Local BCJ numerators for ten-dimensional SYM at one loop

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We obtain local numerators satisfying the BCJ color-kinematics duality at one loop for super-Yang–Mills theory in ten dimensions. This is done explicitly for six points via the field-theory limit of the genus-one open superstring correlators for different color orderings, in an analogous manner to an earlier derivation of local BCJ-satisfying numerators at tree level from disk correlators. These results solve an outstanding puzzle from a previous analysis where the six-point numerators did not satisfy the color-kinematics duality.

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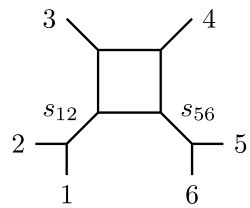


$$\frac{V_{12}T_{3,4,56}}{s_{12}s_{56}}$$

Fig. 1 The multiparticle superfields and pure spinor one-loop building blocks lead to intuitive mappings between one-loop cubic graphs and pure spinor superspace expressions encoding the polarization dependence of ten-dimensional supersymmetric Yang–Mills states [1].

1. Description of the problem and its solution

This paper aims to answer a question left over from the pure spinor construction of one-loop integrands of super-Yang–Mills (SYM) using locality and BRST invariance [1]. Can one find a set of local and supersymmetric numerators for ten-dimensional SYM one-loop integrands at six points satisfying the Bern-Carrasco-Johansson (BCJ)[1] color-kinematics duality? We will see below that the answer is *yes*, and we will also outline the solution for seven-point integrands.

The one-loop integrands of SYM in ten dimensions for five and six points were constructed in [1], where it was shown that the numerators for the five-point amplitude satisfied the color-kinematics duality while those at six points did not. The proposal of [1] was based on two main ingredients: locality and BRST invariance. Using the multiparticle superfields in pure spinor superspace developed in [3], these requirements together with a basic understanding of the zero-mode saturation rules of the pure spinor formalism [4,5] led to intuitive rules mapping one-loop cubic graphs to superspace numerators, see fig. 1. By assembling the numerators of the cubic graphs for all p-gons of a n-point amplitude such that their sum is in the pure spinor BRST cohomology (up to anomalous terms of the form discussed in [6,7]), the amplitudes of the color-ordered five and six-point amplitudes for the canonical color ordering were constructed. The six-point integrand was later successfully used in [8], passing some consistency checks.

[1] A brief review of the BCJ color-kinematics duality sufficient for our purposes will be given below in section 3.1 but a much more in-depth review is contained in [2].

1.1. Genus-one open superstring correlators in pure spinor superspace

In this paper we will also use the same formalism of multiparticle superfields in pure spinor superspace to present local representations of the five-, six- and seven-point amplitudes that do obey the color-kinematics duality. Since we are using the same superfield language, it is therefore important to highlight the differences with respect to the previous analysis of [1]. The difference stems from the knowledge of the open-string one-loop *correlators* recently obtained in [9–11] up to seven points. They are given by

$$\mathcal{K}_4(\ell) = V_1 T_{2,3,4} \mathcal{Z}_{1,2,3,4}\,, \tag{1.1}$$

$$\begin{aligned}\mathcal{K}_5(\ell) = {}& V_1 T^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} \\ &+ \big(V_A T_{B,C,D} \mathcal{Z}_{A,B,C,D} + [A,B,C,D|12345]\big)\,,\end{aligned} \tag{1.2}$$

$$\begin{aligned}\mathcal{K}_6(\ell) = {}& \frac{1}{2} V_1 T^{mn}_{2,3,4,5,6} \mathcal{Z}^{mn}_{1,2,3,4,5,6} \\ &+ \big(V_A T^m_{B,C,D,E} \mathcal{Z}^m_{A,B,C,D,E} + [A,B,C,D,E|123456]\big) \\ &+ \big(V_A T_{B,C,D} \mathcal{Z}_{A,B,C,D} + [A,B,C,D|123456]\big)\,,\end{aligned} \tag{1.3}$$

$$\begin{aligned}\mathcal{K}_7(\ell) = {}& \frac{1}{6} V_1 T^{mnp}_{2,3,4,5,6,7} \mathcal{Z}^{mnp}_{1,2,3,4,5,6,7} \\ &+ \frac{1}{2}\big(V_A T^{mn}_{B,C,D,E,F} \mathcal{Z}^{mn}_{A,B,C,D,E,F} + [A,B,C,D,E,F|1234567]\big) \\ &+ \big(V_A T^m_{B,C,D,E} \mathcal{Z}^m_{A,B,C,D,E} + [A,B,C,D,E|1234567]\big) \\ &+ \big(V_A T_{B,C,D} \mathcal{Z}_{A,B,C,D} + [A,B,C,D|1234567]\big) \\ &- \big(V_1 J^m_{2|3,4,5,6,7} \mathcal{Z}^m_{2|1,3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big) \\ &- \big(\big(V_A J_{B|C,D,E,F} \mathcal{Z}_{B|A,C,D,E,F} + (B \leftrightarrow C,D,E,F)\big) + [A,B,C,D,E,F|1234567]\big) \\ &- \big(\Delta_{1|2|3,4,5,6,7} \mathcal{Z}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big)\,,\end{aligned} \tag{1.4}$$

where the $[A_1, ..., A_m|12...n]$ notation is used to denote a sum over Stirling cycles [11], see the appendix A for more details.[2] The supersymmetric polarizations of ten-dimensional gluons and gluinos are encoded in the pure spinor multiparticle building blocks $V_A T^{m\cdots}_{B,C,D\ldots}$ reviewed in [9]. The various $\mathcal{Z}^{m\cdots}_{A,B,C,\ldots}$ are worldsheet functions elaborated in [10] and they depend on the insertion points of the vertices on the Riemann surface and on the loop momentum ℓ^m.

