ONE-RELATOR GROUPS AND ALGEBRAS RELATED TO POLYHEDRAL PRODUCTS

JELENA GRBIĆ, MARINA ILYASOVA, TARAS PANOV, AND GEORGE SIMMONS

Abstract. We link distinct concepts of geometric group theory and homotopy theory through underlying combinatorics. For a flag simplicial complex K, we specify a necessary and sufficient combinatorial condition for the commutator subgroup RC_K' of a right-angled Coxeter group, viewed as the fundamental group of the real moment-angle complex $\mathcal{R}_K$, to be a one-relator group; and for the Pontryagin algebra $H_*(\Omega\mathcal{Z}_K)$ of the moment-angle complex to be a one-relator algebra. We also give a homological characterisation of these properties. For RC_K', it is given by a condition on the homology group $H_2(\mathcal{R}_K)$, whereas for $H_*(\Omega\mathcal{Z}_K)$ it is stated in terms of the bigrading of the homology groups of $\mathcal{Z}_K$.

1. Introduction

Let K be a flag simplicial complex on vertex set $[m] = \{1, \ldots, m\}$ and K^1 be its 1-skeleton. The *right-angled Coxeter group* corresponding to K is defined as the group RC_K with generators $g_1, \ldots, g_m$ for each vertex in K and relations $g_i^2 = 1$ and $g_i g_j = g_j g_i$ whenever $\{i, j\} \in K^1$. Right-angled Coxeter groups are interesting from a geometric point of view because they arise from reflections in the facets of right-angled polyhedra in hyperbolic space.

For a given group G, we denote by G' the commutator subgroup of G. The *real moment-angle complex* $\mathcal{R}_K = (D^1, S^0)^K$ associated with a flag complex K is a finite-dimensional aspherical space whose fundamental group is the commutator subgroup RC_K' of the right-angled Coxeter group RC_K. In [15] it was shown that $RC_K' = \pi_1(\mathcal{R}_K)$ is free if and only if K^1 is a chordal graph. A graph is called *chordal* if each of its cycles with 4 or more vertices has a chord, an edge joining two vertices that are not adjacent in the cycle. Furthermore, for arbitrary flag K, a minimal generating set for RC_K' was given in terms of iterated commutators of the generators of RC_K [15, Theorem 4.5].

Another space associated with a simplicial complex K is the *moment-angle complex* $\mathcal{Z}_K = (D^2, S^1)^K$. Throughout this paper, all homology groups are considered with coefficients in $\mathbb{Z}$, unless otherwise stated. The Pontryagin algebra $H_*(\Omega\mathcal{Z}_K)$, was studied in [7] when K is a flag complex. It was shown that $H_*(\Omega\mathcal{Z}_K)$ is a graded free associative algebra if and only if the 1-skeleton K^1 is a chordal graph. Furthermore, a minimal generating set for $H_*(\Omega\mathcal{Z}_K)$ with flag K was given in [7, Theorem 4.3] in terms of iterated commutators.

Therefore, for both $\mathcal{R}_K$ and $\mathcal{Z}_K$ the algebraic freeness property, that is, that $\pi_1(\mathcal{R}_K)$ and $H_*(\Omega\mathcal{Z}_K)$ are free as groups and algebras, respectively, is characterised by the same combinatorial condition. More precisely, these algebraic objects are free if and only if the 1-skeleton K^1 of the simplicial complex K is a chordal graph. The question of $H_*(\Omega\mathcal{Z}_K)$ being a free associative algebra is related to the Golodness

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property of a simplicial complex K. A simplicial complex K is *Golod* if all cup products and higher Massey products vanish in $H^*(\mathcal{Z}_K)$. In [7, Theorem 4.6] it was proved that a flag simplicial complex K is Golod if and only if K^1 is a chordal graph.

In this paper we study other properties of objects naturally arising in geometric group theory and homotopy theory that have the same combinatorial characterisation. In particular, we describe a combinatorial condition on a flag complex K under which $\pi_1(\mathcal{R}_K)$ is a one-relator group, and $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra. A 1-dimensional simplicial complex C_p that is the boundary of a p-gon is called a *p-cycle*. In [7] it was shown that when K is a 5-cycle then there is only one relation between the 10 multiplicative generators of $H_*(\Omega\mathcal{Z}_K)$; while in [16], a single relation was again found between the 34 multiplicative generators of $H_*(\Omega\mathcal{Z}_K)$ when K is a 6-cycle. Similarly, in [15], it was noted that if K is a p-cycle for $p \geqslant 4$, then $\pi_1(\mathcal{R}_K)$ is a one-relator group. The one-relator condition places strong restrictions on the form of K, and our main combinatorial characterisation is the following.

Theorem 1.1. *Let K be a flag simplicial complex. Then $\pi_1(\mathcal{R}_K)$ and $H_*(\Omega\mathcal{Z}_K)$ have exactly one relation if and only if the following combinatorial condition holds*

$$(*)\qquad K = C_p \text{ or } K = C_p * \Delta^q \text{ for } p \geqslant 4,\ q \geqslant 0$$

where C_p is a p-cycle, Δ^q is a q-simplex and $*$ denotes the join of simplicial complexes.

For $\mathcal{R}_K$ this is proved in Theorem 3.2 and for $\mathcal{Z}_K$ in Theorem 5.1. The proofs of Theorem 3.2 and Theorem 5.1 are completely different in character. For Theorem 3.2, the key argument comes from geometric group theory. When $K = C_p$ or $K = C_p * \Delta^q$ for $q \geqslant 0$, the space $\mathcal{R}_K$ is homeomorphic to the product $S_g \times D^{q+1}$, where S_g is a closed orientable surface of genus $g = (p-4)2^{p-3} + 1$ and D^{q+1} is a $(q+1)$-dimensional disc, and therefore its fundamental group is a one-relator surface group. The converse statement is proved using the Lyndon Identity Theorem [12] (see [6, Theorem 2.1]) because the group $\pi_1(\mathcal{R}_K) = RC'_K$ is torsion-free.

To prove Theorem 5.1, we study the simply connected space $\Omega\mathcal{Z}_K$ using homotopy-theoretical methods. When $K = C_p$ or $K = C_p * \Delta^q$ for $q \geqslant 0$, by a result of McGavran [13], there is a homotopy equivalence

$$\mathcal{Z}_K \simeq \#_{k=3}^{p-1}(S^k \times S^{p+2-k})^{\#(k-2)\binom{p-2}{k-1}}$$

where $\#$ denotes the connected sum operation on manifolds. Beben and Wu [4] computed the algebra $H_*(\Omega X;\mathbb{Z}_p)$, p prime, where X is a highly-connected manifold obtained by attaching a single cell to a space Y which has the homotopy type of a double suspension. This implies that $H^*(Y;\mathbb{Z}_p)$ has no non-trivial cup products, which places sufficient restrictions on $H^*(X;\mathbb{Z}_p)$ so that $H_*(\Omega X;\mathbb{Z}_p)$ can be studied via a homology Serre spectral sequence. We adapt the Beben–Wu method to study the integral Pontryagin algebra of an arbitrary connected sum of sphere products

$$(1)\qquad M = \#_{i=1}^{k}(S^{d_i} \times S^{d-d_i})$$

where $d_i \geqslant 2$ and $d \geqslant 4$. In this case, the Beben–Wu method reduces to the Adams–Hilton model and the highly-connectedness assumption can be dropped. In Proposition 4.1, we prove that the integral Pontryagin algebra $H_*(\Omega M)$ is isomorphic as a Hopf algebra to the quotient of a graded free associative algebra by a single relation. Proposition 4.1 implies that when $K = C_p$ or $K = C_p * \Delta^q$ for $q \geqslant 0$, $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra. We compute the Poincaré series $P(H_*(\Omega\mathcal{Z}_K);t)$ explicitly in Proposition 4.2.

