in the final D entries. Similarly, an a-type supervector has a-numbers in the first D places and c-numbers in the final D places.

One of the key notations we use involves indices and geometric objects as powers of (-1):

$$(-1)^A, \quad (-1)^{\mathbf{X}}, \quad (-1)^{\mathbf{X}A}. \tag{3.3}$$

When written this way, A should not be seen as an index (in the sense of the Einstein summation convention), but as a label for which $(-1)^A = 1$ for $A = \mu$ and $(-1)^A = -1$ for $A = a$. In addition, we say that $(-1)^{\mathbf{X}} = 1$ for $\mathbf{X}$ c-type and $(-1)^{\mathbf{X}} = -1$ for $\mathbf{X}$ a-type. Generally, any object or index appearing in a power of (-1) should be read as a shorthand for the $\mathbb{Z}_2$ Grassmann grading of the object (0 or 1)[2]. Note that if, for example $(-1)^A$ multiplies an expression with A free, then this is to be read as a multiplier that depends on whether $A = \mu$ or $A = a$. However, if A is summed over, then $(-1)^A$ scans over its possible values. For example,

$$(-1)^A X^A X_A = X^\mu X_\mu - X^a X_a. \tag{3.4}$$

Note that the above is only defined for "pure" supervectors, those that are c-type or a-type. However, any formulae involving these can be multilinearly extended for general supervectors since we can write these (uniquely) as a linear combination of a c-type supervector and an a-type supervector.

We will allow indices to be in four different positions: they can be left/right indices as well as up/down indices. This will denote slightly different transformation properties (see later). We also take the convention that in addition to only contracting indices when one is up and the other is down, that the "natural" contraction is between adjacent indices, with no object or other indices in between them, otherwise an index-dependent sign will appear.

A supervector space is defined over $\mathbb{R}_c^D \times \mathbb{R}_a^D$ in the same manner as one would define a normal vector space, only now left and right multiplication are different maps. From now on, we will use what DeWitt [77] calls a "standard basis" $\{{}_A\mathbf{e}\}$ which, under

[2]Sometimes this would be notated as, e.g., $\epsilon(\mathbf{X})$, but since for our purposes we can proceed without ambiguity, there is no need to introduce this here.

complex conjugation, behaves as

$$ {}_A\mathbf{e}^* = (-1)^A {}_A\mathbf{e}. \tag{3.5}$$

Thus, a real supervector $\mathbf{X} = X^A {}_A\mathbf{e} = \mathbf{X}^*$ must have components which satisfy

$$ X^{A*} = (-1)^{\mathbf{X}A} X^A. \tag{3.6}$$

This may not seem immediately obvious, so is worth spelling out:

$$ \mathbf{X}^* = (X^A {}_A\mathbf{e})^* = {}_A\mathbf{e}^* X^{A*} = (-1)^A {}_A\mathbf{e} X^{A*} = (-1)^A (-1)^{A(\mathbf{X}+A)} X^{A*} {}_A\mathbf{e} \tag{3.7}$$

where equality with $\mathbf{X}$ implies (3.6) using that $A + A$ is zero in $\mathbb{Z}_2$ (equivalently $(-1)^2 = 1$) and $A^2 = A$. We have also used that the Grassmann grading of X^A is $\mathbf{X} + A$ and that of ${}_A\mathbf{e}$ is A.

We are used to situations in which an index being up or down is sufficient to determine its transformation properties. For example, for a vector we have

$$ X^\mu \mapsto X'^\mu = X^\nu K^\mu{}_\nu = X^\nu \frac{\partial x^\mu}{\partial x^\nu} \tag{3.8}$$

which is unambiguous in the case of all objects commuting. However, for a supermanifold with non-commuting objects, we need to specify whether the Jacobian matrix K acts from the left or right. Suppose we start with a basis $\{{}_A\mathbf{e}\}$ and wish to change to a different basis (linearly). The index placement seems to indicate a Jacobian acting from the left, that is

$$ {}_A\mathbf{e} = {}_A K^B {}_B\bar{\mathbf{e}}. \tag{3.9}$$

Since $\mathbf{X} = X^A {}_A\mathbf{e} = \bar{X}^A {}_A\bar{\mathbf{e}}$ doesn't depend on a basis, we are thus led to define

$$ \bar{X}^A = X^B {}_B K^A. \tag{3.10}$$

If $\{{}_A\mathbf{e}\}$ and $\{{}_A\bar{\mathbf{e}}\}$ are standard bases, then it follows that K has the block diagonal form

$$ K = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.11}$$

where A and D are matrices of c-numbers and B and C are matrices of a-numbers. So we see that the degree of ${}_A K^B$ is $(-1)^{A+B}$. More generally, this is the case for c-type matrices, which map c-type supervectors to c-type supervectors and a-type supervectors to a-type supervectors and are the only type of matrix which we will consider.

Similarly to how, in normal differential geometry indices can be up or down regardless of the "natural" placement provided we transform consistently, we wish to define ${}^A X$. Clearly this would be done via transformation properties. Indeed, we would expect to see

$$ {}^A\bar{X} = {}^A K^{\sim}_{B}\, {}^B X \tag{3.12} $$

for some matrix $K^{\sim}$. Indeed, using the respective $\mathbb{Z}_2$ gradings of the components of $\mathbf{X}$ and K, we have

$$ \begin{aligned} \bar{X}^A = X^B{}_B K^A &= (-1)^{(\mathbf{X}+B)(A+B)}{}_B K^A X^B \\ &= (-1)^{\mathbf{X}A}(-1)^{B(A+B)}{}_B K^A (-1)^{\mathbf{X}B} X^B \end{aligned} \tag{3.13} $$

and so we are led to define

$$ {}^A X = (-1)^{\mathbf{X}A} X^A, \text{ and } {}^A K^{\sim}_{B} = (-1)^{B(A+B)}{}_B K^A \tag{3.14} $$

so that (3.12) holds. We define $K^{\sim}$ to be the *supertranspose* of K. Note that if we want ${}^A X$ to be components of a vector, we need a basis for vectors with indices arranged like this, say $\{\mathbf{e}_A\}$. Since we'd like to have $\mathbf{e}_A {}^A X = X^A{}_A\mathbf{e}$, we can use the above and we find

$$ \mathbf{e}_A = (-1)^A{}_A\mathbf{e}. \tag{3.15} $$

We have defined the supertranspose for one index placement, but we'd like to be able to define it more generally. Using similar logic to above, we can define

$$ \begin{aligned} {}_A L^{\sim B} = (-1)^{A(A+B)}{}^B L_A, \quad {}_A M^{\sim}{}_B = (-1)^{A+B+AB}{}_B M_A, \\ {}^A N^{\sim B} = (-1)^{AB}{}^B N^A. \end{aligned} \tag{3.16} $$

Note that in all cases, $K^{\sim\sim} = K$ and may be expected for a generalisation of transposition. We define a *supersymmetric* matrix to be one for which $K = K^{\sim}$.

Now we look at forms. Fortunately, given our basis for vectors $\{{}_A\mathbf{e}\}$, we have a dual basis $\{\mathbf{e}^A\}$ which acts as a basis for forms. Thus we have $\boldsymbol{\omega} = \mathbf{e}^A{}_A\omega$ where the degree of ${}_A\omega$ is $(-1)^{\boldsymbol{\omega}+A}$ and we define $\boldsymbol{\omega}(\mathbf{X}) = X^A{}_A\omega$. We could follow similar steps to above to derive rules for manipulating forms, but these can all be deduced from those for vectors. For instance, insisting on $X^A{}_A\omega = (-1)^{\boldsymbol{\omega}\mathbf{X}}\omega_A {}^A X$ tells us that

$$ \omega_A = (-1)^{A(\boldsymbol{\omega}+1)}{}_A\omega. \tag{3.17} $$

Note that equations (3.14) and (3.17) show that there are different rules for shifting indices depending on whether they are upstairs or downstairs indices. One can show through similar methods that this behaviour is carried to tensors. Since we only care about c-type matrices, we only show the index-shifting conventions for these. We have

$$K_A{}^B = (-1)^A {}_A K^B, \quad L^A{}_B = {}^A L_B, \quad M_{AB} = (-1)^A {}_A M_B, \quad N^{AB} = {}^A N^B \tag{3.18}$$

and this pattern can be extended for c-type tensors: moving an upstairs index is free, whereas moving a downstairs index A comes with a factor of $(-1)^A$. Note that we can only move the leftmost right index and the rightmost left index in this way. For any matrix with both of its indices on the right, we have

$$K^{\sim}_{AB} = (-1)^{AB} K_{BA} \tag{3.19}$$

as one would naively expect. In particular, a supersymmetric matrix must have

$$S_{AB} = (-1)^{AB} S_{BA}. \tag{3.20}$$

Note that (3.18) requires us to be somewhat careful. As an example, let us look at the Kronecker δ. This can be represented by

$$\delta^A{}_B, \quad {}^A\delta_B, \text{ or } {}_B\delta^A \tag{3.21}$$

but not by

$$\delta_B{}^A = (-1)^B {}_B\delta^A. \tag{3.22}$$

In addition, we have to be careful about inverses. For example, if we say N is the inverse of the matrix M, what we mean is

$${}^A N^B {}_B M_C = {}^A \delta_C \tag{3.23}$$

and other expressions must all be derived from this using (3.18) or symmetries of the matrices in question - we can't simply shift indices in the answer and hope that it works. In particular, in general (assuming M, N c-type)

$$N^{AB} M_{BC} = (-1)^B \, {}^A N^B {}_B M_C \neq \delta^A{}_C. \tag{3.24}$$

For a matrix with index positions ${}_A K^B$, we can define the *supertrace*:

$$\mathrm{str} K = (-1)^A {}_A K^A = K_A{}^A \tag{3.25}$$

and similarly for ${}^A L_B$:

$$\mathrm{str} L = (-1)^A \, {}^A L_A = (-1)^A L^A{}_A. \tag{3.26}$$

Note that with these conventions, we have

$$\mathrm{str}(MN) = (-1)^{MN} \mathrm{str}(NM) \tag{3.27}$$

which does not hold for the trace when applied to matrices with non-commuting elements. Note also that where the indices "naturally" live can have an effect on the behaviour (as seen when the indices are shifted to the right).

Using this, we define the superdeterminant, or Berezinian, by analogy with the relationship between the normal trace and determinant:

$$\delta \ln \mathrm{sdet} M = \mathrm{str}(M^{-1}\delta M) \tag{3.28}$$

with the condition that $\mathrm{sdet}\mathbb{I} = 1$. This means that we have

$$\mathrm{sdet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det(D)^{-1} \tag{3.29}$$

and in particular

$$\mathrm{sdet}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{\det(A)}{\det D}. \tag{3.30}$$

3.2.2 Derivatives, the metric and curvature

As always, a vector field is defined by its action on functions:

$$\mathbf{X}(f) = X^A \frac{\overrightarrow{\partial}}{\partial x^A} f = X^A{}_{A,}f. \tag{3.31}$$

Note that we've introduced the comma notation for derivatives. The generalisation is hopefully obvious - any indices further from the object than the comma indicate derivatives. In this instance, the derivative is acting from the left. We have to be careful, since now a common notation $f_{,A}$ means something slightly different:

$$f_{,A} = f\frac{\overleftarrow{\partial}}{\partial x^A} = (-1)^{A(f+1)}{}_{A,}f. \tag{3.32}$$

To define a metric on the supermanifold we require that it is a supersymmetric, real, c-type, non-degenerate, rank (0,2) tensor. This defines a natural inner product, which by supersymmetry satisfies

$$g(\mathbf{X},\mathbf{Y}) = X^A{}_A g_B{}^B Y = (-1)^{\mathbf{XY}} g(\mathbf{Y},\mathbf{X}). \tag{3.33}$$

We also have the inverse metric ${}^A g^B = g^{AB}$ which satisfies

$${}^A g^B{}_B g_C = {}^A\delta_C \text{ and } {}_A g_B{}^B g^C = {}_A\delta^C. \tag{3.34}$$

Now we can raise and lower indices, as well as shift the left and right. However, we must be careful to only use "natural" contractions, i.e., those between adjacent indices, to avoid any index-dependant signs. Thus we have

$$X_A = X^B{}_B g_A, \quad X^A = X_B{}^A g^B, \quad {}_A X = {}_A g_B{}^B X, \quad {}^A X = {}^A g^B{}_B X \tag{3.35}$$

i.e., left indices are raised/lowered with the second metric index whereas right indices are raised/lowered with the first index.

With a metric, we can define a line element. Since we have some non-commutativity, we have to be careful about precisely how to define this. Using only natural contractions is the way to go, and we get

$$ds^2 = dx^A{}_A g_B{}^B dx = dx^A{}_A g_B dx^B = (-1)^A dx^A g_{AB} dx^B. \tag{3.36}$$

Now that we have a metric, we can define a connection. We will use the natural generalisation of the Levi-Cevita connection, which is torsion free[3] and metric compatible:

$$\Gamma^A{}_{BC} = \frac{(-1)^D}{2} g^{AD} \left(g_{DB,C} + (-1)^{BC} g_{DC,B} - (-1)^{D(B+C)} g_{BC,D} \right). \tag{3.37}$$

Using these, we can define the Riemann curvature tensor

$$R^A{}_{BCD} = -\Gamma^A{}_{BC,D} + (-1)^{CD}\Gamma^A_{BD,C} + (-1)^{C(E+B)}\Gamma^A{}_{EC}\Gamma^C{}_{BD} - (-1)^{D(E+B+C)}\Gamma^A{}_{ED}\Gamma^E{}_{BC} \tag{3.38}$$

as well as the Ricci tensor and the Ricci scalar[4]

$$R_{AB} = (-1)^{C(A+1)} R^C{}_{ACB}, \quad R = R_{AB} g^{BA}. \tag{3.39}$$

Now we have all of the required pieces to construct the Einstein-Hilbert action for a supermanifold.

3.3 The super-Einstein-Hilbert action

We start from an action that we know to be superdiffeomorphism invariant, the super-Einstein-Hilbert action:

$$S_{EH} = -\frac{2}{\kappa^2} \int d^D x d^D \theta \sqrt{g} R \tag{3.40}$$

[3] That is to say, supersymmetric on the lower indices.

[4] Note that the Ricci scalar can be written as $R = (-1)^A{}_A R_B g^{BA} = \mathrm{str}(\mathrm{Ric}\ g^{-1})$, so this arrangement of indices is indeed the correct generalisation. Similar arguments hold for the connection coefficients and Riemann tensor.

where $\kappa = \sqrt{32\pi G}$, with G Newton's gravitational constant. The factor -2 will be ignored for now, and we have set the cosmological constant to zero. Since the metric is supersymmetric, we write it as

$$g_{AB} = \begin{pmatrix} g_{\mu\nu} & g_{\mu b} \\ g_{a\nu} & g_{ab} \end{pmatrix} \tag{3.41}$$

where $g_{\mu\nu} = g_{\nu\mu}$, $g_{\mu a} = g_{a\mu}$ and $g_{ab} = -g_{ab}$, as assumed in Section 3.1. Note that from the point of view of the base manifold, that spanned by the x_μ, $g_{\mu\nu}$ is a rank-2 tensor, as normal. However, the $g_{\mu a}$ are seen as 4 fermionic vectors, and g_{ab} are seen as 6 scalars.

3.3.1 Expansion

We wish to find the propagating degrees of freedom, and so we want to expand around a flat spacetime (that satisfies the vacuum equations). Normally, we'd like to use δ_{AB}, but this is not allowed by supersymmetry of g_{AB}. Instead we define

$$\bar{\delta}_{AB} = \begin{pmatrix} \delta_{\mu\nu} & 0 \\ 0 & \epsilon_{ab} \end{pmatrix} \tag{3.42}$$

with ϵ_{ab} some antisymmetric matrix. Note $\epsilon_{ab} = -{}_a\epsilon_b$. We thus expand:

$$g_{AB} = \bar{\delta}_{AB} + \kappa h_{AB} \tag{3.43}$$

and to find the propagating degrees of freedom, we expect to expand the super-Einstein-Hilbert action to $\mathcal{O}(\kappa^0)$, that is up to the bilinear terms in h.

For what follows, we will lower indices with ${}_A\bar{\delta}_B = (-1)^A \bar{\delta}_{AB}$, and raise indices with its inverse, $\bar{\delta}^{AB} = {}^A\bar{\delta}^B$. We thus have, to $\mathcal{O}(\kappa)$, that the inverse metric is

$$g^{AB} = \bar{\delta}^{AB} - \kappa h^{AB}. \tag{3.44}$$

With these conventions, it is consistent to raise bosonic indices $(\mu, \nu, \dots)$ with ${}^\mu\delta^\nu = \delta^{\mu\nu}$ and fermionic indices $(a, b, \dots)$ with ${}^a\epsilon^b = \epsilon^{ab}$, the matrix inverse of ${}_a\epsilon_b = -\epsilon_{ab}$.

From equation (3.28), we can see that, to $\mathcal{O}(\kappa)$, we have

$$\sqrt{g} = 1 + \frac{\kappa}{2}({}^A\bar{\delta}^B{}_B h_C) = 1 + \frac{\kappa}{2}(-1)^A h^A{}_A = 1 + \frac{\kappa}{2}(h^\mu{}_\mu - h^a{}_a). \tag{3.45}$$

Note that both g^{AB} and $\sqrt{g}$ are both only needed to $\mathcal{O}(\kappa)$, despite us wanting the $\mathcal{O}(\kappa^2)$ part of $\sqrt{g}R$. First, we observe that R is $\mathcal{O}(\kappa)$ anyway, so clearly $\sqrt{g}$ is only needed to $\mathcal{O}(\kappa)$. For g^{AB}, we note that this appears in the connection coefficients

(3.37) multiplying the differentiated metric, which is already $\mathcal{O}(\kappa)$. g^{AB} also appears in the Ricci scalar (3.39) multiplying the Riemann tensor, which again is $\mathcal{O}(\kappa)$.

For book-keeping purposes, we talk about metric fluctuations as having "bosonic" indices ($h_{\mu\nu}$), "mixed" indices ($h_{\mu a}$) or "fermionic" indices (h_{ab})[5]. With these 3 types of fluctuations, there are 6 types of bilinears that can be built, and thus we write[6]

$$\frac{\sqrt{g}R}{\kappa^2} = \mathcal{L}_{bb} + \mathcal{L}_{bm} + \mathcal{L}_{bf} + \mathcal{L}_{mm} + \mathcal{L}_{mf} + \mathcal{L}_{ff} + \mathcal{O}(\kappa). \tag{3.46}$$

Now we need to unpack the connection coefficients (3.37). We see that there are 6 different index structures (naively 8, but $\Gamma^{\nu}_{\mu a} = \Gamma^{\nu}{}_{a\mu}$ and $\Gamma^{b}{}_{\mu a} = \Gamma^{b}{}_{a\mu}$) and each of these contains 6 terms. We substitute these into (3.39) which give around a hundred terms, before collection. Note that for both the connection coefficients and the Riemann tensor, care is needed for dealing with the derivatives that arise since we have

$$\begin{aligned} g_{AB,C} &= g_{AB}\overleftarrow{\partial_C} = (-1)^{C(A+B+1)}\partial_C g_{AB} \\ \Gamma^{A}{}_{BC,D} &= \Gamma^{A}{}_{BC}\overleftarrow{\partial_D} = (-1)^{D(A+B+C+1)}\partial_D \Gamma^{A}{}_{BC} \end{aligned} \tag{3.47}$$

and we use this to change all derivatives to "standard" derivatives acting from the left. However, this comes with its own subtlety: although ∂_A looks like a right index, it is in fact a left index ${}_{A,}$. Therefore, we have to be careful with some contractions:

$$\partial_a \partial^a = -\partial^a \partial_a, \text{ as with } h^a{}_a = -h_a{}^a, \text{ but } \partial_a V^a = \partial^a V_a. \tag{3.48}$$

Taking care of all of these details, our final answer is

$$\begin{aligned} \mathcal{L}_{bb} &= \frac{1}{4}\partial_\rho h^\mu{}_\mu \partial^\rho h^\nu{}_\nu + \frac{1}{2}h^\rho{}_\rho \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{2}\partial_\nu h^{\mu\nu}\partial^\rho h_{\mu\rho} \\ &\qquad - \frac{1}{4}\partial_a h^\mu{}_\mu \partial^a h^\nu{}_\nu + \frac{1}{4}\partial_a h_{\mu\nu}\partial^a h^{\mu\nu}, \\ \mathcal{L}_{bm} &= -h^\mu{}_\mu \partial_\nu \partial_a h^{\nu a} - \partial^\nu h_{\mu\nu}\partial_a h^{\nu a}, \\ \mathcal{L}_{bf} &= \frac{1}{2}\partial_\nu \partial^\nu h^\mu{}_\mu h^a{}_a + \frac{1}{2}h^\mu{}_\mu \partial^b \partial_b h^a{}_a - \frac{1}{2}h^a{}_a \partial_\mu \partial_\nu h^{\mu\nu} - \frac{1}{2}h^\mu{}_\mu \partial_a \partial_b h^{ab}, \\ \mathcal{L}_{mm} &= -\frac{1}{2}\partial_\nu h_{\mu a}\partial^\nu h^{\mu a} - \frac{1}{2}\partial_\mu h^{\mu a}\partial^\nu h_{\nu a} - \frac{1}{2}\partial_b h_{\mu a}\partial^b h^{\mu a} + \frac{1}{2}\partial^a h_{\mu a}\partial_b h^{\mu b}, \\ \mathcal{L}_{mf} &= h^a{}_a \partial_\mu \partial_b h^{\mu b} + \partial^\mu h_{\mu a}\partial_b h^{ab}, \\ \mathcal{L}_{ff} &= \frac{1}{4}\partial_\mu h^a{}_a \partial^\mu h^b{}_b - \frac{1}{4}\partial_c h^a{}_a \partial^c h^b{}_b + \frac{1}{2}h^a{}_a \partial_b \partial_c h^{bc} + \frac{1}{4}\partial_\mu h_{ab}\partial^\mu h^{ab} \\ &\qquad - \frac{1}{4}\partial_c h_{ab}\partial^c h^{ab} + \frac{1}{2}\partial^b h_{ab}\partial_c h^{ac}. \end{aligned} \tag{3.49}$$

[5]Note that this does not imply anything about the statistics of the field. For example, the fluctuations h_{ab} have fermionic indices but are bosonic fields.

[6]Note that there are κ^{-1} terms, but these are necessarily total derivatives as there is only one fluctuation field in these terms, so these are discarded - subject to suitable boundary conditions, which are implicitly assumed.

There are a few things to note here. Firstly, these terms are only defined up to integration by parts. Indeed this has been used to simplify expressions in many cases. Thus we have implicitly assumed something about the boundary conditions at infinity. This is worth bearing in mind, but will not be addressed further in this thesis. Also, if we delete all terms with fermionic indices (including derivatives) we recover the standard Einstein-Hilbert action, as we might expect. We also note that not only is every possible contraction represented, but there is a pleasing symmetry between types of indices - e.g., compare $\mathcal{L}_{bb}$ and $\mathcal{L}_{ff}$ (up to some signs, which can be accounted for by arranging terms/indices differently). These both point in the direction of this being the correct answer, and we also have another check in the form the Lie derivative to check that the correct linearised symmetries are in fact respected.

3.3.2 Super-diffeomorphism invariance

First, we must define the Lie derivative and its action on various fields. Indeed, we have [77]

$$\mathcal{L}_\xi f = \xi(f) \tag{3.50}$$

$$\mathcal{L}_\xi X = [\xi, X] \tag{3.51}$$

$$\mathcal{L}_\xi(T(X,Y)) = (\mathcal{L}_\xi T)(X,Y) + (-1)^{\xi T} T(\mathcal{L}_\xi X, Y) + (-1)^{\xi(T+X)} T(X, \mathcal{L}_\xi Y) \tag{3.52}$$

where ξ, X and Y are vector fields, f is a function and T is a rank (0,2) tensor field. From these rules (and a generalisation of the third) the Lie derivative for any tensor field can be calculated.

We wish (3.49) to be invariant under linearised diffeomorpisms:

$$_A h_B \mapsto {}_A h_B + {}_A(\mathcal{L}_\xi \bar{\delta})_B \tag{3.53}$$

for any c-type vector field ξ, as these generate the superdiffeomorphism algebra SDiff [77]. In order to apply this to (3.49), we will need coordinate expressions for the action of the Lie derivative. First we note that we have

$$(\mathcal{L}_\xi X)^A = \xi^B{}_{B,} X^A - X^B{}_{,B}\xi^A \text{ and } {}^A(\mathcal{L}_\xi X) = -{}^A\xi_{,B}{}^B X + {}^A X_{,B}{}^B\xi \tag{3.54}$$

Using (3.50)−(3.52), we see that, for a c-type $(0,2)$ tensor T,

$$\begin{aligned} &\xi^C{}_{C,} X^A{}_A T_B{}^B Y + X^A \xi^C{}_{C,A} T_B{}^B Y + X^A{}_A T_B\, \xi^C{}_{C,}{}^B Y = \\ &X^A{}_A(\mathcal{L}_\xi T)_B{}^B Y + (\xi^C{}_{C,} X^A - X^C{}_{C,}\xi^A)_A T_B{}^B Y + X^A{}_A T_B(-{}^B\xi_{,C}{}^C Y + {}^B Y_{,C}{}^C\xi). \end{aligned} \tag{3.55}$$

Now, the terms with X differentiated evidently cancel. We can also show with our index shifting rules is Section 3.2 that

$$\xi^C{}_{C,}{}^B Y = {}^B Y_{,C}{}^C\xi \tag{3.56}$$

and so the terms in which Y is differentiated cancel. After this, both the LHS and RHS are multiplied on the left by X^A and on the right by ${}^B Y$, with X, Y arbitrary. Therefore, we can strip off these vector fields and we find that

$${}_A(\mathcal{L}_\xi T)_B = \xi^C{}_{C,A} T_B + {}_{A,}\xi^C{}_C T_B + {}_A T_C{}^C \xi_{,B} \tag{3.57}$$

and so we have that

$${}_A(\delta_\xi h)_B = {}_A(\mathcal{L}_\xi \bar{\delta})_B = {}_{A,}\xi_B + {}_{B,}\xi_A. \tag{3.58}$$

Specialising to the different types of fluctuation, we have

$$\begin{aligned} \delta_\xi h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \delta_\xi h_{\mu a} &= \partial_\mu \xi_a - \partial_a \xi_\mu \\ \delta_\xi h_{ab} &= -\partial_a \xi_b + \partial_b \xi_a. \end{aligned} \tag{3.59}$$

Since at the linearised level, (3.58) is still a tensor, we can move the indices with $\bar{\delta}$ (equivalently δ and ϵ) so that we get[7]

$$\begin{aligned} \delta_\xi h^{\mu\nu} &= \partial^\mu \xi^\nu + \partial^\nu \xi^\mu \\ \delta_\xi h^{\mu a} &= \partial^\mu \xi^a + \partial^a \xi^\mu \\ \delta_\xi h^{ab} &= \partial^a \xi^b - \partial^b \xi^a \end{aligned} \tag{3.60}$$

and finally, for the trace terms we have

$$\delta_\xi h^\mu{}_\mu = 2\partial_\mu \xi^\mu, \quad \delta_\xi h^a{}_a = 2\partial_a \xi^a. \tag{3.61}$$

[7] We could alternatively derive these expressions by considering the Lie derivative acting on a (2,0) tensor. The answer obtained in this manner is identical.

All of this put together means that we have (up to integration by parts) the following variation of (3.49)

$$\begin{aligned}
\delta_\xi \mathcal{L}_{bb} &= -\partial_a h^\mu{}_\mu \partial^\nu \partial^a \xi_\nu + \partial_a h_{\mu\nu} \partial^a \partial^\mu \xi^\nu, \\
\delta_\xi \mathcal{L}_{bm} &= -\partial_\mu \xi^\mu \partial_\nu \partial_a h^{\nu a} - h^\mu{}_\mu \partial_\nu \partial_a (\partial^\nu \xi^a + \partial^a \xi^\nu) - \partial_\nu \partial^\nu \xi_\mu \partial_a h^{\mu a} \\
&\qquad - \partial^\nu h_{\mu\nu} \partial_a (\partial^\mu \xi^a + \partial^a \xi^\mu), \\
\delta_\xi \mathcal{L}_{bf} &= \partial_\rho \partial^\rho h^\mu{}_\mu \partial_a \xi^a + \partial_\mu \xi^\mu \partial^b \partial_b h^a{}_a - \partial_a \xi^a \partial_\mu \partial_\nu h^{\mu\nu} - \partial_\mu \xi^\mu \partial_a \partial_b h^{ab}, \\
\delta_\xi \mathcal{L}_{mm} &= -\partial_\nu h_{\mu a} \partial^\nu (\partial^\mu \xi^a + \partial^a \xi^\mu) - \partial_\nu h^{\nu a} \partial^\mu (\partial_\mu \xi_a - \partial_a \xi_\mu) \\
&\qquad - \partial_b h_{\mu a} \partial^b (\partial^\mu \xi^a + \partial^a \xi^\mu) + \partial^a h_{\mu a} \partial_b (\partial^\mu \xi^b + \partial^b \xi^\mu), \\
\delta_\xi \mathcal{L}_{mf} &= \partial_a \xi^a \partial_\mu \partial_b h^{\mu b} + h^a{}_a \partial_\mu \partial_b (\partial^\mu \xi^b + \partial^b \xi^\mu) - \partial^\mu h_{\mu a} \partial_b \partial^b \xi^a \\
&\qquad + \partial^\mu (\partial_\mu \xi_a + \partial_a \xi_\mu) \partial_b h^{ab}, \\
\delta_\xi \mathcal{L}_{ff} &= \partial_\mu h^a{}_a \partial^\mu \partial_b \xi^b + \partial_\mu h_{ab} \partial^\mu \partial^a \xi^b.
\end{aligned} \tag{3.62}$$

and one can see that adding all of these together results in a total derivative, and so (3.49) is invariant under linearised diffeomorphisms. It is worth noting that this linearised super-diffeomorphism invariance is to expected already from the form of the action that we started with, so this calculation mainly functions as a consistency check of the calculation of the linearised action.

3.4 Field Decomposition

Even though we have been expecting from the start to specialise to $D = 4$, all of the above is clearly valid in D dimensions, as can be seen by the fact that the Lie derivative makes no assumptions on generality and hence (3.49) is diffeomorphism invariant in D dimensions. From this point, we will be specialising to $D = 4$.

Up until this point, all of our fields have been dependent on both x and θ. However, if we want to understand our theory from the perspective of the base manifold, we need to perform the integral over the θ coordinates in the action. To do this, it is convenient to Taylor expand:

$$\begin{aligned}
h(x,\theta) = h(x) + M\theta^a h_{|a}(x)x + M^2\theta^a\theta^b h_{|ab}(x) \\
+ M^3\theta^a\theta^b\theta^c h_{|abc}(x) + M^4\theta^a\theta^b\theta^c\theta^d h_{|abcd}(x).
\end{aligned} \tag{3.63}$$

Note that the expansion terminates due to there only being 4 Grassmann-odd coordinates (if we have 5 θ's, at least one is repeated and $(\theta^a)^2 = 0$). Note that we have also suppressed the spacetime indices on h, as this are not relevant here. Where confusion between $h(x,\theta)$ and $h(x)$ is unlikely, the argument of h will not be specified.

Note also that we have introduced a scale M into our theory. This is because $[\theta^a] = -1$ and, anticipating the need to diagonalise the resulting action, we wish to have all of our fields have $[h|_{\dots}] = 1$. In addition it is worth noting that we have absorbed some numerical factors in comparison to the Taylor series coefficients:

$$M^n h_{|a_1 \dots a_n} = \frac{1}{n!} \partial_{a_n} \dots \partial_{a_1} h|_{\theta=0} \tag{3.64}$$

and this difference is purely to make the numbers simpler in what follows.

In order to perform the $d^4\theta$ integral, we need to establish a convention for doing so. Indeed, integrating over a Grassmann variable is the same as differentiating with respect to it up to some constant multiplier, which we are free to choose by convention. Since $\theta^a\theta^b\theta^c\theta^d$ must be totally antisymmetric with respect to its 4 indices, we know that its integral must be proportional to ϵ^{abcd}. The convention that is most convenient for us is to choose

$$\int d^4\theta\, \theta^a\theta^b\theta^c\theta^d = M^{-4}\epsilon^{abcd}. \tag{3.65}$$

This factor ensures that the dimension one fields come out with correctly normalised kinetic terms.