[2] These sums can also be described by all the ways in which $12...n$ can be completely decomposed into m Lyndon words, with every letter appearing in precisely one such word.

The open string amplitudes for supersymmetric states are obtained from these correlators after integration over the vertex insertion points, over the loop momentum, and over the modulus τ of the genus-one Riemann surfaces

$$\mathcal{A}_n = \sum_{\rm top} C_{\rm top} \int_{D_{\rm top}} d\tau \, dz_2 \, dz_3 \, \ldots \, dz_n \int d^D \ell \; |\mathcal{I}_n(\ell)| \, \langle \mathcal{K}_n(\ell) \rangle \, , \tag{1.5}$$

where $\mathcal{I}_n(\ell)$ denotes the Koba-Nielsen factor, $D_{\rm top}$ denotes an ordered region of integration over the insertion points z_i, and $C_{\rm top}$ denotes a group-theory factor which depends on the topology of the genus-one surface (cylinder, Möbius strip or non-planar cylinder) [12]. For simplicity we will consider only the planar cylinder topology in the following. For more details on this setup, see section 2 of [9].

To gain intuition why the one-loop open-string correlators lead to a representation of one-loop SYM numerators that satisfy the color-kinematics duality it will be illustrative to review the quest for local BCJ-satisfying ten-dimensional supersymmetric numerators at tree level, solved in pure spinor superspace in [13].

1.2. BCJ-satisfying local numerators at tree level from string disk correlators

1.2.1. Cohomology analysis: Five-point tree numerators from relabeling

When the tree-level color-ordered amplitudes were first proposed in [14], the construction was based on the principles of locality and BRST invariance of pure spinor superspace expressions using multiparticle superfields. These same principles were later used when proposing SYM one-loop integrands in [1]. The difference between the expressions in [14] and [1] originates from the differences in the pure spinor amplitude prescriptions at tree level [5] and one loop [4]. The n-point tree-level numerators of [14] had to be built from three unintegrated (multiparticle) vertices V following the OPE contractions with $(n-3)$ integrated vertices $U(z)$. For example, at tree level the five-point SYM amplitude in the canonical color ordering was obtained as[3]

$$A^{\rm SYM}(1,2,3,4,5) = \frac{V_{[12,3]}V_4V_5}{s_{12}s_{123}} + \frac{V_{[1,23]}V_4V_5}{s_{23}s_{123}} + \frac{V_{[1,2]}V_{[3,4]}V_5}{s_{12}s_{34}} + \frac{V_1V_{[23,4]}V_5}{s_{23}s_{234}} + \frac{V_1V_{[2,34]}V_5}{s_{34}s_{234}} \tag{1.6}$$

[3] For convenience we shall frequently omit from amplitudes such as (1.6) the pure spinor brackets $\langle \ldots \rangle$ that extract the top element $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ in the cohomology of the pure spinor BRST operator [5]. The component evaluation of ghost-number three expressions uses the identities from the appendix of [6].

where $V_{[A,B]}$ denotes the multiparticle unintegrated vertex operator in the BCJ gauge, see the review on multiparticle superfields in section 3 of [9] and section 4.3 of [15]. The expression (1.6) correctly reproduces the five-point tree amplitude of SYM in the canonical color ordering. The next task is to check whether this representation leads to numerators that satisfy the color-kinematics duality, this is where a subtle point arises.

A triplet of numerators participating in a kinematic Jacobi identity necessarily involves numerators from amplitudes with different color orderings, but the naive relabeling of the amplitude (1.6) does not lead to a representation satisfying the BCJ color-kinematics duality. Let us illustrate this point with an example. Using the parameterization of numerators from [16] where

$$\begin{aligned} A^{\rm SYM}(1,2,3,4,5) &= \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}} \\ A^{\rm SYM}(1,4,3,2,5) &= \frac{n_6}{s_{14}s_{25}} + \frac{n_5}{s_{51}s_{34}} + \frac{n_7}{s_{32}s_{14}} + \frac{n_8}{s_{25}s_{34}} + \frac{n_2}{s_{51}s_{23}} \end{aligned} \tag{1.7}$$

in order to check whether the numerators n_3, n_5 and n_8 satisfy the kinematic Jacobi identity $n_3 - n_5 + n_8 = 0$ one needs to extract the numerator n_8 of the pole in $1/(s_{25}s_{34})$, n_3 of $1/(s_{34}s_{12})$ and n_5 of $1/(s_{51}s_{34})$. While n_3 and n_5 can be read off from the amplitude $A(1,2,3,4,5)$ in (1.6), the numerator n_8 is found in the different color ordering $A(1,4,3,2,5)$. If we assume that this color ordering is given by the relabeling of (1.6) the kinematic Jacobi relating these three numerators is not satisfied,

$$n_3 - n_5 + n_8 = V_{[1,2]}V_{[3,4]}V_5 - V_1V_{[2,34]}V_5 + V_1V_{[43,2]}V_5 \neq 0\,, \tag{1.8}$$

where we used $n_8 = V_1V_{[43,2]}V_5$ obtained from $n_4 = V_1V_{[23,4]}V_5$ via the relabeling $2 \leftrightarrow 4$.

