UNIVERSITY OF SOUTHAMPTON

FACULTY OF ENGINEERING AND PHYSICAL SCIENCES
School of Physics and Astronomy

Aspects of Four-Point Functions in $\mathcal{N} = 4$ SYM at Strong Coupling

by

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ABSTRACT

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In this thesis we focus on two main topics: the double-trace spectrum of strongly-coupled $\mathcal{N} = 4$ SYM theory and the construction of one-loop four-point functions in $AdS_5 \times S^5$. We begin by providing a basic review of $\mathcal{N} = 4$ SYM and its connection to holographic correlators on $AdS_5 \times S^5$ through the AdS/CFT duality. In the second part, we examine the spectrum of double-trace operators at strong coupling, which are dual to two-particle bound states in AdS. At large N, these states are degenerate and to obtain their order $1/N^2$ anomalous dimensions one has to solve a mixing problem. We present a compact formula for all tree-level supergravity anomalous dimensions and we observe an interesting pattern of residual degeneracies. Considering further string corrections, we identify a ten-dimensional principle which dictates the structure of the string corrected spectrum. The third part is devoted to the construction of one-loop corrections to four-point correlation functions. We develop an algorithm for bootstrapping one-loop supergravity correlators for arbitrary Kaluza-Klein modes, which relies solely on implementing the consistency of the OPE to order $1/N^4$. We illustrate the subtle features of this algorithm by constructing new explicit results for multi-channel correlators. Lastly, we consider one-loop string corrections to the $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2 \rangle$ correlator. We find that a transcendental weight three function involving a new type of singularity is required, whose presence is a novelty in the context of AdS amplitudes.

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Declaration of Authorship

I, Hynek Paul, declare that this thesis entitled "Aspects of Four-Point Functions in $\mathcal{N} = 4$ SYM at Strong Coupling" and the work presented in it are my own and have been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
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3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
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6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as: [1], [2], [3], [4], [5], [6], [7], [8], [9], [10].

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To my parents, Martina and Pavel,
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Chapter 1

Introduction

The development of quantum field theory in the course of the last century is doubtlessly one of the greatest achievements of modern theoretical physics. Quantum field theories are mathematical frameworks unifying the theory of special relativity with the principles of quantum mechanics, and even though they were originally developed to describe the interactions between elementary particles, its applications reach from cosmology to condensed matter physics. The most successful such theory however is the standard model of particle physics. More precisely, the standard model is a non-abelian Yang-Mills theory with gauge group $SU(3) \times SU(2) \times U(1)$, which accounts for three out of the four fundamental interactions of nature: electromagnetism, the strong and the weak nuclear force. Thanks to immense experimental efforts, it has been tested to a very high degree of precision.

The theory describing the strong nuclear force, governing the interactions of quarks and gluons, is given by the $SU(3)$ part of the standard model and is known under the name of quantum chromodynamics (QCD). It stands out because of its special property that the interactions become weaker at high energies, a phenomenon called asymptotic freedom. Conversely, at low energies, QCD becomes strongly-coupled and gives rise to the intricate spectrum of hadrons, observed in e.g. collider experiments. While we can use perturbation theory to study the weak coupling limit, understanding quantum field theories at strong coupling remains a big challenge to this day. One possibility is to put the theory on a discrete spacetime instead, an approach called lattice field theory, which has been mostly applied to numerically study the hadronic spectrum of QCD using various simulation techniques. On the other hand, we do not have any systematic *analytic* technique to study the strongly-coupled regime of generic quantum field theories.

For a special class of quantum field theories however, some promising progress has been achieved in recent years. These are so-called conformal field theories (CFTs), which are much more constrained because of their additional symmetries. For example, their two- and three-point correlation functions are entirely fixed by conformal symmetry. One avenue of progress comes from the revival of the old idea of using symmetries and other consistency-conditions to solve a theory. Thanks to significant advances in computational power, the modern (numerical) conformal bootstrap program, initiated in the seminal work [11], has sparked a wealth of new non-perturbative results. Most notably, this lead to a high precision determination of operator dimensions and OPE coefficients in the three-dimensional Ising model [12, 13], surpassing by far any other available method.

Another avenue to study CFTs at strong coupling comes from the celebrated discovery by Maldacena: the AdS/CFT correspondence [14–16], a conjectured duality between strongly-coupled CFTs and weakly-coupled gravitational theories on an AdS background of one more spacetime dimension. As such, it is a prime example of the holographic principle, an idea that gravity is emergent from a lower-dimensional description. This, and the extraordinary property that it is a strong-weak coupling duality, has made the AdS/CFT correspondence one of the most vibrant research areas of present-day theoretical physics. The first and most-studied example of such a correspondence is the duality between four-dimensional $\mathcal{N}=4$ SYM theory with gauge group $SU(N)$ and type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. It is important to mention that this duality has not been proven, but over the years it has passed many non-trivial checks and an enormous amount of evidence in its favour has been gathered. For our purposes, we will assume the duality and use it to study the strongly-coupled regime of $\mathcal{N}=4$ SYM theory, which in turn provides an avenue to study quantum gravity in $\mathrm{AdS}_5\times\mathrm{S}^5$.

Even though $\mathcal{N}=4$ SYM theory is arguably quite far from real-life physics, having maximal supersymmetry and conformal invariance which leads to the loss of asymptotic freedom, it can still be seen as a more symmetric ‘cousin’ of QCD, as their perturbative scattering amplitudes share many qualitative properties.[1] Moreover, $\mathcal{N}=4$ SYM has many further special features which make it worthwhile studying: for example, in a limit where one takes the number of colours N large (the planar limit), this theory becomes integrable.[2] Furthermore, its superconformal symmetry persists at the quantum level, and we can use CFT techniques to study the theory at any value of the coupling. It is also believed that this theory obeys S-duality (Montonen-Olive duality), realised as a $SL(2,\mathbb{Z})$ symmetry of the complexified gauge coupling, which relates the weak and strong coupling regions in a non-trivial manner. Finally, for us the most interesting feature of this theory is its presence in the previously mentioned AdS/CFT correspondence. The central quantities of interest within this duality are the correlation functions of local operators. In particular, we will consider correlators of one-half BPS operators, which are dual to states belonging to the AdS_5 graviton supermultiplet and its Kaluza-Klein modes

[1] See e.g. the recent review [17].

[2] In particular, one finds an infinite number of symmetries in the planar limit, making it possible to compute the spectrum of scaling dimensions exactly. See e.g. [18] for a review on integrability in this regime.

on S^5. While their two- and three-point correlation functions are protected quantities (whose matching served as one of the early tests of the correspondence [19, 20]), the first non-trivial dynamics appears in four-point functions. At weak coupling, many results for such four-point functions have been obtained in perturbation theory: for the simplest, lowest-charge correlator the works [21–26] have lead to results up to ten-loops [27]. For higher-charge correlators, results are known up to three-loop order [28–35], and later even up to five-loops [36]. On the other hand, more interesting for us are the results at strong coupling, where the correlation functions have a dual interpretation as supergravity scattering amplitudes in AdS [37–40]. In a truly heroic effort, the effective type IIB supergravity action on $AdS_5 \times S^5$ has been obtained up to quartic order by Arutyunov and Frolov [41]. In principle, their results prepared the ground for the computation of any four-point supergravity correlator as a sum of Witten diagrams, but the complexity of this traditional method limits the direct computation to correlators of low external charges, see [28, 29, 42–46] for results obtained in this way. It then took almost twenty years and a completely new approach to make further progress: in the groundbreaking work of Rastelli and Zhou [47, 48], a remarkably compact Mellin space formula for all tree-level supergravity correlation functions was obtained by a bootstrap approach, which completely bypasses the diagrammatic expansion in terms of Witten diagrams.[3] Their formula is consistent with all previously known correlators, and has been further checked in many new cases by explicit supergravity computations [58–60]. Finally, as this formula was obtained as a solution of a bootstrap problem, the overall normalisation was left unfixed. Using a physical argument on the absence of certain states, we will be able to determine this normalisation.

Here, we take the general result for the tree-level supergravity correlators from [47, 48] as an input and we will employ CFT techniques, in particular the (super)conformal partial wave expansion developed by Dolan and Osborn [61–65], to systematically analyse the spectrum of exchanged operators in the supergravity limit. To leading order in large N, the only operators in the supergravity spectrum which develop an order $1/N^2$ anomalous dimension are double-trace operators, corresponding to bound two-particle states in AdS. In turn, we can then use the obtained information on the spectrum to bootstrap order $1/N^4$ corrections to the supergravity correlators, which correspond to one-loop amplitudes in the dual gravity theory. Solely by implementing the consistency of the OPE to order $1/N^4$, we can thus learn about quantum corrections to supergravity on $AdS_5 \times S^5$ without any reference to actual one-loop diagrams, and in fact such a direct computation remains extremely challenging. Instead, explicit one-loop computations in the bulk have so far been restricted to much simpler, scalar theories [66–71].

Lastly, let us mention that a considerable amount of the recent work on the AdS/CFT correspondence has focussed on studying general constraints on possible holographic theories, see e.g. [72–85]. We believe that both general considerations and also explorations

[3]Similar methods have subsequently been applied to holographic correlators in other theories and backgrounds, see references [49–54], and also for boundary CFTs [55–57].

of explicit examples, such as discussed in this thesis, are necessary to further advance our understanding of strongly-coupled CFTs, quantum corrections to gravitational theories and ultimately the remarkable AdS/CFT duality itself.

Outline of the Thesis

This thesis is divided into three parts. The first part consists of two introductory chapters, starting with a collection of basic facts about $\mathcal{N}=4$ SYM theory in Chapter 2. In particular, we describe the spectrum of local operators and review the superconformal partial wave (SCPW) expansion of four-point correlation functions of one-half BPS operators. In Chapter 3, we present the statement of the AdS/CFT correspondence and give a precise definition of the operators dual to single-particle states in AdS. We then describe the consequences of the duality for four-point correlation functions and review the results for tree-level supergravity correlators.

In the second part, we describe how to resolve the mixing of exchanged double-trace operators in the supergravity limit by using data from many tree-level correlators. In Chapter 4, this leads to a formula for all tree-level supergravity anomalous dimensions, which is of remarkable simplicity and has an interesting pattern of residual degeneracies. Further string corrections are then addressed in Chapter 5, and we find that their structure follows from a new ten-dimensional principle. We compute the order $\lambda^{-\frac{3}{2}}$ and $\lambda^{-\frac{5}{2}}$ anomalous dimensions and, most notably, the order $\lambda^{-\frac{5}{2}}$ result lifts the supergravity degeneracy in a non-trivial way.

The third part is devoted to the construction of one-loop corrections to four-point correlation functions. Building on the results for the double-trace anomalous dimensions from part two, our strategy is to predict the leading logarithmic discontinuity and complete it to the full one-loop amplitude. In Chapter 6, we develop a general algorithm for bootstrapping one-loop supergravity correlators of generic external Kaluza-Klein states and we illustrate this algorithm by considering many explicit examples. From the constructed one-loop amplitudes, we then extract new subleading anomalous dimensions. In Chapter 7, we then consider string corrections to the one-loop supergravity amplitudes. We find that the string corrections require the presence of a transcendental weight three function involving a new type of singularity, which has not appeared in the context of AdS amplitudes before. Finally, some additional material can be found in the five appendices.

Part I

Setting the Stage

Chapter 2

Basics of $\mathcal{N} = 4$ SYM Theory

As already mentioned in the introduction, on one side of the AdS/CFT duality we have the $\mathcal{N} = 4$ SYM theory. In this chapter, we will collect some basic facts and results about this special theory. We will start by giving a Lagrangian description of the theory and discuss its symmetries. In Section 2.2, we will view this theory as a superconformal field theory and study its spectrum of gauge invariant operators. Up to that point, the reviewed material is standard and we will loosely follow references [86–88]. In Section 2.3, we then introduce four-point correlation functions of one-half BPS operators, which are one of the main objects of interest in this thesis. Finally, in Section 2.4, we describe the $\mathcal{N} = 4$ superconformal operator product expansion (OPE), which gives rise to the superconformal partial wave decomposition used later as our main tool for analysing the spectrum of exchanged operators.

2.1 Lagrangian Description and Superconformal Algebra

A convenient description of the $\mathcal{N} = 4$ SYM theory can be obtained by dimensional reduction from a ten-dimensional Lagrangian. Consider

$$\mathcal{L}_{10} = -\frac{1}{2g_{\mathrm{YM}}^2}\,\mathrm{tr}\left\{F_{MN}F^{MN} - 2i\bar{\Lambda}\Gamma^M D_M \Lambda\right\}, \tag{2.1}$$

which describes a massless vector multiplet in 10 dimensions with $\mathcal{N} = 1$ supersymmetry. In the above, F_{MN} is the field strength tensor for a ten-dimensional gauge field A_M, Λ denotes a Majorana-Weyl spinor with 16 real components, and M, N are ten-dimensional indices. The trace is taken over the gauge group $SU(N)$, under which the fields transform in the adjoint representation.

The four-dimensional Lagrangian $\mathcal{L}_4$ then follows upon Kaluza-Klein compactification on a six-torus T^6. The resulting field content in four dimensions constitutes the full $\mathcal{N} = 4$ supersymmetry gauge multiplet, given by one gauge field A_μ, four chiral fermions λ^A_α

$(A=1,\ldots,4)$ and six real scalar degrees of freedom ϕ^I $(I=1,\ldots,6)$ transforming in the fundamental representation of $SO(6)$. The reason why we end up with (rigid) $\mathcal{N}=4$ supersymmetry in the four-dimensional theory is a consequence of the compactification on T^6 which preserves all 16 supercharges, such that the four-dimensional multiplet has an additional $SU(4)\simeq SO(6)$ global R-symmetry. The four-dimensional Lagrangian with $\mathcal{N}=4$ supersymmetry turns out to be unique, and it is given by

$$\begin{aligned}\mathcal{L}_4 = \operatorname{tr}\Big\{ &-\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} - (D_\mu\varphi^{AB})(D^\mu\bar{\varphi}_{AB}) + 2i\bar{\lambda}_A\bar{\sigma}^\mu D_\mu\lambda^A \\ &- 2g_{\mathrm{YM}}\big(\lambda^A[\bar{\varphi}_{AB},\lambda^B] - \bar{\lambda}_A[\varphi^{AB},\bar{\lambda}_B]\big) + \tfrac{1}{2}g_{\mathrm{YM}}^2[\varphi^{AB},\varphi^{CD}][\bar{\varphi}_{AB},\bar{\varphi}_{CD}]\Big\},\end{aligned} \tag{2.2}$$

where the φ^{AB} describe three complex scalars transforming in the antisymmetric rank-two representation of $SU(4)$. Note that they are simply related to the six real scalars ϕ^I mentioned above through a linear transformation.

By inspection of the mass-dimensions of the terms in the above Lagrangian, we find that $\mathcal{L}_4$ describes a scale invariant theory at the classical level. Combined with the usual Poincaré invariance of $\mathcal{L}_4$, this results in conformal symmetry of the theory, described by the conformal group $SO(4,2)\simeq SU(2,2)$. Together with $\mathcal{N}=4$ supersymmetry with R-symmetry group $SU(4)$, this gives rise to the even larger supergroup $PSU(2,2|4)$ of superconformal symmetries.

An important and very remarkable fact about this theory, which is not directly visible from the Lagrangian formulation, is its superconformal symmetry at the quantum level: perturbative computations of correlation functions in this theory have shown no ultraviolet divergences, resulting in an identically vanishing β-function of the renormalisation group. As a consequence, the superconformal group mentioned above remains an exact symmetry of $\mathcal{N}=4$ SYM even at the quantum level. This allows us to adopt a different point of view on this theory: instead of thinking about it as a framework for perturbative computations in the gauge coupling g_{YM} of e.g. scattering amplitudes (as one does for example in QED or QCD), we will think of $\mathcal{N}=4$ SYM as a (super)*conformal* field theory and apply CFT methods. This will allow us to study the theory in the strong coupling regime, which is not accessible through the traditional perturbative approach. We will elaborate more on this CFT point of view and introduce the necessary terminology in the next section.

Let us now turn to the algebra of the superconformal group $PSU(2,2|4)$, as we will need its generators in order to understand the construction of the operator spectrum of $\mathcal{N}=4$ SYM theory. The $\mathcal{N}=4$ superconformal algebra in four dimensions can be broken down into smaller subalgebras. One of the two bosonic subalgebras is the algebra of the conformal group $SU(2,2)$, an extension of the Poincaré algebra: it is generated by translations P_μ, Lorentz transformations $M_{\mu\nu}$, together with dilatations D and special conformal transformations K_μ. Another extension of the Poincaré algebra is given by adding four fermionic supercharges Q^a_α and $\bar{Q}_{\dot{\alpha}a}$ with $a=1,\ldots,4$, and R-symmetry generators R^A $(A=1,\ldots,15)$, which gives the $\mathcal{N}=4$ supersymmetry algebra. Now, we can combine these two extensions in a consistent way once we add 16 additional fermionic generators in order to close the algebra. These are the so-called conformal supercharges $S_{\alpha a}$ and $\bar{S}^a_{\dot{\alpha}}$, which complete the full $\mathcal{N}=4$ superconformal algebra. To emphasise the structure of the resulting super-algebra, one can organise the generators into the four blocks [86]

$$\begin{pmatrix} M_{\mu\nu},\ P_\mu,\ K_\mu,\ D & Q^a_\alpha,\ \bar{S}^a_{\dot{\alpha}} \\ \bar{Q}_{\dot{\alpha}a},\ S_{\alpha a} & R^A \end{pmatrix}, \tag{2.3}$$

with the generators of the two commuting bosonic subalgebras on the diagonal, and the fermionic generators on the off-diagonal. We will refrain from writing down all (anti)commutation relations of the $\mathcal{N}=4$ superconformal algebra, which can be found in e.g. [87]. For our purposes it will be enough to note the anticommutation relations of the Poincaré supercharges (in the absence of a central charge):

$$\{Q^a_\alpha, Q^b_\beta\}=0, \qquad \{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\}=2\sigma^\mu_{\alpha\dot{\beta}}P_\mu\delta^a_b. \tag{2.4}$$

2.2 The Spectrum of Local Operators

Due to the conformal symmetry of $\mathcal{N}=4$ SYM, there are no massive particles (both, fundamental or composite) in this theory and hence no mass spectrum to study. Nonetheless, there exists an analogue of composite particles and their mass spectrum in a CFT: these are the so-called *local operators*, usually denoted by $\mathcal{O}$, which are gauge-invariant objects built from products of the fundamental fields and their derivatives, inserted at the same point in spacetime. For example, the simplest class of such operators is given by single-trace operators $\mathcal{O}(x)=\operatorname{tr}\left\{F^{(1)}(x)\cdots F^{(n)}(x)\right\}$, where the $F^{(i)}$ can be any of the fundamental fields introduced in the previous section, or covariant derivatives thereof. For reasons of gauge-invariance, rather than using the gauge field A_μ, we take the field strength $F_{\mu\nu}$ as a basic building block instead. It is also possible to have multi-trace operators, which are simply products of single-trace ones. The CFT-analogue of a mass spectrum is then given by the spectrum of *scaling dimensions* of the local operators of a theory, where the scaling dimension (or simply dimension) $\Delta_\mathcal{O}$ of a local operator $\mathcal{O}$ is defined as its eigenvalue under dilatations:

$$[D,\mathcal{O}(0)]=i\Delta_\mathcal{O}\mathcal{O}(0). \tag{2.5}$$

The classical value of $\Delta_\mathcal{O}$ equals the sum of the naive scaling dimensions of the constituent fields, which are given by

$$[\phi^I]=[D_\mu]=1, \qquad [\lambda^A]=\frac{3}{2}, \qquad [F_{\mu\nu}]=2. \tag{2.6}$$

At the quantum level however, the scaling dimensions generically receive quantum corrections. These so-called *anomalous dimensions* are then functions of the coupling g_{YM} and the number of colours N of the gauge group $SU(N)$.

Let us now proceed by describing the classification of local operators in $\mathcal{N} = 4$ SYM theory. First of all, as in any CFT, local operators can be divided into two distinct classes: *primaries* and *descendants*. Conformal primary operators are by definition those which are annihilated by special conformal transformations, that is

$$[K_\mu, \mathcal{O}(0)] = 0, \tag{2.7}$$

whereas descendants can be obtained from linear combinations of derivatives of primary operators (i.e. by action of the translation operator P_μ). In other words, P_μ (K_μ) acts as a raising (lowering) operator with respect to the scaling dimension, and thus primary operators are defined as the operators of lowest dimension within a conformal multiplet. Now, in a superconformal theory, there is a second set of raising and lowering operators given by the supercharges Q and S. To obtain a *super*conformal primary operator, which is the state of lowest dimension in a superconformal multiplet, we thus supplement the condition (2.7) by additionally requiring

$$[S, \mathcal{O}(0)] = 0. \tag{2.8}$$

In fact, since S lowers the dimension by $-1/2$ compared to K_μ which lowers it by -1, equation (2.8) is the stronger condition as it implies the former condition (2.7). A superconformal primary is therefore automatically also a conformal primary, whereas the converse is not true in general.

The descendants of a given superconformal primary operator $\mathcal{O}$ are constructed by applying the raising operators P_μ and Q on $\mathcal{O}$. Recalling the anticommutation relations from equation (2.4), one can obtain P_μ from a combination of Poincaré supercharges Q, and therefore the entire superconformal multiplet can be generated by the action of the Q's alone. Note that this also implies that a superconformal multiplet comprises a *finite* number of conformal primaries, since there are only finitely many ways to apply the supercharges Q before obtaining a derivative operator P_μ. In $\mathcal{N} = 4$ SYM theory there are 16 supercharges, so a generic supermultiplet consists of 16 conformal primaries.[1] In summary, descendants are operators which can be written as Q-commutators of other operators, and in turn the entire supermultiplet can be generated by the action of Q's on a superconformal primary.

Using the above reasoning, let us explicitly construct the superconformal primary operators of $\mathcal{N} = 4$ SYM theory which will be relevant in the context of this thesis. To this

[1] Compared to these so-called long supermultiplets of maximal size, there also exist special classes of supermultiplets which obey shortening conditions. We will elaborate further on this important concept later in this section.

end it is instructive to inspect the action of the Poincaré supercharges on the fundamental building blocks of gauge-invariant local operators. Schematically and omitting all indices, they read [86]

$$
\begin{aligned}
[Q, \phi] &= \lambda, & [Q, F] &= D\lambda, \\
\{Q, \lambda\} &= F + [\phi, \phi], & \{Q, \bar{\lambda}\} &= D\phi,
\end{aligned}
\tag{2.9}
$$

where D stands for the covariant derivative. As we have argued earlier, all quantities which arise from the action of Q's can not be primaries. The only way to obtain a superconformal primary is therefore by a symmetric combination of the scalar fields ϕ^I, all inserted at the same spacetime point x. The simplest such operators are given by single-trace operators

$$
\mathrm{str}\big\{\phi^{\mathrm{I}_1}\phi^{\mathrm{I}_2}\cdots\phi^{\mathrm{I}_\mathrm{p}}\big\}(\mathrm{x}), \tag{2.10}
$$

where str denotes the symmetrised trace over the gauge group $SU(N)$ and the I_k are the $SO(6)$ R-symmetry vector indices. Since $\mathrm{tr}\{\phi^I\} = 0$ for an $SU(N)$ gauge group, we have $p \geq 2$ in the above. These operators transform in the symmetrised product of n vector representations, which in general yields a reducible representation. One way of obtaining an irreducible representation is by removing the traces:[2]

$$
\mathcal{O}_p^{I_1,I_2,\ldots,I_p}(x) = \mathrm{tr}\left\{\phi^{\{I_1}\phi^{I_2}\cdots\phi^{I_p\}}\right\}(x), \tag{2.11}
$$

where the curly brackets denote traceless symmetrisation of the R-symmetry indices. One can conveniently remove the free indices by contracting them with auxiliary $SO(6)$ null-vectors y^I, giving

$$
\mathcal{O}_p(x, y) = y^{I_1} y^{I_2} \cdots y^{I_p}\, \mathrm{tr}\left\{\phi_{I_1}\phi_{I_2}\cdots\phi_{I_p}\right\}(x), \tag{2.12}
$$

where the null condition $y^I y_I = 0$ automatically projects onto the symmetric traceless part, corresponding to the $[0, p, 0]$ representation of $SU(4)$. Furthermore, it turns out that $\mathcal{O}_p$ is annihilated by half of the supercharges, making it a so-called *one-half BPS* operator whose superconformal multiplet is shorter than the generic one. As a consequence, its classical scaling dimension $\Delta = p$ is protected from quantum corrections and hence remains unrenormalised. These operators play a special role within the AdS/CFT correspondence, as they are dual to single-particle states in the bulk theory. Note that this statement is strictly true only in the large N limit, as additional $1/N$ suppressed multi-trace terms to the definition (2.12) need to be considered. We will discuss this issue in more detail in Section 3.2.

A zoo of more complicated superconformal primary operators can be constructed from products of the above single-trace ones, which are accordingly called multi-trace opera-

[2] Another irreducible representation can be obtained by taking the trace over the $SO(6)$ indices. The first such example is given by the so-called Konishi operator $\sum_I \mathrm{tr}\left\{\phi^I\phi^I\right\}$.

tors. Depending on their detailed form, they can give rise to BPS or non-BPS multiplets. For example, a multi-trace one-half BPS operator can be obtained from the product $\mathcal{O}_{p_1}\cdots\mathcal{O}_{p_n}$ after projection onto the $[0,p,0]$ representation:

$$\big(\mathcal{O}_{p_1}\cdots\mathcal{O}_{p_n}\big)\big|_{[0,p,0]}, \qquad \text{with } p=p_1+\ldots+p_n. \tag{2.13}$$

Just like the single-trace operators $\mathcal{O}_p$, they have protected scaling dimension $\Delta=p$.

On the other hand, we can also obtain multi-trace operators in more general $SU(4)$ representations, which give rise to long superconformal multiplets. As such, their scaling dimensions will in general be unprotected and these operators acquire an anomalous dimension. In this thesis, we will mainly focus on double-trace operators which schematically are of the form

$$\mathcal{O}_p\Box^n\partial^{\{\mu_1}\ldots\partial^{\mu_\ell\}}\mathcal{O}_q\big|_{[aba]}, \tag{2.14}$$

such that the operator is in a totally symmetric and traceless irreducible representation with Lorentz-spin ℓ and classical scaling dimension $\Delta^{(0)}=p+q+2n+\ell$. These operators arise as exchanged states in the operator product expansion of four-point functions of one-half BPS operators, which we will introduce next. In the context of the AdS/CFT correspondence, double-trace operators correspond to bound two-particle states in AdS.

2.3 Four-Point Functions of One-Half BPS Operators

As in any CFT, the form of two- and three-point functions in $\mathcal{N}=4$ SYM theory is entirely fixed by conformal symmetry. Furthermore, since the operators $\mathcal{O}_p$ are protected by supersymmetry, their two- and three-point functions are fully described by their free field expressions. The first non-trivial dynamics appears therefore in four-point functions, which are generically coupling dependent because unprotected operators can be exchanged in the operator product expansion. The fact that we take one-half BPS operators as external states means that the four-point functions of any operators in the supermultiplets are uniquely determined in terms of the four-point function of the superconformal primaries,

$$\langle p_1p_2p_3p_4\rangle \equiv \langle\mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\mathcal{O}_{p_3}(x_3,y_3)\mathcal{O}_{p_4}(x_4,y_4)\rangle. \tag{2.15}$$

Such correlation functions are homogeneous polynomials of degree p_i in the y_i variables, while the dependence on the position space variables x_i is in general much more involved. However, the four-point function should really depend on the conformal and $su(4)$ cross-ratios u, v and σ, τ only. This can be achieved by pulling out a dimensionful prefactor,

$$\langle p_1p_2p_3p_4\rangle = \mathcal{P}_{\vec{p}}\;\mathcal{G}_{\vec{p}}\,(u,v;\sigma,\tau), \tag{2.16}$$

where $\mathcal{P}_{\vec{p}}$ carries the conformal and $su(4)$ weights of the correlator such that $\mathcal{G}_{\vec{p}}$ is a (á priori arbitrary) function of the conformal cross-ratios u and v, and a polynomial in the $su(4)$ cross-ratios σ and τ. For future convenience, we introduced the short-hand notation $\vec{p}$ to denote the dependence on the four external charges (p_1, p_2, p_3, p_4). In terms of the original variables, the cross-ratios are given by

$$\begin{aligned} u &= x\bar{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad & v &= (1-x)(1-\bar{x}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \\ \frac{1}{\sigma} &= y\bar{y} = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \qquad & \frac{\tau}{\sigma} &= (1-y)(1-\bar{y}) = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}, \end{aligned} \tag{2.17}$$

with $x_{ij}^2 = (x_i - x_j)^2$ and $y_{ij}^2 = y_i \cdot y_j$. We also introduced the variables x, $\bar{x}$ and y, $\bar{y}$, which we will use interchangeably with u, v and σ, τ, often even including both sets of variables in the same formula. Without loss of generality, we can arrange the external charges such that $p_{43} \geq p_{21} \geq 0$, where $p_{ij} = p_i - p_j$. The prefactor $\mathcal{P}_{\vec{p}}$ is then given by

$$\mathcal{P}_{\vec{p}} = g_{12}^{\frac{p_1+p_2-p_{43}}{2}} g_{14}^{\frac{p_{43}-p_{21}}{2}} g_{24}^{\frac{p_{43}+p_{21}}{2}} g_{34}^{p_3}, \tag{2.18}$$

where g_{ij} denotes the superpropagator defined as

$$g_{ij} = \frac{y_{ij}^2}{x_{ij}^2}, \tag{2.19}$$

obeying $g_{ij} = g_{ji}$ and $g_{ii} = 0$. As mentioned before, in contrast to two- and three-point functions of one-half BPS operators, the four-point functions are not identical to their free field expressions. However, their dependence on the coupling is heavily constrained by superconformal symmetry. To express the constraints imposed by superconformal symmetry, it is useful to separate the correlator into a free field theory and an interacting piece. The so-called *partial non-renormalisation theorem* [89] then constrains the four-point functions to have the following structure

$$\langle p_1 p_2 p_3 p_4 \rangle = \langle p_1 p_2 p_3 p_4 \rangle_{\text{free}} + \mathcal{P}_{\vec{p}} \times \mathcal{I}(x, \bar{x}, y, \bar{y}) \times \mathcal{H}_{\vec{p}}(u, v; \sigma, \tau; g_{\text{YM}}), \tag{2.20}$$

where the first term is the coupling-independent free field correlator and the second term is the interacting piece. The key point is that all non-trivial dependence on the gauge coupling g_{YM} appears in $\mathcal{H}_{\vec{p}}$. Furthermore, the interacting (or dynamical) term admits a decomposition into the three terms shown above, where the factor $\mathcal{I}$ is fixed by the superconformal Ward identities to take the fully factorised form[3]

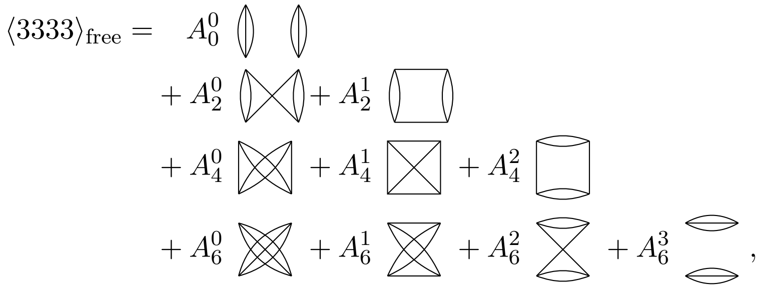
$$\mathcal{I}(x, \bar{x}, y, \bar{y}) = \frac{(x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})}{(y\bar{y})^2}. \tag{2.22}$$

[3]Alternatively, it can be written as the following degree-two polynomial

$$\mathcal{I}(u, v, \sigma, \tau) = v + \sigma^2 uv + \tau^2 u + \sigma v(v-1-u) + \tau(1-u-v) + \sigma\tau u(u-1-v). \tag{2.21}$$

Note that the presence of this factor reduces the polynomial degree of $\mathcal{H}_{\vec{p}}$ in σ and τ by two compared to the free field theory part.

The free field four-point functions can be computed simply by performing Wick contractions between the elementary fields. The result is a sum over the different allowed superpropagator structures accompanied by their colour factors. Graphically, the four external operators $\mathcal{O}_{p_i}$ are represented as vertices with p_i legs and each superpropagator g_{ij} is represented as a line between the points i and j. We arrange the four operators at the corners of a square, labelled clockwise from the bottom left. For example, the free theory correlator of four dimension-three one-half BPS operators $\mathcal{O}_3$ reads



(2.23)

where the coefficients A_γ^k are the associated colour factors. The subscript γ counts the total number of propagators connecting the left half of the graph to the right half, and k is the number of propagators along the top edge of the square. Due to the full crossing symmetry of the above correlator, many colour factors are equal to each other whenever the corresponding graphs are isomorphic. Indeed, in this example there are only three independent colour factors and an explicit computation of the Wick contractions yields the factors

$$\begin{aligned} A_0^0 = A_6^0 = A_6^3 &= \frac{9(N^2-4)^2(N^2-1)^2}{N^2}, \\ A_2^0 = A_2^1 = A_4^0 = A_4^2 = A_6^1 = A_6^2 &= \frac{9}{N^2-1} A_0^0, \\ A_4^1 &= \frac{18(N^2-12)}{(N^2-1)(N^2-4)} A_0^0, \end{aligned} \tag{2.24}$$

which are exact in N. For a general correlator (using our conventions for the arrangement of external charges, i.e. $p_{43} \geq p_{21} \geq 0$), the free theory result reads

$$\langle p_1 p_2 p_3 p_4 \rangle_{\text{free}} = \mathcal{P}_{\vec{p}} \sum_{\substack{\gamma = p_{43} \\ \gamma - p_{43} \in 2\mathbb{Z}}}^{\min\{p_1+p_2, p_3+p_4\}} \left[\left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right)^{\frac{\gamma - p_{43}}{2}} \sum_{k=0}^{\frac{\gamma - p_{43}}{2}} A_\gamma^k \left(\frac{g_{14} g_{23}}{g_{13} g_{24}} \right)^k \right], \tag{2.25}$$

where A_γ^k are the N-dependent colour factors and the prefactor $\mathcal{P}_{\vec{p}}$ has been defined in equation (2.18). Using the definition of the superpropagator from (2.19), we have

$$\frac{g_{13} g_{24}}{g_{12} g_{34}} = \frac{x \bar{x}}{y \bar{y}} = u \sigma, \qquad \frac{g_{14} g_{23}}{g_{13} g_{24}} = \frac{(1-y)(1-\bar{y})}{(1-x)(1-\bar{x})} = \frac{\tau}{v \sigma}. \tag{2.26}$$

Substituting these relations into the general formula (2.25), we find that the free theory correlators are simply rational functions of the conformal cross-ratios (u, v), whereas the (σ, τ) dependence is only polynomial, as expected.

2.4 The $\mathcal{N}=4$ Superconformal OPE

One of the central tools we will use throughout this thesis is the *superconformal partial wave* (SCPW) decomposition (or superconformal block decomposition) of the four-point functions introduced in the previous section. It relies on the notion of an *operator product expansion* (OPE), which describes the product of two operators as a sum over the spectrum of the theory, offering a fully non-perturbative approach to the study of correlation functions. We will consider the OPE obtained in the limit $x_{12}^2 \to 0$, $x_{34}^2 \to 0$, which in cross-ratio variables corresponds to the limit $u \to 0$ with v fixed. The OPE of two one-half BPS operators reads

$$\mathcal{O}_{p_1}(x_1)\,\mathcal{O}_{p_2}(x_2) \sim \sum_{\mathcal{O}} g_{12}^{\frac{p_1+p_2-\Delta}{2}}\, C_{p_1 p_2 \mathcal{O}}\; \mathcal{L}^{(\ell)}(x_{12}, \partial_{x_2}) * \mathcal{O}_{\Delta}^{(\ell)}(x_2), \tag{2.27}$$

where the sum runs over all primary operators $\mathcal{O}_{\Delta}^{(\ell)}$ of dimension Δ and spin ℓ which belong to the $SU(4)$ representations in the tensor product

$$[0,p_1,0]\otimes[0,p_2,0] = \sum_{k_1=0}^{p_1}\sum_{k_2=0}^{p_1-k_1}[k_1, p_2-p_1+2k_2, k_1], \qquad (p_1 \leq p_2). \tag{2.28}$$

The descendants are captured by the action of the derivative operator $\mathcal{L}^{(\ell)}(x_{12}, \partial_{x_2})$ on the primaries $\mathcal{O}_{\Delta}^{(\ell)}$. A manifest $\mathcal{N}=4$ formulation of the OPE can be obtained by reorganising the sum over operators into supermultiplets. Inserting the OPE of $\mathcal{O}_{p_1}(x_1)\mathcal{O}_{p_2}(x_2)$ and $\mathcal{O}_{p_3}(x_3)\mathcal{O}_{p_4}(x_4)$ into the four-point correlator we obtain the representation

$$\langle p_1 p_2 p_3 p_4\rangle = \mathcal{P}_{\vec{p}} \sum_{\{\tau,\ell,\mathfrak{R}\}} A_{\vec{p},\mathfrak{R}}(\tau,\ell)\; \mathbb{S}_{\vec{p},\mathfrak{R}}(\tau,\ell), \tag{2.29}$$

where instead of summing over the dimensions of the exchanged operators we choose to sum over their *twists* $\tau \equiv \Delta - \ell$, and the sum over representations $\mathfrak{R}$ runs over those which belong to $([0,p_1,0]\otimes[0,p_2,0]) \cap ([0,p_3,0]\otimes[0,p_4,0])$. The functions $\mathbb{S}_{\vec{p},\mathfrak{R}}(\tau,\ell)$ are the superconformal blocks which we will describe below. Note that inside correlation functions the OPE is convergent and therefore if we keep all terms in the expansion (as we do in the following discussion) it is valid for all values of u and v inside the radius of convergence. The coefficients $A_{\vec{p},\mathfrak{R}}(\tau,\ell)$ depend explicitly on the charges and are related to the OPE coefficients from equation (2.27) by

$$A_{\vec{p},\mathfrak{R}}(\tau,\ell) \;=\sum_{\mathcal{O}\in\,\mathfrak{R}} C_{p_1p_2\mathcal{O}}C_{p_3p_4\mathcal{O}}\,, \tag{2.30}$$

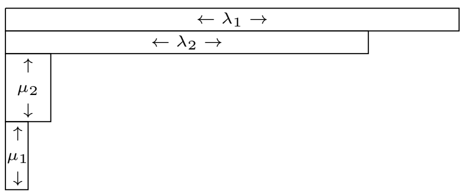
where the sum is over all operators with leading order dimension Δ, spin ℓ and $SU(4)$ representation $\mathfrak{R}$. The block decomposition is invariant under swapping points 1 and 2, points 3 and 4 and swapping the pairs of points (1,2) and (3,4). Using this symmetry we can clearly always ensure that $p_{43}\geq p_{21}\geq 0$, which justifies our choice of conventions for the ordering of the external charges in the definition (2.18) of the prefactor $\mathcal{P}_{\vec{p}}$.

Finally, the superblocks $\mathbb{S}_{\vec{p},\mathfrak{R}}(\tau,\ell)$ can be derived using a variety of approaches. There has been a great deal of work on superblocks in $\mathcal{N}=4$ SYM both from the pioneering work of Dolan and Osborn [61, 63–65] and more recently [90], as well as supergroup theoretic approaches [91, 92]. Here we follow the formalism of [92] and explain the superblocks in a compact fashion in terms of representations of $GL(2|2)$, which provides a group theoretic, manifestly unitary approach and has the great advantage of dealing with all representations in a uniform way.

2.4.1 The $GL(2|2)$ Superconformal Partial Wave Expansion

To address the SCPW expansion we must first describe the conformal blocks for all supermultiplets that might be exchanged in the OPE of one-half BPS operators. Following [92], we label the superconformal primaries $\mathcal{O}_{\gamma,\underline{\lambda}}$ by a number γ and a finite dimensional representation of $GL(2|2)$ which we specify by a Young diagram $\underline{\lambda}\equiv[\lambda_1,\ldots,\lambda_n]$ where λ_i is the length of the ith row.

The Young diagrams do not have an arbitrary shape but have to fit into a 'fat hook' shape, which amounts to the additional constraint that the third row (and hence any subsequent rows) cannot be longer than length two, i.e. $\lambda_3\leq 2$. The number of rows n also satisfies $n\leq(\gamma-p_{43})/2$. For example, a generic such diagram has the form



$$= [\lambda_1,\lambda_2,2^{\mu_2},1^{\mu_1}], \tag{2.31}$$

with first row of length λ_1, second row of length λ_2 and then μ_2 rows of length 2 (which we denote by 2^{μ_2}) and μ_1 rows of length 1 (denoted by 1^{μ_1}). Such a generic Young tableau corresponds to a long multiplet. Short multiplets instead have row 2 of length 1 or 0 and so have the shape of a 'thin hook'. The parameters γ and $\underline{\lambda}$ determine the usual quantum numbers of twist τ, spin ℓ and $SU(4)$ representation which here always takes the form $[a,b,a]$. The dictionary is summarized below in Table 2.1.

$GL(2\|2)$ rep $\underline{\lambda}$	$\tau=\Delta-\ell$	ℓ	$SU(4)$ representation	multiplet type
$[0]$	γ	0	$[0,\gamma,0]$	one-half BPS
$[1^\mu]$	γ	0	$[\mu,\gamma-2\mu,\mu]$	quarter BPS
$[\lambda,1^\mu]$ $(\lambda\geq 2)$	γ	$\lambda-2$	$[\mu,\gamma-2\mu-2,\mu]$	semi-short
$[\lambda_1,\lambda_2,2^{\mu_2},1^{\mu_1}]$ $(\lambda_2\geq 2)$	$\gamma+2\lambda_2-4$	$\lambda_1-\lambda_2$	$[\mu_1,\gamma-2\mu_1-2\mu_2-4,\mu_1]$	long

Table 2.1: Translation between $\mathcal{N}=4$ superconformal representations and superfields $\mathcal{O}_{\gamma,\lambda}$.

Note that the Young tableau representation of a long multiplet is invariant up to the shift-symmetry,

$$\lambda_1\to\lambda_1+1,\quad \lambda_2\to\lambda_2+1,\quad \mu_2\to\mu_2-1,\quad \gamma\to\gamma-2, \tag{2.32}$$

under which the twist τ, spin ℓ and $su(4)$ representation $[a,b,a]$ remain fixed. On the contrary, protected operators require both γ and the Young tableau to be fully specified.

We denote the superconformal block corresponding to the contribution of an operator $\mathcal{O}_{\gamma,\underline{\lambda}}$ to the four-point correlator $\langle p_1p_2p_3p_4\rangle$ by $\mathbb{S}_{\vec{p};\gamma,\underline{\lambda}}$, Long superblocks (those with $\lambda_2=2,3,...$) will occur often and we will also denote them by $\mathbb{L}_{\vec{p};\vec{\tau}}$, where $\vec{\tau}\equiv(\tau,\ell,[a,b,a])$ is a compact notation for the quantum numbers specifying the representation. They have the following factorised structure,

$$\mathbb{L}_{\vec{p};\vec{\tau}}\equiv\mathbb{S}_{\vec{p};\gamma,\underline{\lambda}}=\mathcal{P}_{\vec{p}}\times\mathcal{I}\times\widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}}\,,\qquad \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}}=\frac{\mathcal{B}^{(2+\frac{\tau}{2},\ell)}}{u^{2+\frac{p_{43}}{2}}}\times\Upsilon_{[aba]}\,, \tag{2.33}$$

with $\mathcal{P}_{\vec{p}}$ defined in equation (2.18) and $\mathcal{I}$ in (2.22). Note that the presence of the explicit factor $\mathcal{I}$ in the blocks for long multiplets agrees with the expectation that all quantum corrections appear with such a prefactor in accordance with the partial non-renormalisation theorem (2.20). The dimensions and therefore the twists of such multiplets are coupling dependent and hence generically not integer valued. Likewise, their corresponding OPE coefficients are also explicitly dependent on the coupling. In the above, $\mathcal{B}^{(t,\ell)}$ and $\Upsilon_{[aba]}$ are the ordinary four-dimensional bosonic blocks for conformal and internal symmetries. Explicitly, we have

$$\mathcal{B}^{(t,\ell)}(x,\bar{x})=(-1)^\ell\,u^t\left(\frac{x^{\ell+1}F_{t+\ell}(\bar{x})F_{t-1}(\bar{x})-(x\leftrightarrow\bar{x})}{x-\bar{x}}\right), \tag{2.34}$$

and

$$\Upsilon_{[aba]}(y,\bar{y})=-\frac{P_{n+1}(y)P_m(\bar{y})-(y\leftrightarrow\bar{y})}{y-\bar{y}}\,,\qquad \begin{cases} n=m+a,\\ m=\dfrac{b-p_{43}}{2}\,, \end{cases} \tag{2.35}$$

where we used

$$F_t(x) = {}_2F_1\big(t - \tfrac{p_{12}}{2}, t + \tfrac{p_{34}}{2}, 2t; x\big),$$
$$P_n(y) = \frac{n!\, y}{(n+1+p_{43})_n} J_n^{\left(\frac{p_{43}-p_{21}}{2}, \frac{p_{43}+p_{21}}{2}\right)}\Big(\tfrac{2}{y} - 1\Big), \tag{2.36}$$

with J being the standard Jacobi polynomial.

Explicit formulae for the semi-short, quarter and one-half BPS superconformal blocks were obtained in [92] and can be found in Appendix A. Especially in these cases, the superblock formalism naturally provides manifestly unitary representations.

Since the parameters λ_i are defined by a Young diagram, they are a priori integer valued. However, for long superblocks in the interacting theory the scaling dimension Δ (or equivalently the twist τ) of an operator becomes anomalous and hence non-integer. We can thus allow an analytic continuation of λ_1 and λ_2 such that the spin $\lambda_1 - \lambda_2 = \ell$ remains integer. In such cases we even allow for continuations such that $\lambda_2 < 2$. This means that the labels of such continued long superblocks can coincide with those of short superblocks when $\lambda_2 \to 1, \mu_2 = 0$. To avoid this potential confusion we simply use the notation for long superblocks, $\mathbb{L}_{\vec{p};\vec{\tau}}$, on the LHS of (2.33) and allow $\tau \geq 2a+b+2$ to be non-integer valued.

When long supermultiplets sit exactly on the unitarity bound, $\tau = 2a+b+2$, they become reducible and can be expressed as a sum of two short multiplets

$$\mathbb{L}_{\vec{p};\vec{\tau}} = \mathbb{S}_{\vec{p};\tau,[\ell+2,1^a]} + \mathbb{S}_{\vec{p};\tau+2,[\ell+1,1^{a+1}]}, \qquad \text{for } \tau = 2a+b+2\,. \tag{2.37}$$

The first term on the RHS of the above equation is a semi-short superblock of spin ℓ while the second one is a semi-short superblock of spin $\ell-1$ or a quarter-BPS superblock if $\ell = 0$. We will make use of this reducibility equation when discussing multiplet recombination of semi-short operators at the unitarity bound in Appendix B.

2.5 The SCPW Expansion of the Free Theory

The SCPW of free theory correlators naturally stratifies by the label $\gamma = p_{43}, p_{43}+2, \ldots, \tau^{\min} = \min\{p_1+p_2, p_3+p_4\}$ introduced in (2.25). As mentioned in that context, γ counts the number of propagators connecting operators inserted at points 1 and 2 to operators inserted at points 3 and 4. In the SCPW expansion of a free theory correlator, γ simply corresponds to the number of fundamental fields appearing in the operator $\mathcal{O}_{\gamma,\underline{\lambda}}$ being exchanged in the OPE. Note that this is a good quantum number only in the free theory, and simply reflects the number of Wick contractions which have occurred in the OPE:

$$\begin{aligned}\gamma &= \#\text{ fundamental fields defining } \mathcal{O}_{\gamma,\underline{\lambda}} \\ &= p_1+p_2-(\#\text{ Wick contractions in } \mathcal{O}_{p_1}\mathcal{O}_{p_2}\sim\mathcal{O}_{\gamma,\underline{\lambda}}\text{ OPE }) \\ &= p_3+p_4-(\#\text{ Wick contractions in } \mathcal{O}_{p_3}\mathcal{O}_{p_4}\sim\mathcal{O}_{\gamma,\underline{\lambda}}\text{ OPE }).\end{aligned} \tag{2.38}$$

The general free theory correlator (2.25) then decomposes as

$$\langle p_1p_2p_3p_4\rangle_{\text{free}} = \sum_{\substack{\gamma=p_{43}\\ \gamma-p_{43}\in 2\mathbb{Z}}}^{\min\{p_1+p_2,p_3+p_4\}} \sum_{\underline{\lambda}} A_{\vec{p};\gamma,\underline{\lambda}}\ \mathbb{S}_{\vec{p};\gamma,\underline{\lambda}}, \tag{2.39}$$

where each term in the sum over γ represents the expansion in SCPW of the analogous terms in (2.25). Furthermore, the Young tableaux $\underline{\lambda}$ have at most $(\gamma-p_{43})/2$ rows. Note also that in the free theory all Young tableaux are proper, having both integer rows and correct shape and thus the above decomposition is unambiguous.

However, we do not consider the free theory in isolation, rather we will consider it as the limit of the interacting theory as the coupling vanishes. In the interacting theory, the OPE of two one-half BPS operators contains both operators in short supermultiplets, whose dimensions are protected, and long operators which have anomalous dimensions. Therefore we will split the SCPW expansion (2.39) accordingly and we will distinguish between the short sector *which by definition remains short in the interacting theory*, and a *free long* sector which will then acquire an anomalous dimension in the interacting theory,

$$\langle p_1p_2p_3p_4\rangle_{\text{free}} = \langle p_1p_2p_3p_4\rangle_{\text{short}} + \langle p_1p_2p_3p_4\rangle_{\text{free long}}\,. \tag{2.40}$$

For the short sector we sum over superblocks with the specific form $\mathbb{S}_{\vec{p};\gamma,[\lambda,1^\mu]}$ (given in Appendix A), and for the long sector we sum over the long superblocks $\mathbb{L}_{\vec{p};\vec{\tau}}$ defined in equation (2.33). More explicitly, we introduce the SCPW coefficients $S_{\vec{p};\gamma,[\lambda,1^\mu]}$ and $L^f_{\vec{p};\vec{\tau}}$ as follows:

$$\begin{aligned}\langle p_1p_2p_3p_4\rangle_{\text{short}} &= \sum_{\substack{\gamma=p_{43}\\ \gamma-p_{43}\in 2\mathbb{Z}}}^{\tau^{\min}} \left[S_{\vec{p};\gamma,\emptyset}\ \mathbb{S}_{\vec{p};\gamma,\emptyset} + \sum_{\lambda=1}^{\infty} \sum_{\mu=0}^{\frac{1}{2}(\gamma-p_{43})-1} S_{\vec{p};\gamma,[\lambda,1^\mu]}\ \mathbb{S}_{\vec{p};\gamma,[\lambda,1^\mu]} \right], \\ \langle p_1p_2p_3p_4\rangle_{\text{free long}} &= \sum_{\substack{a,b\in 2\mathbb{Z}\\ 2a+b+4\le\tau^{\min}}} \sum_{\ell=0}^{\infty} \sum_{\substack{\tau>2a+b\\ \tau-b\in 2\mathbb{Z}}} L^f_{\vec{p};\vec{\tau}}\ \mathbb{L}_{\vec{p};\vec{\tau}}\,.\end{aligned} \tag{2.41}$$

It is important to note that this split is non-trivial due to multiplet recombination: in the free theory limit a long multiplet whose twist lies on the unitary bound is indistinguishable from the direct sum of certain short multiplets. A consequence of this is the identity of superblocks (2.37). The challenge then is to relate the SCPW coefficients $S_{\vec{p};\gamma,[\lambda,1^\mu]}$ and $L^f_{\vec{p};\vec{\tau}}$ to the original ones $A_{\vec{p};\gamma,\underline{\lambda}}$ in (2.39).

The simplest SCPW coefficients to identify are the coefficients of one-half BPS operators (corresponding to an empty Young tableau, i.e. $\underline{\lambda}=\emptyset$), which are unchanged:

$$S_{\vec{p};\gamma,\emptyset} = A_{\vec{p};\gamma,\emptyset} \ . \tag{2.42}$$

The next simplest to deal with are the long representations above the unitarity bound. Here we take into account the fact that γ ceases to be a good quantum number for long operators. This is because long operators with different fundamental fields can mix. For example, the double-trace operators $\mathcal{O}_3\mathcal{O}_3$ ($\gamma=6$) mixes with $\mathcal{O}_2\Box\mathcal{O}_2$ ($\gamma=4$), which both have twist 6. This is the origin of the ambiguity in the description of long operators, which manifests itself in the shift-symmetry (2.32) of the Young tableaux corresponding to long operators. Thus we need to collect together all SCPW coefficients with the same quantum numbers $\vec{\tau}$ (but different values of γ) using the shift-symmetry (2.32), giving

$$L^f_{\vec{p};\vec{\tau}} = \sum_{\gamma=2a+b+4}^{\min\{p_1+p_2,p_3+p_4\}} A_{\vec{p};\gamma,[2+\frac{\tau-\gamma}{2}+\ell,2+\frac{\tau-\gamma}{2},2^{\frac{\gamma-b}{2}-a-2},1^a]}, \qquad \text{for } \tau \geq 2a+b+4. \tag{2.43}$$

The most difficult SCPW coefficients to identify in (2.41) are the (non-half BPS) short coefficients $S_{[\lambda,1^\mu]}$ with non-zero λ or μ and the related long coefficients $L_{\vec{p};\vec{\tau}}$ at the unitarity bound $\tau = 2a+b+2$. This is because as we deform away from the free theory, some semi-short blocks combine to become long (as in (2.37)), whereas others remain semi-short. Thus, a single SCPW coefficient A for a semi-short block at the unitarity bound can actually contain the contribution of both short and long multiplets of the interacting theory. More details along with some concrete examples on how to properly disentangle the semi-short and long contributions at the unitarity bound are given in Appendix B. At first sight this may seem like a technical detail, but the correct identification of the long sector will be relevant for the consistency of the SCPW expansion, and ultimately for the construction of one-loop correlators from tree-level data as discussed later in this thesis.

Chapter 3

Basics of Holographic Correlators in $\mathrm{AdS}_5\times\mathrm{S}^5$

After having reviewed the basics of $\mathcal{N}=4$ SYM theory, we turn our attention to the other side of the AdS/CFT duality: holographic correlators on $\mathrm{AdS}_5\times\mathrm{S}^5$. We will start by discussing the basic consequences of the AdS/CFT correspondence, and in particular we will give a precise definition of the single-particle operators $\mathcal{O}_p$, whose four-point correlation functions we have already introduced in the previous chapter. In Section 3.3 we then describe the large N, strong-coupling expansion of the four-point correlators, as well as the spectrum of unprotected exchanged operators which survive in the supergravity limit: the spectrum of double-trace operators. Finally, we will review the progress in computing tree-level supergravity correlators in AdS_5, which culminated in an elegant Mellin space formula for all supergravity correlators of arbitrary external charges.

