Supplemental Material to "How far does a fold go?"

Atul Bhaskar, Kevin Jose

September 2020

1 Plate strip folding: decaying ansatz

The strain energy of stretch-free bending of a thin elastic sheet, when the twist w_{xy} and the Poisson's coupling between the two curvatures $2\nu w_{xx} w_{yy}$ are accounted for, is given by

$$U = \frac{D}{2} \int_{-\infty}^{\infty} \int_{0}^{b} \left(w_{xx}^2 + 2\nu w_{xx} w_{yy} + 2(1-\nu) w_{xy}^2 + w_{yy}^2 \right) \mathrm{d}y \ \mathrm{d}x \tag{S1}$$

Consider the following ansatz for the transverse deflection of the form, which assumes an exponential decay in the x-direction

$$w(x, y) = \exp\left(\frac{\lambda |x|}{b}\right) \cdot \left(\frac{y}{b}\right)^2 \tag{S2}$$

Substituting the above and evaluating the energy expression, after carrying out the integrations, we obtain

$$U = -\frac{D}{2b^2} \left(\frac{\lambda^3}{5} + \frac{4}{3}(2-\nu)\lambda + \frac{4}{\lambda} \right). \tag{S3}$$

The above expression brings out the competition between various terms in the strain energy that determine the equilibrium shape. The cubic term arising from w_{xx} scales as $\sim \lambda^3$, whereas the term arising from w_{yy} scales as per $\sim \lambda^{-1}$. The term that scales according to $\sim \lambda^2$ originates from the Poisson's coupling and the twist combined. The only undetermined parameter in this expression is the decay rate λ, which can be treated as the generalised coordinate of the problem. Minimising the strain energy with respect to λ by setting $\partial U / \partial \lambda = 0$, we obtain a quadratic equation for λ^2,

$$\lambda^4 + \frac{20}{9}(2-\nu)\lambda^2 - \frac{20}{3} = 0. \tag{S4}$$

The discriminant of the solution of the above equation, $\left(\frac{20}{9}(2-\nu)\right)^2 + \frac{80}{3}$, is always positive for all physically possible values of Poisson ratio $-1 < \nu < 0.5$.

Hence λ^2, will always be real. In Figure S1, we see that one of the roots is always positive and the other is always negative. Hence, Equation S4 will always have roots of the form $\lambda = \pm p, \pm qi$ where $p, q \in \mathbb{R}$. But the boundary conditions at ∞ demand exponential growth to be discarded, therefore, only the negative real root is admissible.

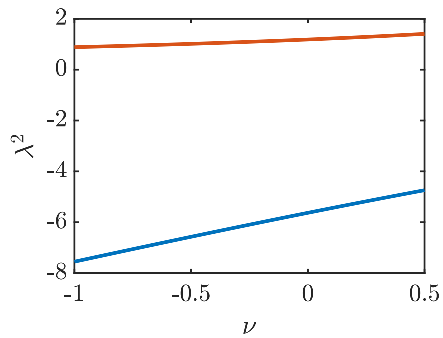

Figure S1: One root of Equation S4 (as quadratic in λ^2) is always positive real while the other is negative real for all possible ν

2 Variational minimisation under separable ansatz

Using the same energy expression as before (Equation S1) consider the ansatz for transverse deflection, which assumes a separable form $w(x, y) = f(x)(y/b)^2$ in the energy expression and evaluating the y integral we obtain,

$$U = \frac{D}{2} \int_{-\infty}^{\infty} \left(\frac{b}{5} f''^2(x) + \frac{4\nu}{3b} f''(x) f(x) + \frac{8(1-\nu)}{3b} f'^2(x) + \frac{4}{b^3} f^2(x) \right) \,\mathrm{d}x \quad \text{(S5)}$$

Non dimensionalize using $\xi = x/b$ & $f(x) = f(b\xi) = F(\xi)$. Change the limits of integration to $[0, \infty)$ given that $F(\xi)$ is and even function. (The argument of F is suppressed from here on in the interest of clarity.) Applying the principle of minimum potential energy, i.e. $\delta U = 0$, we have

$$\delta U = \frac{D}{b^2} \delta \int_0^{\infty} \left(\frac{1}{5} F''^2 + \frac{4\nu}{3} F'' F + \frac{8(1-\nu)}{3} F'^2 + 4F^2 \right) \,\mathrm{d}\xi = 0, \quad \text{(S6)}$$

where the first variation of $(\cdot)$ is written as $\delta(\cdot)$. Since the integral and the variation operator commute, the variation of the integral simplifies after integrating the integrand by parts to

$$\delta U = \frac{D}{b^2}\left[\frac{2}{5}\left((F''\delta F')\Big|_0^\infty - (F'''\delta F)\Big|_0^\infty + \int_0^\infty F''''\delta F \mathrm{d}\xi\right) + \frac{4\nu}{3}\left(\int_0^\infty F''\delta F \mathrm{d}\xi\right)\right. \tag{S7}$$

$$+\frac{4\nu}{3}\left((F\delta F')\Big|_0^\infty - (F'\delta F)\Big|_0^\infty + \int_0^\infty F''\delta F \mathrm{d}\xi\right) \tag{S8}$$

$$\left. + \frac{16(1-\nu)}{3}\left((F'\delta F)\Big|_0^\infty - \int_0^\infty F''\delta F \mathrm{d}\xi\right) + 8\left(\int_0^\infty F\delta F \mathrm{d}\xi\right)\right].$$