1.2.2. Open superstring: Five-point tree numerators from the field-theory limit

The solution to the above problem was found in [13] by utilizing the n-point string disk correlator of [17] to generate different color orderings in its field-theory limit. These orderings follow from the various integration regions over the insertion points z_i ordered along the boundary of a disk. For five points the superstring tree-level correlator is

$$\mathcal{K}_5(z_1,\ldots,z_5) = \frac{V_{123}V_4V_5}{z_{12}z_{23}} + \frac{V_1V_{432}V_5}{z_{43}z_{32}} + \frac{V_{12}V_{43}V_5}{z_{12}z_{43}} + (2\leftrightarrow 3)\,. \tag{1.9}$$

The string tree-level amplitudes with different color orderings are obtained by the different integration regions of the vertex insertion points relative to each other. The corresponding color-ordered SYM amplitudes follow from the field-theory limit $\alpha' \to 0$ of the disk integrals, encoded in the biadjoint scalar amplitudes [18] (see also [19]). More precisely, one can express the field-theory limit of the string correlator (1.9) as follows [20,21]

$$A^{\rm SYM}(\Sigma) = \sum_{XY=23} V_{1X} V_{(n-1)\tilde{Y}} V_n \, m(\Sigma|1,X,n,Y,n-1)(-1)^{|Y|+1} + (2\leftrightarrow 3)\,, \tag{1.10}$$

where $m(\Sigma|\Omega)$ denotes the biadjoint tree amplitudes,

$$m(P,n|Q,n) = s_P \phi_{P|Q} \tag{1.11}$$

and $\phi_{P|Q}$ are the Berends-Giele double currents [21]. They can be computed recursively

$$\phi_{P|Q} = \frac{1}{s_P} \sum_{XY=P} \sum_{AB=Q} \big(\phi_{X|A}\phi_{Y|B} - (X\leftrightarrow Y)\big)\,, \quad \phi_{P|Q} = 0 \text{ if } P\setminus Q \neq \emptyset\,. \tag{1.12}$$

in terms of generalized Mandelstam invariants $s_P = \frac{1}{2} k_P \cdot k_P$ where k_P is a multiparticle momentum defined by $k_P = k_{p_1} + k_{p_2} + \cdots$ (for example $k_{123} = k_1 + k_2 + k_3$).

Extracting the field-theory limit of the string disk integrals computed in the ordering $z_1 \leq z_4 \leq z_3 \leq z_2 \leq z_5$ – corresponding to $\Sigma = 14325$ in (1.10) – leads to the following color-ordered amplitude

$$\begin{aligned} A^{\rm SYM}(1,4,3,2,5) = {} & \frac{1}{s_{14}s_{25}}(V_1V_{432} + V_{12}V_{43} + V_{13}V_{42} + V_{132}V_4)V_5 \\ & + \frac{1}{s_{51}s_{34}} V_1 V_{432} V_5 - \frac{1}{s_{23}s_{14}}(V_1 V_{[4,23]} + V_{[1,23]}V_4)V_5 \\ & + \frac{1}{s_{25}s_{34}}(V_1V_{432} + V_{12}V_{43})V_5 - \frac{1}{s_{51}s_{23}} V_1 V_{[4,23]} V_5\,. \end{aligned} \tag{1.13}$$

One can now read off the numerator $n_8 = V_1V_{432}V_5 + V_{12}V_{43}V_5$ and verify that the BCJ identity $n_3 - n_5 + n_8 = 0$ is identically[4] satisfied [13]

$$n_3 - n_5 + n_8 = V_{[1,2]}V_{[3,4]}V_5 - V_1V_{[2,34]}V_5 + (V_1V_{432}V_5 + V_{12}V_{43}V_5) = 0\,, \tag{1.14}$$

where the bracket notation reviewed in section 3.4.3 of [9] implies $V_{432} = V_{[23,4]}$ and $V_{43} = -V_{[3,4]}$.

The field-theory tree-level SYM numerators are extracted from the knowledge of the singular behavior of the correlator as vertex operators collide as encoded in the biadjoint

[4] *Identically* means that no BRST cohomology identity (of the type discussed in [22]) is required to verify the vanishing of the triplet of numerators; it vanishes at the superfield level.

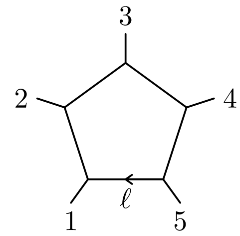


Fig. 2 The cubic graph associated to the pentagon $N^{(5)}_{1|2,3,4,5}(\ell)$ from equation (1.15). The convention for the loop momentum ℓ is to run from the last argument of the numerator to the first.

Berends-Giele currents. But we know that these limits constitute a local property of the Riemann surface and therefore must be independent of its genus. These results together with the analysis of [23] lead to the following expectation:

> *The field-theory limit of the one-loop string correlators integrated along different vertex insertion orderings should give rise to a local representation for SYM one-loop integrands that satisfy the BCJ color-kinematics duality.*

As an illustration of this method – to be fully developed in the next sections – let us apply it in the simplest case of the five-point SYM integrand/amplitude following from the string correlator (1.2).

1.3. BCJ-satisfying local numerators at one loop from string genus-one correlators

1.3.1. Cohomology analysis: Five-point pentagon from relabeling

As mentioned above, the five-point SYM integrand was proposed based on a few constraints such as locality and BRST invariance. The pentagon for the color order $A(1,2,3,4,5)$ was given as [1]

$$N^{(5)}_{1|2,3,4,5}(\ell) = \ell^m V_1 T^m_{2,3,4,5} + \frac{1}{2}\big[V_{12}T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big] + \frac{1}{2}\big[V_1 T_{23,4,5} + (2,3|2,3,4,5)\big] \tag{1.15}$$

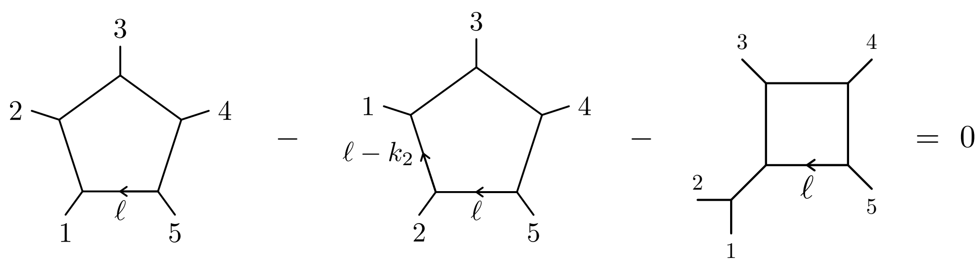
where the notation $+(i,j|2,3,4,5)$ denotes a sum over all possible ways to choose two elements i and j from the set $\{2,3,4,5\}$ while keeping the same order of i and j within the set. The cubic graph associated to this pentagon is displayed in fig. 2. Note the convention of assigning the loop momentum ℓ to the edge between 5 and 1.