The final piece of machinery which will assist this decomposition is a Hodge dual over the θ-space[8]. The formulae that will be most useful to us will be

$$*h = \epsilon^{abcd} h_{|abcd}, \quad *h^{|a} = \epsilon^{abcd} h_{|bcd}, \quad *h^{|ab} = \frac{1}{2}\epsilon^{abcd} h_{|cd} \tag{3.66}$$

and we also define for completeness

$$*h^{|abc} = \frac{1}{6}\epsilon^{abcd} h_{|d}, \quad *h^{|abcd} = \frac{1}{24}\epsilon^{abcd} h. \tag{3.67}$$

In addition, we define the dual with lower index expressions in the same way, just with ϵ_{abcd} instead. What this means is that we have

$$*(*h)_{|a_1 \dots a_n} = (-1)^n h_{|a_1 \dots a_n} \tag{3.68}$$

as would be expected for Hodge dual in an even number of dimensions. We will also need to make use of the following:

$$\epsilon^{i_1 \dots i_k \dots i_n} \epsilon_{i_i \dots i_k j_{k+1} \dots j_n} = k!\, \delta^{i_k \dots i_n}_{j_k \dots j_n} \tag{3.69}$$

where the generalised Kronecker δ is the sums of products of $\delta^{i_n}_{j_m}$ along with sign to get from the i_n to the j_m. In particular, for our purposes we will use

$$\begin{gathered} \epsilon^{abcd}\epsilon_{abcd} = 24, \quad \epsilon^{abcd}\epsilon_{abce} = 6\delta^d_e, \quad \epsilon^{abcd}\epsilon_{abef} = 2(\delta^c_e\delta^d_f - \delta^c_f\delta^d_e), \\ \epsilon^{abcd}\epsilon_{aefg} = \delta^b_e\delta^c_f\delta^d_g + \delta^b_f\delta^c_g\delta^d_e + \delta^b_g\delta^c_e\delta^d_f - \delta^b_e\delta^c_g\delta^d_f - \delta^b_f\delta^c_e\delta^d_e - \delta^b_g\delta^c_e\delta^d_f. \end{gathered} \tag{3.70}$$

[8]Since the space of the θ coordinates is trivial, there are therefore no topological issues that arise from this.

Using these, we see that for two metric components h and h', we have (discarding any terms which are zero or give zero on integration)

$$M^{-4}\partial_\mu h \partial_\nu h' = \partial_\mu h \theta^a\theta^b\theta^c\theta^d \partial_\nu h'_{|abcd} + \theta^a \partial_\mu h_{|a} \theta^b\theta^c\theta^d \partial_\nu h'_{bcd} + \theta^a\theta^b \partial_\mu h_{|ab} \theta^c\theta^d \partial_\nu h_{|cd} + \theta^a\theta^b\theta^c \partial_\mu h_{|abc} \theta^d \partial_\nu h_{|d} + \theta^a\theta^b\theta^c\theta^d \partial_\mu h_{|abcd} \partial_\nu h' \quad (3.71)$$

Pulling the θ factors outside so we can more easily perform the integration gives us

$$M^{-4}\partial_\mu h \partial_\nu h' = \theta^a\theta^b\theta^c\theta^d (\partial_\mu h \partial_n h'_{|abcd} + (-1)^{h+1} \partial_\mu h_{|a} \partial_\nu h'_{|bcd} + \partial_\mu h_{|ab} \partial_\nu h'_{|cd} + (-1)^{h+1} \partial_\mu h_{|abc} \partial_\nu h'_{|a} + \partial_\mu h_{|abcd} \partial_\nu h'). \quad (3.72)$$

Note that $(-1)^h$ should be read as the grading of the *component* of h in question, and *not* the grading of h itself (which would simply give $(-1)^h = 1$). Now, performing the integral and using the Hodge dual, we arrive at[9]

$$\int d^4\theta\, \partial_\mu h \partial_\nu h' = \partial_\mu h * \partial_\nu h' + (-1)^{h+1} \partial_\mu h_{|a} * \partial_\nu h'^{|a} + 2\partial_\mu h_{|ab} * \partial_\nu h'^{|ab} + (-1)^h * \partial_\mu h^{|a} \partial_\nu h'_{|a} + * \partial_\mu h \partial_\nu h'. \quad (3.73)$$

Similarly we can derive similar formulae for the expansion of other types of terms:

$$\int d^4\theta\, \partial_\mu h \partial_a h' = (-1)^{h+1} M \partial_\mu h_{|a} * h' - 2M \partial_\mu h_{|ab} * h'^{|b} + 2(-1)^h M * \partial_\mu h^{|b} h'_{|ab} + M * \partial_\mu h h'_{|a} \quad (3.74)$$

and the related formula which can be derived as above or by symmetry:

$$\int d^4\theta\, \partial_a h \partial_\mu h' = M h_{|a} * \partial_\mu h' + 2(-1)^h M h_{|ab} * \partial_\mu h'^{|b} - 2M * h^{|b} \partial_\mu h'_{|ab} - (-1)^h M * h \partial_\mu h_{|a}. \quad (3.75)$$

Finally, we have

$$\int d^4\theta\, \partial_a h \partial_b h' = 2(-1)^h M^2 h_{|ab} * h' - M^2 \epsilon_{abcd} * h^{|c} * h'^{|d} + 2(-1)^h M^2 * h h'_{|ab}. \quad (3.76)$$

[9]We have implicitly assumed $\partial_\mu * h = * \partial_\mu h$.

Using these, we can now expand our action in terms of fields which depend only on x and not θ. Indeed, we have

$$\begin{aligned}\mathcal{L}_{bb} = &2\partial_\mu\varphi * \partial^\mu\varphi - 2\partial_\mu\varphi_{|a} * \partial^\mu\varphi^{|a} + 2\partial_\mu\varphi_{|ab} * \partial^\mu\varphi^{|ab} - \varphi_{|a} * \partial_\mu\partial_\nu h_{\mu\nu}{}^{|a} + 2\varphi_{|ab} * \partial_\mu\partial_\nu h^{|ab} \\ &+ *\varphi^{|a}\partial_\mu\partial_\nu h^{\mu\nu}{}_{|a} + *\varphi\partial_\mu\partial_\nu h^{\mu\nu} - \frac{1}{2}\partial_\rho h_{\mu\nu} * \partial^\rho h^{\mu\nu} + \frac{1}{2}\partial_\rho h_{\mu\nu|a} * \partial^\rho h^{\mu\nu|a} \\ &- \frac{1}{2}\partial_\rho h_{\mu\nu|ab} * \partial^\rho h^{\mu\nu|ab} + \partial^\nu h_{\mu\nu} * \partial_\rho h^{\mu\rho} - \partial^\nu h_{\mu\nu|a} * \partial_\rho h^{\mu\rho|a} + \partial^\nu h_{\mu\nu|ab} * \partial_\rho h^{\mu\rho|ab} \\ &- \epsilon^{ab}M^2\left(4\varphi_{|ab} * \varphi - \epsilon_{abcd} * \varphi^{|c} * \varphi^{|d} - h_{\mu\nu|ab} * h^{\mu\nu} + \frac{1}{4}\epsilon_{abcd} * h_{\mu\nu}{}^{|c} * h^{\mu\nu|d}\right)\end{aligned} \tag{3.77}$$

where we've defined

$$\varphi = \frac{1}{2}h^\mu{}_\mu. \tag{3.78}$$

Similarly, we have

$$\begin{aligned}\mathcal{L}_{bm} = M\Big(&2\partial_\mu\varphi_{|a} * h^{\mu a} - 4\partial_\mu\varphi_{|ab} * h^{\mu a|b} - 4 * \partial_\mu\varphi^{|b}h^{\mu a}{}_{|ab} + 2 * \partial_\mu\varphi h^{\mu a}{}_{|a} \\ &- \partial^\nu h_{\mu\nu|a} * h^{\mu a} + 2\partial^\nu h_{\mu\nu|ab} * h^{\mu a|b} + 2 * \partial^\nu h_{\mu\nu}{}^{|b}h^{\mu a}{}_{|ab} - *\partial^\nu h_{\mu\nu}h^{\mu a}{}_{|a}\Big),\end{aligned} \tag{3.79}$$

$$\begin{aligned}\mathcal{L}_{bf} = &2\varphi * \Box\chi - 2\varphi_{|a} * \Box\chi^{|a} + 4\varphi_{|ab} * \Box\chi^{|ab} + 2 * \varphi^{|a}\Box\chi_{|a} + 2 * \varphi\Box\chi + \partial_\mu\chi * \partial_\nu h^{\mu\nu} \\ &- \partial_\mu\chi_{|a} * \partial_\nu h^{\mu\nu|a} + 2\partial_\mu\chi_{|ab} * \partial_\nu h^{\mu\nu|ab} + *\partial_\mu\chi^{|a}\partial_\nu h^{\mu\nu}{}_{|a} + *\partial_\mu\chi\partial_\nu h^{\mu\nu} \\ &+ M^2\Big(\epsilon^{ab}\left[4\chi_{|ab} * \varphi - 2\epsilon_{abcd} * \chi^{|c} * \varphi^{|d} + 4 * \chi\varphi_{|ab}\right] \\ &\qquad + 2\varphi_{|ab} * h^{ab} - \epsilon_{abcd} * \varphi^{|c} * h^{ab|d} + 2 * \varphi h^{ab}{}_{|ab}\Big)\end{aligned} \tag{3.80}$$

where we've defined

$$\chi = \frac{1}{2}h^a{}_a \tag{3.81}$$

and we've used the fact that, in general

$$f_{|ab} * g^{|ab} = *f^{|ab}g_{|ab}. \tag{3.82}$$

Looking at the remaining parts of the action, we have

$$\begin{aligned}\mathcal{L}_{mm} = &-\partial_\nu h_{\mu a} * \partial^\nu h^{\mu a} - \partial_\nu h_{\mu a|b} * \partial^\nu h^{\mu a|b} - \partial_\nu h_{\mu a|bc} * \partial^\nu h^{\mu a|bc} \\ &- \partial^\mu h_{\mu a} * \partial^\nu h_{\nu a} - \partial^\mu h_{\mu a|b} * \partial_\nu h^{\nu a|b} - \partial^\mu h_{\mu a|bc} * \partial_\nu h^{\nu a|bc} \\ &- \frac{1}{2}M^2\epsilon^{ab}\left(4 * h^{\mu c}h_{\mu c|ab} - \epsilon_{abcd} * h^{\mu e|c} * h_{\mu e}{}^{|d} - 4h_\mu{}^a{}_{|ab} * h^{\mu b} + \epsilon_{abcd} * h_\mu{}^{a|c} * h^{\mu b|d}\right),\end{aligned} \tag{3.83}$$

$$\mathcal{L}_{mf} = M\Big(2\partial_\mu\chi_{|a} *h^{\mu a} + 4\partial_\mu\chi_{|ab} *h^{\mu a|b} - 4 *\partial_\mu\chi^{|b}h^{\mu a}{}_{|ab} - 2 *\partial_\mu\chi h^{\mu a}{}_{|a}$$
$$+ \partial^\mu h_{\mu a|b} *h^{ab} - 2\partial^\mu h_{\mu a|bc} *h^{ab|c} - 2 *\partial^\mu h_{\mu a}{}^{|c}h^{ab}{}_{|bc} + *\partial^\mu h_{\mu a}h^{ab}{}_{|b}\Big), \quad (3.84)$$

$$\begin{aligned}\mathcal{L}_{ff} = {} & 2\partial_\mu\chi *\partial^\mu\chi - 2\partial_\mu\chi_{|a} *\partial^\mu\chi^{|a} + 2\partial_\mu\chi_{|ab} *\partial^\mu\chi_{|ab} \\ & + \frac{1}{2}\partial_\mu h_{ab} *\partial^\mu h^{ab} - \frac{1}{2}\partial_\mu h_{ab|c} *\partial^\mu h^{ab|c} + \frac{1}{2}\partial_\mu h_{ab|cd} *\partial^\mu h^{ab|cd} \\ & - M^2\epsilon^{ab}\left(4\chi_{|ab} *\chi - \epsilon_{abcd} *\chi^{|c} *\chi^{|d} + h_{cd|ab} *h^{cd} - \frac{1}{4}\epsilon_{abcd} *h_{ef}{}^{|c} *h^{ef|d}\right) \\ & + M^2\Big(2\chi_{|ab} *h^{ab} + 2 *\chi h^{ab}{}_{|ab} - \epsilon_{abcd} *\chi^{|c} *h^{ab|d} \\ & \qquad - 2h_a{}^b{}_{|bc} *h^{ac} + \frac{1}{2}\epsilon_{cdef} *h_a{}^{c|e} *h^{ad|f}\Big).\end{aligned} \quad (3.85)$$

Clearly, as a result of this field decomposition, we have a lot of fields to deal with. However, all is not lost, as many of them are gauge. In order to get a better feel for the degrees of freedom that are genuine, and to be able to compute propagators, we now wish to fix a gauge.

3.5 Gauge fixing

In order to see this system as that of a graviton plus extra fields which are part of the regulator structure, we will hold off on fixing $\xi_\mu(x)$ for the time being, instead focusing on using $\xi_a(x,\theta)$ and the other components of $\xi_\mu(x,\theta)$. In addition, we wish to use as many "unitary gauge" style choices as we can, by way of analogy to spontaneous symmetry breaking. This means we focus on local algebraic elimination, rather than the inversion of differential operators. Using the results of Section 3.3.2 we have that

$$\delta_\xi h_{ab} = 2M\xi_{[a|b]}, \quad \delta_\xi h_{ab|c} = 4M\xi_{[a|b]c}, \quad \delta_\xi h_{ab|cd} = 6M\xi_{[a|b]cd}, \quad \delta_\xi h_{ab|cde} = 8M\xi_{[a|b]cde},$$
$$\delta_\xi h_{ab|cdef} = 0. \quad (3.86)$$

Other than $h_{ab|cdef}$, which usually appears as $*h_{ab}$ and is gauge-invariant, each of the terms on the RHS take then general form for a function that is antisymmetric in a, b. Therefore, we can fix $\xi_{[a|b]}$ so that each of $h_{ab|\ldots} = 0$ (expect $*h_{ab}$). Furthermore, since there is no 3-index (or more) object with symmetry on its first two indices and antisymmetry on its second 2 indices as we'd have

$$T_{abc} = -T_{acb} = -T_{cab} = T_{cba} = T_{bca} = -T_{bac} = -T_{abc} \quad (3.87)$$

and hence $T_{abc} = 0$, we have that $\xi_{(a|b)c\ldots} = 0$. Therefore we only have $\xi_{(a|b)}$ and ξ_a left to fix, as well as $\xi_{\mu|\ldots}$. Looking at the gauge transformation for the mixed index fields, we have

$$\delta_\xi h_{\mu a} = \partial_\mu \xi_a - M\xi_{\mu|a}, \quad \delta_\xi h_{\mu a|b} = \partial_\mu \xi_{a|b} - 2M\xi_{\mu|ab}, \quad \delta_\xi h_{\mu a|bc} = \partial_\mu \xi_{a|bc} - 3M\xi_{\mu|abc},$$
$$\delta_\xi h_{\mu a|bcd} = \partial_\mu \xi_{a|bcd} - 4M\xi_{\mu|abcd}, \quad \delta_\xi h_{\mu a|bcde} = \partial_\mu \xi_{a|bcde} \tag{3.88}$$

and so $\xi_{\mu|a}$ can be fixed to eliminate $h_{\mu a}$. Similarly, we can fix $\xi_{\mu|ab\ldots}$ to eliminate $h_{\mu[a|b]\ldots}$ except for $h_{\mu[a|b]cde}$, equivalently $*h_{\mu a}$. However, by the symmetry argument above, we have that $h_{\mu(a|b)c\ldots} = 0$. Therefore, with the exceptions of $*h_{\mu a}$ and $h_{\mu(a|b)}$, have eliminated $h_{\mu a|\ldots}$. In addition, since $\xi_{a|bcde}$ is already fixed from above, $*h_{\mu a}$ is now gauge invariant. After all of this, we have the residual gauge invariance

$$\delta_\xi h_{\mu(a|b)} = \partial_\mu \xi_{(a|b)}. \tag{3.89}$$

Now, ξ_a has not been fixed and so will still generate a gauge invariance. However, we see from above that, in order to preserve $h_{\mu a} = 0$, any change in $\delta\xi_a = \xi'_a$ requires a corresponding change in $\delta\xi_{\mu|a} = \xi'_{\mu|a}$ such that

$$\xi'_{\mu|a} = \frac{1}{M}\partial_\mu \xi'_a. \tag{3.90}$$

We are still yet to fix $\xi_\mu(x)$ and $\xi_a(x)$ (equivalently $\xi_{\mu|a}$), but it is instructive at this point to see the effect of this gauge fixing on our action. Clearly, $\mathcal{L}_{bb}$ is unaffected, but we have $\mathcal{L}_{mm} = \mathcal{L}_{ff} = 0$, since each bilinear has at least one part which now vanishes. In addition, since now $h^{\mu a}{}_{|a} = h^{\mu}{}_{b|a}\epsilon^{ba} = h^{\mu}{}_{[b|a]}\epsilon^{ba} = 0$, we also have $\mathcal{L}_{mf} = 0$. The remaining parts are therefore $\mathcal{L}_{bb}$ as well as

$$\mathcal{L}_{bm} = M\left(2\partial_\mu \varphi_{|a} *h^{\mu a} - \partial^\nu h_{\mu\nu|a} *h^{\mu a}\right) \tag{3.91}$$

and

$$\mathcal{L}_{bf} = 2\varphi\Box *\chi + *\partial_\mu \chi \partial_\nu h^{\mu\nu} + 4M^2\epsilon^{ab} *\chi\varphi_{|ab} + 2M^2\varphi_{|ab} *h^{|ab}. \tag{3.92}$$

Note that we still have the invariance (3.89), but at this level, $h_{\mu a|b}$ plays no further role with our gauge choice. At the interacting level, however, it may result in Lagrange multipliers, leading to crucial constraints. Keeping $\xi_\mu(x)$ free, the only remaining gauge invariance is

$$\delta_\xi h_{\mu\nu|a} = \partial_\mu \xi_{\nu|a} + \partial_\nu \xi_{\mu|a} = \frac{2}{M}\partial_\mu \partial_\nu \xi_a \tag{3.93}$$

where we've used the condition (3.90). This means in particular that

$$\delta_\xi \varphi_{|a} = \frac{2}{M}\Box\xi_a \tag{3.94}$$

and so, by use of a Green's function (which is inherently non-local), we can set $\varphi_{|a} = 0$. Since we still have the usual $\xi_\mu(x)$ to play with, we can also specialise to the usual transverse and traceless gauge for $h_{\mu\nu}$.

This means that finally, we have our gauge fixed Lagrangian, split into G(rassmann)-even and G-odd fields:

$$\mathcal{L} = \mathcal{L}_e + \mathcal{L}_o \tag{3.95}$$

where

$$\begin{aligned}\mathcal{L}_e = {}& 2\partial_\mu\varphi_{|ab} *\partial^\mu\varphi^{|ab} + 2\varphi_{|ab} *\partial_\mu\partial_\nu h^{\mu\nu|ab} - \frac{1}{2}\partial_\rho h_{\mu\nu} *\partial^\rho h^{\mu\nu} \\ & - \frac{1}{2}\partial_\rho h_{\mu\nu|ab} *\partial^\rho h^{\mu\nu|ab} + \partial^\nu h_{\mu\nu|ab} *\partial_\rho h^{\mu\rho|ab} - 4M^2\epsilon^{ab}\varphi_{|ab} *\varphi \\ & + M^2\epsilon^{ab}h_{\mu\nu|ab} *h^{\mu\nu} + 4M^2\epsilon^{ab}\varphi_{|ab} *\chi + 2M^2\varphi_{|ab} *h^{ab}\end{aligned} \tag{3.96}$$

is the Lagrangian for the bosonic (G-even) sector, and

$$\begin{aligned}\mathcal{L}_o = {}& *\varphi^{|a}\partial^\mu\partial^\nu h_{\mu\nu|a} + \frac{1}{2}\partial_\rho h_{\mu\nu|a} *\partial^\rho h^{\mu\nu|a} - \partial^\nu h_{\mu\nu|a} *\partial_\rho h^{\mu\rho|a} \\ & - M\partial^\nu h_{\mu\nu|a} *h^{\mu a} + M^2\epsilon^{ab}\epsilon_{abcd} *\varphi^{|c} *\varphi^{|d} - \frac{1}{4}M^2\epsilon^{ab}\epsilon_{abcd} *h_{\mu\nu}{}^{|c} *h^{\mu\nu|d}.\end{aligned} \tag{3.97}$$

is the Lagrangian for the fermionic (G-odd) sector.

3.6 Degrees of freedom

3.6.1 Bosonic sector

First, we note that $*h_{ab}$ appears only linearly in our Lagrangian (to this order), so can be seen as a Lagrange multiplier enforcing the constraint (ensuring we recall that $*\chi$ is part of $*h^{ab}$)

$$\varphi_{|ab} + \epsilon_{ab}\epsilon^{cd}\varphi_{|cd} = 0. \tag{3.98}$$

Contracting this with ϵ^{ab} and using $\epsilon_{ab}\epsilon^{ab} = 4$[10] tells us that

$$\epsilon^{ab}\varphi_{|ab} = 0 \tag{3.99}$$

and feeding this into (3.98) tells us that $\varphi_{|ab} = 0$, and as such we can also take $h_{\mu\nu|ab}$ as being traceless on its first two indices. Therefore, we have

$$\mathcal{L}_e = -\frac{1}{2}\partial_\rho h_{\mu\nu} * \partial^\rho h^{\mu\nu} - \frac{1}{2}\partial_\rho h_{\mu\nu|ab} * \partial^\rho h^{\mu\nu|ab} + \partial^\nu h_{\mu\nu|ab} * \partial_\rho h^{\mu\rho|ab} + M^2\epsilon^{ab}h_{\mu\nu|ab} * h^{\mu\nu}. \tag{3.100}$$

In order to diagonalise the remaining terms, it will be helpful to write

$$h_{\mu\nu|ab} = \frac{1}{2}\epsilon_{ab}h^{\parallel}_{\mu\nu} + h^{\perp}_{\mu\nu|ab} \tag{3.101}$$

[10] Note that $\epsilon^{ab}\epsilon_{ab} = -{}^a\epsilon^b\epsilon_{ba} = {}^a\epsilon^b{}_b\epsilon_a = {}^a\delta_a = 4$, not -4 as one might expect from "normal" matrices.

where

$$\epsilon^{ab} h^{\perp}_{\mu\nu|ab} = 0 \quad \Longleftrightarrow \quad \epsilon^{ab} h_{\mu\nu|ab} = 2h^{\|}_{\mu\nu}. \tag{3.102}$$

Now, we can see from (3.100) that this will result in the appearance of

$$*\epsilon^{ab} = \frac{1}{2}\epsilon^{abcd}\epsilon_{cd}, \tag{3.103}$$

the dual of ϵ_{ab}. However, this potential complication is simplified by the fact that the dual is in fact proportional to the inverse of ϵ:

$$*\epsilon^{ab} = s\epsilon^{ab} \tag{3.104}$$

where s is the *Pfaffian* of ϵ:

$$s = \frac{1}{8}\epsilon^{abcd}\epsilon_{ab}\epsilon_{cd}. \tag{3.105}$$

Indeed, (3.104) holds for any 4×4 invertible antisymmetric matrix ${}_a\epsilon_b = -\epsilon_{ab}$ and its inverse ϵ^{ab}. This can be most easily seen by rotating to a basis in which ϵ is block-diagonal:

$$\epsilon = \lambda_1 i\sigma_2 \oplus \lambda_2 i\sigma_2 \quad \Rightarrow \quad s = \lambda_1\lambda_2 \quad \text{and} \quad \epsilon^{-1} = i\sigma_2/\lambda_1 \oplus i\sigma_2/\lambda_2 \tag{3.106}$$

where σ_2 is the second Pauli matrix, and hence $i\sigma_2$ is the totally antisymmetric symbol in 2 dimensions. The final property we will note is that since, for any antisymmetric matrix ϵ, $s^2 = \det\epsilon$, we have $s = \pm 1$ since we normalised $\det\epsilon = 1$ when defining our background metric.

The above can be combined to give us

$$*h_{\mu\nu}{}^{|ab} = \frac{s}{2}\epsilon^{ab}h^{\|}_{\mu\nu} + *h^{\perp}_{\mu\nu}{}^{|ab} \tag{3.107}$$

and therefore, contracting with ϵ_{ab} gives

$$*h^{\perp}_{\mu\nu}{}^{|ab}\epsilon_{ab} = 0 \quad \text{and} \quad *h_{\mu\nu}{}^{|ab}\epsilon_{ab} = 2sh^{\|}_{\mu\nu} \tag{3.108}$$

and thus our action becomes

$$\begin{aligned}\mathcal{L}_e = -\frac{1}{2}\partial_\rho h_{\mu\nu} *\partial^\rho h^{\mu\nu} - \frac{s}{2}\partial_\rho h^{\|}_{\mu\nu}\partial^\rho h^{\|\mu\nu} + s\,\partial^\nu h^{\|}_{\mu\nu}\partial_\rho h^{\|\mu\rho} + 2M^2 h^{\|}_{\mu\nu} *h^{\mu\nu} \\ - \frac{1}{2}\partial_\rho h^{\perp}_{\mu\nu|ab} *\partial^\rho h^{\perp\mu\nu|ab} + \partial^\nu h^{\perp}_{\mu\nu|ab}\partial_\rho h^{\perp\mu\rho|ab}.\end{aligned} \tag{3.109}$$

Now we see from the second line that the perpendicular components $h^{\perp}_{\mu\nu|ab}$ propagate amongst themselves, so can be treated separately. In order to diagonalise these terms, it is useful to define

$$h^{\perp\pm}_{\mu\nu|ab} = \frac{1}{2}\left(h^{\perp}_{\mu\nu|ab} \pm *h_{\mu\nu}{}^{|ab}\right). \tag{3.110}$$

Note that we have identified the covariant and contravariant indices in the θ-space, so this statement depends on the basis chosen. Nevertheless, any change in basis would simply lead to a re-labelling of the same fields, and so the physics will not change. With this definition, we see that the perpendicular components have the Lagrangian

$$-\frac{1}{2}\partial_\rho h^{\perp+}_{\mu\nu|ab}\partial^\rho h^{\perp+\mu\nu|ab}+\partial^\nu h^{\perp+}_{\mu\nu|ab}\partial_\rho h^{\perp+\mu\rho|ab}+\frac{1}{2}\partial_\rho h^{\perp-}_{\mu\nu|ab}\partial^\rho h^{\perp-\mu\nu|ab}-\partial^\nu h^{\perp-}_{\mu\nu|ab}\partial_\rho h^{\perp-\mu\rho|ab} \tag{3.111}$$

and so we see that we have a set of fields propagating normally, and the same number of fields propagating with the wrong sign kinetic term. We would not expect any of these to be physical, and hence we would expect these to gain a regulator mass when spontaneous symmetry breaking is implemented.

Looking at the remainder of (3.109), we see that varying $*h_{\mu\nu}$ results in the equation of motion

$$\Box h_{\mu\nu}+4M^2h^{\|}_{\mu\nu}=0 \tag{3.112}$$

and therefore, since $h_{\mu\nu}$ was gauge fixed to be transverse and traceless, we can deduce that $h^{\|}_{\mu\nu}$ is transverse and traceless also. Then we see that the only piece of $*h_{\mu\nu}$ that is left in our action is therefore the transverse traceless piece $*h^{\mathrm{tt}}_{\mu\nu}$ since it only couples to other fields which are also transverse and traceless. We then diagonalise the kinetic terms by writing

$$h^{\pm}_{\mu\nu}=\frac{1}{2}\left(h_{\mu\nu}\pm *h^{\mathrm{tt}}_{\mu\nu}\right) \tag{3.113}$$

so that the first line of (3.109) becomes

$$-\frac{1}{2}\partial_\rho h^{+}_{\mu\nu}\partial^\rho h^{+\mu\nu}+\frac{1}{2}\partial_\rho h^{-}_{\mu\nu}\partial^\rho h^{-\mu\nu}-\frac{s}{2}\partial_\rho h^{\|}_{\mu\nu}\partial^\rho h^{\|\mu\nu}+2M^2h^{\|}_{\mu\nu}(h^{+\mu\nu}-h^{-\mu\nu}) \tag{3.114}$$

In Section 3.3, we assumed a certain normalisation for the super-Einstein-Hilbert action. However, it is not entirely clear what the correct normalisation should be, especially since we could modify our definition of Berezin integration to change this. As such, it is not clear which of $h^{+}_{\mu\nu}$ or $h^{-}_{\mu\nu}$ propagates with the correct sign, just that exactly one of them will. In addition, we could also fix s to make $h^{\|}_{\mu\nu}$ propagate with either sign. Which of these fills the role of the graviton would be expected to depend on the precise symmetry-breaking mechanism.

The fields $X^T_{\mu\nu}=(h^{+}_{\mu\nu},h^{-}_{\mu\nu},h^{\|}_{\mu\nu})$ are coupled together with what appears to be a mass term. Writing $U^T=(1,-1,0)$, $V^T=(0,0,1)$ and $D=\mathrm{diag}(1,-1,s)$ we can write (3.114) as

$$\frac{1}{2}X^T_{\mu\nu}D\Box X^{\mu\nu}+M^2X^T_{\mu\nu}AX^{\mu\nu} \tag{3.115}$$

where $A=UV^T+VU^T$. The M^2 term could be diagonalised if we had all kinetic terms with the correct sign, however it is the presence of ghosts which prevents this. In fact, if we set the overall normalisation of the action to $-1/\alpha$, (so that in (3.40),

$\alpha = \frac{1}{2}$), the propagator takes the form

$$\langle X_{\mu\nu}(p) X^{T\,\rho\sigma}(-p)\rangle = \alpha \Pi^{\rho\sigma}_{\mu\nu}\Delta \tag{3.116}$$

where $\Pi^{\rho\sigma}_{\mu\nu}$ is the transverse traceless projector on the space of symmetric tensor fields, and we have

$$\Delta = \langle XX^T\rangle = (p^2 D - 2M^2 A)^{-1} = \frac{D}{p^2} + 2M^2\frac{DAD}{p^4} + 4M^4\frac{D(AD)^2}{p^6} \tag{3.117}$$

where the expansion terminates since $(AD)^3 = 0$ (or equivalently $(DA)^3 = 0$). Thus we see that the mass-like term in (3.115) does not in fact result in a physical mass (by shifting the pole in the propagator) but instead leads to further massless propagator-like contributions with improved UV behaviour (i.e., p^{-4} and p^{-6}). If we write $W^T = (1,1,0)$, then we have that $DAD = s(WV^T + VW^T)$ and $D(AD)^2 = sWW^T$ and therefore we have

$$\begin{aligned}
&\langle h^+h^+\rangle = \frac{1}{p^2} + 4s\frac{M^4}{p^6}, \qquad \langle h^-h^-\rangle = -\frac{1}{p^2}, +4s\frac{M^4}{p^6} \qquad \langle h^{\|}h^{\|}\rangle = \frac{s}{p^2},\\
&\langle h^+h^-\rangle = 4s\frac{M^4}{p^6}, \qquad \langle h^+h^{\|}\rangle = 2s\frac{M^2}{p^4}, \qquad \langle h^-h^{\|}\rangle = 2s\frac{M^2}{p^4}.
\end{aligned} \tag{3.118}$$

The fact that M does not result in a physical mass is not in fact surprising. It was introduced manually in order to ensure that all of the fields have the same mass dimension and to aid in the diagonalisation. In fact, we can completely eliminate M by redefining

$$h_{|a_1\ldots a_p} \mapsto M^{-p} h_{|a_1\ldots a_p}, \qquad \alpha \mapsto M^{-4}\alpha. \tag{3.119}$$

This now means that our fields have different dimensions. In particular, $[h] = 1$, $[h^{\|}] = 3$ and $[*h] = 5$. Since we no longer have M to fix dimensions, we must deal with the powers of p that are fixed by the field dimensions. Therefore, we expect each of these fields to have propagators that differ from the usual $1/p^2$ behaviour. In addition, $h^{\pm}_{\mu\nu}$ are no longer well-defined, so we must work in our original basis of fields. We then have

$$\begin{aligned}
&\langle hh\rangle = \frac{W^T\Delta W}{M^4} = 16\frac{s}{p^6}, \quad \langle *h\, *h\rangle = M^4 U^T\Delta U = 0, \quad \langle h^{\|}h^{\|}\rangle = V^T\Delta V = \frac{s}{p^2},\\
&\langle h\, *h^{tt}\rangle = W^T\Delta U = \frac{2}{p^2}, \qquad \langle hh^{\|}\rangle = \frac{W^T\Delta V}{M^2} = 4\frac{s}{p^4}, \quad \langle *hh^{\|}\rangle = M^2U^T\Delta V = 0.
\end{aligned} \tag{3.120}$$

We see now that M has indeed been eliminated, and note that these propagators are indeed dimensionally correct once we account for the fact that now $[\alpha] = 4$.