We extend the equivalences of Theorem 1.1 by determining an equivalent homological criteria on $\mathcal{R}_K$ and $\mathcal{Z}_K$. For $\mathcal{R}_K$, the combinatorial condition $(*)$ is equivalent to the homological condition $H_2(\mathcal{R}_K) = \mathbb{Z}$, and this is proved in Theorem 3.2. The homology groups of $\mathcal{Z}_K$ have a natural bigrading, see [5, § 4.4]. The combinatorial condition (∗) is then equivalent to the homological condition

$$H_{2-j,2j}(\mathcal{Z}_K) = \begin{cases} \mathbb{Z} & \text{if } j = p \\ 0 & \text{otherwise.} \end{cases}$$

This is proved in Theorem 5.1.

Although the homotopy type of a moment-angle complex $\mathcal{Z}_K$ is not accessible in general, various homotopy-theoretical concepts can be described if K is a flag complex. Moreover, many of these homotopy-theoretical characterisations of $\mathcal{Z}_K$ are equivalent. For example, for K a flag complex, $\mathcal{Z}_K$ having the homotopy type of a wedge of spheres is equivalent to $\mathcal{Z}_K$ being a co-H space and these concepts are equivalent to K being Golod. In this paper, we show that $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra if and only if $\mathcal{Z}_K$ has the homotopy type of a connected sum of sphere products, with two spheres in each product. Additionally, these properties are closely related to K being minimally non-Golod, see [11], and we summarise this relationship in Proposition 5.6.

We note that despite the similarity of the results for the moment-angle complex $\mathcal{Z}_K$ and its real analogue $\mathcal{R}_K$ when K is flag, the techniques used in proofs differ significantly. For the case of $\mathcal{Z}_K$, homotopy-theoretical methods are more prevalent, whereas the case of $\mathcal{R}_K$ requires the use of methods in combinatorial and geometric group theory. Given a homotopy-theoretical result related to $\mathcal{Z}_K$, one could predict the corresponding group-theoretical result for $\mathcal{R}_K$, but it is an open and challenging problem to find a systematic way of translating these results directly. This is a problem of interest to both topologists and group theorists.

For non-flag K, all homotopy-theoretical characterisations of moment-angle complexes $\mathcal{Z}_K$ are more complex. For example, for an arbitrary Golod complex K, the moment-angle complex $\mathcal{Z}_K$ is not necessarily a co-H space [10] and its cohomology can contain torsion [7]. Furthermore, describing the Pontryagin algebra $H_*(\Omega\mathcal{Z}_K)$, and in particular determining the class of K for which it is a free or one-relator algebra, is considerably harder in the non-flag case. The problem of determining those K which are Golod or minimally non-Golod is also more involving, and in general distinct from studying $H_*(\Omega\mathcal{Z}_K)$. At the end of Section 5 we expand on the distinction between the properties of K being minimally non-Golod and $H_*(\Omega\mathcal{Z}_K)$ being a one-relator algebra. This complexity is also seen in the real case. In the non-flag case, the real moment-angle complex $\mathcal{R}_K$ is not aspherical, so its topology is not determined by its fundamental group. Therefore, the question of describing $H_*(\Omega\mathcal{R}_K)$ does not lie entirely within combinatorial group theory.

2. Preliminaries

Let K be a *simplicial complex* on the set $[m] = \{1, 2, \dots, m\}$, that is, K is a collection of subsets $I \subseteq [m]$ such that for any $I \in K$ all subsets of I also belong to K. We always assume that K contains $\emptyset$ and all singletons $\{i\} \in [m]$.

Let

$$(\mathbf{X}, \mathbf{A}) = \{(X_1, A_1), \dots, (X_m, A_m)\}$$

be a sequence of m pairs of pointed topological spaces, $pt \in A_i \subseteq X_i$. For each subset $I \subseteq [m]$, we set

$$(\mathbf{X}, \mathbf{A})^I = \Big\{(x_1, \dots, x_m) \in \prod_{k=1}^{m} X_k \mid x_k \in A_k \text{ for } k \notin I\Big\}$$

and define the *polyhedral product* of $(\mathbf{X},\mathbf{A})$ over the complex K as

$$(\mathbf{X},\mathbf{A})^K=\bigcup_{I\in K}(\mathbf{X},\mathbf{A})^I=\bigcup_{I\in K}\Big(\prod_{i\in I}X_i\times\prod_{i\notin I}A_i\Big)\subseteq\prod_{k=1}^m X_k.$$

In the case when $X_i=X$ and $A_i=A$ for all i we use the notation $(X,A)^K$ for $(\mathbf{X},\mathbf{A})^K$.

Example 2.1.
1. Let $(X,A)=(D^1,S^0)$, where D^1 is the closed interval $[-1,1]$ and S^0 is its boundary $\{-1,1\}$. The polyhedral product $(D^1,S^0)^K$ is known as the *real moment-angle complex* and is denoted by $\mathcal{R}_K$,

$$\mathcal{R}_K=(D^1,S^0)^K=\bigcup_{I\in K}(D^1,S^0)^I.$$

Note that $\mathcal{R}_K$ is a cubic subcomplex in the cube $(D^1)^m=[-1,1]^m$.
2. Let $(X,A)=(D^2,S^1)$, where D^2 is the closed unit disc and S^1 is its boundary. The polyhedral product $(D^2,S^1)^K$ is known as the *moment-angle complex* and is denoted by $\mathcal{Z}_K$. If D^2 is considered as a CW-complex with one cell in each dimension zero, one and two, then the moment-angle complex $\mathcal{Z}_K$ is a CW-subcomplex of the CW-product complex $(D^2)^m$.
3. Let $(X,A)=(\mathbb{C}P^\infty,pt)$. The polyhedral product $(\mathbb{C}P^\infty,pt)^K$ is known as the *Davis–Januszkiewicz space* and is denoted by DJ_K.

For any subset $J\subseteq[m]$, the corresponding *full subcomplex* of K is defined by

$$K_J=\{I\in K|I\subseteq J\}.$$

The homology groups of the moment-angle complex $\mathcal{Z}_K$ have a natural bigrading arising from the bigrading in the CW-structure of $\mathcal{Z}_K$, see [5, §4.4],

$$H_k(\mathcal{Z}_K)\cong\bigoplus_{-i+2j=k}H_{-i,2j}(\mathcal{Z}_K). \tag{2}$$

The bigraded components $H_{-i,2j}(\mathcal{Z}_K)$ can be described through the reduced simplicial homology groups of full subcomplexes K_J using Hochster's theorem, see [5, Theorem 4.5.8],

$$H_{-i,2j}(\mathcal{Z}_K)\cong\bigoplus_{J\subseteq[m],\,|J|=j}\widetilde{H}_{j-i-1}(K_J). \tag{3}$$

Similarly, the homology groups of the real moment-angle complex $\mathcal{R}_K$ are given by

$$H_k(\mathcal{R}_K)\cong\bigoplus_{J\subseteq[m]}\widetilde{H}_{k-1}(K_J) \tag{4}$$

for any $k\geqslant 0$, see [5, §4.5].

A *missing face* of K is a subset $I\subseteq[m]$ such that I is not a simplex of K, but every proper subset of I is a simplex of K. A simplicial complex K is called a *flag complex* if each of its missing faces consists of two vertices, that is, any set of vertices of K which are pairwise connected by edges spans a simplex. A *clique* of a graph Γ is a subset I of vertices pairwise connected by edges. For a graph Γ, we define the *clique complex* of Γ as the simplicial complex obtained by filling in each clique of Γ by a simplex. Each flag complex K is the clique complex of its 1-skeleton $\Gamma=K^1$.

A graph Γ is called *chordal* if each of its cycles with 4 or more vertices has a chord, an edge joining two vertices that are not adjacent in the cycle. A *p-cycle* is the same as the boundary of a p-gon. It is a chordal graph only when $p = 3$. The simplicial complex which is a p-cycle is denoted by C_p.

If $K = C_p$, $p \geqslant 4$, by a result of McGavran [13], there is a homeomorphism

$$\mathcal{Z}_K \cong \#_{k=3}^{p-1}(S^k \times S^{p+2-k})^{\#(k-2)\binom{p-2}{k-1}}. \tag{5}$$

The corresponding real moment-angle complex $\mathcal{R}_K$ is an orientable surface of genus $1 + (p-4)2^{p-3}$, see [5, Proposition 4.1.8].