Using BRST cohomology arguments the box and pentagon numerators following from relabelings (while respecting the loop momentum assignment convention and the constraint that leg 1 is contained in a multiparticle unintegrated vertex V) were proposed in [1],

$$N^{(4)}_{A|B,C,D}(\ell) = V_A T_{B,C,D} \tag{1.16}$$

$$N^{(5)}_{A|B,C,D,E}(\ell) = \ell_m V_A T^m_{B,C,D,E} + \frac{1}{2}\big[V_{[A,B]} T_{C,D,E} + (B \leftrightarrow C, D, E)\big] \tag{1.17}$$
$$+ \frac{1}{2}\big[V_A T_{[B,C],D,E} + (B, C|B, C, D, E)\big]\,.$$

For a cubic-graph parameterization of the five-point integrand to obey the BCJ color-kinematics duality the antisymmetric combination of two pentagons in the legs 1 and 2 must give rise to a box [24]



From the figure above we see that the pentagon in the middle must come from the color ordering $A(2,1,3,4,5)$ so as to keep the momenta in the common edges of the participating cubic graphs the same while respecting the loop momentum convention mentioned above. However, the generic expression (1.17) has to ensure that the leg 1 appears in A, so the solution proposed in [1] satisfying both constraints was to assign the pentagon numerator $N_{1|3,4,5,2}(\ell - k_2)$ to the middle diagram, with a shift in the loop momentum. Using that the 12-box numerator is $V_{12}T_{3,4,5}$, the expression (1.17) implies that the numerator translation of the diagrams above is given by

$$\langle N^{(5)}_{1|2,3,4,5}(\ell) - N^{(5)}_{1|3,4,5,2}(\ell - k_2) - N^{(4)}_{12|3,4,5}\rangle = \tag{1.18}$$

$$\langle k^2_m V_1 T^m_{2,3,4,5} + V_{21}T_{3,4,5} + V_1 T_{23,4,5} + V_1 T_{24,3,5} + V_1 T_{25,3,4}\rangle = 0. \tag{1.19}$$

The BCJ color-kinematic identity relating two pentagons with a box is satisfied, but only up to BRST-exact terms in pure spinor superspace that are annihilated by the pure spinor cohomology bracket $\langle \ldots \rangle$. The BRST exactness of the second line was shown in [22].

1.3.2. The BCJ pentagon from the field-theory limit of the string correlator

The five-point analysis of [1] was primarily based on the BRST cohomology properties of the integrands, and as we reviewed above this was enough to obtain a BCJ-satisfying parameterization up to BRST-exact terms. However, using the field-theory limit of the string correlator the resulting numerators for the pentagons improve the BCJ identity to be satisfied identically at the superspace level, requiring no cohomology manipulations.

To see this we consider the five-point correlator (1.2) written in terms of the Eisenstein-Kronecker coefficient functions $g^{(1)}_{ij}$ of [10], namely $\mathcal{Z}^m_{1,2,3,4,5} = \ell^m$ and $\mathcal{Z}_{12,3,4,5} = g^{(1)}_{12}$

$$\mathcal{K}_5(\ell) = V_1 T^m_{2,3,4,5}\ell^m + \big[V_{12}T_{3,4,5}g^{(1)}_{12} + (2 \leftrightarrow 3,4,5)\big] + \big[V_1 T_{23,4,5}g^{(1)}_{23} + (2,3|2,3,4,5)\big]\,. \tag{1.20}$$

In the string-based formalism [25], the field-theory limit of the string propagator (in our case the $g^{(1)}_{ij}$ functions) depends on the relative ordering of how the vertex insertion points are integrated by a term proportional to sgn_{ij}. More precisely, if the color ordering of the resulting SYM integrand is P, the field-theory limit of $g^{(1)}_{ij}$ contains a term $\frac{1}{2}\mathrm{sgn}^P_{ij}$, where sgn^P_{ij} is defined in (2.16). Therefore the pentagons of the integrands in the $A(1,2,3,4,5)$ and $A(2,1,3,4,5)$ orderings differ by a sign in the term coming from $g^{(1)}_{12}$. This gives rise to the following pentagons:

$$\begin{aligned} N_{1|2,3,4,5}(\ell) &= \ell^m V_1 T^m_{2,3,4,5} + \frac{1}{2}V_{12}T_{3,4,5} + \frac{1}{2}V_{13}T_{2,4,5} + \frac{1}{2}V_{14}T_{2,3,5} + \frac{1}{2}V_{15}T_{2,3,4} \\ &\quad + \frac{1}{2}V_1T_{23,4,5} + \frac{1}{2}V_1T_{24,3,5} + \frac{1}{2}V_1T_{25,3,4} + \frac{1}{2}V_1T_{34,2,5} + \frac{1}{2}V_1T_{35,2,4} + \frac{1}{2}V_1T_{45,2,3} \end{aligned} \tag{1.21}$$

$$\begin{aligned} N_{2|1,3,4,5}(\ell) &= \ell^m V_1 T^m_{2,3,4,5} - \frac{1}{2}V_{12}T_{3,4,5} + \frac{1}{2}V_{13}T_{2,4,5} + \frac{1}{2}V_{14}T_{2,3,5} + \frac{1}{2}V_{15}T_{2,3,4} \\ &\quad + \frac{1}{2}V_1T_{23,4,5} + \frac{1}{2}V_1T_{24,3,5} + \frac{1}{2}V_1T_{25,3,4} + \frac{1}{2}V_1T_{34,2,5} + \frac{1}{2}V_1T_{35,2,4} + \frac{1}{2}V_1T_{45,2,3} \end{aligned} \tag{1.22}$$

where we note that the constraint that leg 1 is within V is automatically satisfied because the correlator (1.20) is always the same, what changes is the relative ordering of integration of the vertex positions.