3.6.2 Fermionic sector

Finally we deal with the fermionic fields described by (3.97). First, we note that at this level, $*h^{\mu a}$ acts as a Lagrange multiplier enforcing the transversality constraint

$$\partial^\nu h_{\mu\nu|a} = 0. \tag{3.121}$$

We also define

$$*\epsilon_{ab} = \frac{1}{2}\epsilon_{abcd}\epsilon^{cd} = s\epsilon_{ab} \tag{3.122}$$

by analogy with $*\epsilon^{ab}$. Since this is also what we get from lowering indices on the latter, this is completely unambiguous. It is useful at this point to write our now-transverse fields as a transverse traceful part and a transverse traceless part:

$$h_{\mu\nu|a}(p) = h^{tt}_{\mu\nu|a}(p) + \frac{2}{3}\Pi^t_{\mu\nu}(p)\varphi_{|a}(p) \tag{3.123}$$

and similarly for $*h^{\mu\nu|a}$. Here, we have the transverse traceful projector

$$\Pi^t_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \tag{3.124}$$

and the coefficient of 2/3 simply follows from taking the trace and comparing with $h^\mu{}_{\mu|a} = 2\varphi_{|a}$. We then find that

$$\mathcal{L}_o = -\frac{1}{2}\partial_\rho h^{tt}_{\mu\nu}{}^{|a}\epsilon_{ab} *\partial^\rho h^{tt\,\mu\nu|b} - \frac{s}{2}M^2 *h^{tt}_{\mu\nu}{}^{|a}\epsilon_{ab} *h^{tt\,\mu\nu|b} - \frac{2}{3}\partial_\rho\varphi^{|a}\epsilon_{ab} *\partial^\rho\varphi^{|b} + \frac{4}{3}sM^2 *\varphi^{|a}\epsilon_{ab} *\varphi^{|b} \tag{3.125}$$

and so the traceful and traceless modes decouple. Note that these fields are scalars (φ) and tensors ($h_{\mu\nu}$) which have fermionic statistics. Therefore, they have the wrong statistics and thus they must gain a mass via some symmetry breaking mechanism. As in the bosonic case, we will see that M does not play this role, despite appearances. Writing $Y^{Ta} = (\varphi^{|a}, *\varphi^{|a})$, we can write the transverse traceful part of the action as

$$\frac{1}{3}Y^{Ta}\epsilon_{ab}(\sigma_1\Box + 2sM^2\sigma_-)Y^b \tag{3.126}$$

where $\sigma_\pm = \frac{1}{2}(\mathbb{I} + \sigma_3)$ and the σ_i are the Pauli matrices. With the action normalised to $-1/\alpha$ as above, we see that

$$\langle Y^a(p)Y^{Tb}(-p)\rangle = -\frac{3}{2}\alpha\epsilon^{ab}(p^2\sigma_1 - 2sM^2\sigma_-)^{-1} = -\frac{3}{2}\epsilon^{ab}\left(\frac{\sigma_1}{p^2} + 2sM^2\frac{\sigma_+}{p^4}\right) \tag{3.127}$$

where the $1/p$ expansion terminates because $\sigma_1(\sigma_-\sigma_1)^2 = 0$. We thus have that

$$\langle\varphi^{|a}\varphi^{|b}\rangle = -3\alpha sM^2\frac{\epsilon^{ab}}{p^4}, \quad \langle *\varphi^{|a}\,\varphi^{|b}\rangle = 0, \quad \langle\varphi^{|a} *\varphi^{|b}\rangle = -\frac{3\alpha}{2}\frac{\epsilon^{ab}}{p^2}. \tag{3.128}$$

Similarly to the bosonic case, we see again that M does not act as a physical mass, but instead appears in propagators with improved UV properties. Again, this can be understood by dimensions and the fact that M is arbitrary. Indeed, we can see that once again, (3.119) removes all references to M. We can see that the pattern is very similar for $h^{tt}_{\mu\nu}{}^{|a}$, and we find that

$$\langle h^{tt}_{\mu\nu}{}^{|a} h^{tt\,\rho\sigma|b}\rangle = \alpha s M^2 \Pi^{\rho\sigma}_{\mu\nu}\frac{\epsilon^{ab}}{p^4}, \quad \langle h^{tt}_{\mu\nu}{}^{|a} *h^{tt\,\rho\sigma|b}\rangle = -2\alpha\Pi^{\rho\sigma}_{\mu\nu}\frac{\epsilon^{ab}}{p^2} \tag{3.129}$$

and $\langle *h^{tt}_{\mu\nu}{}^{|a} *h^{tt\,\rho\sigma|b}\rangle = 0$.

3.7 Discussion

As seen in [33], the Parisi-Sourlas regularization works in gauge theory by adding the original gauge field A^1_μ, a pair of complex fermion fields B_μ, $\bar{B}_\mu$ and a ghost copy A^2_μ. In pure $SU(N|N)$ gauge theory, only the two transverse polarizations propagate and they are decoupled from each other.

The above work on the analogous situation in gravity shows that the solution at the free level is already more complex and more subtle. We have to expand around a non-vanishing background (3.42) which can lead to some spontaneous symmetry breaking. However, we found that the resulting mass-like terms do not actually provide masses by modifying the position of the poles in the propagator, but instead provide further massless propagators with improved ultraviolet behaviour (terms with $1/p^4$ and $1/p^6$).

The propagating modes at the free level are not simply the transverse traceless ones that we would expect from the graviton. The bosonic transverse traceless modes form a multiplet, consisting of $h^{\pm}_{\mu\nu}$, which are self-dual anti-self-dual linear combinations of $h_{\mu\nu}$ and $*h^{tt}_{\mu\nu}$, and $h^{\|}_{\mu\nu}$, the part of $h_{\mu\nu|ab}$ proportional to ϵ_{ab}. We see that exactly one of $h^{\pm}_{\mu\nu}$ propagates with the correct sign (depending on the overall normalisation of the action) and by choosing s, $h^{\|}_{\mu\nu}$ can propagate with either sign. The remaining piece of $h_{\mu\nu|ab}$, $h^{\perp}_{\mu\nu|ab}$, is traceless but not transverse. We saw that we can also split this into self-dual and anti-self-dual pieces $h^{\perp\pm}_{\mu\nu|ab}$ which do not mix and propagate with opposite signs; one propagates as a real field while the other propagates as a ghost field.

The propagating fermionic modes, $h_{\mu\nu|a}$ and $*h^{\mu\nu|a}$ have the wrong statistics, so clearly cannot be physical. In fact they are intended to be Pauli-Villars fields. These split into transverse traceless and transverse traceful sectors, each of which forms a doublet with $1/p^2$ and $1/p^4$ propagators, similarly to the bosonic case. This summarizes all of the propagating modes.

Looking now at the non-propagating modes, $h_{ab}(x,\theta)$ (a set of scalars) was algebraically eliminated using linearised superdiffeomorphisms (3.86), except for $*h_{ab}$, which is gauge invariant but at this level becomes a Lagrange multiplier enforcing $\varphi_{|ab} = 0$. Similarly, $h_{\mu a}(x,\theta)$ is eliminated using gauge transformations, with the exception of $*h_{\mu a}$, which is gauge invariant but behaves as a Lagrange multiplier enforcing the transversality of the propagating fermionic modes. A remaining gauge invariance $\xi'_a(x)$ is then used to fix a radiation gauge $\varphi_{|a} = 0$ as seen in (3.94). In addition, we have the usual bosonic gauge invariance generated by $\xi_\mu(x)$ (i.e., the more familiar diffeomorphisms on the base manifold), and so we can fix $h_{\mu\nu}(x)$ to be transverse and traceless.

Finally, we are left with the vector field $h_{\mu(a|b)}$, which does not play a role in the action at the free level, and the corresponding gauge invariance $\xi_{(a|b)}$ is not fixed. At the interacting level we'd expect this field to have a role, most likely as a Lagrange multiplier, potentially giving important constraints on the specific form of the expected spontaneous symmetry breaking.

Finding this symmetry breaking is clearly the next step required in this construction. We can expect to be required to induce all modes, except for the graviton, analogously to what was achieved in $U(1|1)$ gauge theory in [80], since the kind of decoupling seen in gauge theory in [33] is unlikely to work in this case. Due to the similarity of the cosmological constant term to a mass term for the graviton when expanded around flat space, this seems to be promising. However, it is clearly the case that in normal Einstein gravity, the cosmological constant does not provide a mass term for the graviton since diffeomorphism invariance is unbroken, the linearised piece

$$\mathcal{L}_{c.c.} \sim \kappa\varphi \tag{3.130}$$

is more important, since it shows that flat space is no longer a solution to the equations of motion. Since in the supermanifold case the equivalent term is $\sim \kappa \mathrm{str}(h)$, we see that after integrating over θ the cosmological constant only induces curvature in $*h_{\mu\nu}$ and $*h_{ab}$. In the cosmological constant term, $h^{\|}_{\mu\nu}$ appears first only to second order where it already takes the form of a mass term.

It is worth reiterating the differences between the Parisi-Sourlas supergravity construction and more familiar notions of supergravity. In standard ($N = 1$, $D = 4$) supergravity there are four fermionic coordinates but, contrary to our approach, these are cast as a complex conjugate pair of two-component coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$. Crucially, we set the torsion field to vanish, so that the regularising structure can maintain the close similarity to the graviton interactions in the Einstein-Hilbert action. In the standard realisation of supergravity, the torsion field is non-vanishing even in flat space, being related to the Pauli matrices $\sigma^\mu_{\alpha\dot{\alpha}} \sim (i, \boldsymbol{\sigma})$, and the tangent space symmetry of θ^α and $\bar{\theta}^{\dot{\alpha}}$ is then tied to the bosonic vectorial Lorentz representation, as in [79]. The Parisi-Sourlas supergravity presented here could therefore be viewed as a deformation of standard supergravity. Since the expansion over the θ^α leads to component fields carrying antisymmetric vectorial indices (the fermionic $a, b, \ldots$) reminiscent of forms, and this also leading to fields with mixed representations, it has some superficial resemblance to Generalized Geometry [81, 82]. However, in our case, the indices $a, b, \ldots$ are not associated to the cotangent bundle but belong to a new space. This latter property gives the theory an apparent resemblance to Double Field Theory [83, 84], although there is no doubling of the bosonic coordinates here or relation to T-duality.

Chapter 4

Pure Gravity in the BV Formalism

4.1 Introduction

This chapter will develop the perturbative theory of quantum gravity which began in [5, 29, 34, 42] and relies heavily on [4, 7, 35]. The theory is perturbative in $\kappa \sim \sqrt{G}$ (where G is Newton's gravitiational coupling), but non-perturbative in $\hbar$. The theory presented is a logical consequence of combining the Wilsonian RG with the action for free gravitons, while being sure to respect the wrong-sign kinetic term of the conformal factor, as explored in Chapter 2. This leads to a well-defined QFT, which nonetheless leads to a conceptually different theory to many other approaches to quantum gravity.

Figure 4.1 shows the conceptual difference. Normally, we would expect the presence of a cutoff Λ to break diffeomorphism invariance. However, the Slavnov-Taylor identities are replaced with modified Slavnov-Taylor identities (mST) which reduce to the usual ones in the physical limit $\Lambda \to 0$. The difference when compared to the usual approach is that the UV fixed point exists outside of the diffeomorphism-invariant subspace, and it is only the choice of couplings which allows the trajectory to enter the subspace when $\Lambda \ll \Lambda_p$, where Λ_p is a dynamically generated scale determined by the couplings. At this point, the limit $\Lambda_p \to \infty$ is taken and as we will see, diffeomorphism invariance is recovered.

Due to many of the properties seen is Chapter 2, many of the things one might usually take for granted now need more careful treatment. In particular, the direction in which the RG flow is well-posed for φ is the opposite direction to that of $h_{\mu\nu}$, and so care must be taken with the limiting procedures so as not to introduce new divergences.

Section 4.2 introduces the BRST algebra and the Quantum Master Equation (QME), which give a new way of parametrizing diffeomorphism invariance. We see that we are

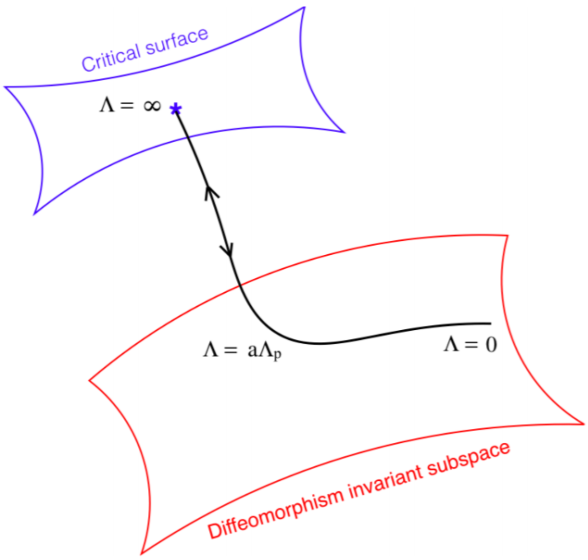

FIGURE 4.1: The renormalized trajectory emnating from the Gaussian fixed point cannot respect diffeomorphism invariance for $\Lambda > a\Lambda_p$, where a and Λ_p are as discussed in Chapter 2. By appropriate choices of couplings, diffeomorphism invariance is recovered at required scales. Figure taken from [4].

specifically interested in the BRST cohomology, which involves removing field reparametrisations as a quotient, and thus we can ignore certain operators which may appear in the action. We also see how the QME is consistent with the Wilsonian RG, in particular introducing cutoffs in order to appropriately define objects in the BRST algebra.

In Section 4.3, we see how, even though many the concepts are most easily explained using the Wilsonian effective action, the Legendre average effective action is in fact more convenient for making calculations. One of the effects of making this change is the fact that the QME now becomes a modified Slavnov-Taylor identity (mST), which can be seen as parametrising the extent to which diffeomorphism invariance is broken by the presence of the cutoff. Then in Section 4.4 we see how the general formalism relates to the case of our interest, quantum gravity.

Section 4.5 then relates some of the results from Chapter 2 to this new framework, and in particular shows how diffeomorphism invariance is recovered at linear order when certain coefficient functions “trivialise”. Section 4.6 then discusses the free quantum BRST algebra and the gradings which make our life easier when constructing solutions. Then Section 4.7 finally constructs the quantum gravity action to first order using this new formalism.

The next few sections will then concentrate on the second order solution. Section 4.8 constructs the second order solution in the classical regime. Section 4.9 next discusses how solutions are best constructed in the diffeomorphism invariant subspace (with large amplitude suppression scale) and uses a different parametrisation in order to solve the second order equations. Section 4.10 finally uses these equations, which provide one-loop Feynman integrals, to calculate the quantum corrections from the classical solution.

Finally, Sections 4.11, 4.12 and 4.13 discuss all of the above, including limitations and of the framework and future possible directions for extension, in particular focusing on the properties of the differential flow equations themselves (and their parabolic nature).

4.2 BRST and QME

4.2.1 Quantum Master Equation

In order set up the framework, we will follow [34] to see how BRST and Wilsonian RG fit together. To construct the BRST algebra, we need a Grassmann odd derivation[1] Q which we call the "BRST charge". Then we have the BRST transformation

$$\delta\Phi^A = \epsilon Q\Phi^A \tag{4.1}$$

where ϵ is some Grassmann number. Here, the Φ^A are the quantum fields, which includes ghost fields and auxiliary fields required to realise BRST invariance off-shell. In order to renormalize a theory with BRST invariance, we need to supplement the bare action $S[\Phi]$ with source terms Φ^*_A for the BRST transformations, and so total action is

$$S[\Phi, \Phi^*] = S[\Phi] - (Q\Phi^A)\Phi^*_A. \tag{4.2}$$

These Φ^*_A have opposite statistics to Φ^A and are called "antifields". Note that we are using the DeWitt notation, so A scans over the fields, but also spacetime position and indices. With this action, the partition function is then

$$\mathcal{Z}[\Phi^*] = \int \mathcal{D}\Phi\, e^{-S[\Phi,\Phi^*]}. \tag{4.3}$$

We now define the Quantum Master Functional (QMF)

$$\Sigma[S] = \frac{1}{2}(S, S) - \Delta S \tag{4.4}$$

where we have defined the antibracket [85, 86] and the measure operator Δ, which on general functionals X and Y take the form

$$(X, Y) = X\left(\frac{\overleftarrow{\partial}}{\partial\Phi^A}\frac{\overrightarrow{\partial}}{\partial\Phi^*_A} - \frac{\overleftarrow{\partial}}{\partial\Phi^*_A}\frac{\overrightarrow{\partial}}{\partial\Phi^A}\right)Y \tag{4.5}$$

[1] This effectively means a Grassmann odd operator which follows the Leibniz rule. That is, acts "like a derivative".

and

$$\Delta X = (-1)^A \frac{\overrightarrow{\partial}}{\partial \Phi^A} \frac{\overrightarrow{\partial}}{\partial \Phi^*_A} X. \tag{4.6}$$

This notation probably bears some explanation. Similarly to Chapter 3, the appearance of an index in a power of (-1) should be read as the Grassmann grading of the associated field, that is 0 (so that $(-1)^A = 1$) for a bosonic field and 1 (so that $(-1)^A = -1$) for a fermionic field. In addition, we have defined left- and right- acting derivatives. These are distinguished by their Leibniz properties:

$$\begin{aligned} \frac{\overrightarrow{\partial}}{\partial \Phi^A}(fg) &= \frac{\overrightarrow{\partial} f}{\partial \Phi^A} g + (-1)^{Af} f \frac{\overrightarrow{\partial} g}{\partial \Phi^A} \\ (fg)\frac{\overleftarrow{\partial}}{\partial \Phi^A} &= f\left(g \frac{\overleftarrow{\partial}}{\partial \Phi^A}\right) + (-1)^{Ag} f \frac{\overleftarrow{\partial}}{\partial \Phi^A} g \end{aligned} \tag{4.7}$$

and they have the same action on single fields:

$$\frac{\overrightarrow{\partial}}{\partial \Phi^A} \Phi^B = \Phi^B \frac{\overleftarrow{\partial}}{\partial \Phi^A} = \delta^B_A. \tag{4.8}$$

Sometimes in the literature one can find $\partial_r X = X\overleftarrow{\partial}$ and $\partial_l X = \overrightarrow{\partial} X = \partial X$, and which is used is down to taste. It is also worth pointing out that since we are using the DeWitt notation, these "partial" derivatives are in fact functional derivatives, and we will often not write integrals for action functionals, instead understanding them to be integrated over spacetime.

If we restore factors of $\hbar$, it is clear to see that ΔS is the quantum part of the QMF. However, without regularisation this term is not well-defined. As we will see, the Wilsonian RG provides a natural regularisation. The QMF is used to test whether the gauge symmetry is successfully incorporated. This is the case if the action is invariant under (4.1). This is true if and only if

$$\Sigma[S] = 0, \tag{4.9}$$

and this is the Quantum Master Equation (QME). Indeed, the vanishing of Σ follows from the observation

$$\int \mathcal{D}\Phi \Sigma e^{-S} = \int \mathcal{D}\Phi \Delta(e^{-S}) \tag{4.10}$$

which vanishes since it is the integral over Φ of a total Φ derivative.

4.2.2 BRST cohomology

We will start from a solution of the QME (in our case, the free graviton action) S, and perturbing it to get another solution. That is, considering some quasi-local operator integrated over spacetime $\mathcal{O}$, we want $S + \epsilon\mathcal{O}$ to also be a solution for the QME.

Substituting this perturbed action into the QME tells us that this is a solution if and only if

$$s\mathcal{O} = 0 \tag{4.11}$$

where s is the full quantum BRST operator, defined by

$$s\mathcal{O} = (S, O) - \Delta\mathcal{O}. \tag{4.12}$$

Algebraically, we write

$$\mathcal{O} \in \ker s \tag{4.13}$$

that is, $\mathcal{O}$ is in the "kernel" of s, which is the set of functionals mapped to 0 under s.

Note that s is distinguished from the previously introduced BRST transformation, which can simply be shown to satisfy[2]

$$Q\Phi^A = (S, \Phi^A) \tag{4.14}$$

We also define the (Grassmann-odd) Koszul-Tate differential Q^- by its action on antifields

$$Q^-\Phi^*_A = (S, \Phi^*_A). \tag{4.15}$$

For consistency and simplicity, we also note that

$$Q\Phi^*_A = 0 = Q^-\Phi^A \tag{4.16}$$

and we also note that $s\Phi^A = Q\Phi^A$ and $s\Phi^*_A = Q\Phi^*_A$ since clearly Δ vanishes on these. Combining (4.14) and (4.15) results in

$$(Q + Q^-)\mathcal{O} = (S, \mathcal{O}) \tag{4.17}$$

which in turn implies that

$$s = Q + Q^- - \Delta. \tag{4.18}$$

Now, it is not immediately obvious that $s^2 = 0$, as we'd like in order to construct the cohomology. Indeed,

$$s^2\mathcal{O} = (\Sigma, \mathcal{O}). \tag{4.19}$$

However, our requirement that the QME is satisfied ($\Sigma = 0$) means that for our purposes, $s^2 = 0$. Therefore, if the action S satisfies the QME, operators $\mathcal{O}$ which are "s-exact", that is

$$\mathcal{O} = sK = (S, K) - \Delta K \tag{4.20}$$

[2]Note that in the DeWitt notation, the loose index indicates that the expression only makes sense when treated as part of an integrated operator, similarly to how the Dirac delta and its derivatives are only defined under an integral.

where K is some quasi-local functional, then satisfy (4.11). Algebraically, we write

$$\mathcal{O} \in \operatorname{im} s \tag{4.21}$$

that is, $\mathcal{O}$ is in the "image" of s, simply meaning it can be written in the form (4.20) for some K. From the definition of the antibracket we can see that such an operator $\mathcal{O} = sK$ corresponds to the infinitesimal field and source redefinitions

$$\delta\Phi^A = \frac{\overrightarrow{\partial}}{\partial\Phi^*_A}K, \quad \delta\Phi^*_A = -\frac{\overrightarrow{\partial}}{\partial\Phi^A}K \tag{4.22}$$

with $-\Delta K$ corresponding to the Jacobian change of variables in the partition function. Indeed, if $\mathcal{O}$ is s-exact and equals sK and $\mathcal{O}_1 \dots \mathcal{O}_n$ are BRST-invariant operators (and all operators have disjoint spacetime support) then their correlator vanishes:

$$\langle \mathcal{O}\,\mathcal{O}_1 \dots \mathcal{O}_n \rangle = \langle sK\mathcal{O}_1 \dots \mathcal{O}_n \rangle = \langle s(K\mathcal{O}_1 \dots \mathcal{O}_n) \rangle = -\frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi \Delta(K\mathcal{O}_1 \dots \mathcal{O}_n) e^{-S} \tag{4.23}$$

which is zero since it is the integral of a total Φ-derivative. Therefore, we regard the addition of any $\mathcal{O} = sK$ as uninteresting, and thus we are only interested in functionals which live in the quotient

$$\frac{\ker s}{\operatorname{im} s}. \tag{4.24}$$

Note that since $s^2 = 0$, we have $\operatorname{im} s \leq \ker s$, and so this quotient is well-defined. All this means is that we are interested only in operators which are annihilated by s, making sure that we treat operators which differ only by a piece sK to be physically the same. This is what is meant by the cohomology of s.

4.2.3 Wilsonian RG

A full derivation the compatibility of the QME and RG is seen in [34], but here the results are presented. As one might expect, the effect of introducing the RG is to regularise the terms in the free action

$$S_0 = \frac{1}{2}\Phi^A (\Delta^\Lambda)^{-1}_{AB} \Phi^B - (Q_0\Phi^A)(C^\Lambda(p))^{-1}\Phi^*_A, \tag{4.25}$$

where Δ^Λ is the regularised propagator proportional to C^Λ. For example, for a scalar field we have $\Delta^\Lambda(p) = C^\Lambda(p)/p^2$. In addition, the QMF is regularised by inserting the cutoff into the antibracket and the measure operator:

$$(X,Y)_{\text{reg}} = X\left(\frac{\overleftarrow{\partial}}{\partial\Phi^A} C^\Lambda \frac{\overrightarrow{\partial}}{\partial\Phi^*_A} - \frac{\overleftarrow{\partial}}{\partial\Phi^*_A} C^\Lambda \frac{\overrightarrow{\partial}}{\partial\Phi^A}\right)Y, \quad \Delta_{\text{reg}}X = (-1)^A \frac{\overrightarrow{\partial}}{\partial\Phi^A} C^\Lambda \frac{\overrightarrow{\partial}}{\partial\Phi^*_A} X \tag{4.26}$$

Note that with these definitions, the free BRST transformations and Koszul-Tate differentials

$$Q_0\Phi^A = (S_0, \Phi^A)_{\text{reg}}, \quad Q_0^- \Phi^*_A = (S_0, \Phi^*_A)_{\text{reg}} \tag{4.27}$$

are independent of the cutoff. Thus the only piece of the free BRST cohomology which depends on the cutoff is Δ, which is now well-defined due to the regularisation.

Now, we suppose that the action can be expanded in a small parameter, and we call this parameter κ (looking ahead somewhat). We write

$$S = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} S_n \tag{4.28}$$

and substituting this into (4.12) results in a similar expansion for s, and we have the relation

$$\hat{s}_0 S_1 = 0, \tag{4.29}$$

the non-trivial solutions of which define all possible interactions to first order which are consistent with BRST invariance[3]. Note that field redefinition ambiguities are defined in terms of the free BRST cohomology. In addition, we similarly find

$$\hat{s}_0 S_2 = -\frac{1}{2}(S_1, S_1), \quad \hat{s}_0 S_3 = -(S_1, S_2), \ldots \tag{4.30}$$

and so we can iteratively construct the action order-by-order to be consistent with BRST invariance.

In order to be consistent with RG, we require in addition that the interacting part of the action S_I, where $S = S_0 + S_I$ satisfies the Polchinski flow equation

$$\dot{S}_I = \frac{1}{2} S_I \frac{\overleftarrow{\partial}}{\partial \Phi^A} (\dot{\Delta}^\Lambda)^{AB} \frac{\overrightarrow{\partial}}{\partial \Phi^B} S_I - \frac{1}{2} (\dot{\Delta}^\Lambda)^{AB} \frac{\overrightarrow{\partial}}{\partial \Phi^B} \frac{\overrightarrow{\partial}}{\partial \Phi^A} S_I. \tag{4.31}$$

Writing this in terms of linear functionals a_0 and a_1, we have

$$\dot{S}_I = \frac{1}{2} a_0[S_I, S_I] - a_1[S_I] \tag{4.32}$$

and in [34] it is shown that

$$\dot{\Sigma} = a_0[S_I, \Sigma] - a_1[\Sigma] \tag{4.33}$$

and so, if $\Sigma = 0$ (i.e., the QME is satsified) at some Λ, then $\dot{\Sigma} = 0$ at all Λ. We say that the QME is compatible with the flow equation.

[3] For consistency with the literature, specifically [4], we reserve s_0 for the free classical BRST transformation, $s_0 = Q_0 + Q_0^-$.

4.3 Legendre effective action and mST

In this section, we follow [4]. We will use the infrared cutoff Legendre effective action Γ^{tot}. This loses elegance in many ways compared to using the continuum Wilsonian action effective action, but will be more convenient for explicit calculations since the limit $\lim_{\Lambda\to 0}\Gamma$ gives direct access to physical amplitudes. It will also mean that instead of the QME, the diffeomorphism invariance is broken and expressed through the modified Slavnov-Taylor identities (mST), and nilpotency of the BRST charge at the interacting level is recovered only in the limit $\Lambda \to 0$. The free charges are still nilpotent, however, and it is their cohomology which is important to solve for the effective action.

For much of what follows, it is convenient to write

$$\Gamma^{\text{tot}} = \Gamma + \frac{1}{2}\Phi^A \mathcal{R}_{AB} \Phi^B \tag{4.34}$$

where $\mathcal{R}_{AB}$ is the infrared cutoff expressed in additive form. We are mainly interested in the "effective average action" part Γ, which we express as

$$\Gamma = \Gamma_0 + \Gamma_I \tag{4.35}$$

where Γ_0 is the free part

$$\Gamma_0 = \frac{1}{2}\Phi^A \Delta^{-1}_{AB} \Phi^B - (Q_0 \Phi^A)\Phi^*_A \tag{4.36}$$

and Γ_I is the interacting part. Note in particular that the free part contains no regularisation, in contrast to the Wilsonian case. The flow equation for the interactions then takes the form (1.18)[4]

$$\dot{\Gamma}_I = -\frac{1}{2}\text{str}\left(\dot{\Delta}_\Lambda \Delta_\Lambda^{-1}\left[1 + \Delta_\Lambda \Gamma_I^{(2)}\right]^{-1}\right). \tag{4.37}$$

Under the Legendre transformation to get from S to Γ, we see that the QMF becomes

$$\Sigma = \frac{1}{2}(\Gamma,\Gamma) - \text{tr}\left(C^\Lambda \Gamma^{(2)}_{I*}\left[1 + \Delta_\Lambda \Gamma_I^{(2)}\right]^{-1}\right), \tag{4.38}$$

the vanishing of which is the modified Slavnov-Taylor identity (mST) and (loosely speaking) describes the extent to which the presence of a cutoff breaks diffeomorphism invariance. In particular, diffeomorphism invariance is restored in the limit $\Lambda \to 0$ and the mST reduces to the Zinn-Justin equation $\frac{1}{2}(\Gamma,\Gamma) = 0$ [87, 88], which gives the usual realisation of BRST invariance the Slavnov-Taylor identities for the corresponding vertices.

[4] The supertrace instead of the trace is related to the fact that we now have Grassmann-odd fields.

Note that in the above we have introduced the notation

$$\left(\Gamma_I^{(2)}\right)_{AB} = \frac{\overrightarrow{\partial}}{\partial \Phi^A} \Gamma_I \frac{\overleftarrow{\partial}}{\partial \Phi^B}, \quad \left(\Gamma_{I*}^{(2)}\right)^A{}_B = \frac{\overrightarrow{\partial}}{\partial \Phi^*_A} \Gamma_I \frac{\overleftarrow{\partial}}{\partial \Phi^B} \tag{4.39}$$

and we use str$M = (-1)^A M^A{}_A$, similarly to Chapter 3, in addition to the usual tr$M = M^A{}_A$, with the usual conventions for "indices" in the DeWitt notation.

We now express the free BRST transformation and Koszul-Tate operator as

$$Q_0 \Phi^A = (\Gamma_0, \Phi^A), \quad Q_0^- \Phi^*_A = (\Gamma_0, \Phi^*_A) \tag{4.40}$$

and note that this is now completely free of regularisation. In addition, when using the formulation in terms of the average effective action, our bracket s are completely free of regularisation, however the BV measure operator is still regularised. We wish to expand our effective action Γ in terms of a formal expansion parameter ϵ:

$$\Gamma_I = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \Gamma_n. \tag{4.41}$$

Note that we are working at all orders in $\hbar$, so there is no loop expansion. In this, ϵ is a formal order-counting parameter which we envisage setting to unity at the end of our procedure. The eventual physical coupling comes about in a more complicated manner.

Expanding to first order, we see that the flow equation (4.37) and the mST (4.38) become

$$\dot{\Gamma}_1 = \frac{1}{2} \mathrm{str}\, \dot{\Delta}_\Lambda \Gamma_1^{(2)} \tag{4.42}$$

$$0 = (\Gamma_0, \Gamma_1) - \mathrm{tr}\, C^\Lambda \Gamma_{1*}^{(2)}. \tag{4.43}$$

and we note that the linearised mST (4.43) can be written as

$$0 = (Q_0 + Q_0^-)\Gamma_1 - \Delta \Gamma_1 = \hat{s}_0 \Gamma_1. \tag{4.44}$$

The flow equation (4.42) tells us that the RG time derivative of the eigenoperators is given by a tadpole operator, and the linearised mST (4.43) tells us that the BV measure operator Δ generates Λ-dependent terms to the terms that we would get in the classical BRST cohomology. In fact, since the flow equation and mST are compatible, these corrections are exactly what is required for the eigenoperators to be simultaneous solutions for these.

4.4 Application to Quantum Gravity

From this point, any expression for an action functional should be understood as being integrated over all of spacetime. The free action is chosen to be[5]

$$\Gamma_0 = \frac{1}{2}(\partial_\rho H_{\mu\nu})^2 - 2(\partial_\mu \varphi)^2 - (\partial_\mu H_{\mu\nu})^2 + 2\partial_\mu \varphi \partial_\nu H_{\mu\nu} - 2\partial_\mu c_\nu H^*_{\mu\nu} \tag{4.45}$$

i.e., the action for the free graviton $H_{\mu\nu}$, where $\varphi = \frac{1}{2}H_{\mu\mu}$, and the graviton antifield $H^*_{\mu\nu}$, which sources the free BRST transformation of the graviton

$$Q_0 H_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu. \tag{4.46}$$

We also note that the free action is what we get from expanding the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{EH} = -\frac{2}{\kappa^2}\sqrt{g}R \tag{4.47}$$

by writing

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa H_{\mu\nu}. \tag{4.48}$$

Similarly, we see that the BRST transformation for $H_{\mu\nu}$ is equivalent to the Lie derivative $\mathcal{L}_c H$ in linearised gravity.