The algebra $H_*(\Omega\mathcal{Z}_K)$ was studied in [14, 7]. The homotopy fibration $\mathcal{Z}_K \to DJ_K \to (\mathbb{C}P^\infty)^m$ gives rise to a short exact sequence of Hopf algebras

$$1 \longrightarrow H_*(\Omega\mathcal{Z}_K) \longrightarrow H_*(\Omega DJ_K) \xrightarrow{\text{Ab}} \Lambda[u_1, \ldots, u_m] \longrightarrow 0 \tag{6}$$

where Ab is the "abelianisation" homomorphism to the graded commutative algebra $\Lambda[u_1, \ldots, u_m] = H_*(\Omega(\mathbb{C}P^\infty)^m)$ with $\deg u_i = 1$. The algebra $H_*(\Omega\mathcal{Z}_K)$ can be viewed as the commutator subalgebra of $H_*(\Omega DJ_K)$. Let $[a, b] = ab + (-1)^{\deg a \deg b + 1}ba$ denote the graded Lie commutator of the elements a and b. In the case that K is flag, there is an algebra isomorphism [14, Theorem 9.3]

$$H_*(\Omega DJ_K) \cong T(u_1, \ldots, u_m)/\langle u_i^2,\ [u_i, u_j] \text{ if } \{i, j\} \in K\rangle \tag{7}$$

where $T(u_1, \ldots, u_m)$ is a graded free associative algebra and $\deg u_i = 1$. A minimal multiplicative generating set for $H_*(\Omega\mathcal{Z}_K)$ is given as in [7, Theorem 4.3]. Namely, $H_*(\Omega\mathcal{Z}_K)$ is multiplicatively generated by $\sum_{J\subseteq[m]} \operatorname{rank} \widetilde{H}_0(K_J)$ iterated commutators of the form

$$[u_j, u_i],\ [u_{k_1}, [u_j, u_i]],\ \ldots,\ [u_{k_1}, [u_{k_2}, \ldots, [u_{k_{l-2}}, [u_j, u_i]] \ldots]] \tag{8}$$

where $k_1 < k_2 < \cdots < k_{l-2} < j > i$, $k_s \neq i$ for any s and i is the smallest vertex in a connected component of $K_{\{k_1,\ldots,k_{l-2},j,i\}}$ not containing j. Additionally, it was shown in [7, Theorem 4.6] that $H_*(\Omega\mathcal{Z}_K)$ is a free associative algebra if and only if the graph K^1 is chordal, in which case $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres.

Parallel results for the real moment-angle complex $\mathcal{R}_K$ were obtained in [15] in the group-theoretical setting. Let $(g, h) = g^{-1}h^{-1}gh$ denote the group commutator of elements g and h.

The *right-angled Coxeter group* RC_K corresponding to K is defined by

$$RC_K = F(g_1, \ldots, g_m)/(g_i^2,\ (g_i, g_j) \text{ if } \{i, j\} \in K)$$

where $F(g_1, \ldots, g_m)$ is a free group with m generators. Note that RC_K depends only on the 1-skeleton $\Gamma = K^1$, which is a graph.

Recall that a path-connected space X is *aspherical* if $\pi_i(X) = 0$ for $i \geqslant 2$. An aspherical space X is an Eilenberg–Mac Lane space $K(\pi, 1)$ with $\pi = \pi_1(X)$. The following facts relating the real moment-angle complex $\mathcal{R}_K$ to the right-angle Coxeter group RC_K are known, see, for example [15, Corollary 3.4]):

(i) $\pi_1(\mathcal{R}_K)$ is isomorphic to the commutator subgroup RC_K';
(ii) $\mathcal{R}_K$ is aspherical if and only if K is flag.

Therefore, in the flag case the algebra $H_*(\Omega\mathcal{R}_K)$ reduces to the non-abelian group $H_0(\Omega\mathcal{R}_K) = RC_K'$ and the analogue of (6) is the short exact sequence of groups

$$1 \longrightarrow RC_K' \longrightarrow RC_K \xrightarrow{\text{Ab}} (\mathbb{Z}_2)^m \longrightarrow 0$$

where $\mathbb{Z}_2$ is an elementary abelian 2-group and Ab is the abelianisation homomorphism.

By analogy with [7], the following combinatorial criterion was obtained in [15, Corollary 4.4]: the commutator subgroup RC_K' is a free group if and only if the graph K^1 is chordal.

An explicit minimal generator set for the commutator subgroup RC'_K is described in [15, Theorem 4.5]. It consists of $\sum_{J\subseteq[m]} \operatorname{rank} \widetilde{H}_0(K_J)$ nested commutators

$$(g_j, g_i),\quad (g_{k_1}, (g_j, g_i)),\ \ldots,\ (g_{k_1}, (g_{k_2}, \ldots, (g_{k_{l-2}}, (g_j, g_i))\ldots)) \tag{9}$$

where $k_1 < k_2 < \cdots < k_{l-2} < j > i$, $k_s \neq i$ for any s, and i is the smallest vertex in a connected component of $K_{\{k_1,\ldots,k_{l-2},j,i\}}$ not containing j.

3. One-Relator Groups

A group G is called *a one-relator group* if G is not a free group and can be presented with a generating set with a single relation.

Let G be a one-relator group, that is, $G = F/R$, where $F = F(x_1, \ldots, x_l)$ is a free group and R is the smallest normal subgroup in F generated by relation r. Consider the space

$$Y(G) = \Big(\bigvee_{i=1}^{l} S^1_i\Big) \cup_{\bar{r}} e^2 \tag{10}$$

obtained by attaching a 2-cell to a wedge of circles via a map $\bar{r}\colon S^1 \to \bigvee S^1_i$ corresponding to the element $r \in F$.

Recall that all homology groups are considered with coefficients in $\mathbb{Z}$. The homology groups of $Y(G)$ are described as follows.

Proposition 3.1. *$H_k(Y(G)) = 0$ for $k \geqslant 3$, $H_1(Y(G)) = \mathbb{Z}^l$ and*

$$H_2(Y(G)) = \begin{cases} \mathbb{Z} & \text{if } r \in [F,F] \\ 0 & \text{otherwise.} \end{cases}$$

□

Lyndon [12] studied cohomology theory of groups with a single relation by considering the corresponding space $Y(G)$. Dyer and Vasquez [6] gave an equivalent formulation of the Lyndon Identity Theorem in the following form (see [6, Theorem 2.1]]): if G is a one-relator group with relation r which is not a proper power, that is, $r \neq u^n$ for $n > 1$, then $Y(G)$ is a $K(G,1)$-space.

Under the conditions of the Lyndon Identity Theorem, we have $H_k(G;\mathbb{Z}) = H_k(Y(G);\mathbb{Z})$, that is, the homological dimension of G is at most 2.

Theorem 3.2. *Let K be a flag simplicial complex on $[m]$. The following conditions are equivalent:*

(a) *$\pi_1(\mathcal{R}_K) = RC'_K$ is a one-relator group;*
(b) *$H_2(\mathcal{R}_K) = \mathbb{Z}$;*
(c) *$K = C_p$ or $K = C_p * \Delta^q$ for $p \geqslant 4$ and $q \geqslant 0$, where C_p is a p-cycle, Δ^q is a q-simplex, and $*$ denotes the join of simplicial complexes.*

If any one of these conditions is met, we have $H_k(\mathcal{R}_K) = 0$ for $k \geqslant 3$.

Proof. (c) ⇒ (b). This implication follows from the implications below, but we include an independent proof as it is simple and illustrative. Suppose that $K = C_p$ or $K = C_p * \Delta^q$ for $p \geqslant 4$ and $q \geqslant 0$. Let the p-cycle C_p be supported on the set of vertices $I = \{i_1, \ldots, i_p\}$. By homology decomposition (4),

$$H_2(\mathcal{R}_K) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}_1(K_J). \tag{11}$$

Since K_I is a p-cycle, we have $\widetilde{H}_1(K_I) = \mathbb{Z}$. Because any subcomplex K_J with $J \neq I$ is contractible, $\widetilde{H}_1(K_J) = 0$ for $J \neq I$. It follows that $H_2(\mathcal{R}_K) = \mathbb{Z}$.