It is easy to see that the numerators (1.21) and (1.22) imply that the BCJ identity is identically satisfied at the superfield level,

$$N_{1|2,3,4,5}(\ell) - N_{2|1,3,4,5}(\ell) - N_{12|3,4,5}(\ell) = 0\,, \tag{1.23}$$

where $N_{12|3,4,5}(\ell) = V_{12}T_{3,4,5}$. We thus see that the derivation of n-gon numerators from the field-theory limit of the open superstring correlator evaluated at different regions of integration implies that the associated BCJ identity is satisfied even before applying the pure spinor cohomology bracket to extract the polarization content of the superfields, unlike the case (1.18) obtained from relabeling. For five points this difference is immaterial as both approaches eventually satisfy the color-kinematics duality in the cohomology. However, we will see below that the field-theory limit technique leads to a six-point representation that satisfies the color-kinematics duality in contrast to the representation of [1].

2. SYM one-loop integrands from string correlators

The field-theory limit of the one-loop string correlators is obtained by shrinking the strings to points with $\alpha' \to 0$ while degenerating the genus-one surface with modular parameter τ to point-particle worldline diagrams with $\mathrm{Im}(\tau) \to \infty$ [26]. In principle this can be done using the tropical limit techniques of [27] or the string-based formalism [25], although the explicit form of the Kronecker-Eisenstein coefficient functions $g^{(n)}(z,\tau)$ lead to subtleties arising from the regular functions with $n \geq 2$. Alternatively, one can combine the strengths of these approaches with the requirement that the field-theory integrands for different color orderings and loop-momentum parameterizations obtained from the string correlators are in the BRST cohomology of the pure spinor BRST charge. Some trial and error led to the combinatorial rules described below.

2.1. Kinematic poles and biadjoint Berends-Giele currents

The kinematic poles arise when the insertion points of the vertex operators approach each other $z_i \to z_j$ on the Riemann surface. The short-distance behavior of the Koba-Nielsen factor and the OPE propagator is independent of the genus of the Riemann surface. This means that the pole structure of the genus-one string correlators can be described by the same combinatorics of tree-level poles, given by the biadjoint scalar amplitudes (1.11). These amplitudes are efficiently computed using the Berends-Giele double currents $\phi_{P|Q}$ of explicit form given in (1.12) where the words P and Q encode the integration region and integrand.

In the one-loop case however, in addition to the tree-level kinematic poles in Mandelstam invariants the field-theory limit of the genus-one string correlators also yield Feynman loop momentum integrands

$$I_{A_{n+1}1A_1,A_2,\ldots,A_n}(\ell) = \frac{1}{(\ell - k_{A_1})^2(\ell - k_{A_1A_2})^2 \cdots (\ell - k_{A_1A_2\ldots A_n})^2} \tag{2.1}$$

to be integrated over a D-dimensional loop momentum ℓ with $\int d^D\ell$. Note the special role played by the label 1 in the above definition; this handling fixes the freedom to shift the loop momentum and is useful in obtaining BRST-closed SYM integrands [1].

In summary, the field-theory limit of genus-one open string correlators will be described by poles in Mandelstam invariants encoded in Berends-Giele double currents multiplied by Feynman loop momentum integrals.

2.1.1. Encoding different integration regions

In the same way as in the tree-level case, the color ordering of the resulting SYM integrand from the field-theory limit of the genus-one open string correlator is associated to the relative ordering of the z_i variables among each other on the boundary of the Riemann surface. For example, the ordering $z_1 \leq z_3 \leq z_5 \leq z_4 \leq z_2$ yields an integrand with color ordering $\sigma = 13542$.

The presence or absence of kinematic poles depend crucially on the region of integration relative to the ordering of the z_{ij} variables being integrated. To encode this information we define a map $\mathrm{Ord}_A(B)$ acting on two words A and B that crops the word A while maintaining the letters it shares with B. That is, we take the word B and return the smallest sequence of consecutive letters in the cyclic-symmetric object A containing every letter in B. For example,

$$\begin{gathered}\mathrm{Ord}_{123456}(32) = 23\,, \quad \mathrm{Ord}_{123456}(13) = 123\,, \quad \mathrm{Ord}_{123456}(15) = 561\,, \\ \mathrm{Ord}_{24856317}(58) = 85\,, \quad \mathrm{Ord}_{24856317}(465) = 4856\,, \quad \mathrm{Ord}_{24856317}(78) = 7248\,. \end{gathered} \tag{2.2}$$

This map can be defined algebraically by

$$\mathrm{Ord}_A(B) = \begin{cases} A_i A_{i+1} \ldots A_{j-1} A_j & \text{if } A_i, A_j \in B, \;\; B \subseteq A_i \ldots A_j, \;\; j-i \leq \frac{|A|}{2} \\ A_j A_{j+1} \ldots A_{|A|} A_1 A_2 \ldots A_i & \text{if } A_i, A_j \in B, \;\; B \subseteq A_i \ldots A_j, \;\; j-i > \frac{|A|}{2} \\ 0 & \text{else} \end{cases} \tag{2.3}$$

This map will be used with the Berends-Giele double current to correctly generate kinematic poles for each integration region σ. It will be convenient to introduce the notation:

$$\hat{\phi}(\sigma|A) \equiv \phi_{\mathrm{Ord}_\sigma(A)|A}\,, \tag{2.4}$$

for an amplitude with color ordering σ.