Looking ahead, this means that we know a priori from the geometry what the classical contributions should be, however, it would be good if we could reconstruct this from the BRST algebra, allowing for more confidence in this approach.

Note that we have introduced c_μ, the fermionic ghost fields. For consistency, we also need the (bosonic) ghost antifields c^*_μ. There are other fields required in order to construct propagators, and this construction is detailed in [4]. For our purposes, however, it will suffice to know that these additional fields are the (fermionic) antighost $\bar{c}_\mu$, the (bosonic) antighost antifield $\bar{c}^*_\mu$, and the (bosonic) auxiliary field b_μ. Not including these in the free action is the same as saying we are using the *minimal gauge fixed basis*, and the only field not in the free action which appears at the interacting level is c^*_μ.

The non-vanishing Koszul-Tate differentials in this basis are given by

$$Q_0^- H_{\mu\nu} = -2G^{(1)}_{\mu\nu}, \quad Q_0^- c_\mu = -2\partial_\nu H^*_{\mu\nu} \tag{4.49}$$

where $G^{(1)}_{\mu\nu}$ is the linearised Einstein tensor

$$G^{(1)}_{\mu\nu} = -R^{(1)}_{\mu\nu} + \frac{1}{2}\delta_{\mu\nu}R^{(1)} \tag{4.50}$$

[5]For our purposes, raising/lowering indices will be done with δ, so has no effect. Therefore, subscripts and superscripts will be used entirely interchangeably.

where we have defined the linearised curvatures

$$R^{(1)}_{\mu\nu} = -\partial_\mu\partial_\nu\varphi + \partial_{(\mu}\partial^\alpha H_{\nu)\alpha} - \frac{1}{2}\Box H_{\mu\nu}, \quad R^{(1)} = \partial_\mu\partial_\nu H_{\mu\nu} - 2\Box\varphi. \tag{4.51}$$

After gauge fixing, the propagators can be calculated. Writing $H_{\mu\nu}$ in terms of its SO(d) irreducible parts

$$H_{\mu\nu} = \frac{2}{d}\delta_{\mu\nu}\varphi + h_{\mu\nu} \tag{4.52}$$

where $h_{\mu\mu} = 0$, the propagators in d-dimensions are then

$$\langle h_{\mu\nu}(p)h_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu} - \frac{1}{d}\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2} \tag{4.53}$$

$$\langle\varphi(p)\varphi(-p)\rangle = -\frac{d}{2(d-2)}\frac{1}{p^2} \tag{4.54}$$

$$\langle c_\mu(p)\bar{c}_\nu(-p)\rangle = -\langle\bar{c}_\mu(p)c_\nu(-p)\rangle = \frac{\delta_{\mu\nu}}{p^2} \tag{4.55}$$

where we have defined

$$\Delta^{AB} = \langle\Phi^A\Phi^B\rangle, \quad \Phi^A(p) = \int_p e^{-ip\cdot x}\Phi^A(x), \quad \int_p = \int\frac{d^dp}{(2\pi)^d}, \tag{4.56}$$

as one might expect. At some points, it may also be useful to have to hand the propagator for the full graviton

$$\langle H_{\mu\nu}(p)H_{\alpha\beta}(-p)\rangle = \frac{\delta_{\mu(\alpha}\delta_{\beta)\nu}}{p^2} - \frac{1}{d-2}\frac{\delta_{\mu\nu}\delta_{\alpha\beta}}{p^2}. \tag{4.57}$$

4.5 Solutions to the linearised equations

This section follows closely the treatment in [7]. The flow equation (4.42) is the equations satisfied by eigenoperators. Due to the conformal factor instability, which is discussed at length in Chapter 2, the eigenoperators we expand in are given by

$$\delta^{(2l+\epsilon)}_\Lambda(\varphi)\sigma(\partial, \partial\varphi, h, c, \Phi^*) + \ldots, \tag{4.58}$$

where σ is some Lorentz-invariant Λ-independent monomial involving some or all of the fields indicated, $l \geq 0$ is an integer, and $\epsilon = 0(1)$ according to the even(odd) φ-amplitude parity [4, 29, 34]. Here we have used notation from Chapter 2, which is re-stated for convenience

$$\delta^{(n)}_\Lambda(\varphi) = \frac{\partial^n}{\partial\varphi^n}\delta^{(0)}_\Lambda(\varphi), \quad \delta^{(0)}_\Lambda(\varphi) = \frac{1}{\sqrt{2\pi\Omega_\Lambda}}\exp\left(-\frac{\varphi^2}{2\Omega_\Lambda}\right) \tag{4.59}$$

where

$$\Omega_\Lambda = |\langle\varphi(x)\varphi(x)\rangle| = \int_q \frac{C^\Lambda(q)}{q^2} = \frac{\Lambda^2}{2a^2} \tag{4.60}$$

is the modulus of the regularised tadpole integral, and we have also defined the non-universal dimensionless constant a. Since Ω_Λ is proportional to $\hbar$, this means that the $\delta_\Lambda^{(n)}$ are non-perturbative in $\hbar$. Note that the expansion over the operators (4.58) only converges in the sense of square integrability under the Sturm-Liouville measure

$$\frac{1}{2\Omega_\Lambda} \exp\left(\varphi^2 - h_{\mu\nu}^2 - c_\mu \bar{c}_\mu\right) \tag{4.61}$$

in the UV, for $\Lambda > a\Lambda_p$, where Λ_p is an *amplitude suppression scale*[6]. Below this scale, as we will see, we recover a solution which is in some sense perturbative in $\hbar$. Now we note the ellipsis in (4.58). The flow equation (4.42) has a tadpole operator, which produces Λ-dependent UV regulated tadpoles, with fewer fields in σ. These are the terms which are denoted by the ellipsis, and the term shown is described as the "top term".

The general solution of (4.42) can be written as $\Gamma_1 = \Gamma(\mu)$, where

$$\Gamma(\mu) = \left(-\frac{1}{2}\Delta^{\Lambda AB}\frac{\overrightarrow{\partial}^2}{\partial\Phi^B\partial\Phi^A}\right)\Gamma_{\text{phys}}(\mu) = \sum_\sigma (\sigma f_\Lambda^\sigma(\varphi,\mu) + \dots) \tag{4.62}$$

which is a linear combination of the eigenoperators (4.58) with constant coefficients $g_{2l+\epsilon}^\sigma(\mu)$, which are subsumed into coefficient functions f_Λ^σ:

$$f_\Lambda^\sigma(\varphi,\mu) = \int_{-\infty}^{\infty}\frac{d\pi}{2\pi}\mathfrak{f}^\sigma(\pi,\mu)e^{-\frac{1}{2}\pi^2\Omega_\Lambda + i\pi\varphi} \tag{4.63}$$

where

$$\mathfrak{f}^\sigma(\pi,\mu) = i^\epsilon \sum_{l=0}^{\infty}(-1)^l g_{2l+\epsilon}^\sigma(\mu)\pi^{2l+\epsilon}. \tag{4.64}$$

The tadpole corrections are those generated by attaching propagators to (4.62). It can be shown that the Taylor series of $\mathfrak{f}^\sigma(\pi,\mu)$ converges absolutely for all π. In addition, $\mathfrak{f}^\sigma(\pi,\mu)$ decays exponentially for $\pi > 1/\Lambda_p$. This solution makes sense for any $\Lambda \geq 0$.

Thus we see that every monomial σ has associated to it an infinite number of couplings. At first order, these couplings can be treated at μ independent, and all of them are relevant, with the exception of one marginal coupling [4, 34]. At higher order, new higher dimension monomials σ appear through quantum corrections. Infinitely many of these couplings are also relevant, but the first few are irrelevant. These irrelevant couplings are not freely variable but are fixed by the requirement that we have a well-defined renormalized trajectory, in keeping with standard RG techniques [29, 89]. At second order there are no new marginal couplings, the first order couplings still do not run, and the new irrelevant couplings that appear are determined by the first order couplings [89].

[6] Indeed, this is the definition of Λ_p.

At first glance, this would seem to be problematic, as the infinite number of relevant couplings seems to destroy any hope of predictivity in the theory. However, this can be restored, and the mST (4.43) satisfied if it the relevant couplings are chosen such that each $f^{\sigma}_{\Lambda}(\varphi,\mu)$ has a common amplitude suppression scale Λ_p, independent of σ. In addition, we arrange for them to "trivialise" in the large Λ_p limit [4]. By this we mean, for α a non-negative integer

$$f^{\sigma}_{\Lambda}(\varphi,\mu) \to A_{\sigma}\left(\frac{\Lambda}{2ia}\right)^{\alpha} H_{\alpha}\left(\frac{ai\varphi}{\Lambda}\right), \quad \text{as} \quad \Lambda_p \to \infty \tag{4.65}$$

where A_{σ} is some (possibly zero) constant, and H_{α} is the Hermite polynomial of degree α, so that

$$\left(\frac{\Lambda}{2ia}\right)^{\alpha} H_{\alpha}\left(\frac{ai\varphi}{\Lambda}\right) = \varphi^{\alpha} + \frac{1}{2}\alpha(\alpha-1)\Omega_{\Lambda}\varphi^{\alpha-2} + \dots. \tag{4.66}$$

This is the unique form for the coefficient functions $f^{\sigma}_{\Lambda}(\varphi,\mu)$ such that the linearised flow equation (4.42) is satisfied and which becomes φ^{α} in the $\Lambda \to 0$ limit.

Since the coefficients in (4.58) are now polynomial, the whole linearised solution is now polynomial. In particular, it is a sum over polynomial operators, which are made up of the Λ-independent $\sigma\varphi^{\alpha}$ along with the Λ-dependent tadpole corrections generated by the RG flow. The solutions are thus practically polynomial in $\hbar$, with its power being given by the loop order of the tadpole corrections. They are effectively no different to what we would write down as the solutions to (4.42) in the standard quantization.

Now, the mST (4.43) says that a linearised solution must be closed under the free quantum BRST charge $\hat{s}_0$. In this framework, BRST invariance is only recovered after the trivialisation above, and hence we take

$$\Gamma_1 \to \kappa(\check{\Gamma}_1 + \check{\Gamma}_{1q1}), \tag{4.67}$$

(see below sections for notation) which is the result of taking $A_{\sigma} = \kappa$ and now $\check{\Gamma}_1 + \Gamma_{1q1}$ is a free quantum BRST cohomology representative, i.e., it is closed under $\hat{s}_0$ but not exact. Thus, the gravitation coupling κ, only appears as the collective result of all of the underlying couplings, and only makes an appearance when we are in the diffeomorphism invariant subspace that we entered with the trivialisation above in the large Λ_p regime. Indeed, in this subspace we can in fact consider the perturbation series of Γ_I as a series in κ, as one might expect normally.

4.6 Free quantum BRST cohomology

As seen in, e.g., [90], finding the $\hat{s}_0$-cohomology is easier if we split the action by various gradings. The main one we will be interested in is the antighost grading, but it will also be useful to know the ghost number and mass (engineering) dimension. A table of all of the relevant weights is shown in Table 4.1. In this table, we have split the BV measure operator into

$$\Delta = \Delta^- + \Delta^= \tag{4.68}$$

where

$$\Delta^- X = \frac{\partial}{\partial H_{\mu\nu}} C^\Lambda \frac{\overrightarrow{\partial}}{\partial H^*_{\mu\nu}} X, \quad \Delta^= X = -\frac{\overrightarrow{\partial}}{\partial c_\mu} C^\Lambda \frac{\partial}{\partial c^*_\mu} X \tag{4.69}$$

are the pieces that lower the antighost number by one or two, respectively[7]. In addition, we have anticipated this grading in our notation for Q_0 and Q_0^-, since Q_0 leaves the antighost grading unchanged and Q_0^- lowers it by one.

	ϵ	gh #	agh #	Dimension
$H_{\mu\nu}$	0	0	0	$(d-2)/2$
c_μ	1	1	0	$(d-2)/2$
$H^*_{\mu\nu}$	-1	0	1	$d/2$
c^*_μ	0	-2	2	$d/2$
Q	1	1	0	1
Q^-	1	0	-1	1
Δ^-	1	1	-1	1
$\Delta^=$	1	1	-2	1

TABLE 4.1: Various weights of fields, antifields and operators. These will be useful since we will construct the action from pieces of definite weights, and so this will tell us which terms are allowed. We include the Grassmann grading (ϵ) for convenience. The mass dimension and ghost number (also "total ghost number") are respected in the action, whereas the antighost number allows for the action to be split by this grading, which also splits the BRST operator.

At a given over in the perturbation series, Γ_n will have a definite mass dimension, so for constructing these we need only consider operators of the appropriate mass dimension. In addition, the action has a whole has a definite ghost number (0), and so we know that all composite operators must be put together in such a way to make this so. However, antighost number is not respected by the action. Thus, we split Γ_n into pieces of definite antighost number:

$$\Gamma_n = \sum_{k=0}^{m} \Gamma_n^k \tag{4.70}$$

for some maximum antighost number m. With this split, and recalling that our free quantum BRST operator is now

$$\hat{s}_0 = Q_0 + Q_0^- - (\Delta^- + \Delta^=), \tag{4.71}$$

[7]Note that where a derivative is taken with respect to a bosonic field, we have not specified whether it is a right or left derivative since these will have the same action either way and so it is entirely unambiguous.

we note that any (integrated) operator with maximum antighost number m $\mathcal{O} = \sum_{k=0}^{m} \mathcal{O}^k$ that satisfies $\hat{s}_0\mathcal{O} = 0$ now satisfies "descent" equations

$$Q_0\mathcal{O}^m = 0, \quad Q_0\mathcal{O}^{m-1} = (\Delta^- - Q_0^-)\mathcal{O}^m, \quad Q_0\mathcal{O}^{m-2} = (\Delta^- - Q_0^-)\mathcal{O}^{m-1} + \Delta^{=}\mathcal{O}^m \tag{4.72}$$

which arise from grading $\hat{s}_0\mathcal{O} = 0$ by antighost number. One might reasonably ask why it is simpler to deal with many equations rather than few. The answer is that Q_0 is much easier to "invert"[8] than $\hat{s}_0$, and so by seeing what operators have antighost number m and are annihilated by Q_0, the remaining parts of the operator are defined by this "top term", up to cohomology ambiguities. Grading $(\hat{s}_0)^2 = 0$ yields the useful identities[9]

$$\begin{gathered}(Q_0)^2 = (Q_0^-)^2 = (\Delta^-)^2 = (\Delta^{=})^2 = 0 \\ \{Q_0, Q_0^-\} = \{Q_0, \Delta^-\} = \{Q_0^-, \Delta^{=}\} = \{\Delta^-, \Delta^{=}\} = 0 \\ \{Q_0^-, \Delta^-\} + \{Q_0, \Delta^{=}\} = 0.\end{gathered} \tag{4.73}$$

4.7 First order gravitational action

Now we wish to construct the action for gravity at first order. In order to follow the procedure above, we need to find a top term from which we can begin the descent equation. It is also convenient to set $\Delta = 0$ temporarily in order to find the solution at the classical level, $\check{\Gamma}_1$. Thus our descent equations are now

$$Q_0\check{\Gamma}_1^{n-1} + Q_0^-\check{\Gamma}_1^n = 0. \tag{4.74}$$

It can be shown that all solutions to (4.74) for $n > 2$ are cohomologically trivial [91, 92], and so our top term will be $\check{\Gamma}_1^2$, satisfying $Q_0\check{\Gamma}_1^2 = 0$. Clearly, there are many solutions to this, but the most useful for our purposes will be

$$\check{\Gamma}_1^2 = -(c^\mu\partial_\mu c^\nu)c^*_\nu \tag{4.75}$$

which represents using the Lie derivative of c^μ (in brackets) without use of a metric to raise/lower indices. Since we envisage higher-order terms being the "covariantised" versions of the free/linearised terms being expanded to higher orders, the need for a metric to raise indices, such as in [34] this will lead to antighost level 2 pieces at higher orders. This parametrization therefore, is more useful as these terms will not be present. Now, using the descent equations results in

$$Q_0\check{\Gamma}_1^1 = -Q_0^-\check{\Gamma}_1^2 = -2c_\alpha\partial_\alpha c_\mu\partial_\nu H^*_{\mu\nu} \tag{4.76}$$

[8] Clearly, since $(Q_0)^2 = 0$ it is not invertible, but by this it is just meant that the pre-image is easier to find.

[9] Note there is a slight abuse of "=" here, since these expressions do not all have the same mass dimension and other gradings, however it is hoped that the intention is clear nonetheless.

and it can be shown that this is solved by

$$\check{\Gamma}^1_1 = -(c^\alpha \partial_\alpha H_{\mu\nu} + 2c^\alpha \partial_\mu H_{\alpha\nu}) H^*_{\mu\nu} \tag{4.77}$$

and once again we see that the term in brackets, combined with the $H^*_{\mu\nu}$ term in the free action gives exactly the Lie derivative form we might expect. Again, the benefits of treating c^μ as a contravariant vector field manifest in the lack of a requirement to raise/lower indices with a metric to express things in terms of a Lie derivative, and so we do not see analogous terms at higher orders. For the antighost number 0 piece, we again use our descent equations

$$Q_0 \check{\Gamma}^0_1 = -Q^-_0 \check{\Gamma}^1_1 \tag{4.78}$$

and we can see that this is solved by

$$\begin{aligned}\check{\Gamma}^0_1 = {} & 2\varphi \partial_\mu H_{\mu\nu} \partial_\nu \varphi - 2\varphi(\partial_\mu \varphi)^2 - 2H_{\mu\nu}\partial_\rho H_{\rho\mu}\partial_\nu\varphi + 2H_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 2H_{\nu\rho}\partial_\rho H_{\mu\nu}\partial_\mu\varphi \\ & + \frac{1}{2}\varphi(\partial_\rho H_{\mu\nu})^2 - \frac{1}{2}H_{\rho\sigma}\partial_\rho H_{\mu\nu}\partial_\sigma H_{\mu\nu} - H_{\mu\rho}\partial_\sigma H_{\nu\rho}\partial_\sigma H_{\mu\nu} + 2H_{\mu\nu}\partial_\sigma H_{\nu\rho}\partial_\mu H_{\rho\sigma} \\ & + H_{\mu\rho}\partial_\sigma H_{\nu\rho}\partial_\nu H_{\mu\sigma} - \varphi\partial_\rho H_{\mu\nu}\partial_\mu H_{\nu\rho} - H_{\mu\nu}\partial_\rho H_{\mu\nu}\partial_\sigma H_{\rho\sigma} + 2H_{\mu\nu}\partial_\rho H_{\mu\nu}\partial_\rho\varphi\end{aligned} \tag{4.79}$$

which is the same as the 3-graviton vertex one would get from expanding the Einstein-Hilbert action to this order. It is worth noting at this stage that the above formulae are valid in d dimensions.

Our problem now is that the classical action $\check{\Gamma}_1$ does not satisfy the linearised flow equation (4.42) or mST (4.43). In order to solve this, we write[10]

$$\Gamma_1 = \check{\Gamma}_1 + \check{\Gamma}_{1q1}. \tag{4.80}$$

Geometrically, we know that the answer is. In Einstein gravity, the only possible terms at first order are the Einstein-Hilbert terms $\sqrt{g}R$, and the cosmological constant term $\sqrt{g}$, which to this order is $\sim \varphi$. Thus we expect a Λ-dependant quantum correction proportional to φ. This is found by writing out the linearised flow equation in full, being sure compute the tadpole corrections in the gauge-invariant basis. As seen in [4], the result of this procedure is to introduce a cosmological constant term

$$\check{\Gamma}_{1q1} = \frac{7}{2} b \Lambda^4 \varphi, \tag{4.81}$$

where we have defined the non-universal dimensionless constant

$$b = \frac{1}{\Lambda^4}\int_p C^\Lambda(p). \tag{4.82}$$

[10] Here $1q$ stands for being part of the solution to the linearised flow equation, and the remaining 1 is due to the order in the perturbation series in κ.

4.8 Second order classical solution

From now until the end of the chapter we will be following [7]. It will be useful to us to somewhat generalise some of the above procedure. Suppose $\check{\Gamma}$ solves the Classical Master Equation[11] (or Zinn-Justin equation)

$$0 = \frac{1}{2}(\check{\Gamma}, \check{\Gamma}) = (Q\Phi^A)\frac{\overrightarrow{\partial}}{\partial \Phi^A}\check{\Gamma}, \tag{4.83}$$

and takes the standard form

$$\check{\Gamma} = \check{\Gamma}^0 - (Q\Phi^A)\Phi^*_A. \tag{4.84}$$

Then we expand this as

$$\check{\Gamma} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!}\check{\Gamma}_n \tag{4.85}$$

so that $\check{\Gamma}_1$ is unchanged from the above. Similarly, we can expand the one-loop quantum piece, while noting that the quantum piece only contributes at antighost level zero

$$\check{\Gamma}^0_{1q} = \sum_{n=1}^{\infty} \frac{\kappa^n}{n!}\check{\Gamma}^0_{1qn}. \tag{4.86}$$

Indeed, since the one-loop quantum part is only level zero, we note that it does not interfere with the parametrization (4.84), and thus the Zinn-Justin identity (4.83) tells us that

$$0 = (\check{\Gamma}_1, \check{\Gamma}^0_{1q}) = (Q\Phi^A)\frac{\overrightarrow{\partial}}{\partial \Phi^A}\check{\Gamma}^0_{1q}. \tag{4.87}$$

In addition, and as noted previously, our choice in cohomology representative means that Q is given exactly: there are no further corrections. This observation can also be deduced by expanding the Zinn-Justin identities (4.83) and (4.87) to tell us

$$\left(\check{\Gamma}^2_1, \check{\Gamma}^2_1\right) = 0, \quad 2\left(\check{\Gamma}^2_1, \check{\Gamma}^1_1\right) + \left(\check{\Gamma}^1_1, \check{\Gamma}^1_1\right) = 0 \tag{4.88}$$

and we also see how the diffeomorpism invariance is expressed on $\check{\Gamma}^0_2$ and $\check{\Gamma}_{1q2}$:

$$Q_0\check{\Gamma}^0_2 = -\left(\check{\Gamma}^1_1, \check{\Gamma}^0_1\right), \quad Q_0\check{\Gamma}_{1q2} = -\left(\check{\Gamma}^1_1, \check{\Gamma}_{1q1}\right). \tag{4.89}$$

Now, since $\check{\Gamma}^0_1$ turned out to be the linearised Einstein-Hilbert action, and since $\check{\Gamma}_{1q1}$ is the $\mathcal{O}(\kappa)$ Λ-dependent cosmological constant term, it is natural to guess an all-orders solution

$$\check{\Gamma}^0 = -\frac{2}{\kappa^2}\sqrt{g}R, \quad \check{\Gamma}^0_{1q} = \frac{7}{2}b\Lambda^4\sqrt{g}. \tag{4.90}$$

[11]That is, it is invariant under the classical BRST charge Q.

In keeping with this guess, we see that expanding both of these to $\mathcal{O}(\kappa^2)$ leads to

$$\begin{aligned}
\check{\Gamma}^0_2 = \varphi^2 &\left(\frac{1}{4}\partial_\mu h_{\mu\nu}\partial_\nu\varphi - \frac{3}{16}(\partial_\mu\varphi)^2 + \frac{1}{8}(\partial_\rho h_{\mu\nu})^2 - \frac{1}{4}\partial_\rho h_{\mu\nu}\partial_\mu h_{\nu\rho}\right) \\
&+ \varphi\Big(h_{\mu\nu}\partial_\rho h_{\rho\mu}\partial_\nu\varphi - \frac{1}{4}\partial_\mu h^2_{\nu\rho}\partial_\mu\varphi - \frac{1}{4}h_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \partial_\rho h_{\rho\sigma}h_{\mu\nu}\partial_\sigma h_{\mu\nu} \\
&+ \frac{1}{2}\partial_\rho h_{\mu\nu}\partial_\sigma h_{\mu\nu}h_{\rho\sigma} - 2\partial_\mu h_{\nu\rho}\partial_\sigma h_{\mu\nu}h_{\rho\sigma} + \partial_\mu h_{\nu\rho}\partial_\mu h_{\nu\sigma}h_{\rho\sigma} - \partial_\mu h_{\nu\rho}\partial_\nu h_{\mu\sigma}h_{\rho\sigma}\Big) \\
&+ \frac{1}{2}\partial_\rho h_{\rho\alpha}h_{\alpha\beta}\partial_\beta h^2_{\mu\nu} + \partial_\rho h_{\rho\alpha}\partial_\alpha h_{\beta\mu}h_{\beta\nu}h_{\mu\nu} + \frac{1}{4}\partial_\rho h_{\rho\sigma}\partial_\sigma\varphi h^2_{\mu\nu} - \frac{1}{8}(\partial_\rho h_{\alpha\beta})^2 h^2_{\mu\nu} \\
&+ \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h_{\alpha\beta}h_{\mu\rho}h_{\nu\rho} + \partial_\alpha h_{\mu\beta}\partial_\alpha h_{\nu\beta}h_{\mu\rho}h_{\nu\rho} + \partial_\alpha h_{\mu\rho}\partial_\beta h_{\nu\rho}h_{\alpha\beta}h_{\mu\nu} \\
&- \partial_\alpha h_{\mu\rho}\partial_\nu h_{\beta\rho}h_{\alpha\beta}h_{\mu\nu} - 2\partial_\alpha h_{\mu\beta}\partial_\nu h_{\alpha\beta}h_{\mu\rho}h_{\nu\rho} - \frac{3}{2}\partial_\mu h_{\nu\rho}\partial_\rho h_{\alpha\beta}h_{\mu\alpha}h_{\nu\beta} \\
&+ \frac{1}{2}\partial_\rho h_{\alpha\beta}\partial_\rho h_{\mu\nu}h_{\mu\alpha}h_{\nu\beta} + \frac{1}{4}\partial_\rho h_{\alpha\beta}\partial_\alpha h_{\rho\beta}h^2_{\mu\nu} + \frac{1}{2}h_{\alpha\beta}\partial_\alpha h_{\beta\rho}\partial_\rho h^2_{\mu\nu} \\
&- \partial_\alpha\varphi\partial_\mu h_{\nu\alpha}h_{\mu\rho}h_{\nu\rho} - \partial_\alpha h_{\mu\beta}\partial_\beta h_{\alpha\nu}h_{\mu\rho}h_{\nu\rho} - \frac{1}{2}\partial_\alpha h_{\mu\rho}\partial_\rho h_{\nu\beta}h_{\alpha\beta}h_{\mu\nu} \\
&+ \partial_\alpha h_{\alpha\mu}\partial_\nu\varphi h_{\mu\rho}h_{\nu\rho} - \frac{1}{8}\left(\partial_\rho h^2_{\mu\nu}\right)^2 - \frac{3}{16}h^2_{\mu\nu}(\partial_\rho\varphi)^2
\end{aligned} \tag{4.91}$$

for the classical part, and the 1-loop quantum part is

$$\check{\Gamma}^0_{1q2} = \frac{7}{2}b\Lambda^4(\varphi^2 - h^2_{\mu\nu}). \tag{4.92}$$

Both of these would be awkward to derive using (4.89), but it can be shown that these do indeed satisfy the required equations for BRST invariance. If we wanted to derive these with (4.89), we would have to write $\check{\Gamma}^0_2$ and $\check{\Gamma}^0_{1q2}$ as general linear combinations of the allowed functionals (i.e., those with correct mass dimension, ghost number etc.) and then use (4.89) along with the known actions that we have to end up with a set of simultaneous equations. Clearly this would be needlessly convoluted when we have a "trial" functional that we know from the geometry and can check very simply. Essentially, it is easier to make an educated guess and apply the BRST transformation and antibracket than it is to attempt to invert these operations.

4.9 Inside the diffeomorphism invariant subspace

At second order, we can write the flow equation (4.37) and mST (4.38) as

$$\dot{\Gamma}_2 - \frac{1}{2}\mathrm{str}\,\dot{\Delta}_\Lambda\Gamma^{(2)}_2 = -\frac{1}{2}\mathrm{str}\,\Delta_\Lambda\Gamma^{(2)}_2\Delta_\Lambda\Gamma^{(2)}_2 \tag{4.93}$$

$$\hat{s}_0\Gamma_2 = -\frac{1}{2}(\Gamma_1,\Gamma_1) - \mathrm{tr}\,C^\Lambda\Gamma^{(2)}_{1*}\Delta_\Lambda\Gamma^{(2)}_1. \tag{4.94}$$

In [89], the general continuum limit to (4.93) was constructed, that is the solution which realises the full renormalized trajectory for $\Lambda \geq 0$. It takes the form

$$\Gamma_2 = \frac{1}{2}\left[1 + \mathcal{P}_\Lambda - (1+\mathcal{P}_\mu)e^{\mathcal{P}^\mu_\Lambda}\right]\Gamma_1\Gamma_1 + \Gamma_2(\mu) \tag{4.95}$$

where

$$\mathcal{P}^\mu_\Lambda = \Delta^{\mu\,AB}_\Lambda \frac{\overrightarrow{\partial}^L}{\partial\Phi_B}\frac{\overrightarrow{\partial}^R}{\partial\Phi_A} \tag{4.96}$$

where the L indicates that the derivative acts only on the left-hand factor, and the R similarly indicates that the derivative acts only on the right-hand factor. In addition, $\mathcal{P}_\Lambda$ and $\mathcal{P}_\mu$ are defined similarly by changing the cutoffs on the propagator to match those attached to $\mathcal{P}$. Clearly, these operators connect two copies Γ_1.

In (4.95), the first term on the RHS is the particular integral and the second term is the complementary solution. The complementary solution takes exactly the form (4.62), except now μ has meaning as the (arbitrary) initial point on the renormalized trajectory $0 < \mu < a\Lambda_p$. Importantly, the solution in (4.95) is finite with all of the divergences absorbed into the underlying couplings, as described in [89].

In what follows, we will assume we are in the diffeomorphism invariant subspace. Thus, we can use κ as our expansion parameter, and the solution (4.67) applies in this case. In particular, this solution can be substituted into the second-order mST (4.94) and into the particular integral, since these are well-defined in the IR and UV thanks to their regularisation, and remain so in the limit of large Λ_p. Since $\check{\Gamma}_1$ contains a maximum of three fields, the particular integral collapses to a one-loop integral (in particular, the exponential simplifies) so that the renormalized trajectory (4.95) can now be written as

$$\Gamma_2 = \Gamma_2(\mu) + \kappa^2(I_{2\Lambda} - I_{2\mu}) \tag{4.97}$$

where

$$I_{2k} = -\frac{1}{4}\mathrm{str}\left[\Delta_k\check{\Gamma}^{(2)}_1\Delta_k\check{\Gamma}^{(2)}_1\right]. \tag{4.98}$$

Note that in principle, $\check{\Gamma}_{1q1}$ appears in (4.98), but since the action that appears here is differentiated twice, this term drops out since this part of the action is linear in φ. In fact, (4.98) is identical to a standard 1-loop contribution in the standard quantization. In addition, since we are now in the diffeomorpism-invariant subspace, we re-label Γ_2 and $\Gamma_2(\mu)$ with factors of κ^2 so that the κ dependence entirely drops out of (4.97).

Note that the particular integral is now polynomial in the fields. In the large Λ_p limit, we also arrange for $\Gamma_2(\mu)$ to trivialise (become polynomial) as in Section 4.5. Thus, from a practical standpoint, we see that the calculation will be very similar to the standard quantization. There is, however, a conceptual difference. The coupling κ is, on the face of it, an irrelevant coupling which cannot be used to construct a continuum QFT. In our construction, however, the continuum limit (and renormalized trajectory) is expressed through the underlying couplings $g^{\sigma}_{2l+\epsilon}$ and it is these (marginally) relevant couplings which are renormalized.

From the viewpoint of the standard quantization, the large Λ_p limit of the renormalized trajectory (4.97) still looks odd due to the difference of two terms in the particular integral $I_{2\Lambda} - I_{2\mu}$. These parts are IR regulated but are seperately UV divergent. It is the case, however, that the difference is UV finite, and thus we regulate each one by, for example, dimensional regularization with $d = 4 - 2\epsilon$, as done in [35]. In addition, we can subtract the UV divergences using a gauge-invariant and cutoff independent scheme, such as $\overline{\text{MS}}$, since the UV divergences will cancel in difference anyway.

Since $\check{\Gamma}_1$ contains only three-point vertices, the particular integral contains only two-point vertices. When derivative-expanded, $I_{2\mu}$ trivially results in polynomial (in the fields) solutions to the linearised flow equation (4.42). These carry no Λ-dependence and the tadpole corrections, where they exist, are field-independent and thus, although they can be calculated, are discarded as they contain no physics. It is tempting, therefore, to dispense with $I_{2\mu}$ by absorbing it into the definition of the complementary solution $\Gamma_2(\mu) \mapsto \Gamma_2(\mu) + I_{2\mu}$. Doing this however, leaves $I_{2\Lambda}$ on its own, which is ambiguous on its own due to the UV divergence - it is the difference that is well-defined in the large-Λ_p limit.