3.1 Statement of the AdS/CFT Correspondence

The first and most-studied example of the AdS/CFT correspondence is the following conjectured duality [14]:

- Type IIB string theory on an $\mathrm{AdS}_5\times\mathrm{S}^5$ background ('gravity side')
- Four-dimensional $\mathcal{N}=4$ SYM with gauge group $SU(N)$ ('CFT side')

The original argument by Maldacena for such a duality is based on considering a stack of N parallel D3-branes in type IIB superstring theory. The D3-brane is a tensionless object in ten-dimensional superstring theory, which incidentally is a maximally-symmetric solution of type IIB supergravity with rotational symmetry group given by $SO(1,3)\times SO(6)$.

In the low-energy limit ($\alpha' \to 0$) and taking the D3-branes to be coincident, the massless excitations of open strings on the brane describe a four-dimensional theory: $\mathcal{N}=4$ SYM theory with $SU(N)$ gauge symmetry.[1] This is the CFT side of the duality.

On the other hand, the same setup allows for another, inherently gravitational description. As mentioned above, the D3-brane is a solution of type IIB supergravity. In the near-horizon limit, this solution gives rise to an $\mathrm{AdS}_5\times\mathrm{S}^5$ background, where both the AdS_5 and the S^5 factors have a common radius L given by

$$L^4 = 4\pi g_s N(\alpha')^2, \tag{3.1}$$

and the string coupling constant g_s is related to the gauge theory coupling g_{YM} via

$$g_s = \frac{g_{\mathrm{YM}}^2}{4\pi}. \tag{3.2}$$

Note that here the supergravity solution is understood as the low-energy limit of the classical type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$. While its quantum completion, the full type IIB superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$, is currently not fully understood because of the great difficulties of string theory quantisation on curved spacetimes, we can still study the proposed duality in an interesting limit, namely in perturbative string theory. This is the case for small string coupling g_s. In order to preserve equation (3.1), we simultaneously need to take the limit $N\to\infty$. In other words, if we define the 't Hooft coupling $\lambda \equiv g_{\mathrm{YM}}^2 N$, this corresponds to the well-known *'t Hooft limit* of $\mathcal{N}=4$ SYM theory, where one takes N large with λ fixed. This limit is well-defined and corresponds to a topological expansion of the gauge theory, where non-planar Feynman diagrams are suppressed.

After taking the 't Hooft limit, the only parameter left is $\lambda = L^4/(\alpha')^2$. Now, we want to take the limit where supergravity is a good approximation to string theory, i.e. the supergravity limit $\alpha'\to 0$. On the CFT side, this corresponds to a strong-coupling expansion around large λ. In summary, taking the supergravity limit in AdS corresponds to a double expansion around large N and large λ. In this picture, the $1/N$ expansion corresponds to a loop-expansion in the bulk, while the $1/\lambda$ expansion corresponds to adding string corrections to supergravity. We will further discuss the implications of this double expansion for four-point correlation functions in Section 3.3.

Not too long after the original conjecture by Maldacena, further details of the AdS/CFT correspondence were made more precise in references [15, 16]: it was explained how the states of the two theories can be identified with each other, and how correlation functions on the CFT side are related to supergravity scattering amplitudes on $\mathrm{AdS}_5\times\mathrm{S}^5$. In particular, it was understood that the one-half BPS single-trace operators of $\mathcal{N}=4$

[1] Technically, the total gauge group is given by $U(N)$. However, the $U(1)$ factor corresponds to the overall position of the D3-branes. As such, it decouples from the dynamics of the theory and the effective gauge group is reduced to $SU(N)$.

SYM theory (as introduced in the previous chapter) correspond to the spectrum of type IIB supergravity, which is given by the graviton supermultiplet and its Kaluza-Klein modes on S^5. In fact, the identification of single-particle supergravity states with single-trace operators only holds strictly in the large N limit, and in the next section we will discuss how to uplift this statement to finite N. Once the states on both sides are correctly identified with each other, we can ask about physical observables, like e.g. correlation functions. The AdS/CFT dictionary states that the CFT generating functional equals the AdS path integral with boundary sources:

$$\left\langle e^{\int_{\partial\mathrm{AdS}} \bar{\varphi}_i \, \mathcal{O}_i} \right\rangle_{\mathrm{CFT}} = \int_{\bar{\varphi}_i} \mathcal{D}\varphi_i \, e^{-S_{\mathrm{AdS}}[\varphi_i]}, \tag{3.3}$$

where the bulk fields φ_i of the AdS path integral equal $\bar{\varphi}_i$ on the conformal boundary of AdS. In this sense, one can think of $\mathcal{N} = 4$ SYM theory living on the boundary of AdS_5, with the boundary states $\bar{\varphi}_i$ acting as classical sources for their dual operators $\mathcal{O}_i$. From the above equality, one can obtain n-point correlation functions by taking n functional derivatives with respect to the boundary sources $\bar{\varphi}_i$. We will elaborate on this procedure further when discussing tree-level supergravity correlators in Section 3.4.

3.2 Single-Particle Operators

We are interested in correlation functions of one-half BPS operators which describe scattering of single-particle supergravity states in $\mathrm{AdS}_5 \times \mathrm{S}^5$. The first task is thus to determine the precise form of the gauge theory operators which are dual to those single-particle states. We will call these operators *single-particle operators*, and from now on we denote them by $\mathcal{O}_p$. It turns out that in the general case these are not simply the single-trace operators T_p which, recalling the discussion in Section 2.2, are schematically of the form $T_p = \mathrm{tr}\left\{\phi^p\right\}$. Instead, they require admixtures of multi-trace operators, i.e. products of single-trace ones, such that $\mathcal{O}_p = T_p + (\text{multi-trace terms})$. This important subtlety was already noticed in the early works [40, 93, 94] and discussed more recently again in references [4, 6, 45, 48, 59, 60, 95]. In particular, it was noticed in [45] that the connected part of $\langle p_1 p_2 p_3 p_4 \rangle_{\mathrm{free}}$ generated via tree-level Witten diagrams disagrees with the free theory four-point functions of single-trace one-half BPS operators. The resolution is that one has to include admixtures of multi-trace operators, and the first order double-trace corrections have recently been worked out directly from supergravity in [59, 60]. However, the non-perturbative nature of the AdS/CFT correspondence points towards a definition of single-particle states which is valid to all orders in N. Such a non-perturbative definition can be given in terms of the following deceptively simple statement:

> Single-particle operators are one-half BPS operators which are orthogonal to all multi-trace operators.

Up to a normalisation, this definition via orthogonality of operators uniquely fixes the single-particle operators $\mathcal{O}_p$. Moreover, it allows us to compute the additional multi-trace terms purely within free field theory (i.e. using Wick contractions), and the results are exact in N. In the strict large N limit, the above definition reduces to the familiar statement that single-particle states correspond to single-trace operators in the $[0, p, 0]$ representation of $su(4)$: $\mathcal{O}_p \to T_p + O(1/N)$. For finite N however, our definition automatically picks the correct multi-trace admixtures which are needed to uplift one-half BPS single-trace operators T_p to single-particle operators $\mathcal{O}_p$.

Note that a similar orthogonal basis for all one-half BPS operators in the $U(N)$ theory was described previously in terms of Schur polynomials [96]. However, this basis does not project onto an orthogonal basis for $SU(N)$, which is the case of interest here. Nevertheless, at large N and with charge close to N, we find that the single-particle operators defined here become proportional to the so-called (sub)-determinant operators. These operators are given by completely antisymmetric Schur polynomials, and in [97] it was argued that they are dual to the giant sphere gravitons predicted in [98].

Let us now consider some explicit examples of single-particle operators with low charges. For the first two cases ($\mathcal{O}_p$ with $p = 2$,3), there are no multi-trace operators to mix with,[2] and the single-particle operator thus equals the single-trace operator even at finite N:

$$\mathcal{O}_2 = T_2, \qquad \mathcal{O}_3 = T_3. \tag{3.4}$$

In the holographic context, the operator with $p = 2$ is the superconformal primary for the energy-momentum multiplet which is dual to the graviton supermultiplet in AdS_5. Operators with $p \geq 3$ then correspond to Kaluza-Klein modes arising from reduction of the graviton supermultiplet on S^5.

For Kaluza-Klein modes with $p \geq 4$ we find non-trivial multi-trace terms. In the case of $p = 4$, the coefficient of the double-trace contribution $[T_2T_2]$ to $\mathcal{O}_4$ is, according to the above definition, determined by the orthogonality condition

$$\langle \mathcal{O}_4(x_1, y_1)[T_2T_2](x_2, y_2)\rangle = 0. \tag{3.5}$$

A computation using Wick contractions yields the result

$$\mathcal{O}_4 = T_4 - \frac{2N^2 - 3}{N(N^2 + 1)}\,[T_2T_2], \tag{3.6}$$

and it is with this identification of $\mathcal{O}_4$ that (the free theory part of) the supergravity result for the $\langle 2244\rangle$ correlator from [45] agrees with the free theory computation. For

[2] This is true only when the gauge group is $SU(N)$. In a $U(N)$ gauge theory, one also has to consider the operator T_1, in which case for example T_2 can mix with the double-trace term $[T_1T_1]$. Interestingly, in turns out that the single-particle operators in the $U(N)$ theory and the $SU(N)$ theory are actually the same, see [9] for more details.

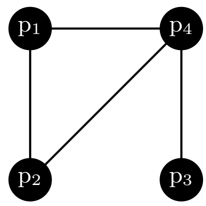


Figure 3.1: Example of a vanishing free theory diagram.

illustration, the next two examples read

$$\mathcal{O}_5 = T_5 - \frac{5(N^2-2)}{N(N^2+5)}\,[T_3T_2], \tag{3.7}$$

$$\begin{aligned}\mathcal{O}_6 = T_6 &- \frac{3N^4-11N^2+80}{N(N^4+15N^2+8)}\,[T_3T_3] - \frac{6(N-2)(N+2)(N^2+5)}{N(N^4+15N^2+8)}\,[T_4T_2] \\ &+ \frac{7(N^2-7)}{N^4+15N^2+8}\,[T_2T_2T_2],\end{aligned} \tag{3.8}$$

where $\mathcal{O}_6$ is the first case with a triple-trace contribution. An explicit formula for all single-particle operators was recently given in [9], together with their two-point functions

$$\langle \mathcal{O}_p(x_1)\mathcal{O}_p(x_2)\rangle = g_{12}^{p}\,R_p\,. \tag{3.9}$$

The N dependent factor R_p can be derived by using the group theoretic approach of [99], and takes the form

$$R_p = p^2(p-1)\left[\frac{1}{(N-p+1)_{p-1}} - \frac{1}{(N+1)_{p-1}}\right]^{-1}. \tag{3.10}$$

As mentioned briefly in the above, the correct identification of single-particle operators has very important non-trivial implications for their four-point correlation functions. In particular, connected free theory diagrams where e.g. $\mathcal{O}_{p_3}$ is joined only to $\mathcal{O}_{p_4}$ (see Figure 3.1) are absent due to the orthogonality property of single-particle operators.[3] Obviously, any topology related by a permutation to the diagram in Figure 3.1 also vanishes. As a consequence of this observation, the colour factors of all extremal and next-to-extremal correlators of single-particle operators vanish identically. These are correlators whose charges satisfy (with our choice of $p_{43} \geq p_{21} \geq 0$):

$$\begin{aligned} p_4 &= p_1 + p_2 + p_3, &&\text{(extremal)}, \\ p_4 &= p_1 + p_2 + p_3 - 2, &&\text{(next-to-extremal)}. \end{aligned} \tag{3.11}$$

[3] To see this, note that at twist p_{43} in the $\mathcal{O}_{p_3} \times \mathcal{O}_{p_4}$ OPE, only a one-half BPS operator $\mathcal{O}_{p_{43}}$ of charge p_{43} could potentially be transferred. By our definition, $\mathcal{O}_{p_4}$ is orthogonal to all multi-trace operators and in particular to the double (or higher) trace operator $[\mathcal{O}_{p_{43}}\mathcal{O}_{p_3}]$. But the vanishing two-point function $\langle [\mathcal{O}_{p_{43}}\mathcal{O}_{p_3}]\mathcal{O}_{p_4}\rangle$ is just a non-singular limit of the three-point function $\langle \mathcal{O}_{p_{43}}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle$, which therefore also vanishes. Hence no operator $\mathcal{O}_{p_{43}}$ can be exchanged and the coefficient of the diagram in Figure 3.1 must vanish. Note that this holds no matter whether $\mathcal{O}_{p_{43}}$ is single-trace, multi-trace or any linear combination thereof.

Note that extremal and next-to-extremal correlators in general do not vanish for single-trace operators as external states, but they do for our definition of single-particle operators. The first single-particle correlators which are non-vanishing are therefore next-to-next-to-extremal, with charges obeying

$$p_4 = p_1 + p_2 + p_3 - 4, \qquad \text{(next-to-next-to-extremal)}. \tag{3.12}$$

More generally, we define the so-called *degree of extremality* $\kappa_{\vec{p}}$ by

$$\kappa_{\vec{p}} = \min\left\{\tfrac{1}{2}(p_1 + p_2 + p_3 - p_4),\, p_3\right\}, \tag{3.13}$$

and we say that a correlator is N$^\kappa$E, according to its degree of extremality. As such, the next-to-next-to-extremal correlators obeying (3.12) have degree of extremality $\kappa_{\vec{p}} = 2$.

3.3 Correlation Functions at Strong 't Hooft Coupling

With the correct single-particle operators in mind, we now describe what the AdS/CFT correspondence implies for their four-point correlation functions in the interacting theory. In particular, we will consider $\mathcal{N} = 4$ SYM in the supergravity regime, where quantum corrections are organised in a double expansion in $1/N^2$ and $1/\lambda$. Furthermore, the decoupling of excited string states in this limit leads to a restricted spectrum of exchanged operators, which at leading order is given by double-trace operators.

3.3.1 The Large N, Strong-Coupling Expansion

Let us recall the general structure of the four-point correlation functions $\langle p_1p_2p_3p_4\rangle$ as introduced in Section 2.3: the partial non-renormalisation theorem (2.20) singles out the factor $\mathcal{H}_{\vec{p}} \equiv \mathcal{H}_{p_1p_2p_3p_4}$ as the only piece of the correlator which depends on the gauge coupling. As such, $\mathcal{H}_{\vec{p}}$ contributes only to the long sector of the SCPW expansion. However, this is not the only contribution to the long sector: recalling the SCPW expansion of the free theory (see Section 2.5), we also have to take into account the free theory contribution to the long sector, denoted by $\langle p_1p_2p_3p_4\rangle_{\text{free long}}$ in equation (2.40).

Now, in the supergravity limit of large N and large 't Hooft coupling λ, the long sector of the free theory admits the expansion

$$\begin{aligned}\langle p_1p_2p_3p_4\rangle_{\text{free long}} = \langle p_1p_2p_3p_4\rangle^{(0)}_{\text{free long}} &+ \mathfrak{a}\,\langle p_1p_2p_3p_4\rangle^{(1)}_{\text{free long}} \\ &+ \mathfrak{a}^2\langle p_1p_2p_3p_4\rangle^{(2)}_{\text{free long}} + O(\mathfrak{a}^3),\end{aligned} \tag{3.14}$$

where for convenience we use $\mathfrak{a} = 1/(N^2-1)$ as our large N expansion parameter. Note that since the free theory does not depend on the gauge coupling, all of the above terms are of leading order in $1/\lambda$. On the other hand, the dynamical part $\mathcal{H}_{\vec{p}}$ admits a double expansion of the form

$$\begin{aligned}\mathcal{H}_{\vec{p}} = \quad & \mathfrak{a}\,\big(\mathcal{H}_{\vec{p}}^{(1,0)} + \lambda^{-\frac{3}{2}}\mathcal{H}_{\vec{p}}^{(1,3)} + \lambda^{-\frac{5}{2}}\mathcal{H}_{\vec{p}}^{(1,5)} + \lambda^{-3}\mathcal{H}_{\vec{p}}^{(1,6)} + \lambda^{-\frac{7}{2}}\mathcal{H}_{\vec{p}}^{(1,7)} + \ldots\big) \\ & + \mathfrak{a}^2\big(\lambda^{\frac{1}{2}}\mathcal{H}_{\vec{p}}^{(2,-1)} + \mathcal{H}_{\vec{p}}^{(2,0)} + \lambda^{-\frac{1}{2}}\mathcal{H}_{\vec{p}}^{(2,1)} + \lambda^{-1}\mathcal{H}_{\vec{p}}^{(2,2)} + \lambda^{-\frac{3}{2}}\mathcal{H}_{\vec{p}}^{(2,3)} + \ldots\big) \\ & + O(\mathfrak{a}^3).\end{aligned} \tag{3.15}$$

The order $\mathfrak{a}$ terms of this double expansion correspond to tree-level amplitudes in AdS_5, with the first term $\mathcal{H}_{\vec{p}}^{(1,0)}$ being the well-studied tree-level supergravity correlator discussed further in Section 3.4. The supergravity term is followed by an infinite tower of $1/\lambda$ suppressed string corrections $\mathcal{H}_{\vec{p}}^{(1,n)}$. The structure of this $1/\lambda$ expansion is related via the flat space limit to the low-energy expansion of the tree-level type IIB string amplitude in 10 dimensions, the so-called Virasoro-Shapiro amplitude. In other words, the $1/\lambda$ expansion arises from contact interaction vertices in the string theory effective action, where the order $\lambda^{-\frac{3}{2}}$ and $\lambda^{-\frac{5}{2}}$ terms descend from dimensional reduction of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ supervertices, respectively. These tree-level terms are most conveniently studied in their Mellin space representation, which will be introduced later.

The order $\mathfrak{a}^2$ terms of the double expansion (3.15) correspond to one-loop amplitudes in AdS_5. Note that the term $\mathcal{H}_{\vec{p}}^{(2,-1)}$ comes with a superleading power of $\lambda^{\frac{1}{2}}$, and is due to the presence of a quadratic divergence at one-loop in ten-dimensional supergravity. This divergence is regulated by a specific $\mathcal{R}^4$ counterterm at one-loop in string theory. The next contribution, $\mathcal{H}_{\vec{p}}^{(2,0)}$, is the one-loop supergravity term which we will address in Chapter 6. Note that a direct computation of such one-loop amplitudes would require the application of a renormalisation procedure. The corresponding counterterms are of the form of contact Witten diagrams, and in our bootstrap program they manifest themselves as a finite set of undetermined free parameters which we call 'ambiguities'. Ultimately, the values of these ambiguities are determined within the full superstring theory, but as our bootstrap approach is not able to fix them we have to rely on other methods, such as e.g. supersymmetric localisation techniques. Next, the term $\mathcal{H}_{\vec{p}}^{(2,1)}$ corresponds to the genus-one contribution to the modular completion of the $\mathcal{H}_{\vec{p}}^{(1,5)}$ term. The existence of such a term follows from the S-duality of $\mathcal{N}=4$ SYM and type IIB string theory, which is realised as a $SL(2,\mathbb{Z})$ symmetry of the complexified gauge coupling. For the correlation functions under consideration, this symmetry is manifest in the 'very strong coupling' limit ($N \to \infty$ with g_{YM} fixed), where the corresponding terms are given by certain modular functions.[4] In particular, the first few terms are given by Eisenstein series, which (apart from an infinite sequence of non-perturbative instanton corrections) receive a finite number of perturbative contributions. In the case of the $\mathcal{H}_{\vec{p}}^{(2,1)}$ term, the corresponding Eisenstein series receives perturbative contributions only at genus zero and genus two [103], and we therefore expect $\mathcal{H}_{\vec{p}}^{(2,1)}$ to vanish.[5] The term $\mathcal{H}_{\vec{p}}^{(2,2)}$ gives

[4] See e.g. the recent series of papers [100–102].

[5] The vanishing of this term is also consistent with the supersymmetric localisation analysis of [104,

rise, in the flat space limit, to the analytic part of the one-loop string amplitude studied in [106]. It is therefore non-vanishing and corresponds to the genus-one contribution to the modular completion of the $\mathcal{H}_{\vec{p}}^{(1,6)}$ term. Finally, at order $\lambda^{-\frac{3}{2}}$, we find the $\mathcal{H}_{\vec{p}}^{(2,3)}$ term which is the genuine one-loop string correction induced by the presence of the $\mathcal{H}_{\vec{p}}^{(1,3)}$ term at tree-level. This term as well as some higher order corrections will be addressed in Chapter 7.

3.3.2 The Double-Trace Spectrum at Strong Coupling

At last, let us combine the strong coupling expansion described above with the SCPW decomposition introduced in Section 2.4. As the short sector of the SCPW expansion can be understood within free field theory only, we will specialise to the long sector. In particular, we have to describe the precise spectrum of exchanged operators in the OPE of single-particle one-half BPS operators $\mathcal{O}_p$.

In the supergravity limit, and in particular after we take the limit of large 't Hooft coupling λ, we expect all operators which are dual to excited string states to decouple as they become infinitely massive. We are thus left with the spectrum of supergravity states, consisting of the protected single-particle operators $\mathcal{O}_p$ themselves as well as multi-particle operators built out of the single-particle ones. Such operators can themselves be either protected or unprotected. The unprotected operators of this type are still present in the spectrum because in the strictly infinite N limit they keep their classical scaling dimensions due to operator factorisation, and hence the corresponding states do not acquire infinite mass. In the supergravity spectrum such operators are 'nearly' protected and receive anomalous dimensions at order $1/N^2$ and higher.

However, all other operators, not built from products of single-particle operators, correspond to the afore-mentioned string states and such operators are therefore absent from the spectrum in the supergravity limit.[6] In fact, the OPE analysis of known supergravity four-point correlators [65, 107] reveals that certain long operators indeed cancel in the sum of the long sector of free theory and the dynamical part,

$$\langle p_1 p_2 p_3 p_4 \rangle^{(1)}_{\text{free}} + \mathcal{P}_{\vec{p}} \times \mathcal{I} \times \mathcal{H}_{\vec{p}}^{(1,0)}, \tag{3.16}$$

resulting in the absence of string states. This is a non-trivial consistency-check of the AdS/CFT correspondence. In Section 3.4.3, we will explain how to make use of this cancellation to determine the normalisation of the supergravity correlators $\mathcal{H}_{\vec{p}}^{(1,0)}$.

The simplest unprotected operators which remain in the supergravity spectrum are the

105].

[6] A simple example of an operator corresponding to a string state is the Konishi operator, which is a twist 2 operator in the $su(4)$ singlet representation. At higher twists, one has to carefully distinguish operators which remain in the supergravity spectrum from excited string states through multiplet recombination, as outlined in detail in Appendix B.

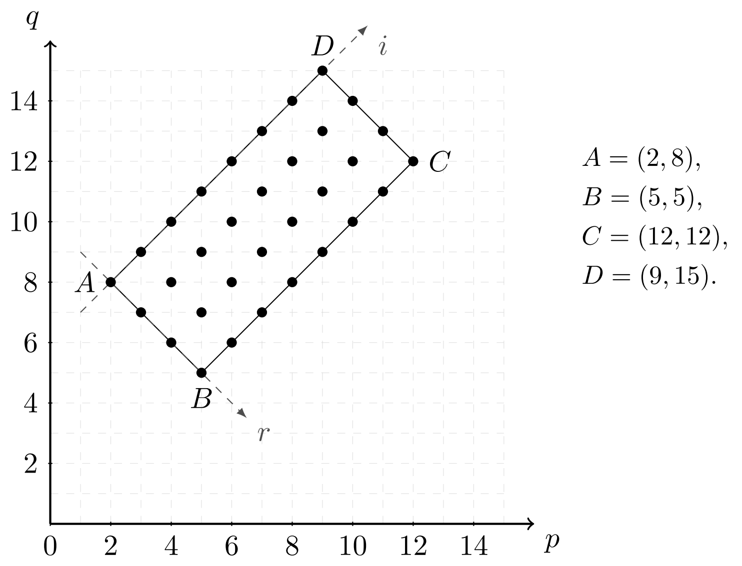


Figure 3.2: The set of double-trace operators $\mathcal{O}_{pq;\vec{\tau}}$ depicted as points in the (p, q) plane. An alternative description is given by the labels (i, r). In that parametrisation, it is easy to see that the 'width' of the rectangle is fixed by the choice of $su(4)$ channel, while its 'length' depends on the twist τ, i.e. the value of t. In this example, we show the set of double-trace operators for twist $\tau = 24$ and $su(4)$ channel $[0, 6, 0]$, corresponding to $t = 9$ and $\mu = 4$. According to equation (3.18), we therefore have $i = 1, \ldots, 8$ and $r = 0, \ldots, 3$.

so-called double-trace operators, which correspond to two-particle bound states in the bulk theory. These double-trace operators are special for two reasons: firstly, their three-point functions are of leading order in the large N expansion, whereas we expect the three-point functions involving triple-trace operators and higher to be suppressed. Secondly, there is a unique double-trace operator of spin ℓ for fixed twist τ and $su(4)$ labels. In contrast, triple- and higher multi-trace operators do not have this property as their number grows with spin. A basis of unprotected double-trace operators of twist τ, spin ℓ and $su(4)$ labels $[a, b, a]$ is of the schematic form

$$\mathcal{O}_{pq;\vec{\tau}} = \left(\mathcal{O}_p \Box^{\frac{1}{2}(\tau-p-q)} \partial^\ell \mathcal{O}_q + \ldots\right)\Big|_{[a,b,a]}, \qquad (p \leq q), \tag{3.17}$$

where the ellipsis denotes similar terms with the spacetime derivatives distributed differently between the two constituent operators. The precise combination is not relevant here, but importantly there is a unique combination yielding a superconformal primary operator. The allowed values for the pairs (p, q) are given by the set $R_{\vec{\tau}}$, defined as

$$R_{\vec{\tau}} := \left\{ (p,q) : \begin{array}{l} p = i + a + 1 + r \\ q = i + a + 1 + b - r \end{array} \quad \text{for} \quad \begin{array}{l} i = 1, \ldots, t-1 \\ r = 0, \ldots, \mu - 1 \end{array} \right\}. \tag{3.18}$$

There are in total $d = \mu(t-1)$ allowed values, where

$$t \equiv \frac{(\tau - b)}{2} - a\,, \qquad \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a + \ell \text{ even}, \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a + \ell \text{ odd}. \end{cases} \tag{3.19}$$

Note that the set of double-trace operators $R_{\vec{\tau}}$ traces out a rectangle in the (p,q) plane. As an example, a pictorial representation with quantum numbers $\vec{\tau} = (24, \ell, [0,6,0])$ is given in Figure 3.2.[7]

Note that the double-trace operators $\mathcal{O}_{pq;\vec{\tau}}$ all have the same classical dimension $\Delta^{(0)} = \tau+\ell$, and hence in general they will mix. We collectively denote the true eigenstates (with well-defined scaling dimensions) by $\mathcal{K}_{pq}$, which are simply certain linear combinations of the operators $\mathcal{O}_{pq;\vec{\tau}}$ from (3.17). Their three-point couplings $C_{p_ip_j\mathcal{K}}$ with two external operators $\mathcal{O}_{p_i}$, $\mathcal{O}_{p_j}$ are related to the three-point functions $\langle \mathcal{O}_{p_i}\mathcal{O}_{p_j}\mathcal{K}\rangle$ and admit the double expansion

$$\begin{aligned} C_{p_ip_j\mathcal{K}} = \quad & \Big(C^{(0,0)}_{p_ip_j\mathcal{K}} + \lambda^{-\frac{3}{2}} C^{(0,3)}_{p_ip_j\mathcal{K}} + \lambda^{-\frac{5}{2}} C^{(0,5)}_{p_ip_j\mathcal{K}} + \ldots \Big) \\ & + \mathfrak{a}\Big(C^{(1,0)}_{p_ip_j\mathcal{K}} + \lambda^{-\frac{3}{2}} C^{(1,3)}_{p_ip_j\mathcal{K}} + \lambda^{-\frac{5}{2}} C^{(1,5)}_{p_ip_j\mathcal{K}} + \ldots \Big) + O(a^2). \end{aligned} \tag{3.20}$$

Similarly, the double expansion of their scaling dimensions reads

$$\begin{aligned} \Delta_{\mathcal{K}} = \tau + \ell + 2\mathfrak{a} & \left(\eta^{(1,0)} + \lambda^{-\frac{3}{2}} \eta^{(1,3)} + \lambda^{-\frac{5}{2}} \eta^{(1,5)} + \ldots \right) \\ & + 2\mathfrak{a}^2 \left(\lambda^{\frac{1}{1}} \eta^{(2,-1)} + \eta^{(2,0)} + \lambda^{-\frac{1}{2}} \eta^{(2,1)} + \lambda^{-1} \eta^{(2,2)} + \lambda^{-\frac{3}{2}} \eta^{(2,3)} + \ldots \right) \\ & + O(\mathfrak{a}^3), \end{aligned} \tag{3.21}$$

where η denotes (half) the anomalous dimension.

In the second part of this thesis, we will explain in detail how the mixing problem of the double-trace spectrum can be resolved by combining data from many correlation functions simultaneously, both at the level of tree-level supergravity (Chapter 4) and including the first string corrections (Chapter 5). Furthermore, certain one-loop anomalous dimensions can be extracted from the explicit results for correlators at one-loop, which we will discuss in the third part of this thesis.

3.4 The Tree-Level Supergravity Correlator

Let us now turn our attention to the supergravity correlators $\mathcal{H}^{(1,0)}_{\vec{p}}$. As mentioned before, the AdS/CFT correspondence predicts, in the regime of strong 't Hooft coupling corresponding to classical supergravity, the leading large N behaviour of the correlation functions $\langle p_1p_2p_3p_4\rangle$. We will start by briefly outlining the traditional computation of supergravity correlators in terms of Witten diagrams. We then focus on the Mellin space representation of these correlators, which turned out to be the right language to write down a surprisingly simple formula for all supergravity correlators with arbitrary external charges.

[7] For general quantum numbers, the four corners of the rectangle defined by (3.18) have the coordinates $A = (a{+}2, a{+}b{+}2)$, $B = (a{+}\mu{+}1, a{+}b{-}\mu{+}3)$, $C = (a{+}t{+}\mu{-}1, a{+}b{+}t{-}\mu{+}1)$ and $D = (a{+}t, a{+}b{+}t)$.

3.4.1 The Traditional Method of Computing Supergravity Correlators

The standard method of computing holographic correlation functions in supergravity relies on the equality (3.3) of the CFT partition function and the AdS path integral. One starts with the action for a collection of scalar Kaluza-Klein modes $\{\varphi_k\}$ on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which can be written as

$$S_{\mathrm{sugra}} = \frac{N^2}{8\pi^2 L^3} \int d\Omega \, \left(\mathcal{L}_{(2)} + \mathcal{L}_{(3)} + \mathcal{L}_{(4)} + \ldots \right), \tag{3.22}$$

with $d\Omega$ being the measure on AdS_5 and L its radius, which can be set to one.[8] We denote the bulk coordinate by z and the boundary coordinates by $\vec{x}$. The index n on $\mathcal{L}_{(n)}$ indicates the number of fields, in particular $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(4)}$ contain cubic and quartic interactions among the Kaluza-Klein modes, which include the graviton and the gauge fields. The above action is known explicitly up to quartic order [41].

Let us focus on one single Kaluza-Klein mode $\bar{\varphi}(z, \vec{x})$ from the infinite tower. In the saddlepoint approximation, valid at large N, the bulk field $\bar{\varphi}(z, \vec{x})$ propagates according to its equation of motion, $(\nabla^2 - m^2)\bar{\varphi} = J[\{\varphi_k\}]$, where the source term J depends on all the fields coupling to $\bar{\varphi}$. The general solution for $\bar{\varphi}(z, \vec{x})$ can be written in terms of the bulk Green's function $\mathbb{G}_{bb}$ and the bulk-to-boundary propagator $\mathbb{G}_{b\partial}$ as follows,

$$\begin{aligned} \bar{\varphi}(z, \vec{x}) &= \varphi^0(z, \vec{x}) + \int dz d^4\vec{x}' \mathbb{G}_{bb}(z, \vec{x}; z', \vec{x}') \; J[\{\varphi_k(z', \vec{x}')\}], \\ \bar{\varphi}^0(z, \vec{x}) &= \int d^4\vec{x}' \; \mathbb{G}_{b\partial}(z, \vec{x}; \vec{x}') \; \mathcal{S}(\vec{x}'), \end{aligned} \tag{3.23}$$

where φ^0 solves the homogeneous equation of motion with boundary conditions $\mathcal{S}(\vec{x}')$. According to the AdS/CFT correspondence, $\mathcal{S}(\vec{x}')$ is identified with the boundary source which couples to the operator dual to $\bar{\varphi}(z, \vec{x})$. The perturbative expansion around the homogeneous solutions $\{\varphi_k^0(z', \vec{x}')\}$ defines the corresponding series expansion for J, i.e. $J = J_{(2)} + J_{(3)} + \ldots$, where the label indicates again the number of boundary fields $\mathcal{S}_k(\vec{x}')$ at each order. Finally, evaluating the action on-shell can be interpreted diagrammatically as summing over tree-level Witten diagrams. The result is the following generating functional for the boundary sources:

$$\begin{aligned} S_{\mathrm{sugra}} = & \int (d^4\vec{x})^2 \; \mathcal{S}_{k_1}(\vec{x}_1)\mathcal{S}_{k_2}(\vec{x}_2) \; \mathcal{D}^{(2)}_{k_1k_2}(\vec{x}_1, \vec{x}_2) \\ & + \int (d^4\vec{x})^3 \; \mathcal{S}_{k_1}(\vec{x}_1)\mathcal{S}_{k_2}(\vec{x}_2)\mathcal{S}_{k_3}(\vec{x}_3) \; \mathcal{D}^{(3)}_{k_1k_2k_3}(\vec{x}_1, \vec{x}_2, \vec{x}_3) \\ & + \int (d^4\vec{x})^4 \; \mathcal{S}_{k_1}(\vec{x}_1)\mathcal{S}_{k_2}(\vec{x}_2)\mathcal{S}_{k_3}(\vec{x}_3)\mathcal{S}_{k_4}(\vec{x}_4) \; \mathcal{D}^{(4)}_{k_1k_2k_3k_4}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) \\ & + \ldots, \end{aligned} \tag{3.24}$$

[8] We can do this since we will not consider any curvature corrections here. Curvature and loop corrections to the supergravity effective action have been discussed in [108].

where the functions $\mathcal{D}^{(i=2,3,4)}(\{\vec{x}_i\})$ are proportional to N^2 according to the action (3.22). Correlators of n operators can then be computed by taking n functional derivatives with respect to the dual sources, i.e.

$$\langle \mathcal{O}_{p_1}(\vec{x}_1)\mathcal{O}_{p_2}(\vec{x}_2)\mathcal{O}_{p_3}(\vec{x}_3)\mathcal{O}_{p_4}(\vec{x}_4)\rangle = \prod_{n=1}^{4} \frac{\delta}{\delta \mathcal{S}_n(\vec{x}_n)}\, e^{-S_{\rm sugra}}\Big|_{\mathcal{S}_n=0}. \tag{3.25}$$

The two- and three-point functions of AdS supergravity obtained in an analogous way manifestly agree with CFT expectations [19, 20, 93]. In the supergravity conventions all two-point functions are normalised to N^2, it is however always possible to redefine the sources in such a way to match the normalisation given in (3.9)-(3.10).

Four-point correlation functions are more interesting and require quite involved manipulations. Explicit Witten diagram computations have been carried out for a number of different cases: $\langle \mathcal{O}_p\mathcal{O}_p\mathcal{O}_p\mathcal{O}_p\rangle$ for $p = 2, 3, 4$ [28, 29, 42], $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_q\mathcal{O}_q\rangle$ [44, 45] and for the two parameter family of next-to-next-to-extremal correlators $\langle \mathcal{O}_{k+2}\mathcal{O}_{k+2}\mathcal{O}_{q-k}\mathcal{O}_{q+k}\rangle$ [46] for arbitrary q and k. Despite highly complicated calculations, the end result for all of the above cases is neat and can be written in terms of a restricted set of functions (the so-called $\overline{D}$-functions), suggesting that a generalisation to arbitrary external charges is feasible. Indeed, it turned out that such a generalisation can be achieved using the Mellin space representation of holographic correlators, which we will review in detail in the next section.

Lastly, it is clear from the form of $S_{\rm sugra}$ in (3.24) that upon taking functional derivatives with respect to the sources, a four-point correlator will get a leading contribution from disconnected two-point functions $\mathcal{D}^{(2)}_{k_1k_2}\mathcal{D}^{(2)}_{k_3k_4}$ (when it exists), followed by the $1/N^2$ suppressed contribution $\mathcal{D}^{(4)}_{k_1k_2k_3k_4}$. The latter will contain a dynamical term with a $\log(u)$ singularity, but also a subset of the corresponding connected free field theory correlator, allowing us to consider the the splitting

$$\mathcal{G}_{\vec{p}}^{sugra} = \mathcal{G}_{\vec{p}}^{\rm free} + \mathcal{G}_{\vec{p}}^{\rm dyna}, \tag{3.26}$$

where $\mathcal{G}_{\vec{p}}^{\rm free}$ can be computed independently from supergravity. Furthermore, the dynamical part of the correlator, $\mathcal{G}_{\vec{p}}^{\rm dyna}$, turns out to be of the form

$$\mathcal{G}_{\vec{p}}^{\rm dyna} = \mathcal{I}(x,\bar{x},y,\bar{y}) \times \mathcal{H}_{\vec{p}}^{\rm dyna}, \tag{3.27}$$

which is in exact agreement with the partial non-renormalisation theorem (2.20) as well as the splitting (3.16) discussed in the previous section, both of which are necessary for the consistency of the AdS/CFT correspondence.

Next, let us introduce the Mellin formalism for holographic correlators, and in particular we review the generalisation of the previously known tree-level supergravity amplitudes mentioned above to arbitrary external charges.

3.4.2 Holographic Correlators in Mellin Space

The explicit position space results mentioned above have a simple representation when expressed in Mellin space. In fact, the Mellin space formalism [74, 109–111] has turned out to be an efficient framework to describe holographic correlators, where in particular tree-level Witten diagrams take a particularly simple form. In the case of tree-level supergravity, they are rational functions of the Mellin variables with a prescribed set of poles which correspond to the exchanged single-trace operators in a given Witten diagram. Beyond supergravity, further string corrections are even simpler when expressed in Mellin space, as their corresponding Mellin amplitudes are only polynomial.[9]

The Mellin representation of the interacting part $\mathcal{H}_{\vec{p}}$ is defined by the integral transform[10]

$$\mathcal{H}_{\vec{p}}(u,v;\sigma,\tau)=\int_{-i\infty}^{i\infty}\frac{ds}{2}\frac{dt}{2}\,u^{\frac{s}{2}-\frac{p_{43}}{2}}v^{\frac{t}{2}-\frac{p_2+p_3}{2}}\mathcal{M}_{\vec{p}}(s,t;\sigma,\tau)\,\Gamma_{\vec{p}}(s,t), \tag{3.28}$$

where the string of six Γ-functions is defined as

$$\Gamma_{\vec{p}}(s,t)=\prod_{i<j}\Gamma\left[c_{ij}\right], \tag{3.29}$$

with the Mellin space parametrisation $c_{ij}=c_{ji}$ given by

$$\begin{aligned} c_{12}&=-\frac{s}{2}+\frac{p_1+p_2}{2}, & c_{13}&=-\frac{\tilde{u}}{2}+\frac{p_1+p_3}{2}, & c_{14}&=-\frac{t}{2}+\frac{p_1+p_4}{2},\\ c_{23}&=-\frac{t}{2}+\frac{p_2+p_3}{2}, & c_{24}&=-\frac{\tilde{u}}{2}+\frac{p_2+p_4}{2}, & c_{34}&=-\frac{s}{2}+\frac{p_3+p_4}{2}. \end{aligned} \tag{3.30}$$

Note that the Mellin space variables $(s,t,\tilde{u})$ satisfy the constraint

$$s+t+\tilde{u}=p_1+p_2+p_3+p_4-4. \tag{3.31}$$

The Mellin amplitude inherits an analogous double expansion at strong coupling, as given for the dynamical part $\mathcal{H}(u,v)$ in equation (3.15). Hence $\mathcal{M}_{\vec{p}}$ admits an expansion of

[9] We will revisit the structure of tree-level string corrections in more detail in Chapter 5.

[10] What we call $\mathcal{M}_{\vec{p}}$ here is in fact the reduced Mellin amplitude (denoted by $\widetilde{\mathcal{M}}_{\vec{p}}$ in [47]), which is related to the full Mellin amplitude $M_{\vec{p}}$ by

$$M_{\vec{p}}(s,t;\sigma,\tau)=\widehat{R}(u,v;\sigma,\tau)\circ\widetilde{\mathcal{M}}_{\vec{p}}(s,t;\sigma,\tau),$$

where $\widehat{R}$ is a difference operator mimicking the action of the factor $\mathcal{I}$ on the interacting part $\mathcal{H}_{\vec{p}}$. See [48] for further details, where also a precise definition of the integration contour is given, such that rational parts of the position space result (corresponding to the long part of free theory) are correctly recovered from the Mellin integrals.

the form

$$\begin{aligned}\mathcal{M}_{\vec{p}} = \ & \mathfrak{a}\left(\mathcal{M}_{\vec{p}}^{(1,0)} + \lambda^{-\frac{3}{2}}\mathcal{M}_{\vec{p}}^{(1,3)} + \lambda^{-\frac{5}{2}}\mathcal{M}_{\vec{p}}^{(1,5)} + \ldots\right) \\ & + \mathfrak{a}^2\left(\lambda^{\frac{1}{2}}\mathcal{M}_{\vec{p}}^{(2,-1)} + \mathcal{M}_{\vec{p}}^{(2,0)} + \lambda^{-\frac{1}{2}}\mathcal{M}_{\vec{p}}^{(2,1)} + \lambda^{-1}\mathcal{M}_{\vec{p}}^{(2,2)} + \lambda^{-\frac{3}{2}}\mathcal{M}_{\vec{p}}^{(2,3)} + \ldots\right) \\ & + O(\mathfrak{a}^3).\end{aligned} \tag{3.32}$$

In this formalism, the supergravity correlator $\mathcal{M}_{\vec{p}}^{(1,0)}$ for arbitrary external charges has been obtained by solving a bootstrap problem in Mellin space [47, 48]. The final result of Rastelli and Zhou takes the form of a simple rational Mellin amplitude and, up to an undetermined overall normalisation $\mathcal{N}_{\vec{p}}$, it is given by

$$\mathcal{M}_{\vec{p}}^{(1,0)} = \mathcal{N}_{\vec{p}} \sum_{i,j\geq 0} \frac{a_{ijk}\,\sigma^i\tau^j}{(s-s_0+2k)(t-t_0+2j)(\tilde{u}-\tilde{u}_0+2i)}, \tag{3.33}$$

where $k = \min\{p_3, \frac{p_1+p_2+p_3-p_4}{2}\} - i - j - 2$ and the range of i, j is such that $k \geq 0$ in the sum. We have used the definitions

$$\begin{aligned} s_0 &= \min\{p_1+p_2, p_3+p_4\} - 2, \\ t_0 &= p_2 + p_3 - 2, \\ \tilde{u}_0 &= p_1 + p_3 - 2, \end{aligned} \tag{3.34}$$

and the coefficients a_{ijk} are given by

$$a_{ijk} = \frac{1}{i!j!k!}\frac{8(M-1)!}{\left(1+\frac{p_{43}+p_{21}}{2}\right)_i\left(1+\frac{p_{43}-p_{21}}{2}\right)_j\left(1+\frac{|p_1+p_2-p_3-p_4|}{2}\right)_k}, \tag{3.35}$$

with $M = \min\{p_3, \frac{p_1+p_2+p_3-p_4}{2}\} - 1$. The result (3.33) is consistent with the various previously known position space results obtained in [28, 29, 42–46], and further indications about the correctness of the general formula come from more recent explicit supergravity computations [58–60].

Note that the overall normalisation $\mathcal{N}_{\vec{p}}$ has been left undetermined in equation (3.33). This is because the result for $\mathcal{M}_{\vec{p}}^{(1,0)}$ has been obtained as the solution of a bootstrap problem, and as such it is not sensitive to the overall normalisation. However, as mentioned earlier, consistency with the AdS/CFT correspondence requires the absence of excited string states in the spectrum of $\mathcal{N} = 4$ SYM at strong coupling which the supergravity correlators must satisfy. We will make use of this in the following.

3.4.3 Determining the Supergravity Normalisation

Let us explain how to determine the normalisation $\mathcal{N}_{\vec{p}}$ in the supergravity Mellin amplitude (3.33), by requiring the cancellation of string states between the free field theory

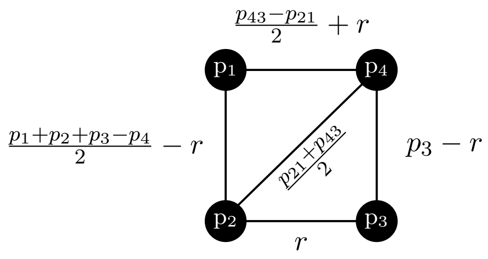


Figure 3.3: Free theory diagrams in the light-like limit.

part of the correlator at order $1/N^2$ and the dynamical part given by $\mathcal{M}^{(1,0)}_{\vec{p}}$. In particular, the normalisation $\mathcal{N}_{\vec{p}}$ can be fixed by imposing the following non-trivial condition:

$$\lim_{u,v\to 0}\frac{\langle p_1p_2p_3p_4\rangle}{\mathcal{P}_{\vec{p}}}\Big|_{\frac{1}{N^2}}=0,\quad \text{with } \frac{u}{v} \text{ fixed,} \tag{3.36}$$

where the limit $u,v\to 0$ whilst keeping the ratio u/v fixed corresponds to taking the points $x_1,\ldots,x_4$ to be sequentially light-like separated.

Examining both the free theory and interacting contributions to the LHS of (3.36), we find that it takes the form $\sum_{r=1}^{M}A_r(u\tau/v)^r$, with

$$A_r=p_1p_2p_3p_4\,\frac{p_{21}+p_{43}+2}{2}-\mathcal{N}_{\vec{p}}\,B_{\vec{p}}. \tag{3.37}$$

The first term comes from $\langle p_1p_2p_3p_4\rangle_{\text{free}}/\mathcal{P}_{\vec{p}}$ and arises from the diagrams shown in Figure 3.3. The normalisation of each of these diagrams in the planar limit can be simply obtained by counting the number of inequivalent planar embeddings: cyclic rotations on each vertex leave the diagram unchanged, hence the factor $p_1p_2p_3p_4$. Additionally, the diagonal propagators can be drawn inside or outside the square, giving $\frac{1}{2}(p_{21}+p_{43})+1$ different possibilities. Note that the multi-trace terms of the single-particle operators $\mathcal{O}_{p_i}$ do not affect the leading large N result for the diagram. The cases $r=0$ and $r=M+1$ correspond to diagrams of Figure 3.1 which are absent due to the orthogonality property of the single-particle operators $\mathcal{O}_{p_i}$.

The second contribution in (3.37) is obtained from $\mathcal{I}\times\mathcal{H}^{(1,0)}_{\vec{p}}$, where $\mathcal{H}^{(1,0)}_{\vec{p}}$ is computed from $\mathcal{M}^{(1,0)}_{\vec{p}}$ by inverting the Mellin transformation. Note that each term in the Mellin amplitude $\mathcal{M}^{(1,0)}_{\vec{p}}$ is proportional to

$$u^{\frac{s}{2}-\frac{p_{43}}{2}}v^{\frac{t}{2}-\frac{p_2+p_3}{2}}\times\frac{a_{ijk}\,\sigma^i\tau^j}{\big(s-p_{43}-2-2(i+j)\big)\big(t-p_2-p_3+2+2j\big)}, \tag{3.38}$$

which upon residue integration will produce a term proportional to $(u\sigma)^i(u/v)^{1+j}\tau^j$. Since $\mathcal{I}=\tau+O(u,v)$, the contribution to A_r comes from taking the simple poles with $i=0$ in the Mellin amplitude. Taking all contributions into account, the residue of the tree-level supergravity amplitude yields

$$B_{\vec{p}} = \big(M-1\big)!\big(\tfrac{|p_1+p_2-p_3-p_4|}{2}\big)!\big(\tfrac{p_{43}-p_{21}}{2}\big)!\big(\tfrac{p_{43}+p_{21}+2}{2}\big)!\,. \tag{3.39}$$

Crucially, the j dependence cancels between $a_{0jk}/(j!k!)$ and $\Gamma_{\vec{p}}(s,t)$ and hence A_r is in fact independent of r. Now the statement (3.36) is clearly equivalent to the statement $A_r = 0$ for all r. Rearranging equation (3.37) for $\mathcal{N}_{\vec{p}}$ we thus obtain the result

$$\mathcal{N}_{\vec{p}} = \frac{p_1p_2p_3p_4}{\big(M-1\big)!\big(\frac{|p_1+p_2-p_3-p_4|}{2}\big)!\big(\frac{p_{43}-p_{21}}{2}\big)!\big(\frac{p_{43}+p_{21}}{2}\big)!}\,. \tag{3.40}$$

The result combines neatly with the coefficients a_{ijk} and we find

$$\mathcal{N}_{\vec{p}}\ a_{ijk} = \frac{1}{i!j!k!}\,\frac{8p_1p_2p_3p_4}{\big(\frac{p_{43}+p_{21}}{2}+i\big)!\big(\frac{p_{43}-p_{21}}{2}+j\big)!\big(\frac{|p_1+p_2-p_3-p_4|}{2}+k\big)!}. \tag{3.41}$$

3.4.4 Proof of the Light-Like Vanishing

The light-like limit projects the common OPE of $(\mathcal{O}_{p_1}\times\mathcal{O}_{p_2})$ and $(\mathcal{O}_{p_3}\times\mathcal{O}_{p_4})$ onto operators with large spin and twist $\tau \leq p_{43}+2M$, i.e. twist $\tau < \min\{p_1+p_2, p_3+p_4\}$. To justify the statement (3.36), let us consider the various contributions to the OPE expected in the supergravity regime. First of all, we have single-particle states corresponding to one-half BPS superconformal primary operators. Such operators have spin zero and do not contribute in the limit $v\to 0$, which receives contributions from large spin. Next, we have (both protected and unprotected) double-trace operators of the form $[\mathcal{O}_p\partial^\ell\Box^n\mathcal{O}_q]$ or mixtures thereof. The leading large N contribution to three-point functions of the form $\langle\mathcal{O}_p\mathcal{O}_q[\mathcal{O}_{p'}\partial^\ell\Box^n\mathcal{O}_{q'}]\rangle \sim O(N^{p+q})$ arises when $p=p'$ and $q=q'$, which is when the three-point function factorises into a product of two-point functions. The twist τ of the double-trace operator therefore must obey $\tau\geq p+q$, otherwise the three-point function will be suppressed by $1/N^2$. The exchanged operators surviving the light-like limit (3.36) all have twists less than both p_1+p_2 and p_3+p_4 and hence their contributions will be suppressed by at least $1/N^4$ and will not contribute to the LHS of (3.36). Higher multi-trace operators are even more suppressed and we conclude that no operators in the supergravity spectrum can contribute in the light-like limit, justifying the vanishing of the light-like limit (3.36).

3.4.5 A Hidden Ten-Dimensional Conformal Symmetry

Lastly, let us point out an unexpected feature of the tree-level supergravity correlators $\mathcal{H}^{(1,0)}_{\vec{p}}$. In [112], these correlators were found to exhibit a hidden conformal symmetry in 10 dimensions.[11] One of the main consequences of this hidden symmetry is the existence

[11] Subsequently, a similar hidden conformal symmetry has been observed for tree-level supergravity on $AdS_3\times S^3$, in which case the symmetry is of six-dimensional origin, see references [113, 114].

of a ten-dimensional generating functional, from which one can obtain all four-point functions of arbitrary external charges by the action of certain differential operators on the stress-tensor correlator:

$$\mathcal{H}^{(1,0)}_{\vec{p}} = \widehat{\mathcal{D}}_{\vec{p}} \cdot u^2\, \mathcal{H}^{(1,0)}_{2222}\,, \tag{3.42}$$

As such, the stress-tensor correlator $\mathcal{H}^{(1,0)}_{2222} \propto u^2\, \overline{D}_{2422}(u,v)$ acts as a seed-correlator for the general correlator $\mathcal{H}^{(1,0)}_{\vec{p}}$. In Mellin space, the differential operators $\widehat{\mathcal{D}}_{\vec{p}}$ act as shift-operators on the Mellin variables. Importantly, in [112] it has been shown how to re-derive the Mellin amplitudes $\mathcal{M}^{(1,0)}_{\vec{p}}$ as given in (3.33), thus proving consistency of the above (3.42) with the Mellin space results of [47, 48].

For concreteness, let us give a couple of examples of the differential operators $\widehat{\mathcal{D}}_{\vec{p}}$ for low external charges. We have

$$\begin{aligned}
\widehat{\mathcal{D}}_{2233} &= \tfrac{9}{4}(4-u\partial_u),\\
\widehat{\mathcal{D}}_{2323} &= \tfrac{9}{4}(u\partial_u+v\partial_v),\\
\widehat{\mathcal{D}}_{2244} &= 2\,(5-u\partial_u)(4-u\partial_u),\\
\widehat{\mathcal{D}}_{2424} &= 2\,(1+u\partial_u+v\partial_v)(u\partial_u+v\partial_v),\\
\widehat{\mathcal{D}}_{2334} &= \tfrac{9}{2}(4-u\partial_u)(u\partial_u+v\partial_v),\\
\widehat{\mathcal{D}}_{3324} &= \tfrac{9}{2}v\partial_v(-u\partial_u-v\partial_v),\\
\widehat{\mathcal{D}}_{3335} &= \tfrac{135}{16}(4-u\partial_u)v\partial_v(-u\partial_u-v\partial_v),\\
\widehat{\mathcal{D}}_{4424} &= 8\,(4-u\partial_u)v\partial_v(-u\partial_u-v\partial_v),\\
\widehat{\mathcal{D}}_{3333} &= \tfrac{81}{16}\left[(4-u\partial_u)^2+\frac{u\tau}{v}(v\partial_v)^2+u\sigma(u\partial_u+v\partial_v)^2\right],\\
\widehat{\mathcal{D}}_{4444} &= 4\left[(5-u\partial_u)^2+4\frac{u\tau}{v}(v\partial_v)^2+4u\sigma(u\partial_u+v\partial_v)^2\right](4-u\partial_u)^2,\\
&\quad+4\left[\frac{(u\tau)^2}{v^2}(1-v\partial_v)^2(v\partial_v)^2+(u\sigma)^2(u\partial_u+v\partial_v)^2(1+u\partial_u+v\partial_v)^2\right]\\
&\quad+16\,\frac{u\tau}{v}(u\sigma)(v\partial_v)^2(v\partial_v+u\partial_u)^2,
\end{aligned} \tag{3.43}$$

where we note that $\widehat{\mathcal{D}}_{3335}$ and $\widehat{\mathcal{D}}_{4424}$ are proportional to each other. This (accidental) degeneracy is a non-trivial consequence of the hidden ten-dimensional symmetry, and it will be lifted by the one-loop corrections as discussed in Chapter 6.

Part II

The Double-Trace Spectrum at Strong Coupling

Chapter 4

Supergravity Anomalous Dimensions

The result for supergravity correlators of arbitrary external charges presented above opens up the systematic study of the leading anomalous dimensions of exchanged double-trace operators. In this chapter, we will focus solely on the supergravity correction which is of leading order in the $1/\lambda$ expansion. Further string corrections to the spectrum will be addressed in the next chapter.