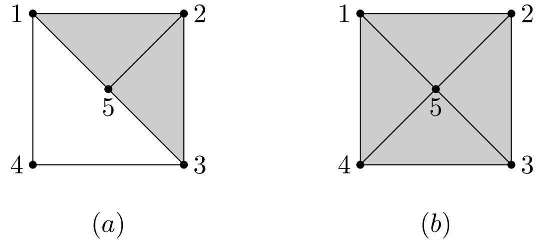

FIGURE 1.

(b) $\Rightarrow$ (c). Suppose $H_2(\mathcal{R}_K) = \mathbb{Z}$. Then only one summand in (11) is $\mathbb{Z}$, and all other summands are zero. Since K is a flag complex, this implies that there exists a set of vertices $I = \{i_1, \dots, i_p\}$ such that K_I is a p-cycle with $p \geqslant 4$. Since $\widetilde{H}_1(K_J) = 0$ for any proper subset $J \subseteq I$, any two vertices which are not adjacent in the p-cycle are not connected by an edge. If there exists a vertex $j \notin I$ in the complex K, then $\widetilde{H}_1(K_{I\cup\{j\}}) = 0$ implies that the vertex j is connected to each vertex in the p-cycle I. If K has two vertices $j_1, j_2 \notin I$ which are not connected by an edge, then the subcomplex $K_{\{i_1,i_3\}\cup\{j_1,j_2\}}$ is a 4-cycle and $\widetilde{H}_1(K_{\{i_1,i_3\}\cup\{j_1,j_2\}}) = \mathbb{Z}$, which contradicts the assumption. Hence, all vertices of K which are not in the set I are connected to each other and to all vertices of I. Since K is a flag complex, we obtain $K = C_p * \Delta^q$ for some $p \geqslant 4$ and $q \geqslant 0$.

(c) $\Rightarrow$ (a). First let K be a p-cycle C_p. In this case the complex $\mathcal{R}_K$ is homeomorphic to a closed orientable surface of genus $(p-4)2^{p-3}+1$ (see [5, Proposition 4.1.8]). Also, $\pi_1(\mathcal{R}_K) \cong RC'_K$. Hence, RC'_K is a one-relator group.

Now let $\widetilde{K} = K * \Delta^q$, where K is a p-cycle. Then $\mathcal{R}_{\widetilde{K}} = \mathcal{R}_K \times D^{q+1}$ and $RC'_{\widetilde{K}} = \pi_1(\mathcal{R}_{\widetilde{K}}) = \pi_1(\mathcal{R}_K) = RC'_K$ is a one-relator group.

(a) $\Rightarrow$ (b). Since $\mathcal{R}_K$ is an aspherical finite cell complex, the group $\pi_1(\mathcal{R}_K)$ is torsion-free (for example, see [9, Proposition 2.45]). So if $\pi_1(\mathcal{R}_K) = F/R$ is a one-relator group with a relation r, then r is not a proper power u^n for $n > 1$, as otherwise the element u would be of finite order.

Consider the space $Y(RC'_K)$, constructed as in (10). According to the Lyndon Identity Theorem, $Y(RC'_K)$ is homotopy equivalent to $K(RC'_K, 1)$, so its homology groups coincide with the homology groups of the space $\mathcal{R}_K$. Proposition 3.1 implies that $H_2(\mathcal{R}_K)$ is either $\mathbb{Z}$ or 0. The group RC'_K is not free, so the graph K^1 is not chordal, that is, there exists a chordless cycle on I of length $p \geqslant 4$. Therefore, one of the summands on the right hand side of (11) is equal to $\mathbb{Z} = \widetilde{H}_1(K_I)$. Thus, $H_2(\mathcal{R}_K) = \mathbb{Z}$.

It remains to prove that (c) implies that $H_k(\mathcal{R}_K) = 0$ for $k \geqslant 3$. Considering homology decomposition (4),

$$H_k(\mathcal{R}_K) \cong \bigoplus_{J\subseteq[m]} \widetilde{H}_{k-1}(K_J)$$

we claim that all summands with $k \geqslant 3$ on the right hand side are equal to 0. Indeed, let $I = \{i_1, \dots, i_p\}$ be the set of vertices of K forming a p-cycle. Then $\widetilde{H}_{k-1}(K_I) = 0$ for $k \geqslant 3$. Since any full subcomplex K_J with $J \neq I$ is contractible, we get $\widetilde{H}_{k-1}(K_J) = 0$. Hence, $H_k(\mathcal{R}_K) = 0$ for $k \geqslant 3$. □

The following examples illustrate Theorem 3.2.

Example 3.3.
1. Let K be the flag complex in Figure 1 (a). Generator set (9) for the commutator

subgroup RC'_K is

$$(g_3, g_1),\ (g_4, g_2),\ (g_5, g_4),\ (g_2, (g_5, g_4)).$$

These satisfy the relations

$$(g_3, g_1)^{-1}(g_4, g_2)^{-1}(g_3, g_1)(g_4, g_2) = 1, \quad (g_3, g_1)^{-1}(g_5, g_4)^{-1}(g_3, g_1)(g_5, g_4) = 1$$

and

$$(g_3, g_1)^{-1}(g_2, (g_5, g_4))^{-1}(g_3, g_1)(g_2, (g_5, g_4)) = 1.$$

Indeed, since each of g_1 and g_3 commutes with each of g_2 and g_4, the commutators $(g_4, g_2)^{-1}$ and (g_3, g_1) commute too. We therefore obtain

$$(g_3, g_1)^{-1}(g_4, g_2)^{-1}(g_3, g_1)(g_4, g_2) = (g_3, g_1)^{-1}(g_3, g_1)(g_4, g_2)^{-1}(g_4, g_2) = 1.$$

The other two relations are proved similarly. Using homology decomposition (4), we get $H_2(\mathcal{R}_K) = \mathbb{Z}^3$.

2. Let K be the flag complex in Figure 1 (b). Generator set (9) for RC'_K is

$$(g_3, g_1),\ (g_4, g_2),$$

which satisfy a single relation $(g_3, g_1)^{-1}(g_4, g_2)^{-1}(g_3, g_1)(g_4, g_2) = 1$. Here RC'_K is a one-relator group and $H_2(\mathcal{R}_K) = \mathbb{Z}$.

4. Connected Sums of Sphere Products

Let $M = \#_{i=1}^k \left(S^{d_i} \times S^{d-d_i}\right)$, where $d_i \geqslant 2$, $d \geqslant 4$ and $\#$ denotes the connected sum operation on manifolds. Topologically, such connected sums are obtained by attaching a single cell to a wedge of spheres, that is, there is a cofibration sequence

$$S^{d-1} \xrightarrow{\ w\ } \bigvee_{i=1}^k S^{d_i} \vee S^{d-d_i} \xrightarrow{\ i\ } \#_{i=1}^k (S^{d_i} \times S^{d-d_i}) \tag{12}$$

where w is the sum of Whitehead products $w_i\colon S^{d-1} \to S^{d_i} \vee S^{d-d_i}$. Denote by $\overline{M}$ the wedge $\bigvee_{i=1}^d S^{d_i} \vee S^{d-d_i}$. Then by the Bott–Samelson theorem $H_*(\Omega\overline{M}) \cong T(a_1, b_1, \ldots, a_k, b_k)$, where $\deg(a_i) = d_i - 1$ and $\deg(b_i) = d - d_i - 1$. The looped inclusion $\Omega i\colon \Omega\overline{M} \to \Omega M$ induces a map of algebras

$$(\Omega i)_*\colon T(a_1, b_1, \ldots, a_k, b_k) \longrightarrow H_*(\Omega M).$$

The adjoint $\overline{w}\colon S^{d-2} \to \Omega\left(\bigvee_{i=1}^k S^{d_i} \vee S^{d-d_i}\right)$ of the sum of Whitehead products w induces a map $\overline{w}_*\colon H_{d-2}(S^{d-2}) \to H_{d-2}(\Omega\overline{M})$ which sends the canonical generator to the element $\chi = [a_1, b_1] + \cdots + [a_k, b_k]$. In particular, χ is primitive and $(\Omega i)_*(\chi) = 0$ in $H_*(\Omega M)$. Then the algebra

$$\frac{T(a_1, b_1, \ldots, a_k, b_k)}{\langle [a_1, b_1] + \cdots + [a_k, b_k] \rangle} \tag{13}$$

is a primitively generated Hopf algebra, where the quotient ideal is two-sided, and the algebra map $(\Omega i)_*$ factors as a map of Hopf algebras

$$\begin{array}{ccc} T(a_1, b_1, \ldots, a_k, b_k) & \xrightarrow{(\Omega i)_*} & H_*(\Omega M) \\ \downarrow & \nearrow_{\theta} & \\ \dfrac{T(a_1, b_1, \ldots, a_k, b_k)}{\langle [a_1, b_1] + \cdots + [a_k, b_k] \rangle} & & \end{array} \tag{14}$$

defining the map θ.