Writing the solution to (4.93) in terms of its classical and one-loop parts $\Gamma_2 = \Gamma_{2\text{cl}} + \Gamma_{2q}$, we now write

$$\Gamma_{2\text{cl}} = \Gamma_{2\text{cl}}(\mu) \tag{4.99}$$

$$s_0\Gamma_{2\text{cl}} = -\frac{1}{2}\left(\check{\Gamma}_1, \check{\Gamma}_1\right) \tag{4.100}$$

$$\Gamma_{2q} = \Gamma_{2q}(\mu) + I_{2\Lambda} - I_{2\mu} \tag{4.101}$$

$$s_0\Gamma_{2q} - \Delta\Gamma_{2\text{cl}} = -\left(\check{\Gamma}_1, \check{\Gamma}_{1q}\right) - \operatorname{tr} C^{\Lambda}\check{\Gamma}^{(2)}_{1*}\Delta_{\Lambda}\check{\Gamma}^{(2)}_{1} \tag{4.102}$$

where we've similarly split the complementary solution in terms of its classical and one-loop parts $\Gamma_2(\mu) = \Gamma_{2\text{cl}}(\mu) + \Gamma_{2q}(\mu)$. Note that in addition, we have split the free quantum BRST operator $\hat{s}_0 = s_0 - \Delta$ into its classical part $s_0 = Q_0 + Q_0^-$ and its quantum part $\Delta = \Delta^- + \Delta^=$. As discussed in [7], (4.99)-(4.102) are a complete set of equations for $\mathcal{O}(\kappa^2)$.

The classical flow equation (4.99) says that $\Gamma_{2\text{cl}}$ must be Λ-independent. If $I_{2\mu}$ is absorbed into $\Gamma_2(\mu)$ as discussed above, then the remaining equations are simply what we'd get from the standard quantization at one-loop [35]. Now, since $I_{2\Lambda}$ is defined using dimensional regularization and a gauge invariant subtraction scheme, such as $\overline{\text{MS}}$, it will have a $\ln\mu_R$ ambiguity, where μ_R is the mass scale which is introduced due to the analytic continuation of mass dimension in the couplings. The insertion of the cutoff Λ leads to the mST (4.102), but for a gauge-invariant subtraction scheme, this will already be automatically satisfied.

These ambiguities, however are cancelled in the combination $I_{2\Lambda} - I_{2\mu}$, which is simply a reflection of the fact that the quantum part of the solution (4.101) is a well-defined expression. Thus, the mass parameter μ fulfils the role of being the arbitrary initial point on the renormalized trajectory. We can therefore choose to absorb $I_{2\mu}$ except for exchanging μ_R for μ. Then the $\overline{\text{MS}}$ scheme essentially amounts to imposing a renormalization condition at $\mu = \mu_R$.

The failure point of the standard quantization is usually seen as having the need to introduce new bare couplings in order to absorb UV divergences. In the standard quantization, these new couplings multiply new non-trivial BRST cohomology representatives order by order in perturbation theory, and so new couplings are needed at each order. But we do not need access to the UV divergences to see the problem. The freedom to change the scheme from $\overline{\text{MS}}$ to any other gauge invariant scheme is contained in the freedom to add suitable local terms in the finite parts of these divergences. These finite scheme ambiguities would force the introduction of new couplings on their own in standard quantization. Phrased in this way, however, the new couplings are finite. Even if we remain with the $\overline{\text{MS}}$ scheme, imposing μ_R independence would force the introduction of new couplings.

In this quantization, the UV divergences have already been absorbed into the underlying couplings $g^{\sigma}_{2l+\epsilon}$, and the ambiguities which arise from defining integrals cancel since we use the difference $I_{2\Lambda} - I_{2\mu}$. However, the equivalent freedom still exists order by order in κ. Indeed, the requirement that our solution for the renormalized trajectory (4.95) is independent of the initial point μ will force the existence of new effective couplings in the same way. More generally, we have the freedom to add a function to Γ_2 as long as it satisfies the LHS of the flow equation and mST (4.93,4.94), that is essentially the linearised equations. This corresponds to a change in complementary solution $\Gamma(\mu)$, which simply represents a change in our quantum BRST cohomology representative. Thus, once we have one solution for Γ_2, we know that other solutions will only differ by a change in the BRST cohomology representative. Since $I_{2\Lambda}$ on its own (defined with a suitable gauge-invariant scheme) will solve the equations, then we know that scheme ambiguities will be contained in changes to the complementary solution.

4.10 Vertices at second order

4.10.1 Antighost level one

We now wish to see how the vertices are calculated in practice. As already noted, (4.99) is Λ-independent, and the BRST invariance (4.100) then implies that the solution is

$$\Gamma_{2\text{cl}} = \Gamma_{2\text{cl}}(\mu) = \check{\Gamma}^0_2, \tag{4.103}$$

which is in fact independent of μ. There is only a one-loop tadpole (as anticipated), which is generated by the exponential in (4.62), which gives $\Gamma_{2q}(\mu) \ni \Gamma_{2q2}$, where

$$\Gamma_{2q2} = \Omega_\Lambda \left(\frac{3}{2}(\partial_\mu\varphi)^2 - 2\partial_\mu h_{\mu\nu}\partial_\nu\varphi - (\partial_\rho h_{\mu\nu})^2 + 2(\partial_\mu h_{\mu\nu})^2 \right) - \frac{3}{4} b\Lambda^4(\varphi^2 + h^2_{\mu\nu}). \tag{4.104}$$

Note that if we had absorbed $I_{2\mu}$ into $\Gamma_{2q}(\mu)$, this would be the full expression for $\Gamma_{2q}(\mu)$, being the unique $\mathcal{O}(\kappa^2)$ tadpole formed from the classical action.

Now we turn our attention to the particular integral. It is clear by inspection that the particular integral (4.97) and the RHS of the mST (4.102) cannot contribute above antighost level 2. In addition, it is possible to show that there is no contribution even at this level. In the particular integral, an antighost level 2 contribution would require either propagators from $\check{\Gamma}^2_1$ to $\check{\Gamma}^1_1$, or between two copies of $\check{\Gamma}^1_1$ with both antifields intact, but it is not possible to join the propagators in this way. Now, since $\check{\Gamma}_{1q}$ only has a level zero contribution, the antibracket in (4.102), while the correction term (trace term) at level two would require $\check{\Gamma}^{2(2)}_1$, and there is no way to join this by a propagator to $\check{\Gamma}^{(2)}_{1*}$. Thus we have $\Gamma^{n\geq 2}_{2q} = \Gamma^{n\geq 2}_{2q}(\mu) = 0$. This can also be seen by simply expanding out the relevant traces and seeing that no contributions of this type are possible.

Similarly, it can be shown that, at level one, the mST (4.102) collapses to

$$Q_0\Gamma^1_{2q} = 0 \tag{4.105}$$

since the correction term requires $\check{\Gamma}^{1(2)}_1$ with its antifield intact, but no such contributions are possible. The particular integral at this level becomes

$$I^1_{2\Lambda} = -\frac{1}{4}\text{str}\left(\Delta_\Lambda\check{\Gamma}^{1(2)}_1\Delta_\Lambda\check{\Gamma}^{1(2)}_1\right). \tag{4.106}$$

Now we note that, as discussed in [4], quantum correction should be computed in the gauge-fixed basis, which is related to the gauge invariant basis that we've been using by

$$H^*_{\mu\nu}|_{\text{gi}} = H^*_{\mu\nu}|_{\text{gf}} + \partial_{(\mu}\bar{c}_{\nu)} - \frac{1}{2}\delta_{\mu\nu}\partial_\rho\bar{c}_\rho. \tag{4.107}$$

In this basis, we see from the expression for $\check{\Gamma}^1_1$ (4.42) that the non-zero second derivatives (recalling that these must be with respect to fields, not antifields) are $\left(\check{\Gamma}^1_1\right)_{Hc}$ and $\left(\check{\Gamma}^1_1\right)_{H\bar{c}}$. Thus an example non-zero term in the particular integral is

$$(\Delta_\Lambda)_{HH}\left(\check{\Gamma}^1_1\right)_{Hc}(\Delta_\Lambda)_{c\bar{c}}\left(\check{\Gamma}^1_1\right)_{\bar{c}H}. \tag{4.108}$$

Note the rules for contracting indices are the same as those for "normal" matrices, and that the (super)trace ensures that the first and last indices match. In addition, the remaining terms are reached from this due to swapping c for $\bar{c}$ and from swapping the order of derivatives. Thus, the particular integral at level one is given by

$$I^1_{2\Lambda} = i \int_p H^*_{\mu\nu}(p) \mathcal{B}^I_{\mu\nu\alpha}(p,\Lambda) c_\alpha(-p) \tag{4.109}$$

where, in $d = 4$, we have

$$\begin{aligned} \mathcal{B}^I_{\mu\nu\alpha}(p,\Lambda) = -\int_q \frac{C_\Lambda(q)C_\Lambda(p+q)}{q^2(p+q)^2} \Big(&\frac{3}{2} p_\alpha q_{(\mu} q_{\nu)} + \frac{3}{2} p_\mu p_\nu q_\alpha + 3 p_{(\mu} q_{\nu)} q_\alpha \\ &+ p^2 p_{(\mu}\delta_{\nu)\alpha} + (p+q)^2 \left[2\delta_{\alpha(\mu} p_{\nu)} + 2\delta_{\alpha(\mu} q_{\nu)} + \delta_{\mu\nu} q_\alpha \right] \Big) \end{aligned} \tag{4.110}$$

which is a formal expression in $d = 4$, since there are both quadratic and logarithmic divergences. It is convenient to set the complementary solution to have the same form as (4.109), with kernal $\mathcal{B}^c(p,\mu)$, since this means that Γ^1_{2q} also has this form, with kernal

$$\mathcal{B}_{\mu\nu\alpha}(p,\Lambda) = \mathcal{B}^c_{\mu\nu\alpha}(p,\mu) + \mathcal{B}^I_{\mu\nu\alpha}(p,\Lambda) - \mathcal{B}^I_{\mu\nu\alpha}(p,\mu). \tag{4.111}$$

We define (4.109) using dimensional regularisation which automatically subtracts the quadratic divergence, and using $\overline{\text{MS}}$, we subtract the logarithmic divergence, leaving only the usual $\log \mu_R$ ambiguity. Expanding the momentum integral (equivalent to a derivative expansion) up to $\mathcal{O}(p^4)$ results in

$$\begin{aligned} (4\pi)^2 I^1_{2\Lambda} = \Lambda^2 \int_0^\infty du\, C(C-2) \left(\frac{1}{2}\varphi^* \partial \cdot c - \frac{9}{8} \hat{s}_0 (c^*_\mu c_\mu) \right) - \frac{1}{2} \varphi^* \Box \partial \cdot c \\ + \hat{s}_0 \left(\frac{1}{4} H^*_{\mu\nu} \partial_\mu \partial_\nu \varphi + \frac{5}{16} c^*_\mu \Box c_\mu \right) + \frac{1}{2} \int_0^\infty du\, u (C')^2\, \hat{s}_0 \left(H^*_{\mu\nu} \partial_\mu \partial_\nu \varphi - \frac{5}{4} c^*_\mu \Box c_\mu \right) \\ + \frac{1}{2} \left(\log\left(\frac{\mu_R^2}{\Lambda^2} \right) + \int_0^1 \frac{du}{u} (1-C)^2 + \int_0^\infty \frac{du}{u} C(C-2) \right) \hat{s}_0 \left(H^*_{\mu\nu} \partial_\mu \partial_\nu \varphi + \frac{3}{4} c^*_\mu \Box c_\mu \right) \end{aligned} \tag{4.112}$$

up to $\mathcal{O}(\partial^5)$ terms. Here, $C = C(u)$ is the cutoff function. Recall that $C^\Lambda(p) = C(u)$ where $u = p^2/\Lambda^2$, and we also have $C_\Lambda(p) = 1 - C^\Lambda(p) = \overline{C}(u)$. Note that we also have instances of Ω_Λ and b, which in these terms are given by

$$\Omega_\Lambda = \frac{\Lambda^2}{(4\pi)^2} \int_0^\infty du\, C(u), \quad b = \frac{1}{(4\pi)^2} \int_0^\infty du\, u C(u). \tag{4.113}$$

A method for the derivation of this is sketched in Chapter 5, along with explicit examples, since these will be easier to present. As previously noted, if we absorbed $I_{2\mu}$ into $\Gamma_2(\mu)$, then $\Gamma^1_{2q} = I^1_{2\Lambda}$ would already be a solution. In addition, the Λ-independent $\hat{s}_0$-exact pieces could be discarded by changing $\Gamma_2(\mu)$, but we keep them for consistency with $\overline{\text{MS}}$. Recognising that the final result must be independent of μ_R, we set the one-loop complementary solution to be

$$\Gamma^1_{2q}(\mu) = i\int_p H^*_{\mu\nu}(p)\mathcal{B}^c_{\mu\nu\alpha}(p,\mu)c_\alpha(-p) = I^1_{2\mu} + Z^1_2(\mu)\hat{s}_0\left(H^*_{\mu\nu}\partial_\mu\partial_\nu\varphi + \frac{3}{4}c^*_\mu\Box c_\mu\right), \tag{4.114}$$

which is independent of μ/μ_R, since this cancels between $I_{2\mu}$ and

$$Z^1_2(\mu) = \frac{1}{(4\pi)^2}\log\frac{\mu}{\mu_R} + z^1_2. \tag{4.115}$$

We there see that $\kappa^2 Z^1_2(\mu)$ induces a change in BRST cohomology represented, much as had been anticipated. In this case, the change is $\hat{s}_0$-exact and thus corresponds to a canonical reparametrization, as discussed in [7], and so Z^1_2 is a wavefunction-like parameter. Its presence is necessary as it ensures Γ^1_2 is independent of the initial point μ of the renormalized trajectory, since a change of $\mu \mapsto \alpha\mu$ in $\mathcal{B}_{\mu\nu\alpha}(p,\Lambda)$ can be absorbed by a change $\delta Z^1_2 = -\log\alpha/(4\pi)$. Finally, the one-loop solution all put together is

$$\begin{aligned}(4\pi)^2\Gamma^1_{2q} = \Lambda^2\int_0^\infty du\, C(C-2)\left(\frac{1}{2}\varphi^*\partial\cdot c - \frac{9}{8}\hat{s}_0(c^*_\mu c_\mu)\right) - \frac{1}{2}\varphi^*\Box\partial\cdot c \\ + \hat{s}_0\left(\frac{1}{4}H^*_{\mu\nu}\partial_\mu\partial_\nu\varphi + \frac{5}{16}c^*_\mu\Box c_\mu\right) + \frac{1}{2}\int_0^\infty du\, u(C')^2\,\hat{s}_0\left(H^*_{\mu\nu}\partial_\mu\partial_\nu\varphi - \frac{5}{4}c^*_\mu\Box c_\mu\right) \\ + \frac{1}{2}\left((4\pi)^2 Z^1_2(\Lambda) + \int_0^1\frac{du}{u}(1-C)^2 + \int_0^\infty\frac{du}{u}C(C-2)\right)\hat{s}_0\left(H^*_{\mu\nu}\partial_\mu\partial_\nu\varphi + \frac{3}{4}c^*_\mu\Box c_\mu\right).\end{aligned} \tag{4.116}$$

If we work with scaled variables, where Λ is absorbed according to dimensions, then Γ^1_{2q} depends on Λ only indirectly through Z^1_2. The scaled result is thus of the same self-similar form one would expect of a renormalized trajectory [2]. Renormalization schemes then follow from renormalization conditions (initial conditions) on Z^1_2. For example, $\overline{\text{MS}}$ is recovered using the condition

$$Z(\mu) = 0 \quad \text{at} \quad \mu = \mu_R \tag{4.117}$$

which sets $z^1_2 = 0$.

We can evaluate the physical limit $\lim_{\Lambda\to 0}\mathcal{B}_{\mu\nu\alpha}(p,\Lambda) = \mathcal{B}_{\mu\nu\alpha}(p)$ by evaluating (4.110) with the cutoffs set to 1 and dealing with this in the same manner as a standard Feynman integral to give us (in the above scheme)

$$(4\pi)^2\mathcal{B}_{\mu\nu\alpha}(p) = \left(\frac{3}{4}p^2 p_\mu\delta_{\nu\alpha} - \frac{1}{2}p_\mu p_\nu p_\alpha\right)\log\left(\frac{p^2}{\mu^2}\right) + \frac{2}{3}p_\mu p_\nu p_\alpha - \frac{5}{6}p^2 p_\mu\delta_{\nu\alpha} + \frac{1}{6}\delta_{\mu\nu}p^2 p_\alpha \tag{4.118}$$

where the net effect of the complementary solution and renormalization condition is to replace μ_R by μ.

4.10.2 Antighost level zero

At level zero, we write the one-loop solution (4.101) as

$$\Gamma^0_{2q} = \check{\Gamma}^0_{2q2} + \delta\Gamma^0_{2q}(\mu) + I^0_{2\Lambda} - I^0_{2\mu}, \tag{4.119}$$

where the first two terms are the complementary solution where we have split off the one-loop tadpole. Using a similar notation to above, we write

$$I^0_{2\Lambda} = \frac{1}{2}\int_p H_{\mu\nu}(p)\mathcal{A}^I_{\mu\nu\alpha\beta}(p,\Lambda)H_{\alpha\beta}(-p). \tag{4.120}$$

Here, $\mathcal{A}^I_{\mu\nu\alpha\beta}(p,\Lambda)$ has two contributions: one from two copies of $\check{\Gamma}^1_1$ connected by ghost propagators and one from two copies of $\check{\Gamma}^0_1$ connected by graviton propagators. In the following we will understand $\mathcal{A}_{\mu\nu\alpha\beta}$ to be symmetrised on (μ,ν) and on (α,β), and indeed symmetrised under the interchange of these pairs. In $d=4$, the formal integral we get is

$$\begin{aligned}
&\mathcal{A}^I_{\mu\nu\alpha\beta}(p,\Lambda) = \\
&-\int_q \frac{C_\Lambda(q)C_\Lambda(p+q)}{q^2(p+q)^2}\Big(p_\mu p_\nu p_\alpha p_\beta + 2p_\alpha p_\beta p_\mu q_\nu + 2p_\alpha p_\beta q_\mu q_\nu + p_\alpha p_\beta q_\nu q_\beta + 2p_\alpha q_\beta q_\mu q_\nu \\
&\quad + q_\alpha q_\beta q_\mu q_\nu - p^2\delta_{\alpha\mu}p_\nu p_\beta - \frac{1}{2}p^2\delta_{\mu\nu}(p_\alpha p_\beta + 3p_\alpha q_\beta + 3q_\alpha q_\beta) + \frac{1}{16}p^4\delta_{\mu\nu}\delta_{\alpha\beta} + \frac{1}{2}\delta_{\mu\alpha}\delta_{\nu\beta}\Big) \\
&+\int_q \frac{C_\Lambda(q)C_\Lambda(p+q)}{q^2}\Big(\frac{1}{8}p^2\delta_{\alpha\beta}\delta_{\mu\nu} + \frac{5}{4}p\cdot q\,\delta_{\alpha\beta}\delta_{\mu\nu} - p\cdot(p+q)\delta_{\mu\alpha}\delta_{\nu\beta} \\
&\quad + 2\delta_{\alpha\mu}(p+q)_\beta(p+q)_\nu - \delta_{\mu\nu}(p_\alpha p_\beta + 3p_\alpha q_\beta + q_\alpha q_\beta)\Big) + \frac{1}{4}\delta_{\mu\nu}\delta_{\alpha\beta}\int_q C_\Lambda(q)C_\Lambda(p+q).
\end{aligned} \tag{4.121}$$

Once again, this is defined using dimensional regularization and $\overline{\text{MS}}$. We thus find that, Taylor expanding in p,

$$\begin{aligned}
(4\pi)^2 I^0_{2\Lambda} = {}& \Lambda^4\int_0^\infty du\, u\, C(C-2)\left(\frac{5}{24}h^2_{\mu\nu} + \frac{1}{8}\varphi^2\right) \\
&+ \Lambda^2\int_0^\infty du\, C(C-2)\left(\frac{5}{24}\varphi\partial_\mu\partial_\nu h_{\mu\nu} + \frac{5}{8}(\partial_\mu h_{\mu\nu})^2 - \frac{19}{48}(\partial_\rho h_{\mu\nu})^2 - \frac{5}{32}(\partial_\mu\varphi)^2\right) \\
&+ \Lambda^2\int_0^\infty du\, u^2(C')^2\left(\frac{1}{12}\varphi\partial_\mu\partial_\nu h_{\mu\nu} + \frac{1}{8}(\partial_\mu h_{\mu\nu})^2 + \frac{7}{96}(\partial_\rho h_{\mu\nu})^2 + \frac{1}{16}(\partial_\mu\varphi)^2\right) + \mathcal{O}(\partial^4).
\end{aligned} \tag{4.122}$$

This is a unique result, but acquires a $\log\mu_R$ dependence in the $\mathcal{O}(\partial^4)$ terms, which are not shown here as there are too many of them. As before, the complementary solution is chosen such that this is a solution (ignoring the tadpole) up to converting the $\log \mu_R$ dependence to $\log \mu$ dependence. Similarly to above, we find

$$\delta\Gamma^0_{2q}(\mu) = I^0_{2\mu} + Z^0_{2a}\left(R^{(1)}_{\mu\nu\alpha\beta}\right)^2 + Z^0_{2b}\left(R^{(1)}\right)^2 \tag{4.123}$$

where, to one loop we have

$$Z^0_{2a}(\mu) = -\frac{1}{(4\pi)^2}\frac{61}{120}\log\frac{\mu}{\mu_R} + z^0_{2a}, \quad Z^0_{2b}(\mu) = -\frac{1}{(4\pi)^2}\frac{23}{120}\log\frac{\mu}{\mu_R} + z^0_{2b}. \tag{4.124}$$

As before, $\kappa^2 Z_{2a,b}$ ensure that the full solution is independent of μ at this order, and also ensures that the scaled result is a self-similar solution [2]. Since the only other $\log\mu$ dependence, which is in $\Gamma^1_{2q}(\mu)$, is $\hat{s}_0$-closed, this addition must also be $\hat{s}_0$-closed (since the total must be), but it clearly is by virtue of being invariant under linearised diffeomorphisms. Indeed, they are also cohomologically trivial:

$$\hat{s}_0(\varphi^* R^{(1)}) = Q^-_0(\varphi^* R^{(1)}) = -\left(R^{(1)}\right)^2 \tag{4.125}$$

$$\hat{s}_0(H^*_{\mu\nu}R^{(1)}_{\mu\nu}) = 2G^{(1)}_{\mu\nu}R^{(1)}_{\mu\nu} = \frac{1}{2}\left(R^{(1)}_{\mu\nu\alpha\beta}\right)^2 - \frac{1}{2}\left(R^{(1)}\right)^2 \tag{4.126}$$

and thus, due to the action of the Koszul-Tate differential, these terms vanish when the free equations of motion are satisfied (i.e., on-shell) and so the $Z^0_{2a,b}$ are again wavefunction-like [25]. Indeed, this could be seen directly, using the Gauss-Bonnet identity

$$4\left(R^{(1)}_{\mu\nu}\right)^2 = \left(R^{(1)}_{\mu\nu\alpha\beta}\right)^2 + \left(R^{(1)}\right)^2 \tag{4.127}$$

to eliminate the square of the linearised Riemann tensor, and noting that the free equations of motion read $R_{\mu\nu} = 0$[12]. As previously, the $\overline{\text{MS}}$ scheme is recovered by choosing $z^0_{2a,b} = 0$ as our renormalization condition. In the physical limit, the tadpole correction Γ_{2q2} vanishes, so once again the net effect of the renormalization condition is to swap μ_R for μ. For the physical ($\Lambda \to 0$) Γ^0_2 two-point vertex we have (where $\mathcal{A}_{\mu\nu\alpha\beta}$ is understood to be appropriately symmetrized)

$$\begin{aligned}(4\pi)^2\mathcal{A}_{\mu\nu\alpha\beta}(p) = \bigg(&\frac{7}{10}p_\alpha p_\beta p_\mu p_\nu - \frac{23}{60}p^2\delta_{\alpha\beta}p_\mu p_\nu - \frac{61}{60}p^2\delta_{\alpha\mu}p_\beta p_\nu + \frac{23}{120}p^4\delta_{\alpha\beta}\delta_{\mu\nu} \\ &+ \frac{61}{120}p^4\delta_{\alpha\mu}\delta_{\beta\nu}\bigg)\log\left(\frac{p^2}{\mu^2}\right) + \frac{19}{75}p_\alpha p_\beta p_\alpha p_\beta - \frac{1229}{1800}p^2\delta_{\alpha\beta}p_\mu p_\nu \\ &- \frac{283}{1800}p^2\delta_{\alpha\mu}p_\beta p_\nu + \frac{1829}{3600}p^4\delta_{\alpha\beta}\delta_{\mu\nu} + \frac{283}{3600}p^4\delta_{\alpha\mu}\delta_{\beta\nu}\end{aligned} \tag{4.128}$$

where the quartic multiplying the log term is the same as appears in (4.123).

[12]Note that our coefficients do not agree with those in [25], which are computed using the background field method. Indeed, agreement is only required on-shell, which is trivial in this case since all of the terms vanish.

Finally, we look at the one-loop second order mST (4.102) at antighost level zero. We can write this as, using (4.89) to eliminate the antibracket,

$$Q_0\left(\Gamma^0_{2q}-\check{\Gamma}^0_{1q2}\right)+Q_0^-\Gamma^1_{2q}=-\operatorname{tr}C^\Lambda\check{\Gamma}^{(2)}_{1*}\Delta_\Lambda\check{\Gamma}^{(2)}_1\Big|^0 \tag{4.129}$$

since $\Delta\Gamma_2$ trivially vanishes. This last term has three contributions: one with $\check{\Gamma}^2_1$ and $\check{\Gamma}^1_1$ differentiated with respect to (anti)ghosts, and the other two using $\check{\Gamma}^1_1$ differentiated with respect to H^*. One of these has a second copy of $\check{\Gamma}^1_1$ differentiated with respect to H and $\bar{c}$, or $\check{\Gamma}^1_0$ where the differentials are both with respect to H. The result is then

$$-\operatorname{tr}C^\Lambda\check{\Gamma}^{(2)}_{1*}\Delta_\Lambda\check{\Gamma}^{(2)}_1\Big|^0=i\int_p H_{\mu\nu}(p)\mathcal{F}_{\mu\nu\alpha}(p,\Lambda)c_\alpha(-p), \tag{4.130}$$

where

$$\mathcal{F}_{\mu\nu\alpha}(p,\Lambda)=\int_q\frac{C_\Lambda(q)C^\Lambda(p+q)}{q^2}\Big(\delta_{\mu\nu}q^2p_\alpha+3\delta_{\mu\nu}q^2q_\alpha+2q_\mu q_\nu(p+q)_\alpha+4p_\mu p_\nu q_\alpha \\ -2p\cdot q(p+q)_{(\mu}\delta_{\nu)\alpha}+p\cdot q\delta_{\mu\nu}(p+q)_\alpha-4\delta_{\mu\nu}q_\alpha p^2\Big). \tag{4.131}$$

The above $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}$ vertices are analogous to vertices in Yang-Mills theory, which were labelled similarly in [35]. Note that since $\mathcal{F}_{\mu\nu\alpha}$ is regulated in both the UV and IR (and hence has no $1/\epsilon$ divergences), it is unaffected by $\overline{\text{MS}}$.

If we write $G^{(1)}_{\mu\nu}$ in momentum space as

$$G^{(1)}_{\mu\nu}(p)=-G^{(1)}_{\alpha\beta\mu\nu}(p)H_{\alpha\beta}(p), \tag{4.132}$$

we now see that (4.129) is a modified Slavnov-Taylor identity for two-point vertices:

$$\mathcal{A}_{\mu\nu\alpha\beta}p_\beta+G^{(1)}_{\mu\nu\rho\sigma}\mathcal{B}_{\rho\sigma\alpha}=\frac{7}{8}b\Lambda^4\left(\delta_{\mu\nu}p_\alpha-2p_{(\mu}\delta_{\nu)\alpha}\right)+\frac{1}{2}\mathcal{F}_{\mu\nu\alpha}. \tag{4.133}$$

Note that in the physical limit $\Lambda\to 0$, the RHS vanishes, and this equation simply because the normal (unmodified) Slavnov-Taylor identity: it simply says that, on-shell ($G^{(1)}_{\mu\nu}=0$), the amplitude $\mathcal{A}$ is gauge invariant. One can check that the physical vertices do indeed satisfy the $\Lambda\to 0$ limit of this equation. Further manipulation (involving the identity $C_\Lambda=1-C^\Lambda$, similar to those seen in [35] then shows that (4.133) also holds at general Λ. In fact, the Bianchi identity $p_\mu G^{(1)}_{\mu\nu}$, ensures that only the last term in the physical $\mathcal{B}$ vertex (4.118) makes a contribution. Thus, (4.133) states that the part of the physical $\mathcal{A}$ vertex which is dependent on renormalization conditions, that is the $\log p^2/\mu^2$ part of (4.128), is transverse, which in fact was also seen in (4.123). The derivative expansion of $\mathcal{F}$ gives us

$$\begin{aligned}
&-(4\pi)^2 \,\mathrm{tr}\, C^\Lambda \check{\Gamma}^{(2)}_{1*} \Delta_\Lambda \check{\Gamma}^{(2)}_1 \Big|^0 = \\
&\Lambda^4 \int_0^\infty du\, u\, C(C-2) \left(\frac{5}{6} h_{\mu\nu}\partial_\mu c_\nu + \frac{1}{4}\varphi \partial\cdot c\right) - b\Lambda^4 \left(\frac{7}{6} h_{\mu\nu}\partial_\mu c_\nu + \frac{15}{4}\varphi\partial\cdot c\right) \\
&\qquad + \Lambda^2 \int_0^\infty du\, C(C-2)\left(\frac{1}{3} h_{\mu\nu}\Box\partial_\mu c_\nu + \frac{11}{8}\varphi\Box\partial\cdot c - \frac{11}{12} h_{\mu\nu}\partial_\mu\partial_\nu\partial_\rho c_\rho\right) \\
&\qquad\qquad + \Lambda^2 \int_0^\infty du\, u^2 (C')^2 \left(\frac{1}{24} h_{\mu\nu}\partial_\mu\partial_\nu\partial_\rho c_\rho + \frac{13}{24}\Box\partial_\mu c_\nu\right) + \mathcal{O}(\partial^5)
\end{aligned} \tag{4.134}$$

and thus we can see that the mST (4.129) is satisfied up to this order. In particular we see that the tadpole contributions are exactly such that the $\mathcal{O}(\partial^0)$ and $\mathcal{O}(\partial^2)$ terms match.

4.11 Discussion

We have seen that, at second order in perturbation theory, the end result is the standard one for the one-particle irreducible effective action at $\mathcal{O}(\kappa^2)$, which is therefore a one loop contribution. Since so far we have dealt with pure gravity with vanishing cosmological constant, the logarithmic running within the diffeomorphism-invariant subspace is due to wavefunction-like reparametrisations. This is true in the standard quantisation [25] but is also seen in the new quantisation, as we might expect within the diffeomorphism-invariant subspace. Outside of this subspace, however, these reparametrisations are not purely wavefunction-like, but are accompanied by coefficient functions, which at antighost level zero take the form

$$\delta H_{\mu\nu} = R^{(1)}_{\mu\nu} f^a_\Lambda(\varphi,\mu) + \delta_{\mu\nu} R^{(1)} f^b_\Lambda(\varphi,\mu) \tag{4.135}$$

where

$$f^i_\Lambda(\varphi,\mu) \to c_i \kappa^2 \log\mu \quad \text{as} \quad \Lambda_p \to 0 \tag{4.136}$$

with c_i $(i = a, b)$ being numerical constants. There are also infinitely many perturbative reparameterisations possible of the form

$$\delta\varphi = f_\Lambda(h_{\mu\nu}, \varphi). \tag{4.137}$$

Some combinations will correspond to redundant operators [93, 94], which will lead to the kinds of reparametrisations that would lead to a demonstration of the quantum equivalence of unimodular gravity and ordinary gravity [34, 95] within the new quantisation.

Note that the logarithmic running encapsulated by $Z^1_2(\mu)$ and $Z^0_{2a,b}(\mu)$ is not the only logarithmic running in the theory. Infinitely more cases will be generated in the derivative expansion of the general solution for the second-order renormalized trajectory (4.95) [89]. However, all of these other cases vanish as a power of Λ_p in the amplitude suppression scale limit.

We would expect that once we add matter and/or a cosmological constant, it would no longer be the case that the logarithmic running within the diffeomorphism invariant subspace is attributable to a reparametrisation. Instead we would expect them to be attributed to new diffeomorphism invariant effective couplings. The reason for this is simply because, in $\delta\Gamma_{2q}(\mu)$, we now have that $R_{\mu\nu} \neq 0$ on-shell due to now having a non-zero energy-momentum tensor. These new couplings are precisely the same couplings as those which need to be introduced in the standard quantisation [25].