Here, we will finally put to use all of the technical machinery introduced in the two introductory chapters. By combining the superconformal block decomposition of the long part of four-point functions with the strong coupling expansion and the knowledge of the supergravity spectrum, we arrive at a set of 'unmixing equations' whose solution yields both the supergravity anomalous dimensions $\eta^{(1,0)}$ and the leading order three-point couplings of two single-particle operators with an unprotected double-trace operator. After having introduced the unmixing equations, we start by first discussing in detail the singlet channel results, which we then generalise to the series of channels of the form $[n, 0, n]$. In Section 4.4, we present the general formula for all $su(4)$ representations, which is the main result of this chapter. The general formula turns out to be of a remarkably simple structure and exhibits an interesting pattern of residual degeneracies.

4.1 The Unmixing Equations

Let us recall the SCPW decomposition introduced in Section 2.4, and in particular the contribution of long multiplets

$$\langle p_1 p_2 p_3 p_4 \rangle_{\text{long}} = N^{\Sigma} \sum_{\{\tau, \ell, \mathfrak{R}\}} L_{\vec{p},\vec{\tau}} \, \mathbb{L}_{\vec{p},\vec{\tau}}, \tag{4.1}$$

where the external operators are the single-particle operators $\mathcal{O}_p$ normalised as in (3.9), and $\Sigma = (p_1 + p_2 + p_3 + p_4)/2$ such that the disconnected free theory part (when it exists) is of order 1 in the large N expansion. Due to operator mixing, the long SCPW coefficients $L_{\vec{p},\vec{\tau}}$ are not in one-to-one correspondence with the CFT data. Instead, they are given as a sum over the exchanged operators, i.e.

$$L_{\vec{p},\vec{\tau}} = \sum_{\mathcal{O}\in\mathcal{K}} C_{p_1p_2\mathcal{O}}\, C_{p_3p_4\mathcal{O}}. \tag{4.2}$$

Expanding both the dimensions and OPE coefficients up to first order in $1/N^2$ (and disregarding any $1/\lambda$ corrections for now), we have

$$\Delta_{\mathcal{O}} = \Delta^{(0)} + 2\mathfrak{a}\,\eta_{\mathcal{O}} + \ldots, \quad C_{p_1p_2\mathcal{O}} = C^{(0)}_{p_1p_2\mathcal{O}} + \mathfrak{a}\, C^{(1)}_{p_1p_2\mathcal{O}} + \ldots, \tag{4.3}$$

where we simplified the notation of the supergravity anomalous dimensions $\eta^{(1,0)}$ to simply η. Upon substituting the above expansions into the partial wave decomposition (4.1), we obtain the following refinement

$$\begin{aligned} \langle p_1p_2p_3p_4\rangle_{\text{long}} = N^{\Sigma}\Bigg(&\sum_{\vec{\tau}_0} \mathcal{A}_{\vec{p},\vec{\tau}_0}\ \mathbb{L}_{\vec{p},\vec{\tau}_0} \\ &+ \mathfrak{a}\log(u) \sum_{\vec{\tau}_0} \mathcal{M}_{\vec{p},\vec{\tau}_0}\ \mathbb{L}_{\vec{p},\vec{\tau}_0} + \ldots \Bigg), \end{aligned} \tag{4.4}$$

where at order $\mathfrak{a}$ we omitted analytic terms in u which will not be relevant four our discussion. By $\vec{\tau}_0$ we collectively denote the quantum numbers $\vec{\tau}_0 \equiv (\tau_0, \ell, [a, b, a])$ of the exchanged double-trace operators, with τ_0 being their classical (integer valued) twist $\tau_0 = \Delta^{(0)} - \ell$. In the above, we have used the definitions

$$\mathcal{A}_{\vec{p},\vec{\tau}_0} = \sum_{\mathcal{O}\in\mathcal{K}} C^{(0)}_{p_1p_2\mathcal{O}}\, C^{(0)}_{p_3p_4\mathcal{O}}, \tag{4.5}$$

$$\mathcal{M}_{\vec{p},\vec{\tau}_0} = \sum_{\mathcal{O}\in\mathcal{K}} C^{(0)}_{p_1p_2\mathcal{O}}\, \eta_{\mathcal{O}}\, C^{(0)}_{p_3p_4\mathcal{O}}, \tag{4.6}$$

which constitute the set of 'unmixing equations' which we will analyse in detail in the following. The data on the LHS of these equations will be obtained from the explicit form of the correlators. In particular, disconnected free theory determines $\mathcal{A}_{\vec{p},\vec{\tau}_0}$, whereas $\mathcal{M}_{\vec{p},\vec{\tau}_0}$ is obtained from the leading $\log(u)$ singularity of the supergravity correlator $\mathcal{H}^{(1,0)}_{\vec{p}}$, which we obtain from $\mathcal{M}^{(1,0)}_{\vec{p}}$ by inverting the Mellin transform.

As discussed before, we expect the set of double-trace operators (denoted by $\mathcal{K}$ in the above sums) to be the only long operators $\mathcal{O}$ to have non-vanishing leading order three-point functions $C^{(0)}_{p_1p_2\mathcal{O}}$. By solving the above unmixing equations, we wish to obtain their anomalous dimensions $\eta_{\mathcal{O}}$ in all $su(4)$ channels $[a, b, a]$ as well as their leading order three-point functions $C^{(0)}_{p_1p_2\mathcal{O}}$. To illustrate the general computation, we will start by first considering the singlet channel representation.

4.2 Unmixing the Singlet Channel $[0,0,0]$

Specialising to the singlet representation, the set of exchanged singlet operators in question have the following schematic form

$$\mathcal{K}_{t,\ell,i}|_{[0,0,0]} = \{\mathcal{O}_2\Box^{t-2}\partial^\ell\mathcal{O}_2,\ \mathcal{O}_3\Box^{t-3}\partial^\ell\mathcal{O}_3,\ \dots\ ,\ \mathcal{O}_t\Box^0\partial^\ell\mathcal{O}_t\}, \tag{4.7}$$

where we label the different degenerate operators by $i = 1,\dots,t-1$, and in the singlet channel t is simply half the classical twist, i.e. $t = \tau_0/2$. First note that at leading order in the large N limit the OPE of $\mathcal{O}_p \times \mathcal{O}_p$ contains the operators $\mathcal{K}_{t,\ell,i}$ for all $t \geq p$. Thus for fixed t, the four-point correlators $\langle ppqq\rangle$ with $p \leq q$ contain information about operators $\mathcal{K}_{t,\ell,i}$ for all $q \leq t$. Noting the $p \leftrightarrow q$ symmetry we deduce that there are $t(t-1)/2$ such independent correlators. We can then organize the information $\mathcal{A}_{ppqq,\vec{\tau}_0}$ coming from each correlator in the free theory at leading order into the symmetric matrix

$$\widehat{\mathcal{A}}(t,\ell)|_{[0,0,0]} = \begin{pmatrix} \mathcal{A}_{2222} & \mathcal{A}_{2233} & \dots & \mathcal{A}_{22tt} \\ & \mathcal{A}_{3333} & \dots & \mathcal{A}_{33tt} \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{tttt} \end{pmatrix}. \tag{4.8}$$

In fact, from the form of the large N free theory correlators one can see immediately that the above matrix $\widehat{\mathcal{A}}$ is actually diagonal. Likewise, we can organise the information $\mathcal{M}_{ppqq,\vec{\tau}_0}$ coming from the $\log(u)$ term at order a of each correlator into another symmetric matrix

$$\widehat{\mathcal{M}}(t,\ell)|_{[0,0,0]} = \begin{pmatrix} \mathcal{M}_{2222} & \mathcal{M}_{2233} & \dots & \mathcal{M}_{22tt} \\ & \mathcal{M}_{3333} & \dots & \mathcal{M}_{33tt} \\ & & \ddots & \vdots \\ & & & \mathcal{M}_{tttt} \end{pmatrix}, \tag{4.9}$$

where both in $\widehat{\mathcal{A}}(t,\ell)$ and $\widehat{\mathcal{M}}(t,\ell)$ we have just given the independent entries in the upper triangular part explicitly.

Consider now the $t-1$ independent operators $\mathcal{K}_{t,l,i}$. They are associated with $(t-1)^2$ three-point functions $C^{(0)}_{pp\mathcal{K}_{t,\ell,i}}$, with $i = 1,\dots,t-1$ and $p = 2,\dots,t$, and $t-1$ anomalous dimensions $\eta_{t,\ell,i}$. This results in a total of $t(t-1)$ unknowns that need to be determined. Thus the matrices (4.8) and (4.9) contain the precise amount of data needed! The reason for this exact matching of degrees of freedom is that the operators $\mathcal{K}_{t,\ell,i}$ are in one-to-one correspondence with bilinears of single-particle operators. The matching is thus a remarkable feature of the strong coupling regime of large 't Hooft coupling and large N, as in general there will be many other types of operators contributing.

Let us now examine the unmixing equations (4.5)-(4.6) in detail, beginning with low twist cases. To simplify notation a little, we redefine $C^{(0)}_{pp\mathcal{K}_{t,\ell,i}}$ in favour of c_{pi} by taking out a universal factor which we find is always present,

$$\left(C^{(0)}_{pp\mathcal{K}_{t,\ell,i}}\right)^2 = \frac{(\ell+t+1)!^2}{(2\ell+2t+2)!}\, c_{pi}^2, \tag{4.10}$$

and note that at fixed twist we expect c_{pi} to depend non-trivially on the spin ℓ.

4.2.1 Twist 4 (t=2)

At twist 4, there is only one operator contributing and it only appears in the simplest correlator $\langle 2222 \rangle$. Extracting the relevant superblock coefficients

$$\begin{aligned} (C_{22\mathcal{K}_{2,\ell,1}})^2 &= \mathcal{A}_{2222} \quad \Rightarrow \quad & c_{21}^2 &= \frac{4}{3}(\ell+1)(\ell+6), \\ \eta_1 (C_{22\mathcal{K}_{2,\ell,1}})^2 &= \mathcal{M}_{2222} \quad \Rightarrow \quad & c_{21}^2\eta_1 &= -64. \end{aligned} \tag{4.11}$$

This clearly yields

$$\eta_1 = -\frac{48}{(\ell+1)(\ell+6)}, \qquad c_{21} = \sqrt{\frac{4(\ell+1)(\ell+6)}{3}}, \tag{4.12}$$

which has been known for a long time [63]. Note the symmetry under $\ell \to -7-\ell$ of this result.

4.2.2 Twist 6 (t=3)

The situation becomes more interesting when we move to twist 6. Here there are two operators contributing, $\mathcal{K}_{3,\ell,1}$ and $\mathcal{K}_{3,\ell,2}$. The disconnected free theory results give:

$$\begin{aligned} c_{21}^2 + c_{22}^2 &= \frac{2}{5}(\ell+1)(\ell+8), \\ c_{31}^2 + c_{32}^2 &= \frac{9}{40}(\ell+1)(\ell+2)(\ell+7)(\ell+8), \\ c_{21}c_{31} + c_{22}c_{32} &= 0. \end{aligned} \tag{4.13}$$

It is interesting at this point to compare this briefly with the free theory at large N. The relevant disconnected free theory correlator is *exactly the same* as the one we are discussing here at strong coupling. However, despite this one should not be tempted to assume the leading large N three-point functions are also the same at strong and weak coupling. Recall that in the free theory at large N the two operators are explicitly given as $\mathcal{K}_{3,\ell,1} = \mathcal{O}_2\Box\partial^\ell\mathcal{O}_2 + \ldots$ and $\mathcal{K}_{3,\ell,2} = \mathcal{O}_3\partial^\ell\mathcal{O}_3 + \ldots$. Although in general other operators contribute at weak coupling (single-trace etc.), at large N only these two contribute (the OPE can be easily performed explicitly via Wick contractions to verify this). Furthermore, the weak coupling three-point functions c_{22}^{weak} and c_{31}^{weak} are suppressed at this order and thus the solution of the above equations simply reads

$$\begin{aligned}(c_{21}^{\text{weak}})^2 &= \frac{2}{5}(\ell+1)(\ell+8),\\ c_{22}^{\text{weak}} &= c_{31}^{\text{weak}} = 0,\\ (c_{32}^{\text{weak}})^2 &= \frac{9}{40}(\ell+1)(\ell+2)(\ell+7)(\ell+8),\end{aligned} \tag{4.14}$$

and the three-point functions c_{pi}^{weak} are diagonal.

The strong coupling interpretation of the disconnected free theory equations (4.13) turns out to be very different however, even though it arises from the same free theory correlators. The dynamical parts of the correlators give

$$\begin{aligned}c_{21}^2\eta_1 + c_{22}^2\eta_2 &= -96,\\ c_{31}^2\eta_1 + c_{32}^2\eta_2 &= -54(\ell^2+9\ell+44),\\ c_{21}c_{31}\eta_1 + c_{22}c_{32}\eta_2 &= 432,\end{aligned} \tag{4.15}$$

and in particular the last equation means that here the three-point functions c_{pi} cannot be diagonal. Instead, we can straightforwardly solve the above equations and obtain the solution

$$\begin{aligned}\eta_1 &= -\frac{240}{(\ell+1)(\ell+2)}, \quad \eta_2 = -\frac{240}{(\ell+7)(\ell+8)},\\ c_{21} &= -\sqrt{\frac{2(\ell+1)(\ell+2)(\ell+8)}{5(2\ell+9)}}, \quad c_{22} = -\sqrt{\frac{2(\ell+1)(\ell+7)(\ell+8)}{5(2\ell+9)}},\\ c_{31} &= \sqrt{\frac{9(\ell+1)(\ell+2)(\ell+7)^2(\ell+8)}{40(2\ell+9)}}, \quad c_{32} = -\sqrt{\frac{9(\ell+1)(\ell+2)^2(\ell+7)(\ell+8)}{40(2\ell+9)}}.\end{aligned} \tag{4.16}$$

4.2.3 Generalisation to All Twists

The first task in attempting to understand the general structure is to generalise the equations we obtain from the correlators via the superconformal block expansion. At leading order the situation is simpler, since off-diagonal correlators $\langle ppqq\rangle$ with $p \neq q$ are suppressed and therefore the matrix $\widehat{\mathcal{A}}(t,\ell)$ is diagonal. We have computed a number of explicit examples and spotted the pattern that leads to the general formula

$$\mathcal{A}_{pppp}|_{[0,0,0]} = \frac{24(\ell+1)(t-2)!(t!)^2(\ell+2t+2)(\ell+t-1)!((\ell+t+1)!)^2(p+t)!(\ell+p+t+1)!}{(p+1)(p-2)!((p-1)!)^3(2t)!(t+2)!(\ell+t+3)!(2\ell+2t+2)!(t-p)!(\ell-p+t+1)!}, \tag{4.17}$$

and let us notice that $\mathcal{A}_{pppp}$ has a completely factorized form.[1] For a fixed twist t, we define the matrix of leading order three-point function coefficients $\mathbb{C}$ by

$$\mathbb{C}(t,\ell)|_{[0,0,0]} = \begin{pmatrix} C_{22\mathcal{K}_{t,\ell,1}} & C_{22\mathcal{K}_{t,\ell,2}} & \cdots & C_{22\mathcal{K}_{t,\ell,t-1}} \\ C_{33\mathcal{K}_{t,\ell,1}} & C_{33\mathcal{K}_{t,\ell,2}} & \cdots & C_{33\mathcal{K}_{t,\ell,t-1}} \\ \vdots & \vdots & \ddots & \vdots \\ C_{tt\mathcal{K}_{t,\ell,1}} & C_{tt\mathcal{K}_{t,\ell,2}} & \cdots & C_{tt\mathcal{K}_{t,\ell,t-1}} \end{pmatrix}, \tag{4.18}$$

and rewrite the first unmixing equation (4.5) in matrix form,

$$\widetilde{\mathbb{C}} \cdot \widetilde{\mathbb{C}}^T = \mathbb{1}, \qquad \text{where} \quad \widetilde{\mathbb{C}}(t,\ell) \equiv \widehat{\mathcal{A}}^{-\frac{1}{2}} \cdot \mathbb{C}, \tag{4.19}$$

where the orthonormality property of the matrix $\widetilde{\mathbb{C}}(t,\ell)$ is manifest. The second unmixing equation (4.6) then becomes

$$\widetilde{\mathbb{C}} \cdot \mathrm{diag}\,(\eta_1, \ldots, \eta_{t-1}) \cdot \widetilde{\mathbb{C}}^T = \widehat{\mathcal{A}}^{-\frac{1}{2}} \cdot \widehat{\mathcal{M}}(t,\ell) \cdot \widehat{\mathcal{A}}^{-\frac{1}{2}}. \tag{4.20}$$

The columns of $\widetilde{\mathbb{C}}(t,\ell)$ are then eigenvectors of the matrix $\widehat{\mathcal{A}}^{-\frac{1}{2}} \cdot \widehat{\mathcal{M}}(t,\ell) \cdot \widehat{\mathcal{A}}^{-\frac{1}{2}}$ and the anomalous dimensions are the corresponding eigenvalues. Notice from the structure of equation (4.20) (recalling that $\widehat{\mathcal{A}}$ is diagonal) the remarkable property that $\det(\widehat{\mathcal{M}})$ will factorise. From the explicit expressions for $\mathcal{M}_{ppqq}$ obtained upon decomposing the $\log(u)$ part of the supergravity correlators $\mathcal{H}^{(1,0)}_{ppqq}$ into superblocks this property is completely obscure. In particular, $\mathcal{M}_{ppqq}$ is found to be proportional to a polynomial in ℓ of degree $2(p-2)$, with $p \leq q$, which does not admit real roots. Their expressions are cumbersome and thus we will not display them explicitly.

Let us rewrite the solutions at twist four and six from equations (4.11) and (4.16) in this new notation. In these two cases, the $\widetilde{\mathbb{C}}$ matrices read

$$\widetilde{\mathbb{C}}(2,\ell) = \big(1\big), \qquad \widetilde{\mathbb{C}}(3,\ell) = \begin{pmatrix} \sqrt{\frac{\ell+2}{2\ell+9}} & \sqrt{\frac{\ell+7}{2\ell+9}} \\ -\sqrt{\frac{\ell+7}{2\ell+9}} & \sqrt{\frac{\ell+2}{2\ell+9}} \end{pmatrix}, \tag{4.21}$$

and it can easily be verified that $\widetilde{\mathbb{C}}(3,\ell) \cdot \widetilde{\mathbb{C}}(3,\ell)^T = \mathbb{1}$. For later convenience, we also repeat the formulae for the anomalous dimensions

$$\eta_{2,\ell,i} = \left\{ -\frac{48}{(\ell+1)(\ell+6)} \right\}, \quad \eta_{3,\ell,i} = \left\{ -\frac{240}{(\ell+1)(\ell+2)}, -\frac{240}{(\ell+7)(\ell+8)} \right\}. \tag{4.22}$$

We now proceed by performing the superconformal block decomposition to find $\widehat{\mathcal{M}}(t,\ell)$ up to twists $t \leq 12$, and solve for the anomalous dimensions $\eta_{t,\ell,i}$ and $\widetilde{\mathbb{C}}(t,\ell)$. From the

[1] In more detail, we first computed the cases with $p = t$ up to 6 and spotted a pattern for these which we then confirmed at $p = 7$. Next we considered cases for fixed p with general t, some of which were already known [63, 92]. We spotted a pattern for these up to a numerical p-dependent coefficient using results up to $p = 5$. This final numerical factor we can then fix as a function of p uniquely by comparison with the $p = t$ case.

solution at twist 8 we obtain

$$\widetilde{\mathbb{C}}(4,\ell) = \begin{pmatrix} \sqrt{\frac{7(\ell+2)(\ell+3)}{6(2\ell+9)(2\ell+11)}} & \sqrt{\frac{5(\ell+3)(\ell+8)}{3(2\ell+9)(2\ell+13)}} & \sqrt{\frac{7(\ell+8)(\ell+9)}{6(2\ell+11)(2\ell+13)}} \\ -\sqrt{\frac{2(\ell+2)(\ell+8)}{(2\ell+9)(2\ell+11)}} & -\sqrt{\frac{35}{(2\ell+9)(2\ell+13)}} & \sqrt{\frac{2(\ell+3)(\ell+9)}{(2\ell+11)(2\ell+13)}} \\ \sqrt{\frac{5(\ell+8)(\ell+9)}{6(2\ell+9)(2\ell+11)}} & -\sqrt{\frac{7(\ell+2)(\ell+9)}{3(2\ell+9)(2\ell+13)}} & \sqrt{\frac{5(\ell+2)(\ell+3)}{6(2\ell+11)(2\ell+13)}} \end{pmatrix}, \tag{4.23}$$

and

$$\eta_{4,\ell,i} = \left\{ -\frac{720(\ell+7)}{(\ell+1)(\ell+2)(\ell+3)}, -\frac{720}{(\ell+3)(\ell+8)}, -\frac{720(\ell+4)}{(\ell+8)(\ell+9)(\ell+10)} \right\}. \tag{4.24}$$

For higher twists the solutions become quite lengthy so we find it helpful to introduce a more compact notation for the square root factors, and we define

$$(n) = \sqrt{\ell+n}\,, \qquad [n] = \sqrt{2\ell+n}\,. \tag{4.25}$$

With this more compact notation the solution at twist 10 takes the form,

$$\widetilde{\mathbb{C}}(5,\ell) = \begin{pmatrix} \sqrt{\frac{3}{2}}\frac{(2)(3)(4)}{[9][11][13]} & \sqrt{\frac{5}{2}}\frac{(3)(4)(9)}{[9][13][15]} & \sqrt{\frac{5}{2}}\frac{(4)(9)(10)}{[11][13][17]} & \sqrt{\frac{3}{2}}\frac{(9)(10)(11)}{[13][15][17]} \\ -\sqrt{\frac{27}{8}}\frac{(2)(3)(9)}{[9][11][13]} & -\sqrt{\frac{5}{8}}\frac{(\ell+18)(3)}{[9][13][15]} & \sqrt{\frac{5}{8}}\frac{(\ell-5)(10)}{[11][13][17]} & \sqrt{\frac{27}{8}}\frac{(4)(10)(11)}{[13][15][17]} \\ \sqrt{\frac{5}{2}}\frac{(2)(9)(10)}{[9][11][13]} & -\sqrt{\frac{3}{2}}\frac{(\ell-3)(10)}{[9][13][15]} & -\sqrt{\frac{3}{2}}\frac{(\ell+16)(3)}{[11][13][17]} & \sqrt{\frac{5}{2}}\frac{(3)(4)(11)}{[13][15][17]} \\ -\sqrt{\frac{5}{8}}\frac{(9)(10)(11)}{[9][11][13]} & \sqrt{\frac{27}{8}}\frac{(2)(10)(11)}{[9][13][15]} & -\sqrt{\frac{27}{8}}\frac{(2)(3)(11)}{[11][13][17]} & \sqrt{\frac{5}{8}}\frac{(2)(3)(4)}{[13][15][17]} \end{pmatrix} \tag{4.26}$$

with

$$\begin{aligned} \eta_{5,\ell,i} = \Big\{ &-\frac{1680(\ell+7)(\ell+8)}{(\ell+1)(\ell+2)(\ell+3)(\ell+4)}, -\frac{1680}{(\ell+3)(\ell+4)}, \\ &-\frac{1680}{(\ell+9)(\ell+10)}, -\frac{1680(\ell+5)(\ell+6)}{(\ell+9)(\ell+10)(\ell+11)(\ell+12)} \Big\}. \end{aligned} \tag{4.27}$$

We begin to see an intriguing structure in the entries of the matrix as well as in the anomalous dimensions. Note the symmetry $\ell \to -2t-\ell-3$, which is an invariance of the set of anomalous dimensions and an invariance up to signs of the $\widetilde{\mathbb{C}}$ matrix under a flip about the vertical axis. Note also that at twist 10 we see for the first time the appearance of polynomials in ℓ (without a square root) in the numerators of the central entries of (4.26). At twist 10 these polynomials are all linear, but their degrees increase as we increase the twist further.

Indeed, proceeding to compute the next few examples one gets a better idea of the structure. The anomalous dimensions reveal a fairly simple structure that is consistent with the formula

$$\eta_{t,\ell,i}|_{[0,0,0]} = -\frac{2(t-1)_4(t+\ell)_4}{(\ell+2i-1)_6}, \tag{4.28}$$

where $(x)_n = x(x+1)\dots(x+n-1)$ is the familiar Pochhammer symbol (rising factorial). Note that the anomalous dimensions are all negative for all physical values of spin ℓ.

On the other hand, the general form of the $\widetilde{\mathbb{C}}(t,\ell)$ matrices is trickier to understand. Already from the results up to twist 10 we recognise a pattern of square roots of linear factors of ℓ. In addition, we have seen that in the entries towards the centre one finds fewer square root factors in the numerator, and polynomials in ℓ without a square root. Note that the entries of the matrix always have a finite (but possibly vanishing) limit as $\ell \to \infty$. In fact, we can deduce the structure of $\widetilde{\mathbb{C}}(t,\ell)$ for a given twist in terms of an ansatz with some undetermined free parameters,

$$\begin{aligned}\tilde{c}_{pi} = &\sqrt{\frac{2^{1-t}(2\ell+4i+3)\left((\ell+i+1)_{t-i-p+1}\right)^{\sigma_1}\left((t+\ell+p+2)_{i-p+1}\right)^{\sigma_2}}{\left(\ell+i+\frac{5}{2}\right)_{t-1}}} \\ &\times \sum_{k=0}^{\min\{i-1,p-2,t-i-1,t-p\}} \ell^k\, a(p,i,k).\end{aligned} \tag{4.29}$$

The powers of the Pochhammer factors inside the square root are signs given explicitly by

$$\sigma_1 = \mathrm{sgn}(t-p-i+1)\,, \qquad \sigma_2 = \mathrm{sgn}(i-p+1)\,. \tag{4.30}$$

where $p = 2,\dots,t$ and $i = 1,\dots,t-1$. We notice that the square root structure in $\tilde{c}_{pi}$ follows from complicated combinatorics, which nevertheless can be captured by the two (non-analytic) sign functions σ_1 and σ_2. Around the outer frame of the matrix, the unfixed polynomial has degree 0, i.e. it is simply a constant. Its degree increases as we move towards the inside of the matrix. One can readily check (4.29) is consistent with the explicit examples given above and we have tested the structure up to $t = 12$.

Given the ansatz (4.29), we have reduced the problem to that of finding the constants $a(p,i,k)$. Quite surprisingly, enforcing orthonormality of $\widetilde{\mathbb{C}}(t,\ell)$ uniquely fixes the solution.[2] In more detail, we first insist that the first row has unit norm, $\sum_i \tilde{c}_{2i}^2 = 1$, which is a linear equation in $a(2,i,0)^2$ with a unique solution. In fact, the constraint is a rational function of ℓ and so this single equation can fix more than one constant. Then, orthogonality of the rows $\sum \tilde{c}_{pi}\tilde{c}_{qi} = 0$ for $p \neq q$ gives a linear system in the remaining variables and uniquely fixes them, up to an overall scale which is fixed by the unit norm condition.

We find it remarkable both that there exist such orthonormal matrices with the structure (4.29) and that the matrix is uniquely fixed by orthonormality as a linear system. The fact that the problem is essentially linear means it can be solved quickly and we have complete data up to $t = 24$. This enables us to spot patterns and write down general formulae.

[2]We have checked this up to twist 48 ($t = 24$).

We do not have a completely general formula for the full matrix $\widetilde{\mathbb{C}}$ but we do have various cases in closed form. In particular, the top row of the matrix is given by the formula[3]

$$a(2,i,0) = \frac{2^{t-1}(2i+2)!(t-2)!(2t-2i+2)!}{3(i-1)!(i+1)!(t+2)!(t-i-1)!(t-i+1)!}, \quad i = 1, \ldots, t-1. \tag{4.31}$$

4.3 A First Generalisation: from $[0,0,0]$ to $[n,0,n]$

Before giving the solution for general $su(4)$ channels $[a,b,a]$, let us discuss a first simple generalisation of the above singlet channel results. Specifically, we can investigate operators in the series of representations $[n,0,n]$ which also arise in the OPE of correlation functions of the form $\langle ppqq\rangle$. For each channel of the form $[n,0,n]$ the structure of this problem is analogous to that of the singlet channel. In particular, at twist $2t$ a basis of double-trace operators in the $[n,0,n]$ representation will have the schematic form

$$\mathcal{K}_{t,\ell,i}|_{[n,0,n]} = \left\{\mathcal{O}_{2+n}\Box^{t-n-2}\partial^\ell\mathcal{O}_{2+n},\ \mathcal{O}_{3+n}\Box^{t-n-3}\partial^\ell\mathcal{O}_{3+n}, \ldots,\ \mathcal{O}_t\Box^0\partial^\ell\mathcal{O}_t\right\}, \tag{4.32}$$

with $t-n-1$ degenerate operators labelled by i, where t is again half the twist.

The analysis of the $[n,0,n]$ channel for fixed n follows a very similar logic to that presented in the singlet case. Once again we conclude that the series of correlators $\langle ppqq\rangle$ for $n+2 \le p \le q \le t$ provides precisely the right amount of information needed in order to solve for the anomalous dimensions and three-point functions of the exchanged double-trace operators. From the general form of the long superconformal blocks (2.33), it is straightforward to isolate the appropriate channel and organise the data from the SCPW expansion into the symmetric matrices $\widehat{\mathcal{A}}(t,\ell)|_{[n,0,n]}$ and $\widehat{\mathcal{M}}(t,\ell)|_{[n,0,n]}$. Let us go through some explicit examples in the $[1,0,1]$ channel before presenting results for general n.

4.3.1 Unmixing the $[1,0,1]$ Channel

In this channel, the matrices $\widehat{\mathcal{A}}(t,\ell)|_{[1,0,1]}$ and $\widehat{\mathcal{M}}(t,\ell)|_{[1,0,1]}$ take the form

$$\widehat{\mathcal{A}}(t,\ell)|_{[1,0,1]} = \begin{pmatrix} \mathcal{A}_{3333} & \mathcal{A}_{3344} & \cdots & \mathcal{A}_{33tt} \\ & \mathcal{A}_{4444} & \cdots & \mathcal{A}_{44tt} \\ & & \ddots & \vdots \\ & & & \mathcal{A}_{tttt} \end{pmatrix}, \tag{4.33}$$

[3]This formula, together with equations (4.10) and (4.17), completely specifies the leading order three-point functions of the form $C^{(0)}_{22\mathcal{K}_{t,\ell,i}}$, which is an essential ingredient in the prediction of the one-loop supergravity correction to the $\langle 2222\rangle$ correlator, as we will discuss in Chapter 6.

and

$$\widehat{\mathcal{M}}(t,\ell)|_{[1,0,1]} = \begin{pmatrix} \mathcal{M}_{3333} & \mathcal{M}_{3344} & \dots & \mathcal{M}_{33tt} \\ & \mathcal{M}_{4444} & \dots & \mathcal{M}_{44tt} \\ & & \ddots & \vdots \\ & & & \mathcal{M}_{tttt} \end{pmatrix}, \tag{4.34}$$

where $\widehat{\mathcal{A}}(t,\ell)$ is diagonal with entries

$$\mathcal{A}_{pppp}|_{[1,0,1]} = \frac{15(p-2)(t-1)(t+2)(\ell+t)(\ell+t+3)}{(p+2)(t-2)(t+3)(\ell+t-1)(t+\ell+4)}\,\mathcal{A}_{pppp}|_{[0,0,0]}, \tag{4.35}$$

with $\mathcal{A}_{pppp}|_{[0,0,0]}$ given in equation (4.17). Analogously to the singlet channel, we can then introduce the orthonormal matrix $\widetilde{\mathbb{C}}(t,\ell)$ and start solving the mixing problem twist by twist. For illustration, let us look at the first three cases:

- At twist 6 there is only one operator, giving

$$\widetilde{\mathbb{C}}(3,\ell) = \big(1\big), \quad \eta_{3,\ell,i} = \left\{-\frac{144}{(3+\ell)(6+\ell)}\right\}. \tag{4.36}$$

- At twist 8 there are two operators, and we find

$$\widetilde{\mathbb{C}}(4,\ell) = \begin{pmatrix} \sqrt{\frac{\ell+2}{2\ell+11}} & \sqrt{\frac{\ell+9}{2\ell+11}} \\ -\sqrt{\frac{\ell+9}{2\ell+11}} & \sqrt{\frac{\ell+2}{2\ell+11}} \end{pmatrix}, \tag{4.37}$$

  with anomalous dimensions

$$\eta_{4,\ell,i} = \left\{-\frac{560(\ell+8)}{(\ell+2)(\ell+4)(\ell+7)}, -\frac{560(\ell+3)}{(\ell+4)(\ell+7)(\ell+9)}\right\}. \tag{4.38}$$

- At twist 10, it becomes evident that the structure of eigenvectors and anomalous dimension found in the singlet case generalises to $[1,0,1]$ with minor modifications. In particular, we find

$$\widetilde{\mathbb{C}}(5,\ell) = \begin{pmatrix} \sqrt{\frac{9(\ell+2)(\ell+3)}{8(2\ell+11)(2\ell+13)}} & \sqrt{\frac{7(\ell+3)(\ell+10)}{4(2\ell+11)(2\ell+15)}} & \sqrt{\frac{9(\ell+10)(\ell+11)}{8(2\ell+13)(2\ell+15)}} \\ -\sqrt{\frac{2(\ell+2)(\ell+10)}{(2\ell+11)(2\ell+13)}} & -\frac{3\sqrt{7}}{\sqrt{(2\ell+11)(2\ell+15)}} & \sqrt{\frac{2(\ell+3)(\ell+11)}{(2\ell+13)(2\ell+15)}} \\ \sqrt{\frac{7(\ell+10)(\ell+11)}{8(2\ell+11)(2\ell+13)}} & -\sqrt{\frac{9(\ell+2)(\ell+11)}{4(2\ell+11)(2\ell+15)}} & \sqrt{\frac{7(\ell+2)(\ell+3)}{8(2\ell+13)(2\ell+15)}} \end{pmatrix}, \tag{4.39}$$

  while the anomalous dimensions are given by

$$\eta_{5,\ell,i} = \left\{-\frac{1440(\ell+9)}{(\ell+2)(\ell+3)(\ell+5)}, -\frac{1440}{(\ell+5)(\ell+8)}, -\frac{1440(\ell+4)}{(\ell+8)(\ell+10)(\ell+11)}\right\}. \tag{4.40}$$

The solutions to the mixing problem up to $t = 12$ can be found straightforwardly and lead to the expression

$$\eta_{t,\ell,i}|_{[1,0,1]} = \frac{(\ell+2i-1)(t-2)(t+3)(t+\ell-1)(t+\ell+4)}{(\ell+2i+5)(t-1)(t+2)(t+\ell)(t+\ell+3)}\ \eta_{t,\ell,i}|_{[0,0,0]} \tag{4.41}$$

for the anomalous dimensions, and

$$\begin{aligned}\tilde{c}_{pi} =& \sqrt{\frac{2^{1-t}(2\ell+4i+5)\left((\ell+i+1)_{t-i-p+1}\right)^{\sigma_1}\left((t+\ell+p+2)_{i-p+2}\right)^{\sigma_2}}{\left(\ell+i+\frac{7}{2}\right)_{t-2}}} \\ &\times \sum_{k=0}^{\min\{i-1,p-3,t-i-2,t-p\}} \ell^k\ a(p,i,k).\end{aligned} \tag{4.42}$$

for the entries of the $\widetilde{\mathbb{C}}(t,\ell)$ matrix, where $\sigma_1 = \mathrm{sgn}(t-p-i+1)$, $\sigma_2 = \mathrm{sgn}(i-p+2)$, with $p = 3,\dots,t$ and $i = 1,\dots,t-2$. The orthogonality condition of the matrix again determines completely the values of $a(p,i,k)$ at any twist.

4.3.2 From $[2,0,2]$ to $[n,0,n]$

After the detailed study of the $[1,0,1]$ channel, let us present the generalisations of the matrices $\widehat{\mathcal{A}}(t,\ell)$ and $\widehat{\mathcal{M}}(t,\ell)$, anomalous dimensions and matrices of three-point functions $\widetilde{\mathbb{C}}(t,\ell)$ to the $[n,0,n]$ channels.

We begin with the disconnected part of free theory, where we have obtained the result

$$\begin{aligned}&\mathcal{A}_{pppp}|_{[n,0,n]} = \\ &\quad \frac{p^2(t!)^2}{n!p!(p-1)!}\frac{(n+2)_{n+3}}{(p+1+n)!(p-2-n)!}\times\frac{(\ell+1)((1+\ell+t)!)^2(\ell+2t+2)}{(2t)!(2\ell+2t+2)!}\times \\ &\quad (\ell+t-p+2)_{p-2-n}(\ell+t+4+n)_{p-2-n}(\ell+1+t-n)_n(\ell+1+t+2)_n\times \\ &\quad (t-p+1)_{p-2-n}(t+3+n)_{p-2-n}(t-n)_n(t+2)_n.\end{aligned} \tag{4.43}$$

Introducing the $\widetilde{\mathbb{C}}(t,\ell)|_{[n,0,n]}$ matrices and computing $\mathcal{M}(t,\ell)|_{[n,0,n]}$ for a large number of twists and several values of n, we have been able to fit and test both the anomalous dimensions and the entries of $\widetilde{\mathbb{C}}(t,\ell)$ with the following formulae: for the anomalous dimensions we find

$$\eta_{t,\ell,i}|_{[n,0,n]} = -\frac{2(t)_2(t-n-1)(t+n+2)(t+\ell+1)_2(t+\ell-n)(t+\ell+n+3)}{(\ell+2i+n-1)_6}, \tag{4.44}$$

with the degeneracy label i running over $i = 1,\dots,t-n-1$. For the entries of the

$\widetilde{\mathbb{C}}(t,\ell)$ matrix we have

$$\tilde{c}_{pi} = \sqrt{\frac{2^{1-t}(2\ell+4i+3+2n\,((\ell+i+1)_{t-i-p+1})^{\sigma_1}\,((t+\ell+p+2)_{i-p+n+1})^{\sigma_2}}{\left(\ell+i+n+\frac{5}{2}\right)_{t-n-1}}} \times \sum_{k=0}^{\min\{i-1,p-n-2,t-n-i-1,t-p\}} \ell^k\; a(p,i,k), \tag{4.45}$$

where $\sigma_1 = \mathrm{sgn}(t-i-p+1)$ and $\sigma_2 = \mathrm{sgn}(i-p+n+1)$. All unspecified coefficients $a(p,i,k)$ can again be determined by imposing orthogonality of the $\widetilde{\mathbb{C}}$ matrix.

4.4 Generalisation to All $su(4)$ Channels

Let us finally describe how to determine the anomalous dimensions of the true double-trace eigenstates $\mathcal{K}_{pq}$ for general $su(4)$ representations $[a,b,a]$. Recall from the previous chapter that the set of exchanged operators $\mathcal{K}_{pq}$ is parametrised by pairs $(p,q) \in R_{\vec{\tau}}$, where

$$R_{\vec{\tau}} = \left\{ (p,q) : \begin{array}{c} p = i+a+1+r \\ q = i+a+1+b-r \end{array} \quad \text{for} \quad \begin{array}{c} i = 1,\ldots,t-1 \\ r = 0,\ldots,\mu-1 \end{array} \right\}, \tag{4.46}$$

and

$$t = \frac{(\tau-b)}{2} - a\,, \qquad \mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a+\ell \text{ even,} \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a+\ell \text{ odd.} \end{cases} \tag{4.47}$$

As in the previous cases, we will have to assemble matrices of correlators from which we extract the necessary data to resolve the mixing of the set of double-trace operators described above. We will therefore consider the correlators $\langle p_1p_2p_3p_4\rangle$ in which the pairs (p_1,p_2) and (p_3,p_4) are both drawn from the set $R_{\vec{\tau}}$, resulting in a symmetric $d\times d$ matrix of SCPW coefficients, with $d = \mu(t-1)$. From disconnected free field theory we obtain $\widehat{\mathcal{A}}(t,\ell)$, whereas $\widehat{\mathbb{M}}(t,\ell)$ follows from the $\log(u)$ part of the supergravity correlators $\mathcal{H}^{(1,0)}_{\vec{p}}$. These two matrices contain the CFT data for the operators $\mathcal{K}_{pq}$ as follows:

$$\begin{aligned} \widehat{\mathcal{A}}(t,\ell)|_{[a,b,a]} &= \mathbb{C}_{[a,b,a]}(t,\ell)\cdot\mathbb{C}^T_{[a,b,a]}(t,\ell), \\ \widehat{\mathbb{M}}(t,\ell)|_{[a,b,a]} &= \mathbb{C}_{[a,b,a]}(t,\ell)\cdot\widehat{\eta}\cdot\mathbb{C}^T_{[a,b,a]}(t,\ell), \end{aligned} \tag{4.48}$$

where we again used the matrix notation of the unmixing equations (4.5)-(4.6), with $\mathbb{C}_{[a,b,a]}$ being the matrix of leading order three-point functions and $\widehat{\eta} = \mathrm{diag}(\eta_{pq})$ the $d\times d$ diagonal matrix of anomalous dimensions of the operators $\mathcal{K}_{pq}$. As we have previously seen for the $[n,0,n]$ channels, the above equations define an eigenvalue problem for the anomalous dimensions $\widehat{\eta}$. A simple counting reveals again that the number of unknowns in $\widehat{\mathcal{A}}(t,\ell)$ and $\widehat{\mathbb{M}}(t,\ell)$ exactly equals the number of unknown three-point couplings $C_{p_i p_j \mathcal{K}_{pq}}$ and anomalous dimensions η_{pq}, and thus the eigenvalue problem (4.48) is well defined.

Let us comment on the structure of the matrices $\widehat{\mathcal{A}}(t,\ell)$ and $\widehat{\mathcal{M}}(t,\ell)$. The SCPW expansion of disconnected free theory has the following compact expression:

$$\widehat{\mathcal{A}}(t,\ell)|_{[a,b,a]} = \mathrm{diag}\big(\mathcal{F}_{1+a+i+r,b-2r,r,a,t+a+r}\big)_{\substack{1\le i\le t-1\\ 0\le r\le \mu-1}}\,, \tag{4.49}$$

where the function $\mathcal{F}$ is given by

$$\begin{aligned}\mathcal{F}_{p,h,m,a,s} = &\frac{p(p+h)(1+\delta_{h0})(1+a)(2m+2+h+a)(l+1)(l+2s+2+h)}{(p-1-m)!(p-2-m-a)!(p+m+h)!(p+m+h+1+a)!}\\ &\times\frac{(m+1+h)_{m+1}}{m!}\frac{(m+2+a+h)_{m+2+a}}{(m+1+a)!}\,\Pi_s\,\Pi_{l+s+1},\end{aligned} \tag{4.50}$$

with

$$\begin{aligned}\Pi_s = &\frac{((s+h)!)^2}{(2s+h)!}(s+1-m)_m(s+1+h)_m(s-m-a)_a\\ &\times(s+2+h+m)_a(s+1-p)_{p-2-m-a}(s+3+h+m+a)_{p-2-m-a}.\end{aligned} \tag{4.51}$$

The matrix elements of $\widehat{\mathcal{M}}(t,\ell)$ are of the form

$$\frac{\big(\ell+1+t+a+r+\frac{p_{43}-p_{21}}{2}\big)!\big(\ell+1+t+a+r+p_{43}\big)!}{\big(2(\ell+1+t+r+a)+p_{43}\big)!}\times P_n(\ell) \tag{4.52}$$

where $P_n(\ell)$ is a polynomial in ℓ of degree $n=\min\{p_1+p_2,p_3+p_4\}-(p_{43}-p_{21})-4$, and r labels the pairs (p_3,p_4). We determine this polynomial case by case, and solve the eigenvalue problem as outlined in the previous sections.

This leads us to the main result of this chapter: the general formula for all supergravity anomalous dimensions in all $su(4)$ channels, given by[4]

$$\eta^{(1,0)}_{pq}|_{[a,b,a]} = -\frac{2M^{(4)}_t M^{(4)}_{t+\ell+1}}{\Big(\ell+2(p-1)-a-\frac{1+(-1)^{a+\ell}}{2}\Big)_6}\,, \tag{4.53}$$

where the twist is parametrised by t, see equation (4.47), and $M^{(4)}_t$ is defined as

$$M^{(4)}_t = (t-1)(t+a)(t+a+b+1)(t+2a+b+2). \tag{4.54}$$

Note that this formula is consistent with the previously discussed cases, namely the singlet channel anomalous dimensions (4.28) and its generalisation to the $[n,0,n]$ channel given in (4.44).

[4]We have verified that our conjecture (4.53) holds systematically in the $su(4)$ channels $[a,b,a]$ with $0\le a\le 3$, $0\le b\le 6$ up to twist 24 for both even and odd spins. In particular, we have been able to perform non-trivial tests on the pattern of residual degeneracies which we describe below.

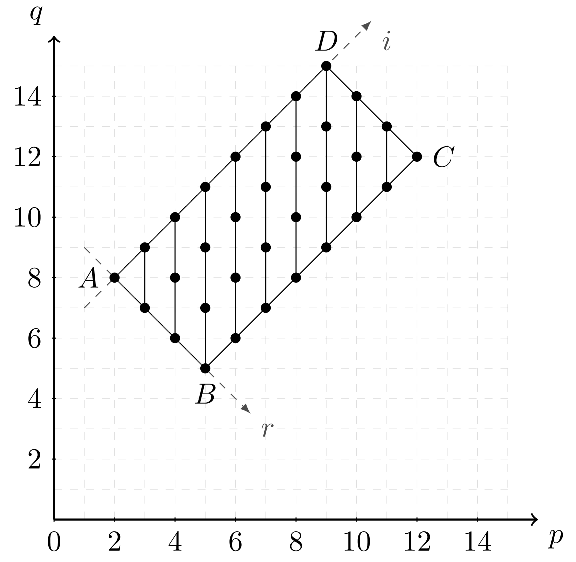


Figure 4.1: Degeneracy in the supergravity anomalous dimensions $\eta^{(1,0)}_{pq}$: dots connected by vertical lines in the (p,q) plane represent operators of common anomalous dimensions. In this example, we again depict the rectangle $R_{\vec{\tau}}$ with quantum numbers $\vec{\tau} = (24, \ell, [0, 6, 0])$.

A few comments are in order. We find it remarkable that the supergravity anomalous dimensions (4.53) take such a simple, fully factorised and rational form. This is not at all obvious from the form of the eigenvalue problem (4.48), which features an intricate spin dependence in the matrices $\widehat{\mathcal{M}}(t,\ell)$ with non-factorised polynomials in ℓ. Furthermore, note that there is an interesting residual degeneracy in the supergravity spectrum: since $\eta^{(1,0)}_{pq}$ is in fact independent of q, the anomalous dimensions are in general partially degenerate. This is the case when $\mu > 1$ and $t > 2$, and we display this property in Figure 4.1, where states which lie on the same vertical line have the same anomalous dimensions.[5] This residual degeneracy in the supergravity spectrum arises as a consequence of a surprising property of the tree-level supergravity correlators $\mathcal{H}^{(1,0)}_{\vec{p}}$, from which we extract the relevant data: the hidden ten-dimensional conformal symmetry predicts a degeneracy among some of the tree-level correlators, see the discussion below (3.43), which in turn prevents the spectrum from being fully unmixed at tree-level [112]. This means that although the eigenvalue problem (4.48) is well-defined, the leading order three-point functions $\mathbb{C}_{[a,b,a]}$ are not fully determined when there is a residual degeneracy.

Furthermore, we observe that the supergravity spectrum exhibits (at least) two interesting symmetries. Firstly, the anomalous dimensions (4.53) are left invariant under the

[5]The first instance of residual degeneracy occurs in the $[0,2,0]$ channel at twist 8 ($t = 3$). There are four operators in total, and the two operators labelled by $(p,q) = (3,3)$ and $(3,5)$ remain degenerate (corresponding to the states with labels $(i,r) = (1,1)$ and $(2,0)$, such that their sum $i + r = 2$ in both cases).

discrete shift

$$t \to -t - \ell - 2a - b - 2, \tag{4.55}$$

which exchanges the two factors in its numerator. As we will see later, we find this symmetry to be present also in the string corrected anomalous dimensions at orders $\lambda^{-\frac{3}{2}}$ and $\lambda^{-\frac{5}{2}}$, and we believe that this symmetry extends to all string corrections. It is not known whether this symmetry persists to higher orders in $1/N$, or whether it is broken by quantum corrections. It would be interesting to answer this question to one-loop order, $O(1/N^4)$, but due to the mixing with triple-trace operators we can only extract the one-loop anomalous dimensions for low twists (where there is no degeneracy) and we are thus not able to test this symmetry beyond tree-level.

Secondly, the supergravity spectrum exhibits a non-trivial $\mathbb{Z}_2$ symmetry. The statement of this so-called *reciprocity symmetry*[6] is that under the map

$$\ell \to -\ell - 2t - a - 3, \tag{4.56}$$

the anomalous dimensions of certain families of operators are mapped into each other. In the $[n, 0, n]$ channel for example, the anomalous dimensions are labelled by only one degeneracy label $i = 1, \ldots, t - n - 1$, and one can check that

$$\eta^{(1,0)}_{t,\ell,i}|_{[n,0,n]} \to \eta^{(1,0)}_{t,-\ell-2t-3,i}|_{[n,0,n]} = \eta^{(1,0)}_{t,\ell,j}|_{[n,0,n]}, \quad \text{where } j = t - i - n. \tag{4.57}$$

Under this symmetry, the family of operators with labels (t, i) maps to operators with $(t, j = t - i - n)$ and the symmetry simply reverses the list of operators at a given twist. This symmetry is believed to persist to all orders in $1/N$, and it manifests itself as an invariance of the full anomalous twist of these operators. Up to order $1/N^4$, we indeed observe that it is obeyed by the one-loop anomalous dimensions in a non-trivial manner, see Section 6.5. On the other hand, as we will discuss in detail in the next chapter, further string corrections to the double-trace spectrum truncate at finite values of the spin ℓ. This leads to formulae for the string corrected anomalous dimensions which are non-analytic in spin, and thus the symmetry (4.56) is broken by the $1/\lambda$ corrections.

[6] This symmetry was first noted in [115] and explored further in [116]. As argued in [116, 117], under certain assumptions about the analyticity of the spectrum, the reciprocity symmetry (4.56) is ultimately a consequence of conformal symmetry.

Chapter 5

Adding String Corrections to the Double-Trace Spectrum

String corrections to the supergravity result arise from higher derivative interaction terms in the $\text{AdS}_5 \times S^5$ effective action. At tree-level, the structure of these $1/\lambda$ corrections is related to the well-known Virasoro-Shapiro amplitude via the flat space limit, which we will review shortly. As mentioned earlier, these terms are most conveniently studied in their Mellin space representation. Focussing on tree-level terms, recall that the $1/\lambda$ expansion of the Mellin amplitude to order $\mathfrak{a}$ reads

$$\mathcal{M}_{\vec{p}} = \mathfrak{a}\left(\mathcal{M}^{(1,0)}_{\vec{p}} + \lambda^{-\frac{3}{2}}\mathcal{M}^{(1,3)}_{\vec{p}} + \lambda^{-\frac{5}{2}}\mathcal{M}^{(1,5)}_{\vec{p}} + \lambda^{-3}\mathcal{M}^{(1,6)}_{\vec{p}} + \ldots\right) + O(\mathfrak{a}^2), \tag{5.1}$$

where $\mathcal{M}^{(1,0)}_{\vec{p}}$ is the supergravity amplitude given previously in equation (3.33). In the following, we will consider the corrections to the double-trace spectrum due to the first two terms $\mathcal{M}^{(1,3)}_{\vec{p}}$ and $\mathcal{M}^{(1,5)}_{\vec{p}}$, which descend from dimensional reduction of the $\mathcal{R}^4$ and $\partial^4\mathcal{R}^4$ supervertices, respectively. Note that the analytic structure of tree-level Witten diagrams dictates that for a general term of the schematic form $\partial^{2n}\mathcal{R}^4$, the corresponding Mellin amplitude is simply a polynomial of degree n, together with all subleading polynomial amplitudes coming from terms in 10 dimensions which have legs on S^5 [77, 111, 118–120].[1] As such, the tower of string corrections gives rise to a spin truncated spectrum of exchanged states, which motivates us to introduce the notion of an effective 10d spin ℓ_{10}. As we will discuss in Section 5.3, the notion of ℓ_{10} will provide a useful method to constrain which double-trace states will receive a string correction to their anomalous dimensions.

[1]The tree-level corrections to the supergravity Mellin amplitude are polynomial since they correspond to corrections due to unprotected double-trace operators, whose poles are already correctly accounted for by the gamma functions $\Gamma_{\vec{p}}$ in the definition of the Mellin representation (3.28). The bound on the polynomial comes from considering the flat space limit and moreover the coefficients of the leading Mellin amplitudes can be fixed by comparing against the ten-dimensional type IIB closed superstring amplitude in flat space, as we will discuss in the next section.

Before introducing the unmixing equations and presenting their solution for the string corrected double-trace spectrum at orders $\lambda^{-\frac{3}{2}}$ and $\lambda^{-\frac{5}{2}}$, we will first take a digression and review the flat space limit for Mellin amplitudes. This will allow us to determine the order $\lambda^{-\frac{3}{2}}$ correlator for arbitrary external charges in a simple fashion.

5.1 Digression: the Flat Space Limit

Let us start by reviewing the general flat space limit formula for four-particle Mellin amplitudes. This relation between AdS Mellin amplitudes and flat space physics was motivated in [111] and explored further in [118]. This method was first applied to the $\langle 2222\rangle$ correlator in reference [119], and more recently extended to the $\langle 22pp\rangle$ family of correlators [100, 120]. Their discussion is based on previous work in $\mathrm{AdS}_7\times S^4$ [121, 122], whose logic we will follow here to extend the previous results to the general correlator $\langle p_1p_2p_3p_4\rangle$ with non-trivial (σ,τ) dependence. In four dimensions, the relation reads[2]

$$\lim_{L\to\infty} M(L^2s,L^2t)=\frac{L^{-1}}{\Gamma(\Sigma-2)}\int_0^\infty d\beta\beta^{\Sigma-3}e^{-\beta}\mathcal{A}_{\mathrm{Flat}}\Big(\frac{2\beta s}{L^2},\frac{2\beta t}{L^2}\Big), \tag{5.3}$$

where L is the radius of AdS, Σ is half the sum of external charges, $\Sigma=\frac{p_1+p_2+p_3+p_4}{2}$, and $\mathcal{A}_{\mathrm{Flat}}$ is the corresponding flat space amplitude, which we describe in detail below.