The loop homology Hopf algebra $H_*(\Omega X; \mathbb{Z})$ of a simply connected CW-complex X can be calculated as homology of the *cobar construction* $\operatorname{Cobar} C_*(X)$ of the reduced singular chains $C_*(X)$ [1], or as homology of the *Adams–Hilton model* [2] based on cells and attaching maps.

The cobar construction Cobar is a functor

$$\mathrm{Cobar}\colon \mathrm{DGC}_1 \longrightarrow \mathrm{DGA}$$

from the category DGC_1 of simply connected differential graded (dg) coalgebras to dg algebras. It assigns to a dg coalgebra (C,∂) with $C_0=\mathbb{Z}$ and $C_1=0$ the dg algebra

$$\mathrm{Cobar}\, C = (F(C), d)$$

where $F(C)=T(s^{-1}\overline{C})$ is the free associative algebra on the desuspended module $\overline{C}=C/\mathbb{Z}$, the cokernel of the coaugmentation $\mathbb{Z}\to C$. The differential d is given by

$$dc = -\partial c + \sum_{i=2}^{p-2}(-1)^i \Delta_{i,p-i}c \tag{15}$$

where $c\in s^{-1}\overline{C}_p$ with comultiplication $\Delta c = c\otimes 1 + 1\otimes c + \sum_{i=2}^{p-2}\Delta_{i,p-i}c$.

Adams [1] proved that for a simply connected CW-complex X there is an isomorphism of Hopf algebras

$$H_*(\Omega X)\cong H(\mathrm{Cobar}\, C_*(X), d) = \mathrm{Cotor}_{C_*(X)}(\mathbb{Z},\mathbb{Z})$$

where $C_*(X)$ is the reduced singular chain coalgebra of X.

The Adams–Hilton model [2] is a smaller dg algebra $AH_*(X)$ quasi-isomorphic to $\mathrm{Cobar}\, C_*(X)$; it has generators corresponding to the cells of X and differential defined via the attaching maps.

The following statement generalises [2, Corollary 2.4].

Proposition 4.1. *For $d_i\geqslant 2$ and $d\geqslant 4$, there is an isomorphism of Hopf algebras*

$$H_*\left(\Omega\left(\#_{i=1}^k S^{d_i}\times S^{d-d_i}\right)\right)\cong \frac{T(a_1,b_1,\ldots,a_k,b_k)}{\langle [a_1,b_1]+\cdots+[a_k,b_k]\rangle}$$

where $\deg a_i = d_i-1$, $\deg b_i = d-d_i-1$, *and* $[a_i,b_i]=a_i\otimes b_i + (-1)^{\deg a_i \deg b_i+1}b_i\otimes a_i$ *is the graded commutator.*

Proof. We consider the Adams–Hilton model of $M=\#_{i=1}^k S^{d_i}\times S^{d-d_i}$. The cofibration sequence (12) gives a CW-structure on M consisting of cells $e^0, e_i^{d_i}, e_i^{d-d_i}$, $1\leqslant i\leqslant k$, each attached trivially, and a single cell e^d attached by the sum of Whitehead products $w_i\colon S^{d-1}\to S^{d_i}\vee S^{d-d_i}$. The Adams–Hilton model $AH_*(M)$ can be identified with the cobar construction on the coalgebra generated by positive-dimensional cells, in which the differential is zero, $e_i^{d_i}$ and $e_i^{d-d_i}$ are primitives, and

$$\Delta e^d = e^d\otimes 1 + 1\otimes e^d + \sum_{i=1}^k\big(e_i^{d_i}\otimes e_i^{d-d_i} + (-1)^{d_i(d-d_i)}e_i^{d-d_i}\otimes e_i^{d_i}\big). \tag{16}$$

The Adams–Hilton model is therefore

$$AH_*(M) = \big(T(a_1,b_1,\ldots,a_k,b_k,z), d\big)$$

where $a_i=(-1)^{d_i}s^{-1}e_i^{d_i}$, $b_i=s^{-1}e_i^{d-d_i}$, $z=s^{-1}e^d$ and $\deg a_i = d_i-1$, $\deg b_i = d-d_i-1$ and $\deg z = d-1$. Differential (15) is given by $d(a_i)=d(b_i)=0$ and

$$\begin{aligned} d(z) &= \sum_{i=1}^k\big((-1)^{d_i}s^{-1}e_i^{d_i}\otimes s^{-1}e_i^{d-d_i} + (-1)^{d-d_i}(-1)^{d_i(d-d_i)}s^{-1}e_i^{d-d_i}\otimes s^{-1}e_i^{d_i}\big)\\ &= \sum_{i=1}^k\big(a_i\otimes b_i + (-1)^{(d_i+1)(d-d_i)+d_i}b_i\otimes a_i\big) = \sum_{i=1}^k [a_i,b_i]. \end{aligned}$$

A nonzero $x \in AH_*(M)$ is a cycle if and only if x is not in the two-sided ideal $\langle z\rangle$, and x is a boundary if and only if $x \in \langle d(z)\rangle$. Therefore, homology of ΩM is as stated. □

For a graded vector space V, denote by $P(V;t)$ the Poincaré series of V.

Proposition 4.2. *There is the following identity for the Poincaré series*

$$P\left(H_*\left(\Omega\left(\#_{i=1}^k S^{d_i}\times S^{d-d_i}\right);t\right)\right)=\frac{1}{1-\sum_{i=1}^k(t^{d_i-1}+t^{d-d_i-1})+t^{d-2}}.$$

Proof. Let $A=H_*\left(\Omega\left(\#_{i=1}^k S^{d_i}\times S^{d-d_i}\right)\right)$. By Proposition 4.1, A is the quotient of the free associative algebra on the graded set $S=\{a_1,b_1,\dots,a_k,b_k\}$, where $\deg a_i=d_i-1$ and $\deg b_i=d-d_i-1$, by the two-sided ideal generated by the element

$$\chi=\sum_{i=1}^k[a_i,b_i]=a_1b_1+(-1)^{\deg a_1\deg b_1+1}b_1a_1+\sum_{i=2}^k[a_i,b_i].$$