Indeed, these similarities between the new and old quantisation are inevitable. Once we are in the diffeomorphism invariant subspace, we are obeying both the RG flow equation and the mST, and so the solution in fact *must* correspond to an RG flow in the standard quantisation. The difference is purely in how one views the couplings, i.e., whether we define the theory using κ as a coupling or not. The problem in standard quantisation is that the flows have an infinite number of parameters, with new ones appearing at each loop order. In the standard quantisation they are identified with renormalized couplings, and the corresponding bare couplings are required to absorb the UV divergences. Thus we cannot construct a reormalized trjactory. This is as one might expect in the Wilsonian framework for the irrelevant coupling κ, with $[\kappa] = -1$.

In the new quantisation we have found a way around this and constructed a genuine perturbative renormalized trajectory. It has been shown (in the above and [4, 34, 89]) that this works at both first and second order. It emnates from the Gaussian fixed point along directions provided by the (marginally) relevant couplings $g^\sigma_{2l+\epsilon}$ and it is these which absorb the UV divergences. Once inside the diffeomorphism invariant subspace, this renormalized trajectory must coincide with a subset of the RG flows derived within the standard quantisation. The only question is which subset this will be. Since we send $\Lambda_p \to \infty$ to recover diffeomorphism invariance, we know that these flows must exist to $\Lambda \to \infty$ within the diffeomorphism invariant subspace, despite the fact that within this subspace they do not qualify as part of a perturbative renormalized trajectory.

Once inside the diffeomorphism invariant subspace, the underlying couplings $g^\sigma_{2l+\epsilon}$ disappear and the trajectory is then parametrised by diffeomorphism invariant effective couplings. One possibility is that there is no restriction: the subset in is the whole set and the effective couplings are in one-to-one correspondence with the couplings required in standard quantization. Clearly, this would be disastrous for the predictivity of the theory, however there is currently nothing to suggest that there are any inconsistencies with such a scenario.

Even if this is indeed the outcome, the new quantisation nevertheless provides a different perspective. For example, it is not true that the introduction of these higher order couplings require a loss of unitarity, provided that their signs are chosen to avoid wrong sign poles in the full propagators. In standard quantisation, the assumption is that once couplings are introduced for e.g. the curvature-squared terms, these couplings must be part of some “fundamental” bare action, and thus from the beginning turn the theory into one with higher derivatives even at the free (bilinear) level. Here, however, the bare action lies outside the diffeomorphism invariant subspace. The higher derivative interactions there must always be accompanied by a $\delta_\Lambda^{(n)}(\varphi)$ operator, and thus cannot alter the kinetic terms. In other words, the bilinear action maintains its two-derivative form [34].

It still remains the case that the perturbative development of the theory is organised in powers of κ and therefore by dimensions, accompanied by increasing numbers of spacetime derivatives at higher order. But since we are dealing with a theory with a genuine continuum limit, the fact that perturbation theory breaks down when $\kappa\partial > 1$ (∂ here stands for the typical magnitude of spacetime derivatives) simply indicates that the theory becomes non-perturbative in this regime and is not a signal of the breakdown of an effective quantum field theory description.

It is clear that it is the logarithmically running terms and the finite part ambiguities (necessarily BRST invariant) that demand the introduction of new couplings order by order in perturbative quantum gravity. In contrast, the power-law Λ-dependence is computed unambiguously. Nothing in perturbation theory demands that any new couplings be associated to such Λ^{2n} terms (with $n > 0$ an integer). We also note that the field dependence associated to Λ^{2n} is intimately related to the modifications of the Slavnov-Taylor identities, which tell us to what extent BRST invariance is violated. Thus the problem in quantum gravity is to find the *mechanism*, if there is one, that determines (all or some of) the finite parts associated to the $\log(\Lambda/\mu)$ terms that appear at the perturbative level. If, for example, all these parameters are fixed by such a mechanism, we would be left with only one new parameter at the quantum level: the mass scale that arises by dimensional transmutation from the very existence of the RG (the equivalent to Λ_{QCD} in QCD).

In fact we know that at third order, the first-order couplings will run with Λ [89]. It is conceivable that this running and the required subsequent matching into the diffeomorphism invariant subspace plays a role in providing this missing mechanism. Below, another possibility is discussed, which hints that this mechanism arises solely from insisting that the RG flow within the diffeomorphism invariant subspace remains non-singular all the way to $\Lambda \to \infty$. One well-studied possibility is a non-perturbative

UV fixed point: the asymptotic safety scenario [96–98]. However note that the current construction was born of attempts to solve issues with the degeneration of the fixed points and eigenoperator spectrum that are seen in that scenario if one goes (sufficiently carefully) beyond truncations involving just a finite number of operators (see, e.g., [4, 99]). Below a mechanism for fixing the parameter is explored which follows from the same mathematical properties of the partial differential flow equations that lead to these problems in the first place.

4.12 A possible non-perturbative mechanism

In the conformal sector, the infinite number of couplings $g^{\sigma}_{2l+\epsilon}$ lead to a new effect, namely the fact that almost always, even at the linearised level, RG flows towards the IR become singular and then cease to exist [5, 29]. The subsequent development, seen above and in [4, 34, 89] makes use of this in a fundamental way. Indeed it is for this reason that the construction requires the initial point μ for the renormalized trajectory (4.95) to lie below Λ_p, with most of the trajectory then being safely developed from the IR to the UV. This is due to the fact that we are dealing with solutions of a parabolic differential equation that are non-polynomial in the amplitude: such solutions are only guaranteed when flowing from the IR to the UV.

Of course, these comments apply equally to the $h_{\mu\nu}$ sector but with the crucial difference that there the equation is reverse parabolic, with solutions only guaranteed when flowing from the UV to the IR. The problem is not seen for polynomial linearised solutions, because these solutions are a finite sum of eigenoperators (Hermite polynomials) with constant coefficients. However, diffeomorphism invariance, which is imposed in the IR (i.e., in the diffeomorphism invariant subspace) requires us to use solutions which are non-polynomial in the amplitude of $h_{\mu\nu}$, since there are terms which depend on both $g_{\mu\nu}$ and $g^{\mu\nu}$. Thus diffeomorphism invariance forces us to consider solutions which are non-polynomial in $h_{\mu\nu}$, evolving from the IR to the UV. Such solutions almost always fail at some critical scale $\Lambda_{\rm cr}$ before we reach $\Lambda \to \infty$, in exactly the same way that the flows in the conformal sector fail in the IR.

In reality the solution must exist simultaneously in both the $h_{\mu\nu}$ and φ sectors. Looking again at the linearised flow equation (4.42), we consider a solution $\delta\Gamma$. Isolating the $h_{\mu\nu}$ and φ amplitude dependence, we can expand $\delta\Gamma$ over monomials $\zeta_{\mu_1\dots\mu_n}$:

$$\delta\Gamma = \sum_{\zeta} \zeta(\partial, \partial\varphi, \partial h, c, \Phi^*) f^{\zeta}_{\Lambda\,\mu_1\dots\mu_n}(h_{\alpha\beta}, \varphi) + \dots \tag{4.138}$$

where once again we use the ellipsis to denote tadpole corrections to ensure that the above is indeed a solution of the flow equation. These new coefficient functions f^{ζ}_{Λ} are necessarily non-polynomial in its arguments as argued above. Now, using the linearised flow equation (4.42) it can be shown that the coefficient functions have a flow equation

$$\dot{f}^{\zeta}_{\Lambda\,\mu_1\dots\mu_n} = \Omega_\Lambda \left(\frac{\partial^2}{\partial h^2_{\mu\nu}} - \frac{\partial^2}{\partial \varphi^2} \right) f^{\zeta}_{\Lambda\,\mu_1\dots\mu_n} \tag{4.139}$$

and hence the problem is clearly on display. The equation in either the φ or the $h_{\mu\nu}$ sectors are parabolic, but in opposite directions. Therefore, the Cauchy initial value problem is not well-defined for either direction. Evidently, one or the other sector will generically develop singularities (cusps) in finite time, unless very special initial conditions are used. In our language, this means that the RG flows will be a heavily restricted subspace of all possible flows one could construct within the diffeomorphism invariant subspace. It is worth noting that in this property is not solved by moving to the full non-linear flow equations, it merely obscures this property. In this section, hints are uncovered that the only independent couplings that are allowed to exist are κ and the cosmological constant.

It is worth noting that this issue only applies to the fields which are differentiated in the flow equation, i.e., the quantum fields, whose second order derivatives together with the RG time derivative make the equations (reverse) parabolic. It does not apply to the antifields, nor does it apply to background fields if the background field method is used. Indeed, it also does not apply to the ghost fields, since these are Grassmann-odd and therefore their dependence is necessarily polynomial. Thus the issue only arises for the quantum fluctuation fields $h_{\mu\nu}$ and φ.

To take this further, we recall that the finite part ambiguity $\delta\Gamma_{(l)}$ that appears at l-loop order is a local Λ-independent operator, and its dimension is determined by factors of κ

$$[\delta\Gamma_{(l)}] = 2(l+1). \tag{4.140}$$

We note that for the mST (4.38) to be satisfied within the diffeomorphism invariant subspace, we require $(\Gamma_0, \delta\Gamma_{(l)}) = 0$ (since all the other parts are higher loop order). Thus, at l-loop order the ambiguous parts $\delta\Gamma_{(l)}$ must be invariant under the full classical BRST transformations, reflecting standard treatments [87, 88, 100]. In particular, the level zero part $\delta\Gamma^0$ must be diffeomorphism invariant, and thus at one-loop are curvature-squared terms, as seen in $\delta\Gamma^0_{2q}$ (4.123). At two-loop, they must be κ^2 times curvature squared or higher-derivative terms such as $\kappa^2 R\nabla^2 R$, and so on. Thus they are indeed non-polynomial in $h_{\mu\nu}$ (and indeed φ as imposed by the new quantisation).

At higher loop orders than l, $\delta\Gamma_{(l)}$ gets altered by the flow equation (4.37) and mST (4.38) in non-straightforward ways. If we model the situation by simply using the linearised flow equation (4.42) and imposing $\delta\Gamma = \delta\Gamma_{(l)}$ at $\Lambda = 0$, then the perturbation will no longer satisfy BRST invariance or the mST when $\Lambda > 0$. However, we will still find restrictions that arise from the fact that the flows are typically singular. We see that the equation (4.139) is formally solved by Fourier transform:

$$f^{\zeta}_{\Lambda\,\mu_1\ldots\mu_n}(h_{\alpha\beta},\varphi) = \int \frac{d^9\pi_{\alpha\beta}d\pi}{(2\pi)^{10}} \mathfrak{f}^{\zeta}_{\mu_1\ldots\mu_n}(\pi_{\alpha\beta},\pi) e^{\frac{1}{2}\Omega_\Lambda(\pi^2_{\mu\nu}-\pi^2)+i\pi_{\mu\nu}h_{\mu\nu}+i\pi\varphi} \tag{4.141}$$

where $\pi_{\mu\nu}$ and π and the conjugate momenta to $h_{\mu\nu}$ and φ respectively (hence $\pi_{\mu\nu}$ traceless). Note that $\mathfrak{f}^{\zeta}_{\mu_1\ldots\mu_n}$ is the Fourier transform of

$$f^{\zeta}_{\mu_1\ldots\mu_n} = \lim_{\Lambda\to 0} f^{\zeta}_{\Lambda\,\mu_1\ldots\mu_n} \tag{4.142}$$

as can be seen by setting $\Lambda = 0$ in the above. However, for the above to be more than a formal solution, we need the Fourier integral to converge. Note that as Λ increases from zero, convergence in the φ sector only improves, as it is weighted by $e^{-\frac{\pi^2}{2}\Omega_\Lambda}$, reflecting the fact that the Cauchy initial value problem is well defined for this sector when we flow from the IR to the UV. By contrast, the $h_{\mu\nu}$ sector has an exponentially growing weight, and thus we see that, at fixed π, $\mathfrak{f}^{\zeta}$ must decay faster than an exponential of $\pi^2_{\mu\nu}$, or the solution will be singular at some critical scale $\Lambda = \Lambda_{\rm cr} \geq 0$, above which the flow will cease to exist.

We thus see that the flows will only exist for carefully chosen parametrisations of the metric in terms of $h_{\mu\nu}$ and φ. Now we show that solutions of the form (4.141) cannot exist simultaneously for all of the $\delta\Gamma$ that match the diffeomorphism invariant $\delta\Gamma_{(l)}$ at $\Lambda = 0$. If we take the Einstein-Hilbert action as an example and expand over monomials as in (4.138), the required strong suppression of $\pi_{\mu\nu}$ in $\mathfrak{f}^{\zeta}$ means the for the above to be a solution, there must be no rapid variation of the Einstein-Hilbert action under changes in the $h_{\mu\nu}$ amplitude. At a minimum, we need a parametrisation that exists for all amplitudes. This is not true of the standard linear split of $g_{\mu\nu}$, which is not positive definite for all $h_{\mu\nu}$ and φ, and is singular at $\kappa\varphi = -2$ and whenever $\kappa h_{\mu\nu}$ has -1 as an eigenvalue. This can be cured by parametrising the metric in terms of an exponential of $\kappa h_\mu{}^\nu$ (seen as a matrix), which would also ensure that the measure $\sqrt{g}$ does not lead to any branch cuts [101–105].

Such a parametrisation is not yet enough to allow a solution of the form (4.141). Note that we already require faster than exponential decay, we thus have $|\mathfrak{f}^{\zeta}_{\mu_1\ldots\mu_n}|^2$ integrable. Therefore, we know that, by Parseval's theorem, the squared coefficient functions $f^{\zeta}_{\Lambda\,\mu_1\ldots\mu_n}$ must also be integrable over $d^9h_{\alpha\beta}d\varphi$. This therefore means that the coefficient functions $f^{\zeta}_{\Lambda\,\mu_1,\ldots\mu_n}$ vanish as $h_{\alpha\beta}\to\infty$. Now, since $\sqrt{g}R \mapsto \alpha\sqrt{g}R$ under scaling $g_{\mu\nu}\mapsto \alpha g_{\mu\nu}$ (where α is some constant), we see that this final condition will hold for the Einstein-Hilbert action if and only if $g_{\mu\nu}$ itself vanishes in this limit.

A Fourier solution of the form (4.141) for the cosmological constant term is then not ruled out by this condition, since $\sqrt{g}\mapsto\alpha^2\sqrt{g}$ and thus will also vanish in the limit $h_{\alpha\beta}\to\infty$. However, all higher derivative terms are then ruled out from having such solutions, since curvature squared terms go like α^0, while higher orders behave as negative powers of α and thus diverge in the limit $h_{\alpha\beta} \to \infty$.

Note that despite the fact we are modelling using only linearised solutions, these argument are non-perturbative in κ, because the breakdown in the solutions happens at finite or diverging $\kappa h_{\mu\nu}$. In general, the level zero part satisfies $\delta\Gamma_{(l)} \mapsto \alpha^{1-l}\delta\Gamma_{(l)}$, and thus if these perturbations had to extend to solutions $\delta\Gamma$ of Fourier type (4.141), we would have shown that, despite the apparent freedom to change individually the new effective couplings that appear at each loop order, non-perturbatively in κ the requirement that the renormalized trajectory is non-singular actually rules out all such infinitesimal changes $\delta\Gamma_{(l)}$. We would therefore like to conclude that the only freely variable couplings are in fact κ itself and the cosmological constant.

However, such a dramatic conclusion cannot be drawn from this. These arguments can only really be interpreted as hints. Firstly, we note that there are solutions to the linearised flow equation (4.139) that are not of the Fourier form (4.141). For example, there are solutions which are polynomial in the graviton, but $\mathfrak{f}^\varsigma$ is then distributional. That is, a sum $\delta(\pi_{\alpha\beta})$ and its derivatives. In addition, $\delta\Gamma_{(l)}$ do not in fact satisfy the linearised equation (4.139), but instead

$$\delta\dot{\Gamma}_{(l)} = \frac{1}{2}\mathrm{str}\left(\dot{\Delta}_\Lambda\Delta_\Lambda^{(-1)}\left[1+\Delta_\Lambda\Gamma_I^{(2)}\right]^{-1}\Delta_\Lambda\delta\Gamma_{(l)}^{(2)}\left[1+\Delta_\Lambda\Gamma_I^{(2)}\right]^{-1}\right) \tag{4.143}$$

which is evidently much more involved than the simple linearised equations. However, it does share the property that the Cauchy initial value problem is not well defined in either direction.

Indeed, if we were to expect any other terms to arise from a more complete analysis, we would guess that some curvature-squared terms are also present. These are almost always seen in other studies of quantum gravity, and were only "marginally excluded" using our crude analysis above (i.e., they scaled like α^0 or logarithmically). A more detailed calculation should shed light on whether the coefficient of R^2 is freely variable.

4.13 Summary, conclusions and outlook

Everything in this chapter relates back to the observations of Chapter 2, and noting that flows close to the Gaussian fixed point which involve the conformal factor φ remain well-defined only if they are expanded over a countably infinite set of increasingly relevant operators $\delta_\Lambda^{(n)}(\varphi)$. The conceptual result is best summarized in Figure 4.1. The key difference between this and the standard quantisation is that the UV part of the trajectory is outside the diffeomorphism invariant subspace. Thus, the quantisation is defined "off space-time". In the UV, the traceless fluctuations $h_{\mu\nu}$ and conformal factor φ act as separate fields. The dynamical metric $g_{\mu\nu}$, which combines these as required by diffeomorphism invariance then only comes to be within the diffeomorphism invariant subspace.

The other key difference is that the renormalized trajectory is now parametrised in terms of an infinite set of couplings $g^{\sigma}_{2l+\epsilon}$. These parametrise the flow outside the diffeomorphism invariant subspace. Within this subspace, we see that diffeomorphism invariance is recovered if we insist that certain "trivialisation" conditions hold. In this case, we see that the infinite number of couplings collapse to form one effective coupling, κ. Thus, within the diffeomorphism invariant subspace, the QFT can be constructed using κ despite the fact that $[\kappa] = -1$, simply because κ does not exist near the Gaussian fixed point, where the power counting arguments are usually made.

This dependence on an infinite set of couplings somewhat resembles the split Ward identity. This is an identity used in many studies of background independence and more complete discussions can be found in [105–118]. The split Ward identity arises simply from the observation that when a metric is split into a background and fluctuation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{4.144}$$

then all physical quantities cannot depend on $\bar{g}$ and h individually, but must depend on their sum. This is what is meant by background independence. Essentially, the split Ward identity is to background independence as the mST is to diffeomorphism invariance.

The breaking of the mST is somewhat similar to the breaking of the split Ward identity, in that it results in a diffeomorphism invariant operator splitting into an infinitely many terms, each of which has couplings which flow independently. However, there are some crucial differences. Firstly, there is the fact that the split Ward identity is broken by gauge-fixing terms, at which point the correct thing to use is the Slavnov-Taylor identity, i.e., the identity which ensures BRST invariance for the quantum fields (that is, the appropriate generalisation of diffeomorphism invariance). This is then broken by the cutoff, at which point we use the mST which, as we have seen, ensures that BRST invariance is respected in the limit in which the cutoff is removed.

More strikingly, although the breaking of both the split Ward identity and the mST result in operators splitting into infinitely many parts, only in the current formalism are infinitely many of these are relevant near the Gaussian fixed point. In the standard treatment using the split Ward identity however, only the h, h^2 and h^3 terms are relevant, arising from the splitting of the cosmological constant term.

It is worth emphasising here why the use of $\mathcal{L}_-$ and allowing flows to exist *off* the diffeomorphism subspace are both required for this construction to work. Without the requirement that functions of φ lie in $\mathcal{L}_-$ (and its generalization when other fields are included), the eigenoperator spectrum degenerates and becomes continuous, and it is no longer possible to unambiguously divide a perturbation into its relevant and irrelevant parts [99]. This problem went unnoticed for a long time simply because of the fact that to see it, one must work with solutions that involve an infinite number of operators. However, since truncations have long been a large part of studying behaviour of theories under Wilsonian RG, investigations into quantum gravity involving the RG have involved truncations with only a finite number of operators retained, with few exceptions [43, 119, 120].

One may expect that restricting flows to the diffeomorphism invariant subspace (in contrast to Figure 4.1) may help matters since diffeomorphism invariance at the classical level restricts the functional dependence of the conformal factor to just a few operators at any given order in the derivative expansion. However, upon a more careful analysis, the $f(R)$ approximations [105, 106, 121–129], which are diffeomorphism invariant and keep an infinite number of operators, also display the degeneration of the eigenoperator spectrum.

Using the BRST formalism to express the diffeomorphism invariance, this chapter has outlined how to construct the first and second order gravitational action. It was shown in [89] that the renormalized trajectory can be solved at second order, and indeed that the required trivialisation can take place. In [7] it was shown that it is indeed possible for the renormalized trajectory to enter the diffeomorphism invariant subspace. The subsequent evolution was then solved, in particular in the physical limit $\Lambda \to 0$ where one obtains the physical amplitudes. It is worth noting that the result is equivalent to solving for quantum gravity at one loop and $\mathcal{O}(\kappa^2)$ in standard perturbation theory. Therefore, as in the standard case, we find that the effective parameters left behind are associated to logarithmically running terms at this order, and that for pure quantum gravity these are not physical because they can be absorbed by reparametrisations.

Finally, we note that the flow equations for $h_{\mu\nu}$ and φ sectors are parabolic in opposite directions (towards IR and UV, respectively). This means in particular that when we want a solution that involves both (and of course, physically we do) that the Cauchy initial value problem for the flow equation is not well-posed. In particular, if we use solutions which are non-polynomial in $h_{\mu\nu}$ and φ (which is required by diffeomorphism invariance) then the flows are typically singular. Some hints were found that this property would give a mechanism by which the free parameters in the theory are just κ and the cosmological constant. However, this would require a much more careful treatment, and indeed more study of how the presence of a cosmological constant affects the quantum properties of gravity within this new quantisation.

Chapter 5

Coupling a Scalar to Gravity

5.1 Introduction

Following Chapter 4, we wish to extend the formalism by including matter in the theory. As such, we will consider the simplest form of matter, a complex scalar χ. In addition, this chapter will include more detailed calculations than in the pure gravity case, since the methods used are easier to show in this case since the expressions tend to be somewhat simpler. We do note however, that the treatment in this chapter is rather simplistic and takes place entirely within the diffeomorphism invariant subspace of the theory. In order to gain a full understanding of the effects of adding matter to the theory, one would have to construct the theory off space-time, and then show that the required trivialisation can take place with a scalar present.

In addition, the notion of a perturbation series itself is potentially a problem. Previously, the perturbation series was ordered by powers of κ, and this made things easier since the terms at each order have definite mass dimension. However, once one includes other dimensionful couplings, such as a mass term, then $M\kappa$ is a dimensionless combination, and many of our notions of ordering in the perturbation series become significantly less powerful as an organisational tool. Furthermore, with additional couplings comes additional power series, and this leads to significant complications when trying to construct such theories. Nevertheless, we will look at some of the results within the diffeomorphism invariant subspace in the most "naive" way, and see what alterations this makes to the theory when constructed entirely within the diffeomorphism invariant subspace.

We first see in Section 5.2 how the BRST operator behaves for the scalar and how the free action is required to be modified. Then in Section 5.3 we see how this is used to construct the first order action, and we see how the "standard" classical solution (as one would get from an appropriately covariantised kinetic term) is modified by quantum corrections to ensure that the action is a solution of both the flow equation and mST to this order. Sections 5.4 & 5.5 show this process in action for the second order part of the action, which is where the logarithmic divergences start to appear and we follow the pure gravity example to absorb these with wave-function like corrections to the complementary function. Finally, Section 5.6 shows how the action derived in previous sections is compatible with the expression of diffeomorphism invariance in the mST, and how the tadpole corrections are required to make it so.

5.2 Free action and BRST algebra

We start with the free action for the graviton, and then we add the kinetic term for a complex scalar field χ. Note that in the BRST formalism, we also need to add an antifield with a BRST source term $-(Q\chi)\chi^*$. Using our conventions from the previous chapter, we treat c^μ as a contravariant vector field and we will use upstairs/downstairs indices interchangeably. Now, since we treat κc^μ as the small diffeomorphism when defining the transformation of the graviton, we are required by consistency to also do so here. As before, we are treating $\delta_{\mu\nu}$ as our background metric, and therefore we will not draw a distinction between upstairs and downstairs indices. Thus we have, from the standard transformation of a scalar field under diffeomorphisms,

$$Q\chi = \kappa c_\mu \partial_\mu \chi \tag{5.1}$$

and similarly for the complex conjugate $\bar{\chi}$. In particular, we note that this term is $\mathcal{O}(\kappa)$, and thus must not appear in the free action. Thus the free action for a graviton plus a complex scalar is

$$\Gamma_0 = \frac{1}{2} H_{\mu\nu} (\Delta_{\mu\nu,\alpha\beta})^{-1} H_{\alpha\beta} - 2\partial_\mu c_\nu H^*_{\mu\nu} + \partial_\mu \chi \partial_\mu \bar{\chi} \tag{5.2}$$

where $\Delta_{\mu\nu,\alpha\beta}$ is the graviton propagator. Even though Γ_0 is well-defined in its own right (i.e., without reference to a metric or gravity), the interpretation we have in mind is that $H_{\mu\nu}$ is the metric fluctuation[1]

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa H_{\mu\nu} \tag{5.3}$$

and we will bear this in mind when we check our algebraic answers using the geometry. All of the previous free BRST transformations and Koszul-Tate differentials are unchanged, but we repeat them here for convenience including the new transformations of the scalar:

$$Q_0 c_\mu = 0, \quad Q_0 H_{\mu\nu} = 2\partial_{(\mu} c_{\nu)}, \quad Q_0 \chi = 0, \quad Q_0 \bar{\chi} = 0 \tag{5.4}$$

[1] Note that this also explains interchanging upstairs and downstairs indices: we have $\delta_{\mu\nu}$ as a background metric.

$$Q_0^- c_\mu^* = 2\partial_\nu H_{\mu\nu}^*, \quad Q_0^- H_{\mu\nu}^* = -2G_{\mu\nu}^{(1)}, \quad Q_0^- \chi^* = -\Box\bar{\chi}, \quad Q_0^- \bar{\chi}^* = -\Box\chi \tag{5.5}$$

where we've defined the linearised Einstein tensor

$$G_{\mu\nu}^{(1)} = \frac{1}{2}\Box H_{\mu\nu} - \delta_{\mu\nu}\Box\varphi + \partial_\mu\partial_\nu\varphi + \frac{1}{2}\partial_\alpha\partial_\beta H_{\alpha\beta} - \partial_{(\mu}\partial^\alpha H_{\nu)\alpha}. \tag{5.6}$$

One important thing to note is that for a general field Φ with antifield Φ^*, we have

$$\bar{\Phi}^* = -\overline{\Phi^*} \tag{5.7}$$

in order to make the action real. This will be a useful fact to consider when constructing the action at first order.

5.3 First order action

We write the first order classical action as

$$\check{\Gamma}_1 = \check{\Gamma}_1^H + \check{\Gamma}_1^\chi \tag{5.8}$$

where $\check{\Gamma}_1^H$ is the action that appears in Section 4.7. In general, the H superscript will mean the pure gravity action and the χ superscript will mean the additional terms that arise from the presence of a scalar. We also assume that the full action due to the scalar Γ^χ has been expanded in κ and graded by antighost number in a notation which is hopefully clear. Now, since $\check{\Gamma}_1^2$ was found uniquely, there is no contribution from the χ sector. Thus the BRST descent equations for this part of the action reduce to

$$Q_0\check{\Gamma}^\chi{}_1^1 = 0, \quad Q_0\check{\Gamma}^\chi{}_1^0 + Q_0^-\check{\Gamma}^\chi{}_1^1 = 0. \tag{5.9}$$

Now we search for terms with which to build $\check{\Gamma}^\chi{}_1^1$. Clearly the terms allowed must contain χ or χ^* (and/or complex conjugates thereof), have mass dimension 5, vanishing total ghost number and antighost number 1. The most general form that is annihilated by Q_0 is

$$\check{\Gamma}^\chi{}_1^1 = \alpha\left([\partial_\mu\chi c_\mu\chi^* + \partial_\mu\bar{\chi}c_\mu\bar{\chi}^*] - [\chi c_\mu\partial\chi^* + \bar{\chi}c_\mu\partial_\mu\bar{\chi}^*]\right) \tag{5.10}$$

where α is some constant. Note that we have excluded symmetric combinations of the terms in square brackets since these would be equivalent (after integration by parts) a piece which is Q_0-exact, and thus ignored since we are dealing with the cohomology[2]. In fact, after integration by parts, we can remove another Q_0-exact piece and have

$$\check{\Gamma}^\chi{}_1^1 = \beta(\partial_\mu\chi c_\mu\chi^* + \partial_\mu\bar{\chi}c_\mu\bar{\chi}^*) \tag{5.11}$$

[2]In principle there are terms linear in χ such as $c_\mu\partial_\nu\chi H_{\mu\nu}^*$ which can be made real by adding the complex conjugate, but it can be shown that these terms cannot by consistent with the descent equations.

for some other constant, β. This constant will not be fixed until second order, and represents the relative strength of the gravitational coupling of the scalar compared to the graviton. In addition, we see that this form is as expected, since

$$\kappa\check{\Gamma}^{\chi}{}^{1}_{1} = \beta(Q\chi)\chi^{*} + \beta(Q\bar{\chi})\bar{\chi}^{*} \tag{5.12}$$

which corresponds to the BRST source term if $\beta = -1$. We will see various hints to this value of β at first order, even if the value is not yet fixed.

Next we use (5.9) to find the antighost level zero part. We find that

$$Q_0\check{\Gamma}^{\chi}{}^{0}_{1} = -\beta\left(\partial_\mu\chi c_\mu\Box\bar{\chi} + \partial_\mu\bar{\chi}c_\mu\Box\chi\right). \tag{5.13}$$

Now, since χ is inert under Q_0, it follows that each term in $\check{\Gamma}^{\chi}{}^{0}_{1}$ must have exactly one χ and one $\bar{\chi}$. Since $H_{\mu\nu}$ is the only other field allowed, then it follows that our general ansatz (which allows for integration by parts) is

$$\check{\Gamma}^{\chi}{}^{0}_{1} = A\partial_\mu\chi\partial_\nu\bar{\chi}H_{\mu\nu} + B\partial_\mu\partial_\mu\bar{\chi}H_{\nu\nu} + C\partial_\mu(\chi\bar{\chi})\partial_\nu H_{\mu\nu} + D\partial_\mu(\chi\bar{\chi})\partial_\mu H_{\nu\nu} \tag{5.14}$$

for constants A, B, C and D that we expect to find in terms of β. One can then see that this is solved by $A = \beta$, $B = -\frac{\beta}{2}$, $C = 0$ and $D = 0$, and thus we have

$$\check{\Gamma}^{\chi}{}^{0}_{1} = \beta\partial_\mu\chi\partial_\nu\bar{\chi}H_{\mu\nu} - \frac{\beta}{2}\partial_\mu\chi\partial_\mu\bar{\chi}H_{\nu\nu}. \tag{5.15}$$

Note that once again, we have uncovered a hint of $\beta = -1$, since in this case we would have exactly the $\mathcal{O}(\kappa)$ piece of

$$\sqrt{g}\partial_\mu\chi g^{\mu\nu}\partial_\nu\bar{\chi}. \tag{5.16}$$

As before, we note that $\Gamma_1 = \check{\Gamma}_1$ is not a solution to the linearised flow equation (repeated here for convenience)

$$\dot{\Gamma}_1 = -\frac{1}{2}\text{str}\,\dot{\Delta}^{\Lambda}\Gamma^{(2)}_1. \tag{5.17}$$

Similarly to the pure gravity case, we know that the only possible tadpole corrections have single fields, and thus these do not contribute to the RHS. Thus we have

$$\dot{\Gamma}^{\chi}_1 = -\beta\int_p \frac{\dot{C}^{\Lambda}(p)}{p^2}\frac{\delta^2}{\delta\chi(p)\delta\bar{\chi}(-p)}\left(\partial_\mu\chi\partial_\nu\bar{\chi}H_{\mu\nu} - \frac{1}{2}\partial_\mu\chi\partial_\mu\bar{\chi}H_{\nu\nu}\right). \tag{5.18}$$

Also similarly to the pure gravity case, we reason that the classical part of the solution does not run, and therefore, evaluating the functional derivatives, we have

$$\check{\Gamma}^{\chi}_{1q1} = \frac{\beta}{2}b\Lambda^4\varphi \tag{5.19}$$

where, as before, $\varphi = \frac{1}{2}H_{\mu\nu}$ and

$$b\Lambda^4 = \int_p C^\Lambda(p). \tag{5.20}$$

5.4 Second order classical action

To deal with the second order contributions, we refer to the formulae in Section 4.9. These are repeated below for convenience

$$\Gamma_{2\mathrm{cl}} = \Gamma_{2\mathrm{cl}}(\mu), \tag{5.21}$$

$$s_0\Gamma_{2\mathrm{cl}} = -\frac{1}{2}\left(\check{\Gamma}_1, \check{\Gamma}_1\right), \tag{5.22}$$

$$\Gamma_{2q} = \Gamma_{2q}(\mu) + I_{2\Lambda} - I_{2\mu}, \tag{5.23}$$

$$s_0\Gamma_{2q} - \Delta\Gamma_{2\mathrm{cl}} = -\left(\check{\Gamma}_1, \check{\Gamma}_{1q}\right) - \mathrm{tr}\, C^\Lambda \check{\Gamma}^{(2)}_{1*}\Delta_\Lambda \check{\Gamma}^{(2)}_I, \tag{5.24}$$

recalling that $s_0 = Q_0 + Q_0^-$ is the free classical BRST charge.