Here we will follow the logic of [122] and extend this formula to four-point functions with arbitrary Kaluza-Klein modes as external operators. Starting from the above ten-dimensional expression in flat space, we need to restrict the kinematics to the five-plane $\mathbb{R}^5\simeq \mathrm{AdS}_5|_{L\to\infty}$ by integrating over the S^5 wavefunctions of the Kaluza-Klein modes dual to $\mathcal{O}_p$, where the integration over S^5 yields an additional factor of L^5. Denoting the ten-dimensional amplitude in transverse kinematics by $\mathcal{A}^{(10)}_\perp(s,t;\sigma,\tau)$, the relation (5.3) can be inverted to give

$$\begin{aligned}\mathcal{A}^{(10)}_\perp(s,t;\sigma,\tau)=&\frac{\Theta_4^{\mathrm{flat}}(s,t;\sigma,\tau)}{16\,\mathcal{N}_{\mathcal{A}}}\;\Gamma(\Sigma-2)\\&\cdot\lim_{L\to\infty}L^{14}\int_{-i\infty}^{+i\infty}\frac{d\alpha}{2\pi i}\;\alpha^{-(\Sigma+2)}e^{\alpha}\;\mathcal{M}_{\vec{p}}\Big(\frac{L^2s}{2\alpha},\frac{L^2t}{2\alpha};\sigma,\tau\Big),\end{aligned} \tag{5.4}$$

where we made use of equation (5.2) to replace $M(s,t)$ with the reduced Mellin amplitude $\mathcal{M}(s,t;\sigma,\tau)$, and $\mathcal{N}_{\mathcal{A}}$ is a normalisation factor which we need to fix. Note that in our case the relevant ten-dimensional amplitude we want to recover in the flat space limit

[2] Note that the above relation (5.3) requires the use of the full Mellin amplitude $M(s,t;\sigma,\tau)$ which is related to the *reduced* Mellin amplitude $\mathcal{M}_{\vec{p}}(s,t;\sigma,\tau)$ as defined in (3.28) through the action of a difference operator $\widehat{R}$ corresponding to the factor $\mathcal{I}$. In the flat space limit $s,t\to\infty$, the relation is given by

$$M_{\vec{p}}(s,t;\sigma,\tau)\simeq\frac{1}{16}\Theta_4^{\mathrm{flat}}(s,t;\sigma,\tau)\,\mathcal{M}_{\vec{p}}(s,t;\sigma,\tau),\quad\text{with }\Theta_4^{\mathrm{flat}}(s,t;\sigma,\tau)=(tu+st\sigma+su\tau)^2. \tag{5.2}$$

is given by the four-graviton scattering amplitude in type IIB superstring theory. This amplitude admits the genus-expansion

$$\mathcal{A}_{\text{Flat}} = \kappa_{10}^2 \, g_s^2 \, \mathcal{R}^4 \left(\mathcal{A}^{\text{tree}} + 2\pi g_s^2 \, \mathcal{A}^{\text{genus-1}} + O(g_s^4) \right), \tag{5.5}$$

where $\kappa_{10}^2 = 64\pi^7 (\alpha')^4$ and $\mathcal{R}^4$ is a universal kinematic factor, which is fixed by supersymmetry and hence factors out from the entire amplitude. In the case of four external gravitons, it is given by a specific tensor-contraction of four Weyl curvatures [106]. When expanded, this kinematic factor encodes the amplitudes of all the states within the ten-dimensional supermultiplet. Most importantly for our purposes, when restricted to transverse kinematics as required by the flat space limit formula, it reduces to [122]

$$\mathcal{R}_\perp^4 = \frac{\Theta_4^{\text{flat}}(s,t;\sigma,\tau)}{16}, \tag{5.6}$$

and therefore it precisely cancels against the identical overall factor on the RHS of (5.4).

The tree-level amplitude $\mathcal{A}^{\text{tree}}$ in the above (5.5) is given by the well-known Virasoro-Shapiro amplitude

$$\mathcal{A}^{\text{tree}} = \frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)}, \tag{5.7}$$

where s, t, u are the usual ten-dimensional Mandelstam invariants obeying $s+t+u=0$. The string theory parameter $\alpha' = l_s^2$ can be converted into CFT quantities using the relation $\alpha' = \lambda^{-\frac{1}{2}} L^2$. The low-energy expansion of the amplitude (5.7) then corresponds to an expansion in $1/\lambda$:

$$\mathcal{A}^{\text{tree}} = -\frac{64}{stu\,(\alpha')^3} \left(1 + \frac{stu\,\zeta_3}{32} \cdot \lambda^{-\frac{3}{2}} L^6 + \frac{stu\,\zeta_5}{1024} (s^2+t^2+u^2) \cdot \lambda^{-\frac{5}{2}} L^{10} + \dots \right). \tag{5.8}$$

We can then fix the normalisation $\mathcal{N}_{\mathcal{A}}$ in the flat space limit formula (5.4) by plugging in the AdS supergravity Mellin amplitude $\mathcal{M}_{\vec{p}}^{(1,0)}$ from equation (3.33), and comparing it to the first term in the above expansion (5.8), giving

$$\mathcal{N}_{\mathcal{A}} = \frac{(\alpha')^3}{32\pi^5} \, \frac{B_{\vec{p}}^{\text{sugra}}(\sigma,\tau)}{\Sigma - 2}. \tag{5.9}$$

Note that $\mathcal{N}_{\mathcal{A}}$ has a non-trivial dependence on the $su(4)$ cross-ratios through the factor $B_{\vec{p}}^{\text{sugra}}(\sigma,\tau)$, which follows from the large s, t limit of the tree-level supergravity amplitude and is explicitly given by

$$B_{\vec{p}}^{\text{sugra}}(\sigma,\tau) = \sum_{i,j\geq 0} \frac{1}{i!j!k!} \frac{8 p_1 p_2 p_3 p_4 \, \sigma^i \tau^j}{\left(\frac{p_{43}+p_{21}}{2}+i\right)! \left(\frac{p_{43}-p_{21}}{2}+j\right)! \left(\frac{|p_1+p_2-p_3-p_4|}{2}+k\right)!}, \tag{5.10}$$

with $k = p_3 + \min\left\{0, \frac{p_1+p_2-p_3-p_4}{2}\right\} - i - j - 2$, and the range of i,j is such that $k \geq 0$ in the sum.

With the normalisation factor fixed by the supergravity result, we can turn our attention to the subsequent tower of string corrections. At each order in the $1/\lambda$ expansion, the flat space limit of the tree-level Mellin amplitude has to match the corresponding term in the expansion of the Virasoro-Shapiro amplitude (5.8). As a consequence, the structure of tree-level string corrections in AdS is constrained by the structure of the expansion (5.8), which we already implicitly assumed in equation (5.1). Furthermore, the flat space limit completely determines the leading polynomial term in the Mellin amplitudes $\mathcal{M}_{\vec{p}}^{(1,n)}$, leaving only the subleading polynomial terms unconstrained. For the first string correction at order $\lambda^{-\frac{3}{2}}$, this turns out to fix the entire amplitude, as we will discuss next.

5.1.1 The Order $\lambda^{-\frac{3}{2}}$ Correlator for Arbitrary External Charges

The first string correction arises from the order $\lambda^{-\frac{3}{2}}$ term in the Virasoro-Shapiro amplitude, corresponding to an $\mathcal{R}^4$ interaction vertex in the string theory effective action. As such, the Mellin amplitude $\mathcal{M}_{\vec{p}}^{(1,3)}$ is given by a polynomial of degree zero and therefore has only one contribution, the constant term. It is due to this simplicity that the first string correction is entirely fixed by the flat space limit only: using the flat space limit formula (5.4) for a constant Mellin amplitude and matching the $\lambda^{-\frac{3}{2}}$ term in the expansion (5.8), we are led to the compact result

$$\mathcal{M}_{\vec{p}}^{(1,3)} = \frac{(\Sigma-1)_3\,\zeta_3}{4}\; B_{\vec{p}}^{\text{sugra}}(\sigma,\tau), \tag{5.11}$$

with $B_{\vec{p}}^{\text{sugra}}$ given in (5.10). We can straightforwardly obtain the explicit position space expression by performing the inverse Mellin transform of the above amplitude, giving

$$\mathcal{H}_{\vec{p}}^{(1,3)} = \frac{(\Sigma-1)_3\,\zeta_3}{4}\; B_{\vec{p}}^{\text{sugra}}(\sigma,\tau)\; u^{\frac{p_1+p_2+p_3-p_4}{2}}\;\overline{D}_{p_1+2,p_2+2,p_3+2,p_4+2}(u,v). \tag{5.12}$$

The above formula is consistent with the results for $\langle 2222\rangle$ [119] and $\langle 22pp\rangle$ [120], and by construction obeys the correct crossing transformation properties. We checked explicitly for many cases that, upon decomposing into superconformal blocks, our general result (5.12) contributes to spin 0 only, as expected from the $\mathcal{R}^4$ correction term. Furthermore, note that the above correlator contributes only to $su(4)$ channels with $a=0$, which is a non-trivial property of the polynomial $B^{\text{sugra}}(\sigma,\tau)$. This will be of particular relevance when studying the order $\lambda^{-\frac{3}{2}}$ anomalous dimensions in the next section.

Lastly, let us note that the above result for $\mathcal{H}_{\vec{p}}^{(1,3)}$ can also be obtained from a ten-dimensional generating functional. In particular, we observe that the hidden ten-dimensional conformal symmetry of the supergravity correlators $\mathcal{H}_{\vec{p}}^{(1,0)}$ (see Section 3.4.5) remains unbroken by the first string correction! Due to the simplicity of the Mellin amplitude $\mathcal{M}_{\vec{p}}^{(1,3)}$ the same construction for a generating functional from [112] goes through, and we have

$$\mathcal{H}^{(1,3)}_{\vec{p}} = \widehat{\mathcal{D}}_{\vec{p}} \cdot u^2\, \mathcal{H}^{(1,3)}_{2222}\,, \tag{5.13}$$

where $\widehat{\mathcal{D}}_{\vec{p}}$ is the same differential operator as in the supergravity case.[3] In this way, all correlators of arbitrary external charges are neatly repackaged and descend from the same seed-correlator $\mathcal{H}^{(1,3)}_{2222} \propto u^2\, \overline{D}_{4444}$.

5.2 The Double-Trace Spectrum at Order $\boldsymbol{\lambda^{-\frac{3}{2}}}$

With the general correlator $\mathcal{H}^{(1,3)}_{\vec{p}}$ at hand, we can now employ the same methods which allowed us to solve the supergravity mixing problem to the first string correction. In this section, we extend the set of unmixing equations encountered in the last chapter to order $\lambda^{-\frac{3}{2}}$. We then proceed by first solving the equations in the singlet channel, before generalising the results to all channels $[0, b, 0]$. These string corrected anomalous dimensions are of a surprisingly simple structure, for which we will provide an intuitive ten-dimensional explanation in the next section.

5.2.1 The Unmixing Equations at Order $\boldsymbol{\lambda^{-\frac{3}{2}}}$

The unmixing equations are most compactly presented in their matrix form. We will thus use the same matrix notation as we did in the supergravity case: for a given $su(4)$ channel $[a, b, a]$, we denote by $\mathbb{C}_{[a,b,a]}(t, \ell)$ the matrix of three-point functions $C_{pq\mathcal{K}}$ with $(p, q) \in R_{\vec{\tau}}$. Keeping only the leading terms in the large N expansion, we have

$$\mathbb{C}_{[a,b,a]}(t, \ell) = \left(\mathbb{C}^{(0)} + \lambda^{-\frac{3}{2}}\mathbb{C}^{(3)} + \lambda^{-\frac{5}{2}}\mathbb{C}^{(5)} + \dots\right) + O(a). \tag{5.14}$$

Analogously, by $\widehat{\eta}^{(n)}$ we denote the diagonal matrix of the corresponding set of tree-level anomalous dimensions $\eta^{(1,n)}_{pq}$, where n denotes the order in the $\lambda^{-\frac{1}{2}}$ expansion. The unmixing equations then follow by plugging in the above expansions into the superconformal block decomposition of the interacting part of the correlator. Keeping terms up to order $\mathfrak{a}\lambda^{-\frac{3}{2}}$ and omitting all arguments for simplicity, we have

$$O(1): \qquad \widehat{\mathcal{A}} = \mathbb{C}^{(0)} \left(\mathbb{C}^{(0)}\right)^T, \tag{5.15}$$

$$O(\mathfrak{a}): \qquad \widehat{\mathbb{M}}^{(1,0)} = \mathbb{C}^{(0)}\, \widehat{\eta}^{(0)} \left(\mathbb{C}^{(0)}\right)^T, \tag{5.16}$$

$$O(\lambda^{-\frac{3}{2}}): \qquad 0 = \mathbb{C}^{(0)} \left(\mathbb{C}^{(3)}\right)^T + \mathbb{C}^{(3)} \left(\mathbb{C}^{(0)}\right)^T, \tag{5.17}$$

$$O(\mathfrak{a}\lambda^{-\frac{3}{2}}): \qquad \widehat{\mathbb{M}}^{(1,3)} = \mathbb{C}^{(0)}\, \widehat{\eta}^{(3)} \left(\mathbb{C}^{(0)}\right)^T + \mathbb{C}^{(0)}\, \widehat{\eta}^{(0)} \left(\mathbb{C}^{(3)}\right)^T + \mathbb{C}^{(3)}\, \widehat{\eta}^{(0)} \left(\mathbb{C}^{(0)}\right)^T, \tag{5.18}$$

[3]See equation (3.43) for some explicit examples of $\widehat{\mathcal{D}}_{\vec{p}}$ with low external charges.

where in the first two lines we repeat the supergravity unmixing equations given already in (4.48). Recall that the leading order SCPW coefficients $\widehat{\mathcal{A}}(t,\ell)$ are obtained from disconnected free field theory, while the matrices $\widehat{\mathbb{M}}^{(1,0)}(t,\ell)$ and $\widehat{\mathbb{M}}^{(1,3)}(t,\ell)$ are extracted from the $\log(u)$ parts of the known tree-level correlators $\mathcal{H}^{(1,0)}_{\vec{p}}$ and $\mathcal{H}^{(1,3)}_{\vec{p}}$, respectively. Also, note that the zero on the LHS of equation (5.17) comes from the fact that there are no $1/\lambda$ corrections to the free theory.

In the last chapter we solved the supergravity unmixing equations (5.15)-(5.16), obtaining $\mathbb{C}^{(0)}$ and $\widehat{\eta}^{(0)}$. With this data at hand, we can turn our attention to the next two equations, with $\mathbb{C}^{(3)}$ and $\hat{\eta}^{(3)}$ as our unknowns. Note that, unlike the supergravity case, the order $\lambda^{-\frac{3}{2}}$ equations are linear in the unknowns. As before, we can verify that the number of equations equals the number of unknowns, and as such this is a well defined set of equations with a unique solution. In the following, we will first consider the singlet channel and reveal a surprising simplicity in the first string corrections to the spectrum, before generalising to all $su(4)$ channels.

5.2.2 Singlet Channel Results

Let us start by describing the singlet channel solution to the above unmixing equations. We obtain the necessary data, namely the $\log(u)$ part of the correlators $\mathcal{H}^{(1,3)}_{ppqq}$ in the singlet channel, from our new result (5.12). As expected, the conformal block decomposition yields only spin 0 contributions. Solving the unmixing equations (5.17)-(5.18) twist by twist, we find the surprisingly simple solution

$$\widehat{\eta}^{(3)}_{[0,0,0]} = \left\{\eta^{(3)}_1, 0, \ldots, 0\right\}, \qquad \mathbb{C}^{(3)}_{[0,0,0]} = 0, \tag{5.19}$$

with $\eta^{(3)}_1$ being consistent with the formula

$$\eta^{(3)}_1 = -\frac{\zeta_3}{840}(t-1)^2 t^3 (t+1)^4 (t+2)^3 (t+3)^2 \cdot \delta_{\ell,0}. \tag{5.20}$$

Some comments are in order:

- The leading $1/\lambda$ correction to the matrix of three-point functions $\mathbb{C}^{(3)}$ is identically zero. A priori, such a correction is not forbidden by consistency of the OPE and its vanishing is a very non-trivial result.

- The pattern of anomalous dimensions turns out to be remarkably simple: only the operators with degeneracy label $i = 1$ (note that $r = 0$ in the singlet channel) receives a $\lambda^{-\frac{3}{2}}$ correction to its dimension, all other anomalous dimensions vanish. This pattern follows from a ten-dimensional principle, which we will describe later in Section 5.3.

- For large t, the anomalous dimension has the asymptotic behaviour

$$\eta_1^{(3)} \to -\frac{\zeta_3}{840}\, t^{14}, \tag{5.21}$$

a fact which will become important when comparing with the anomalous dimensions for other $su(4)$ channels. Furthermore, $\eta_1^{(3)}$ has the discrete symmetry $t \to -t-2$, which we already observed in the supergravity anomalous dimensions.

- Lastly, together with the leading order three-point functions $\mathbb{C}^{(0)}_{[0,0,0]}$, our result for $\widehat{\eta}^{(3)}_{[0,0,0]}$ correctly reproduces the averages of squared anomalous dimensions derived in [120], see equation (3.11) therein.

5.2.3 General Solution for All $su(4)$ Channels

The above singlet channel results can be straightforwardly generalised to all $su(4)$ channels. First however, let us note a non-trivial fact about the $\mathcal{H}^{(1,3)}_{\vec{p}}$ correlator from equation (5.12): its dependence on the $su(4)$ cross-ratios σ and τ, which is determined entirely by the polynomial $B^{\text{sugra}}_{\vec{p}}(\sigma,\tau)$, is such that it has support only on channels of the form $[0,b,0]$, i.e. all channels $[a,b,a]$ with $a>0$ are absent from $\mathcal{H}^{(1,3)}_{\vec{p}}$. As a direct consequence, we therefore have

$$\widehat{\eta}^{(3)}_{[a,b,a]} = 0, \qquad \mathbb{C}^{(3)}_{[a,b,a]} = 0, \quad \text{for } a>0. \tag{5.22}$$

For the remaining channels with $a=0$, we repeat the computation described above and find

$$\widehat{\eta}^{(3)}_{[0,b,0]} = \left\{\eta^{(3)}_{1,0}, 0, \ldots, 0\right\}, \qquad \mathbb{C}^{(3)}_{[0,b,0]} = 0, \tag{5.23}$$

where only the first anomalous dimension with degeneracy labels $(i,r)=(1,0)$ is non-zero, and we find it is consistent with the formula

$$\eta^{(3)}_{1,0} = -\frac{\zeta_3}{840} M^{(4)}_t M^{(4)}_{t+1} (t-1)_3 (t+b+1)_3 \cdot \delta_{\ell,0}, \tag{5.24}$$

where we set $a=0$ in $M^{(4)}_t = (t-1)(t+a)(t+a+b+1)(t+2a+b+2)$, which is the same factor as in the supergravity anomalous dimensions, and note that we observe again the discrete symmetry $t \to -t-b-2$. The large t limit of the above anomalous dimension is independent of the quantum number b,

$$\eta^{(3)}_{1,0} \to -\frac{\zeta_3}{840}\, t^{14}, \tag{5.25}$$

and thus necessarily matching the singlet channel value (5.21). The non-trivial vanishing of the three-point functions $\mathbb{C}^{(3)}$ together with the simple pattern of the string anomalous dimensions, as well as the above matching of their large twist behaviour across different

$su(4)$ channels, motivates us to consider a ten-dimensional principle behind these simple patterns which we discuss next.

5.3 Constraints from a New Ten-Dimensional Principle

In Section 3.4.5, we briefly reviewed the recently observed hidden ten-dimensional conformal symmetry of the supergravity correlators $\mathcal{H}_{\vec{p}}^{(1,0)}$, which coincidentally extends to the $\lambda^{-\frac{3}{2}}$ correction in the form of equation (5.13). The discovery of this hidden symmetry in [112] was in part motivated by the following observation: the supergravity anomalous dimensions $\eta_{pq}^{(1,0)}$ and the partial-wave coefficients of the flat ten-dimensional $2 \to 2$ scattering amplitude of axi-dilatons in type IIB supergravity share a common Pochhammer structure in their denominators:

$$\frac{1}{(\ell_{10}+1)_6} \sim \frac{1}{\left(\ell + 2(i+r) + a - \frac{1+(-1)^{a+\ell}}{2}\right)_6}, \tag{5.26}$$

where the LHS depends on an effective ten-dimensional spin $\ell_{10} = 0, 2, \ldots$ and the RHS is the denominator of the supergravity anomalous dimensions (4.53), which depends on the usual four-dimensional spin ℓ, the $su(4)$ channel $[a, b, a]$ and the degeneracy labels (i, r), whose definition we repeat for convenience:

$$i = 1, \ldots, t-1, \qquad r = 0, \ldots, \mu - 1, \qquad \mu = \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a+\ell \text{ even}, \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a+\ell \text{ odd}. \end{cases} \tag{5.27}$$

The correspondence (5.26) assigns a value of the effective ten-dimensional spin ℓ_{10} to each long double-trace operator in the supergravity spectrum by the identification

$$\ell_{10} \equiv \ell + 2(i+r) + a - 1 - \frac{1+(-1)^{a+\ell}}{2}. \tag{5.28}$$

Note that in general this identification allows for many four-dimensional operators to be assigned the same effective ten-dimensional spin ℓ_{10}. Heuristically, this observation can be motivated as follows: in the ten-dimensional four-point correlator considered in [112], the exchanged operators are built from a single ten-dimensional scalar field Φ and are given by bilinears of the schematic form $[\Phi\, \partial^{\ell_{10}}\, \Phi]$. As such, there is only one primary operator for each even spin ℓ_{10}, which upon dimensional reduction results in multiple four-dimensional double-trace operators descending from the same 10d primary. Furthermore, their effective spin ℓ_{10} is simply related to the number of derivatives in the ten-dimensional theory.

Now consider the first string correction at order $\lambda^{-\frac{3}{2}}$, for which we have computed the spectrum of anomalous dimensions, with the result given in (5.23). Unexpectedly, we found that only the first anomalous dimensions with degeneracy labels $(i, r) = (1, 0)$ in channels $[a,b,a]$ with $a=0$ are non-vanishing. A neat interpretation of this result can be given by the assignment of ten-dimensional spin: as the order $\lambda^{-\frac{3}{2}}$ string correction descends from the $\mathcal{R}^4$ supervertex, its ten-dimensional partial-wave decomposition contributes only to spin $\ell_{10}=0$. Considering the identification (5.28), there is a *unique* choice of the four-dimensional quantum numbers satisfying that equation, namely

$$\ell_{10}=0 \quad \Rightarrow \quad (\ell,i,r,a)=(0,1,0,0), \tag{5.29}$$

which exactly coincides with our explicit results for the spectrum.

The heuristic assignment (5.28) thus seems to correctly give a prediction for which four-dimensional double-trace operators acquire an anomalous dimension, depending on the allowed ten-dimensional spin ℓ_{10}. Note that this interpretation is also consistent with the supergravity case, where operators of any even spin ℓ_{10} are exchanged. As such, equation (5.28) does not give any restrictions on the four-dimensional quantum numbers (ℓ,i,r,a), and indeed all operators are found to acquire a non-zero supergravity anomalous dimension. On the other hand, by assuming this ten-dimensional interpretation remains valid when considering further string corrections (which due to their polynomial amplitudes yield only finite spin contributions to the partial-wave expansion), we can deduce constraints on the spectrum of anomalous dimensions, as shown above for the $\lambda^{-\frac{3}{2}}$ case.

As a second example, let us consider the next string correction at order $\lambda^{-\frac{5}{2}}$. The $\lambda^{-\frac{5}{2}}$ term descends from the $\partial^4\mathcal{R}^4$ supervertex, allowing for ten-dimensional spins up to $\ell_{10}=2$. Using again the assignment (5.28), we find the allowed values

$$\begin{aligned}\ell_{10}=2 \quad \Rightarrow \quad (\ell,i,r,a)= \;& (2,1,0,0),\ (1,1,0,0),\ (1,1,0,1),\\ & (0,2,0,0),\ (0,1,1,0),\ (0,1,0,1),\ (0,1,0,2).\end{aligned} \tag{5.30}$$

Together with the spin $\ell_{10}=0$ contribution (5.29), we therefore expect the 8 states with four-dimensional spins $\ell=0,1,2$ in the various channels $[0,b,0]$, $[1,b,1]$ and $[2,b,2]$ to be the only non-vanishing contributions to the $\lambda^{-\frac{5}{2}}$ spectrum.

There is one further implication of the relation to 10 dimensions, which concerns the observed coincidence of the large twist behaviour of the order $\lambda^{-\frac{3}{2}}$ anomalous dimensions, recall equations (5.21) and (5.25). Note that for finite spin the large twist asymptotics accesses the flat space limit, which can be understood from the inverse Mellin transform (3.28): the flat space limit tells us to look at the large s,t behaviour, which in particular translates into large powers of u in position space. Restricting ourselves to finite spin contributions, we then see that large twist indeed corresponds to the flat space limit. Schematically, for finite spin we thus have the correspondence

$$\text{flat space limit} \;\sim\; \text{large twist asymptotics}. \tag{5.31}$$

Therefore, at a given order in the $1/\lambda$ expansion, we expect the same large twist asymptotics for all four-dimensional operators which descend from a common ten-dimensional primary according to the identification (5.28), regardless of their four-dimensional quantum numbers.

The above observations thus motivate the following general proposal:

- $\eta_{pq}^{(n)} = 0, \quad \text{for } \ell_{10}(p) > n-3\,,$ (5.32)
- $\eta_{pq}^{(n)}$ only depends on $\ell_{10}(p)$ in the limit $t \to \infty\,,$ (5.33)
- $\eta_{i=1,r=0}^{(n)}$ is polynomial in t of degree $8+2n\,,$ (5.34)
- $C_{pq\mathcal{K}_{\tilde{p}\tilde{q}}}^{(n)} = 0, \quad \text{for } \ell_{10}(\tilde{p}) > n-3\,.$ (5.35)

The first constraint (5.32) says that ℓ_{10} dictates the non-zero contributions to η, and generalises the conditions from examples (5.29) and (5.30). The second condition (5.33) is related to the restoration of ten-dimensional Lorentz symmetry in the flat space limit (corresponding to the limit $t \to \infty$, as discussed above). Next, the condition (5.34) is an assumption on the anomalous dimension in the case of no partial degeneracy. Furthermore, we expect the polynomial to obey the discrete symmetry (4.55). Lastly, the fourth condition (5.35) demands that the columns of $\mathbb{C}^{(n)}$ corresponding to operators with too high ten-dimensional spin vanish. Note that in the $n=3$ case this implies $\mathbb{C}^{(3)} = 0$, since the first unmixing equation (5.15) implies up to rescaling that $\mathbb{C}^{(0)}$ is an orthogonal matrix. Its first correction then leads to the equation

$$\mathbb{C}^{(3)}\left(\mathbb{C}^{(0)}\right)^T + \mathbb{C}^{(0)}\left(\mathbb{C}^{(3)}\right)^T = 0, \tag{5.36}$$

and therefore, after a change of basis, $\mathbb{C}^{(3)}$ is antisymmetric. If all but the first column vanishes then the whole matrix vanishes, in agreement with the explicit results described before.

In the next section, we will consider the second string correction at order $\lambda^{-\frac{5}{2}}$ and describe how the above constraints can be used to set up a bootstrap problem, allowing us to solve for both the correlator and the spectrum of anomalous dimensions at the same time.

5.4 Bootstrapping the Order $\boldsymbol{\lambda^{-\frac{5}{2}}}$ String Correction

The second string correction at order $\lambda^{-\frac{5}{2}}$ descends from the $\partial^4\mathcal{R}^4$ term in the string theory effective action, and as such gives rise to exchanged operators up to ten-dimensional spin $\ell_{10} = 2$. This is a particularly interesting case to study, as it allows us to address the partial degeneracy of the supergravity anomalous dimensions: according to equation (5.30), in $su(4)$ channels of the form $[0, b, 0]$ with $b \geq 2$ we expect the $(i, r) = (2, 0)$ and $(1, 1)$ anomalous dimensions to be non-zero, and the order $\lambda^{-\frac{5}{2}}$ string correction will thus determine the resolution of the first case of partial degeneracy.

Compared to the $\lambda^{-\frac{3}{2}}$ correction however, the necessary tree-level correlators $\mathcal{M}_{\vec{p}}^{(1,5)}$ are not known for arbitrary external charges. Prior to our work, the only known correlators were of the form $\langle 22pp \rangle$ [100, 120], which is not enough to solve the full mixing problem without further assumptions.[4] We thus have to come up with a set of conditions which allow us to determine *both* the correlator $\mathcal{M}_{\vec{p}}^{(1,5)}$ and the corresponding spectrum of anomalous dimensions $\eta_{pq}^{(5)}$. In the following, we will describe the bootstrap problem and the conditions which achieve both of the above.

5.4.1 A Bootstrap Problem

Let us start with an ansatz for the general $\lambda^{-\frac{5}{2}}$ correlator in Mellin space. By using the flat space limit of Section 5.1, we see that $\mathcal{M}_{\vec{p}}^{(1,5)}$ is a degree two polynomial in the Mellin variables and furthermore it determines the quadratic terms, such that we are left with

$$\begin{aligned} \mathcal{M}_{\vec{p}}^{(1,5)} = \zeta_5 \Big(&\tfrac{1}{32} (\Sigma - 1)_5 \, B_{\vec{p}}^{\text{sugra}}(\sigma, \tau) \, (s^2 + t^2 + \tilde{u}^2) \\ &+ \alpha_{\vec{p}}(\sigma, \tau) \, s + \beta_{\vec{p}}(\sigma, \tau) \, t + \gamma_{\vec{p}}(\sigma, \tau) \Big), \end{aligned} \tag{5.37}$$

where the additional linear and constant contributions are left unspecified. The SCPW coefficients $\mathfrak{M}_{\vec{p}}^{(1,5)}$ of the above correlators are related to the spectrum of anomalous dimensions through the unmixing equations, which in our matrix notation read

$$0 = \mathbb{C}^{(0)} \left(\mathbb{C}^{(5)} \right)^T + \mathbb{C}^{(5)} \left(\mathbb{C}^{(0)} \right)^T, \tag{5.38}$$

$$\widehat{\mathfrak{M}}^{(1,5)} = \mathbb{C}^{(0)} \, \widehat{\eta}^{(5)} \left(\mathbb{C}^{(0)} \right)^T + \mathbb{C}^{(0)} \, \widehat{\eta}^{(0)} \left(\mathbb{C}^{(5)} \right)^T + \mathbb{C}^{(5)} \, \widehat{\eta}^{(0)} \left(\mathbb{C}^{(0)} \right)^T, \tag{5.39}$$

exactly mirroring the order $\lambda^{-\frac{3}{2}}$ unmixing equations (5.17)-(5.18). Let us now spell out the set of constraints we impose on the correlators $\mathcal{M}_{\vec{p}}^{(1,5)}$ and the OPE data:[5]

- We make an ansatz with finitely many coefficients for each of the unknown functions $\alpha_{\vec{p}}(\sigma, \tau)$, $\beta_{\vec{p}}(\sigma, \tau)$ and $\gamma_{\vec{p}}(\sigma, \tau)$, constrained by consistency with the known $\langle 22pp \rangle$ results and crossing symmetry.
- On the OPE data, we impose the previously introduced conditions (5.32)-(5.35) with $n = 5$, and hence the maximal ten-dimensional spin is $\ell_{10} = 2$.

[4] One can, however, solve the mixing problem in the singlet and $[0, 1, 0]$ channel under the (erroneous) assumption $\mathbb{C}^{(5)} = 0$, see Section 5 in reference [5]. This does not shed any light on the resolution of the partial degeneracy, but allows one to correctly determine the $\langle 23p-1p \rangle$ family of correlators up to one free parameter, despite the incorrect assumption $\mathbb{C}^{(5)} = 0$.

[5] For more details on the individual points, we refer the reader to [8].

By imposing these two sets of conditions we find a unique consistent solution for the Mellin amplitude and the spectrum. We emphasise that the existence of a solution consistent with the ansatz for the Mellin amplitude, crossing symmetry and the spectrum constraints is highly non-trivial. The final form of the coefficients $\alpha_{\vec{p}}(\sigma,\tau)$, $\beta_{\vec{p}}(\sigma,\tau)$ and $\gamma_{\vec{p}}(\sigma,\tau)$ is not very illuminating and can be found in [8]. Instead, let us focus on the spectrum of anomalous dimensions.

5.4.2 The Order $\lambda^{-\frac{5}{2}}$ Anomalous Dimensions

According to condition (5.32) the anomalous dimensions $\eta^{(5)}$ are non-vanishing only for $\ell_{10} \leq 2$, constraining the possible values of their quantum numbers (ℓ, i, r, a) as shown in (5.30). In order to explicitly give the individual anomalous dimensions, we use the notation $\eta^{(5)}_{i,r|\ell,a}$ and we define the polynomial $\mathcal{T}$ as follows

$$\begin{aligned} \mathcal{T}_{t,\ell,a,b} &= \frac{\zeta_5}{166320} M_t^{(4)} M_{t+\ell+1}^{(4)} N_t^{(3)} N_{-t-2a-b-\ell-2}^{(3)}, \\ N_t^{(3)} &= (t-1)(t+a)(t+a+b+1), \end{aligned} \tag{5.40}$$

where $M_t^{(4)} = (t-1)(t+a)(t+a+b+1)(t+2a+b+2)$ is the same factor as in the numerator of the supergravity anomalous dimensions and note that $\mathcal{T}_{t,\ell=0,a=0,b} \sim \eta^{(3)}_{1,0}$, see equation (5.24). For spin $\ell = 2$ we must have $i = 1$, $r = 0$, $a = 0$ and we find

$$\eta^{(5)}_{1,0|2,0} = \mathcal{T}_{t,2,0,b}\,(t+1)(t+2)(t+b+2)(t+b+3). \tag{5.41}$$

For spin $\ell = 1$ we have $i = 1$, $r = 0$ with either $a = 0$ or $a = 1$:

$$\eta^{(5)}_{1,0|1,0} = \tfrac{1}{2}\mathcal{T}_{t,1,0,b}\;(t+1)(t+b+2)(2t(3+b+t)+b), \tag{5.42}$$

$$\eta^{(5)}_{1,0|1,1} = \mathcal{T}_{t,1,1,b}\;t(t+2)(t+b+3)(t+b+5). \tag{5.43}$$

The spin zero anomalous dimensions have support on $a = 0, 1, 2$. For $a = 1, 2$ we have only $i = 1$ and $r = 0$:

$$\eta^{(5)}_{1,0|0,1} = \tfrac{1}{2}\mathcal{T}_{t,0,1,b}\;t(t+b+4)(2t^2+2(4+b)t+b+6), \tag{5.44}$$

$$\eta^{(5)}_{1,0|0,2} = \mathcal{T}_{t,0,2,b}\;t(1+t)(5+b+t)(6+b+t). \tag{5.45}$$

In all the above cases we have $\mathbb{C}^{(5)} = 0$. Finally, the case $a = 0$ allows for two or three components depending on the values of t and b. Using the definition $\theta \equiv 2t + 2 + b$, the $i = 1$ component reads

$$\begin{aligned} \eta^{(5)}_{1,0|0,0} &= \tfrac{77}{18}\mathcal{T}_{t,0,0,b}\,f_{b,t}, \\ f_{b,t} &= \tfrac{9}{4}(\theta^2-b_0)^2 - 35(\theta^2-b_0) - 34b_0 + 639, \end{aligned} \tag{5.46}$$

with $b_0 = b(b+4)$. Lastly and most interestingly, we arrive at the first case of partial degeneracy: the $(i,r) = (2,0)$ and $(1,1)$ components read

$$\eta^{(5)}_{2,0|0,0} = \tfrac{1}{9}\mathcal{T}_{t,0,0,b}\left(j_{b,t} - 10\sqrt{k_{b,t}}\right), \qquad \eta^{(5)}_{1,1|0,0} = \tfrac{1}{9}\mathcal{T}_{t,0,0,b}\left(j_{b,t} + 10\sqrt{k_{b,t}}\right), \tag{5.47}$$

with the quartic polynomials $j_{b,t}$ and $k_{b,t}$ given by

$$j_{b,t} = \tfrac{1}{4}f_{b,t} - \tfrac{15}{4}(\theta^2 + b_0 + 21), \qquad k_{b,t} = j_{b,t} + (\theta^2 + b_0)(\theta^2 + b_0 - 10). \tag{5.48}$$

Note that the residual partial degeneracy observed in supergravity is lifted by the square root term, and in the $\ell = a = 0$ case we have $\mathbb{C}^{(5)} \neq 0$.

5.4.3 Comments on the Resolution of Partial Degeneracy

The results of the previous section provide the full spectrum at order $\lambda^{-\frac{5}{2}}$.[6] In the first case where residual degeneracy is present in the supergravity spectrum, the $\lambda^{-\frac{5}{2}}$ corrections resolve it. Due to the residual two-fold mixing problem, the appearance of square roots in the anomalous dimension is to be expected; this did not happen in supergravity due to the hidden ten-dimensional conformal symmetry. However, in some special cases the square roots in (5.47) have to disappear:

- When $t = 2$, there is no degeneracy and only two states acquire an anomalous dimension. The above results for the two degenerate anomalous dimensions are consistent with this, since $k_{b,2} = j^2_{b,2}/100$ such that $\eta^{(5)}_{2,0|0,0}$ vanishes and $\eta^{(5)}_{1,1|0,0}$ becomes rational.
- When $b = 0$ or $b = 1$, there is no degeneracy for any t (since $\mu = 1$ in (5.27) for those cases) and the square roots disappear again.
- In the flat space limit $t \to \infty$ the square root terms are suppressed and degeneracy is restored, respecting the ten-dimensional Lorentz symmetry.

The disappearance of the square roots in these cases is a strong check of the consistency of the solution. Finally, all the anomalous dimensions have some shared features.

- When expressed in terms of the twist τ (or $\theta = 2t + 2a + b + \ell + 2$) instead of t, they really depend on the $su(4)$ labels only through the Casimir combination $b_a = b(b + 4 + 2a)$.
- They enjoy the discrete supergravity symmetry $t \to -t - \ell - 2a - b - 2$: this in turns means that all the quartic polynomials f, j and k are actually quadratic in

[6] Note that the anomalous dimensions given here differ from those conjectured in [5], since we have found here that $\mathbb{C}^{(5)} \neq 0$ in general.

θ^2. We partly imposed this property as a part of condition (5.34), but in many examples it was found to follow from the other assumptions.

5.5 Conclusions

In the last two chapters, we have presented a detailed analysis of the double-trace spectrum of $\mathcal{N}=4$ SYM theory in the supergravity limit. Firstly, we have shown that the known tree-level supergravity results contain all the necessary information to resolve the degeneracy of the double-trace operators in the large N limit. Our results for the leading order OPE coefficients are surprisingly simple, and we find it remarkable that the result for all supergravity anomalous dimensions admits such a simple and fully factorised formula as (4.53). Furthermore, the fact that the orthogonal $\widetilde{\mathbb{C}}$ matrices of the universal structure (4.45) exist is surprising, considering that modifications of the square root factors typically lead to no orthogonal solutions at all.

Secondly, we have found a surprisingly simple structure in the string corrections to the double-trace spectrum at orders $\lambda^{-\frac{3}{2}}$ and $\lambda^{-\frac{5}{2}}$, which is related to an effective ten-dimensional spin ℓ_{10}. At order $\lambda^{-\frac{3}{2}}$, the flat space limit completely determines the correlators $\mathcal{H}^{(1,3)}_{\vec{p}}$ for all external charges, and we observed that the ten-dimensional conformal symmetry extends to the first string correction. Furthermore, we have found that only anomalous dimensions with $\ell_{10}=0$ are non-zero, and as a consequence the string corrected three-point functions vanish identically, i.e. $\mathbb{C}^{(3)}=0$. On the other hand, the order $\lambda^{-\frac{5}{2}}$ correction allows for ten-dimensional spins up to $\ell_{10}=2$, allowing for a richer spectrum. Nevertheless, we were able to set up a bootstrap problem which determines both the correlator $\mathcal{H}^{(1,5)}_{\vec{p}}$ and the spectrum of anomalous dimensions at the same time. It turns out that $\mathbb{C}^{(5)}$ is non-vanishing in general, and more importantly we have found that the residual degeneracy in the supergravity spectrum is resolved by the order $\lambda^{-\frac{5}{2}}$ correction, thus explicitly breaking the ten-dimensional conformal symmetry. Notably, in the large twist limit (corresponding to the flat space limit), the anomalous dimensions recombine such that degeneracy and ten-dimensional Lorentz symmetry is restored. The explicit formulae for the order $\lambda^{-\frac{3}{2}}$ and $\lambda^{-\frac{5}{2}}$ anomalous dimensions derived here will enter the computation of string corrections to one-loop correlators described in Section 7.

We believe that the methods for bootstrapping string-corrected tree-level correlators described here will continue to be effective at higher orders in $1/\lambda$, with the next case being the order λ^{-3} correction. However, one caveat is the number of free parameters which grows with the order in $1/\lambda$, as higher order Mellin polynomials (corresponding to a larger effective spin ℓ_{10}) are allowed in the ansatz for the Mellin amplitude. In particular, there will be a growing number of undetermined coefficients which are left unfixed by the bootstrap constraints (5.32)-(5.35). Although these coefficients are proportional to previously found solutions to the bootstrap problem and thus correspond to known amplitudes from lower orders in the $1/\lambda$ expansion, we nevertheless can not independently fix their values. For example, there are two such free parameters in the order $\lambda^{-\frac{5}{2}}$ amplitude: one corresponding to a constant Mellin amplitude (i.e. the order $\lambda^{-\frac{3}{2}}$ result), the other one corresponding to a linear Mellin amplitude (this would be the order λ^{-2} amplitude, which however comes with coefficient zero in the expansion of the Virasoro-Shapiro amplitude, see equation (5.8)). One method of fixing those two remaining parameters is to make use of the constraints obtained by supersymmetric localisation techniques in [100, 104, 105]. These localisation results can in principle be expanded to any order in $1/\lambda$, but they will provide only a finite number of constraints. A naive counting reveals that there are enough independent constraints to go at least two orders further in the $1/\lambda$ expansion than the results presented in this thesis. We hope that our future investigations will bring us a step closer towards the ultimate goal of our tree-level bootstrap program: the construction of the analogue of the Virasoro-Shapiro amplitude (5.7) on a curved background, i.e. the full tree-level amplitude of type IIB superstring theory on $\mathrm{AdS}_5\times\mathrm{S}^5$.

It would also be fascinating if the results for the double-trace spectrum at strong coupling discussed here could be compared with the corresponding results obtained in perturbation theory. There has been a lot of progress in pushing the weak coupling results to finite values of the coupling using methods based on integrability [123–127], but there is a conceptual obstacle which prevents the direct comparison of results: the spectrum of exchanged operators at finite coupling is much richer than in the supergravity limit studied here. In particular, there are unprotected single-trace operators (corresponding to excited string states which decouple in the supergravity limit) which are also present in the spectrum and mix with the familiar double-trace operators. As such, one would require additional information in the form of correlators with more general external operators to solve the mixing problem. Nevertheless, from the structure of string corrections to the double-trace spectrum discussed in this chapter, we expect that for finite values of the coupling λ the residual degeneracy of the supergravity anomalous dimensions will be completely lifted.

Finally, while we have focussed on $\mathcal{N}=4$ SYM theory here, the phenomenon of large N degeneracy and the associated problem of operator mixing is presumably common to many holographic theories. Essentially, the mixing problem arises because of the presence of a compact factor in the gravity background (here given by S^5), which leads to a tower of Kaluza-Klein modes related to the massless graviton multiplet. For fixed twist and spin, one will then typically have many double-trace operators with equal classical dimensions, which generically will mix. It would be interesting to consider both other models and the generic structure of the spectrum of large N CFTs further.

Part III

Bootstrapping One-Loop Correlators

Chapter 6

One-loop Correlators in Supergravity

This third part is devoted to the construction of order $1/N^4$ corrections, corresponding to one-loop amplitudes in AdS, using information from the tree-level data we discussed previously. In this chapter we will focus on one-loop supergravity correlators, leaving the discussion of one-loop string corrections for the final chapter. Our approach will solely rely on implementing the consistency of the SCPW decomposition to order $1/N^4$, and as such our results are naturally written in their position space representation. A complementary approach using the Mellin space formalism has recently been employed to obtain the Mellin amplitudes for the one-loop $\langle 2222 \rangle$ correlator [128, 129], and a first generalisation to the $\langle 22pp \rangle$ family of correlators [95]. In contrast, we will describe an algorithm which solves the analytic bootstrap program for one-loop supergravity correlators of generic external single-particle Kaluza-Klein states. Compared to correlators with two graviton multiplets as external states, which receive contributions from a restricted set of exchanged operators and thus are of some physical simplicity, considering the general case presents a network of new complications which we will address in the following.

6.1 General Outline

Let us recall the result of the partial non-renormalisation theorem, which restricts the general four-point correlator to take the form

$$\langle p_1 p_2 p_3 p_4 \rangle = \langle p_1 p_2 p_3 p_4 \rangle_{\text{free}} + \mathcal{P}_{\vec{p}}\, \mathcal{I}(x, \bar{x}, y, \bar{y})\ \mathcal{H}_{\vec{p}}(u, v; \sigma, \tau; \lambda), \tag{6.1}$$

where, contrary to the free theory, the interacting (dynamical) part $\mathcal{H}_{\vec{p}}$ depends both on N and the 't Hooft coupling λ. In this chapter, we will focus solely on the order zero terms in the $1/\lambda$ strong coupling expansion, and hence we drop the λ dependence for now.

In Section 2.5 and together with Appendix B, we studied the SCPW decomposition of the free theory. In particular, we split the free theory correlator $\langle p_1p_2p_3p_4\rangle_{\text{free}}$ into a part with only (semi-)short contributions to the SCPW expansion, and a part with only long contributions which we denoted by $\langle p_1p_2p_3p_4\rangle_{\text{free long}}$, see equation (2.40). We will now incorporate the dynamical part $\mathcal{H}_{\vec{p}}$ and specialise to the long sector. It will be convenient to distinguish the two $1/N$ expansions,

$$\begin{aligned}\langle p_1p_2p_3p_4\rangle_{\text{free long}} &= \langle p_1p_2p_3p_4\rangle^{(0)}_{\text{free long}} + \frac{1}{N^2}\langle p_1p_2p_3p_4\rangle^{(1)}_{\text{free long}} + \dots,\\ \mathcal{H}_{\vec{p}} &= \frac{1}{N^2}\mathcal{H}^{(1)}_{\vec{p}} + \frac{1}{N^4}\mathcal{H}^{(2)}_{\vec{p}} + \dots,\end{aligned} \tag{6.2}$$

where for general external charges we choose the expansion parameter to be $1/N^2$ instead of the previously introduced $\mathfrak{a} = 1/(N^2-1)$.[1] The notation we will use to refer to the SCPW expansion of the long sector of $\langle p_1p_2p_3p_4\rangle$ (i.e. the long sector of the free theory together with the dynamical part), up to order $1/N^4$, is

$$\langle p_1p_2p_3p_4\rangle\big|_{\text{long}} = \log^0(u)\sum_{\vec{\tau}}\left(L^{(0)}_{\vec{p};\vec{\tau}} + \frac{1}{N^2}L^{(1)}_{\vec{p};\vec{\tau}} + \frac{1}{N^4}L^{(2)}_{\vec{p};\vec{\tau}}\right)\mathbb{L}_{\vec{p};\vec{\tau}} + \dots \tag{6.3}$$

$$+\log^1(u)\sum_{\vec{\tau}}\left(\frac{1}{N^2}M^{(1)}_{\vec{p};\vec{\tau}} + \frac{1}{N^4}M^{(2)}_{\vec{p};\vec{\tau}}\right)\mathbb{L}_{\vec{p};\vec{\tau}} + \dots \tag{6.4}$$

$$+\log^2(u)\sum_{\vec{\tau}}\left(\frac{1}{N^4}N^{(2)}_{\vec{p};\vec{\tau}}\right)\mathbb{L}_{\vec{p};\vec{\tau}}\,, \tag{6.5}$$

where the ellipses stand for omitted terms with τ-derivatives of the blocks, which are not important for our purpose here. In the above equations we clustered together various contributions within each $\log(u)$ stratum, and we did not specify the ranges of summation. In fact, understanding the precise ranges of summation for different contributions needs extra explanations, which we will provide shortly.

First however, note that the $\log(u)$ and $\log^2(u)$ terms receive contributions only from the dynamical function $\mathcal{H}_{\vec{p}}$, whereas the non-log projection (6.3) is subject to non-trivial interplay between the free theory and the dynamical part, since beyond leading order both contribute to the $1/N$ expansion:

$$\begin{aligned}L^{(0)}_{\vec{p};\vec{\tau}} &= L^{f(0)}_{\vec{p};\vec{\tau}}\,,\\ L^{(i)}_{\vec{p};\vec{\tau}} &= L^{f(i)}_{\vec{p};\vec{\tau}} + L^{\mathcal{H}(i)}_{\vec{p};\vec{\tau}}\,, \qquad \text{for } i\geq 1.\end{aligned} \tag{6.6}$$

The SCPW coefficients in equations (6.3)-(6.5) are predicted by the OPE, and in particular they depend on the spectrum of exchanged operators. As discussed in detail in

[1] For correlators of the form $\langle 22pp\rangle$, we will however still use $\mathfrak{a}$ as the expansion parameter, since with this choice the free theory correlator contributes only to the first two orders in $\mathfrak{a}$, i.e. $\langle 22pp\rangle^{(2)}_{\text{free long}} = 0$.

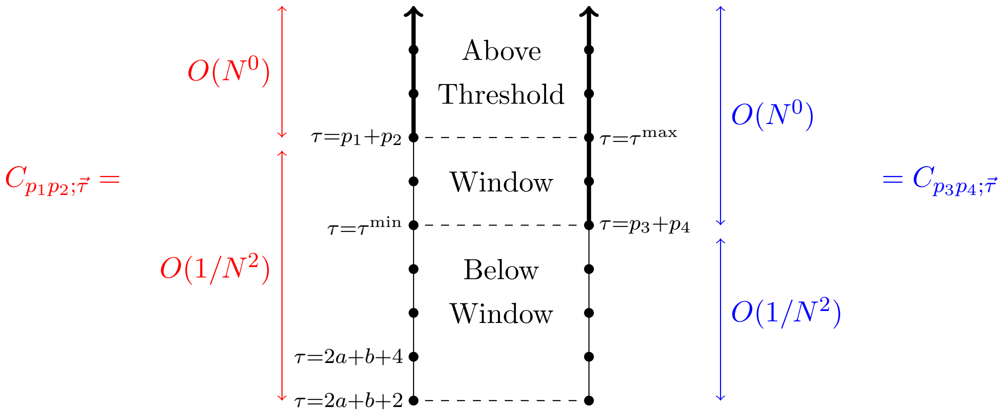


Figure 6.1: The large N structure of $C_{p_1p_2;\vec{\tau}}\, C_{p_3p_4;\vec{\tau}}$ for two-particle operators $\mathcal{K}_{pq}$ in an $su(4)$ representation $[a,b,a]$ and varying twist τ.

previous chapters, the supergravity spectrum consists of the single-particle states $\mathcal{O}_p$ and multi-particle states built from products of the single-particle ones. Importantly, multi-particle operators corresponding to bound states of more than two particles do not have leading order three-point functions with the external operators, and therefore do not appear in the OPE at leading order. In other words, only the three-point couplings $C_{p_ip_j\mathcal{K}_{pq}}$, where $\mathcal{K}_{pq}$ denotes the set of true two-particle scaling eigenstates introduced in Section 3.3.2, have a leading contribution in the large N expansion. Since the operators $\mathcal{K}_{pq}$ with leading order quantum numbers $\vec{\tau}_0 = (\tau_0, \ell, [a,b,a])$ are mixtures containing some contribution from every double-trace operator $\mathcal{O}_{pq;\vec{\tau}}$, they will have leading order three-point couplings at twist $\tau \geq p_i + p_j$. On the other hand, exchanged two-particle operators with twist in the range $2a+b+2 \leq \tau < p_i + p_j$ do not receive any contribution of the form $\mathcal{O}_{p_ip_j;\vec{\tau}}$, and therefore necessarily have $1/N^2$ suppressed three-point couplings, i.e.

$$C_{p_ip_j\mathcal{K}_{pq}} = C^{(0)}_{p_ip_j\mathcal{K}_{pq}} + \frac{1}{N^2} C^{(1)}_{p_ip_j\mathcal{K}_{pq}} + \dots, \tag{6.7}$$

with $C^{(0)}_{p_ip_j\mathcal{K}_{pq}} \neq 0$ only for $\tau \geq p_i + p_j$. Note again that we have disregarded any $1/\lambda$ terms in the above expansion.

The exchange of two-particle operators in the common OPE of a four-point correlator $\langle p_1p_2p_3p_4 \rangle$ gives a contribution of the form $C_{p_1p_2;\vec{\tau}} \times C_{p_3p_4;\vec{\tau}}$, for different values of the quantum numbers $\vec{\tau}$. Putting together two of these, we obtain a representation of the common OPE coefficient as in Figure 6.1, leading to the following three regions:

- Above Threshold: for $\tau \geq \tau^{\max} \equiv \max\{p_1+p_2, p_3+p_4\}$, we find exchanged operators for which both three-point couplings are leading order, i.e. $C^{(0)}_{p_1p_2}$ and $C^{(0)}_{p_3p_4}$ are both non-zero. In particular, $\tau^{\max}$ is the threshold twist for the exchange of long two-particle operators in disconnected free theory, giving rise to $L^{(0)}_{\vec{p};\vec{\tau}}$ in equation (6.3).

- Window: in the window region $\tau^{\max} > \tau \geq \tau^{\min} \equiv \min\{p_1+p_2, p_3+p_4\}$, we find exchanged operators which have leading order three-point couplings with one pair of external operators, but $1/N^2$ suppressed three-point couplings with the other pair of external operators, e.g. we have $C^{(0)}_{p_1p_2} = 0$ but $C^{(0)}_{p_3p_4}$ non-zero.
- Below Window: in the below-window region $\tau < \tau^{\min}$ we have $C^{(0)}_{p_1p_2} = C^{(0)}_{p_3p_4} = 0$ and the OPE contains contributions which only involve products of $1/N^2$ suppressed three-point couplings. These contributions give rise to a genuine $1/N^4$ effect which enters the SCPW coefficients $L^{(2)}_{\vec{p};\vec{\tau}}$ in equation (6.3).

Note that for any arrangement of external charges, there is always a threshold twist such that a tower of long operators is exchanged, whereas the window itself might be empty (if $\tau^{\min} = \tau^{\max}$). Similarly, the location of the unitarity bound in Figure 6.1 depends on the external charges. Generically, the unitarity bound $\tau = 2a+b+2$ is below the window, but there are two other situations which do occur. Firstly, the unitarity bound can coincide with $\tau^{\min}$, i.e $\tau^{\min} = 2a+b+2$, in which case there is no below window region. Secondly, the unitarity bound can coincide with $\tau^{\max}$, in which case there is an empty window and $\tau^{\max} = \tau^{\min} = 2a+b+2$.