2.2. p-gon loop momentum integrands

Frequently we will need the Feynman loop momentum integrands (2.1) with a general shift in the loop momentum $\ell \to \ell + a^i k_i$. This will be indicated by superscripts

$$I^{a_1,a_2,\ldots,a_m}_{A_{n+1}1A_1,A_2,\ldots,A_n}(\ell) = I_{A_{n+1}1A_1,A_2,\ldots,A_n}(\ell + a_1k_1 + a_2k_2 + \cdots + a_mk_m) \qquad (2.5)$$

Explicitly we have

$$I^{a_1,a_2,\ldots,a_m}_{A_{n+1}1A_1,A_2,\ldots,A_n} = \frac{1}{(\ell + f_{a_1\ldots a_m} - k_{A_1})^2 \ldots (\ell + f_{a_1\ldots a_m} - k_{A_1A_2\ldots A_n})^2}\,, \qquad (2.6)$$

where we defined for convenience

$$f_{a_1,\ldots,a_m} = a_1k_1 + a_2k_2 + \ldots + a_mk_m\,. \qquad (2.7)$$

In the event of an a_i being zero, we will omit it from the notation. Note that the words characterizing the integrands (2.6) are totally symmetric e.g. $I_{1,342,5,6} = I_{1,234,5,6}$.

We will sometimes simplify the notation for the loop momentum integrands by dropping all indices which are single letters, and dropping the shifts in the loop momentum. When this is done it should always be clear the color ordering of the amplitude. For example, in the canonical ordering $A(1,2,\ldots,n;\ell)$ we have

$$\begin{aligned} I_\emptyset = I = I^{a_1,\ldots,a_n}_{1,2,\ldots,n}\,, \qquad & I_{234} = I^{a_1,\ldots,a_n}_{1,234,5,6,\ldots,n}\,, \\ I_{23,56} = I^{a_1,\ldots,a_n}_{1,23,4,56,7,8,\ldots,n}\,, \qquad & I_{n1,34} = I^{a_1,\ldots,a_n}_{n1,2,34,5,6,\ldots,n-1}\,. \end{aligned} \qquad (2.8)$$

In a few instances, we may wish to use this notation when it is not immediately clear what the underlying color ordering is. In these circumstances we will include it as a superscript in the I. So, for example

$$I^{235416}_{\emptyset} = I^{235416} = I_{2,3,5,4,1,6}\,, \quad I^{235416}_{53} = I_{2,35,4,1,6}\,, \quad I^{235416}_{612} = I_{162,3,5,4}\,. \qquad (2.9)$$

2.3. Field-theory limit of Kronecker-Eisenstein coefficients

We are now ready to give the field theory limits. These are:

$$g^{(p)}_{ij} \to b^{(p)}_{ij} P + c^{(p)}_{ij} P(ij) \qquad (2.10)$$

$$g^{(p)}_{ij} g^{(q)}_{kl} \to b^{(p)}_{ij} b^{(q)}_{kl} P + b^{(p)}_{ij} c^{(q)}_{kl} P(kl) + c^{(p)}_{ij} b^{(p)}_{kl} P(ij) + c^{(p)}_{ij} c^{(q)}_{kl} P(ij,kl) \qquad (2.11)$$

$$\begin{aligned} g^{(p_1)}_{i_1j_1} g^{(p_2)}_{i_2j_2} g^{(p_3)}_{i_3j_3} \to\; & b^{(p_1)}_{i_1j_1} b^{(p_2)}_{i_2j_2} b^{(p_3)}_{i_3j_3} P + b^{(p_1)}_{i_1j_1} b^{(p_2)}_{i_2j_2} c^{(p_3)}_{i_3j_3} P(i_3j_3) \\ & + b^{(p_1)}_{i_1j_1} c^{(p_2)}_{i_2j_2} b^{(p_3)}_{i_3j_3} P(i_2j_2) + c^{(p_1)}_{i_1j_1} b^{(p_2)}_{i_2j_2} b^{(p_3)}_{i_3j_3} P(i_1j_1) \\ & + b^{(p_1)}_{i_1j_1} c^{(p_2)}_{i_2j_2} c^{(p_3)}_{i_3j_3} P(i_2j_2, i_3j_3) + c^{(p_1)}_{i_1j_1} b^{(p_2)}_{i_2j_2} c^{(p_3)}_{i_3j_3} P(i_1j_1, i_3j_3) \\ & + c^{(p_1)}_{i_1j_1} c^{(p_2)}_{i_2j_2} b^{(p_3)}_{i_3j_3} P(i_1j_1, i_2j_2) + c^{(p_1)}_{i_1j_1} c^{(p_2)}_{i_2j_2} c^{(p_3)}_{i_3j_3} P(i_1j_1, i_2j_2, i_3j_3) \end{aligned} \qquad (2.12)$$