The field terms and the boundary terms can be separated. Collecting all the field terms, we get Assuming that all boundary terms, the equation reduces to,

$$\delta U = \frac{D}{b^2}\int_0^\infty \left(\frac{2}{5}F'''' + \frac{8}{3}(3\nu-2)F'' + 8F\right)\delta F \;\mathrm{d}\xi + \text{ Boundary terms} = 0. \tag{S9}$$

The boundary terms must vanish separately as and so must the field terms, as they are associated with independent variations. Because the variation δF is arbitrary, the field term vanishes only when the integrand must be *identically* zero regardless of the variations in F, i.e.

$$F'''' + \frac{20}{3}(3\nu - 2)F'' + 20F = 0. \tag{S10}$$

The boundary terms contain *geometric* as well as *natural* boundary conditions. Since the strip is symmetric about $x = 0$, the geometric conditions at this point are $F'(0) = 0$. Additionally, the value of the function $F(0) = 1$ is prescribed, as this follows from the imposed y-wise shape at the origin $w(0, y) = (y/b)^2$. Solutions of the form $F(\xi) = \exp(\lambda\xi)$ are admissible. Hence, the characteristic equation is the following quadratic in λ^2

$$\lambda^4 + \frac{20}{3}(3\nu - 2)\lambda^2 + 20 = 0, \tag{S11}$$

which can be solved analytically. The discriminant of this quadratic is $\mathcal{D} = (20/3(3\nu - 2))^2 - 80$. Consider the regime of ν where $\mathcal{D} < 0$, i.e.

$$\begin{aligned}\mathcal{D} = \left(\frac{20}{3}(3\nu-2)\right)^2 - 80 < 0 \\ \implies (3\nu-2)^2 < \frac{9}{5} \\ \implies \frac{1}{3}\left(2 - \frac{3}{\sqrt{5}}\right) < \nu < \frac{1}{3}\left(2 + \frac{3}{\sqrt{5}}\right) \\ \implies 0.2195 < \nu < 1.1139\end{aligned}$$

Since, for real materials, $\nu \leq 0.5$ we may define $\nu_{\text{crit}} = 0.2195$ such that, for $\nu > \nu_{\text{crit}}$, Equation S11 (which is a quadratic in λ^2) has two complex solutions that are conjugates of each other and have a positive real component. Hence the solutions for λ are four complex numbers of which two are conjugate and the other two are negative of the first two. Of these, two will have negative real parts and will be conjugates of each other. These two are admissible λs. For $\nu \leq \nu_{\text{crit}}$, Equation S11 (as a quadratic in λ^2) has two roots, both positive real. Hence, the solutions for λ will be four numbers, two distinct (except when $\nu = \nu_{\text{crit}}$) real numbers and their negatives. The two negative numbers are admissible solutions.

3 Finite element model and persistence length extraction

Finite element (FE) simulations were carried out on the commercial code ANSYS Mechanical[1] using SHELL63 element (KEYOPT(1) = 2, to retain only bending stiffness). The plate strip domain of $[-30b, 30b] \times [0, b]$ was meshed with square elements of side length $b/30$. This mesh density was found to be sufficient to obtain convergent result. For the simulations b was taken to be 1cm. Young modulus of 200 GPa and thickness of $b/100$ was used. The choice of these parameters is obviously arbitrary as demonstrated in our work.

To extract ℓ_{P} values from FE we fit the logarithm of the absolute value of the deflection of the free edge to logarithm of the absolute value of RHS of Equation (6) or (7) depending on monotonicity. The fitting was carried out on the logartihm of the function since fitting weighted sum of exponentials is known to be ill-posed[2]. MATLAB™[3] 's `fit` function is used with `'NonlinearLeastSquares'` method to do the fitting. Further, the non-monotonicity of some of the deflection profiles necessitates use of absolute value to avoid imaginary values when taking the logarithm. $\ell_{1/e}$ is found simply by finding the horizontal location where the deflection crosses $1/e \approx 0.368$.