With the above discussion in mind, let us revisit the $\log(u)$-stratification from equations (6.3)-(6.5). The OPE predicts the following form of the SCPW coefficients for the indicated ranges of the twist of exchanged operators:

$$\tau \geq \tau^{\max}: \quad L^{(0)}_{\vec{p};\vec{\tau}} = \sum_{(pq)\in R_{\vec{\tau}}} C^{(0)}_{p_1p_2\mathcal{K}_{pq}} C^{(0)}_{p_3p_4\mathcal{K}_{pq}}, \tag{6.8}$$

$$M^{(1)}_{\vec{p};\vec{\tau}} = \tfrac{1}{2} \sum_{(pq)\in R_{\vec{\tau}}} C^{(0)}_{p_1p_2\mathcal{K}_{pq}} \eta_{\mathcal{K}_{pq}} C^{(0)}_{p_3p_4\mathcal{K}_{pq}}, \tag{6.9}$$

$$N^{(2)}_{\vec{p};\vec{\tau}} = \tfrac{1}{8} \sum_{(pq)\in R_{\vec{\tau}}} C^{(0)}_{p_1p_2\mathcal{K}_{pq}} \eta^2_{\mathcal{K}_{pq}} C^{(0)}_{p_3p_4\mathcal{K}_{pq}}, \tag{6.10}$$

$$\tau^{\max} > \tau \geq \tau^{\min}: \quad L^{(1)}_{\vec{p};\vec{\tau}} = \sum_{(pq)\in R_{\vec{\tau}}} C^{(1)}_{p_1p_2\mathcal{K}_{pq}} C^{(0)}_{p_3p_4\mathcal{K}_{pq}} + C^{(0)}_{p_1p_2\mathcal{K}_{pq}} C^{(1)}_{p_3p_4\mathcal{K}_{pq}}, \tag{6.11}$$

$$M^{(2)}_{\vec{p};\vec{\tau}} = \tfrac{1}{2} \sum_{(pq)\in R_{\vec{\tau}}} C^{(1)}_{p_1p_2\mathcal{K}_{pq}} \eta_{\mathcal{K}_{pq}} C^{(0)}_{p_3p_4\mathcal{K}_{pq}} + C^{(0)}_{p_1p_2\mathcal{K}_{pq}} \eta_{\mathcal{K}_{pq}} C^{(1)}_{p_3p_4\mathcal{K}_{pq}}, \tag{6.12}$$

$$\tau < \tau^{\min}: \quad L^{(2)}_{\vec{p};\vec{\tau}} = \sum_{(pq)\in R_{\vec{\tau}}} C^{(1)}_{p_1p_2\mathcal{K}_{pq}} C^{(1)}_{p_3p_4\mathcal{K}_{pq}}, \tag{6.13}$$

where the set of exchanged double-trace operators $R_{\vec{\tau}}$ has been defined in (3.18), and $\eta_{\mathcal{K}_{pq}} = 2\eta^{(1,0)}_{pq}$ is the supergravity anomalous dimension derived in Chapter 4, see (4.53).

Our next task is to leverage data coming from the known tree-level four-point functions, specifically the SCPW coefficients $L^{(0)}, L^{(1)}$ and $M^{(1)}$, in order to obtain information about the one-loop four-point function, in particular the entire double-log discontinuity,

$N^{(2)}$, but also pieces of the single-log part $M^{(2)}$ and analytic part $L^{(2)}$.[2] In order to have better control over the various phenomena taking place in equations (6.8)-(6.13), let us recall the three regions shown pictorially in Figure 6.1:

- Above Threshold: the leading-log SCPW coefficients $L^{(0)}$, $M^{(1)}$ and $N^{(2)}$ have contributions above the threshold only. Similarly to the unmixing of the supergravity anomalous dimensions discussed in Chapter 4, we will use the knowledge of $L^{(0)}$ and $M^{(1)}$ from many different correlators to bootstrap the double-log contributions $N^{(2)}$. However, it is important to point out that for general correlators the knowledge of the double-log alone is not sufficient to fully determine the one-loop correlator consistently, since there are non-trivial parts of the correlator which will have to be determined by data below the threshold.

- Window: the significance of this region is that the leading order three-point functions are vanishing on one side, but not on the other. This allows us to use the tree-level SCPW coefficients $L^{(1)}$ to predict a part of the single-log contributions $M^{(2)}$.

- Below Window: in this range of twists one can predict a piece of the analytic or non-log(u) contribution to the one-loop correlator. Again, we will use lower order data, specifically results from $L^{(1)}$ from many correlators, to determine the below window contributions $L^{(2)}$.

The precise details on how to obtain these predictions will be described in the following.

6.2 Predicting the One-Loop Double-Log

Let us begin with the $\log^2(u)$ discontinuity. Because of the mixing of exchanged double-trace operators we will again need data from many different correlators, whose SCPW coefficients we will conveniently package into matrices. To this end, we will adopt a similar matrix notation as in Section 4.4, where hatted quantities stand for matrices of SCPW coefficients from correlators $\langle p_1 p_2 p_3 p_4 \rangle$ with (p_1, p_2) and $(p_3, p_4) \in R_{\vec{\tau}}$. Recall that the set of exchanged operators $\mathcal{K}_{pq}$ is parametrised by pairs (p, q) in the set $R_{\vec{\tau}}$, defined previously in (4.46), and its dimension is $d = \mu(t-1)$. Promoting equation (6.8) into matrix notation, we have

$$\widehat{L}^{(0)}_{\vec{\tau}} = \mathbb{C}^{(0)}_{\vec{\tau}} \left(\mathbb{C}^{(0)}_{\vec{\tau}}\right)^T, \tag{6.14}$$

[2]Note that correlators with general external charges will generically get contributions from $su(4)$ channels $[a, b, a]$ with $b \geq 2$, where the supergravity anomalous dimensions $\eta_{\mathcal{K}_{pq}}$ exhibit a residual degeneracy. This degeneracy means it is not possible to fully unmix and determine all leading order three-point functions in those channels. Nevertheless, it is possible to overcome this problem and we will discuss how to bootstrap one-loop data from tree-level correlators even in $su(4)$ channels with a remaining partial degeneracy.

where $\mathbb{C}^{(0)}_{\vec{\tau}}$ is the $d \times d$ matrix of leading order three-point functions $C^{(0)}_{p_1 p_2 \mathcal{K}_{p_3 p_4}}$ with both (p_1, p_2) and $(p_3, p_4) \in R_{\vec{\tau}}$. Furthermore, the matrix of SCPW coefficients $\widehat{L}^{(0)}_{\vec{\tau}}$ is diagonal and explicitly given by formula (4.49). Similarly, converting equation (6.9) into matrix notation yields

$$\widehat{M}^{(1)}_{\vec{\tau}} = \tfrac{1}{2}\, \mathbb{C}^{(0)}_{\vec{\tau}}\ \widehat{\eta}_{\vec{\tau}}\ \left(\mathbb{C}^{(0)}_{\vec{\tau}}\right)^T, \tag{6.15}$$

with $\widehat{\eta}_{\vec{\tau}}$ being the (diagonal) matrix of supergravity anomalous dimensions. Using this matrix form it is now straightforward to see that the SCPW coefficients contributing to the $\log^2(u)$ discontinuity at one-loop are given by

$$\begin{aligned} \widehat{N}^{(2)}_{\vec{\tau}} &= \tfrac{1}{8}\, \mathbb{C}^{(0)}_{\vec{\tau}}\ \widehat{\eta}^{2}_{\vec{\tau}}\ \left(\mathbb{C}^{(0)}_{\vec{\tau}}\right)^T \\ &= \tfrac{1}{2}\, \widehat{M}^{(1)}_{\vec{\tau}}\ \left(\widehat{L}^{(0)}_{\vec{\tau}}\right)^{-1}\ \widehat{M}^{(1)}_{\vec{\tau}}, \end{aligned} \tag{6.16}$$

where the second equality follows from equations (6.14) and (6.15), allowing us to predict the one-loop leading-log coefficients $N^{(2)}_{\vec{\tau}}$ from tree-level data only. Note that the above formula does not require us to explicitly find the anomalous dimensions $\widehat{\eta}_{\vec{\tau}}$ nor the three-point functions $\mathbb{C}^{(0)}_{\vec{\tau}}$ themselves. This fact is important as it allows us to deal with $su(4)$ channels in which the anomalous dimensions remain partially degenerate, in which case the matrices $\mathbb{C}^{(0)}_{\vec{\tau}}$ are not fully determined.

Once the coefficient matrices $\widehat{N}^{(2)}_{\vec{\tau}}$ are assembled using formula (6.16), we can simply extract the entry corresponding to the $\langle p_1 p_2 p_3 p_4 \rangle$ correlator and obtain the full one-loop double-log discontinuity by performing the sum

$$\mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^2(u)} = \sum_{\vec{\tau}}\ N^{(2)}_{\vec{p};\vec{\tau}}\ \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}}, \tag{6.17}$$

with the sum restricted to twists $\tau \geq \tau^{\max}$, and we recall that the long blocks take the form $\mathbb{L}_{\vec{p};\vec{\tau}} = \mathcal{P}_{\vec{p}} \times \mathcal{I} \times \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}}$, see (2.33). By explicit computation of (6.17) to high order in the twist we obtained the resummation of the leading-log discontinuity in a number of cases. It turns out that, as a function of the external charges, it always has the structure

$$\begin{aligned} \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^2(u)} = &\frac{P_{2,1}(x,\bar{x};\sigma,\tau)}{(x-\bar{x})^{d_{\vec{p}}+8}}\big[\mathrm{Li}_2(x) - \mathrm{Li}_2(\bar{x})\big] + \frac{P_{1,2}(x,x;\sigma,\tau)}{(x-\bar{x})^{d_{\vec{p}}+8}}\big[\mathrm{Li}_1^2(x) - \mathrm{Li}_1^2(\bar{x})\big] \\ &+ \frac{P_{1,-}(x,\bar{x};\sigma,\tau)}{(x-\bar{x})^{d_{\vec{p}}+8}}\big[\mathrm{Li}_1(x) - \mathrm{Li}_1(\bar{x})\big] + \frac{P_{1,+}(x,\bar{x};\sigma,\tau)}{(x-\bar{x})^{d_{\vec{p}}+7}}\log(v) \\ &+ \frac{P_0(x,\bar{x};\sigma,\tau)}{(x-\bar{x})^{d_{\vec{p}}+7}}\frac{1}{v^{\kappa_{\vec{p}}-2}}, \end{aligned} \tag{6.18}$$

where $\kappa_{\vec{p}}$ is the degree of extremality defined in equation (3.13), $d_{\vec{p}}$ is given by

$$d_{\vec{p}} \equiv p_1 + p_2 + p_3 + p_4 - 1, \tag{6.19}$$

and the coefficient functions P in the above are certain polynomials depending implicitly on the set of external charges $\vec{p}$. These polynomials are obtained by matching the series expansion in small x and $\bar{x}$ of (6.18) against the sum over long blocks in formula (6.17), where each conformal block (of twist τ and spin ℓ) has a series expansion of the form $u^\tau(1-v)^\ell f(x,\bar{x})$, with $f(x,\bar{x})$ being a symmetric function. We call an expression of the form (6.18) a *two-variable resummation.* Since the $\log^2(u)$ discontinuity only receives contributions from long operators above the threshold twist $\tau \geq \tau^{\max}$, we expect $\mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^2(u)}$ to go like $u^{(\tau^{\max}-p_{43})/2}$ for small u.

Finally, let us mention that there is another way to directly obtain the double-log discontinuity of $\mathcal{H}^{(2)}_{\vec{p}}$, without the need to perform the two-variable resummation described above. In fact, by exploiting the hidden ten-dimensional symmetry of [112] one can find an explicit formula for the one-loop leading discontinuities by making use of the differential operators $\widehat{\mathcal{D}}_{\vec{p}}$. We have checked for many cases that the results obtained from these two different methods agree. More details and explicit formulae for this very neat alternative way can be found in e.g. [6, 112].

6.3 The $\langle 2222 \rangle$ Correlator

Before continuing the general discussion of the window and below window contributions, let us consider a first explicit example. In this section, we will consider the $\langle 2222\rangle$ correlator, which is of particular physical significance and is the simplest correlator to study. It has degree of extremality $\kappa_{2222}=2$, and as such its interacting part $\mathcal{H}_{2222}$ has only one $su(4)$ channel, namely the singlet channel. As a consequence, $\mathcal{H}_{2222}$ does not depend on the internal $su(4)$ cross-ratios, and furthermore the full crossing symmetry of the correlator implies the relations

$$\mathcal{H}_{2222}(u,v) = \frac{1}{v^2}\,\mathcal{H}_{2222}(u/v,1/v) = \frac{u^2}{v^2}\,\mathcal{H}_{2222}(v,u). \tag{6.20}$$

In the following, we will discuss how the knowledge of the $\log^2(u)$ part together with constraints from the above crossing relations and the absence of unphysical poles allows us to determine the one-loop contribution $\mathcal{H}^{(2)}_{2222}$.

6.3.1 Resummation of the $\log^2(u)$ Discontinuity

By explicitly assembling the SCPW coefficients of the $\log^2(u)$ discontinuity according to equation (6.16) and performing the two-variable resummation of the sum (6.17), we obtain

$$\begin{aligned}\mathcal{H}^{(2)}_{2222}\big|_{\log^2(u)} =& \frac{u}{v}\Big[p(u,v)\frac{\mathrm{Li}_1(x)^2-\mathrm{Li}_1(\bar{x})^2}{x-\bar{x}} + 2\Big[p(u,v)+p\Big(\frac{1}{v},\frac{u}{v}\Big)\Big]\frac{\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x})}{x-\bar{x}}\\ &+ q(u,v)(\mathrm{Li}_1(x)+\mathrm{Li}_1(\bar{x})) + r(u,v)\frac{\mathrm{Li}_1(x)-\mathrm{Li}_1(\bar{x})}{x-\bar{x}} + s(u,v)\Big]\,. \end{aligned}\tag{6.21}$$

where p, q, r, s are rational functions of u, v with denominator $(x-\bar{x})^{14}$. Note that the above expression agrees with the general form given in equation (6.18). This double discontinuity was also obtained in references [130, 131].

The coefficient function p is symmetric $p(u,v) = p(v,u)$ as required by crossing since the double discontinuity in both u and v comes only from the first term in (6.21) which contributes $p(u,v)\log^2 u \log^2 v$ and hence must be symmetric in u and v. As we will see, the fact that the coefficient of the Li_2 term is related simply to the same function $p(u,v)$ is a hint at an additional simplicity in the final amplitude. It is possible to write the coefficient $p(u,v)$ in quite a simple form,

$$p(u,v) = 96uv\partial_x^2\partial_{\bar{x}}^2\left[\frac{u^2v^2(1-u-v)[(1-u-v)^4+20uv(1-u-v)^2+30u^2v^2]}{(x-\bar{x})^{10}}\right], \tag{6.22}$$

whereas the other coefficients are more complicated and we will not give their explicit expressions. Instead, we will proceed to construct a fully crossing symmetric function $\mathcal{H}^{(2)}_{2222}(u,v)$ with the correct $\log^2(u)$ discontinuity. The remaining coefficients in (6.21) can then be obtained from the full function by taking the double discontinuity.

6.3.2 Completion to a Crossing Symmetric Amplitude

Having obtained the double discontinuity from resumming the OPE, we make an ansatz for the form of the full crossing invariant contribution to supergravity at one loop. In order to construct a suitable ansatz we note that the tree-level supergravity function $\mathcal{H}^{(1)}_{2222}(u,v)$ is expressible in terms of a $\overline{D}$-function, which itself is a particular combination of derivatives acting on the one-loop box function $\Phi^{(1)}(u,v)$. This means that it is expressible as a combination of single-valued polylogarithms of weights 2, 1 and 0 with rational functions of x and $\bar{x}$ as coefficients. The particular class of single-valued polylogarithms of interest here are linear combinations of polylogarithms constructed on the singularities (or 'letters') $\{x, 1-x, \bar{x}, 1-\bar{x}\}$ such that they are single-valued when $\bar{x}$ is taken to be the complex conjugate of x. They are constructed in general in [132] and appear in many contexts, such as in perturbative contributions to the correlation functions $\langle p_1p_2p_3p_4\rangle$ [33, 35], in multi-Regge kinematics of scattering amplitudes [133, 134] as well as Feynman integral calculations [135, 136].

Since our result for the double discontinuity $\mathcal{H}^{(2)}_{2222}\big|_{\log^2(u)}$ is expressible in terms of logarithms and dilogarithms, it seems a natural choice to construct an ansatz for the full function $\mathcal{H}^{(2)}_{2222}(u,v)$ from the same class of single-valued polylogarithms, but this time of weights 4, 3, 2, 1 and 0 with rational functions as their coefficients. We then impose crossing symmetry and the fact that the double discontinuity must match our result given in equation (6.21).

The constraints described in the previous paragraph fix completely the weight 4 and weight 3 parts of the result with rational coefficients which are determined by the coefficients appearing in $\mathcal{H}^{(2)}_{2222}\big|_{\log^2(u)}$. On the other hand, the weight 2, 1 and 0 parts are not fixed completely by matching to the double discontinuity. Since the double discontinuity has a total of 15 powers of $(x-\bar{x})$ in the denominator, so do the rational coefficients in the weight 4 and weight 3 parts. This leaves the possibility that the resulting function has unphysical poles at $x=\bar{x}$. In order to make sure that poles at $x=\bar{x}$ are in fact absent, we have to arrange the weight 2, 1 and 0 parts so that they cancel those of the weight 4 and weight 3 pieces. We then allow a maximum of 15 powers of $(x-\bar{x})$ in the denominators of the coefficients of the weight 2, 1 and 0 parts of the ansatz to match the denominators in the weight 4 and weight 3 parts and demand that all poles at $x=\bar{x}$ cancel. We also demand that the twist-two sector is completely absent from $\mathcal{H}^{(2)}_{2222}(u,v)$.[3] These constraints completely fix the answer within our ansatz up to a single free coefficient.

We find that we can express the final crossing symmetric result in terms of the so-called ladder integrals [137, 138]. These are a particular subset of the single-valued polylogarithms under consideration here. They are given by

$$\Phi^{(l)}(u,v) = -\frac{1}{x-\bar{x}}\phi^{(l)}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right), \tag{6.23}$$

where

$$\phi^{(l)}(x,\bar{x}) = \sum_{r=0}^{l}(-1)^r\frac{(2l-r)!}{r!(l-r)!l!}\log^r(x\bar{x})\left(\mathrm{Li}_{2l-r}(x)-\mathrm{Li}_{2l-r}(\bar{x})\right). \tag{6.24}$$

The functions $\Phi^{(l)}$ obey the symmetry

$$\Phi^{(l)}(u,v) = \Phi^{(l)}(v,u), \tag{6.25}$$

while $\Phi^{(1)}$ also obeys

$$\frac{1}{u}\Phi^{(1)}\left(\frac{1}{u},\frac{v}{u}\right) = \Phi^{(1)}(u,v). \tag{6.26}$$

Our final result for the one-loop correction contains a single unfixed parameter within the ansatz outlined above. We first quote a particular solution where we set the free parameter α to zero, and we will come back to the ambiguity later. Our particular solution is given by the crossing symmetric combination

$$\mathcal{H}^{(2)}_{2222}(u,v) = \frac{u}{v}\left[f(u,v)+\frac{1}{u}f\left(\frac{1}{u},\frac{v}{u}\right)+\frac{1}{v}f\left(\frac{1}{v},\frac{u}{v}\right)\right]. \tag{6.27}$$

[3] Recall that the twist-two long operators are absent from the supergravity spectrum and the cancellation of such contributions between $\langle 2222\rangle^{(1)}_{\text{free}}$ and $\langle 2222\rangle^{(1)}_{\text{int}}$ is complete due to the choice of $\mathfrak{a}=1/(N^2-1)$ as our expansion parameter. Therefore there should be no twist-two contributions in $\mathcal{H}^{(2)}_{2222}$.

To simplify the presentation of the function $f(u,v)$ we write

$$f(u,v) = \Delta^{(4)} g(u,v)\,, \qquad \Delta^{(4)} = (x-\bar{x})^{-1} uv \partial_x^2 \partial_{\bar{x}}^2 (x-\bar{x})\,. \tag{6.28}$$

Furthermore we can decompose the function g into pieces according to the transcendental weight of the polylogarithmic contributions

$$g = (x-\bar{x})^{-10}[g^{(4)} + g^{(3)} + g^{(2)} + g^{(1)} + g^{(0)}]\,. \tag{6.29}$$

The pieces of given weight are then as follows,

$$\begin{aligned}
g^{(4)}(u,v) &= P_-^{(4)}(u,v)\Phi^{(2)}(u,v),\\
g^{(3)}(u,v) &= P_+^{(3)}(u,v)\Psi(u,v) + P_-^{(3)}(u,v)\log(uv)\Phi^{(1)}(u,v),\\
g^{(2)}(u,v) &= P_+^{(2)}(u,v)\log u \log v + P_-^{(2)}(u,v)\Phi^{(1)}(u,v),\\
g^{(1)}(u,v) &= P_+^{(1)}(u,v)\log(uv),\\
g^{(0)}(u,v) &= P_+^{(0)}(u,v),
\end{aligned} \tag{6.30}$$

where the function $\Psi(u,v)$ is a particular derivative of the two-loop ladder integral:

$$\begin{aligned}
\Psi(u,v) &= (x-\bar{x})(u\partial_u + v\partial_v)\big[(x-\bar{x})\Phi^{(2)}(u,v)\big]\\
&= [x(1-x)\partial_x - \bar{x}(1-\bar{x})\partial_{\bar{x}}]\,\phi^{(2)}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right).
\end{aligned} \tag{6.31}$$

The coefficients $P_\pm^{(r)}(u,v)$ in (6.30) are symmetric polynomials in u and v. The subscripts $\pm$ correspond to the symmetry properties under $x \leftrightarrow \bar{x}$ of the pure transcendental factor that each coefficient $P^{(r)}$ multiplies (antisymmetric for the ladder functions and symmetric for constants, for logarithms of u and v and for $\Psi(u,v)$). Note that the weight four piece is entirely expressible in terms of $\Phi^{(2)}(u,v)$, whose transcendental part is antisymmetric in x and $\bar{x}$. In principle there could have been a symmetric part, e.g. $\Phi^{(1)}(u,v)^2$, but in fact our function does not have such a contribution. The fact that the weight four piece is given by $\Phi^{(2)}(u,v)$ only implies the relationship between the coefficients of the Li_2 terms and the Li_1^2 terms in the double discontinuity (6.21).

To express the coefficient polynomials it is helpful to introduce symmetric variables

$$\bar{s} = 1-u-v, \qquad p = uv. \tag{6.32}$$

The coefficient polynomials are then given by

$$P_-^{(4)}(u,v) = 384p^2\bar{s}[\bar{s}^4 + 20p\bar{s}^2 + 30p^2]\,, \tag{6.33}$$

$$P_+^{(3)}(u,v) = \frac{32}{5}p^2[137\bar{s}^4 + 1214p\bar{s}^2 + 512p^2]\,, \tag{6.34}$$

$$P_-^{(3)}(u,v) = 1344p^2[\bar{s}(1-\bar{s})(6-6\bar{s}+\bar{s}^2) + 2p(3-14\bar{s}+4\bar{s}^2) - 16p^2]\,, \tag{6.35}$$

$$\begin{aligned}P_+^{(2)}(u,v) = 8[&(1-\bar{s})^2\bar{s}^6 - 2p\bar{s}^4(20-33\bar{s}+14\bar{s}^2)\\ &+8p^2(756-1323\bar{s}+601\bar{s}^2-54\bar{s}^3+30\bar{s}^4)\\ &-32p^3(583-25\bar{s}+26\bar{s}^2)+1024p^4]\,,\end{aligned} \tag{6.36}$$

$$\begin{aligned}P_-^{(2)}(u,v) = 224p^2[&-\bar{s}^2(2-\bar{s})(18-18\bar{s}+5\bar{s}^2)\\ &+2p(108-144\bar{s}+128\bar{s}^2-11\bar{s}^3)-8p^2(63-\bar{s})]\,,\end{aligned} \tag{6.37}$$

$$\begin{aligned}P_+^{(1)}(u,v) = \frac{4}{3}[&5\bar{s}^7(2-3\bar{s})-2p\bar{s}^5(158-193\bar{s})\\ &+16p^2\bar{s}(378-567\bar{s}+233\bar{s}^2-147\bar{s}^3)\\ &+32p^3(378-139\bar{s}+129\bar{s}^2)+256p^4]\,,\end{aligned} \tag{6.38}$$

$$\begin{aligned}P_+^{(0)}(u,v) = \frac{8}{15}(x-\bar{x})^2[&20(1-\bar{s})\bar{s}^6-5p\bar{s}^4(102-75\bar{s}-4\bar{s}^2)\\ &+8p^2(630-630\bar{s}+481\bar{s}^2-255\bar{s}^3-30\bar{s}^4)\\ &-16p^3(217-215\bar{s}-60\bar{s}^2)-1280p^4]\,.\end{aligned} \tag{6.39}$$

The terms involving $P_-^{(4)}, P_\pm^{(3)}, P_+^{(2)}$ contribute to the double discontinuity and therefore the coefficients are related to those appearing in (6.21). In particular we have

$$p(u,v) = \frac{1}{4}(x-\bar{x})\Delta^{(4)}\left[\frac{P_-^{(4)}(u,v)}{(x-\bar{x})^{11}}\right]. \tag{6.40}$$

The ambiguity in the result is much simpler. In fact all terms proportional to the single free parameter α can be expressed in a similar way to the tree-level amplitude,

$$\alpha\,\frac{u}{v}[(1+u\partial_u+v\partial_v)u\partial_u v\partial_v]^2\Phi^{(1)}(u,v)\,. \tag{6.41}$$

At this stage our solution is given by the particular solution $\mathcal{H}^{(2)}_{2222}(u,v)$, as described in equations (6.27)-(6.39), plus the ambiguity in equation (6.41) above. When written out in terms of single-valued polylogarithms with rational coefficients, the above ambiguity has 13 powers of $(x-\bar{x})$ in the denominator. In terms of $\overline{D}$-functions it can be expressed as $u^4\overline{D}_{4444}$. Note that the ambiguity (6.41) has no double discontinuity, has no unphysical poles, is fully crossing symmetric and has no twist-two contribution. As such, our bootstrap approach is not able to fix its coefficient α and we need to rely on different methods to determine it. One such method is given by supersymmetric localisation, which (in the conventions used here) determines the value of α to be [104]

$$\alpha = 60. \tag{6.42}$$

In principle there are further ambiguities we could add within the class of single-valued polylogarithms multiplied by rational functions. However, these all have higher powers of $(x-\bar{x})$ in the denominator than the 15 we allowed above and they correspond to crossing symmetric $\overline{D}$-functions with higher weights. Indeed such functions have arisen in the context of tree-level string corrections, see e.g. [5, 77, 100, 120].

We should sound a note of caution that what we have presented is not strictly a derivation of the one-loop correction. It is possible that the true answer differs from the expression we have constructed above by a function that itself has no double discontinuity, no unphysical poles, no twist-two sector and is fully crossing symmetric on its own. Finally, it is also possible that there are functions which do not sit in the class of single-valued polylogarithms that we have allowed. However, it is highly non-trivial that we are able to find a solution, unique up to a single free parameter within the simplest class of functions we are led to consider, and we take this as a very strong encouragement that our amplitude is in fact correct.

6.4 The $\langle 2233 \rangle$ Correlator

The next simplest case to consider is the one-loop correction to the $\langle 2233 \rangle$ correlator. As before, its degree of extremality is 2 and there is only a single $su(4)$ channel to consider. On the other hand, this correlator does not have full crossing symmetry as in the previous case, and we will have to consider its two orientations $\langle 2233 \rangle$ and $\langle 2323 \rangle$. The remaining crossing symmetries read

$$\mathcal{H}_{2233}(u,v) = \frac{1}{v^2}\,\mathcal{H}_{2233}(u/v,1/v), \qquad \mathcal{H}_{2323}(u,v) = \frac{u^2}{v^2}\,\mathcal{H}_{2323}(v,u), \tag{6.43}$$

and the two orientations are related via

$$\mathcal{H}_{2233}(u,v) = \mathcal{H}_{2323}(1/u,v/u). \tag{6.44}$$

Furthermore, the $\langle 2233 \rangle$ correlator is the first case featuring a non-trivial window contribution (at twist $\tau = 4$), which allows us to show an example where additional tree-level information is necessary to fix the one-loop correlator.

We begin again by performing the two-variable resummation of the sum (6.17), obtaining the $\log^2(u)$ parts for both of the two orientations. They take the form,

$$\begin{aligned}\mathcal{H}^{(2)}_{2233}\big|_{\log^2(u)} = \hat{P}(u,v)\frac{\mathrm{Li}_1(x)^2-\mathrm{Li}_1(\bar{x})^2}{x-\bar{x}} + 2\left[\frac{1}{u^2v}\hat{P}\left(\frac{u}{v},\frac{1}{v}\right)+\hat{P}(u,v)\right]\frac{\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x})}{x-\bar{x}} \\ + \hat{Q}(u,v)\big(\mathrm{Li}_1(x)+\mathrm{Li}_1(\bar{x})\big) + \hat{R}(u,v)\frac{\mathrm{Li}_1(x)-\mathrm{Li}_1(\bar{x})}{x-\bar{x}} + \hat{S}(u,v),\end{aligned} \tag{6.45}$$

and

$$\begin{aligned}\mathcal{H}^{(2)}_{2323}\big|_{\log^2(u)} = P(u,v)\frac{\mathrm{Li}_1(x)^2-\mathrm{Li}_1(\bar{x})^2}{x-\bar{x}} + 2\left[\frac{1}{u^2v}\hat{P}\left(\frac{1}{v},\frac{u}{v}\right)+P(u,v)\right]\frac{\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x})}{x-\bar{x}} \\ + Q(u,v)\big(\mathrm{Li}_1(x)+\mathrm{Li}_1(\bar{x})\big) + R(u,v)\frac{\mathrm{Li}_1(x)-\mathrm{Li}_1(\bar{x})}{x-\bar{x}} + S(u,v),\end{aligned} \tag{6.46}$$

where the coefficient functions P, Q, R, S and similarly the hatted quantities are rational functions of x and $\bar{x}$ with denominators of the form $(x-\bar{x})^{16}$, and are symmetric under $x \leftrightarrow \bar{x}$. Note that the symmetry of the full correlation function, $\mathcal{H}_{2323}(u,v) = u^2/v^2\, \mathcal{H}^{(2)}_{2323}(v,u)$, is visible in the double discontinuity $\mathcal{H}^{(2)}_{2323}\big|_{\log^2(u)}$ for the term proportional to $\log^2(u)\log^2(v)$. Indeed we can verify that $P(u,v) = u^2/v^2\, P(v,u)$. On the other hand, we are able to express the coefficient function of Li_2 in terms of $\hat{P}(u,v)$ and $P(1/v,u/v)$. This non-trivial fact will be important in the next step, when we uplift the double discontinuity to the full correlation function.

6.4.1 Uplifting to the Full Function

The structure of the double discontinuities (6.45) and (6.46) is very similar to the double discontinuity for the $\langle 2222\rangle$ correlator discussed before. In fact, they again follow the general structure (6.18). This suggests that the transcendental functions appearing in the full one-loop contributions of $\langle 2233\rangle$ and $\langle 2323\rangle$ will also be given by the same one-loop and two-loop ladder functions which arise in the case of $\langle 2222\rangle$, i.e. $\phi^{(1)}(x,\bar{x})$ and $\phi^{(2)}(x,\bar{x})$ as given in (6.24), with symmetry properties

$$\phi^{(l)}\left(\frac{1}{x},\frac{1}{\bar{x}}\right) = -\phi^{(l)}(x,\bar{x}), \tag{6.47}$$

while the one-loop function also obeys

$$\phi^{(1)}(1-x,1-\bar{x}) = -\phi^{(1)}(x,\bar{x}). \tag{6.48}$$

We proceed very much as in the previous case: we make an ansatz for $\mathcal{H}^{(2)}_{2233}(u,v)$ (or equivalently $\mathcal{H}^{(2)}_{2323}(u,v)$) in terms of single-valued harmonic polylogarithms with coefficients wich are rational functions of x and $\bar{x}$ with denominators of the form $(x-\bar{x})^{17}$, to match the double discontinuities (6.45) and (6.46). We demand that our ansatz reproduces correctly both double discontinuities and furthermore that the resulting function does not have any unphysical poles at $x=\bar{x}$. This set of constraints produces a particular solution with four free parameters. To express the dependence we first quote the particular solutions $\widehat{\mathcal{H}}^{(2)}_{2323}(u,v)$ and $\widehat{\mathcal{H}}^{(2)}_{2233}(u,v)$, and then describe the four remaining degrees of freedom. Without trying to further simplify the final expressions, let us for convenience first quote the form of $\widehat{\mathcal{H}}^{(2)}_{2323}(u,v)$,

$$\begin{aligned}
\widehat{\mathcal{H}}^{(2)}_{2323} = u^2\Big[&A_1(x,\bar{x})\phi^{(2)}(x',\bar{x}') + A_2(x,\bar{x})\phi^{(2)}(x,\bar{x}) + A_2(1-x,1-\bar{x})\phi^{(2)}(1-x,1-\bar{x})\\
&+\big[A_3(x,\bar{x})x(1-x)\partial_x\phi^{(2)}(x',\bar{x}') + (x\leftrightarrow\bar{x})\big]\\
&+\big[A_4(x,\bar{x})x\partial_x\phi^{(2)}(x,\bar{x}) + (x\leftrightarrow\bar{x})\big]\\
&-\big[A_4(1-x,1-\bar{x})(1-x)\partial_x\phi^{(2)}(1-x,1-\bar{x}) + (x\leftrightarrow\bar{x})\big]\\
&+A_5(x,\bar{x})\log^2(u/v) + A_6(x,\bar{x})\log^2(u) + A_6(1-x,1-\bar{x})\log^2(v)\\
&+A_7(x,\bar{x})\phi^{(1)}(x,\bar{x}) + A_8(x,\bar{x})\log(u) + A_8(1-x,1-\bar{x})\log(v) + A_9(x,\bar{x})\Big],
\end{aligned} \tag{6.49}$$

where we have used the notation $x' = \frac{x}{x-1}$. The explicit expressions for the coefficient functions $A_1, \ldots, A_9$ are rather cumbersome and we provide them in an attached `Mathematica` notebook. These functions obey,

$$\begin{aligned}
A_1(\bar{x},x) &= -A_1(x,\bar{x}), & A_1(1-x,1-\bar{x}) &= -A_1(x,\bar{x}),\\
A_2(\bar{x},x) &= -A_2(x,\bar{x}), & &\\
A_3(\bar{x},x) &= +A_3(x,\bar{x}), & A_3(1-x,1-\bar{x}) &= +A_3(x,\bar{x}),\\
A_5(\bar{x},x) &= +A_5(x,\bar{x}), & A_5(1-x,1-\bar{x}) &= +A_5(x,\bar{x}),\\
A_6(\bar{x},x) &= +A_6(x,\bar{x}), & &\\
A_7(\bar{x},x) &= -A_7(x,\bar{x}), & A_7(1-x,1-\bar{x}) &= -A_7(x,\bar{x}),\\
A_8(\bar{x},x) &= +A_8(x,\bar{x}), & &\\
A_9(\bar{x},x) &= +A_9(x,\bar{x}), & A_9(1-x,1-\bar{x}) &= +A_9(x,\bar{x}).
\end{aligned} \tag{6.50}$$

The above properties are necessary for $\widehat{\mathcal{H}}^{(2)}_{2323}(u,v)$ to be symmetric under $x \leftrightarrow \bar{x}$ and for the crossing property $\widehat{\mathcal{H}}^{(2)}_{2323}(u,v) = u^2/v^2\, \widehat{\mathcal{H}}^{(2)}_{2323}(v,u)$ to hold. Part of the weight-four function in the first line of (6.49) can be immediately related to $\mathcal{H}^{(2)}_{2323}|_{\log^2(u)}$ from equation (6.46). In particular, we recognise

$$\frac{P(u,v)}{x-\bar{x}} = -\frac{u^2 A_1(x,\bar{x})}{4}, \quad \frac{1}{u^2 v}\hat{P}\left(\frac{1}{v},\frac{u}{v}\right) + P(u,v) = \frac{u^2}{4}\big(A_2(x,\bar{x}) - A_1(x,\bar{x})\big). \tag{6.51}$$

whereas the remaining coefficient functions Q, R and S enter non trivially into the set of coefficient functions $A_i(x,\bar{x})$.

The particular solution $\widehat{\mathcal{H}}^{(2)}_{2233}(u,v)$ is given by applying the crossing transformation (6.44) to the function (6.49), resulting in

$$\begin{aligned}
\widehat{\mathcal{H}}^{(2)}_{2233} = \frac{1}{u^2}\Big[&-\hat{A}_2(x',\bar{x}')\phi^{(2)}(x',\bar{x}') - \hat{A}_2(x,\bar{x})\phi^{(2)}(x,\bar{x}) - \hat{A}_1(x,\bar{x})\phi^{(2)}(1-x,1-\bar{x})\\
&+ \big[\hat{A}_4(x',\bar{x}')x(1-x)\partial_x\phi^{(2)}(x',\bar{x}') + (x\leftrightarrow\bar{x})\big]\\
&+ \big[\hat{A}_4(x,\bar{x})x\partial_x\phi^{(2)}(x,\bar{x}) + (x\leftrightarrow\bar{x})\big]\\
&- \big[\hat{A}_3(x,\bar{x})(1-x)\partial_x\phi^{(2)}(1-x,1-\bar{x}) + (x\leftrightarrow\bar{x})\big]\\
&+ \hat{A}_6(x',\bar{x}')\log^2(u/v) + \hat{A}_6(x,\bar{x})\log^2(u) + \hat{A}_5(x,\bar{x})\log^2(v)\\
&- \hat{A}_7(x,\bar{x})\phi^{(1)}(x,\bar{x}) - \big[\hat{A}_8(x,\bar{x}) + \hat{A}_8(x',\bar{x}')\big]\log(u)\\
&+ \hat{A}_8(x',\bar{x}')\log(v) + \hat{A}_9(x,\bar{x})\Big],
\end{aligned} \tag{6.52}$$

where the functions $\hat{A}_1, \ldots, \hat{A}_9$ are related to $A_1, \ldots, A_9$ via $\hat{A}_i(x,\bar{x}) = A_i(1/x, 1/\bar{x})$.

Lastly, let us describe the four ambiguities. We find that they can be described in terms of the following four $\overline{D}$-functions,

$$\begin{aligned}
\mathcal{H}^{(2)}_{2323}(u,v) = \widehat{\mathcal{H}}^{(2)}_{2323}(u,v) &+ \alpha\, u^2\overline{D}_{4444}(u,v) + \beta\, u^2\overline{D}_{4545}(u,v)\\
&+ \gamma\, u^2\overline{D}_{4646}(u,v) + \delta\, u^2 v\overline{D}_{4565}(u,v).
\end{aligned} \tag{6.53}$$

6.4.2 The Window of $\langle 2233 \rangle$: the Twist 4 Sector

Within our ansatz we have obtained a one-loop solution for the $\langle 2233 \rangle$ correlator with 4 free parameters. Can we further constrain these coefficients? The answer is affirmative, and in fact there are further consistency conditions that our one-loop result must satisfy, which are exactly the window-contributions mentioned earlier. Recall that in the long sector at twist 4 there exists a single double-trace operator $K_{22,\ell} \sim \mathcal{O}_2 \partial^\ell \mathcal{O}_2$. Since the three-point function $\langle \mathcal{O}_3 \mathcal{O}_3 K_{22,\ell} \rangle$ vanishes at leading order, i.e. $C^{(0)}_{33K_{22,\ell}} = 0$, the OPE coefficients at twist 4 read

$$L^{(1)}_{2233,4,\ell} = C^{(0)}_{22K_{22,\ell}} \, C^{(1)}_{33K_{22,\ell}}, \tag{6.54}$$

$$M^{(2)}_{2233,4,\ell} = \tfrac{1}{2} C^{(0)}_{22K_{22,\ell}} \, \eta_{K_{22,\ell}} \, C^{(1)}_{33K_{22,\ell}}, \tag{6.55}$$

where the supergravity anomalous dimension $\eta_{K_{22,\ell}} = -96/((\ell+1)(\ell+6))$ has been derived in Section 4.2. The coefficients $L^{(1)}_{2233,4,\ell}$ can be straightforwardly obtained from the analytic part of the tree-level supergravity amplitude $\mathcal{H}^{(1)}_{2233}$, and are given by

$$L^{(1)}_{2233,4,\ell} = 240 \, \frac{\big((\ell+3)!\big)}{(2\ell+6)!}. \tag{6.56}$$

Thus the twist 4 sector of the $\log(u)$ part of the one-loop correlator is fully determined by the knowledge of the above tree-level OPE coefficients and the anomalous dimensions $\eta_{K_{22,\ell}}$. It is interesting to notice in (6.56) that the contributions from the long sector of free theory and supergravity have the same ℓ-dependence but differ in the overall coefficient, 24 and 216, respectively. Very nicely we find that this OPE constraint is consistent with our one-loop result and fixes two of the four remaining constants, namely

$$\alpha = 0, \qquad \delta = 0, \tag{6.57}$$

and we thus have a solution with two free parameters. The remaining two parameters are genuine ambiguities, which our bootstrap method is not able to fix. We will comment on the general form of these ambiguities later in Section 6.7.4.

6.5 Digression: One-Loop Anomalous Dimensions

Now that we have explicitly constructed the order $1/N^4$ corrections to the $\langle 2222 \rangle$ and $\langle 2233 \rangle$ correlators, we can obtain further CFT data up to this order: namely, the one-loop anomalous dimensions $\eta^{(2)}_{K_{pq}}$ can be extracted from the $\log(u)$ stratum of the computed one-loop correlators. More precisely, they appear in the SCPW coefficients $M^{(2)}_{\vec{p};\vec{\tau}}$ for twists above the threshold $\tau \geq \tau^{\max}$:

$$M^{(2)}_{\vec{p};\vec{\tau}} = \tfrac{1}{2} \sum_{(pq)\in R_{\vec{\tau}}} \Big[C^{(1)}_{p_1p_2\mathcal{K}_{pq}}\, \eta^{(1)}_{\mathcal{K}_{pq}}\, C^{(0)}_{p_3p_4\mathcal{K}_{pq}} + C^{(0)}_{p_1p_2\mathcal{K}_{pq}}\, \eta^{(1)}_{\mathcal{K}_{pq}}\, C^{(1)}_{p_3p_4\mathcal{K}_{pq}} + C^{(0)}_{p_1p_2\mathcal{K}_{pq}}\, \eta^{(2)}_{\mathcal{K}_{pq}}\, C^{(0)}_{p_3p_4\mathcal{K}_{pq}} \Big], \tag{6.58}$$

where we already subtracted the terms with τ-derivatives on the blocks, whose coefficients are simply given by the known quantity $2 \times N^{(2)}_{\vec{p};\vec{\tau}}$. In order to successfully isolate $\eta^{(2)}_{\mathcal{K}_{pq}}$ from the above expression, we will also need information about the subleading three-point functions $C^{(1)}_{p_1p_2\mathcal{K}_{pq}}$, which can be obtained from the analytic part of the tree-level supergravity correlators $\mathcal{H}^{(1)}_{\vec{p}}$. They are contained in the SCPW coefficients $L^{(1)}_{\vec{p};\vec{\tau}}$, given by

$$L^{(1)}_{\vec{p};\vec{\tau}} = \sum_{(pq)\in R_{\vec{\tau}}} C^{(1)}_{p_1p_2\mathcal{K}_{pq}}\, C^{(0)}_{p_3p_4\mathcal{K}_{pq}} + C^{(0)}_{p_1p_2\mathcal{K}_{pq}}\, C^{(1)}_{p_3p_4\mathcal{K}_{pq}}, \tag{6.59}$$

after subtraction of τ-derivatives on the blocks with known coefficients $M^{(1)}_{\vec{p};\vec{\tau}}$.

However, it is important to remember that due to the potential mixing with triple-trace operators we can isolate the one-loop anomalous dimensions only in cases where there is no degeneracy and thus a single operator for each spin. This is the case for the unique twist 4 operator $K_{22,\ell} \sim \mathcal{O}_2\partial^\ell\mathcal{O}_2$ in the singlet channel, as well as the twist 5 operator $K_{23,\ell} \sim \mathcal{O}_2\partial^\ell\mathcal{O}_3$ in the $[0,1,0]$ representation.

6.5.1 The Twist 4, $[0,0,0]$ One-Loop Anomalous Dimension

We can now extract the twist 4 singlet channel anomalous dimension from the long sector of the $\langle 2222\rangle$ correlator. From the non-log part of the tree-level $\langle 2222\rangle$ correlator we obtain the coefficients

$$L^{(1)}_{2222;4,\ell} = \left[\frac{-64\big(2\ell+9+2(2\ell+7)(H_{\ell+3}-H_{2\ell+7})\big)}{2\ell+7} + \frac{16}{3} \right] \frac{\big((\ell+3)!\big)^2}{(2\ell+6)!}, \tag{6.60}$$

where H_n denotes the harmonic numbers and in the above we have split the coefficients into their contributions from supergravity and connected part of the free theory, respectively. Note that the above formula is consistent with the derivative relation observed in [72, 139], namely

$$L^{(1)}_{2222;4,\ell} = \frac{\partial}{\partial\tau}\, M^{(1)}_{2222;\tau,\ell}\,\big|_{\tau=4}\,. \tag{6.61}$$

Next, from the explicit result for the one-loop correlator from equations (6.27)-(6.39), we obtain the coefficients $M^{(2)}_{2222;\vec{\tau}}$ at twist 4. Since there is only one operator which contributes at twist 4, the sum in equation (6.58) has only one term and we can isolate the one-loop anomalous dimension $\eta^{(2)}_{K_{22,\ell}}$ by simply subtracting the combination $L^{(1)}_{2222;4,\ell} \times \eta^{(1)}_{K_{22,\ell}}$ from the coefficients $M^{(2)}_{2222;4,\ell}$, where we know the tree-level anomalous dimension $\eta^{(1)}_{K_{22,\ell}} = -96/\big((\ell+1)(\ell+6)\big)$. We find

$$\eta^{(2)}_{K_{22,\ell}} = \begin{cases} \frac{2688(\ell-7)(\ell+14)}{(\ell-1)(\ell+1)^2(\ell+6)^2(\ell+8)} - \frac{4608(2\ell+7)}{(\ell+1)^3(\ell+6)^3}, & \ell = 2,4,\dots \\ -\frac{36}{7}\,\alpha + \frac{2296}{3}, & \ell = 0, \end{cases} \tag{6.62}$$

where $\alpha = 60$ is the coefficient of the one-loop ambiguity. Note that the non-zero value of α breaks the analyticity in spin at spin zero, in agreement with the arguments from the Lorentzian inversion formula [140].

We can now also test the expected reciprocity symmetry (4.56) at one-loop order, which now should really be thought of in terms of an expansion in $\mathfrak{a}$. The full anomalous twist of the operators $K_{22,\ell}$ to one-loop order reads

$$\tau(\ell;\mathfrak{a}) = 4 + \mathfrak{a}\,\eta^{(1)}_{\ell} + \mathfrak{a}^2\,\eta^{(2)}_{\ell} + O(\mathfrak{a}^3), \tag{6.63}$$

and to the relevant order the symmetry (4.56) becomes $\ell \to -l - 7 - \mathfrak{a}\,\eta^{(1)}_{\ell} + O(\mathfrak{a}^2)$. Under this transformation the above quantity should remain invariant, and indeed one can check that

$$\begin{aligned} \tau(-\ell-7-\mathfrak{a}\,\eta^{(1)}_{\ell};\mathfrak{a}) &= 4 + \mathfrak{a}\,\eta^{(1)}_{-\ell-7-\mathfrak{a}\,\eta^{(1)}_{\ell}} + \mathfrak{a}^2\eta^{(2)}_{-\ell-7} + O(\mathfrak{a}^3) \\ &= 4 + \mathfrak{a}\,\eta^{(1)}_{-\ell-7} - \mathfrak{a}^2\,\eta^{(1)}_{\ell}\frac{\partial}{\partial\ell}\eta^{(1)}_{-\ell-7} + \mathfrak{a}^2\eta^{(2)}_{-\ell-7} + O(\mathfrak{a}^3) \\ &= \tau(\ell;\mathfrak{a}) + O(\mathfrak{a}^3). \end{aligned} \tag{6.64}$$

Note that the last equality arises from the identities

$$\eta^{(1)}_{-\ell-7} = \eta^{(1)}_{\ell}, \qquad \eta^{(2)}_{-\ell-7} = \eta^{(2)}_{\ell} - \frac{1}{2}\frac{\partial}{\partial\ell}\big(\eta^{(1)}_{\ell}\big)^2, \tag{6.65}$$

where the first equation is simply the statement of reciprocity symmetry of the tree-level anomalous dimensions, observed previously in (4.57). The second equation states that the antisymmetric part of $\eta^{(2)}_{\ell}$ is given by $\frac{1}{4}\frac{\partial}{\partial\ell}\big(\eta^{(1)}_{\ell}\big)^2$, which means it is predicted from the tree-level anomalous dimension $\eta^{(1)}_{\ell}$ by the full symmetry. This is a non-trivial property of the one-loop anomalous dimensions, and we believe that the higher-order corrections will preserve the reciprocity symmetry in a similar fashion to (6.64).

6.5.2 The Twist 5, $[0,1,0]$ One-Loop Anomalous Dimension

The twist 5 anomalous dimension can be extracted from our result for the one-loop $\langle 2323\rangle$ correlator. Note that the long part of the $\langle 2323\rangle$ correlator contributes only to the $[0,1,0]$ channel and admits both even and odd spins. The non-log part of the tree-level correlator yields

$$L^{(1)}_{2323;5,\ell} = \begin{cases} \frac{\left(144(\ell+4)(\ell+7)(H_{\ell+3}-H_{2\ell+7})+\frac{54}{5}(7\ell^2+97\ell+296)\right)((\ell+3)!)^2}{(2\ell+7)!}, & \ell \text{ even}, \\ \frac{\left(144(\ell+1)(\ell+4)(H_{\ell+3}-H_{2\ell+7})+\frac{18}{5}(\ell+1)(21\ell+104)\right)((\ell+3)!)^2}{(2\ell+7)!}, & \ell \text{ odd}, \end{cases} \tag{6.66}$$

which is again consistent with the derivative relation

$$L^{(1)}_{2323;5,\ell} = \frac{\partial}{\partial\tau} M^{(1)}_{2323;\tau,\ell}\big|_{\tau=5}\,, \tag{6.67}$$

where one has to treat the even and odd spin formulae as independent cases. Next, we consider the SCPW expansion of the $\log(u)$ part of the one-loop result (6.49). After subtracting the combination $L^{(1)}_{2323;5,\ell} \times \eta^{(1)}_{K_{23,\ell}}$, where $\eta^{(1)}_{K_{23,\ell}} = -160/\big((\ell+1)(\ell+4)\big)$ and $\eta^{(1)}_{K_{23,\ell}} = -160/\big((\ell+4)(\ell+7)\big)$ for even and odd spins respectively, we find the one-loop anomalous dimension to be given by

$$\eta^{(2)}_{K_{23,\ell}} = \begin{cases} \frac{640\left(9\ell^4+68\ell^3-1151\ell^2-5738\ell-3688\right)}{(\ell-1)(\ell+1)^3(\ell+4)^3(\ell+8)}, & \ell = 2,4,\ldots, \\ \frac{640\left(9\ell^4+140\ell^3-487\ell^2-11262\ell-29400\right)}{\ell(\ell+4)^3(\ell+7)^3(\ell+9)}, & \ell = 3,5,\ldots, \\ 4610 - \frac{30}{7}\beta - \frac{250}{21}\gamma, & \ell = 0, \\ -41 + \frac{8}{3}\gamma, & \ell = 1, \end{cases} \tag{6.68}$$

where β and γ are the two unfixed parameters of the one-loop correlator. We observe again that for $\beta, \gamma \neq 0$ the spectrum is not analytic in spin for $\ell = 0, 1$.

As before, we can check that the above one-loop results continue to obey the reciprocity symmetry, which in this case is an invariance of the full anomalous twist of the operators $K_{23,\ell}$ under the transformation $\ell \to -\ell - 8 - \mathfrak{a}\,\eta^{(1)}_{\ell} + O(\mathfrak{a}^2)$. The symmetry swaps the even and odd spin families, and is satisfied non-trivially thanks to the following order by order relations which can be readily checked:

$$\begin{aligned} \eta^{(1)\text{even}}_{-\ell-8} &= \eta^{(1)\text{odd}}_{\ell}, & \eta^{(2)\text{even}}_{-\ell-8} &= \eta^{(2)\text{odd}}_{\ell} - \frac{1}{2}\frac{\partial}{\partial\ell}\big(\eta^{(1)\text{odd}}_{\ell}\big)^2, \\ \eta^{(1)\text{odd}}_{-\ell-8} &= \eta^{(1)\text{even}}_{\ell}, & \eta^{(2)\text{odd}}_{-\ell-8} &= \eta^{(2)\text{even}}_{\ell} - \frac{1}{2}\frac{\partial}{\partial\ell}\big(\eta^{(1)\text{even}}_{\ell}\big)^2. \end{aligned} \tag{6.69}$$

6.6 Back to the Bootstrap: the Below-Threshold Region

Now that we have described in great detail the construction of one-loop correlators for two explicit cases, let us outline a general algorithm which can be applied to correlators of higher external charges. First, we will discuss how to obtain predictions for the two below-threshold regions: the window and below-window. Second, we need to incorporate the results of multiplet recombination at the unitarity bound into our algorithm. The structure of general one-loop correlators will then be addressed in the following section.

6.6.1 Predictions for the Below-Threshold Region

A feature of four-point correlators of single-particle operators with generic charges is that one can bootstrap the below-threshold pieces of the $\log(u)$ and analytic part of the correlator. Similarly to the double discontinuity (discussed in Section 6.2), which lies entirely above the threshold, there is information from within and below the window which further constrains the four-point function. Remarkably, using all of this available lower order data always fixes the one-loop four-point function up to certain well understood ambiguities which only have finite spin contributions to the SCPW expansion.

To begin with consider the long SCPW coefficients of the analytic part of the tree-level correlator $L^{(1)}$, arising from operators in the window region $\tau^{\min} \leq \tau < \tau^{\max}$ (see Figure 6.1). For simplicity assume $p_1+p_2 \geq p_3+p_4$ (the other case is similar), then (6.11) becomes

$$L^{(1)}_{\vec{p};\vec{\tau}} = \sum_{(pq)\in R_{\vec{\tau}}} C^{(1)}_{p_1p_2\mathcal{K}_{pq}} C^{(0)}_{p_3p_4\mathcal{K}_{pq}}, \qquad \text{for } p_1+p_2 \geq p_3+p_4. \tag{6.70}$$

The key point here is that there are new, leading three-point functions at order $O(1/N^2)$, $C^{(1)}_{p_1p_2\mathcal{K}_{pq}}$, with below threshold twist $\tau < p_1+p_2$.

Fixing the pair (p_1p_2) and $\vec{\tau}$, let us consider all values of $(p_3p_4) \in R_{\vec{\tau}}$ and rewrite (6.70) as a vector equation[4]

$$\mathbf{L}^{(1)}_{(p_1p_2);\vec{\tau}} = \mathbf{C}^{(1)}_{(p_1p_2);\vec{\tau}} \, \big(\mathbb{C}^{(0)}_{\vec{\tau}}\big)^T. \tag{6.71}$$

Here we have defined the row-vector $\mathbf{C}^{(1)}_{(p_1p_2);\vec{\tau}}$ with entries

$$\Big(\mathbf{C}^{(1)}_{(p_1p_2);\vec{\tau}}\Big)_{(pq)} = C^{(1)}_{p_1p_2\mathcal{K}_{(pq);\vec{\tau}}}, \quad \forall (pq) \in R_{\vec{\tau}}, \tag{6.72}$$

and the row-vector of the analytic $O(1/N^2)$ SCPW coefficients, $\mathbf{L}^{(1)}_{(p_1p_2);\vec{\tau}}$, with entries

$$\Big(\mathbf{L}^{(1)}_{(p_1p_2);\vec{\tau}}\Big)_{(p_3p_4)} = L^{(1)}_{\vec{p};\vec{\tau}}, \quad \forall (p_3p_4) \in R_{\vec{\tau}}. \tag{6.73}$$

The other ingredient is the matrix of leading three-point couplings $\mathbb{C}^{(0)}_{\vec{\tau}}$ encountered in previous chapters.