These limits always have the same form; we take the subscripts of the $g_{ij}^{(p)}$, and sum over the possible ways to assign these to either a $b^{(p)}$ or a $c^{(p)}$ (to be defined below), and whenever we assign them to a $c^{(p)}$ they are also entered into the P function. In turn these are defined by

$$
\begin{aligned}
P &= I \\
P(ij) &= \hat{\phi}(\sigma|ij) I_{ij} \\
P(ij,kl) &= \begin{cases} \hat{\phi}(\sigma|ijl) I_{ijl} & \text{if } j=k \\ \hat{\phi}(\sigma|ij)\hat{\phi}(\sigma|kl) I_{ij,kl} & \text{if all } i \text{ unique} \end{cases} \\
P(ij,kl,mn) &= \begin{cases} \hat{\phi}(\sigma|ijln) I_{ijln} & \text{if } j=k, \quad l=m \\ \hat{\phi}(\sigma|ijl)\hat{\phi}(\sigma|mn) I_{ijl,\,mn} & \text{if } j=k, \quad m,n \notin \{i,j,k,l\} \\ \hat{\phi}(\sigma|ijn)\hat{\phi}(\sigma|kl) I_{ijn,\,kl} & \text{if } j=m, \quad k,l \notin \{i,j,m,n\} \\ \hat{\phi}(\sigma|ij)\hat{\phi}(\sigma|kln) I_{ij,\,kln} & \text{if } l=m, \quad i,j \notin \{k,l,m,n\} \\ \hat{\phi}(\sigma|ij)\hat{\phi}(\sigma|kl)\hat{\phi}(\sigma|mn) I_{ij,\,kl,mn} & \text{if all } i \text{ unique} \end{cases}
\end{aligned}
\tag{2.13}
$$

where we used the notation (2.4). The cases provided above will be sufficient for our purposes.

Finally, the coefficients $b^{(p)}$ and $c^{(p)}$ for an integrand $A(\sigma; \ell + \sum_{i=1}^{n} a_i k_i)$ are given by

$$
b_{ij}^{(p)} = \sum_{m=0}^{p} \left(\mathrm{sgn}_{ij}^{\sigma}\right)^m \frac{B_m (a_j - a_i)^{p-m}}{m!(p-m)!} \tag{2.14}
$$

$$
c_{ij}^{(p)} = \frac{1}{2(p-1)!} \big((a_j - a_i) + \mathrm{sgn}_{ij}^{\sigma} \mathrm{dist}_4^{\sigma}(i,j)\big)^{p-1} \tag{2.15}
$$

where B_n denotes the n^{th} Bernoulli number[5] and

$$
\mathrm{sgn}_{ij}^{B} = \begin{cases} +1 & : i \text{ is left of } j \text{ in } B \\ -1 & : i \text{ is right of } j \text{ in } B \end{cases} \tag{2.16}
$$

The function $\mathrm{dist}_a^B(i,j)$ measures the distance between i and j in the word B and returns $+1$ if it is larger than a and 0 otherwise,

$$
\mathrm{dist}_a^B(i,j) = \begin{cases} +1 & : \text{if } i \text{ is } a \text{ or more letters to the left or right of } j \text{ in } B \\ 0 & : \text{if } i \text{ is fewer than } a \text{ letters to the left or right of } j \text{ in } B \end{cases} \tag{2.17}
$$

Note that when $a_i = 0 \; \forall i$, we must take $0^0 = 1$ in the above.

[5] The amplitudes up to seven points require up to B_3: $B_0 = 1\,, B_1 = \frac{1}{2}\,, B_2 = \frac{1}{6}\,, B_3 = 0$.

2.3.1. A seven-point example

The field-theory limit of the term $g^{(1)}_{25}g^{(1)}_{57}g^{(1)}_{76}V_1T_{2576,3,4}$ in the seven-point string correlator (1.4) for the SYM integrand with color ordering $A(1,2,3,4,5,6,7;\ell+4k_4-6k_5)$ follows from (2.12) with $a_4=4$ and $a_5=-6$,

$$\begin{aligned}g^{(1)}_{25}g^{(1)}_{57}g^{(1)}_{76} \to\ & b^{(1)}_{25}b^{(1)}_{57}b^{(1)}_{76}P + b^{(1)}_{25}b^{(1)}_{57}c^{(1)}_{76}P(76)\\ &+ b^{(1)}_{25}c^{(1)}_{57}b^{(1)}_{76}P(57) + c^{(1)}_{25}b^{(1)}_{57}b^{(1)}_{76}P(25)\\ &+ b^{(1)}_{25}c^{(1)}_{57}c^{(1)}_{76}P(57,76) + c^{(1)}_{25}b^{(1)}_{57}c^{(1)}_{76}P(25,76)\\ &+ c^{(1)}_{25}c^{(1)}_{57}b^{(1)}_{76}P(25,57) + c^{(1)}_{25}c^{(1)}_{57}c^{(1)}_{76}P(25,57,76)\,.\end{aligned} \tag{2.18}$$

Many of these terms vanish. For instance using (2.13) the factor $P(57)$ is proportional to $\hat\phi(1234567|57) = \phi_{57|\mathrm{Ord}_{1234567}(57)} = \phi_{57|567} = 0$. Similarly, we find

$$P(25) = P(25,76) = P(25,57) = P(25,57,76) = 0\,. \tag{2.19}$$

The non-zero terms are then given by

$$\begin{aligned}P = I &= I^{a_4,a_5}_{1,2,3,4,5,6,7} \\ P(76) = \hat\phi(1234567|76)I_{76} &= \phi_{76|67}I_{76} = -\frac{1}{s_{67}}I^{a_4,a_5}_{1,2,3,4,5,76} \\ P(57,76) = \hat\phi(1234567|576)I_{576} &= \phi_{576|567}I_{1,2,3,4,576} = -\frac{1}{s_{67}s_{567}}I^{a_4,a_5}_{1,2,3,4,576}\end{aligned} \tag{2.20}$$

The various $b^{(1)}_{ij}$ and $c^{(1)}_{ij}$ terms are given by (2.14) and (2.15). In the $g^{(1)}_{25}$ case, these are given by (recall that $a_4=4, a_5=-6$)

$$\begin{aligned}b^{(1)}_{25} &= \frac{B_0(-6)^1}{0!\,1!} + \frac{B_1(-6)^0}{1!\,0!} = -6+\frac{1}{2} = -\frac{11}{2} \\ c^{(1)}_{25} &= \frac{\big(a_5-a_2+\mathrm{sgn}^{1234567}_{25}\mathrm{dist}^{1234567}_4(2,5)\big)^{1-1}}{2(1-1)!} = \frac{\big(-6+(-1)^0\times 0\big)^0}{2} = \frac{1}{2}\end{aligned} \tag{2.21}$$