Let B be the graded free monoid on S. Then B_n, the nth graded component of B, is a generating set for A_n, the nth graded component of A. For any monomial $x\in A\setminus\{1\}$, write $x=sy$ for some unique $s\in S$ and $y\in B_{n-\deg s}$. If $x=a_1b_1y'$ then using relation χ we rewrite

$$x=\Big((-1)^{\deg a_1\deg b_1}b_1a_1-\sum_{i=2}^k[a_i,b_i]\Big)y'.$$

Let B'_n be the set of all elements in B_n which do not start with a_1b_1. By induction, B'_n is a minimal generating set for A_n. Define $c_n=|B'_n|=\operatorname{rank}A_n$ for $n\geqslant 1$, $c_n=0$ for $n<0$, and $c_0=1$. From the above description, c_n satisfies the recurrence formula

$$c_n=\sum_{i=1}^k(c_{n-d_i+1}+c_{n-d+d_i+1})-c_{n-d+2}$$

for $n\geqslant 1$. Multiplying by t^n and summing over $n>0$ gives

$$\begin{aligned}
P(A;t)-1&=\sum_{n=1}^\infty c_nt^n\\
&=\sum_{n=1}^\infty\left(\sum_{i=1}^k(c_{n-d_i+1}+c_{n-d+d_i+1})-c_{n-d+2}\right)t^n\\
&=\sum_{i=1}^k\sum_{n=2-d_i}^\infty c_nt^{n+d_i-1}+\sum_{i=1}^k\sum_{n=2-d+d_i}^\infty c_nt^{n+d-d_i-1}-\sum_{n=3-d}^\infty c_nt^{n+d-2}\\
&=\left(\sum_{i=1}^k(t^{d_i-1}+t^{d-d_i-1})-t^{d-2}\right)\sum_{n=0}^\infty c_nt^n\\
&=\left(\sum_{i=1}^k(t^{d_i-1}+t^{d-d_i-1})-t^{d-2}\right)P(A;t)
\end{aligned}$$

which is rearranged to give the claimed identity. □

5. One-Relator Algebras

An algebra is a *one-relator algebra* if it is not free and can be written as the quotient of a free associative algebra by a two-sided ideal generated by a single element.

We recall the bigraded decomposition (3) of the integral homology of the moment-angle complex $\mathcal{Z}_K$.

Theorem 5.1. *Let K be a flag simplicial complex on $[m]$. The following conditions are equivalent:*

(a) *$H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra;*

(b) $H_{2-j,2j}(\mathcal{Z}_K) = \begin{cases} \mathbb{Z} & \text{if } j = p \text{ for some } p,\ 4 \leqslant p \leqslant m \\ 0 & \text{otherwise;} \end{cases}$

(c) *$K = C_p$ or $K = C_p * \Delta^q$ for $p \geqslant 4$ and $q \geqslant 0$, where C_p is a p-cycle and Δ^q is a q-simplex.*

If any one of these conditions is met, we have $H_{-i,2j}(\mathcal{Z}_K) = 0$ for $j - i \geqslant 3$.

To prove the Theorem, we start by showing that if K is a flag complex which is not of the form given in (c), then either $H_*(\Omega\mathcal{Z}_K)$ is free, or it has at least two relations. The following result gives a condition for $H_*(\Omega\mathcal{Z}_K)$ to have at least two relations.

Lemma 5.2. *Let K be a simplicial complex and suppose that K_I and K_J are distinct full subcomplexes of K such that both $H_*(\Omega\mathcal{Z}_{K_I})$ and $H_*(\Omega\mathcal{Z}_{K_J})$ have at least one relation. Then $H_*(\Omega\mathcal{Z}_K)$ is not a one-relator algebra.*

Proof. Note that each of $\mathcal{Z}_{K_I}$ and $\mathcal{Z}_{K_J}$ retracts off $\mathcal{Z}_K$ as K_I and K_J are full subcomplexes. Therefore each of $\Omega\mathcal{Z}_{K_I}$ and $\Omega\mathcal{Z}_{K_J}$ retracts off $\Omega\mathcal{Z}_K$ and we obtain a commutative diagram of algebras

$$\begin{array}{ccc} H_*(\Omega\mathcal{Z}_{K_I}) & \longrightarrow & H_*(\Omega\mathcal{Z}_K) \\ & \searrow^{=} & \downarrow \\ & & H_*(\Omega\mathcal{Z}_{K_I}) \end{array}$$

and similarly for K_J. In particular, each relation of $H_*(\Omega\mathcal{Z}_{K_I})$ appears as a relation of $H_*(\Omega\mathcal{Z}_K)$ under the induced inclusion map and similarly for K_J, and the induced relations are distinct since K_I and K_J are. ☐

Let K be a simplicial complex on $[m]$. Suppose that $j \in K$ is a vertex and define the link

$$\mathrm{lk}_K(j) = \{I \in K \mid j \cup I \in K,\ j \notin I\}$$

and the star

$$\mathrm{st}_K(j) = \{I \in K \mid j \cup I \in K\} = \mathrm{lk}_K(j) * j$$

and assume that $\mathrm{lk}_K(j)$ is on the first l vertices of K. Decompose $K = \mathrm{st}_K(j) \cup_{\mathrm{lk}_K(j)} K_{[m]\setminus j}$. Then there is a homotopy pushout of moment-angle complexes

$$\begin{array}{ccc} \mathcal{Z}_{\mathrm{lk}_K(j)} \times T^{m-l} & \xrightarrow{i\times\pi} & \mathcal{Z}_{K_{[m]\setminus j}} \times S^1 \\ \downarrow{\scriptstyle \mathrm{id}\times\pi} & & \downarrow \\ \mathcal{Z}_{\mathrm{lk}_K(j)} \times T^{m-l-1} & \longrightarrow & \mathcal{Z}_K. \end{array} \tag{17}$$

Lemma 5.3. *If the map $i\colon \mathcal{Z}_{\mathrm{lk}_K(j)} \to \mathcal{Z}_{K_{[m]\setminus j}}$ is nullhomotopic, then there is a homotopy equivalence*

$$\mathcal{Z}_K \simeq \Sigma^2(\mathcal{Z}_{\mathrm{lk}_K(j)} \times T^{m-l-1}) \vee (\mathcal{Z}_{[m]\setminus j} \rtimes S^1).$$

Here the half-smash $X \rtimes Y$ of pointed spaces is defined by $X \times Y/(pt \times Y)$.

Proof. This is a particular case of [8, Lemma 3.3]. □

The following result shows that when $\mathcal{Z}_{[m]\setminus j}$ has the homotopy type of a connected sum of sphere products, $H_*(\Omega\mathcal{Z}_K)$ is not a one-relator algebra.

Lemma 5.4. *Suppose that $M = \#_{i=1}^k \bigl(S^{d_i} \times S^{d-d_i}\bigr)$ where $d_i \geqslant 2$ and $d \geqslant 4$. Then $H_*(\Omega(M \rtimes S^1))$ is not a one-relator algebra.*

Proof. As in Proposition 4.1, we apply the Adams–Hilton model. A cell structure on $M \rtimes S^1$ is given by the image under the quotient map $M \times S^1 \to M \rtimes S^1$, and therefore consists of cells e^0, $e_i^{d_i}, e_i^{d-d_i}, e_i^{d_i+1}, e_i^{d-d_i+1}$, $1 \leqslant i \leqslant k$, along with two cells e^d and e^{d+1}. The Adams–Hilton model $AH_*(M \rtimes S^1)$ can be identified with the cobar construction on the coalgebra generated by positive-dimensional cells, in which the differential is zero, $e_i^{d_i}, e_i^{d-d_i}, e_i^{d_i+1}, e_i^{d-d_i+1}$ are primitives, Δe^d is given by (16) and

$$\Delta e^{d+1} = e^{d+1} \otimes 1 + 1 \otimes e^{d+1} + \sum_{i=1}^k \bigl(e_i^{d_i} \otimes e_i^{d-d_i+1} + (-1)^{d_i(d-d_i+1)} e_i^{d-d_i+1} \otimes e_i^{d_i} \\ + (-1)^{d-d_i} e_i^{d_i+1} \otimes e_i^{d-d_i} + (-1)^{d_i(d-d_i)} e_i^{d-d_i} \otimes e_i^{d_i+1}\bigr).$$