Consider now the $\log(u)$-part of the one-loop correlator with SCPW coefficients $M^{(2)}$ given by (6.12). In direct analogy to $\mathbf{L}^{(1)}_{(p_1p_2);\vec{\tau}}$ above, we define the corresponding vector $\mathbf{M}^{(2)}_{(p_1p_2);\vec{\tau}}$. We can turn the OPE predictions in the window (for varying $(p_3p_4) \in R_{\vec{\tau}}$)

[4] In the following, all boldface quantities refer to row-vectors of SCPW coefficients.

into the vector equation

$$\mathbf{M}^{(2)}_{(p_1p_2);\vec{\tau}} = \tfrac{1}{2}\,\mathbf{C}^{(1)}_{(p_1p_2);\vec{\tau}}\ \widehat{\eta}_{\vec{\tau}}\ \big(\mathbb{C}^{(0)}_{\vec{\tau}}\big)^T. \tag{6.74}$$

Knowing $\mathbb{C}^{(0)}_{\vec{\tau}}$ we can explicitly solve for $\mathbf{C}^{(1)}_{(p_1p_2);\vec{\tau}}$ using (6.71) and plug it in the above equation to get the one-loop SCPW coefficients $\mathbf{M}^{(2)}_{(p_1p_2);\vec{\tau}}$. However, even in cases where we do not have $\mathbb{C}^{(0)}_{\vec{\tau}}$, because of the degeneracy of the anomalous dimensions, we see that by combining equations (6.14), (6.15) and (6.71) we obtain $\mathbf{M}^{(2)}_{(p_1p_2);\vec{\tau}}$ purely in terms of tree-level SCPW data:

$$\mathbf{M}^{(2)}_{(p_1p_2);\vec{\tau}} = \mathbf{L}^{(1)}_{(p_1p_2);\vec{\tau}}\ \big(\widehat{L}^{(0)}_{\vec{\tau}}\big)^{-1}\ \widehat{M}^{(1)}_{\vec{\tau}}. \tag{6.75}$$

We thus obtain a piece of the single-log coefficient of the one-loop correlator from tree-level data.

In a very similar manner, pieces of the analytic part of the one-loop correlator, namely the coefficients $L^{(2)}$ for twists below the window, can be determined purely in terms of tree-level SCPW coefficients. From equation (6.13) we find

$$\begin{aligned} L^{(2)}_{\vec{p};\vec{\tau}} &= \mathbf{C}^{(1)}_{(p_1p_2);\vec{\tau}}\ \big(\mathbf{C}^{(1)}_{(p_3p_4);\vec{\tau}}\big)^T \\ &= \mathbf{L}^{(1)}_{(p_1p_2);\vec{\tau}}\ \big(\widehat{L}^{(0)}_{\vec{\tau}}\big)^{-1}\ \big(\mathbf{L}^{(1)}_{(p_3p_4);\vec{\tau}}\big)^T, \quad \text{for } 4{+}2a{+}b \le \tau < p_3 + p_4. \end{aligned} \tag{6.76}$$

Recall that the SCPW coefficients $L^{(1)}$ appearing in the above (6.76) are determined by summing contributions from tree-level supergravity $\mathcal{H}^{(1)}_{\vec{p}}$ and the connected part of free theory, as shown in (6.6). A general formula for the connected free theory at order $1/N^2$ was presented already in [112], and we we record it in our notation in Appendix C.

In summary, from all the results given above we can determine the following pieces of the $O(1/N^4)$ four-point functions (besides the double discontinuities discussed previously in Section 6.2):

- $\log^1(u)$ stratum obtained from a finite number of twists:

$$\mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^1(u)} = \tfrac{1}{2}\,\log^1(u) \sum_{\ell,a,b}\ \sum_{\tau=\tau^{\min}}^{\tau^{\max}-2} \Big(\mathbf{M}^{(2)}_{(p_1p_2);\vec{\tau}}\Big)_{(p_3p_4)}\ \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}} + \dots, \tag{6.77}$$

  where the coefficients $\mathbf{M}^{(2)}$ are given in (6.75) and we are omitting terms contributing to twists $\tau \ge \tau^{\max}$.

- $\log^0(u)$ stratum obtained from a finite number of twists:

$$\mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^0(u)} = \mathcal{H}^{(2)}_{\text{bound}} + \sum_{\ell,a,b}\ \sum_{\tau=2a+b+4}^{\tau^{\min}-2} L^{(2)}_{\vec{p};\vec{\tau}}\ \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}} + \dots, \tag{6.78}$$

with $L^{(2)}$ given in (6.76) and omitting terms contributing to twists $\tau \geq \tau^{\min}$.

There is an extra subtlety which needs to be tackled in order to fully determine the $\log^0(u)$ stratum: it enters the contribution called $\mathcal{H}^{(2)}_{\text{bound}}$ above and has to do with multiplet recombination at the unitarity bound, $\tau = 2a + b + 2$, in each channel. We will address this next.

6.6.2 Predictions for the Semi-Short Sector

We now come back to the delicate point of multiplet recombination at the unitarity bound. In Appendix B, we discussed multiplet recombination in the free theory, whose results we now need to incorporate.

In equation (6.78) we gave the one-loop non-log predictions which originate from twists *above* the unitary bound, i.e. for twists $2a + b + 4 \leq \tau < \tau^{\min}_{\vec{p}}$. In addition, we claim that the dynamical one-loop function must contain a contribution *at* the unitarity bound $\tau = 2a + b + 2$, which we are also able to predict, namely we have

$$L^{(2)}_{\vec{p};\vec{\tau}} = L^{(2)f}_{\vec{p};\vec{\tau}} + L^{(2)\mathcal{H}}_{\vec{p};\vec{\tau}} = 0, \qquad \text{for } \tau = 2a + b + 2, \tag{6.79}$$

$$L^{(2)f}_{\vec{p};2a+b+2,\ell,[a,b,a]} = \sum_{k=0}^{a} (-1)^k A_{[\ell+2+k,1^{a-k}]}\Big|_{\frac{1}{N^4}} - \sum_{k=0}^{a} (-1)^k S_{[\ell+2+k,1^{a-k}]}, \tag{6.80}$$

where the coefficient $L^{(2)f}_{\tau=2+2a+b}$ was obtained in (B.14). Its first term is given by the SCPW of the order $1/N^4$ connected free theory $A_{\vec{p};2a+b+2-2k,[\ell+2+k,1^{a-k}]}\big|_{\frac{1}{N^4}}$. Its second term is given by summing over the new coefficients $S_{\vec{p};2a+b+2-2k,[\ell+2+k,1^{a-k}]}$, and it follows non-trivially from the analysis of the semi-short sector, which by construction is of order $1/N^4$. The contributions to the analytic (i.e. $\log^0(u)$) part of $\mathcal{H}^{(2)}$ which come from twists at the unitarity bound combine to give the function denoted by $\mathcal{H}^{(2)}_{\text{bound}}$ in (6.78), which itself is of the form

$$\mathcal{H}^{(2)}_{\text{bound}} = -\sum_{\ell,a,b} L^{(2)f}_{\vec{p};\vec{\tau}}\ \widetilde{\mathbb{L}}_{\vec{p};\vec{\tau}}\,. \tag{6.81}$$

The reason for the cancellation in equation (6.79) is the following: the OPE of $\mathcal{O}_{p_i}\mathcal{O}_{p_j}$ in the free theory runs by definition over all operators of $\mathcal{N} = 4$ SYM, but supergravity states correspond only to operators built from one-half BPS operators, i.e. they are either one-half BPS operators themselves or multi-particle operators. Other single-trace operators at the unitarity bound, which are present in the free theory, therefore correspond to excited string states and should be absent from the OPE in the supergravity regime.

Simple examples of such operators which correspond to excited string states are e.g. the Konishi operator $\text{tr}(\phi^2)$ in the [000] representation, or the twist 3 superconformal primary of the form $\mathrm{tr}(\phi^3)$ in the [010] representation. However, these two cases are special because there are no other existing operators with such quantum numbers. In particular, there will be no S-type contribution in (6.80). On the other hand, beyond twist 3 one has to distinguish carefully between multi-trace semi-short operators, which do remain in the spectrum of supergravity, and excited string states, as done in Appendix B.

It is very instructive to compare the new features at order $1/N^4$ with the corresponding tree-level terms. Let us begin from the analogue of equation (6.79) at tree level. It reduces to

$$L^{(1)}_{\vec{p};2a+b+2,\ell,[a,b,a]} = \underbrace{\sum_{k=0}^{a}(-1)^k A_{[\ell+2+k,1^{a-k}]}\big|_{\frac{1}{N^2}}}_{=L^{(1)f}_{\vec{p};2a+b+2,\ell,[a,b,a]}} + L^{(1)\mathcal{H}}_{\vec{p};2a+b+2,\ell,[a,b,a]} = 0. \tag{6.82}$$

The difference compared to equation (6.79) is precisely the difference between performing multiplet recombination with SCPW coefficients of connected free theory – assuming all below threshold ($\tau = 2a+b+2 < \tau^{\min}$) semi-short operators are absent – and performing multiplet recombination with remaining below-threshold semi-short operators. This is simply because the semi-short three-point functions are all of order $O(1/N^2)$, and thus are only visible in the SCPW decomposition at order $1/N^4$.

Indeed, the leading order three-point functions $C^{(0)}_{p_ip_j\mathcal{K}_{pq}} = 0$ whenever $p_i + p_j > \tau$, and thus this vanishing condition extends to the non-semi-short below-window sector $\tau < \tau^{\min}$ at tree-level. We therefore have

$$L^{(1)}_{\vec{p};\vec{\tau}} = L^{(1)f}_{\vec{p};\vec{\tau}} + L^{(1)\mathcal{H}}_{\vec{p};\vec{\tau}} = 0, \qquad \text{for } \tau < \tau^{\min}, \tag{6.83}$$

with the free theory part $L^{(1)f}_{\vec{p};\vec{\tau}}$ given in (2.43) when the twist is above the unitarity bound $\tau \geq 2a+b+4$ and (6.82) when at the bound $\tau = 2a+b+2$.

6.7 The Structure of General One-Loop Correlators

In the previous sections we have explained how to bootstrap predictions about the dynamical one-loop function $\mathcal{H}^{(2)}_{\vec{p}}$ from tree-level results. Summarising, we have obtained the leading $\log^2(u)$ discontinuity, see Section 6.2. Then, after considering the two explicit examples $\mathcal{H}^{(2)}_{2222}$ and $\mathcal{H}^{(2)}_{2233}$, we described the new below-threshold features of correlators with general external charges. In particular, we have obtained pieces of the single $\log^1(u)$ from exchanged operators in the window (see discussion around equations (6.75) and (6.77)), and also pieces of the analytic $\log^0(u)$ part of the correlator from below-window data (see discussion around (6.76) and (6.78)). Finally, we understood how to deal with the SCPW coefficients of long operators at the unitarity bound in (6.80). Let us emphasize that even though the leading-log discontinuity can be obtained more elegantly by using the hidden symmetry of [112], our approach here allows us to go beyond that and compute $M^{(2)}$ and $L^{(2)}$, which are crucial ingredients to our general one-loop bootstrap program.

All the OPE predictions discussed in the above are organised according to the $\log(u)$ stratification of the correlators given in (6.3)-(6.5). Before describing a precise ansatz for the general one-loop correlators, let us point out in the following that the structure of the order $1/N^4$ dynamical function $\mathcal{H}^{(2)}_{\vec{p}}$ admits a further refinement.

6.7.1 A Further Refinement

Consider first the following observation: looking at below-threshold physics at tree-level we found that the analytic sector of the dynamical function $\mathcal{H}^{(1)}_{\vec{p}}$ is subject to the constraint (6.83), i.e

$$L^{(1)\mathcal{H}}_{\vec{p};\vec{\tau}} = -L^{(1)f}_{\vec{\tau}}, \qquad \text{for } 2a+b+2 < \tau < \tau^{\min}, \tag{6.84}$$

augmented by a similar constraint at the unitarity bound given in (6.82). We claim (and we will show this in Section 6.8) that $\mathcal{H}^{(1)}$ is entirely fixed by these constraints, together with the requirement that its $\log(u)$-discontinuity has threshold twist $\tau^{\max}$.

Considering now the analytic sector at one-loop, we find instead

$$L^{(2)\mathcal{H}}_{\vec{p};\vec{\tau}} = -L^{(2)f}_{\vec{p};\vec{\tau}} + L^{(2)}_{\vec{p};\vec{\tau}}, \quad \text{for } 2a+b+2 < \tau < \tau^{\min} \ . \tag{6.85}$$

where $L^{(2)}_{\vec{p};\vec{\tau}}$ is the new $O(1/N^4)$ prediction (6.76) arising from tree-level data via the OPE. It is clear then that the analytic part of $\mathcal{H}^{(2)}$ has two separate contributions: one is cancelling the free theory contribution, i.e $-L^{(2)f}_{\vec{\tau}}$, and the other one is linked to predictions from tree-level data $L^{(2)}_{\vec{p};\vec{\tau}}$. Furthermore, at the unitarity bound we find a similar split into a piece depending directly on free theory SCPW coefficients and a non-trivial prediction arising from correlators of different charges, recall equation (6.80). Since the double- and single-log strata of $\mathcal{H}^{(2)}$ are determined uniquely by tree-level data via the OPE and have no free theory contributions, it is natural to split the one-loop function accordingly into

$$\mathcal{H}^{(2)}_{\vec{p}} = \mathcal{T}^{(2)}_{\vec{p}} + \mathcal{D}^{(2)}_{\vec{p}}, \tag{6.86}$$

where $\mathcal{T}^{(2)}_{\vec{p}}$ and $\mathcal{D}^{(2)}_{\vec{p}}$ have a different interplay with the connected part of the free theory.

The function $\mathcal{T}^{(2)}_{\vec{p}}$ generalises the tree-level function $\mathcal{H}^{(1)}_{\vec{p}}$, and it is defined by the following properties: it has a $\log^1(u)$ discontinuity with threshold twist $\tau^{\max}$, no $\log^2(u)$ double discontinuity, and it fully cancels all long below-window contributions coming from recombined free theory, hence the name of generalised tree-level function. Indeed, for twists $2a+b+2<\tau<\tau^{\min}$ strictly above the unitary bound, we expect

$$L^{(2)\mathcal{T}}_{\vec{p};\vec{\tau}} = -L^{(2)f}_{\vec{p};\vec{\tau}}\,, \tag{6.87}$$

with $L^{(2)f}_{\vec{p};\vec{\tau}}$ being the order $1/N^4$ part of the free-theory SCPW coefficients as defined in (2.43), whereas at the unitarity bound we have

$$L^{(2)\mathcal{T}}_{\vec{p};2a+b+2,\ell,[a,b,a]} = -\sum_{k=0}^{a}(-1)^k A_{[\ell+2+k,1^{a-k}]}\Big|_{\frac{1}{N^4}}. \tag{6.88}$$

It follows that the below-window OPE predictions (6.76) will be encoded only in the function $\mathcal{D}^{(2)}_{\vec{p}}$, i.e.

$$L^{(2)\mathcal{D}}_{\vec{p};\vec{\tau}} = L^{(2)}_{\vec{p};\vec{\tau}}, \qquad \text{for } 2a+b+2<\tau<\tau^{\min}, \tag{6.89}$$

and at the unitarity bound

$$L^{(2)\mathcal{D}}_{\vec{p};2a+b+2,\ell,[a,b,a]} = \sum_{k=0}^{a}(-1)^k S_{[\ell+2+k,1^{a-k}]}. \tag{6.90}$$

Our task now is to construct the full one-loop correlators $\mathcal{H}^{(2)}_{\vec{p}}$ consistently with the OPE predictions. We will see that the splitting $\mathcal{H}^{(2)}_{\vec{p}} = \mathcal{T}^{(2)}_{\vec{p}} + \mathcal{D}^{(2)}_{\vec{p}}$ is also strongly motivated by features of the $\log^2(u)$ discontinuity. In fact, we will discover that $\mathcal{D}^{(2)}_{\vec{p}}$ is the minimal one-loop function which consistently emanates from the leading $\log^2(u)$ discontinuity. Furthermore, we will find that $\mathcal{T}_{\vec{p}}$ can be constructed as an exact function of N. The interplay of $\mathcal{D}^{(2)}_{\vec{p}}$ with the semi-short prediction (6.90) is very remarkable, and when we think of it as descending from the double-logarithmic discontinuity it is a tangible triumph of supergravity within our $\mathcal{N}=4$ bootstrap program.

6.7.2 Ansatz for Minimal One-Loop Functions

We are finally at the stage where we can introduce an ansatz for the minimal one-loop function $\mathcal{D}^{(2)}_{\vec{p}}$, which will accommodate all of the various OPE-predictions discussed in the above. To understand this ansatz and impose as many constraints as possible, we will first consider the consequences of the OPE as well as crossing symmetry on the structure of one-loop correlators.

From the OPE we expect different parts of the correlator to possess contributions from operators of different twists: the $\log^2(u)$ discontinuity has contributions only from operators above threshold $\tau \geq \tau^{\max}$. The $\log^1(u)$ part can have contributions from the window, $\tau \geq \tau^{\min}$, and finally the analytic $\log^0(u)$ part can have contributions starting from the semi-short operators with $\tau \geq p_{43}+2$. Because a long operator of twist τ gives a contribution to the correlator which for small u goes like $u^{\frac{1}{2}(\tau-p_{43})}$, the OPE then dictates that

$$\begin{aligned} \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^2(u)} &= O(u^{\frac{1}{2}(\tau^{\max}-p_{43})}), \\ \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^1(u)} &= O(u^{\frac{1}{2}(\tau^{\min}-p_{43})}), \\ \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^0(u)} &= O(u), \end{aligned} \tag{6.91}$$

with

$$\begin{aligned} \tfrac{1}{2}(\tau^{\max} - p_{43}) &= \max\left\{\tfrac{p_1+p_2+p_3-p_4}{2}, p_3\right\}, \\ \tfrac{1}{2}(\tau^{\min} - p_{43}) &= \min\left\{\tfrac{p_1+p_2+p_3-p_4}{2}, p_3\right\} = \kappa_{\vec{p}}, \end{aligned} \tag{6.92}$$

where the latter is precisely the degree of extremality.

Consider now the splitting $\mathcal{H}^{(2)}_{\vec{p}} = \mathcal{T}^{(2)}_{\vec{p}} + \mathcal{D}^{(2)}_{\vec{p}}$, as discussed in the previous section. We claim that only $\mathcal{T}^{(2)}_{\vec{p}}$ has a contribution at $O(u)$ whereas $\mathcal{D}^{(2)}_{\vec{p}} = O(u^2)$. The reason for this follows again from the detailed understanding of the semi-short sector: the contributions at $O(u)$ arise from semi-short operators with twist $p_{43}+2$ in the $[0, p_{43}, 0]$ representation of $su(4)$. In this case there is a single A-type contribution in the sum of (B.14), which has to be dealt with by $\mathcal{T}^{(2)}_{\vec{p}}$, and a single S contribution, to be dealt with by $\mathcal{D}^{(2)}_{\vec{p}}$. Recall that we deal with the split $\mathcal{H}^{(2)}_{\vec{p}} = \mathcal{T}^{(2)}_{\vec{p}} + \mathcal{D}^{(2)}_{\vec{p}}$ by using equations (6.88) and (6.90). Then notice that the S contribution itself is obtained in (B.13) in terms of the SCPW coefficients $S_{q_r\tilde{q}_r p_3 p_4}$, where $q_r + \tilde{q}_r = p_{43}+2$. But these correlators are next-to-extremal and, according to the discussion around (3.11), they completely vanish when we use the correct definition of single-particle operators, so the S contribution vanishes at that twist. We therefore have

$$\mathcal{T}^{(2)}_{\vec{p}}\big|_{\log^0(u)} = O(u), \qquad \mathcal{D}^{(2)}_{\vec{p}}\big|_{\log^0(u)} = O(u^2). \tag{6.93}$$

Under the crossing transformation $u \leftrightarrow v$, the analysis of the small u expansion in (6.91) translates into predictions for the small v expansion, which is then useful to understand how to constrain the ansatz for the full function.

For the correlator itself crossing symmetry simply implies that

$$\langle \mathcal{O}_{p_1}(x_1)\mathcal{O}_{p_2}(x_2)\mathcal{O}_{p_3}(x_3)\mathcal{O}_{p_4}(x_4)\rangle = \langle \mathcal{O}_{p_{\sigma_1}}(x_{\sigma_1})\mathcal{O}_{p_{\sigma_2}}(x_{\sigma_2})\mathcal{O}_{p_{\sigma_3}}(x_{\sigma_3})\mathcal{O}_{p_{\sigma_4}}(x_{\sigma_4})\rangle\ , \tag{6.94}$$

for any permutation $\sigma \in S_4$. The implications of this while taking into account the prefactor $\mathcal{P}_{\vec{p}}$ requires a little care. When defining the prefactor we always made the choice $0 \leq p_{21} \leq p_{43}$, which should therefore be maintained under the permutation whilst exchanging $u \leftrightarrow v$. This requires considering a number of different cases for the relative values of the charges p_i. In all cases however, there is a unique permutation σ

satisfying the above requirements and one finds that for this permutation

$$\mathcal{H}^{(2)}_{p_1p_2p_3p_4}(u,v) = \left(\frac{u\tau}{v}\right)^{\kappa_{\vec{p}}} \mathcal{H}^{(2)}_{p_{\sigma_1}p_{\sigma_2}p_{\sigma_3}p_{\sigma_4}}(v,u). \tag{6.95}$$

The small u behaviour of $\mathcal{H}^{(2)}_{p_{\sigma_1}p_{\sigma_2}p_{\sigma_3}p_{\sigma_4}}(u,v)$ given in (6.91) then yields the following small v behaviour of $\mathcal{H}^{(2)}_{\vec{p}}(u,v)$

$$\begin{aligned} \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^2(v)} &= O(v^{\frac{1}{2}(p_1+p_4-p_2-p_3)}), \\ \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^1(v)} &= O(v^0), \\ \mathcal{H}^{(2)}_{\vec{p}}\big|_{\log^0(v)} &= O(1/v^{\kappa_{\vec{p}}-1}). \end{aligned} \tag{6.96}$$

Furthermore, the different small u behaviour of $\mathcal{T}^{(2)}_{\vec{p}}$ and $\mathcal{D}^{(2)}_{\vec{p}}$ in (6.93) implies a different small v limit:

$$\mathcal{T}^{(2)}_{\vec{p}}\big|_{\log^0(v)} = O(1/v^{\kappa_{\vec{p}}-1}), \qquad \mathcal{D}^{(2)}_{\vec{p}}\big|_{\log^0(v)} = O(1/v^{\kappa_{\vec{p}}-2})\ . \tag{6.97}$$

Note that the differences in the small v behaviour between $\mathcal{T}^{(2)}_{\vec{p}}$ and $\mathcal{D}^{(2)}_{\vec{p}}$ will be crucial in the determination of our ansatz.

We now have all the relevant information to write an ansatz for the minimal one-loop function $\mathcal{H}^{(2)}$, which is consistent with crossing symmetry and matches the two-variable resummation of the leading $\log^2(u)$ discontinuity. In analogy with the results for the one-loop corrections to the $\langle 2222\rangle$ and $\langle 2233\rangle$ correlators discussed previously, we consider single-valued transcendental functions of up to weight 4. The bound on the overall transcendental weight follows from the explicit form of the two-variable resummations (6.18), in which we find an overall $\log^2(u)$ paired with an at most weight-two antisymmetric transcendental function. We therefore need a basis of weight-four antisymmetric transcendental functions, together with their lower weight completions. As before, we will make use of the series of ladder integrals [137, 138]

$$\phi^{(l)}(x,\bar{x}) = \sum_{r=0}^{l} (-1)^r \frac{(2l-r)!}{r!(l-r)!l!} \log^r(u) \left(\mathrm{Li}_{2l-r}(x) - \mathrm{Li}_{2l-r}(\bar{x})\right). \tag{6.98}$$

Our proposed basis then has the form

$$\begin{aligned} W_{4-} &= h_1\, \phi^{(2)}(x_1', x_2') + h_2\, \phi^{(2)}(x,\bar{x}) + h_3\, \phi^{(2)}(1-x, 1-\bar{x}), \\ W_{3-} &= h_4\, x\partial_x \phi^{(2)}(x,\bar{x}) + h_5\,(x-1)\partial_x\phi^{(2)}(1-x,1-\bar{x}) - (x \leftrightarrow \bar{x}), \\ W_{3+} &= (x-\bar{x})\big[h_6\, \partial_v \phi^{(2)}(x,\bar{x}) + h_7\, \partial_u \phi^{(2)}(1-x,1-\bar{x})\big] + h_8\, \zeta_3, \\ W_{2+} &= h_9 \log(u)\log(v) + h_{10}\log^2(v) + h_{11}\log^2(u), \end{aligned} \tag{6.99}$$

and

$$\begin{aligned} W_{2-} &= h_{\Box}\,\phi^{(1)}(x,\bar{x}), & W_0 &= h_0, \\ W_{1u} &= h_u \log(u), & W_{1v} &= h_v \log(v). \end{aligned} \tag{6.100}$$

The basis at weight four and three is written in terms of the double-box function, which is the $l = 2$ integral in the ladder series (6.98). The weight-two antisymmetric element is instead the $l = 1$ one-loop box-function. Finally, the coefficient functions $h_{i=1,..,11,\Box,u,v,0}$ will be polynomials in the variables x, $\bar{x}$, σ, τ.

From considerations about crossing in equation (6.97) and the structure of the two-variable resummations from (6.18), we conclude that the ansatz for the minimal one-loop function $\mathcal{D}^{(2)}_{\vec{p}}$ is given by

$$\begin{aligned} \mathcal{D}^{(2)}_{\vec{p}} = \frac{W_{4-}+W_{3-}}{(x-\bar{x})^{d_{\vec{p}}+8}} + \frac{1}{(x-\bar{x})^{d_{\vec{p}}+7}}\left[W_{3+} + \frac{W_{2+}}{v^{\kappa_{\vec{p}}-2}}\right] \\ + \frac{1}{v^{\kappa_{\vec{p}}-2}}\left[\frac{W_{2-}}{(x-\bar{x})^{d_{\vec{p}}+8}} + \frac{W_{1v}+W_{1u}}{(x-\bar{x})^{d_{\vec{p}}+7}} + \frac{W_0}{(x-\bar{x})^{d_{\vec{p}}+5}}\right], \end{aligned} \tag{6.101}$$

where we recall the definitions

$$d_{\vec{p}} = p_1+p_2+p_3+p_4-1, \qquad \kappa_{\vec{p}} = \min\left\{\tfrac{p_1+p_2+p_3-p_4}{2},\, p_3\right\}. \tag{6.102}$$

For future convenience, we will refer to the smaller basis (6.100), consisting of the one-loop box-function $\phi^{(1)}(x,\bar{x})$ together with its weight-one and weight-zero completions, as *tree-like*. For example, any $\overline{D}$-function can be decomposed in such a basis.

6.7.3 The Bootstrap Algorithm

Next, we will describe in detail our bootstrap algorithm: step by step, we will go through the sequence of constraints we impose in order to determine the free parameters in the above ansatz (6.101) for $\mathcal{D}^{(2)}_{\vec{p}}$.

Crossing symmetry and matching of the leading-log

For any orientation of the external charges $\vec{p}$, we consider the $\log^2(u)$ projection of the ansatz and match it against the explicit two-variable resummation described in (6.18). This fixes combinations of the coefficient functions W_{4-}, $W_{3\pm}$ and W_{2+}. Note that the power $v^{\kappa_{\vec{p}}-2}$ in the denominator of W_{2+} in equation (6.101) is consistent with the weight-zero part of the leading-log as given in (6.18). Matching all independent leading-log discontinuities actually fixes completely the polynomials h_i for $i = 1, 2, 7, 9, 11$. When $\kappa_{\vec{p}} = 2$, the correlators are next-to-next-to extremal (examples are the $\langle 2222\rangle$ and $\langle 2233\rangle$ correlators considered previously), in which case there is no singular v behaviour in the ansatz. In these cases the general ansatz (6.101) reduces to the ansätze considered in Sections 6.3 and 6.4.

Absence of unphysical poles

Any leading-log discontinuity has itself no poles at $x = \bar{x}$. However, this only accounts for the $\log^2(u)$ projection of the function $\mathcal{D}^{(2)}_{\vec{p}}$. In order for the ansatz to yield a well defined function, we have to ensure that globally there are no unphysical poles. In this way, lower-weight coefficient functions become entangled with those at weights 4, 3 and 2. In particular, both the powers of $(x - \bar{x})$ in the denominators and the coefficient functions of W_{2-}, W_{1u}, W_{1v} and W_0 have the right structure such that all $x = \bar{x}$ poles coming from the symmetric coefficient functions at weights 4, 3 and 2 can be cancelled. For this reason the 'tree-like' coefficient functions of $\mathcal{D}^{(2)}_{\vec{p}}$, i.e. $h_\Box$, h_u, h_v, and h_0, have quite different features compared to their counterparts at tree-level. During this process we can keep $v^{\kappa_{\vec{p}}-2}$ as the maximum singular power in the denominator.

Matching the OPE predictions in and below the window

At this stage of the algorithm we have found a well defined ansatz with the correct $\log^2(u)$ discontinuities. It differs from $\mathcal{D}^{(2)}_{\vec{p}}$ because we have not yet imposed the remaining predictions in and below the window, which we have to compute explicitly by using the strategy outlined in Section 6.6. These OPE predictions come as SCPW coefficients at fixed twist with varying spin, i.e. from sums of the form

$$\sum_{\ell} c_{\tau_0,\ell}\, \mathcal{B}^{(\tau_0,\ell)} + \ldots + \sum_{\ell} c_{\tau_k,\ell}\, \mathcal{B}^{(\tau_k,\ell)}, \tag{6.103}$$

where $c_{\tau,\ell}$ stands for $M^{(2)}_{\tau,\ell}$ or $L^{(2)}_{\tau,\ell}$, and $\tau_k < \tau^{\max}$ is finite. Given the analytic representation of the conformal blocks $\mathcal{B}^{(\tau,\ell)}$, we can series expand the sum (6.103) in the form

$$u^{\tau_0} \sum_{n=0}^{\tau_k-\tau_0} \sum_{m=0}^{\infty} d_{nm}\, x^n \bar{x}^m, \tag{6.104}$$

and then resum it as

$$x^{\tau_0} \sum_{n=0}^{\tau_k-\tau_0} x^n g_n(\bar{x})\,. \tag{6.105}$$

where the functions g_n contain transcendental functions in the variable $\bar{x}$. In fact, the ansatz for the g_n descends from the full two-variable ansatz (6.101), upon performing the same series expansion as in (6.105). We call an expression of the above form a *one-variable resummation.*

correlator	initial free coeffs. in $h_\square, h_u, h_v, h_1$	after leading-log matching and pole cancellation	after OPE predictions in and below window
$\langle 2222\rangle$	1×378	1	1
$\langle 2233\rangle$	1×496	16	2
$\langle 2244\rangle$	1×579	20	2
$\langle 3333\rangle$	3×579	20	2
$\langle 4444\rangle$	6×946	68	4

Table 6.1: Number of tree-like free coefficients across the three steps of our algorithm.

The initial number of free coefficients grows with $p_1+p_2+p_3+p_4$, because of the denominator factors $(x-\bar{x})$ in (6.101), and obviously with the number of $su(4)$ channels. Cancelling $x=\bar{x}$ poles alone still leaves a large number of free coefficients. Imposing OPE predictions in and below the window is indeed crucial to finally obtain the minimal one-loop functions $\mathcal{D}^{(2)}_{\vec{p}}$, as can be seen in Table 6.1.

6.7.4 The (Finite) Set of Ambiguities

Imposing predictions in and below the window fixes the majority of the free coefficients in the ansatz. A sample of this process for a couple of correlators is illustrated in Table 6.1. The free parameters left are associated to a restricted class of tree-like functions, which we call *ambiguities*. By construction, such ambiguities do not contribute to the $\log^2(u)$ discontinuity in any channel, obey the correct crossing transformations by themselves, have no $x=\bar{x}$ poles and contribute only above the window, i.e for twists $\tau\geq\tau^{\max}$. Furthermore, we find the special feature that their SCPW coefficients have finite spin support: they contribute only to spins $\ell=0,1$.

The Mellin amplitude corresponding to these ambiguities is very simple, since it can be at most linear in the Mellin variables (s,t). This is for two reasons: firstly, it cannot be rational, as any additional pole would spoil the OPE predictions in and below the window. Therefore it has to be polynomial. Secondly, this polynomial cannot be higher order than linear, as it would generate tree-like terms with a higher degree denominator than allowed by our ansatz (6.101) for the minimal one-loop function $\mathcal{D}^{(2)}_{\vec{p}}$.

For a generic correlator without any crossing symmetries, we can parametrise the full set of ambiguities by

$$\mathcal{D}^{(2)}_{\vec{p}}\big|_{\text{ambiguity}}=\frac{u^{-\frac{p_{43}}{2}}}{v^{\frac{p_2+p_3}{2}}}\oint u^{\frac{s}{2}}v^{\frac{t}{2}}\,\Gamma_{\vec{p}}(s,t)\sum_{i=0}^{\kappa_{\vec{p}}-2}\sum_{j=0}^{\kappa_{\vec{p}}-2-i}\left(\alpha^{(1,ij)}_{\vec{p}}+\alpha^{(s,ij)}_{\vec{p}}s+\alpha^{(t,ij)}_{\vec{p}}t\right)\sigma^i\tau^j, \tag{6.106}$$

where $\Gamma_{\vec{p}}(s,t)$ is the usual string of six Γ-functions given by

$$\Gamma_{\vec{p}} = \Gamma\left[\tfrac{p_1+p_2-s}{2}\right]\Gamma\left[\tfrac{p_3+p_4-s}{2}\right]\Gamma\left[\tfrac{p_1+p_4-t}{2}\right]\Gamma\left[\tfrac{p_2+p_3-t}{2}\right]\Gamma\left[\tfrac{p_1+p_3-\tilde{u}}{2}\right]\Gamma\left[\tfrac{p_2+p_4-\tilde{u}}{2}\right]. \tag{6.107}$$

Thus, for a generic correlator, we find $\frac{3(\kappa_{\vec{p}}-1)\kappa_{\vec{p}}}{2}$ undetermined ambiguities. In cases in which the correlator has some crossing symmetry, we have to count only crossing symmetric combinations. Let us illustrate this by means of a few explicit examples:

- $\langle 2222 \rangle$: The only fully crossing symmetric combination one can build is the constant Mellin amplitude 1, so there can only be a single ambiguity: $\alpha^{(1)}_{2222}$.[5]

- $\langle 22pp \rangle$: This family of correlators is not fully crossing symmetric if $p > 2$. The remaining crossing symmetry can be understood as an invariance under $t \leftrightarrow \tilde{u}$. As a result, we are left with two out of three ambiguities,

$$\alpha^{(1)}_{22pp}, \qquad \text{and} \qquad \alpha^{(2)}_{22pp}\, s. \tag{6.108}$$

- $\langle 3333 \rangle$: This correlator admits up to linear terms in σ and τ, but crossing symmetry only allows two (fully symmetric) ambiguities,

$$\alpha^{(1)}_{3333}\,(1+\sigma+\tau), \qquad \text{and} \qquad \alpha^{(2)}_{3333}\,(s+\tilde{u}\sigma+t\tau). \tag{6.109}$$

  Other correlators with $\kappa_{\vec{p}} = 3$ but no crossing symmetries would admit a total of 9 ambiguities.

- $\langle 4444 \rangle$: The full crossing symmetry of this correlator greatly reduces the number of ambiguities. With at most quadratic terms in σ and τ, one can construct four independent ambiguities: two ambiguities with constant Mellin amplitudes

$$\alpha^{(1)}_{4444}\,(1+\sigma^2+\tau^2), \qquad \text{and} \qquad \alpha^{(2)}_{4444}\,(\sigma+\tau+\sigma\tau), \tag{6.110}$$

  and two other ambiguities with linear terms

$$\alpha^{(3)}_{4444}\,(s+\tilde{u}\sigma^2+t\tau^2), \qquad \text{and} \qquad \alpha^{(4)}_{4444}\,(t\sigma+\tilde{u}\tau+s\sigma\tau). \tag{6.111}$$

  Correlators with $\kappa_{\vec{p}} = 4$ but no crossing symmetries would otherwise admit 18 ambiguities.

Notice that our analysis here is already in agreement with the observed number of ambiguities, as shown in Table 6.1.

[5] As mentioned in Section 6.3, the value of $\alpha^{(1)}_{2222} = 60$ was found by using a supersymmetric localisation computation [104]. To our knowledge, this is the only case where a value for a one-loop ambiguity has been found. As such, the values of all other ambiguities remain undetermined.

With this we conclude the general discussion of the construction of one-loop supergravity correlators. In order to illustrate all the different features described at length in the above, we will now consider a few more concrete examples: in the next section, we will consider the minimal one-loop functions for the correlators $\langle 2244\rangle$, $\langle 3335\rangle$ and $\langle 4424\rangle$. These are of the same degree of extremality as the two examples $\langle 2222\rangle$ and $\langle 2233\rangle$ discussed before, and therefore their dynamical functions contribute to only one $su(4)$ channel, namely the $[0, p_{43}, 0]$ representation. Later, we also discuss the $\langle 3333\rangle$ correlator, which compared to the previous examples features multiple $su(4)$ channels. As a final and even more complicated example, the $\langle 4444\rangle$ correlator is discussed in Appendix D.

6.7.5 Examples of Next-to-Next-to-Extremal Correlators

In this section, we will consider a few next-to-next-to-extremal correlators, which are defined by the condition $\kappa_{\vec{p}} = 2$, i.e. their external charges are such that either $p_3 = 2$ or $p_1 + p_2 + p_3 - p_4 = 4$. In particular, we will consider the examples $\mathcal{D}^{(2)}_{3335}$, $\mathcal{D}^{(2)}_{4424}$ and $\mathcal{D}^{(2)}_{2244}$.

For such N^{2}E correlators there are no below-window OPE predictions, since the predictions for the semi-short sector $S_{p_1p_2p_3p_4}$ vanish because they are determined through (B.13) in terms of SCPW coefficients $S_{p(r)q(r)p_3p_4}$, where $p(r)+q(r) = p_{43}+2$. These correlators are next-to-extremal, and thus vanish identically as a consequence of our definition of external single particles states (see Section 3.2). Because of the split $\mathcal{T}^{(2)}_{\vec{p}} + \mathcal{D}^{(2)}_{\vec{p}}$, it then follows that $L^{(2)\mathcal{D}}_{2+p_{43},\,[0,p_{43},0]} = 0$.

An additional peculiarity of the $\langle 3335\rangle$ and $\langle 4424\rangle$ correlators (which generalises to other N^{2}E correlators) is the fact that the corresponding tree-level functions $\mathcal{H}^{(1)}_{3335}$ and $\mathcal{H}^{(1)}_{4424}$ are proportional to each other.[6] This implies that, up to a normalisation, both correlators have the same one-loop $\log^2(u)$ discontinuity. Therefore, an ansatz having the correct crossing symmetries, constructed by matching the leading-log discontinuity and imposing absence of $x = \bar{x}$ poles, cannot distinguish between $\mathcal{D}^{(2)}_{3335}$ and $\mathcal{D}^{(2)}_{4424}$. Very interestingly, this type of degeneracy is actually lifted at one-loop, because of the different OPE predictions in the window! This illustrates another important aspect of the OPE predictions in and below the window. In general, we expect the situation to be as follows: pairs of correlators which are degenerate at tree-level will instead have different minimal one-loop functions, and thus are distinguished by the OPE predictions in and below the window.

Concretely, in the case of the $\langle 3335\rangle$ and $\langle 4424\rangle$ correlators, we have different twist 6

[6] This can be understood from the hidden ten-dimensional conformal symmetry of the tree-level correlators $\mathcal{H}^{(1)}_{\vec{p}}$: the differential operators $\widehat{\mathcal{D}}_{3335}$ and $\widehat{\mathcal{D}}_{4424}$ differ only in their overall normalisations, see equation (3.43).

$\log(u)$ predictions in the $[0,2,0]$ channel, given by

$$\begin{aligned} M^{(2)}_{3335;6,\ell,[0,2,0]} &= \frac{Y^{(0)}_{3335} + Y^{(2)}_{3335}(\ell+\frac{9}{2})^2}{(\ell+1)(\ell+4)(\ell+5)(\ell+8)} \frac{(\ell+4)!(\ell+5)!}{(2\ell+8)!}, \\ M^{(2)}_{4424;6,\ell,[0,2,0]} &= \frac{Y^{(0)}_{4424} + Y^{(2)}_{4424}(\ell+\frac{9}{2})^2}{(\ell+1)(\ell+4)(\ell+5)(\ell+8)} \frac{(\ell+4)!(\ell+5)!}{(2\ell+8)!}, \end{aligned} \tag{6.112}$$

where the values of the free Y coefficients above, obtained from the OPE predictions, are

$$\begin{aligned} Y^{(0)}_{3335} &= -4762800, \qquad & Y^{(2)}_{3335} &= \tfrac{4}{35} Y^{(0)}_{3335}, \\ Y^{(0)}_{4424} &= -4628736, \qquad & Y^{(2)}_{4424} &= \tfrac{55}{2009} Y^{(0)}_{4424}. \end{aligned} \tag{6.113}$$

We now proceed according to our previously described bootstrap algorithm. In a first step, we match the ansatz (6.101) for the minimal one-loop functions $\mathcal{D}^{(2)}_{\vec{p}}$ against the computed double-logs and impose the correct crossing symmetries for the correlators. Secondly, we impose absence of unphysical poles at $x = \bar{x}$ on the ansatz. In a third and last step, we then impose the below-threshold OPE predictions.

The results for $\langle 3335 \rangle$ and $\langle 4424 \rangle$ can be obtained in the following instructive way. We initially normalize both correlators in a way that the leading double-logs are the same. After cancelling the unphysical poles, we still have one identical ansatz for both correlators, which has six free coefficients. We now insist that the SCPW coefficients of the ansatz at $\tau = 6$ have the form (6.112), where we do not specify the values of $Y^{(0)}_{\vec{p}}$ and $Y^{(2)}_{\vec{p}}$ yet. This constraint returns a one-parameter ansatz with one additional ambiguity. We go back to the correct normalisations for the correlators, and we keep $Y^{(0)}_{\vec{p}}$ as a free parameter, isolating the tree-like function it multiplies. Then, we can write the minimal one-loop functions in the form

$$\mathcal{D}^{(2)}_{\vec{p}} = \mathcal{N}_{\vec{p}}\, \mathcal{D}'^{(2)}_{\vec{p}} + \tfrac{1}{882} Y^{(0)}_{\vec{p}}\, u^2\, \overline{D}_{4444}, \quad \text{for } \vec{p} = 3335 \text{ and } 4424, \tag{6.114}$$

with $\mathcal{N}_{3335} = 135$ and $\mathcal{N}_{4424} = 128$. Because $Y^{(0)}_{3335} \neq Y^{(0)}_{4424}$, we ultimately find that $\mathcal{D}^{(2)}_{3335}$ and $\mathcal{D}^{(2)}_{4424}$ are not proportional to each other. Differently from the degeneracy at tree-level, the minimal one-loop functions are thus distinct.

The result for the $\langle 2244 \rangle$ correlator is more straightforward to obtain. In the window, we have twist 4 and 6 predictions in the $[0,0,0]$ representation, given by

$$M^{(2)}_{2244;4,\ell,[0,0,0]} = \frac{X^{(0)}_{2244}}{(\ell+1)(\ell+6)} \frac{((\ell+3)!)^2}{(2\ell+6)!}, \tag{6.115}$$

$$M^{(2)}_{2244;6,\ell,[0,0,0]} = \frac{Y^{(0)}_{2244} + Y^{(2)}_{2244}(\ell+\frac{9}{2})^2}{(\ell+1)(\ell+2)(\ell+7)(\ell+8)} \frac{((\ell+4)!)^2}{(2\ell+8)!}, \tag{6.116}$$

with predicted values,

$$X_{2244}^{(0)} = -8 \times 1920, \qquad Y_{2244}^{(0)} = 8 \times 176400, \qquad Y_{2244}^{(2)} = \tfrac{76}{245} Y_{2244}^{(0)}. \tag{6.117}$$

In the other orientation of the correlator, $\langle 2424 \rangle$, the window is empty. The bootstrap algorithm returns $\mathcal{D}_{2244}^{(2)}$ leaving only two ambiguities, in agreement with (6.108).

The minimal one-loop functions corresponding to $\langle 3335 \rangle$, $\langle 4424 \rangle$ and $\langle 2244 \rangle$ are given in an ancillary file. For $\langle 3335 \rangle$ and $\langle 4424 \rangle$ we have only included $\mathcal{D}'^{(2)}_{\vec{p}}$, see equation (6.114). In all cases, we have fixed a particular value of the ambiguities and we have checked that their SCPW coefficients have only finite spin support above the respective threshold twists.

6.7.6 The $\langle 3333 \rangle$ Correlator

We continue to illustrate our bootstrap algorithm with the $\langle 3333 \rangle$ correlator. The solutions for the polynomial coefficients $h_1, \ldots h_{11}, h_\Box, h_u, h_v, h_0$ are listed in an attached `Mathematica` notebook, where for simplicity the ancillary file contains $\mathcal{D}_{3333}^{(2)}$ with a particular value of the ambiguities.

The $\langle 3333 \rangle$ correlator has degree of extremality $\kappa_{3333} = 3$ and it is fully crossing symmetric. The long sector decomposes into the three representations $[0,0,0]$, $[1,0,1]$ and $[0,2,0]$, with threshold twist $\tau^{\max} = 6$. We start by obtaining the two-variable resummations of the $\log^2(u)$ discontinuities in the three channels. We then match the ansatz against the computed double-logs in all channels and impose crossing symmetry. Secondly, we impose absence of $x = \bar{x}$ poles on the ansatz, which leaves us with 20 free parameters. Finally, we have to impose the OPE predictions in and below the window. Since all the external charges are equal in this example, the window region is empty. This implies that upon projecting the ansatz onto the $\log(u)$ stratum, we have to set to zero the one-variable expansion up to order $O(u^3)$. On the other hand, the OPE predictions below the window are non-trivial: in the singlet channel the unitary bound lies at twist $\tau = 2$, where no long supergravity states contribute since there are only string states present at that twist. We thus have

$$L^{(2)\mathcal{D}}_{3333;2,\ell,[0,0,0]} = 0. \tag{6.118}$$

A non-trivial prediction comes in at twist $\tau = 4$. Here there is only one double-trace operator $K_{22;4,\ell,[0,0,0]} \sim \mathcal{O}_2 \partial^\ell \mathcal{O}_2$. Using (6.76) we thus get a prediction for $L^{(2)\mathcal{D}}_{3333;4,\ell,[0,0,0]}$ and we find

$$L^{(2)\mathcal{D}}_{3333;4,\ell,[0,0,0]} = \frac{9 \times 4800}{(\ell+1)(\ell+6)} \, \frac{((\ell+3)!)^2}{(2\ell+6)!} \, \frac{1+(-1)^\ell}{2}. \tag{6.119}$$

Performing the one-variable resummation of the above coefficients (6.119), i.e. performing the sum of equation (6.78), we have

$$\mathcal{D}^{(2)}_{3333}\big|_{[0,0,0],\,\log^0(u)} = 9\,\frac{6!x^2}{\bar{x}^4}\,\left[5(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x}) + \tfrac{5}{3}(6-6\bar{x}+\bar{x}^2)\mathrm{Li}_1^2(\bar{x})\right] + O(x^3). \quad (6.120)$$

In the $[1,0,1]$ and $[0,2,0]$ channels, the unitary bound is at twist $\tau = 4$, and there are no predictions descending from the long sectors at tree-level. Instead, this is the first case where we need to consider the consequences of protected semi-short operators through our formula (6.90), using the results for $S_{4;\ell+2,[1]}$ and $S_{4;\ell+2}$ given in (B.16). More precisely, there is an $S_{4;\ell+2,[1]}$ contribution to the $[1,0,1]$ channel, which implies

$$L^{(2)\mathcal{D}}_{3333;4,\ell,[1,0,1]} = \frac{9\times 576}{(\ell+2)(\ell+5)}\,\frac{((\ell+3)!)^2}{(2\ell+6)!}\,\frac{1-(-1)^\ell}{2}, \quad (6.121)$$

with corresponding one-variable resummation

$$\mathcal{D}^{(2)}_{3333}\big|_{[1,0,1],\,\log^0(u)} = 9\,\frac{6!x^2}{\bar{x}^4}\,\left[3(\bar{x}-2)\bar{x} + (6-6\bar{x}+\tfrac{7}{5}\bar{x}^2)\mathrm{Li}_1(\bar{x}) + \tfrac{1}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1^2(\bar{x})\right] + O(x^3). \quad (6.122)$$

Similarly, in the $[0,2,0]$ channel we have a contribution from $S_{4,\ell+2,[0]}$ which gives

$$L^{(2)\mathcal{D}}_{3333;4,\ell,[0,2,0]} = \frac{9\times 288}{(\ell+3)(\ell+4)}\,\frac{((\ell+3)!)^2}{(2\ell+6)!}\,\frac{1+(-1)^\ell}{2}, \quad (6.123)$$

with one-variable resummation

$$\mathcal{D}^{(2)}_{3333}\big|_{[0,2,0],\,\log^0(u)} = 9\frac{6!x^2}{\bar{x}^4}\left[\tfrac{6}{5}\bar{x}^2 + \tfrac{3}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x}) + \tfrac{1}{10}\bar{x}^2\mathrm{Li}_1^2(\bar{x})\right] + O(x^3). \quad (6.124)$$

Note that there is an important logical distinction between $L^{(2)\mathcal{D}}_{3333;4,\ell,[1,0,1]}$ and $L^{(2)\mathcal{D}}_{3333;4,\ell,[0,2,0]}$ we should highlight: in the $[0,2,0]$ channel, the twist 4 contribution lies at the bottom of multiplet recombination, in the sense that $\tau = 2a+b+2$ with $b=2$ and $a=0$. This means that the corresponding SCPW coefficients do not get shifted by multiplet recombination in another $su(4)$ representation. In fact, our formula (6.90) makes explicit that there is no extra summation over a that needs to be taken into account. This is not the case for the twist 4 contribution in the $[1,0,1]$ channel, where instead the SCPW coefficients receive a contribution due to multiplet recombination from twist 2 in the singlet channel. However, there is no $S_{2;\ell+2,[0]}$ contribution, and therefore $L^{(2)\mathcal{D}}_{3333;4,\ell,[1,0,1]} = S_{4;\ell+2,[1]}$ holds exactly.

Coming back to our ansatz, we match the resummations from the above equations (6.120), (6.122) and (6.124). Recall that we had 20 free coefficients which were not fixed by imposing absence of unphysical poles at $x=\bar{x}$. Now, after matching the OPE predictions below the window, we are left with only two free coefficients, which are exactly the two ambiguities described in (6.109). Upon inspection, their SCPW coefficients contribute only to spin $\ell = 0$ for twists above threshold.

Lastly, one further example of even higher complexity, the $\langle 4444 \rangle$ correlator, is described in Appendix D.

6.8 Generalised Tree-Level Mellin Amplitudes

In the previous sections, we argued that the one-loop function $\mathcal{H}^{(2)}_{\vec{p}}$ admits the splitting $\mathcal{H}^{(2)}_{\vec{p}} = \mathcal{T}^{(2)}_{\vec{p}} + \mathcal{D}^{(2)}_{\vec{p}}$, where the minimal one-loop function $\mathcal{D}^{(2)}_{\vec{p}}$ encodes all the non-trivial OPE predictions at order $1/N^4$, whereas $\mathcal{T}^{(2)}_{\vec{p}}$ is a generalised tree-level function without $\log^2(u)$ contribution. The purpose of this section and our final task is to bootstrap $\mathcal{T}_{\vec{p}}$.

We define the generalised tree-level function $\mathcal{T}_{\vec{p}}$ as the unique function within the ansatz

$$\mathcal{T}_{\vec{p}} = \frac{P_{\Box}\, \phi^{(1)}(x,\bar{x})}{(x-\bar{x})^{d_{\vec{p}}+2}} + \frac{P_v \log(v)}{(x-\bar{x})^{d_{\vec{p}}+1}} + \frac{1}{v^{\kappa_{\vec{p}}-1}}\Big[\frac{P_u \log(u)}{(x-\bar{x})^{d_{\vec{p}}+1}} + \frac{P_1}{(x-\bar{x})^{d_{\vec{p}}-1}}\Big], \qquad (6.125)$$

such that:

(a) the threshold twist for the $\log(u)$ discontinuity is $\tau = \tau^{\max}$.

(b) the SCPW expansion below the window *completely cancels* the free theory contributions as described in equations (6.87) and (6.88).

(c) there are no unphysical poles at $x = \bar{x}$ in the ansatz (6.125).

The coefficient functions denoted by P are polynomials in x, $\bar{x}$ and σ, τ. As functions of the variables x and $\bar{x}$, these polynomials have a Taylor expansion of the form $x^n\bar{x}^m$ with $m + n \leq p_1 + p_2 + p_3 + p_4$. The function $\mathcal{T}_{\vec{p}}$ is symmetric under $x \leftrightarrow \bar{x}$, and therefore a given polynomial P has the same symmetry as the transcendental function it multiplies. Note that the $su(4)$ decomposition of $\mathcal{T}_{\vec{p}}$ is the same as for the full dynamical function $\mathcal{H}_{\vec{p}}$. Implementing condition (a) implies

$$\begin{aligned} P_{\Box} &= O(u^{-\frac{p_{43}}{2}+\frac{\max\{p_1+p_2,\, p_3+p_4\}}{2}}), \\ P_u &= O(u^{-\frac{p_{43}}{2}+\frac{\max\{p_1+p_2,\, p_3+p_4\}}{2}}). \end{aligned} \qquad (6.126)$$

The above conditions in fact define a generalised tree-level function $\mathcal{T}_{\vec{p}}$ which is *exact* in N. We can define it in terms of the coefficients A^k_γ in front of each propagator structure from equation (2.25), which we can leave completely arbitrary (in contrast to the relations between them arising from imposing crossing symmetry). The polynomials P in (6.125) then become functions of the free theory propagator coefficients, $P = P[\{A^k_\gamma\}]$, and the precise value of these A^k_γ does not affect any step of this algorithm. Furthermore, the condition (b) is overconstraining, and therefore the solution we find is unique, i.e. $\mathcal{T}_{\vec{p}}$ is unique.