We have (in hopefully obvious notation) from (5.22)

$$s_0\left(\Gamma^H_{2\mathrm{cl}} + \Gamma^\chi_{2\mathrm{cl}}\right) = -\frac{1}{2}\left(\check{\Gamma}^H_1 + \check{\Gamma}^\chi_1, \check{\Gamma}^H_1 + \check{\Gamma}^\chi_1\right) \tag{5.25}$$

but we also know that (5.22) must hold for the gravitational action alone, and thus we have

$$s_0\Gamma^\chi_{2\mathrm{cl}} = -\frac{1}{2}\left(\check{\Gamma}^\chi_1, \check{\Gamma}^\chi_1\right) - \left(\check{\Gamma}^H_1, \check{\Gamma}^\chi_1\right). \tag{5.26}$$

Grading by antighost number, we get $\Gamma^{\chi}{}^2_{2\mathrm{cl}} = 0$ as a consistent solution (this is also zero by arguments in Chapter 4), and at antighost level one gives us

$$Q_0\Gamma^{\chi}{}^1_{2\mathrm{cl}} = -(\beta^2 + \beta)\left(\partial_\mu\chi\chi^* c_\nu\partial_\nu c_\mu + \partial_\mu\bar{\chi}\bar{\chi}^* c_\nu\partial_\nu c_\mu\right) \tag{5.27}$$

but the only way that the RHS can by in the image of Q_0 is if the derivatives are symmetrised, but this can't be done without removing the terms entirely. Thus we need $\beta^2 + \beta = 0$. This means that the only consistent choices are $\beta = 0$, which is pure gravity, and $\beta = -1$, as the hints from first order suggested, and has now been proven. We thus take $\beta = -1$ from now on, and we choose

$$\Gamma^{\chi}{}^1_{2\mathrm{cl}} = 0 \tag{5.28}$$

so that $\Gamma^{\chi}_{2\mathrm{cl}} = \Gamma^{\chi\,0}_{2\mathrm{cl}}$. Finally, we have

$$\begin{aligned} Q_0\Gamma^{\chi}_{2\mathrm{cl}} = \partial_{(\mu}\chi\partial_{\nu)}\bar{\chi}\left(c_\alpha\partial_\alpha H_{\mu\nu} + 2\partial_\mu c_\alpha H_{\alpha\nu}\right) - \frac{1}{2}\partial_\mu\chi\partial_\mu\bar{\chi}\left(c_\alpha\partial_\alpha H_{\nu\nu} + 2\partial_\nu c_\alpha H_{\alpha\nu}\right) \\ - c_\mu\partial_\mu\chi\partial_\rho(\partial_\sigma\bar{\chi}H_{\rho\sigma}) + \frac{1}{2}c_\mu\partial_\mu\chi\partial_\rho(\partial_\rho\bar{\chi}H_{\sigma\sigma}) \\ - c_\mu\partial_\mu\bar{\chi}\partial_\rho(\partial_\sigma\chi H_{\rho\sigma}) + \frac{1}{2}c_\mu\partial_\mu\bar{\chi}\partial_\rho(\partial_\rho\chi H_{\sigma\sigma}) \end{aligned} \quad (5.29)$$

which, after integrating by parts and making use of the fact that Q_0 is a left derivation, we find

$$\Gamma^{\chi}_{2\mathrm{cl}} = \left(H_{\mu\rho}H_{\rho\nu} - \frac{1}{2}H_{\mu\nu}H_{\rho\rho} + \delta_{\mu\nu}\left(\frac{1}{8}H^2_{\rho\rho} - \frac{1}{4}H_{\rho\sigma}H_{\rho\sigma}\right)\right)\partial_\mu\chi\partial_\nu\bar{\chi} \quad (5.30)$$

which can be shown to be the second order part of $\sqrt{g}\partial_\mu\chi g^{\mu\nu}\partial_\nu\bar{\chi}$, as we again would expect. Thus at both first and second order we see that the classical action is exactly the same as what one would obtain from the perturbative expansion of the covariant kinetic term for χ.

5.5 Quantum corrections at second order

In order to calculate the second order quantum corrections, we will need to use (5.23), where

$$I_{2k} = -\frac{1}{4}\mathrm{str}\,\Delta_k\check{\Gamma}^{(2)}_1\Delta_k\check{\Gamma}^{(2)}_1. \quad (5.31)$$

It can be shown that the only new contributions arise when both of the copies of the action come from the new scalar sector, since there is no way to have one differentiated with respect to χ without the other being differentiated with respect to $\bar{\chi}$, and since all new terms are $\sim \chi\bar{\chi}H$, there must be at least one χ derivative to make this contribute. In addition, the only way to join propagators consistently between the two copies of the action is in the level zero piece; there is no level one correction. The contributions are of two types. One involves both copies being differentiated with respect to H and then χ or $\bar{\chi}$ which will lead to corrections to the χ propagator, and the other involves both copies being differentiated with respect to both χ and $\bar{\chi}$, which will lead to corrections to the graviton propagator. We note that since we use the dimensional regularisation to define our integrals, we must take care to ensure that we work in d dimensions.

5.5.1 Corrections to the scalar propagator

Terms which contribute to the scalar propagator have the form

$$\left(\Delta_\Lambda\right)_{\chi\bar{\chi}}\left(\check{\Gamma}_1\right)_{\bar{\chi}H}\left(\Delta_\Lambda\right)_{HH}\left(\check{\Gamma}_1\right)_{H\chi} \quad (5.32)$$

and the related terms with $\chi \leftrightarrow \bar{\chi}$ and the order of derivatives changed. Thus we have

$$I^{\chi\chi}_{2\Lambda} = -\int_p \chi(p)\mathcal{A}^\chi(p,\Lambda)\bar{\chi}(-p) \tag{5.33}$$

where

$$\mathcal{A}^\chi = -\int_q \frac{C_\Lambda(q)C_\Lambda(p+q)}{q^2}\left(\frac{1}{2}\delta_{\mu\nu}(p\cdot q) - p_{(\mu}q_{\nu)}\right)\Delta_{\mu\nu,\alpha\beta}(p+q)\left(\frac{1}{2}\delta_{\alpha\beta}(p\cdot q) - p_{(\alpha}q_{\beta)}\right) \tag{5.34}$$

and

$$\Delta_{\mu\nu,\alpha\beta}(p) = \frac{1}{p^2}\left(\delta_{\mu(\alpha}\delta_{\beta)\nu} - \frac{1}{d-2}\delta_{\mu\nu}\delta_{\alpha\beta}\right) \tag{5.35}$$

is the graviton propagator. Therefore, after some manipulations we obtain

$$\mathcal{A}^\chi = -\frac{p^2}{2}\int_q \frac{C_\Lambda(q)C_\Lambda(p+q)}{q^2}. \tag{5.36}$$

Clearly, in the physical limit this will be automatically subtracted as part of the dimensional regularisation. Proceeding as previously, we Taylor expand $C_\Lambda(p) = \bar{C}(p^2/\Lambda^2) = 1 - C(p^2/\Lambda^2)$ up to second order (in order to get the derivative expansion to order $\mathcal{O}(\partial^4)$, for consistency with the previous work):

$$C_\Lambda(p+q) = \bar{C}(u) + \frac{\bar{C}'(u)}{\Lambda^2}\left(2(p\cdot q) + p^2\right) + \frac{\bar{C}''(u)}{2\Lambda^4}\left(2(p\cdot q) + p^2\right)^2 \tag{5.37}$$

where $u = q^2/\Lambda^2$. Now, we use a general result which proves useful in calculating many of these integrals. If $f(q)$ is a spherically symmetric function, then we have

$$\int_q f(q)q_{\mu_1}\dots q_{\mu_{2n}} = \int_q f(q)q^{2n}\prod_{k=1}^{n}\frac{1}{d+2(k-1)}\sum_{\text{pairs}}\delta_{\mu_{\sigma_1}\mu_{\sigma_2}}\dots\delta_{\mu_{\sigma_{2n-1}}\mu_{\sigma_{2n}}} \tag{5.38}$$

where the sum is over the ways of dividing the 2n indices into pairs. Thus, for example it allows us to make replacements such as

$$q_\mu q_\nu \longmapsto \frac{1}{d}\delta_{\mu\nu}q^2 \tag{5.39}$$

$$q_\mu q_\nu q_\rho q_\sigma \longmapsto \frac{1}{d(d+2)}(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})q^4. \tag{5.40}$$

In addition, whenever there is a product of an odd number of q_{μ_i}, the integral is zero (odd integrand over symmetric region). Thus we have

$$\mathcal{A}^\chi(p,\Lambda) = -\frac{p^2}{2}\int_q \frac{\bar{C}(u)^2}{q^2} - \frac{p^4}{2\Lambda^2}\int_q \frac{\bar{C}(u)\bar{C}'(u)}{q^2} - \frac{p^4}{d\Lambda^4}\int_q \bar{C}(u)\bar{C}''(u). \tag{5.41}$$

Note that these are all IR regulated, since $\bar{C}(u)$ is the IR regulator. In addition, since $\bar{C}'(u) = -C'(u)$ this means that any expression with a differentiated cutoff is also UV regulated. Thus the only types of integrals that require a touch of care are the integrals of $\bar{C}(u)^2$. To see how this is done, we first need to cast these integrals in terms of u.

Since all of the integrals involved are spherically symmetric, we have

$$\int_q = \frac{1}{\Gamma\left(\frac{d}{2}\right)(4\pi)^{\frac{d}{2}}}\Lambda^d \int_0^\infty du\, u^{\frac{d}{2}-1} \tag{5.42}$$

where the prefactor comes from the integration over the $d-1$ dimensional sphere. In the dimensional regularization we will take $d = 4-2\epsilon$, and at one-loop we will have at worst a $1/\epsilon$ pole, and so we are safe to limit ourselves to $\mathcal{O}(\epsilon)$ expansions. To this order, we have

$$\int_q = (\mu_R^2)^{-\epsilon}\frac{\Lambda^4}{(4\pi)^2}\left(1+\epsilon\left(1-\gamma-\log\left(\frac{\Lambda^2}{\mu_R^2(4\pi)}\right)\right)\right)\int_0^\infty du\, u^{1-\epsilon} \tag{5.43}$$

where γ is the Euler-Mascheroni constant and μ_R the scale included as standard to ensure that the argument of the logarithm is dimensionless. With this, we have, after writing the part in brackets as $(\mu_R^2)^\epsilon R(\epsilon)$:

$$\begin{aligned}(4\pi)^2\mathcal{A}^\chi(p,\Lambda) = &-\frac{p^2\Lambda^2}{2}R(\epsilon)\int_0^\infty du\, u^{-\epsilon}\bar{C}(u)^2 - \frac{p^4}{2}R(\epsilon)\int_0^\infty du\, u^{-\epsilon}\bar{C}(u)\bar{C}'(u)\\ &-\frac{p^4}{d}R(\epsilon)\int_0^\infty du\, u^{1-\epsilon}\bar{C}(u)\bar{C}''(u). \end{aligned} \tag{5.44}$$

First we see how the $\bar{C}(u)^2$ terms are regulated. Taking direction from the Appendix in [7], we write, for n an integer

$$\begin{aligned}\int_0^\infty du\, u^{n-\epsilon}\bar{C}(u)^2 &= \int_0^1 du\, u^{n-\epsilon}\bar{C}(u)^2 + \int_1^\infty du\, u^{n-\epsilon}(\bar{C}(u)^2-1) + \int_1^\infty du\, u^{n-\epsilon}\\ &= \int_0^1 du\, u^n\bar{C}(u)^2 + \int_1^\infty du\, u^n C(C-2) - \frac{1}{n+1-\epsilon} + \mathcal{O}(\epsilon)\end{aligned}$$

where we've taken the $\epsilon \to 0$ limit in the first and second terms since they are already IR and UV finite (recall $C = 1-\bar{C}$ is the UV cutoff). Note that we have also discarded the upper limit of the final integral by analytic continuation of ϵ. Importantly we see that for $n \neq -1$, we can safely take the $\epsilon \to 0$ limit at this point and cancel the last part with a contribution from the first integral by writing it in a way such that it can be combined with the second integral:

$$\int_0^\infty du\, u^{n-\epsilon}\bar{C}(u)^2 = \int_0^\infty du\, u^n C(C-2) + \mathcal{O}(\epsilon), \quad n \neq -1. \tag{5.45}$$

For $n=-1$, we simply don't take the $\epsilon\to 0$ in the third integral above, and thus we have

$$\int_0^\infty du\, u^{-1-\epsilon}\bar{C}(u)^2 = \int_0^1 \frac{du}{u}(1-C)^2 + \int_1^\infty \frac{du}{u}C(C-2) + \frac{1}{\epsilon} + \mathcal{O}(\epsilon). \tag{5.46}$$

To remove ambiguity as much as possible, all integrands with a differentiated cutoff will be manipulated via integration by parts until they are the sum of terms where both cutoffs have the same number of derivatives. Thus, for example

$$\begin{aligned}\int_0^\infty du\, u^{n-\epsilon}\bar{C}(u)\bar{C}''(u) &= -\int_0^\infty du\, u^{n-\epsilon}(\bar{C}'(u))^2 - (n-\epsilon)\int_0^\infty du\, u^{n-1-\epsilon}\bar{C}(u)\bar{C}'(u)\\ &= -\int_0^\infty du\, u^{n-\epsilon}(C')^2 - \frac{1}{2}(n-\epsilon)\int_0^\infty du\, u^{n-1-\epsilon}\frac{d}{du}(\bar{C}^2)\end{aligned}$$

and thus, integrating by parts on the final term again, and taking $\epsilon\to 0$ since as stated earlier the integral is automatically regulated in the UV and the IR:

$$\int_0^\infty du\, u^{n-\epsilon}\bar{C}(u)\bar{C}''(u) = \frac{1}{2}n(n-1)\int_0^\infty u^{n-2}C(C-2) - \int_0^\infty du\, u^n (C')^2 + \mathcal{O}(\epsilon) \tag{5.47}$$

unless $n=1$, but in this case the answer is still finite, but requires a touch more care to reach it (see above).

With all of these rules in place, we see that there is no $1/\epsilon$ divergence from the integrals we have (at least, none which are not subsequently multiplied by ϵ) and thus in $\mathcal{A}^\chi$, we can set $R(\epsilon)=1$. In particular, this means that there will be no logarithmic divergence, as we might expect from an expression which gives zero in the physical limit. Thus, we have

$$(4\pi)^2\mathcal{A}^\chi(p,\Lambda) = -\frac{p^2\Lambda^2}{2}\int_0^\infty du\, C(C-2) - \frac{p^4}{4} - \frac{p^4}{4}\left(-\frac{1}{2} - \int_0^\infty du\, u(C')^2\right) \tag{5.48}$$

and finally, we find that

$$(4\pi)^2 I^{\chi\chi}_{2\Lambda} = -\frac{\Lambda^2}{2}\hat{s}_0(\chi\chi^*)\int_0^\infty du\, C(C-2) + \frac{1}{8}\hat{s}_0(\chi\Box\chi^*) - \frac{1}{4}\hat{s}_0(\chi\Box\chi^*)\int_0^\infty du\, u(C')^2 \tag{5.49}$$

so there are no logarithmic divergences leading to new couplings, and the terms which do contribute are all $\hat{s}_0$ exact, and in addition do not contribute to the mST, so we will pay no further attention to them. It is worth noting at this point that this will in fact be the case in general. With our BRST transformation as defined, the corrections to the χ two-point function will always be $\hat{s}_0$ exact to all orders in the derivative expansion, simply due to the observation that

$$\chi p^{2n}\bar{\chi} = \chi\Box^n\bar{\chi} = \hat{s}_0(\chi\Box^{n-1}\chi^*). \tag{5.50}$$

5.5.2 Corrections to the graviton propagator

The terms in the trace (5.31) which contribute to the graviton propagator must leave the H terms in $\check{\Gamma}_1^{\chi}$ intact, and therefore take the form

$$(\Delta_\Lambda)_{\chi\bar{\chi}}\left(\check{\Gamma}_1\right)_{\bar{\chi}\chi}(\Delta_\Lambda)_{\chi\bar{\chi}}\left(\check{\Gamma}_1\right)_{\bar{\chi}\chi} \tag{5.51}$$

and there is an additional contribution from $\chi \leftrightarrow \bar{\chi}$, so that

$$I_{2\Lambda}^{\chi H} = -\frac{1}{2}\int_p H_{\mu\nu}(p)\mathcal{A}^H_{\mu\nu\alpha\beta}(p,\Lambda)H_{\alpha\beta}(-p) \tag{5.52}$$

where

$$(4\pi)^2\mathcal{A}^H_{\mu\nu\alpha\beta}(p,\Lambda) = \int_q \frac{C_\Lambda(q)C_\Lambda(p+q)}{q^2(p+q)^2}\left(\frac{1}{2}q\cdot(p+q)\delta_{\alpha\beta} - q_{(\alpha}(p+q)_{\beta)}\right) \times\left(\frac{1}{2}q\cdot(p+q)\delta_{\mu\nu} - q_{(\mu}(p+q)_{\nu)}\right). \tag{5.53}$$

Taylor expanding this to $\mathcal{O}(p^4)$ and using the dimensional regularization, we have

$$\begin{aligned}(4\pi)^2\mathcal{A}^H_{\mu\nu\alpha\beta}(p,\Lambda) &= \frac{\Lambda^4}{24}\left(\delta_{\mu\nu}\delta_{\alpha\beta}+\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha}\right)\int_0^\infty du\, uC(C-2)\\ &+\Lambda^2\left(-\frac{1}{24}p^2\delta_{\mu\nu}\delta_{\alpha\beta}-\frac{1}{24}p^2\delta_{\mu(\alpha}\delta_{\beta)\nu}+\frac{1}{12}p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)}\right)\int_0^\infty du\, C(C-2)\\ &+\Lambda^2\left(-\frac{5}{96}p^2\delta_{\mu\nu}\delta_{\alpha\beta}-\frac{1}{48}p^2\delta_{\mu(\alpha}\delta_{\beta)\mu}-\frac{1}{12}p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)}\right)\int_0^\infty du\, u^2(C')^2\\ &+\frac{\Lambda^2}{16}\left(p_\mu p_\nu\delta_{\alpha\beta}+p_\alpha p_\beta\delta_{\mu\nu}\right)\int_0^\infty du\, u^2(C')^2+\mathcal{A}^H_{\mu\nu\alpha\beta}(p)_{\text{quart}}\\ &+\mathcal{A}^H_{\mu\nu\alpha\beta}(p)_{\log}\left(\log\left(\frac{\mu_R^2}{\Lambda^2}\right)+\int_0^1\frac{du}{u}(1-C)^2+\int_1^\infty\frac{du}{u}C(C-2)+\int_0^\infty du\, u(C')^2\right)\end{aligned} \tag{5.54}$$

where we've defined

$$\mathcal{A}^H_{\mu\nu\alpha\beta}(p)_{\log} = \frac{1}{30}p_\mu p_\nu p_\alpha p_\beta - \frac{p^2}{40}\left(p_\mu p_\nu\delta_{\alpha\beta}+p_\alpha p_\beta\delta_{\mu\nu}\right) - \frac{p^2}{60}p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)} + \frac{p^4}{40}\delta_{\mu\nu}\delta_{\alpha\beta}+\frac{p^4}{120}\delta_{\mu(\alpha}\delta_{\beta)\nu} \tag{5.55}$$

and

$$\mathcal{A}^H_{\mu\nu\alpha\beta}(p)_{\text{quart}} = \frac{47}{1800}p_\mu p_\nu p_\alpha p_\beta - \frac{1}{1200}p^2(p_\mu p_\nu \delta_{\alpha\beta} + p_\alpha p_\beta \delta_{\mu\nu}) - \frac{31}{1800}p^2 p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)}$$
$$- \frac{17}{1600}p^4\delta_{\mu\nu}\delta_{\alpha\beta} + \frac{47}{7200}p^4\delta_{\mu(\alpha}\delta_{\beta)\nu}$$
$$+ \left(\frac{1}{120}p_\mu p_\nu p_\alpha p_\beta - \frac{1}{60}p^2(p_\mu p_\nu \delta_{\alpha\beta} + p_\alpha p_\beta \delta_{\mu\nu}) + \frac{1}{60}p^2 p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)}\right.$$
$$\left. + \frac{11}{960}p^4\delta_{\mu\nu}\delta_{\alpha\beta} + \frac{1}{480}p^4\delta_{\mu(\alpha}\delta_{\beta)\nu}\right)\int_0^\infty du\, u^3 (C'')^2. \quad (5.56)$$

Therefore, our our integral is given by

$$(4\pi)^2 I^{\chi H}_{2\Lambda} = -\frac{\Lambda^4}{24}\left(2\varphi^2 + H_{\mu\nu}H_{\mu\nu}\right)\int_0^\infty du\, uC(C-2)$$
$$+ \Lambda^2\left(-\frac{1}{12}\varphi\Box\varphi - \frac{1}{48}H_{\mu\nu}\Box H_{\mu\nu} - \frac{1}{24}\partial_\mu H_{\mu\nu}\partial_\rho H_{\nu\rho}\right)\int_0^\infty du\, C(C-2)$$
$$+ \Lambda^2\left(-\frac{5}{48}\varphi\Box\varphi - \frac{1}{96}H_{\mu\nu}\Box H_{\mu\nu} + \frac{1}{24}\partial_\mu H_{\mu\nu}\partial_\rho H_{\nu\rho} + \frac{1}{8}\varphi\partial_\mu\partial_\nu H_{\mu\nu}\right)\int_0^\infty du\, u^2 (C')^2$$
$$+ \mathcal{O}(\partial^4). \quad (5.57)$$

The $\mathcal{O}(\partial^4)$ terms are not shown here since there are too many (although they can be clearly worked out from the above). The only piece that is of interest at this order is

$$\left(-\frac{1}{120}\left(R^{(1)}\right)^2 - \frac{1}{60}\left(R^{(1)}_{\mu\nu}\right)^2\right)\log\left(\frac{\mu_R^2}{\Lambda^2}\right). \quad (5.58)$$

Taking the lead from the previous chapter, we write

$$\Gamma^\chi_{2q} = \Gamma^\chi_{2q2} + \delta\Gamma^\chi_{2q}(\mu) + I^{\chi H}_{2\Lambda} - I^{\chi H}_{2\mu} \quad (5.59)$$

and we write

$$\delta\Gamma^\chi_{2q}(\mu) = I_{2\mu} + Z^\chi_{2a}\left(R^{(1)}\right)^2 + Z^\chi_{2b}\left(R^{(1)}_{\mu\nu}\right)^2 \quad (5.60)$$

where

$$Z^\chi_{2a} = -\frac{1}{(4\pi)^2}\frac{1}{60}\log\left(\frac{\mu}{\mu_R}\right), \quad Z^\chi_{2b} = -\frac{1}{(4\pi)^2}\frac{1}{30}\log\left(\frac{\mu}{\mu_R}\right) \quad (5.61)$$

and so we can once again remove the μ_R dependence and replace it with μ using wavefunction-like parameters. Note that we have already used a renormalization condition of the same form as used in Section 4.10. Looking at the physical limit, we have

$$\mathcal{A}^H_{\mu\nu\alpha\beta} = -\mathcal{A}_{\mu\nu\alpha\beta}(p)_{\log}\log\left(\frac{p^2}{\mu^2}\right) + \frac{47}{900}p_\mu p_\nu p_\alpha p_\beta - \frac{2}{75}p^2(p_\mu p_\nu\delta_{\alpha\beta} + p_\alpha p_\beta\delta_{\mu\nu})$$
$$- \frac{23}{450}p^2 p_{(\mu}\delta_{\nu)(\alpha}p_{\beta)} + \frac{2}{75}p^4\delta_{\mu\nu}\delta_{\alpha\beta} + \frac{23}{900}p^4\delta_{\mu(\alpha}\delta_{\beta)\nu} \quad (5.62)$$

which results in a physical vertex of

$$\Gamma^{\chi}_{2q\text{phys}} = \left(\frac{1}{120}\left(R^{(1)}\right)^2 + \frac{1}{60}\left(R^{(1)}_{\mu\nu}\right)^2\right)\log\left(\frac{p^2}{\mu^2}\right) - \frac{1}{1800}\left(R^{(1)}\right)^2 - \frac{23}{450}\left(R^{(1)}_{\mu\nu}\right)^2. \tag{5.63}$$

5.6 Tadpole corrections and mST

Returning to (5.59), we see that we need to calculate the 2nd-order one-loop tadpole. Since this is given by the linear equation

$$\Gamma^{\chi}_{2q2} = -\frac{1}{2}\text{str}\,\Delta^{\Lambda}\check{\Gamma}^{\chi(2)}_{2} \tag{5.64}$$

it is regulated in the UV, and thus can be calculated in $d = 4$ (without dimensional regularization), where

$$\check{\Gamma}^{\chi}_{2} = \left(h_{\mu\rho}h_{\rho\nu} - \frac{1}{4}h_{\rho\sigma}h_{\rho\sigma}\delta_{\mu\nu}\right)\partial_\mu\chi\partial_\nu\bar{\chi}. \tag{5.65}$$

Therefore, the tadpole in fact vanishes, as can be readily seen fro m the expressions for the h propagator in $d = 4$. However, this is not the only propagator which contributes. Indeed, recall that the mST can be written as

$$Q_0(\Gamma^0_{2q} - \check{\Gamma}^0_{1q2}) + Q^-_0\Gamma^1_{2q} = -\,\text{tr}\,C^{\Lambda}\check{\Gamma}^{(2)}_{1*}\Delta_{\Lambda}\check{\Gamma}^{(2)}_{1}\Big|^0. \tag{5.66}$$

Since the LHS is linear, we can consider only the contributions due to χ (i.e., the ones in this chapter only) and the trace on the RHS can be seen to only contribute new terms when both copies of the action involve the new χ terms. In addition, since Γ^{χ}_{2q} has only antighost level zero pieces, we can drop one of the terms from the mST. Now, using the covariantisation of (5.19), we write

$$\check{\Gamma}^{\chi}_{1q} = -\frac{1}{2}b\Lambda^4\sqrt{g} \tag{5.67}$$

and therefore, we have

$$\check{\Gamma}^{\chi}_{1q2} = -\frac{1}{2}b\Lambda^4\left(\varphi^2 - h^2_{\mu\nu}\right). \tag{5.68}$$

Note that as previously, this is in fact a guess at this point and it is the mST that will verify that it is indeed the correct contribution. Now we wish to evaluate the trace:

$$-\,\text{tr}\,C^{\Lambda}\check{\Gamma}^{\chi\,(2)}_{1*}\Delta_{\Lambda}\check{\Gamma}^{\chi(2)}_{1} = -2i\int_p H_{\mu\nu}(p)\mathcal{F}^{\chi}_{\mu\nu\alpha}(p,\Lambda)c_\alpha(-p) \tag{5.69}$$

where

$$\mathcal{F}^{\chi}_{\mu\nu\alpha}(p,\Lambda) = \int_q \frac{C^{\Lambda}(p+q)C_{\Lambda}(q)}{q^2}\left((p+q)_{(\mu}q_{\nu)} - \frac{1}{2}(p+q)\cdot q\delta_{\mu\nu}\right)q_\alpha \tag{5.70}$$

which, when expanded in powers of momentum leads us to

$$\begin{aligned}\mathcal{F}^{\chi}_{\mu\nu\alpha}(p,\Lambda) = &-\Lambda^4 b\left(\frac{1}{4}p_{(\mu}\delta_{\nu)\alpha} - \frac{1}{8}p_\alpha\delta\right)\\ &-\Lambda^4\left(\frac{1}{12}p_{(\mu}\delta_{\nu)\alpha} + \frac{1}{24}p_\alpha\delta_{\mu\nu}\right)\int_0^\infty du\, uC(C-2)\\ &+\Lambda^2\left(\frac{1}{24}p_\alpha\delta_{\mu\nu}p^2 - \frac{1}{24}p_\mu p_\nu p_\alpha\right)\int_0^\infty du\, C(C-2)\\ &+\Lambda^2\left(\frac{1}{16}p^2 p_{(\mu}\delta_{\nu)\alpha} - \frac{p^2}{96}p_\alpha\delta_{\mu\nu} - \frac{1}{48}p_\mu p_\nu p_\alpha\right)\int_0^\infty du\, u^2(C')^2\\ &+\mathcal{O}(p^5)\end{aligned} \tag{5.71}$$

and this can easily be shown to give the required RHS to make the mST true. In particular, note that the b-dependent terms come entirely from our guess for $\check{\Gamma}^{\chi}_{1q2}$, which has now been vindicated.

The mST can be recast in terms of these vertices:

$$\mathcal{A}^H_{\mu\nu\alpha\beta}p_\beta = \frac{b\Lambda^4}{2}\left(\delta_{\mu\nu}p_\alpha - 2p_{(\mu}\delta_{\nu)\alpha}\right) - \mathcal{F}_{\mu\nu\alpha} \tag{5.72}$$

which, in the physical limit, simply states that

$$p_\mu \mathcal{A}^H_{\mu\nu\alpha\beta}(p) = 0 \tag{5.73}$$

i.e., that the physical vertex must be transverse. This is indeed seen to be the case since the physical graviton correction can be written exclusively as the square of linearised curvature invariants.

5.7 Discussion

There is a lot which is not explored in this chapter. One of the more obvious avenues of generalisation would be to give the scalar a mass term by adding this to the free action. However, this is potentially dangerous. While it is true that if one does this, the appropriate terms for $\sqrt{g}M^2\chi\bar{\chi}$ appear, it is clear that adding such a term to the free action will both change the free BRST cohomology, and we see that in fact we would no longer be working from the Gaussian fixed point.

This brings up another potential issue. The Gaussian fixed point in our framework is outside the diffeomorphism invariant subspace. All of the analysis of this chapter takes place within the diffeomorphism invariant subspace with the associated perturbation series in κ being assumed to be valid. While this appears safe for now, further analysis along the lines of that in [89] in needed to ascertain whether or not the trajectory in gravity-scalar theory enters the diffeomorphism invariant subspace, and what conditions are required to make it so. Thus the inclusion of a mass (and further couplings) must await further analysis. As noted in the introduction, the presence of other couplings makes it so that if we wish to keep the perturbation series order by order in κ, then we must consider any other couplings to all orders.

That is not to say that the above chapter is entirely useless. It shows that should a theory exist involving both the graviton and a scalar, then we know exactly what it must look like within the diffeomorphism invariant subspace. Similar calculations have already been carried out [25], but it is useful to see that the established results are reproduced in the BRST formalism, and thus can be carried over should it be shown that there is a renormalized trajectory from the Gaussian fixed point into the diffeomorphism invariant subspace.

In addition to a mass or quartic couplings one may wish to add at $\mathcal{O}(\kappa^0)$, one may also wish to introduce a conformal coupling term

$$\xi\sqrt{g}R\chi\bar{\chi}. \tag{5.74}$$

This first appears at $\mathcal{O}(\kappa^1)$ as $\xi R^{(1)}\chi\bar{\chi}$, which is a Q_0-exact piece which we are free to add to $\check{\Gamma}_1^{\chi 0}$. Adding this results in an analogous term being added to $\Gamma_{2\mathrm{cl}}^{\chi}$, and this does not seem to interfere with the results thus far. In addition, there appears to be no preferred value for ξ as yet, but analysis to further orders may change this.

All in all, the above chapter has shown that the formalism used to develop a continuum limit for gravity still has a chance of working when (scalar) matter is involved. That is to say, once in the diffeomorphism invariant subspace it was in fact required that the known diffeomorphism invariant possibilities were reproduced for the sake of consistency. What remains to be seen however, is whether, and under what conditions the renormalized trajectory from the Gaussian fixed point enters the diffeomorphism invariant subspace, and this would hopefully also have something to say about the way in which we extend the above to include interacting and massive scalars, as well as vector fields.