4 Higher order effects in sheet folding

The simple analysis of the folding problem using stretch-free strain energy ignores several higher order effects. They include those due to the inevitable stretch (membrane effect) as well as ignoring through-the-thickness shear. The first is due to the change in Gauss curvature required, as the initial Gaussian curvature of the sheet is zero. This effect is likely to be less significant in the present case, because the edges are free, so the mid-surface is not forced to stretch significantly, or tear. Non-linear components of the von Karman membrane strain $\epsilon_{xx} = (1/2)w_x^2$, $\epsilon_{yy} = (1/2)w_y^2$, and $\epsilon_{xy} = (1/2)w_x w_y$ are also

[1] *Ansys® Academic Research Mechanical, Release 19.2*

[2] A. A.Istratov, and O. F. Vyvenko, Exponential analysis in physical phenomena, Review of Scientific Instruments **70**, 2 (1999)

[3] MATLAB. version 9.8 (R2020a). The MathWorks Inc., Natick, Massachusetts

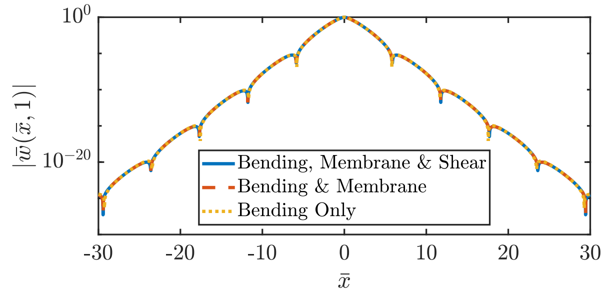

Figure S2: Free edge of the plate strip with different stiffness terms. We see that the bending stiffness captures the physics adequately in the linear regime.

negligible, as we are interested in small rotation situations. Shear could be significant when the thickness to width ratio is not small. This can be accounted for by shear correction, which in plate theory is due to Mindlin, which is a 2D analog of the shear correction in flexural mechanics introduced by Timoshenko. The role of these higher order effects were examined computationally, mainly to verify if the strain energy expression used is capable of capturing the essential physics of the problem, or not.

The effect of ignoring other energies in the linear regime is brought out by comparing the profile of the free edge of the fold when both bending and membrane stiffnesses are included in the simulations (SHELL63, KEYOPT(1) = 0). To study the case where bending, membrane and shear stiffnesses are included SHELL181 element (KEYOPT(1) = 0) is used. We see in Figure S2 that the profile of the free edge deflection is practically identical in all cases indicating that bending stiffness dominates and is adequate to model deflection in the linear regime of thin sheets.

5 The effect of thickness

Within the applicability of thin plate bending mechanics, the effect of thickness is absent as expected. This is because thickness appears as a part of the bending rigidity D. Indeed for the same reason, there is no effect of the Young's modulus on the persistence length. Both of these conclusions follow from simple dimensional analysis and can be verified numerically. Profiles obtained from finite element calculations for three different ratios of $t/b = 1/1000, 1/100, 1/50$ are presented in Figure S3 below. All the profiles overlap exactly within the tolerance of numerical calculations as expected.

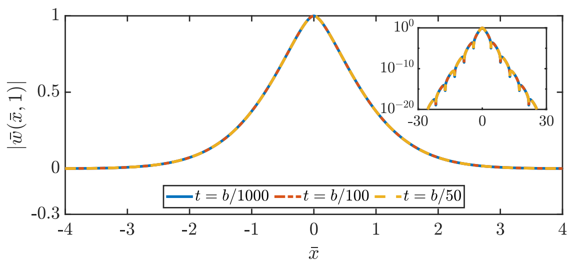

Figure S3: Profile of the leading edge for various values of sheet thickness obtained from FE are plotted. We see that variation of the thickness has no effect on the profile. This is expected since the element used for modeling is a linear thin plate element with only bending energies. This also therefore also agrees with our model for persistence length which is thickness independent.

6 Contribution of Curvature, Twist and Poisson coupling

There are three effects that contribute to the plate behaviour of a folded elastic strip as a 2D surface structure vis-a-vis 1D beam bending: they are curvatures in the two directions, twist, and Poisson's ratio mediated coupling between the two curvatures. The effect of each of these three in the persistence behaviour is brought out next. Persistence length estimated from finite element calculations are compared with three different simplifications in Figure S4 when (a) all three terms are retained in the strain energy expression, (b) only the two curvature terms and the twist term are kept in the expression for U, and (c) only the curvature terms and the Poisson's coupling are kept in the strain energy expression while the twist term is ignored. The monotonically increasing trend (shown in purple line) refers to ignoring twist – this does not match well with numerical calculations (as well as with trends obtained from the other two simplifications). This shows the greater role of twist in determining the persistence of a folded strip than any other aspect of elastic sheet folding.

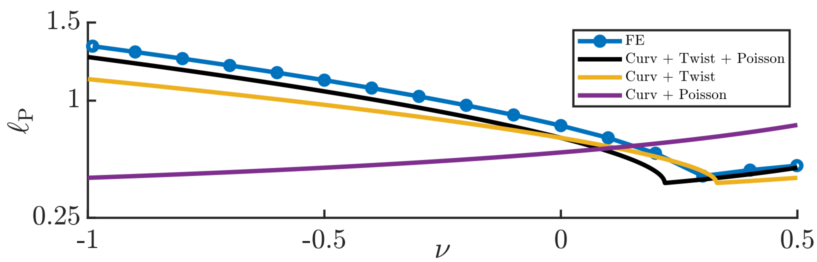

Figure S4: The plot quantities the effect of curvature, twist and Poisson coupling contribution in the persistence length predicted by our model. It is compared with results from FE.