Because of this uniqueness, we expect our function $\mathcal{T}_{\vec{p}}$ to reduce to the known results at tree-level, and we can take the propagator coefficients A^k_γ to take on their free theory values. Indeed, when the external charges are equal, our conditions are precisely those imposed in [43], and for arbitrary charges we expect to recover the tree-level correlators $\mathcal{H}^{(1)}_{\vec{p}}$ of Rastelli and Zhou [47]. Notice that in position space the function of [47] is described by the same ansatz as in (6.125), except for the change $d_{\vec{p}} \to d_{\vec{p}} - 2$. In fact, we find that all polynomials $P_{\square,u,v,1}[\{A^k_\gamma\}]$ non-trivially factor out an extra double-zero $(x-\bar{x})^2$ when we restrict the A^k_γ to their tree-level values, and $\mathcal{T}_{\vec{p}}$ precisely reduces to the tree-level correlators $\mathcal{H}^{(1)}_{\vec{p}}$ when the coefficients A^k_γ are truncated to order $1/N^2$.

Restricted to tree-level, the free theory coefficients $A_\gamma\big|_{1/N^2}$ are all proportional to each other, and thus satisfy linear relations. Therefore, we can understand the tree-level degeneration as the result of imposing these linear relations on the coefficients $P_{\square,u,v,1}[\{A^k_\gamma\}]$. However, beyond order $1/N^2$, the non-planar values of the A^k_γ are not as simple and the corresponding relations become non-linear.

Similarly to the tree-level functions $\mathcal{H}^{(1)}_{\vec{p}}$, the most transparent representation of $\mathcal{T}_{\vec{p}}$ is given in Mellin space. We thus define the corresponding Mellin amplitudes $\mathcal{M}[\mathcal{T}_{\vec{p}}](s,t)$ of the generalised tree-level functions similarly to that of the tree-level functions $\mathcal{H}^{(1)}_{\vec{p}}$. In fact, all the generalised tree-level functions $\mathcal{T}_{\vec{p}}$, as defined by the above conditions (a), (b) and (c), can be written in this form with a simple rational Mellin amplitude with only simple poles. The specific form of $\mathcal{M}[\mathcal{T}_{\vec{p}}]$, i.e. finiteness and rationality, translates into the observation that the entire function $\mathcal{T}_{\vec{p}}$ is determined uniquely in terms of the coefficient $P_\square$ in front of the one-loop box function. This can be understood from the fact that the box function contains a $\log(u)\log(v)$ term, which arises only from double-poles in both s and t in the Mellin transform, and it turns out that this is entirely determined in terms of the Mellin amplitude $\mathcal{M}[\mathcal{T}_{\vec{p}}]$ and vice versa. In the following, we illustrate the above discussions by explicitly considering the generalised tree-level amplitudes $\mathcal{T}_{\vec{p}}$ for next-to-next-to-extremal correlators and the case $\mathcal{T}_{3333}$. As a further example, $\mathcal{T}_{4444}$ is described in Appendix D.2.

6.8.1 Next-to-Next-to-Extremal Correlators

Next-to-next-to-extremal correlators have extremality $\kappa_{\vec{p}} = 2$. As such, their free theory correlators have a total of six propagator structures and their long sectors admit only a single $su(4)$ channel, namely the $[0, p_{43}, 0]$ representation. The definition of single-particle states has two non-trivial consequences: firstly, as we have seen in Section 3.2, the number of connected propagator structures actually reduces to three. Secondly, the connected part of next-to-next-to-extremal correlators in the free theory is given by the *exact* formula,

$$\frac{\langle p_1p_2p_3p_4\rangle_{\text{free conn.}}}{\mathcal{P}_{\vec{p}}} = p_1p_2p_3p_4\ F_{\vec{p}}(N^2) \times \Big[\big(1+\tfrac{p_{43}+p_{21}}{2}\big)\frac{u\tau}{v} + \big(1+\tfrac{p_{13}+p_{42}}{2}\big)u\sigma + \big(1+\tfrac{|p_{23}+p_{14}|}{2}\big)\frac{u^2\sigma\tau}{v}\Big], \tag{6.127}$$

where $F_{\vec{p}}$ scales like $N^{(p_1+p_2+p_3+p_4-4)/2}$ in the large N limit. For example, we have

$$F_{2244} = F_{3324} = \frac{\prod_{k=1}^{3}(N^2-k^2)}{(N^2+1)}, \qquad F_{3335} = F_{3524} = \frac{\prod_{k=1}^{4}(N^2-k^2)}{N(N^2+5)},$$
$$F_{4424} = \frac{\prod_{k=1}^{3}(N^2-k^2)(N^4-20N^2+9)}{N(N^2+1)^2}. \tag{6.128}$$

Thus, for next-to-next-to-extremal correlators, the non-planar result (6.127) is the factorised product of the order $1/N^2$ connected free theory uplifted to all N by the factor $F_{\vec{p}}(N^2)$. It follows that the all N relative coefficients among the three propagator structures is already captured by the order $1/N^2$ result, such that $\mathcal{M}[\mathcal{T}_{\vec{p}}]$ is simply proportional to $\mathcal{M}^{(1,0)}_{\vec{p}}$ as given by formula (3.33). Notice also that the factors $F_{\vec{p}}(N^2)$ manifestly vanish when the number of colours N is less than the charge of any of the external operators. Both these statements would be false if we replaced our single-particle operators $\mathcal{O}_p$ with the corresponding bare single-trace one-half BPS operators, i.e. dropping the multi-trace admixtures.

The particular structure of the connected free theory in (6.127) implies the following exact relations on the SCPW coefficients,

$$L^{f}_{\vec{p};p_{43}+2,\ell,[0,p_{43},0]} = F_{\vec{p}}(N^2) \times \Big[A_{[\ell+2]}\big|_{\frac{1}{N^2}}\Big],$$
$$L^{f}_{\vec{p};\tau>p_{43}+2,\ell,[0,p_{43},0]} = F_{\vec{p}}(N^2) \times L^{(1)f}_{\vec{p};\tau,\ell,[0,p_{43},0]}. \tag{6.129}$$

Therefore, for the purpose of constructing generalised tree-level functions, the defining condition (b) simply becomes

$$L^{\mathcal{T}}_{\vec{p};p_{43}+2,\ell,[0,p_{43},0]} + F_{\vec{p}}(N^2)\Big[A_{[\ell+2]}\big|_{\frac{1}{N^2}}\Big] = 0, \tag{6.130}$$

and by uniqueness we conclude that for next-to-next-to-extremal correlators the generalised tree-level function $\mathcal{T}_{\vec{p}}$ equals the supergravity tree-level result $\mathcal{H}^{(1)}_{\vec{p}}$ multiplied by the all N factor $F_{\vec{p}}(N^2)$.

6.8.2 ⟨3333⟩

Let us now consider a slightly more involved example: the ⟨3333⟩ correlator. The result for the free theory correlator was previously given pictorially in (2.23). In terms of conformal and $su(4)$ cross-ratios, its connected part reads

$$\frac{\langle 3333\rangle_{\text{free conn.}}}{g_{12}^3 g_{34}^3} = \frac{9(N^2-4)^2(N^2-1)}{N^2}\Big[\frac{18(N^2-12)}{(N^2-4)}\frac{u^2\sigma\tau}{v} + 9\Big(u\sigma + \frac{u\tau}{v} + u^2\sigma^2 + \frac{u^2\tau^2}{v^2} + \frac{u^3\sigma^2\tau}{v^2} + \frac{u^3\sigma\tau^2}{v^2}\Big)\Big]. \tag{6.131}$$

Crossing invariance of the $\langle 3333\rangle$ correlator restricts the total number of connected coefficients $\{A_2^k, A_4^k, A_6^k\}$ in the generic sum over propagator structures (2.25) to only two independent ones, and we have

$$A_2^0 = A_2^1 = A_4^0 = A_4^2 = A_6^1 = A_6^2 = \frac{9}{(N^2-1)}\,A_0^0, \tag{6.132}$$

$$A_4^1 = \frac{18(N^2-12)}{(N^2-4)(N^2-1)}\,A_0^0, \tag{6.133}$$

with $A_0^0 = \frac{9(N^2-4)^2(N^2-1)^2}{N^2}$. The generalised tree-level function in Mellin space is given by

$$\mathcal{T}_{3333} = u^3 \oint u^s v^t\ \Gamma[-s]^2\Gamma[-t]^2\Gamma\big[s+t+5\big]^2\mathcal{M}[\mathcal{T}_{3333}], \tag{6.134}$$

with Mellin amplitude

$$\begin{aligned}\mathcal{M}[\mathcal{T}_{3333}] = &-\frac{1}{(s+2)(t+1)(s+t+4)}\ \big[A_2^0 - \tfrac{1}{4}(s+2)(A_4^1 - 2A_2^0)\big]\\ &-\frac{\tau}{(s+1)(t+2)(s+t+4)}\ \big[A_2^0 - \tfrac{1}{4}(t+2)(A_4^1 - 2A_2^0)\big]\\ &-\frac{\sigma}{(s+1)(t+1)(s+t+3)}\ \big[A_2^0 - \tfrac{1}{4}(s+t+3)(A_4^1 - 2A_2^0)\big].\end{aligned} \tag{6.135}$$

After a shift in the Mellin variables s and t, the Mellin amplitude $\mathcal{M}_{3333}^{(1,0)}$ of Rastelli and Zhou would correspond only to the term multiplied by A_2^0 in the above (6.135). Indeed, the new contribution, proportional to $(A_4^1 - 2A_2^0)$, vanishes when we plug in the relations (6.132) and (6.133) and expand to order $1/N^2$.

Lastly, the generalised tree-level correlator $\mathcal{T}_{4444}$ and some more comments about the patterns in the Mellin amplitudes $\mathcal{M}[\mathcal{T}_{\vec{p}}]$ can be found in Appendix D.2.

Chapter 7

One-Loop String Corrections

In this final chapter, we will discuss how to construct further string corrections to the previously presented one-loop supergravity results. Recall that the tower of string corrections descends from contact terms in the string theory effective action, leading to the double expansion

$$\begin{aligned}\mathcal{H}_{\vec{p}} = \;& \mathfrak{a}\left(\mathcal{H}_{\vec{p}}^{(1,0)} + \lambda^{-\frac{3}{2}}\mathcal{H}_{\vec{p}}^{(1,3)} + \lambda^{-\frac{5}{2}}\mathcal{H}_{\vec{p}}^{(1,5)} + \lambda^{-3}\mathcal{H}_{\vec{p}}^{(1,6)} + \lambda^{-\frac{7}{2}}\mathcal{H}_{\vec{p}}^{(1,7)} + \ldots\right) \\ &+ \mathfrak{a}^2\left(\lambda^{\frac{1}{2}}\mathcal{H}_{\vec{p}}^{(2,-1)} + \mathcal{H}_{\vec{p}}^{(2,0)} + \lambda^{-\frac{1}{2}}\mathcal{H}_{\vec{p}}^{(2,1)} + \lambda^{-1}\mathcal{H}_{\vec{p}}^{(2,2)} + \lambda^{-\frac{3}{2}}\mathcal{H}_{\vec{p}}^{(2,3)} + \ldots\right) \\ &+ O(\mathfrak{a}^3).\end{aligned} \tag{7.1}$$

While we have discussed the one-loop supergravity term $\mathcal{H}_{\vec{p}}^{(2,0)}$ in the previous chapter, we will now consider the tower of $1/\lambda$ corrections. In particular, we will focus on the genuine one-loop contributions $\mathcal{H}_{\vec{p}}^{(2,k)}$ with $k \geq 3$, which are induced by the tree-level terms $\mathcal{H}_{\vec{p}}^{(1,3)}$ etc.[1] These terms have been addressed before in Mellin space for the simplest example, the $\langle 2222 \rangle$ correlator [128], and more recently generalised to the $\langle 22pp \rangle$ family of correlators at order $\lambda^{-\frac{3}{2}}$ [10]. Here we will describe the structure of the corresponding position space representation, and also provide new results both in spacetime and in Mellin space for higher orders in the $1/\lambda$ expansion. We find that, in some sense, the one-loop string corrections are simpler than the supergravity term $\mathcal{H}_{\vec{p}}^{(2,0)}$, as the transcendental weight in the spacetime ansatz is actually lower: the supergravity amplitudes require functions up to transcendental weight four, while the string corrections (essentially due to the finite spin support of the string corrected double-trace spectrum) require only weights up to three. On the other hand, we find that we necessarily need a new ingredient, a weight-three function $f^{(3)}$ with a more general set of singularities (or 'letters').

[1] Recall that the terms $\mathcal{H}_{\vec{p}}^{(2,-1)}$, $\mathcal{H}_{\vec{p}}^{(2,1)}$ and $\mathcal{H}_{\vec{p}}^{(2,2)}$ correspond to the genus-one modular completions of the $\mathcal{H}_{\vec{p}}^{(1,3)}$, $\mathcal{H}_{\vec{p}}^{(2,5)}$ and $\mathcal{H}_{\vec{p}}^{(2,6)}$ terms, respectively. Some of those coefficients have been recently fixed by supersymmetric localisation techniques, see references [104, 105]. The first genuine one-loop contribution is then the order $\lambda^{-\frac{3}{2}}$ term $\mathcal{H}_{\vec{p}}^{(2,3)}$.

For simplicity we will consider only the $\langle 2222 \rangle$ correlator, for which we do not need to worry about any subtleties of below-threshold effects discussed at length in the last chapter.[2] Recall that due to the full crossing symmetry of this correlator, its dynamical function $\mathcal{H}_{2222}$ obeys the crossing relations

$$\mathcal{H}_{2222}(u,v) = \frac{1}{v^2}\mathcal{H}_{2222}(u/v, 1/v) = \frac{u^2}{v^2}\mathcal{H}_{2222}(v,u), \tag{7.2}$$

which will enter as one of the constraints in our bootstrap algorithm described in the following.

7.1 Bootstrap Method in Position Space

We begin by reviewing how to obtain the double discontinuity (i.e. the $\log^2(u)$ part of the correlator) from tree-level data only. In Section 7.1.2, this will then allow us to pose a well-defined bootstrap problem, whose solution completely determines the one-loop correlators $\mathcal{H}^{(2,k)}_{2222}$ from a given double discontinuity $\mathcal{H}^{(2,k)}_{2222}|_{\log^2(u)}$, up to a finite number of well understood ambiguities. Notably, a new ingredient enters our ansatz of transcendental functions: it turns out that a certain function of transcendental weight three with a new type of singularity has to be included. We describe this new ingredient in Section 7.1.3.

7.1.1 Predicting the String Corrected Double-Log

Let us start by discussing the specific form of the double discontinuities which arise in the $1/\lambda$ expansion at one-loop order. As is the case for the one-loop supergravity correlators, the string corrected double discontinuity is fully determined by tree-level data through the conformal block decomposition. More explicitly, we can compute the $\log^2(u)$ part of $\mathcal{H}^{(2,k)}_{2222}$ from spectral data at order $\mathfrak{a}$:

$$\mathcal{H}^{(2,k)}_{2222}(x,\bar{x})\big|_{\log^2(u)} = \sum_{m+n=k} \mathcal{D}^{m|n}(x,\bar{x}), \tag{7.3}$$

where for $m \neq n$ we need to include the two identical contributions $\mathcal{D}^{m|n}$ and $\mathcal{D}^{n|m}$, which are defined through the SCPW expansion by

$$\mathcal{D}^{m|n}(x,\bar{x}) = \frac{1}{2}\sum_{t,\ell}\sum_{i=1}^{t-1}\left(C^{(0)}_{22\mathcal{K}_{t,\ell,i}}\right)^2 \eta_i^{(1,m)}\eta_i^{(1,n)}\ \frac{\mathcal{B}^{(t+2,\ell)}(x,\bar{x})}{u^2}, \tag{7.4}$$

[2] See reference [10] for more details on how to deal with below-threshold predictions in the context of one-loop string corrections, where the window region of $\langle 22pp \rangle$ correlators is described in detail.

where $\mathcal{B}^{(t,\ell)}$ denotes the conformal block from equation (2.34), the $\eta_i^{(1,m)}$ are (half) the tree-level anomalous dimensions at order $\lambda^{-\frac{m}{2}}$, and i labels the set of exchanged singlet channel double-trace operators $\mathcal{K}_{t,\ell,i}$, see (4.7).[3] Recalling the double expansion (7.1), the general structure of the $1/\lambda$ expansion at tree-level demands that the integers k, m, n in the above equation are drawn from the set $\{0,3\} \cup \{5,6,7,8\ldots\}$, with the constraint $m+n=k$. Note that when k is large enough to accommodate for different partitions into (m,n), we get more than one contribution to the double discontinuity at that order in $1/\lambda$.[4] See also Table 7.1 for the first few one-loop terms in the $1/\lambda$ expansion.

We have discussed the general form of the supergravity double discontinuities in detail in the previous chapter, and we have found that the two-variable resummation of $\mathcal{D}^{0|0}$ gives rise to transcendental functions of up to weight two. In contrast to the supergravity case however, it turns out that the string corrected double discontinuities ($\mathcal{D}^{m|n}$ with $m,n \neq 0$) resum into expressions of up to transcendental weight one only. The reason for this is the spin truncation in the string corrected spectrum. To be explicit, the double discontinuities $\mathcal{D}^{m|n}$ are of the general form

$$\mathcal{D}^{m|n}(x,\bar{x}) = u^2 \left(\frac{p_1^{m|n}(x,\bar{x})}{(x-\bar{x})^{q-1}} + \frac{p_2^{m|n}(x,\bar{x})\big(\log(1-x)-\log(1-\bar{x})\big)}{(x-\bar{x})^{q}} \right), \tag{7.5}$$

where the denominator powers are given in term of $q = 2(m+n)+15$ and p_1, p_2 are symmetric polynomials in $(x,\bar{x})$ of the same degree as their respective denominator. This simple structure for the double discontinuities was already obtained in [120], and we find complete agreement with their results by explicitly performing the sum (7.4) for different cases.

Note that the double discontinuities have a symmetry under the $1 \leftrightarrow 2$ crossing transformation, which acts on the cross-ratios as $x \to x' \equiv x/(x-1)$, and similarly for $\bar{x}$. This symmetry is inherited from the full crossing symmetry of the $\langle 2222 \rangle$ correlator, and is preserved by the s-channel OPE decomposition. As a formula, we have

$$\mathcal{D}^{m|n}(x',\bar{x}') = v^2\, \mathcal{D}^{m|n}(x,\bar{x}). \tag{7.6}$$

In the following, we provide an algorithm on how to uplift the double discontinuity $\mathcal{D}^{m|n}$ to the corresponding fully crossing symmetric function $\mathcal{H}_{2222}^{(2,k)}(u,v)$.

[3] As outlined in Section 6.2, one can also obtain the SCPW coefficients of the leading double-log through 'squaring' the SCPW coefficients tree-level correlators, without explicitly solving the double-trace mixing-problem. This method simply extends to string corrected double discontinuities for the $\langle 2222 \rangle$ correlator at any given order in $1/\lambda$, which can be obtained from the $\langle 22pp \rangle$ family of correlators up to that order (see e.g. [120] for an application of this complementary approach). In this case however, since we have the precise form of the unmixed three-point functions $C^{(0)}_{22\mathcal{K}_{t,\ell,i}}$ as well as the anomalous dimensions, we prefer to explicitly perform the two-variable resummation of the sums (7.4).

[4] The first instance of this happens already for $k=6$ at order λ^{-3}, for which there are the two distinct possibilities $(m,n) = (0,6)$ or $(3,3)$. These contributions correspond to the insertions of $S|\partial^6\mathcal{R}^4$ and $\mathcal{R}^4|\mathcal{R}^4$ vertices, respectively.

7.1.2 The Bootstrap Problem

In order to simplify the crossing transformations (7.2) of the interacting part $\mathcal{H}_{2222}(u,v)$, we introduce an auxiliary function $\mathcal{F}$ by

$$\mathcal{F}(u,v) = \frac{(x-\bar{x})^4}{u^2}\mathcal{H}_{2222}(u,v), \tag{7.7}$$

such that $\mathcal{F}(u,v)$ transforms without picking up any prefactors under crossing:

$$\mathcal{F}(u,v) = \mathcal{F}(u/v,1/v) = \mathcal{F}(v,u). \tag{7.8}$$

Evidently, $\mathcal{F}$ inherits an analogous double expansion as $\mathcal{H}$ in (7.1). Guided by the explicit form of the double discontinuities as given in (7.5), we propose the following structure for the functions $\mathcal{F}^{(2,k)}$ $(k>0)$:

$$\begin{aligned}\mathcal{F}^{(2,k)}(u,v) = {} & A_1(x,\bar{x})f^{(3)}(x,\bar{x}) + \big(A_2(x,\bar{x})\log(u) - A_2(1-x,1-\bar{x})\log(v)\big)\phi^{(1)}(x,\bar{x}) \\ & + A_3(x,\bar{x})\log^2(u) + A_3\Big(\frac{x-1}{x},\frac{\bar{x}-1}{\bar{x}}\Big)\log^2\Big(\frac{u}{v}\Big) + A_3\Big(\frac{1}{1-x},\frac{1}{1-\bar{x}}\Big)\log^2(v) \\ & + A_4(x,\bar{x})\phi^{(1)}(x,\bar{x}) + \big(A_5(x,\bar{x})\log(u) + A_5(1-x,1-\bar{x})\log(v)\big) + A_6(x,\bar{x}).\end{aligned} \tag{7.9}$$

The main new feature of this ansatz is the presence of $f^{(3)}(x,\bar{x})$, which is an antisymmetric single-valued function of transcendental weight three. As this function is new in the context of AdS amplitudes, involving a new type of singularity compared to the supergravity case, we will describe it in more detail in the next section (see also Appendix E).

On the other hand, the function $\phi^{(1)}(x,\bar{x})$ is the well-known one-loop massless box-integral in four-dimensions, which we already encountered in the ansatz for the minimal one-loop functions in Section 6.7. It is an antisymmetric weight-two function given by

$$\phi^{(1)}(x,\bar{x}) = 2\big(\mathrm{Li}_2(x) - \mathrm{Li}_2(\bar{x})\big) + \log(u)\,\big(\log(1-x) - \log(1-\bar{x})\big), \tag{7.10}$$

and obeys the symmetries

$$\phi^{(1)}(x,\bar{x}) = -\phi^{(1)}(\bar{x},x) = -\phi^{(1)}(1-x,1-\bar{x}) = -\phi^{(1)}(1/x,1/\bar{x}). \tag{7.11}$$

Let us highlight the two main differences of the above ansatz (7.9) to the one-loop supergravity case:

- The ansatz for $\mathcal{F}^{(2,k)}(u,v)$ has maximal transcendental weight three, compared to up to weight-four contributions in supergravity. This difference is ultimately a consequence of the spin truncation of the string corrected spectrum. A truncation to finite spin produces resummed double discontinuities of the form depicted in (7.5), which has terms of maximal weight one. In contrast, the supergravity spectrum has infinite spin support, resulting in up to weight-two contributions to the corresponding double discontinuity.

- As mentioned before, the presence of the function $f^{(3)}(x,\bar{x})$ is a novelty in the context of AdS amplitudes. However, one can already see from the structure of the double discontinuities $\mathcal{D}^{m|n}$ that a new ingredient is required: as we will discuss shortly, an ansatz with ladder functions only would enforce a structure on the polynomial $p_2^{m|n}$ in $\mathcal{D}^{m|n}$ which is not observed from direct resummations. We are therefore led to conclude that we need a new contribution in our ansatz, which we denote by $f^{(3)}(x,\bar{x})$ and whose full characterisation we postpone to Section 7.1.3.

Finally, in order to ensure both the exchange symmetry $x \leftrightarrow \bar{x}$ as well as the full crossing symmetries (7.8) of the ansatz $\mathcal{F}^{(2,k)}(u,v)$, the coefficient functions $A_i(x,\bar{x})$ obey the following relations:

$$
\begin{aligned}
&A_1(x,\bar{x}) = -A_1(\bar{x},x), \qquad && A_1(x,\bar{x}) = -A_1\left(\frac{1}{x},\frac{1}{\bar{x}}\right) = -A_1(1-x,1-\bar{x}),\\
&A_2(x,\bar{x}) = -A_2(\bar{x},x), && A_2(x,\bar{x}) = -A_2\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right),\\
&A_3(x,\bar{x}) = A_3(\bar{x},x), && A_3(x,\bar{x}) = A_3\left(\frac{1}{x},\frac{1}{\bar{x}}\right),\\
&A_4(x,\bar{x}) = -A_4(\bar{x},x), && A_4(x,\bar{x}) = -A_4\left(\frac{1}{x},\frac{1}{\bar{x}}\right) = -A_4(1-x,1-\bar{x}),\\
&A_5(x,\bar{x}) = A_5(\bar{x},x), && A_5(x,\bar{x}) = A_5\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right),\\
&A_6(x,\bar{x}) = A_6(\bar{x},x), && A_6(x,\bar{x}) = A_6\left(\frac{1}{x},\frac{1}{\bar{x}}\right) = A_6(1-x,1-\bar{x}).
\end{aligned}
\tag{7.12}
$$

Additionally, each of the coefficients $A_2(x,\bar{x})$ and $A_5(x,\bar{x})$ obey one more constraint:

$$
\begin{aligned}
A_2(x,\bar{x}) + A_2\left(\frac{1}{1-x},\frac{1}{1-\bar{x}}\right) - A_2(1-x,1-\bar{x}) = 0,\\
A_5(x,\bar{x}) + A_5\left(\frac{1}{1-x},\frac{1}{1-\bar{x}}\right) + A_5(1-x,1-\bar{x}) = 0.
\end{aligned}
\tag{7.13}
$$

The above transformation properties as well as the form of the double discontinuities constrain the coefficient functions to be of the general form

$$
A_i(x,\bar{x}) = \frac{1}{(x-\bar{x})^d} \sum_{r=0}^{d} \sum_{s=0}^{d-r} a_{rs}^{(i)}\, u^r v^s, \tag{7.14}
$$

where $d = 2k+11$ ($d = 2k+10$) in case A_i is antisymmetric (symmetric) under $x \leftrightarrow \bar{x}$. Recall the relation $k = m+n$, where m, n label the double discontinuity $\mathcal{D}^{m|n}$ at order $\lambda^{-\frac{k}{2}}$. Note that the difference between the denominator powers d here and q in (7.5) is due to the explicit $(x-\bar{x})^4$ factor in the definition of $\mathcal{F}$, according to equation (7.7).

This completes the description of our ansatz for the one-loop string amplitudes $\mathcal{F}^{(2,k)}$. Next, we continue by describing the conditions we impose in order to constrain the free parameters $a_{rs}^{(i)}$ in the coefficient functions $A_i(x,\bar{x})$.

Constraining the free parameters

In analogy with our bootstrap algorithm for one-loop supergravity correlators in position space, there are two steps in constraining the free parameters $a_{rs}^{(i)}$ in our ansatz:

1. **Matching the double discontinuity:**
The contribution of our ansatz (7.9) to the $\log^2(u)$ term is given by

$$\begin{aligned}\mathcal{F}^{(2,k)}(u,v)\big|_{\log^2(u)} = &\Big(-\frac{1}{2}A_1(x,\bar{x}) + A_2(x,\bar{x})\Big)\big(\log(1-x)-\log(1-\bar{x})\big)\\ &+ A_3(x,\bar{x}) + A_3\Big(\frac{x-1}{x},\frac{\bar{x}-1}{\bar{x}}\Big).\end{aligned} \tag{7.15}$$

Matching this against the corresponding double discontinuity $\mathcal{D}^{m|n}$ fully fixes the coefficient functions $A_i(x,\bar{x})$ for $i=1,2,3$.

It is a fact that the polynomials $p_2^{m|n}$ in the resummed double discontinuities do not obey the first line of (7.13), and hence we require a non-zero contribution from the new weight-three function $f^{(3)}(x,\bar{x})$ with coefficient $A_1(x,\bar{x})$.

2. **Pole cancellation:**
The ansatz for the function $\mathcal{H}_{2222}^{(2,k)} = \frac{u^2}{(x-\bar{x})^4}\mathcal{F}^{(2,k)}$ contains explicit denominator factors, potentially giving rise to up to $q = 2k+15$ poles at $x=\bar{x}$. Demanding that the full function $\mathcal{H}_{2222}^{(2,k)}$ is free from such unphysical poles is what we mean by pole cancellation. Concretely, by imposing as many zeroes between the functions in the numerator of $\mathcal{F}^{(2,k)}$ as there are poles, we find further non-trivial constraints amongst the remaining free parameters in $A_4(x,\bar{x})$, $A_5(x,\bar{x})$ and $A_6(x,\bar{x})$.

Carrying out the above two steps yields a definite answer for $\mathcal{H}_{2222}^{(2,k)}$, and we are left with only a small number of remaining free parameters. We call these functions which pass all of the above constraints, and whose coefficients we therefore are not able to determine, ambiguities.

By construction, the ambiguities do not contribute to the double discontinuity, are fully crossing symmetric by themselves and free of unphysical poles.[5] They are given by (linear combinations of) $\overline{D}$-functions with their $(x-\bar{x})$ denominator power bounded by the corresponding denominator in the ansatz for $A_4(x,\bar{x})$ in (7.14): $d = 2k+11$. We find that the ambiguities have finite spin support, and hence they are most conveniently

[5] Using the terminology from the previous chapter, we may call the ambiguities to be 'tree-like' in the sense that they are of the form of $\overline{D}$-functions.

$1/\lambda$ order	corresponding supervertices	$\frac{1}{(x-\bar{x})^q}$	N_{amb}	$\ell_{\max}$
1	$S\|S$	15	1	0
$\lambda^{-\frac{3}{2}}$	$S\|\mathcal{R}^4$	21	4	4
$\lambda^{-\frac{5}{2}}$	$S\|\partial^4\mathcal{R}^4$	25	7	6
λ^{-3}	$S\|\partial^6\mathcal{R}^4$, $\mathcal{R}^4\|\mathcal{R}^4$	27	8	6
$\lambda^{-\frac{7}{2}}$	$S\|\partial^8\mathcal{R}^4$	29	10	8
λ^{-4}	$S\|\partial^{10}\mathcal{R}^4$, $\mathcal{R}^4\|\partial^4\mathcal{R}^4$	31	12	8
$\lambda^{-\frac{9}{2}}$	$S\|\partial^{12}\mathcal{R}^4$, $\mathcal{R}^4\|\partial^6\mathcal{R}^4$	33	14	10
λ^{-5}	$S\|\partial^{14}\mathcal{R}^4$, $\mathcal{R}^4\|\partial^8\mathcal{R}^4$, $\partial^4\mathcal{R}^4\|\partial^4\mathcal{R}^4$	35	16	10

Table 7.1: List of one-loop terms in the $1/\lambda$ expansion and their corresponding vertices in the effective string theory action, where S stands for an insertion of the supergravity anomalous dimension. We give the denominator powers $q = 2k + 15$ of the spacetime functions $\mathcal{H}^{(2,k)}_{2222}$, the total number of ambiguities N_{amb} as well as their maximal spin support $\ell_{\max}$. Note that in general there can be more than one term contributing to the same order in $1/\lambda$, the first occurrence of this being at order λ^{-3}.

described in Mellin space because their Mellin amplitudes are only polynomial. We therefore postpone the general discussion of ambiguities to Section 7.2.1, where we give a full classification in terms of polynomial Mellin amplitudes. For now, we simply list the total number of ambiguities and their maximal spin contributions $\ell_{\max}$ for the first couple of orders in the $1/\lambda$ expansion, see Table 7.1.

Results

Following the bootstrap algorithm outlined above, we explicitly computed the full one-loop amplitudes $\mathcal{H}^{(2,3)}_{2222}$ and $\mathcal{H}^{(2,5)}_{2222}$, as well as the contributions to $\mathcal{H}^{(2,6)}_{2222}$, $\mathcal{H}^{(2,8)}_{2222}$ and $\mathcal{H}^{(2,10)}_{2222}$ which descend from the double discontinuities $\mathcal{D}^{3|3}$, $\mathcal{D}^{3|5}$ and $\mathcal{D}^{5|5}$, respectively. In all cases the results are in agreement with the general patterns described in Table 7.1. For the amplitudes $\mathcal{H}^{(2,k)}_{2222}$ with $k = 3, 5, 6$, we attach the corresponding lists of polynomials $A_i(x, \bar{x})$ in an ancillary `Mathematica` file.

From these position space results, one can then extract further subleading spectral data. However, as a consequence of the degeneracy in the double-trace spectrum, this is possible only for the lowest twist contributions at twist $\tau = 4$, where there is a single operator. We computed the order $a^2\lambda^{-\frac{3}{2}}$ and $a^2\lambda^{-\frac{5}{2}}$ one-loop anomalous dimensions $\eta^{(2,3)}$ and $\eta^{(2,5)}$ at twist four, finding agreement with the results of [100].

7.1.3 A New Ingredient: $f^{(3)}$

In finding a suitable crossing symmetric function which matches the structure of the double discontinuities $\mathcal{D}^{m|n}$, we encountered the need to include the new function $f^{(3)}$, which is beyond the ladder class encountered in the one-loop supergravity results described in Chapter 6. This new function is also single-valued in the same sense as the ladder functions, e.g. $\phi^{(1)}(x,\bar{x})$ given in equation (7.10), but involves a new singularity ('letter') of the form $x-\bar{x}$ not found in the ladder series. In fact, $f^{(3)}$ is the unique single-valued antisymmetric function at weight 3 which involves this new type of singularity. Together with the other functions in our general ansatz (7.9), which are given by combinations of log's and $\phi^{(1)}$, the ansatz thus consists of a complete basis of functions within the space of single-valued transcendental functions up to weight 3, built from the set of letters $\{x,\bar{x},1-x,1-\bar{x},x-\bar{x}\}$.

We may characterise $f^{(3)}$ by its total derivative,

$$\begin{aligned} df^{(3)}(x,\bar{x}) = &\ \big[-2\phi^{(1)}(x,\bar{x})+\tfrac{1}{2}\log^2(v)-\log(u)\log(v)\big]d\log x\\ &+\big[-2\phi^{(1)}(x,\bar{x})-\tfrac{1}{2}\log^2(v)+\log(u)\log(v)\big]d\log\bar{x}\\ &+\big[-2\phi^{(1)}(x,\bar{x})-\tfrac{1}{2}\log^2(u)+\log(u)\log(v)\big]d\log(1-x)\\ &+\big[-2\phi^{(1)}(x,\bar{x})+\tfrac{1}{2}\log^2(u)-\log(u)\log(v)\big]d\log(1-\bar{x})\\ &+\big[6\phi^{(1)}(x,\bar{x})\big]d\log(x-\bar{x}), \end{aligned} \tag{7.16}$$

together with its symmetry property,

$$f^{(3)}(x,\bar{x}) = -f^{(3)}(\bar{x},x) \tag{7.17}$$

which implies $f^{(3)}(x,x)=0$. It also obeys antisymmetry under the crossing transformations

$$f^{(3)}(1-x,1-\bar{x}) = -f^{(3)}(x,\bar{x}) = f^{(3)}(1/x,1/\bar{x})\,. \tag{7.18}$$

Up to to adding a linear combination of single-valued HPLs it can be identified with the weight-three function called $\mathcal{Q}_3$ in [141]. Functions with the same type of singularities are also needed in perturbation theory to describe the correlators of one-half BPS operators at three-loop order [33, 35].

We may make the $\log(u)$-discontinuities of $f^{(3)}(x,\bar{x})$ more transparent by writing

$$f^{(3)}(x,\bar{x}) = \log^2(u)\,\tilde{f}^{(1)}(x,\bar{x})+\log(u)\,\tilde{f}^{(2)}(x,\bar{x})+\tilde{f}^{(3)}(x,\bar{x}), \tag{7.19}$$

where the $\tilde{f}^{(k)}$ have no $\log(u)$-discontinuities. Its double $\log(u)$-discontinuity is given by

$$\tilde{f}^{(1)}(x,\bar{x}) = -\frac{1}{2}\big[\log(1-x)-\log(1-\bar{x})\big], \tag{7.20}$$

as already indicated in equation (7.15). The single $\log(u)$-discontinuity can also be simply integrated to obtain

$$\begin{aligned}\tilde{f}^{(2)}(x,\bar{x}) = &+6\,\mathrm{Li}_2\left(\frac{\bar{x}-x}{1-x}\right) + 2\big(\mathrm{Li}_2(x) - \mathrm{Li}_2(\bar{x})\big) \\ &+ \tfrac{5}{2}\log^2(1-x) - 3\log(1-x)\log(1-\bar{x}) + \tfrac{1}{2}\log^2(1-\bar{x}).\end{aligned} \tag{7.21}$$

The non-log term can be integrated in terms of hyperlogarithms (or Goncharov polylogs). We discuss this further in Appendix E, where we also describe various techniques for writing the function in a form suitable for comparison with the Mellin representations of the one-loop string amplitudes, which we address in the next section.

7.2 Comparison with Mellin Space

In previous chapters we have argued that the Mellin space formalism has led to a wealth of new results for tree-level correlators. As it turns out, the Mellin space representation of the string corrected one-loop correlators $\mathcal{H}^{(2,k)}_{2222}$ is of a simple structure which we will describe here. In particular, we will verify that our position space results are in agreement with the Mellin amplitudes found in [128], and we furthermore provide a number of new explicit Mellin amplitudes at higher orders in $1/\lambda$.

Recall that the Mellin space amplitude $\mathcal{M}_{2222}(s,t)$ of the dynamical function $\mathcal{H}_{2222}(u,v)$ is defined by integral transform

$$\mathcal{H}_{2222}(u,v) = \int_{-i\infty}^{i\infty} \frac{ds}{2}\frac{dt}{2}\; u^{\frac{s}{2}} v^{\frac{t}{2}-2}\; \mathcal{M}_{2222}(s,t)\; \Gamma^2\Big(\frac{4-s}{2}\Big)\Gamma^2\Big(\frac{4-t}{2}\Big)\Gamma^2\Big(\frac{4-\tilde{u}}{2}\Big), \tag{7.22}$$

which is a specialisation of the general formula (3.28). In this case, the Mellin variables s, t and $\tilde{u}$ satisfy the constraint equation $s+t+\tilde{u}=4$. In order to obey the crossing transformations from equation (7.2), the Mellin amplitude $\mathcal{M}_{2222}(s,t)$ has the symmetries

$$\mathcal{M}_{2222}(s,t) = \mathcal{M}_{2222}(s,\tilde{u}) = \mathcal{M}_{2222}(t,s). \tag{7.23}$$

Furthermore, the Mellin amplitude inherits the same strong coupling expansion from $\mathcal{H}_{\vec{p}}(u,v)$, see equation (7.1). Hence $\mathcal{M}_{\vec{p}}(s,t)$ admits a double expansion of the form

$$\begin{aligned}\mathcal{M}_{\vec{p}} = \;& \mathfrak{a}\left(\mathcal{M}^{(1,0)}_{\vec{p}} + \lambda^{-\frac{3}{2}}\mathcal{M}^{(1,3)}_{\vec{p}} + \lambda^{-\frac{5}{2}}\mathcal{M}^{(1,5)}_{\vec{p}} + \dots\right) \\ &+ \mathfrak{a}^2\left(\lambda^{\frac{1}{2}}\mathcal{M}^{(2,-1)}_{\vec{p}} + \mathcal{M}^{(2,0)}_{\vec{p}} + \lambda^{-\frac{1}{2}}\mathcal{M}^{(2,1)}_{\vec{p}} + \lambda^{-1}\mathcal{M}^{(2,2)}_{\vec{p}} + \lambda^{-\frac{3}{2}}\mathcal{M}^{(2,3)}_{\vec{p}} + \dots\right) \\ &+ O(\mathfrak{a}^3),\end{aligned} \tag{7.24}$$

where the tree-level terms have been described in previous chapters. We will next give a Mellin space description of the ambiguities which our position space bootstrap method is not able to fix. We will then describe the general structure of the genuine one-loop

Mellin amplitudes in the above expansion, providing a new result for $\mathcal{M}^{(2,5)}_{2222}$ and new partial results at orders $k = 8, 10$.

7.2.1 One-Loop Ambiguities

Before discussing the structure of one-loop Mellin amplitudes, let us describe the ambiguities which are left unfixed by our bootstrap method. They are exactly of the form of tree-level string amplitudes, which can be written in terms of a crossing symmetric basis of monomials given by $\sigma_2^p \sigma_3^q$, with $\sigma_n \equiv s^n + t^n + \tilde{u}^n$ [77]. The only difference is an overall shift in the large λ expansion: along with an additional factor of $\mathfrak{a} = 1/(N^2-1)$, the $1/\lambda$ expansion at one-loop order is shifted by a power of λ^2 compared to the tree-level expansion. This results in a super-leading term at order $\mathfrak{a}^2\lambda^{\frac{1}{2}}$, see equation (7.24), whose coefficient was fixed in [104].

As a consequence, at one-loop order $\mathfrak{a}^2\lambda^{-\frac{k}{2}}$, one finds contributions of monomials $\sigma_2^p\sigma_3^q$ with $2p + 3q \leq k + 1$, in comparison to $2p + 3q \leq k - 3$ at tree-level. This allows us to fully characterise the one-loop ambiguities which arise in the string corrected correlators $\mathcal{H}^{(2,k)}$. According to the counting mentioned above these ambiguities are enumerated by pairs of integers (p, q) such that $2p + 3q \leq k + 1$. Thus we can parametrise the set of ambiguities at any order $\lambda^{-\frac{k}{2}}$ by the sum

$$\sum_{p,q\geq 0}{}' \, \alpha^{(k)}_{p,q} \; \sigma_2^p \sigma_3^q, \tag{7.25}$$

where the primed sum is over integers p and q such that $2p+3q \leq k+1$. We found that the total number of ambiguities $N_{\text{amb}}(k)$ can be computed by expanding the generating function

$$\frac{1}{y(1-y)(1-y^2)(1-y^3)} - \frac{1}{y} = \sum_{k=0}^{\infty} N_{\text{amb}}(k)\, y^k. \tag{7.26}$$

For the first few orders in the $1/\lambda$ expansion, we give the total number of ambiguities and their maximal spin support $\ell_{\max}$ in Table 7.1.

7.2.2 One-Loop Mellin Amplitudes for String Corrections

Let us now turn our attention to the general structure of string corrected one-loop Mellin amplitudes $\mathcal{M}^{(2,k)}_{2222}$. For the genuine one-loop terms with $k \geq 3$, the Mellin representation was proposed to be of the form [128]

$$\mathcal{M}^{(2,k)}_{2222}(s,t) = \sum_{m+n=k} f^{m|n}(s,t)\,\widetilde{\psi}_0\Big(2-\frac{s}{2}\Big) + f^{m|n}(t,s)\,\widetilde{\psi}_0\Big(2-\frac{t}{2}\Big) + f^{m|n}(\tilde{u},t)\,\widetilde{\psi}_0\Big(2-\frac{\tilde{u}}{2}\Big), \tag{7.27}$$

with the constraint

$$f^{m|n}(s,t) = f^{m|n}(s,\tilde{u}), \tag{7.28}$$

to ensure the crossing symmetries (7.23) of the full Mellin amplitude. Instead of using the usual digamma function $\psi_0(w)$ as in [128], we define a shifted digamma function $\widetilde{\psi}_0(w) \equiv \psi_0(w) + \gamma_E$, such that the unphysical Euler-Mascheroni constant γ_E does not appear in the position space representation after performing the Mellin integration of the amplitudes $\mathcal{M}^{(2,k)}_{2222}(s,t)$. Note that for integer values n, $\widetilde{\psi}_0(n)$ is then simply related to the harmonic numbers by $\widetilde{\psi}_0(n) = H_{n-1}$.

In the above formula (7.27), the coefficient functions $f^{m|n}(s,t)$ are simply polynomials in s and t. The order in s of this polynomial is bounded by $m+n+1$, while the order in t is determined by the maximal spin contribution $\ell_{\max}$ of the corresponding double discontinuity $\mathcal{D}^{m|n}$, given by [6]

$$\begin{aligned} \mathcal{D}^{0|n}: &\qquad \ell_{\max} = 2\left\lfloor \frac{n-3}{2} \right\rfloor, \\ \mathcal{D}^{m|n}: &\qquad \ell_{\max} = 2\left\lfloor \frac{\min\{m,n\}-3}{2} \right\rfloor. \end{aligned} \tag{7.29}$$

By matching the computed double discontinuities $\mathcal{D}^{(0|3)}$ and $\mathcal{D}^{(3|3)}$ at orders $\lambda^{-\frac{3}{2}}$ and λ^{-3} against the Mellin space ansatz (7.27), one can determine the corresponding polynomials $f^{m|n}(s,t)$ to be given by [128]

$$f^{0|3}(s) = -16\zeta_3\big(63s^4 - 644s^3 + 2772s^2 - 5776s + 4800\big), \tag{7.30}$$

$$\begin{aligned} f^{3|3}(s) = -\frac{1080\zeta_3^2}{7}\big(&462s^7 - 11627s^6 + 134274s^5 - 908180s^4 \\ &+ 3841208s^3 - 10071488s^2 + 15053056s - 9838080\big), \end{aligned} \tag{7.31}$$

where we made the overall normalisations consistent with our conventions. Note that both of the above amplitudes do not depend on t, in agreement with the spin truncation of the $\mathcal{R}^4$ vertex to spin $\ell_{\max} = 0$. By explicitly performing the Mellin integration in a series expansion around small (u,v), we have verified that the above Mellin amplitudes are in agreement with our position space results obtained by the bootstrap approach described in Section 7.1.2, thus confirming the appearance of the weight-three function $f^{(3)}(x,\bar{x})$.

By making use of the order $\lambda^{-\frac{5}{2}}$ tree-level data, see references [5, 8, 100, 120], we can furthermore provide some new results. For example, we can compute the double

[6] We have checked that our discussion on the orders of the polynomials $f^{m|n}(s,t)$ is in agreement with the 'basis of polynomial Mellin amplitudes' described in [128].

discontinuity $\mathcal{D}^{0|5}$, resulting in

$$
\begin{aligned}
f^{0|5}(s,t) = -2\zeta_5\big(&10890s^6 + 45s^5(11t - 4669) + 9s^4(55t^2 - 640t + 204358)\\
&- 4s^3(945t^2 - 7173t + 2285717) + 36s^2(377t^2 - 2208t + 745066)\\
&- 16s(1575t^2 - 7488t + 2722522) + 576(33t^2 - 132t + 52682)\big),
\end{aligned}
\tag{7.32}
$$

which in fact appears before the $f^{3|3}(s)$ contribution in the $1/\lambda$ expansion and is the first case with non-trivial t-dependence. At order λ^{-4}, we can similarly compute the contribution

$$
\begin{aligned}
f^{3|5}(s,t) = -90\zeta_3\zeta_5\big(&28028s^9 - 1075074s^8 + 19321302s^7 - 211238951s^6\\
&+ 1535536842s^5 - 7645987076s^4 + 25938244248s^3\\
&- 57543276224s^2 + 75453134080s - 44400268800\big),
\end{aligned}
\tag{7.33}
$$

and finally the order λ^{-5} contribution from $\mathcal{D}^{5|5}$ is given by

$$
\begin{aligned}
f^{5|5}(s,t) = -\frac{45\zeta_5^2}{22}\big(&57657600s^{11} + 30030s^{10}(16t - 104093)\\
&+ 12012s^9(40t^2 - 1445t + 6689071)\\
&- 572s^8(26985t^2 - 531356t + 2242111079)\\
&+ 22s^7(11008816t^2 - 151917584t + 638025985123)\\
&- 77s^6(30823520t^2 - 327881344t + 1429188184721)\\
&+ 14s^5(1125229952t^2 - 9688637728t + 44851775822225)\\
&- 28s^4(2593858960t^2 - 18612610496t + 92780493961669)\\
&+ 56s^3(4118587328t^2 - 25104138112t + 135924547490919)\\
&- 64s^2(7551065200t^2 - 39625690048t + 234345782828097)\\
&+ 256s(2355357312t^2 - 10748615808t + 69677906818663)\\
&- 53760(6319936t^2 - 25279744t + 180000568369)\big).
\end{aligned}
\tag{7.34}
$$

Note that the orders of the polynomials $f^{m|n}$ given above all fit into the general pattern described earlier. Before concluding, let us mention once more that we checked agreement between our position space results and the Mellin space amplitudes described here. Such a comparison can be easily performed in a series expansion around small (u,v) by using the explicit representation (E.9) of the function $f^{(3)}(x,\bar{x})$ which is suitable for this expansion.

7.3 Discussion and Outlook

In the final two chapters of this thesis, we have considered the problem of constructing one-loop four-point amplitudes on $\mathrm{AdS}_5\times\mathrm{S}^5$ at the level of supergravity and also its first string corrections. Our results rely solely on imposing consistency with the OPE to order $1/N^4$ on the dual CFT side.

In the supergravity case, we have first derived the one-loop corrections to the $\langle 2222\rangle$ and $\langle 2233\rangle$ correlators in terms of transcendental functions up to weight 4, from which we extracted the one-loop anomalous dimensions of the unique twist 4 and 5 double-trace operators. For higher twist operators, we expect mixing with triple-trace operators to spoil predictability of the double-trace spectrum.[7] Next, we have presented a general algorithm for constructing one-loop correlators which works for arbitrary external charges. This algorithm is based on extracting all relevant data from many tree-level correlators, rearranging it into combinations which appear at one-loop and finally feeding this information into an ansatz for the full one-loop function. The final result is then obtained by demanding no unphysical poles at $x=\bar{x}$. We have illustrated our algorithm for the correlators $\langle 2244\rangle$, $\langle 3335\rangle$, $\langle 4424\rangle$, $\langle 3333\rangle$ and $\langle 4444\rangle$, which we fix up to a finite number of tree-like ambiguities given explicitly by equation (6.106). Recall that due to their at most linear Mellin amplitudes, these ambiguities correspond to contact Witten diagrams of effective ten-dimensional spin $\ell_{10}=0$. They can be thought of as counterterms to regulate the one-loop divergencies, and their values are ultimately fixed within string theory. The only ambiguity whose value has been determined so far is the single ambiguity of the $\langle 2222\rangle$ correlator, which has been fixed by a supersymmetric localisation computation [104], while the values of the ambiguities in other correlators still await to be determined.

Recently, another method of reconstructing one-loop amplitudes from the double discontinuity based on a dispersive inversion integral has been developed [131, 140, 143, 144], and furthermore a systematic unitarity method for computing the double discontinuity of AdS amplitudes has been proposed in [83, 85]. Note that in that context the term ‘double discontinuity’ has a broader meaning than used throughout this thesis: in addition to the double-log part (the $\log^2(u)$ coefficient) it moreover includes terms which are singular in v. As such, it also receives contributions from both the free theory and from tree-level correlators, and in this way the ‘double discontinuity’ already encodes the relevant information about the below-threshold OPE data. In principle, this method provides a direct way to compute one-loop correlators from their double discontinuities (in the broader sense). However, as it relies on the ability to solve highly complicated dispersion integrals, explicit one-loop correlators have been computed with this method only for simpler theories than the full $\mathcal{N}=4$ SYM. In contrast, our bootstrap program

[7] We believe that additional data in the form of higher-point correlation functions could remedy this situation. The first five-point function in supergravity, the $\langle 22222\rangle$ correlator, has been recently computed in [142].

starts with an explicit function in the first place, and the subsequent steps of the algorithm implement the consistency of the OPE, fixing all free parameters up to the set of ambiguities. Furthermore, since our algorithm begins by matching the predicted $\log^2(u)$ coefficient only, constraints from the below-threshold sector are not automatically taken care of and are crucial to obtain the correct one-loop correlators. In particular, these below-threshold predictions ultimately lift the tree-level degeneracy of the $\langle 3335\rangle$ and $\langle 4424\rangle$ correlators, which have the same $\log^2(u)$ discontinuities but are distinguished at one-loop because of different predictions in the window. Two further novel and notable features of our algorithm are the natural splitting of the one-loop dynamical function into two independent pieces, $\mathcal{H}^{(2)}_{\vec{p}} = \mathcal{T}^{(2)}_{\vec{p}} + \mathcal{D}^{(2)}_{\vec{p}}$, as well as the need of a proper understanding of multiplet recombination of semi-short operators at order $1/N^4$. Interestingly, the problem of multiplet recombination can be solved within the free theory only, and it is truly remarkable that these independent predictions are consistent with our one-loop ansatz. As such, we find this is a non-trivial validation of the AdS/CFT correspondence within the $\mathcal{N} = 4$ bootstrap program.

We then addressed string corrections to the one-loop supergravity amplitude, focussing on the $\langle 2222\rangle$ correlator. As in the supergravity case, we use the knowledge of tree-level data to predict the $\log^2(u)$ discontinuity of the one-loop amplitude. Starting from an ansatz of up to weight-three transcendental functions, we constrain the free parameters by matching the computed double-log's and cancellation of unphysical poles. While we provide explicit results for the first few orders in the $1/\lambda$ expansion, we want to stress that the form of our one-loop ansatz is in fact valid to all orders.[8] As a consequence of the spin truncated spectrum of the string corrections, the ansatz is built from functions which are one degree lower in transcendentality compared to the supergravity case. On the other hand, the weight-three function $f^{(3)}$ is required as a new ingredient in the ansatz. This function involves a new type of logarithmic singularity, $x - \bar{x}$, which is not present in the supergravity case, and the appearance of this new letter provides a first understanding of what type of functions will generally appear in loop amplitudes of string theory on AdS.

Our explicit position space results complement the Mellin space approach of [128]. Therefore, at least for one-loop string corrections,[9] the position and Mellin space representations are essentially interchangeable by comparison of their small $(x, \tilde{x})$ and small (u, v) expansions. Note that each of the two representations have their advantages and disadvantages. For example, comparably simple structures emerge when considering the string corrected one-loop amplitudes in Mellin space, whereas their position space

[8] The current restriction is the limited knowledge of higher order (in $1/\lambda$) tree-level amplitudes, which at the moment is limited to order $\lambda^{-\frac{5}{2}}$. Recall that, due to mixing of double-trace operators, one needs tree-level data from $\langle 22pp\rangle$ correlators to construct the one-loop $\langle 2222\rangle$ correlator.

[9] In the supergravity case, there are currently two different representations for the one-loop Mellin amplitudes, which may turn out to be equivalent: the authors of [95, 128] propose an ansatz in terms of an infinite double sum (which requires regularisation), whereas more recently a finite Mellin amplitude was proposed in reference [129].

equivalents turn out to be rather involved. Also, the connection to ten-dimensional physics is given very directly in Mellin space through the flat space limit. On the other hand, the ansatz of transcendental functions for the position space amplitudes makes their singularity-structure very explicit, while this is quite obscure from the Mellin space point of view. In particular, any analytic continuation or kinematic expansion (e.g. the OPE) that one may wish to perform is straightforward from the spacetime representation.

In conclusion, the position space bootstrap algorithm for one-loop correlation functions described in this thesis can be regarded as a proof of principle that it is possible to derive explicit one-loop amplitudes on a curved space (here AdS_5) and in a complicated theory such as supergravity, where direct computations are extremely difficult if not currently out of reach. This progress was only possible because of the remarkable AdS/CFT duality, the simplicity of the supergravity spectrum and moreover the power of CFT techniques which made it possible to tackle the strong coupling regime of $\mathcal{N} = 4$ SYM theory. While it is too early to ask about the implications for the full theory of quantum gravity, we nevertheless provide an understanding of the general analytic structure of quantum corrections to supergravity, at least to one-loop order. It would be very interesting to try to go beyond one-loop amplitudes and explore the possibility of extending our bootstrap program to higher loops. In particular, from the structure of the leading logarithmic discontinuity of the two-loop correlator (the $\log^3(u)$ part, which is predicted from tree-level data) it seems plausible that the set of singularities found so far, namely $\{x, \bar{x}, 1-x, 1-\bar{x}\}$ for the supergravity case and $\{x, \bar{x}, 1-x, 1-\bar{x}, x-\bar{x}\}$ for string corrections, is sufficient for the description of the two-loop amplitude.[10] It remains to be seen whether our methods will provide enough constraints to fix the entire amplitude, since starting from two-loops we expect also triple-trace operators to contribute to the SCPW decomposition. Future directions also include a further detailed investigation of the Mellin space representation of our supergravity one-loop functions, building on the previous works [95, 128, 129].