The Adams–Hilton model is therefore given by

$$AH_*(M \rtimes S^1) = \bigl(T(a_1, b_1, x_1, y_1, \ldots, a_k, b_k, x_k, y_k, z, w), d\bigr)$$

where we set $a_i = (-1)^{d_i} s^{-1} e_i^{d_i}$, $b_i = s^{-1} e_i^{d-d_i}$, $x_i = (-1)^{d+1} s^{-1} e_i^{d_i+1}$, $y_i = s^{-1} e_i^{d-d_i+1}$, $z = s^{-1} e^d$, $w = s^{-1} e^{d+1}$, so that $\deg a_i = d_i - 1$, $\deg b_i = d - d_i - 1$, $\deg x_i = d_i$, $\deg y_i = d - d_i$, $\deg z = d-1$, $\deg w = d$. Differential (15) is given by $d(a_i) = d(b_i) = d(x_i) = d(y_i) = 0$,

$$d(z) = \sum_{i=1}^k [a_i, b_i]$$

and

$$\begin{aligned} d(w) &= \sum_{i=1}^k \bigl((-1)^{d_i} s^{-1} e_i^{d_i} \otimes s^{-1} e_i^{d-d_i+1} + (-1)^{d-d_i+1} (-1)^{d_i(d-d_i+1)} s^{-1} e_i^{d-d_i+1} \otimes s^{-1} e_i^{d_i} \\ &\quad + (-1)^{d_i+1} (-1)^{d-d_i} s^{-1} e_i^{d_i+1} \otimes s^{-1} e_i^{d-d_i} + (-1)^{d-d_i} (-1)^{d_i(d-d_i)} s^{-1} e_i^{d-d_i} \otimes s^{-1} e_i^{d_i+1}\bigr) \\ &= \sum_{i=1}^k \bigl(a_i \otimes y_i + (-1)^{(d_i+1)(d-d_i+1)+d_i} y_i \otimes a_i + x_i \otimes b_i + (-1)^{d_i(d-d_i+1)+1} b_i \otimes x_i\bigr) \\ &= \sum_{i=1}^k \bigl([a_i, y_i] + [x_i, b_i]\bigr). \end{aligned}$$

Therefore any element in $\langle d(z)\rangle$ or $\langle d(w)\rangle$ is trivial in homology since it is a boundary. This induces two relations in $H_*(\Omega(M \rtimes S^1))$, as claimed. □

Proof of Theorem 5.1. (c) ⇒ (a). Suppose that $K = C_p$ or $K = C_p * \Delta^q$ for $p \geqslant 4$, $q \geqslant 0$. Since $\mathcal{Z}_K$ is homotopy equivalent to a connected sum of sphere products (5), the implication follows from Proposition 4.1.

(a) ⇒ (c). Suppose that K is a flag complex on $[m]$ such that $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra. If K^1 is a chordal graph, then $\mathcal{Z}_K$ has the homotopy type of a wedge of spheres [7, Theorem 4.6], and thus $H_*(\Omega\mathcal{Z}_K)$ is a graded free associative algebra, which is a contradiction.

Therefore assume that K^1 is not chordal. In particular, there exists a set of vertices $I \subseteq [m]$ such that the full subcomplex K_I is a p-cycle, and we enumerate $I = \{b_1, b_2, \ldots, b_p\}$. If $I = [m]$, that is $K = C_p$, then $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra by Proposition 4.1.

Assume that $[m] \setminus I \neq \emptyset$. First, we show that each $j \in [m] \setminus I$ is connected to each vertex in I. Consider the full subcomplex $K_{I\cup j}$ of K, and observe that

$$K_{I\cup j} = K_I \cup_{\mathrm{lk}_{I\cup j}(j)} \mathrm{st}_{I\cup j}(j).$$

Suppose that $K_{I\cup j} \neq K_I * j$. Since K is flag, there exists $b_l \in I$ such that there is no edge from j to b_l. Form the sequence of adjacent vertices $b_{l+1}, b_{l+2}, \ldots, b_{l+n_1}$, with the convention that $b_{p+1} = b_1$, where $n_1 \geqslant 1$ is the smallest index such that there is an edge from j to b_{l+n_1}. Similarly, form sequence of adjacent vertices $b_{l-1}, b_{l-2}, \ldots, b_{l-n_2}$, where $n_2 \geqslant 1$ is again the smallest index such that there is an edge from j to b_{l-n_2}. We consider four cases.

(i) Assume that there are no indices n_1 and n_2 as described above. In this case, there are no edges between j and any vertex in I, and so $\mathrm{lk}_{I\cup j}(j) = \emptyset$. Then (17) takes the form

$$\begin{array}{ccc} T^{p+1} & \xrightarrow{i\times \mathrm{id}} & \mathcal{Z}_{K_I} \times S^1 \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ T^p & \longrightarrow & \mathcal{Z}_{K_{I\cup j}} \end{array}$$

where the map $i\colon T^p \to \mathcal{Z}_{K_I}$ is nullhomotopic and therefore $\mathcal{Z}_{K_{I\cup j}} \simeq \Sigma^2 T^p \vee (Z_{K_I} \rtimes S^1)$ by Lemma 5.3. Since $\mathcal{Z}_{K_I}$ is homeomorphic to a connected sum of sphere products, Lemma 5.4 gives that $H_*(\Omega(\mathcal{Z}_{K_I} \rtimes S^1))$ is not a one-relator algebra, and hence neither is $H_*\left(\Omega\mathcal{Z}_{K_{I\cup j}}\right)$.

(ii) If $b_{l+n_1} = b_{l-n_2}$, then $\mathrm{lk}_{I\cup j}(j) = b_{l+n_1}$, and $\mathcal{Z}_{K_{I\cup j}} \simeq \Sigma^2 T^{p-1} \vee (Z_{K_I} \rtimes S^1)$ by Lemma 5.3. Thus $H_*\left(\Omega\mathcal{Z}_{K_{I\cup j}}\right)$ is not a one-relator algebra.

(iii) When b_{l+n_1} and b_{l-n_2} are adjacent in K_I, the link $\mathrm{lk}_{I\cup j}(j) = \{(b_{l+n_1}, b_{l-n_2})\}$, and so $\mathcal{Z}_{K_{I\cup j}} \simeq \Sigma^2 T^{p-2} \vee (Z_{K_I} \rtimes S^1)$ by Lemma 5.3. Thus $H_*\left(\Omega\mathcal{Z}_{K_{I\cup j}}\right)$ is not a one-relator algebra.

(iv) Finally, let b_{l+n_1} and b_{l-n_2} be distinct and not adjacent in K_I. Then by construction the full subcomplex $K_{\{j, b_{l-n_2}, \ldots, b_{l-1}, b_l, b_{l+1}, \ldots, b_{l+n_1}\}}$ of K is a (n_1+n_2+2)-cycle, which is distinct from K_I. Therefore by Lemma 5.2, $H_*\left(\Omega\mathcal{Z}_{K_{I\cup j}}\right)$ is not a one-relator algebra.

In all of the above cases, since the full subcomplex $K_{I\cup j}$ retracts off K and $H_*\left(\Omega\mathcal{Z}_{K_{I\cup j}}\right)$ is not a one-relator algebra, then neither is $H_*(\Omega\mathcal{Z}_K)$. This is a contradiction. We therefore conclude that j is connected to each vertex in K_I and therefore $K_{I\cup j} = K_I * j$.

Second, we show that if $j_1, j_2 \in [m] \setminus I$, then j_1 and j_2 are connected by an edge. If not, since both j_1 and j_2 are connected to each vertex in I, then the full subcomplex $K_{\{j_1, b_{i_1}, j_2, b_{i_3}\}}$ is a 4-cycle distinct from the p-cycle K_I. Therefore, Lemma 5.2 implies that since $H_*\left(\Omega\mathcal{Z}_{I\cup\{j_1,j_2\}}\right)$ is not a one-relator algebra, neither is $H_*(\Omega\mathcal{Z}_K)$, which is a contradiction.

Therefore any vertex in $[m] \setminus I$ is connected to every vertex in I and to every other vertex in $[m] \setminus I$. Since K is flag, $K = K_I * \Delta^q$ for some $q \geqslant 0$.