Chapter 6

Summary and outlook

This thesis is the product of an attempt to understand quantum gravity. At its heart, a redefinition of what is meant by quantisation. Chapter 1 reviews where we start from, which is an understanding of the Wilsonian renormalization group. Clearly, there are many more comprehensive reviews, e.g. [1–3] but the content in Chapter 1 is intended to be as "bare bones" as possible. The most novel part here is the treatment of the scalar field potential, as done in [29], which exposes the RG flow equation as having similarities to diffusion in RG time. In addition, the eigenoperators satisfy a Sturm-Liouville equation, the mathematical properties of which are widely known. This provides a way in which to deal with the major scalar field one is interested in when it comes to studies of quantum gravity: the conformal factor.

In Chapter 2 the conformal factor is studied in detail. This material is based on [5, 29]. By expanding the Einstein-Hilbert action, we see that the kinetic term (in an appropriate gauge) for the conformal factor is exactly as for a "normal" scalar field, but with the wrong sign of the kinetic term. This leads to the path integral being ill-defined due to a growing exponential for sufficiently rapidly varying kinetic terms. However, the RG flow equation is still well-defined, and we take this, rather than the path integral, to define the theory. For simplicity, Chapter 2 takes the scalar field in isolation for the remainder in order to study its properties and eigenoperators, which will prove useful when dealing with the full quantum theory of gravity. The Sturm-Liouville form is somewhat altered to give a weight function that is a growing exponential, and as such the space of solutions is hugely restricted, and is shown to be spanned by the (countably) infinite set of operators $\delta_\Lambda^{(n)}(\varphi)$. Note that this is an alternative treatment to that given in [36], in which the conformal factor is analytically continued to the imaginary axis in order to restore the "correct" sign of the kinetic term.

The properties of these operators are then explored. Defining the QFT of a scalar with a negative kinetic term on a manifold with a non-trivial metric proves rather difficult and potentially futile without a full theory of gravity to work with, so tori are chosen as the playground to explore the properties of the regularized tadpole operator Ω_p and, more specifically, the shape function $\mathcal{S}$. Both the case of $\mathbb{T}^4$ [29] and $\mathbb{T}^3 \times \mathbb{R}$ [5] were studied, and in both cases it was seen that the further the fundamental lengths of the tori differed (i.e., the more inhomogeneous the manifold), the larger the universe was allowed to be. Phrased differently (and for a better catchphrase), small universes are constrained to be highly symmetric. Twisted tori had a far richer structure, but the general trend is very much the same. Indeed, the positions of local extrema of $\mathcal{S}$ for twisted tori seemed to indicate a somewhat different definition of "most symmetric", but that is largely a semantic point.

Clearly this would be an excellent phenomenology if the trend of small universes being symmetric were to carry over into, for example, FLRW universes as it could then explain various cosmological questions. Most obviously, it provides a mechanism for the extremely homogeneous initial conditions for inflation, and could explain the "why now?" problem. However it is evidently far too early to make sweeping claims of this sort, since the mechanism has only really been studied for flat, toric universes. More pertinently to this particular thesis, however, is that many of these effects are predicated on a finite amplitude suppression scale Λ_p, and this is required to diverge in order to restore diffeomorphism invariance when the conformal factor is considered as part of a metric as opposed to a scalar field in isolation. Nevertheless, there are examples of similar calculations in lattice field theory [45], and to the present author's knowledge they have not been done for twisted tori, so the work may still bear fruit in this direction.

Chapter 3 represents something of a change in direction. It follows ideas from [33] and the work of [6]. The idea is somewhat simple. To take a regularisation structure that works for gauge theory, in the sense that it both regularises gauge theory and preserves the gauge invariance at all times, and apply the same framework to gravity. Of course, the history of physics is littered with examples of the phrase "...just do that, but for gravity" being met with failure, but theoretical physicists are evidently gluttons for punishment.

For gauge theory, the $SU(N)$ algebra is extended to a $SU(N|N)$ superalgebra (although, for technical reasons not specified here, even this is non-trivial, see [33]). In addition to the standard gauge field A^1_μ, there is a bosonic copy with the wrong sign kinetic term A^2_μ and a complex pair of fermionic fields B_μ, $\bar{B}_\mu$. Note that the fermionic vector fields are in violation of the spin-statistics relation, and thus cannot be physical at the end of the regularisation process. The $SU(N|N)$ is then broken by a scalar field which results in the fermionic fields gaining a mass and thus decoupling from the theory. Then, the A^2_μ couples to the physical vector field only by irrelevant operators, and so in the physical limit, we are left with $SU(N)$ gauge theory, with gauge invariance preserved at all points on the RG flow.

To extend this to gravity, one must extend the manifold to a supermanifold by adding 4 fermionic coordinates to the 4 standard (bosonic) coordinates that we are already familiar with. This in itself is non-trivial, but fortunately the topic of supermanifolds is well-known to mathematicians, for example the conventions in this thesis come from [77]. The introduction of non-commuting coordinates and geometric objects means that notions of coordinate transformations, vector fields, derivatives and tensors must be re-examined. In particular, indices can now be left or right as well as up or down, to indicate whether the Jacobian (or its transpose) multiplies the object from the left or the right under a coordinate transformation. Despite this apparently difficult set-up, many of the formulae are generalised in a very natural way.

We then see that the Einstein-Hilbert action can be suitably generalised. In order to get access to the propagating degrees of freedom, the action is expanded to $\mathcal{O}(\kappa^2)$ and is seen to be invariant under the suitable generalisation of linearised diffeomorphism invariance. The gauge is then fixed in order to get a handle on the propagating fields. In order to diagonalise the kinetic terms that arise from integration over the Grassmann coordinates, a mass parameter M is introduced in the Taylor expansion of $h_{AB}(x, \theta)$ over θ. Despite appearances, however, this does not play to role of a physical mass as it does not result in any shift in the poles of the propagators. In retrospect, this could have been seen simply from the fact it was introduced by hand and is somewhat arbitrary.

There is clearly a lot to be done before the work [6] and Chapter 3 can lead to a regularisation structure for gravity. Indeed, these should be seen as a starting point. The key part that is missing is a method by which the fermionic degrees of freedom gain a mass analogously to the situation for gauge theory. In particular, this likely involves a symmetry-breaking scalar. Finding the form of the symmetry breaking which leads to unbroken bosonic diffeomorphism invariance at all points of the RG flow is therefore the name of the game for all future work.

Based on [4, 7, 34, 35], Chapter 4 includes some of the concepts from Chapter 2 and includes them in a fully perturbative theory of quantum gravity. The key is the fact that the Quantum Master Equation (QME) $\Sigma = 0$, which is the requirement that gauge symmetry is preserved in the BRST formalism, is compatible with the RG flow equation. That is, the QME being satisfied at some scale Λ is sufficient for it to be satisfied for the RG flow.

After introducing the BRST algebra in relation to the Wilsonian effective action S, the step is taken to trade off the conceptual simplicity and elegance that comes with it for the practical simplicity of the Legendre average effective action Γ. One of the key issues is that fact that the conformal factor operators $\delta^{(n)}_\Lambda$ result in any perturbation from the Gaussian fixed point being parametrized by an infinite number of couplings. In particular, each monomial $\sigma(\partial, \partial\varphi, h, c)$ comes with an infinite number of couplings, and each has its own *amplitude suppression scale* Λ_σ. The expression of diffeomorphism invariance, the QME, becomes the modified Slavnov-Taylor identity (mST) when we pass the the Legendre average effective action description of the QFT, and it turns out that to satisfy this (and hence diffeomorphism invariance) in the physical limit it is convenient to set $\Lambda_\sigma = \Lambda_p$ (i.e., the same for all monomials) and the coefficients are then arranged in such a way that they "trivialize" in the large Λ_p limit. Thus we see that it is in this limit, and this limit only, that diffeomorphism invariance is in fact respected.

Indeed, Figure 4.1 now shows the conceptual difference between this quantisation and the standard procedure. We see that the renormalized trajectory near the Gaussian fixed point is outside the diffeomorphism invariant subspace, and is parametrised by an infinite number of underlying couplings. As the amplitude suppression scale is taken to the UV, we not only retrieve diffeomorphism invariance, but we also have all of the couplings subsumed into a single, effective, diffeomorphism invariant coupling κ. It is worth reminding ourselves that in the standard quantisation, many of the problems of renormalisability come from the fact that $[\kappa] = -1$. However, we have circumvented this issue since the argument that causes the negative mass dimension to cause a problem is based on proximity to the Gaussian fixed point. However, in this case, the trajectory near the Gaussian fixed point is parametrised by only relevant couplings, albeit an infinite number of them. Thus we see that in the diffeomorphism invariant subspace, we are free to organise a perturbation series order by order in κ.

We then saw how the first and second order classical gravitational action exactly coincided with what one would expect from expanding the Einstein-Hilbert action $-2\sqrt{g}R/\kappa^2$ to the appropriate order. In addition, it was found that, in keeping with the compatibility of the RG flow equation and mST, the tadpole corrections required to make the interacting action an eigenoperator of the flow equations are precisely the same as those required to satisfy the mST. In addition, we see that the one-loop corrections can be absorbed by wavefunction-like parameters which vanish on-shell.

The conclusion of Chapter 4 discusses the fact that the properties of the differential equations which cause the problems of renormalising gravity may also be the key to solving them. In particular, the fact that the linearised flow equation is parabolic for both φ and $h_{\mu\nu}$, but in different directions. Since we must solve for both of these sectors simultaneously, we have that the Cauchy initial value problem is generically not well-defined. In addition, there appears to be a problem in the sense that as we enter the diffeomorphism invariant subspace, there appears to be nothing to stop any number of higher order or higher derivative couplings being added. A possible mechanism for excluding all operators but the cosmological constant and the Einstein-Hilbert term is discussed, but clearly requires a more detailed analysis before any firm conclusions can be drawn.

It is worth comparing this approach to work done in asymptotic safety. As we have the seen, this treatment of the negative sign kinetic term in this thesis leads to a natural reversal of the direction in which the RG flow in this sector is well-posed. In [97], it can be seen that in the standard approach to asymptotic safety, one is required to take the (additive) cutoff for the conformal mode as having the opposite sign to that in the graviton sector. Failure to do this will result in spurious poles in the propagator. This change in sign results again in the reversal of the natural direction of the flow, as can be seen in equations (2.32) and (4.16) of [97], where we can clearly see a change in sign between the two sectors. We see here that since the sign of the second derivative is what determines the natural direction of the RG flow (c.f. diffusion), then any implementation of the flow equations will show this change of direction in the conformal sector.

In addition, the issue with the conformal sector being well-posed only towards the UV was noted in [43]. Here, it was found that one runs into numerical difficulties when attempting to solve the RG flow equation in the conformal sector towards the IR. As noted in [43], the issue is somewhat obscured in the literature as it does not become apparent until one attempts to move beyond polynomial truncations.

Finally, Chapter 5 (which is not based on any published work) shows how ne might add a scalar interacting with the graviton field in the BRST formalism. Clearly the treatment contained here is somewhat simplistic as it does not treat the theory as emanating from the Gaussian fixed point, but rather we treat it simply as already being within the diffeomorphism invariant subspace. It is found that, in the free massless scalar case at least, that the extra terms we get in the classical action are what we would get from a covariant kinetic term for the scalar. Even though the same can be said if mass and conformal coupling terms are added (in the sense that the extra terms are given by an appropriate covariantisation), it is also clear that the treatment is incomplete.

One thing in particular to note is the question of how one treats the mass term. The terms that match the covariant mass term arise when the bare mass term is added to the free action, but clearly at such a point we are deviating from the Gaussian fixed point. On the other hand, the theory of a scalar coupled to gravity has (up to now) not been connected with a trajectory which begins at the Gaussian fixed point and flows into the diffeomorphism invariant subspace. It is hoped that such a treatment may give hints on how to deal with a mass term, quartic coupling and conformal couplings. Note in addition that if we wish to keep the perturbation series for κ intact, then we must consider all orders of all other couplings.

Evidently, there is a lot of work to do with regards to the topics covered. The aesthetic advantages of the approaches above are simply that the Einstein-Hilbert action (which has been phenomenologically very accurate) is the starting point, and that the avenue for quantisation was in fact always there, hidden in the conformal factor instability which, if taken seriously, opens the door for a new method of quantising gravity. This is in no way proof that the above is the way forward, and there are a huge number of other approaches, many of which also show promise. However, until theories are developed to the point of making falsifiable predictions (and the technology required to test these is developed) we are only guided by aesthetic and mathematical arguments. The only thing to be said with any confidence is that quantum gravity, at least for the time being, is inextricably linked to the renormalization group, and so any approach would have to address the problems that each of these poses.

Bibliography

[1] Oliver J. Rosten. Fundamentals of the Exact Renormalization Group. *Phys. Rept.*, 511:177–272, 2012.

[2] Tim R. Morris. Elements of the continuous renormalization group. *Prog. Theor. Phys. Suppl.*, 131:395–414, 1998.

[3] Tim R. Morris. The Exact renormalization group and approximate solutions. *Int. J. Mod. Phys. A*, 9:2411–2450, 1994.

[4] Alex Mitchell and Tim R. Morris. The continuum limit of quantum gravity at first order in perturbation theory. *JHEP*, 06:138, 2020.

[5] Matthew P. Kellett and Tim R. Morris. Renormalization group properties of the conformal mode of a torus. *Class. Quant. Grav.*, 35(17):175002, 2018.

[6] Matthew Kellett and Tim R. Morris. Parisi-Sourlas supergravity. *Class. Quant. Grav.*, 37(19):195018, 2020.

[7] Matthew Kellett, Alex Mitchell, and Tim R. Morris. The continuum limit of quantum gravity at second order in perturbation theory. arXiv:2006.16682 2020.

[8] S.F. Novaes. Standard model: An Introduction. In *10th Jorge Andre Swieca Summer School: Particle and Fields*, pages 5–102, 1 1999.

[9] Sheldon L. Glashow. The renormalizability of vector meson interactions. *Nucl. Phys.*, 10:107–117, 1959.

[10] Abdus Salam and John Clive Ward. Weak and electromagnetic interactions. *Nuovo Cim.*, 11:568–577, 1959.

[11] Steven Weinberg. A Model of Leptons. *Phys. Rev. Lett.*, 19:1264–1266, 1967.

[12] Jogesh C. Pati and Abdus Salam. Lepton Number as the Fourth Color. *Phys. Rev. D*, 10:275–289, 1974. [Erratum: Phys.Rev.D 11, 703–703 (1975)].

[13] Graham G. Ross. *Grand Unified Theories.* Westview Press, 1 1985.

[14] Stuart Raby. Grand Unified Theories. In *2nd World Summit: Physics Beyond the Standard Model*, 8 2006.

[15] Kazunori Akiyama et al. First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole. *Astrophys. J.*, 875(1):L1, 2019.

[16] B.P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.*, 116(6):061102, 2016.

[17] Stephen Hawking. Occurrence of singularities in open universes. *Phys. Rev. Lett.*, 15:689–690, 1965.

[18] Roger Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14:57–59, 1965.

[19] Kenneth Eppley and Eric Hannah. The necessity of quantizing the gravitiational field. *Found. Phys.*, 7:51–68, 1977.

[20] Don N. Page and C. D. Geilker. Indirect evidence for quantum gravity. *Phys. Rev. Lett.*, 47:979–982, Oct 1981.

[21] Mark Albers, Claus Kiefer, and Marcel Reginatto. Measurement Analysis and Quantum Gravity. *Phys. Rev. D*, 78:064051, 2008.

[22] Adrian Kent. Simple refutation of the Eppley–Hannah argument. *Class. Quant. Grav.*, 35(24):245008, 2018.

[23] L.P. Kadanoff. Scaling laws for Ising models near T(c). *Physics Physique Fizika*, 2:263–272, 1966.

[24] Enrico Fermi. Tentativo di una teoria dell'emissione dei raggi beta. *Ric. Sci.*, 4:491–495, 1933.

[25] Gerard 't Hooft and M.J.G. Veltman. One loop divergencies in the theory of gravitation. *Ann. Inst. H. Poincare Phys. Theor. A*, 20:69–94, 1974.

[26] K.G. Wilson and John B. Kogut. The Renormalization group and the epsilon expansion. *Phys. Rept.*, 12:75–199, 1974.

[27] Joseph Polchinski. Renormalization and Effective Lagrangians. *Nucl. Phys. B*, 231:269–295, 1984.

[28] Christof Wetterich. Exact evolution equation for the effective potential. *Phys. Lett. B*, 301:90–94, 1993.

[29] Tim R. Morris. Renormalization group properties in the conformal sector: towards perturbatively renormalizable quantum gravity. *JHEP*, 08:024, 2018.

[30] Astrid Eichhorn. An asymptotically safe guide to quantum gravity and matter. *Front. Astron. Space Sci.*, 5:47, 2019.

[31] Kevin G. Falls, Daniel F. Litim, and Jan Schröder. Aspects of asymptotic safety for quantum gravity. *Phys. Rev. D*, 99(12):126015, 2019.

[32] R. Loll. Quantum Gravity from Causal Dynamical Triangulations: A Review. *Class. Quant. Grav.*, 37(1):013002, 2020.

[33] Stefano Arnone, Yuri A. Kubyshin, Tim R. Morris, and John F. Tighe. Gauge invariant regularization via SU(N—N). *Int. J. Mod. Phys. A*, 17:2283–2330, 2002.

[34] Tim R. Morris. Quantum gravity, renormalizability and diffeomorphism invariance. *SciPost Phys.*, 5(4):040, 2018.

[35] Yuji Igarashi, Katsumi Itoh, and Tim R. Morris. BRST in the exact renormalization group. *PTEP*, 2019(10):103B01, 2019.

[36] G.W. Gibbons, S.W. Hawking, and M.J. Perry. Path Integrals and the Indefiniteness of the Gravitational Action. *Nucl. Phys. B*, 138:141–150, 1978.

[37] Job Feldbrugge, Jean-Luc Lehners, and Neil Turok. No rescue for the no boundary proposal: Pointers to the future of quantum cosmology. *Phys. Rev. D*, 97(2):023509, 2018.

[38] Alfio Bonanno and Martin Reuter. Modulated Ground State of Gravity Theories with Stabilized Conformal Factor. *Phys. Rev. D*, 87(8):084019, 2013.

[39] M. Pospelov and M. Romalis. Lorentz invariance on trial. *Phys. Today*, 57N7:40–46, 2004.

[40] T. Pruttivarasin, M. Ramm, S.G. Porsev, I.I. Tupitsyn, M. Safronova, M.A. Hohensee, and H. Haeffner. A Michelson-Morley Test of Lorentz Symmetry for Electrons. *Nature*, 517:592, 2015.

[41] Nick Evans, Tim R. Morris, and Marc Scott. Translational symmetry breaking in field theories and the cosmological constant. *Phys. Rev. D*, 93(2):025019, 2016.

[42] Tim R. Morris. Perturbatively renormalizable quantum gravity. *Int. J. Mod. Phys. D*, 27(14):1847003, 2018.

[43] Alfio Bonanno and Filippo Guarnieri. Universality and Symmetry Breaking in Conformally Reduced Quantum Gravity. *Phys. Rev. D*, 86:105027, 2012.

[44] P. Hasenfratz and H. Leutwyler. Goldstone Boson Related Finite Size Effects in Field Theory and Critical Phenomena With O(N) Symmetry. *Nucl. Phys. B*, 343:241–284, 1990.

[45] Masashi Hayakawa and Shunpei Uno. QED in finite volume and finite size scaling effect on electromagnetic properties of hadrons. *Prog. Theor. Phys.*, 120:413–441, 2008.

[46] Bruce C. Berndt and Aiyangar Srinivasa Ramanujan. *Ramanujan's notebooks. Part 3.* Springer, 1991.

[47] J.P. Serre. *A Course in Arithmetic*. Graduate Texts in Mathematics. Springer New York, 2012.

[48] Joel A. Shapiro. Loop graph in the dual tube model. *Phys. Rev. D*, 5:1945–1948, 1972.

[49] Joseph Polchinski. Evaluation of the One Loop String Path Integral. *Commun. Math. Phys.*, 104:37, 1986.

[50] Tim R. Morris and Anthony W. H. Preston. Manifestly diffeomorphism invariant classical Exact Renormalization Group. *JHEP*, 06:012, 2016.

[51] Tim R. Morris. Noncompact pure gauge QED in 3-D is free. *Phys. Lett. B*, 357:225–231, 1995.

[52] Tim R. Morris. A Manifestly gauge invariant exact renormalization group. In *Workshop on the Exact Renormalization Group*, pages 1–40, 10 1998.

[53] Stefano Arnone, Tim R. Morris, and Oliver J. Rosten. Manifestly gauge invariant QED. *JHEP*, 10:115, 2005.

[54] Tim R. Morris and Oliver J. Rosten. Manifestly gauge invariant QCD. *J. Phys. A*, 39:11657–11681, 2006.

[55] Oliver J. Rosten. A Resummable beta-Function for Massless QED. *Phys. Lett. B*, 662:237–243, 2008.

[56] Tim R. Morris. A Gauge invariant exact renormalization group. 1. *Nucl. Phys. B*, 573:97–126, 2000.

[57] Tim R. Morris. A Gauge invariant exact renormalization group. 2. *JHEP*, 12:012, 2000.

[58] Tim R. Morris. An Exact RG formulation of quantum gauge theory. *Int. J. Mod. Phys. A*, 16:1899–1912, 2001.

[59] S. Arnone, Yu.A. Kubyshin, T.R. Morris, and J.F. Tighe. Gauge invariant regularization in the ERG approach. In *15th International Workshop on High-Energy Physics and Quantum Field Theory (QFTHEP 2000)*, pages 297–304, 9 2000.

[60] S. Arnone, Yu.A. Kubyshin, T.R. Morris, and J.F. Tighe. A Gauge invariant regulator for the ERG. *Int. J. Mod. Phys. A*, 16:1989, 2001.

[61] Stefano Arnone, Antonio Gatti, and Tim R. Morris. Exact scheme independence at one loop. *JHEP*, 05:059, 2002.

[62] Stefano Arnone, Antonio Gatti, and Tim R. Morris. Towards a manifestly gauge invariant and universal calculus for Yang-Mills theory. *Acta Phys. Slov.*, 52:621–634, 2002.

[63] Stefano Arnone, Antonio Gatti, and Tim R. Morris. A Manifestly gauge invariant exact renormalization group. In *5th International Conference on Renormalization Group 2002 (RG 2002)*, 7 2002.

[64] Stefano Arnone, Antonio Gatti, Tim R. Morris, and Oliver J. Rosten. Exact scheme independence at two loops. *Phys. Rev. D*, 69:065009, 2004.

[65] Stefano Arnone, Tim R. Morris, and Oliver J. Rosten. A Generalised manifestly gauge invariant exact renormalisation group for SU(N) Yang-Mills. *Eur. Phys. J. C*, 50:467–504, 2007.

[66] Tim R. Morris and Oliver J. Rosten. A Manifestly gauge invariant, continuum calculation of the SU(N) Yang-Mills two-loop beta function. *Phys. Rev. D*, 73:065003, 2006.

[67] Stefano Arnone, Antonio Gatti, and Tim R. Morris. Manifestly gauge invariant computations. In *5th International Conference on Renormalization Group 2002 (RG 2002)*, 7 2002.

[68] Antonio Gatti. A Gauge invariant flow equation. Phd thesis, Southampton U., 10 2002.

[69] Oliver J. Rosten, Tim R. Morris, and Stefano Arnone. The Gauge invariant ERG. In *13th International Seminar on High-Energy Physics: Quarks 2004*, 9 2004.

[70] Oliver Jacob Rosten. The Manifestly gauge invariant exact renormalisation group. Phd thesis, Southampton U., 3 2005.

[71] Oliver J. Rosten. A Primer for manifestly gauge invariant computations in SU(N) Yang-Mills. *J. Phys. A*, 39:8699–8726, 2006.

[72] Oliver J. Rosten. Scheme independence to all loops. *J. Phys. A*, 39:8141–8156, 2006.

[73] Oliver J. Rosten. A Manifestly gauge invariant and universal calculus for SU(N) Yang-Mills. *Int. J. Mod. Phys. A*, 21:4627–4762, 2006.

[74] Oliver J. Rosten. General Computations Without Fixing the Gauge. *Phys. Rev. D*, 74:125006, 2006.

[75] Oliver J. Rosten. Universality From Very General Nonperturbative Flow Equations in QCD. *Phys. Lett. B*, 645:466–469, 2007.

[76] G. Parisi and N. Sourlas. Random Magnetic Fields, Supersymmetry and Negative Dimensions. *Phys. Rev. Lett.*, 43:744, 1979.

[77] Bryce DeWitt. *Supermanifolds.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2 edition, 1992.

[78] Daniel Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara. Progress Toward a Theory of Supergravity. *Phys. Rev. D*, 13:3214–3218, 1976.

[79] S.J. Gates, Marcus T. Grisaru, M. Rocek, and W. Siegel. *Superspace Or One Thousand and One Lessons in Supersymmetry*, volume 58. 1983.

[80] Kevin Falls and Tim R. Morris. Conformal anomaly from gauge fields without gauge fixing. *Phys. Rev. D*, 97(6):065013, 2018.

[81] N. Hitchin. Generalized calabi-yau manifolds. *The Quarterly Journal of Mathematics*, 54(3):281–308, Sep 2003.

[82] Marco Gualteri. Generalized Complex Geometry. Phd thesis, Oxford U., 2004.

[83] W. Siegel. Superspace duality in low-energy superstrings. *Phys. Rev. D*, 48:2826–2837, 1993.

[84] Chris Hull and Barton Zwiebach. Double Field Theory. *JHEP*, 09:099, 2009.

[85] I.A. Batalin and G.A. Vilkovisky. Gauge Algebra and Quantization. *Phys. Lett. B*, 102:27–31, 1981.

[86] I.A. Batalin and G.A. Vilkovisky. Quantization of Gauge Theories with Linearly Dependent Generators. *Phys. Rev. D*, 28:2567–2582, 1983. [Erratum: Phys.Rev.D 30, 508 (1984)].

[87] Jean Zinn-Justin. Renormalization of Gauge Theories. *Lect. Notes Phys.*, 37:1–39, 1975.

[88] Jean Zinn-Justin. Renormalization Problems in Gauge Theories. In *12th Annual Winter School of Theoretical Physics*, pages 433–453, 1 1975.

[89] Tim R. Morris. The continuum limit of the conformal sector at second order in perturbation theory. 6 2020.

[90] Glenn Barnich and Marc Henneaux. Consistent couplings between fields with a gauge freedom and deformations of the master equation. *Phys. Lett. B*, 311:123–129, 1993.

[91] Glenn Barnich, Friedemann Brandt, and Marc Henneaux. Local BRST cohomology in the antifield formalism. 1. General theorems. *Commun. Math. Phys.*, 174:57–92, 1995.

[92] Nicolas Boulanger, Thibault Damour, Leonardo Gualtieri, and Marc Henneaux. Inconsistency of interacting, multigraviton theories. *Nucl. Phys. B*, 597:127–171, 2001.

[93] F J Wegner. Some invariance properties of the renormalization group. *Journal of Physics C: Solid State Physics*, 7(12):2098–2108, jun 1974.

[94] Juergen A. Dietz and Tim R. Morris. Redundant operators in the exact renormalisation group and in the f(R) approximation to asymptotic safety. *JHEP*, 07:064, 2013.

[95] R. Percacci. Unimodular quantum gravity and the cosmological constant. *Found. Phys.*, 48(10):1364–1379, 2018.

[96] Steven Weinberg. *ULTRAVIOLET DIVERGENCES IN QUANTUM THEORIES OF GRAVITATION*, pages 790–831. 1 1980.

[97] M. Reuter. Nonperturbative evolution equation for quantum gravity. *Phys. Rev. D*, 57:971–985, 1998.

[98] Alfio Bonanno, Astrid Eichhorn, Holger Gies, Jan M. Pawlowski, Roberto Percacci, Martin Reuter, Frank Saueressig, and Gian Paolo Vacca. Critical reflections on asymptotically safe gravity. *Front. in Phys.*, 8:269, 2020.

[99] Juergen A. Dietz, Tim R. Morris, and Zoe H. Slade. Fixed point structure of the conformal factor field in quantum gravity. *Phys. Rev. D*, 94(12):124014, 2016.

[100] Jean Zinn-Justin. Quantum field theory and critical phenomena. *Int. Ser. Monogr. Phys.*, 113:1–1054, 2002.

[101] Hikaru Kawai, Yoshihisa Kitazawa, and Masao Ninomiya. Ultraviolet stable fixed point and scaling relations in (2+epsilon)-dimensional quantum gravity. *Nucl. Phys. B*, 404:684–716, 1993.

[102] Astrid Eichhorn. On unimodular quantum gravity. *Class. Quant. Grav.*, 30:115016, 2013.

[103] Andreas Nink. Field Parametrization Dependence in Asymptotically Safe Quantum Gravity. *Phys. Rev. D*, 91(4):044030, 2015.

[104] Roberto Percacci and Gian Paolo Vacca. Search of scaling solutions in scalar-tensor gravity. *Eur. Phys. J. C*, 75(5):188, 2015.

[105] Roberto Percacci and Gian Paolo Vacca. The background scale Ward identity in quantum gravity. *Eur. Phys. J. C*, 77(1):52, 2017.

[106] Tim R. Morris. Large curvature and background scale independence in single-metric approximations to asymptotic safety. *JHEP*, 11:160, 2016.

[107] Juergen A. Dietz and Tim R. Morris. Background independent exact renormalization group for conformally reduced gravity. *JHEP*, 04:118, 2015.

[108] I. Hamzaan Bridle, Juergen A. Dietz, and Tim R. Morris. The local potential approximation in the background field formalism. *JHEP*, 03:093, 2014.

[109] Jan M. Pawlowski. Aspects of the functional renormalisation group. *Annals Phys.*, 322:2831–2915, 2007.

[110] Daniel F. Litim and Jan M. Pawlowski. Wilsonian flows and background fields. *Phys. Lett. B*, 546:279–286, 2002.

[111] M. Reuter and C. Wetterich. Gluon condensation in nonperturbative flow equations. *Phys. Rev. D*, 56:7893–7916, 1997.

[112] Daniel F. Litim and Jan M. Pawlowski. On gauge invariant Wilsonian flows. In *Workshop on the Exact Renormalization Group*, pages 168–185, 9 1998.

[113] Daniel F. Litim and Jan M. Pawlowski. Renormalization group flows for gauge theories in axial gauges. *JHEP*, 09:049, 2002.

[114] Elisa Manrique and Martin Reuter. Bimetric Truncations for Quantum Einstein Gravity and Asymptotic Safety. *Annals Phys.*, 325:785–815, 2010.

[115] Elisa Manrique, Martin Reuter, and Frank Saueressig. Matter Induced Bimetric Actions for Gravity. *Annals Phys.*, 326:440–462, 2011.

[116] Elisa Manrique, Martin Reuter, and Frank Saueressig. Bimetric Renormalization Group Flows in Quantum Einstein Gravity. *Annals Phys.*, 326:463–485, 2011.

[117] Daniel Becker and Martin Reuter. En route to Background Independence: Broken split-symmetry, and how to restore it with bi-metric average actions. *Annals Phys.*, 350:225–301, 2014.

[118] Peter Labus, Tim R. Morris, and Zoë H. Slade. Background independence in a background dependent renormalization group. *Phys. Rev. D*, 94(2):024007, 2016.

[119] Martin Reuter and Holger Weyer. Background Independence and Asymptotic Safety in Conformally Reduced Gravity. *Phys. Rev. D*, 79:105005, 2009.

[120] Juergen A. Dietz and Tim R. Morris. Asymptotic safety in the f(R) approximation. *JHEP*, 01:108, 2013.

[121] Pedro F. Machado and Frank Saueressig. On the renormalization group flow of f(R)-gravity. *Phys. Rev. D*, 77:124045, 2008.

[122] Alessandro Codello, Roberto Percacci, and Christoph Rahmede. Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation. *Annals Phys.*, 324:414–469, 2009.

[123] Dario Benedetti and Francesco Caravelli. The Local potential approximation in quantum gravity. *JHEP*, 06:017, 2012. [Erratum: JHEP 10, 157 (2012)].

[124] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. RG flows of Quantum Einstein Gravity in the linear-geometric approximation. *Annals Phys.*, 359:141–165, 2015.

[125] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. A proper fixed functional for four-dimensional Quantum Einstein Gravity. *JHEP*, 08:113, 2015.

[126] Nobuyoshi Ohta, Roberto Percacci, and Gian Paolo Vacca. Flow equation for $f(R)$ gravity and some of its exact solutions. *Phys. Rev. D*, 92(6):061501, 2015.

[127] Nobuyoshi Ohta, Roberto Percacci, and Gian Paolo Vacca. Renormalization Group Equation and scaling solutions for f(R) gravity in exponential parametrization. *Eur. Phys. J. C*, 76(2):46, 2016.

[128] Kevin Falls and Nobuyoshi Ohta. Renormalization Group Equation for $f(R)$ gravity on hyperbolic spaces. *Phys. Rev. D*, 94(8):084005, 2016.

[129] Nobuyoshi Ohta. Background Scale Independence in Quantum Gravity. *PTEP*, 2017(3):033E02, 2017.