Furthermore, our results for the double-trace anomalous dimensions have unexpectedly motivated the discovery of a new ten-dimensional symmetry of the tree-level supergravity amplitudes [112], which manifests itself in the partial residual degeneracy of the supergravity spectrum. Although this symmetry is broken by the tree-level string corrections beyond order $\lambda^{-\frac{3}{2}}$, its breaking is controlled by the effective ten-dimensional spin ℓ_{10}, which also dictates the stepwise lifting of the residual degeneracy in the spectrum. Inspired by the flat space Virasoro-Shapiro amplitude, the order by order investigation of the string corrections might pave the way for the construction of the full tree-level string theory amplitude on $\mathrm{AdS}_5\times\mathrm{S}^5$.

Finally, the wealth of new results for tree-level supergravity correlators in other back-

[10] A first step towards exploring the leading-log discontinuity at higher loop orders in supergravity has been recently taken in [145, 146].

grounds, e.g. 10d supergravity on $AdS_3 \times S^3$ [52, 113, 114, 147, 148] and 11d supergravity on $AdS_7 \times S^4$ [49, 53], can be used to study the spectrum of anomalous dimensions in those theories. Furthermore, these results open up the avenue for the construction of one-loop amplitude using similar techniques as we described here for the case of $AdS_5 \times S^5$. For example, various first steps have already been taken in the case of M-theory on $AdS_7 \times S^4$, dual to the 6d (2,0) theory: some aspects of the anomalous dimensions have been studied in [149, 150], string corrections have been addressed in [122, 151] and finally the first one-loop correction has been computed in [152]. We hope to extend the great success of our bootstrap program in AdS_5 to those other cases in the future.

Appendix A

Superconformal Blocks

Here we give the explicit definition of the superblocks $\mathbb{S}_{\vec{p};\gamma,\underline{\lambda}}$, which are defined by a determinantal formula following [92]. Let us introduce first the function

$$F^{\alpha\beta\gamma\underline{\lambda}} = (-1)^{p+1}\frac{(x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})}{(x-\bar{x})(y-\bar{y})} \det\begin{pmatrix} F^X_{\underline{\lambda}} & R \\ K_{\underline{\lambda}} & F^Y \end{pmatrix}, \tag{A.1}$$

where the determinant is taken on the $(p+2)\times(p+2)$ matrices (with $p=\min\{\alpha,\beta\}$) given by

$$\begin{aligned} (F^X_{\underline{\lambda}})_{ij} &= \Big([x_i^{\lambda_j-j}{}_2F_1(\lambda_j+1-j+\alpha,\lambda_j+1-j+\beta;2\lambda_j+2-2j+\gamma;x_i)]\Big)_{1\le i\le 2,\,1\le j\le p}, \\ (F^Y)_{ij} &= \Big((y_j)^{i-1}{}_2F_1(i-\alpha,i-\beta;2i-\gamma;y_j)\Big)_{1\le i\le p,\,1\le j\le 2}, \\ (K_{\underline{\lambda}})_{ij} &= \Big(-\delta_{i;\,j-\lambda_j}\Big)_{1\le i\le p,\,1\le j\le p}, \\ (R)_{ij} &= \Big(\tfrac{1}{x_i-y_j}\Big)_{1\le i\le 2\,,1\le j\le 2}. \end{aligned} \tag{A.2}$$

The brackets in the definition of $F^X_{\underline{\lambda}}$ mean deletion of the singular terms in the Taylor expansion in x_i around $x_i=0$ when $\lambda_j<j$, and we have defined here $x_i=(x,\bar{x})$ $y_j=(y,\bar{y})$ in the matrix. With this at hand, the semi-short, quarter and one-half BPS superblocks are given by the formula

$$\mathbb{S}^{p_1p_2p_3p_4}_{\vec{p};\gamma,\underline{\lambda}} = \mathcal{P}_{\vec{p}}\left(\frac{x\bar{x}}{y\bar{y}}\right)^{\frac{1}{2}(\gamma-p_4+p_3)} F^{\alpha\beta\gamma\underline{\lambda}}, \qquad \text{with} \begin{cases} \alpha=\frac{1}{2}(\gamma-p_1+p_2), \\ \beta=\frac{1}{2}(\gamma+p_3-p_4), \end{cases} \tag{A.3}$$

where the prefactor $\mathcal{P}_{\vec{p}}$ is that of equation (2.18).

Appendix B

Multiplet Recombination

The purpose of this appendix is to settle a task left undone in Section 2.5: to explain how to properly disentangle physical semi-short contributions from the SCPW coefficients of the free theory and to find the short coefficients $S_{[\lambda,1^\mu]}$. We will approach this problem in an $1/N$ expansion around the large N limit, where the leading contributions come from the disconnected part of the free theory correlator. Let us note that separating the short coefficients S from the long coefficients L at the unitary bound is actually straightforward to order $1/N^2$. In particular, we will show that apart from the case $S_{\vec{p};\gamma,[\lambda,1^\mu]}$ with $\gamma = \min\{p_1 + p_2, p_3 + p_4\}$, i.e when $\tau = \tau^{\min}$, all other coefficients $S_{\vec{p};\gamma,[\lambda,1^\mu]}$ vanish. Thus the values of L will be trivially fixed by multiplet recombination. This special feature at $O(1/N^2)$ has lead various people to the assumption that the same would be true for all N, see [153] for a discussion on this point. However, beyond $O(1/N^2)$ the separation of coefficients S from L is a non-trivial problem.

We will now show how to solve this problem to $O(1/N^4)$ and determine the genuine semi-short sector of the single-particle correlators $\langle p_1p_2p_3p_4\rangle$ in the full interacting theory, using only the free theory correlators and knowledge about the form of the semi-short operators. In particular, we provide formulae for all SCPW coefficients – split according to operators which remain short in the interacting theory and those which are long – in terms of the free theory coefficients $A_{\vec{p};\gamma,\underline{\lambda}}$ defined in equation (2.39).

Recall that for long blocks at the unitary bound $\tau = 2a + b + 2$ we need to resolve the ambiguity which follows from a reducibility condition (i.e. that a long SCPW is a sum of two semi-short SCPWs), which we repeat here for convenience:

$$\mathbb{L}_{\vec{p};\vec{\tau}} = \mathbb{S}_{\vec{p};\tau,[\ell+2,1^a]} + \mathbb{S}_{\vec{p};\tau+2,[\ell+1,1^{a+1}]}, \qquad \text{for } \tau = 2a + b + 2\,. \tag{B.1}$$

Comparing the two pieces of the SCPW expansion (2.41), and equating the coefficient of $\mathbb{S}_{\vec{p};\tau,[\ell+2,1^a]}$ using (B.1) yields

$$A_{\vec{p};\tau,[\ell+2,1^a]} = S_{\vec{p};\tau,[\ell+2,1^a]} + L^f_{\vec{p};\vec{\tau}} + L^f_{\vec{p};\tau-2,\ell+1,[a-1,b,a-1]}, \quad \text{for } \tau = 2a+b+2\,. \quad \text{(B.2)}$$

One of the key points allowing us to resolve the ambiguity at the unitarity bound is the following (already tacitly assumed in (2.41)) statement: *a long operator at the unitarity bound necessarily has twist less than* $\tau^{min} = \min\{p_1+p_2, p_3+p_4\}$*, i.e.* $L^f_{\vec{p};\vec{\tau}} = 0$ *if* $\tau = 2a+b+2 \geq \tau^{\min}$. This is a non-perturbative statement and a non-trivial consequence of superconformal symmetry for the corresponding three-point functions [154, 155].

This fact allows us to use equation (B.2) to determine the SCPW coefficients of semi-short operators of twist $\tau^{\min} = \min\{p_1+p_2, p_3+p_4\}$ in terms of lower twist coefficients

$$\begin{aligned} S_{\vec{p};\tau,[\ell+2,1^a]} &= A_{\vec{p};\tau,[\ell+2,1^a]} - L^f_{\vec{p};\tau-2,\ell+1,[a-1,b,a-1]}, && \text{for } \tau = 2a+b+2 = \tau^{\min},\ a \geq 1 \\ S_{\vec{p};\tau,[\ell+2]} &= A_{\vec{p};\tau,[\ell+2]}, && \text{for } \tau = b+2 = \tau^{\min}. \end{aligned} \quad \text{(B.3)}$$

It is useful to understand the $1/N$ expansion[1] of $S_{\vec{p};\tau^{\min},[\ell+2,1^a]}$ first, since it will play a role in our later formulas. Referring to Figure 6.1, when $\tau^{\min} = 2a+b+2$ the two lines coincide, i.e. the lower dashed line sits on top of the middle dashed line, and thus we find that $S_{\vec{p};\tau^{\min},[\ell+2,1^a]}$ in (B.3) is non trivial at $O(1/N^2)$. In particular, it gets a contribution from leading order connected propagator structures. In the special case of the $\langle ppqq\rangle$ correlators, $\tau^{\min} = \tau^{\max}$ and the free theory starts with an $O(1)$ contribution from disconnected diagrams. For all $su(4)$ channels $[a,b,a]$ such that $\tau^{\min} = 2a+b+2$ we find then that all three dashed lines of Figure 6.1 coincide and $S_{\vec{p};\tau^{\max},[\ell+2,1^a]}$ indeed has an $O(1)$ contribution from disconnected free theory diagrams.

What about CPW coefficients of semi-short operators of twist less than $\tau^{\min}$? Semi-short operators generically will sit in the range of twists $\tau \leq \min\{p_1+p_2, p_3+p_4\}$, therefore at the bottom dashed line in Figure 6.1, i.e. below the window. It follows that the corresponding SCPW coefficient is $O(1/N^4)$,

$$S_{\vec{p};\tau,[\ell+2,1^a]} = O(1/N^4), \qquad \text{for } \tau = 2a+b+2 < \tau^{\min}. \quad \text{(B.4)}$$

This is the well known statement that at $O(1/N^2)$ there are no semi-short operators in the spectrum below the window, which implies a cancellation between free theory and the interacting part. Using this information we can solve for $S_{\vec{p};\tau^{\min},[\ell+2,1^a]}$ in (B.3) and for $L^f_{\vec{p};\vec{\tau}}$ in (B.2) explicitly up to order $1/N^2$. First we solve (B.2) recursively, thus obtaining the long SCPW coefficients

$$L^f_{\vec{p};\vec{\tau}} = \sum_{k=0}^{a} (-1)^k A_{\vec{p};\tau-2k,[\ell+2+k,1^{a-k}]} + O(1/N^4), \qquad \text{for } \tau = 2a+b+2 < \tau^{\min}. \quad \text{(B.5)}$$

[1]Note that here and below, 'order $1/N^k$', really means $N^{\frac{1}{2}(p_1+p_2+p_3+p_4)}O(1/N^k)$ because we have not normalised our external operators.

Then, we plug this result into equation (B.3) to give the genuine semi-short coefficients at threshold,

$$S_{\vec{p};\tau,[\ell+2,1^a]} = \sum_{k=0}^{a} (-1)^k A_{\vec{p};\tau-2k,[\ell+2+k,1^{a-k}]} + O(1/N^4), \quad \tau = 2a+b+2 = \tau^{\min}. \tag{B.6}$$

Note that when $a = 0$, we correctly obtain $S_{\vec{p};\tau,[\ell+2]}$ given above in (B.3).

Now, can we determine the $1/N^4$ SCPW coefficients of semi-short operators of twist less than $\tau^{\min}$? The answer is affirmative. We first need to use some non-trivial information about the spectrum of semi-short operators, and then we can determine these SCPW coefficients unambiguously by combining data from many different correlators.

The key point here is that we know the explicit form of the double-trace semi-short operators – or more importantly the number of them. They are twist τ, spin ℓ operators in the $[a, b, a]$ representation of the form

$$\mathcal{O}_{q\tilde{q};\vec{\tau}} = \mathcal{O}_q \partial^\ell \mathcal{O}_{\tilde{q}} \,|_{[a,b,a]}, \tag{B.7}$$

with $\tau = q + \tilde{q} = 2a + b + 2$. For fixed twist and $su(4)$ representation we can enumerate the independent operators as

$$q_r = a+1+r, \qquad \tilde{q}_r = a+1+b-r, \qquad \text{with} \;\; r = \delta_{a,0}, \ldots, \mu{-}1, \tag{B.8}$$

where

$$\mu \equiv \begin{cases} \left\lfloor \frac{b+2}{2} \right\rfloor & a+\ell \text{ even}, \\ \left\lfloor \frac{b+1}{2} \right\rfloor & a+\ell \text{ odd}. \end{cases} \tag{B.9}$$

Unlike the case of long operators (discussed in Section 3.3.2), semi-short operators receive no anomalous dimension. The operators enumerated in (B.8) are therefore degenerate and we may directly choose the $\mathcal{O}_{q\tilde{q}}$ from (B.7) as our basis. The SCPW coefficients of such operators are then expressed in terms of the products of three-point couplings,

$$S_{\vec{p};\tau,[\ell+2,1^a]} = \sum_{r,s} C_{p_1p_2}(\mathcal{O}_{q_r\tilde{q}_r})(M^{-1})_{rs} C_{p_3p_4}(\mathcal{O}_{q_s\tilde{q}_s}), \tag{B.10}$$

where M is the matrix of two-point functions (which is diagonal at leading order in large N),

$$M_{rs} = \langle \mathcal{O}_{q_r\tilde{q}_r} \mathcal{O}_{q_s\tilde{q}_s} \rangle = Y_r \, \delta_{rs} + O(1/N^2). \tag{B.11}$$

We also recall the fact that the only couplings with a leading order contribution in the large N expansion are the ones of the form $C_{pq}(\mathcal{O}_{pq})$. From this it follows that at leading order in large N we have a diagonal structure for the following three-point couplings,

$$C_{q_r\tilde{q}_r}(\mathcal{O}_{q_s\tilde{q}_s}) = X_r\,\delta_{rs} + O(1/N^2). \tag{B.12}$$

With this information at hand, we can now predict the SCPW coefficients of semi-short operators $S_{\vec{p};\tau,[\ell+2,1^a]}$ of twist $\tau < \tau^{\min}$ in terms of SCPW coefficients with $\tau = \tau^{\min}$, which in turn are known from equation (B.6). Finally, the general formula for $S_{\vec{p};\tau,[\ell+2,1^a]}$, correct up to and including order $1/N^4$, is given by

$$S_{\vec{p};\tau,[\ell+2,1^a]} = \sum_{r=0}^{\mu-1} \frac{S_{p_1p_2q_r\tilde{q}_r}S_{q_r\tilde{q}_rp_3p_4}}{S_{q_r\tilde{q}_rq_r\tilde{q}_r}} + O(1/N^6), \qquad \text{for } \tau = 2a+b+2 < \tau^{\min}. \tag{B.13}$$

For simplicity, we have omitted the labels τ and $[\ell+2,1^a]$ in the SCPW coefficients on the RHS above. The two factors in the numerator of the above equation are both $O(1/N^2)$, whereas the factor in the denominator is of leading order in large N, and thus the RHS is $O(1/N^4)$ as we stated already in (B.4). The formula (B.13) may be proven by simply using (B.10) on both sides and then using (B.12) and (B.11) on the RHS to cancel the denominator.

Lastly, with equation (B.13) at hand, we can improve $L^f_{\vec{p};\vec{\tau}}$ in (B.5) and $S_{\vec{p};\tau^{\min},[\ell+2,1^a]}$ in (B.6) up to order $1/N^4$. The results are

$$L^f_{\vec{p};\vec{\tau}} = \sum_{k=0}^{a}(-1)^k A_{\vec{p};\tau-2k,[\ell+2+k,1^{a-k}]} - \sum_{k=0}^{a}(-1)^k S_{\vec{p};\tau-2k,[\ell+2+k,1^{a-k}]} + O(1/N^6), \tag{B.14}$$

for twists $\tau = 2a+b+2 < \tau^{\min}$, and

$$S_{\vec{p};\tau,[\ell+2,1^a]} = \sum_{k=0}^{a}(-1)^k A_{\vec{p};\tau-2k,[\ell+2+k,1^{a-k}]} - \sum_{k=1}^{a}(-1)^k S_{\vec{p};\tau-2k,[\ell+2+k,1^{a-k}]} + O(1/N^6), \tag{B.15}$$

when $\tau = 2a+b+2 = \tau^{\min}$.

In summary, all SCPW coefficients of $\langle p_1p_2p_3p_4\rangle_{\text{short}}$ and $\langle p_1p_2p_3p_4\rangle_{\text{long}}$ from equation (2.41) have been obtained to order $O(1/N^4)$ and therefore we have successfully split the free theory correlators into a protected contribution and an unprotected one. In general, we can not go further in the $1/N$ expansion since to do so would require input from triple-trace (and higher multi-trace) operators.

Let us conclude this section by illustrating our formulas (B.13) and (B.15) for the semi-short sectors of the $\langle 3333\rangle$ correlator, which has been already examined in detail in [92], and the $\langle 4444\rangle$ correlator, which is a new case.

In the case of $\langle 3333\rangle$, we have below threshold semi-short predictions for twists $\tau = 2$ and 4. This semi-short sector is special because no multi-trace mixing occurs in the large

N expansion and therefore we can give formulas which are exact in N. Very explicitly, we find that

$$
\begin{aligned}
S^{\langle 3333\rangle}_{2,[\lambda]} &= 0,\\
S^{\langle 3333\rangle}_{4,[\ell+2]} &= \frac{\big(S^{\langle 2233\rangle}_{4,[\lambda]}\big)^2}{S^{\langle 2222\rangle}_{4,[\lambda]}} = \frac{288((\ell+3)!)^2}{(2\ell+6)!((\ell+3)(\ell+4)+\frac{4}{(N^2-1)})}\frac{A^0_0}{(N^2-1)},\\
S^{\langle 3333\rangle}_{4,[\ell+2,1]} &= \frac{\big(S^{\langle 2233\rangle}_{4,[\lambda,1]}\big)^2}{S^{\langle 2222\rangle}_{4,[\lambda,1]}} = \frac{576((\ell+3)!)^2}{(2\ell+6)!((\ell+2)(\ell+5)-\frac{12}{(N^2-1)})}\frac{A^0_0}{(N^2-1)},
\end{aligned}
\tag{B.16}
$$

where $A^0_0 = (3(N^2-1)(N^2-4)/N)^2$. The structure of the SCPW coefficients of operators at threshold, i.e. at twist 6, follows straightforwardly from equation (B.15):

$$
\begin{aligned}
S_{6,[\lambda]} &= A_{6,[\lambda]},\\
S_{6,[\lambda,1]} &= A_{6,[\lambda,1]} - A_{4,[\lambda+1]} + S_{4,[\lambda+1]},\\
S_{6,[\lambda,1,1]} &= A_{6,[\lambda,1,1]} - A_{4,[\lambda+1,1]} + A_{2,[\lambda+2]} + S_{4,[\lambda+1,1]}.
\end{aligned}
\tag{B.17}
$$

In the case of the $\langle 4444\rangle$ correlator, we have predictions for twists $\tau = 2$, 4 and 6. The computations at twist 2 and twist 4 are analogous to the case of $\langle 3333\rangle$. We find

$$
\begin{aligned}
S^{\langle 4444\rangle}_{2,[\lambda]} &= 0,\\
S^{\langle 4444\rangle}_{4,[\ell+2]} &= \frac{16\times 1152}{(\ell+3)(\ell+4)}\frac{((\ell+3)!)^2}{(2\ell+6)!}\frac{1+(-1)^\ell}{2}\frac{1}{N^4},\\
S^{\langle 4444\rangle}_{4,[\ell+2,1]} &= \frac{6\times 1600}{(\ell+2)(\ell+5)}\frac{((\ell+3)!)^2}{(2\ell+6)!}\frac{1-(-1)^\ell}{2}\frac{1}{N^4}.
\end{aligned}
\tag{B.18}
$$

The twist 6 results are new and given by

$$
\begin{aligned}
S^{\langle 4444\rangle}_{6,[\ell+2]} &= \frac{16\times 384(29+3(2\ell+9)^2)}{(\ell+3)(\ell+6)}\frac{(\ell+4)!^2}{(2\ell+8)!}\frac{1+(-1)^\ell}{2}\frac{1}{N^4},\\
S^{\langle 4444\rangle}_{6,[\ell+2,1]} &= \frac{16\times 72(401+174(2\ell+9)^2+(2\ell+9)^4)}{(\ell+2)(\ell+4)(\ell+5)(\ell+7)}\frac{(\ell+4)!^2}{(2\ell+8)!}\frac{1-(-1)^\ell}{2}\frac{1}{N^4},\\
S^{\langle 4444\rangle}_{6,[\ell+2,1,1]} &= \frac{16\times 2400(\ell+2)(\ell+7)}{(\ell+3)(\ell+6)}\frac{(\ell+4)!^2}{(2\ell+8)!}\frac{1+(-1)^\ell}{2}\frac{1}{N^4}.
\end{aligned}
\tag{B.19}
$$

We insisted on the correlators $\langle 3333\rangle$ and $\langle 4444\rangle$ as examples since these two correlators capture generic features of our discussion about the semi-short sector, and furthermore because they are investigated in Section 6.7.6 and Appendix D, respectively, where we explicitly construct their one-loop completions. This underlines the importance of the information from the semi-short sector for our one-loop bootstrap program.

Appendix C

Connected Free Theory at Order $1/N^2$

We quote a formula for connected free theory at order $1/N^2$, of a generic four-point function $\langle p_1 p_2 p_3 p_4 \rangle_{\text{free}}$ in the free field theory. The same formula is described in a different notation in [112].

Each propagator structure in free theory is labelled by monomials of the form $\mathcal{P}_{\vec{p}}\, \sigma^{i-j}\tau^j$ where i, j belong to the set $T = \{(i,j) \mid 0 \le i \le \kappa_{\vec{p}},\, 0 \le j \le i\}$, and the bound $\kappa_{\vec{p}}$ is precisely the degree of extremality. The lattice of points described by T is schematically depicted in Figure C.1 below.

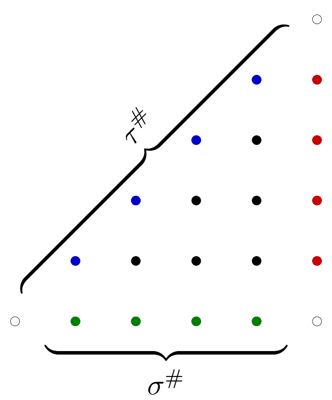


Figure C.1: The set of monomials $\sigma^{i-j}\tau^j$ in a free field theory correlator.

We shall distinguish the three edges from the interior. Vertices at the intersection of the edges correspond to disconnected diagrams (when they exist according to our definition of single-particle states). In [4] we determined the value of the connected propagator structure

$$\frac{\langle p_1p_2p_3p_4\rangle_{\rm free}}{\mathcal{P}_{\vec{p}}} \supset \frac{p_1p_2p_3p_4}{N^2}\left(1+\tfrac{p_{43}+p_{21}}{2}\right)\frac{u\tau}{v}. \tag{C.1}$$

Comparing to the diagram of T in Figure C.1, the above contribution corresponds to the coefficient associated to the point $(1,1)$ on the diagonal edge of the triangle. By crossing on the other edges we find

$$\begin{aligned}\frac{\langle p_1p_2p_3p_4\rangle_{\rm free}}{\mathcal{P}_{\vec{p}}} \supset \frac{p_1p_2p_3p_4}{N^2}\Big[&\left(1+\tfrac{p_{43}+p_{21}}{2}\right)\sum_{k=1}^{\mathtt{t}-1}\left(\frac{u\tau}{v}\right)^k+\left(1+\tfrac{p_{13}+p_{42}}{2}\right)\sum_{k=1}^{\mathtt{t}-1}(u\sigma)^k\\ &+\left(1+|\tfrac{p_{23}+p_{14}}{2}|\right)\sum_{k=1}^{\mathtt{t}-1}(u\sigma)^k\left(\frac{u\tau}{v}\right)^{\mathtt{t}+1-k}\Big].\end{aligned} \tag{C.2}$$

By including the propagator structures in the interior of T we finally obtain the general formula

$$\begin{aligned}\frac{\langle p_1p_2p_3p_4\rangle_{\rm free}}{\mathcal{P}_{\vec{p}}} = \frac{p_1p_2p_3p_4}{N^2}\Big[&\left(1+\tfrac{p_{43}+p_{21}}{2}\right)\sum_{k=1}^{\mathtt{t}-1}\left(\frac{u\tau}{v}\right)^k+\left(1+\tfrac{p_{43}-p_{21}}{2}\right)\sum_{k=1}^{\mathtt{t}-1}(u\sigma)^k\\ &+\left(1+|\tfrac{p_1+p_2-p_3-p_4}{2}|\right)\sum_{k=1}^{\mathtt{t}-1}(u\sigma)^k\left(\frac{u\tau}{v}\right)^{\mathtt{t}+1-k}\\ &+2\sum_{(n_1,n_2)\in T\backslash\text{edges}}(u\sigma)^{n_1}\left(\frac{u\tau}{v}\right)^{n_2}\Big],\end{aligned} \tag{C.3}$$

where the double-sum in the last line is over the interior of the set T (without the edges).

Appendix D

The $\langle 4444 \rangle$ Correlator at One-Loop

In this appendix we describe the construction of the minimal one-loop function for the $\langle 4444 \rangle$ correlator, as well as the generalised tree-level amplitude $\mathcal{T}_{4444}$.

D.1 The Minimal One-Loop Function for $\langle 4444 \rangle$

The final example we study in detail is the $\langle 4444 \rangle$ correlator. The solution of our bootstrap problem for $\mathcal{D}^{(2)}_{4444}$, given in the basis (6.101) and for simplicity with particular values of the ambiguities, is appended in an ancillary file.

This correlator is again fully crossing symmetric and has degree of extremality 4, which results in the long sector containing a total of six $su(4)$ channels: $[0,0,0]$, $[1,0,1]$, $[0,2,0]$ and $[2,0,2]$, $[1,2,1]$, $[0,4,0]$. In this example, the threshold twist is $\tau^{\max} = 8$. After obtaining the two-variable resummations of the leading $\log^2(u)$ discontinuities for all channels, we then initiate our algorithm: we match our ansatz against the double-logs, impose crossing symmetry and absence of $x = \bar{x}$ poles. Lastly, we consider the OPE predictions in and below the window. With the window being empty for this correlator, we project the ansatz onto the $\log(u)$ stratum and we set to zero the one-variable expansions up to order $O(u^4)$. Instead, in the below-window region we find non-trivial physics. For the representations $[0,0,0]$, $[1,0,1]$ and $[0,2,0]$, the discussion is similar to the twist 4 case for the $\langle 3333 \rangle$ correlator, and continues at twist 6 by including predictions coming from the long sector at tree-level. On the other hand, for the $[2,0,2]$, $[1,2,1]$ and $[0,4,0]$ channels we will have to consider non-trivial multiplet recombination, taking into account the predictions arising from the semi-short sector. Let us proceed channel by channel in the following.

In the singlet channel, there is an empty twist 2 sector, whereas the $1/N^2$ subleading three-point couplings $C^{(1)}_{44;4,\ell,[0,0,0]}$ and $C^{(1)}_{44;6,\ell,[0,0,0]}$ give rise to non-trivial predictions for twists 4 and 6. Recall that there are two double-trace operators at twist 6, and therefore the twist 6 computation yields a vector of three-point functions. Using (6.76), we find

$$L^{(2)\mathcal{D}}_{4444;2,\ell,[0,0,0]} = 0, \tag{D.1}$$

$$L^{(2)\mathcal{D}}_{4444;4,\ell,[0,0,0]} = \frac{16\times 4800}{(\ell+1)(\ell+6)}\frac{((\ell+3)!)^2}{(2\ell+6)!}\frac{1+(-1)^\ell}{2}, \tag{D.2}$$

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[0,0,0]} = \frac{16\times 360(2\ell+9)^2(119+(2\ell+9)^2)}{(\ell+1)(\ell+2)(\ell+7)(\ell+8)}\frac{((\ell+4)!)^2}{(2\ell+8)!}\frac{1+(-1)^\ell}{2}. \tag{D.3}$$

The corresponding one-variable resummation yields

$$\begin{aligned}\mathcal{D}^{(2)}_{4444}\big|_{[0,0,0],\,\log^0(u)} = 16\cdot 6!\Bigg[&\frac{x^2}{\bar{x}^4}\Big(5(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x})+\tfrac{5}{3}(6-6\bar{x}+\bar{x}^2)\mathrm{Li}_1^2(\bar{x})\Big)\\ &+\frac{x^3}{\bar{x}^5}\Big((3440-5590\bar{x}+\tfrac{7484}{3}\bar{x}^2-244\bar{x}^3)\mathrm{Li}_1^2(\bar{x})\\ &\quad-\bar{x}(1230-1210\bar{x}+211\bar{x}^2)\mathrm{Li}_1(\bar{x})\\ &\quad-300\bar{x}(6-6\bar{x}+\bar{x}^2)\mathrm{Li}_2(\bar{x})\\ &\quad+\bar{x}^2(\bar{x}-2)\big(\tfrac{4\bar{x}^2}{\bar{x}-1}+205\big)\Big)\Bigg]+O(x^4).\end{aligned} \tag{D.4}$$

In the channels $[1,0,1]$ and $[0,2,0]$, there are twist 4 predictions coming from semi-short operators at the unitarity bound, similarly to the case of ⟨3333⟩. In particular, there is an $S_{4;\ell+2,[1]}$ contribution to the $[1,0,1]$ channel and an $S_{4;\ell+2,[0]}$ to $[0,2,0]$, computed in (B.18). In addition we have $1/N^2$ three-point couplings $C^{(1)}_{44;6,\ell,[1,0,1]}$ and $C^{(1)}_{44;6,\ell,[0,2,0]}$ which give predictions at twist 6. Note that this is the first twist available for a long operator in the $[0,2,0]$ representation, but there is doubling of operators, i.e. $\mu=2$.

The list of results for the $[1,0,1]$ representation reads

$$L^{(2)\mathcal{D}}_{4444;4,\ell,[1,0,1]} = \frac{16\times 1600}{(\ell+2)(\ell+5)}\frac{((\ell+3)!)^2}{(2\ell+6)!}\frac{1-(-1)^\ell}{2}, \tag{D.5}$$

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[1,0,1]} = \frac{16\times 7200(\ell+1)(\ell+8)}{(\ell+3)(\ell+6)}\frac{((\ell+4)!)^2}{(2\ell+8)!}\frac{1-(-1)^\ell}{2}, \tag{D.6}$$

with one-variable resummation

$$\begin{aligned}\mathcal{D}^{(2)}_{4444}\big|_{[1,0,1],\,\log^0(u)} = 16\cdot\tfrac{5^2 6!}{3^2}\Bigg[&\frac{x^2}{\bar{x}^4}\Big(3(\bar{x}-2)\bar{x}+(6-6\bar{x}+\tfrac{7}{5}\bar{x}^2)\mathrm{Li}_1(\bar{x})+\tfrac{1}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1^2(\bar{x})\Big)\\ &+\frac{x^3}{\bar{x}^5}\Big(\tfrac{1}{10}(4-176\bar{x}+87\bar{x}^2)\bar{x}\mathrm{Li}_1^2(\bar{x})+\tfrac{2}{5}(4+4\bar{x}-3\bar{x}^2)\bar{x}\mathrm{Li}_2(\bar{x})\\ &\quad+\tfrac{16}{75}(746-766\bar{x}+201\bar{x}^2)\bar{x}\mathrm{Li}_1(\bar{x})+\tfrac{9(\bar{x}-2)\bar{x}^4}{5(\bar{x}-1)}\\ &\quad+\tfrac{4}{75}(1527\bar{x}-3014)\bar{x}^2\Big)\Bigg]+O(x^4).\end{aligned} \tag{D.7}$$

The list of results in the $[0,2,0]$ channels reads

$$L^{(2)\mathcal{D}}_{4444;4,\ell,[0,2,0]} = \frac{16\times 1152}{(\ell+3)(\ell+4)}\frac{((\ell+3)!)^2}{(2\ell+6)!}\frac{1+(-1)^\ell}{2}, \tag{D.8}$$

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[0,2,0]} = \frac{16\times 864(40817+16702(2\ell+9)^2+81(2\ell+9)^4)}{80(\ell+1)(\ell+4)(\ell+5)(\ell+8)}\frac{((\ell+4)!)^2}{(2\ell+8)!}\frac{1+(-1)^\ell}{2}, \tag{D.9}$$

with one-variable resummation

$$\begin{aligned}\mathcal{D}^{(2)}_{4444}\big|_{[0,2,0],\,\log^0(u)} = 16\cdot 2^2 6!\Bigg[&\frac{x^2}{\bar{x}^4}\Big(\tfrac{6}{5}\bar{x}^2+\tfrac{3}{5}(\bar{x}-2)\bar{x}\mathrm{Li}_1(\bar{x})+\tfrac{1}{10}\bar{x}^2\mathrm{Li}_1^2(\bar{x})\Big)\\ &+\frac{x^3}{\bar{x}^5}\Big(\tfrac{1}{25}(164640-246960\bar{x}+98794\bar{x}^2-8667\bar{x}^3)\mathrm{Li}_1^2(\bar{x})\\ &-\tfrac{3}{5}(11358-11342\bar{x}+1981\bar{x}^2)\bar{x}\mathrm{Li}_1(\bar{x})-72\bar{x}^3\mathrm{Li}_2(\bar{x})\\ &+\tfrac{3}{5}(382-175\bar{x})+\tfrac{243(\bar{x}-2)\bar{x}^2}{25(\bar{x}-1)}\Big)\Bigg]+O(x^4).\end{aligned} \tag{D.10}$$

Finally we arrive at the representations $[2,0,2]$, $[1,2,1]$ and $[0,4,0]$. The unitarity bound for all these representations is at twist 6, and the semi-short predictions can be found in equation (B.19). Note that in the $[0,4,0]$ channel, twist 6 lies at the bottom of the multiplet recombination, since $\tau = 2a+b+2$ with $b=4$ and $a=0$. The prediction for $L^{(2)\mathcal{D}}_{4444;6,\ell,[0,4,0]}$ is thus straightforward. The predictions for the $[1,2,1]$ and $[2,0,2]$ channels involve further shifts, which we now describe. From equation (6.90) we find

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[0,4,0]} = -S_{4444;6,\ell+2;[0]}, \tag{D.11}$$

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[1,2,1]} = +S_{4444;6,\ell+2;[1]} - S_{4444;4,\ell+3;[0]},, \tag{D.12}$$

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[2,0,2]} = -S_{4444;6,\ell+2;[2]} + S_{4444;4,\ell+3;[1]}, \tag{D.13}$$

where in the last line we already implemented the absence of $S_{4444;2,\ell+2;[0]}$. Note that equations (D.12) and (D.13) correctly include the shifts due to multiplet recombination at twist 4 in the $[0,2,0]$ and $[1,0,1]$ channels, respectively. Let us give the explicit expressions here below.

In the $[0,4,0]$ representation, we find

$$L^{(2)\mathcal{H}}_{4444;6,\ell,[0,4,0]} = \frac{16\times 384(29+3(2\ell+9)^2)}{(\ell+3)(\ell+6)}\frac{(\ell+4)!^2}{(2\ell+8)!}\frac{1+(-1)^\ell}{2}, \tag{D.14}$$

with resummation

$$\begin{aligned}\mathcal{D}^{(2)}_{4444}\big|_{[0,4,0],\,\log^0(u)} = 16\cdot 2^2 6!\Bigg[&\frac{x^3}{\bar{x}^5}\Big(\big(208-\tfrac{16\bar{x}^2}{5(\bar{x}-1)}+\tfrac{112}{15}\mathrm{Li}_1^2(\bar{x})\big)(2-\bar{x})\bar{x}^2\\ &-\tfrac{16}{3}(78-78\bar{x}+17\bar{x}^2)\bar{x}\mathrm{Li}_1(\bar{x})\Big)\Bigg]+O(x^4).\end{aligned} \tag{D.15}$$

The results in the $[2,0,2]$ channel read

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[2,0,2]} = \frac{16\times 200\left(-83+3(2\ell+9)^2\right)}{(\ell+3)(\ell+6)}\frac{(\ell+4)!^2}{(2\ell+8)!}\frac{1+(-1)^\ell}{2}, \tag{D.16}$$

and

$$\mathcal{D}^{(2)}_{4444}\Big|_{[2,0,2],\,\log^0(u)} = 16\cdot 6!\Bigg[\frac{x^3}{\bar{x}^5}\Big(\left(\tfrac{25}{3}+\tfrac{5\bar{x}^2}{3(\bar{x}-1)}+\tfrac{35}{9}\mathrm{Li}_1^2(\bar{x})\right)(\bar{x}-2)\bar{x}^2 + \tfrac{5}{9}(30-30\bar{x}+13\bar{x}^2)\mathrm{Li}_1(\bar{x})\Big)\Bigg] + O(x^4). \tag{D.17}$$

Lastly, for the $[1,2,1]$ representation we have

$$L^{(2)\mathcal{D}}_{4444;6,\ell,[1,2,1]} = \frac{16\times 72(3+(2\ell+9)^2)(167+(2\ell+9)^2)}{(\ell+2)(\ell+4)(\ell+5)(\ell+7)}\frac{(\ell+4)!^2}{(2\ell+8)!}\frac{1-(-1)^\ell}{2}, \tag{D.18}$$

and its one-variable resummation gives

$$\begin{aligned}\mathcal{D}^{(2)}_{4444}\Big|_{[1,2,1],\,\log^0(u)} = 16\cdot 6!\Bigg[\frac{x^3}{\bar{x}^5}\Big(&-\left(2184-3276\bar{x}+\tfrac{7104}{5}\bar{x}^2-\tfrac{822}{5}\bar{x}^3\right)\mathrm{Li}_1(\bar{x})\\ &+\tfrac{14}{5}(48-48\bar{x}+7\bar{x}^2)\mathrm{Li}_1^2(\bar{x})+\tfrac{1874}{5}\bar{x}^3\\ &+\tfrac{4\bar{x}^3(2-2\bar{x}+\bar{x}^2)}{5(\bar{x}-1)}-2184\bar{x}(\bar{x}-1)\Big)\Bigg]+O(x^4).\end{aligned} \tag{D.19}$$

Incredibly, all these predictions are consistent with the minimal one-loop ansatz (6.101) and uniquely fix the remaining coefficients, leaving only four unfixed parameters. These are precisely the four ambiguities from equations (6.110) and (6.111), which have once again only spin $\ell=0$ support for above-threshold twists.

D.2 Generalised Tree-Level Amplitude for ⟨4444⟩

Lastly, let us describe the generalised tree-level amplitude for this more complex example. There are in total $3+12$ propagator structures in the free theory, where the first three are disconnected and not relevant here. The result for the connected part is given by

$$\begin{aligned}\frac{\langle 4444\rangle_{\text{free conn.}}}{g_{12}^4 g_{34}^4} = \frac{16(N^2-9)^2(N^2-4)^2(N^2-1)^2}{(N^2+1)^2}\Bigg[&\\ \tfrac{16}{N^2-1}\Big(u\sigma+\frac{u\tau}{v}+u^3\sigma^3+\frac{u^3\tau^3}{v^3}+\frac{u^4\sigma^3\tau}{v}+\frac{u^4\sigma\tau^3}{v^3}\Big)&\\ +8\left(\tfrac{27}{N^2(N^2-9)}+\tfrac{9}{N^2+1}-\tfrac{7N^2+4}{(N^2-4)(N^2-1)}\right)\Big(u^2\sigma^2+\frac{u^2\tau^2}{v^2}+\frac{u^4\sigma^2\tau^2}{v^2}\Big)&\\ +16\left(\tfrac{54}{N^2(N^2-9)}+\tfrac{18}{N^2+1}-\tfrac{16N^2+25}{(N^2-4)(N^2-1)}\right)\Big(\frac{u^2\sigma\tau}{v}+\frac{u^3\sigma^2\tau}{v}+\frac{u^3\sigma\tau^2}{v^2}\Big)\Bigg].&\end{aligned} \tag{D.20}$$

Written as a sum over propagator structures as in (2.25), the connected part of the free theory correlator is constrained by crossing symmetry to have only three independent classes:

$$
\begin{aligned}
A_2^0 &= A_2^1 = A_6^0 = A_6^1 = A_8^1 = A_8^3 = \frac{16}{N^2-1}, \\
A_4^0 &= A_4^2 = A_8^2 = 8\left(\tfrac{27}{N^2(N^2-9)} + \tfrac{9}{N^2+1} - \tfrac{7N^2+4}{(N^2-4)(N^2-1)}\right), \\
A_6^1 &= A_6^2 = A_4^1 = 16\left(\tfrac{54}{N^2(N^2-9)} + \tfrac{18}{N^2+1} - \tfrac{16N^2+25}{(N^2-4)(N^2-1)}\right),
\end{aligned}
\tag{D.21}
$$

from which we pick $\{A_2^0, A_4^0, A_6^1\}$ as the set of independent coefficients.

The generalised tree-level function $\mathcal{T}_{4444}$ can be conveniently written in terms of just two independent functions, $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, in the following way:

$$
\begin{aligned}
\mathcal{T}_{4444} = &\frac{1}{u^2}\Big[\mathcal{F}(u,v) + \sigma^2 u^6 \mathcal{F}(1/v, u/v) + \frac{\tau^2 u^6}{v^6}\mathcal{F}(v,u)\Big] \\
&+ \frac{1}{u^2}\Big[\sigma\tau \widetilde{\mathcal{F}}(u,v) + \frac{\sigma u^6}{v^6}\widetilde{\mathcal{F}}(v,u) + \tau u^6 \widetilde{\mathcal{F}}(1/v, u/v)\Big],
\end{aligned}
\tag{D.22}
$$

where both $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are invariant under the crossing transformation $(u,v) \to (u/v, 1/v)$. Given the Mellin transform

$$
\mathcal{T}_{4444} = u^4 \oint u^s v^t \; \Gamma[-s]^2 \Gamma[-t]^2 \Gamma\big[s+t+6\big]^2 \mathcal{M}[\mathcal{T}_{4444}],
\tag{D.23}
$$

with

$$
\mathcal{M}[\mathcal{T}_{4444}] = m_{4444}^1 + \sigma^2\, m_{4444}^{\sigma^2} + \tau^2\, m_{4444}^{\tau^2} + \sigma\tau\, m_{4444}^{\sigma\tau} + \sigma\, m_{4444}^{\sigma} + \tau\, m_{4444}^{\tau},
\tag{D.24}
$$

we will specify $m_{4444}^1(s,t)$ and $m_{4444}^{\sigma\tau}(s,t)$, which are the Mellin transforms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, respectively. One can then reconstruct $\mathcal{M}[\mathcal{T}_{4444}]$ by using symmetries, analogously to equation (D.22).

The Mellin transforms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are

$$
\begin{aligned}
m_{4444}^1 &= -\tfrac{A_2^0}{(s+3)(t+1)(s+t+5)} - \tfrac{\mathcal{L}_{4444}^1}{2(s+2)(t+1)(s+t+5)} - \tfrac{\mathcal{L}_{4444}^1 + \mathcal{L}_{4444}^2 (s+1)}{6(s+1)(t+1)(s+t+5)}, \\
m_{4444}^{\sigma\tau} &= -\tfrac{A_2^0}{(s+1)(t+2)(s+t+4)} + \tfrac{\mathcal{L}_{4444}^1}{2(s+1)(t+1)(s+t+4)} + \tfrac{\mathcal{L}_{4444}^1}{2(s+1)(t+2)(s+t+5)} \\
&\quad - \tfrac{\mathcal{L}_{4444}^1 (s+3)}{3(s+1)(t+1)(s+t+5)} - \tfrac{\mathcal{L}_{4444}^1 - \mathcal{L}_{4444}^2 + (2\mathcal{L}_{4444}^1 - \mathcal{L}_{4444}^2)(s+1)}{3(s+1)(t+2)(s+t+4)},
\end{aligned}
\tag{D.25}
$$

where

$$
\mathcal{L}_{4444}^1 = A_2^0 - A_4^0, \qquad \mathcal{L}_{4444}^2 = \tfrac{3}{2}A_2^0 - A_4^0 - \tfrac{1}{4}A_6^1.
\tag{D.26}
$$

Finally, the other coefficients are determined by the crossing relations

$$\begin{aligned} m^{\sigma}_{4444}(s,t) &= m^{\sigma\tau}_{4444}(t,s), \qquad & m^{\tau}_{4444}(s,t) &= m^{\sigma\tau}_{4444}(-s-t-6,s), \\ m^{\tau^2}_{4444}(s,t) &= m^{1}_{4444}(t,s), \qquad & m^{\sigma^2}_{4444}(s,t) &= m^{1}_{4444}(-s-t-6,s). \end{aligned} \tag{D.27}$$

The terms in m^1_{4444} and $m^{\sigma\tau}_{4444}$ which are proportional to A^0_2 correspond precisely to the supergravity tree-level amplitude $\mathcal{M}^{(1,0)}_{4444}$ of Rastelli and Zhou. Note that the combinations $\mathcal{L}^{i=1,2}_{4444}$ in (D.26) vanish at order $1/N^2$.

In conclusion, let us highlight some new features of $\mathcal{M}[\mathcal{T}_{4444}]$ beyond tree-level. Recall that the supergravity result from [47, 48] can be obtained by considering an ansatz in Mellin space such that each monomial $\sigma^i\tau^j$ is accompanied by a *single* pole in the (s,t)-plane. In comparison, the generalised tree-level amplitude has more structure than this. In particular, poles like $(s+2)(t+1)$ and $(s+1)(t+1)$, corresponding to powers of u^2 and u^3 in the small u expansion, and therefore corresponding to allowed twists below the window, are also turned on. We see now that their residue is proportional to the linear constraints $\mathcal{L}^{i=1,2}_{4444}$, which indeed vanish at order $1/N^2$. We also notice that by writing each pole in the form $\frac{1}{(s+n_1)(t+n_2)(s+t+n_3)}$ with integers $n_{i=1,2,3}$, the numerator is at most linear in s and t. Therefore, we infer that the limit $s\to\beta s$ and $t\to\beta t$ with large β scales like $1/\beta^2$, i.e. one more power than the $1/\beta^3$ scaling of the supergravity tree-level functions $\mathcal{M}^{(1,0)}_{\vec{p}}$ given in (3.33).

The case of $\mathcal{M}[\mathcal{T}_{4444}]$ exemplifies well what is the general pattern of $\mathcal{M}[\mathcal{T}_{\vec{p}}]$ in Mellin space. In fact, we expect $\mathcal{M}[\mathcal{T}_{\vec{p}}]$ to be a rational function in which all allowed poles in the (s,t)-plane are turned on, eventually decorated by a non-trivial numerator, which is nevertheless constrained by the large s and t behaviour. Similarly to our position space algorithm, the free coefficients in this ansatz will be fixed by demanding that the SCPW expansion in the below-window region *completely cancels* the free theory contributions as described in equations (6.87) and (6.88).

Appendix E

Analytic Properties of $f^{(3)}(x, \bar{x})$

The purpose of this appendix is to give more details on the structure of the function $f^{(3)}$ which makes an appearance in the one-loop string amplitudes. We recall that its total derivative is defined in equation (7.16). By successively stripping off the leading $\log(u)$ discontinuity we arrive at the form (7.19), with $\tilde{f}^{(1)}$ obtained very simply and with the total derivative of $\tilde{f}^{(2)}$ obtained in the form

$$\begin{aligned} d\tilde{f}^{(2)}(x,\bar{x}) = & -\big[2\log(1-x)\big]d\log(x) \\ & +\big[2\log(1-\bar{x})\big]d\log(\bar{x}) \\ & -\big[\log(1-x)-3\log(1-\bar{x})\big]d\log(1-x) \\ & +\big[\log(1-\bar{x})-3\log(1-x)\big]d\log(1-\bar{x}) \\ & +\big[6\log(1-x)-6\log(1-\bar{x})\big]d\log(x-\bar{x}). \end{aligned} \tag{E.1}$$

The form (7.21) agrees with the above and obeys $\tilde{f}^{(2)}(x,x)=0$, as it should by anti-symmetry. Finally, we obtain the total derivative of $\tilde{f}^{(3)}$ in the form

$$\begin{aligned} d\tilde{f}^{(3)}(x,\bar{x}) = & \quad \big[-4(\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x}))+\tfrac{1}{2}\log^2(v)-\tilde{f}^{(2)}(x,\bar{x})\big]d\log(x) \\ & +\big[-4(\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x}))-\tfrac{1}{2}\log^2(v)-\tilde{f}^{(2)}(x,\bar{x})\big]d\log(\bar{x}) \\ & +\big[-4(\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x}))\big]d\log(1-x) \\ & +\big[-4(\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x}))\big]d\log(1-\bar{x}) \\ & +\big[12(\mathrm{Li}_2(x)-\mathrm{Li}_2(\bar{x}))\big]d\log(x-\bar{x}). \end{aligned} \tag{E.2}$$

We can easily integrate this in a form suitable for expansion in small x and $\bar{x}$. However, for comparison with the Mellin space representation it is more convenient to make the change of variables $\tilde{x}=1-\bar{x}$, so that

$$u = x(1-\tilde{x}), \qquad v = \tilde{x}(1-x). \tag{E.3}$$

Then the small x and $\tilde{x}$ expansion can easily be compared to a small u and v expansion as obtained from Mellin space. To this end we first pull (E.2) back to the line $x = 0$,

$$d\tilde{f}^{(3)}(0, \bar{x}) = \left[-12\mathrm{Li}_2(\bar{x}) - \log^2(1-\bar{x})\right] d\log(\bar{x}) + \left[4\,\mathrm{Li}_2(\bar{x})\right] d\log(1-\bar{x}). \tag{E.4}$$

This can then be easily integrated in terms of weight three harmonic polylogarithms [156] with the condition that $\tilde{f}^{(3)}(0,0) = 0$:

$$\tilde{f}^{(3)}(0, \bar{x}) = -12H_3(\bar{x}) - 4H_{1,2}(\bar{x}) - 2H_{2,1}(\bar{x}). \tag{E.5}$$

Performing our change of variables from $\bar{x}$ to $\tilde{x} = 1-\bar{x}$, we have in the small $\tilde{x}$ expansion

$$\tilde{f}^{(3)}(0, 1-\tilde{x}) = -6\zeta_3 + 4\zeta_2 \log(\tilde{x}) + O(\tilde{x}). \tag{E.6}$$

Now using

$$\begin{aligned} \phi^{(1)}(x, 1-\tilde{x}) = &-\log(u)\log(v) - 2\left[\mathrm{Li}_1(x)\log(u) + \mathrm{Li}_1(\tilde{x})\log(v)\right] \\ &- 2\left[\zeta_2 + \mathrm{Li}_1(x)\mathrm{Li}_1(\tilde{x}) - \mathrm{Li}_2(x) - \mathrm{Li}_2(\tilde{x})\right], \end{aligned} \tag{E.7}$$

we may write

$$\begin{aligned} df^{(3)}(x, 1-\tilde{x}) = \;& \left[-2\phi^{(1)}(x, 1-\tilde{x}) + \tfrac{1}{2}\log^2(v) - \log(u)\log(v)\right] d\log(x) \\ &+ \left[-2\phi^{(1)}(x, 1-\tilde{x}) - \tfrac{1}{2}\log^2(v) + \log(u)\log(v)\right] d\log(1-\tilde{x}) \\ &+ \left[-2\phi^{(1)}(x, 1-\tilde{x}) - \tfrac{1}{2}\log^2(u) + \log(u)\log(v)\right] d\log(1-x) \\ &+ \left[-2\phi^{(1)}(x, 1-\tilde{x}) + \tfrac{1}{2}\log^2(u) - \log(u)\log(v)\right] d\log(\tilde{x}) \\ &+ \left[6\phi^{(1)}(x, 1-\tilde{x})\right] d\log(1-x-\tilde{x}). \end{aligned} \tag{E.8}$$

We can then make manifest all the logarithmic singularities in $\log(u)$ and $\log(v)$ as follows,

$$\begin{aligned} f^{(3)}(x, 1-\tilde{x}) = \;& \tfrac{1}{2}\log^2(u)\log(v) + \tfrac{1}{2}\log(u)\log^2(v) - \log^2(u)\log(1-x) - \log^2(v)\log(1-\tilde{x}) \\ &+ \log(u)\log(v)\ \left[2\log(1-x) + 2\log(1-\tilde{x}) - 6\log(1-x-\tilde{x})\right] \\ &+ \log(u)\ g^{(2)}(x, \tilde{x}) + \log(v)\ g^{(2)}(\tilde{x}, x) + 6g^{(3)}(x, \tilde{x}), \end{aligned} \tag{E.9}$$

where the function $g^{(2)}$ can be expressed as

$$g^{(2)}(x, \tilde{x}) = 4\zeta_2 + 2\,\mathrm{Li}_2(x) + 2\,\mathrm{Li}_2(\tilde{x}) - 6\,\mathrm{Li}_2\left(\frac{\tilde{x}}{1-x}\right) - 2\log(1-x)\log\frac{(1-x)^3(1-\tilde{x})}{(1-x-\tilde{x})^6}. \tag{E.10}$$

To write a formula for $g^{(3)}$ it is helpful to use hyperlogarithms $G_w(t)$ which depend on a word $w = a_1 a_2 \ldots a_n$ in letters a_i and a variable t. The function whose word is just a string of n zeros is a power of $\log(t)$,

$$G_{0^n}(t) = \frac{1}{n!}\log^n(t). \tag{E.11}$$

The other functions are defined recursively,

$$G_{aw}(t) = \int_0^t \frac{ds}{s-a} G_w(s). \tag{E.12}$$

Using these hyperlogarithms we can write an expression for $g^{(3)}$ by integrating the total derivative and fixing the term proportional to ζ_3 from (E.6),

$$\begin{aligned} g^{(3)}(x,\tilde{x}) = \; & G_1(\tilde{x})G_{0,1}(x) - 2G_{1-x}(\tilde{x})G_{0,1}(x) - 2G_1(\tilde{x})G_{1,1}(x) \\ & + G_1(x)G_{1,1}(\tilde{x}) - 2G_1(x)G_{1,1-x}(\tilde{x}) - 2G_1(x)G_{1-x,1}(\tilde{x}) \\ & + G_{0,0,1}(x) + G_{0,0,1}(\tilde{x}) - 2G_{0,1,1}(x) + G_{0,1,1}(\tilde{x}) - G_{0,1,1-x}(\tilde{x}) \\ & - 2G_{0,1-x,1}(\tilde{x}) - 2G_{1,0,1}(x) + G_{1,0,1}(\tilde{x}) - G_{1,0,1-x}(\tilde{x}) \\ & - 2G_{1-x,0,1}(\tilde{x}) - 2\zeta_2 \log(1-x-\tilde{x}) - \zeta_3. \end{aligned} \tag{E.13}$$

Although it is not manifest from the above formula, $g^{(3)}$ is symmetric, i.e. $g^{(3)}(x,\tilde{x}) = g^{(3)}(\tilde{x},x)$. The apparent asymmetry is simply due to a choice of the contour of integration (first in the x direction, then the $\tilde{x}$ direction).

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