(c) ⇒ (b). Suppose that $K = C_p$ or $K = C_p * \Delta^q$ for $p \geqslant 4$ and $q \geqslant 0$. Let the p-cycle C_p of K be supported on the set of vertices $I = \{b_1, \dots, b_p\}$. By (3),

$$H_{2-j,2j}(\mathcal{Z}_K) \cong \bigoplus_{J \subseteq [m],\, |J|=j} \widetilde{H}_1(K_J). \tag{18}$$

Since K_I is a p-cycle, we have $\widetilde{H}_1(K_I) = \mathbb{Z}$. Because any subcomplex K_J with $J \neq I$ is contractible, $\widetilde{H}_1(K_J) = 0$ for $J \neq I$. It follows that $H_{2-p,2p}(\mathcal{Z}_K) = \mathbb{Z}$ and $H_{2-j,2j}(\mathcal{Z}_K) = 0$ for $j \neq p$.

(b) ⇒ (c). Suppose that $H_{2-j,2j}(\mathcal{Z}_K)$ is as described in (b). Then only one summand on the right hand side of (18) is $\mathbb{Z}$, and all other summands are zero. The same argument as in the proof of implication (b) ⇒ (c) of Theorem 3.2 shows that $K = C_p * \Delta^q$ for some $p \geqslant 4$ and $q \geqslant 0$.

It remains to prove that if $K = C_p$ or $K = C_p * \Delta^q$ for $q \geqslant 0$ then $H_{-i,2j}(\mathcal{Z}_K) = 0$ for $j - i \geqslant 3$. Considering bigraded decomposition (3),

$$H_{-i,2j}(\mathcal{Z}_K) \cong \bigoplus_{J \subseteq [m],\, |J|=j} \widetilde{H}_{j-i-1}(K_J).$$

We claim that all summands on the right hand side with $j - i \geqslant 3$ are equal to 0. Indeed, let $I = \{b_1, \dots, b_p\}$ be the set of vertices of K forming a p-cycle, $p \geqslant 4$. Then $\widetilde{H}_{j-i-1}(K_I) = 0$ for $j - i \geqslant 3$. Since any full subcomplex K_J with $J \neq I$ is contractible, we get $\widetilde{H}_{j-i-1}(K_J) = 0$. Hence, $H_{-i,2j}(\mathcal{Z}_K) = 0$ for $j - i \geqslant 3$. □

The following examples illustrate Theorem 5.1.

Example 5.5.
1. Let K be the flag complex in Figure 1 (a). Generator set (8) for $H_*(\Omega\mathcal{Z}_K)$ is

$$[u_3, u_1],\ [u_4, u_2],\ [u_5, u_4],\ [u_2, [u_5, u_4]].$$

These satisfy the relations

$$[u_3, u_1][u_4, u_2] - [u_4, u_2][u_3, u_1] = 0, \quad [u_3, u_1][u_5, u_4] - [u_5, u_4][u_3, u_1] = 0$$

and

$$[u_3, u_1][u_2, [u_5, u_4]] - [u_2, [u_5, u_4]][u_3, u_1] = 0$$

which are derived by using the commutativity relations given in (7). By formula (3) we obtain $H_{-2,8}(\mathcal{Z}_K) = \mathbb{Z}^2$ and $H_{-3,10}(\mathcal{Z}_K) = \mathbb{Z}$. Hence, the homological condition of Theorem 5.1 (b) is not satisfied.

2. Let K be the flag complex in Figure 1 (b). Generator set (8) for $H_*(\Omega\mathcal{Z}_K)$ is

$$[u_3, u_1],\ [u_4, u_2]$$

with a single relation $[u_3, u_1][u_4, u_2] - [u_3, u_1][u_4, u_2] = 0$. Here $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra, and formula (3) gives $H_{-2,8}(\mathcal{Z}_K) = \mathbb{Z}$.

Recall that a simplicial complex K is *Golod* if all cup products and higher Massey products vanish in $H^*(\mathcal{Z}_K)$. A simplicial complex K is *minimally non-Golod* if K is not Golod itself, but for every vertex $\{i\} \in K$ the deletion subcomplex $K - \{i\} = K_{[m]\setminus\{i\}}$ is Golod. In the flag case, the properties of $H_*(\Omega\mathcal{Z}_K)$ being a one-relator algebra and K being minimally non-Golod are related as follows.

Proposition 5.6. *Let K be a flag simplicial complex on $[m]$. The following conditions are equivalent:*

(a) *$\mathcal{Z}_K$ is homotopy equivalent to a connected sum of sphere products, with two spheres in each product;*
(b) *$H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra;*

(c) *K is minimally non-Golod, or $K = L * \Delta^q$ where L is a minimally non-Golod complex and $q \geqslant 0$.*

Proof. (a) ⇒ (b). This follows from Proposition 4.1.

(b) ⇒ (c). This follows from Theorem 5.1, because a p-cycle C_p with $p \geq 4$ is a minimally non-Golod complex.

(c) ⇒ (a). This follows from the fact that if K is minimally non-Golod and flag, then $K = C_p$ with $p \geqslant 4$. Indeed, let K^1 be one-skeleton of K. If K^1 is a chordal graph, then $\mathcal{Z}_K$ has the homotopy type of a wedge of spheres [7, Theorem 4.6], so K is Golod and therefore not minimally non-Golod. Hence, K contains a chordless cycle C_p with $p \geqslant 4$. If $\{i\} \in K$ is a vertex not in C_p, then $K - \{i\}$ still contains a chordless cycle and therefore is not Golod. Therefore $K = C_p$, and the result follows from formula (5). □

In the non-flag case the three properties in Proposition 5.6 are all different. The implication (a) ⇒ (c) holds in the non-flag case by a result of Amelotte [3, Theorem 1.2], and (a) ⇒ (b) is Proposition 4.1. We illustrate the failure of other implications in the next example.

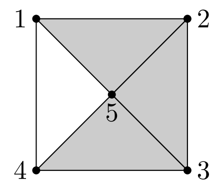

FIGURE 2.

Example 5.7. Let K be the simplicial complex in Figure 2. A calculation similar to Proposition 4.1 using a cellular chain complex for $\mathcal{Z}_K$ shows that $H_*(\Omega\mathcal{Z}_K)$ is a one-relator algebra given by

$$H_*(\Omega\mathcal{Z}_K) \cong \frac{T(a_{13}, a_{24}, b_{145}, b_{1452}, b_{1453}, b_{14523})}{\langle [a_{13}, a_{24}] \rangle}.$$

Moreover, a calculation similar to [5, Example 8.4.5] shows that the homomorphism $H_*(\Omega\mathcal{Z}_K) \longrightarrow H_*(\Omega DJ_K)$ from (6) maps the generators a_{13}, a_{24}, b_{145}, b_{1452}, b_{1453}, b_{14523} to the commutators $[u_1, u_3]$, $[u_2, u_4]$, $[u_1, u_4, u_5]$, $[[u_1, u_4, u_5], u_2]$, $[[u_1, u_4, u_5], u_3]$, $[[[u_1, u_4, u_5], u_2], u_3]$, respectively, where $[u_1, u_4, u_5]$ is the higher bracket corresponding to the missing face $\{145\}$.

Observe that K is not minimally non-Golod, as $K - \{5\}$ is a 4-cycle, so implication (b) ⇒ (c) of Proposition 5.6 fails in the non-flag case. Furthemore, (b) ⇒ (a) also fails here, which is seen from the isomorphism of the cohomology ring of $\mathcal{Z}_K$ with that of $(S^3 \times S^3) \vee S^5 \vee S^6 \vee S^6 \vee S^7$.

The implication (c) ⇒ (a) also fails in the non-flag case. Examples of minimally non-Golod complexes K for which $\mathcal{Z}_K$ is not homotopy equivalent to a connected sum of sphere products were constructed by Limonchenko in [11, Theorem 2.6].

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School of Mathematical Sciences, University of Southampton, UK
E-mail address: j.grbic@soton.ac.uk

Department of Mathematics and Mechanics, Moscow State University, Russia
E-mail address: marina_ilyasova@bk.ru

Department of Mathematics and Mechanics, Moscow State University, Russia; Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia;
Scientific Educational Mathematical Centre, Kazan Federal Univerisity, Russia
E-mail address: tpanov@mech.math.msu.su

School of Mathematical Sciences, University of Southampton, UK
E-mail address: g.j.h.simmons@soton.ac.uk