The others are given by

$$b^{(1)}_{57} = \frac{13}{2}\,, \quad c^{(1)}_{57} = \frac{1}{2}\,, \quad b^{(1)}_{76} = -\frac{1}{2}\,, \quad c^{(1)}_{76} = \frac{1}{2}\,. \tag{2.22}$$

Putting everything together, we see that the limit is given by

$$g^{(1)}_{25}g^{(1)}_{57}g^{(1)}_{76} \to \frac{143}{8}I^{a_4,a_5}_{1,2,3,4,5,6,7} + \frac{143}{8}\frac{1}{s_{67}}I^{a_4,a_5}_{1,2,3,4,5,67} + \frac{11}{8}\frac{1}{s_{67}s_{567}}I^{a_4,a_5}_{1,2,3,4,567} \tag{2.23}$$

Doing this analysis for the full seven point correlator leads to a BRST closed expression up to anomalous terms (the explicit expression is available to download from [28]). In the appendix D the BRST variation of this numerator is worked out and shown to have the desired property of canceling propagators.

2.4. The one-loop SYM field-theory integrands

The one-loop correlators of the open superstring are integrated over the vertex insertions z_i ordered along the boundary of a genus one surface. After taking the field-theory limit, the color ordering of the resulting SYM integrand corresponds to the ordering of the insertions z_i. As alluded to in section 2.1, the field-theory limit of the open-string correlators will be written as a field-theory *integrand* depending on the loop momentum ℓ^m. The parameterization of the one-loop graphs by Feynman loop integrals is notoriously plagued with the "labelling problem": arbitrary shifts of the loop momentum must not affect the integrated amplitude. This will be indicated by labelling a color-ordered SYM integrand with the explicit parameterization of the loop momentum as

$$A(1,2,...,n;\ell+a_1k_1+\cdots+a_nk_n) \tag{2.24}$$

This refers to the amplitude with color ordering $1,2,...,n$, constructed such that the momentum going from the nth leg to the 1st leg is $\ell+a_1k_1+...+a_nk_n$. For example, the field-theory limit of the five-point correlator with insertion points ordered according to $z_1\leq z_3\leq z_5\leq z_2\leq z_4$ and loop momentum ℓ running between legs 4 and 1 is represented by the SYM integrand[6] $A(1,3,5,2,4;\ell)$. The statement of cyclicity – proven in the appendix B – in the color ordering becomes

$$A(1,2,...,n;\ell+a_1k_1+...+a_nk_n)=A(2,3,...,n,1;\ell+(a_1-1)k_1+a_2k_2...+a_nk_n) \tag{2.25}$$

Using this, one can always choose to fix the color ordering of the SYM integrand to start with a leading 1.

2.4.1. The field-theory numerators

The field-theory limit of the open superstring n-point correlator for will be parameterized by a sum over p-gon cubic graphs ranging from $p=4$ (boxes) to $p=n$:

$$A(i_1i_2\dots i_n;\ell+a^jk_j)=\sum_{p=4}^{n}\sum_{A_1\dots A_{p+1}=i_2\dots i_n}\mathcal{N}^{a_1,a_2,\dots,a_n}_{A_{p+1}i_1A_1|A_2,\dots,A_p}(\ell)\,I^{a_1,a_2,\dots,a_n}_{i_1A_1,A_2,\dots,A_p} \tag{2.26}$$

where $\mathcal{N}^{a_1,a_2,\dots,a_n}_{A_1|A_2,\dots,A_p}(\ell)$ denotes the kinematic Berends-Giele numerator of a p-gon constructed as described in the appendix A and $I^{a_1,a_2,\dots,a_n}_{A_1,A_2,\dots,A_p}$ represents the p-gon integrand. We note that in extracting a local numerator $N_{\dots}$ from (2.26) there will be a factor of 1/2 for each inverse Mandelstam invariant, see the definition (A.4).

[6] For simplicity we will consider only the planar topology.

2.4.2. Four points

The extraction of the field theory limit at four points is trivial as there is no propagator function [4]. The only limit to consider is the Koba-Nielsen factor and we get

$$A(\sigma_1, \sigma_2, \sigma_3, \sigma_4|\ell + a_1 k_{\sigma_1} + ... + a_4 k_{\sigma_4}) = V_1 T_{2,3,4} I_{1,2,3,4} \,. \tag{2.27}$$

2.4.3. Five points

The five-point genus-one superstring correlator is given by [11]

$$\begin{aligned} \mathcal{K}_5(\ell) = V_1 T^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} + \big[V_{12} T_{3,4,5} \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big] \\ + \big[V_1 T_{23,4,5} \mathcal{Z}_{23,1,4,5} + (2,3|2,3,4,5)\big] \end{aligned} \tag{2.28}$$

with the worldsheet functions [10]

$$\mathcal{Z}^m_{1,2,3,4,5} = \ell^m \,, \qquad \mathcal{Z}_{12,3,4,5} = g^{(1)}_{12} \,. \tag{2.29}$$

This correlator gives rise to five terms with non-vanishing poles in the canonical color ordering, namely $g^{(1)}_{12}, g^{(1)}_{23}, g^{(1)}_{34}, g^{(1)}_{45}$, and $g^{(1)}_{51}$. The parameterization of the integrand $A(1,2,3,4,5;\ell + a^i k_i)$ from (2.26) is given by

$$\begin{aligned} A(1,2,3,4,5;\ell + a^i k_i) = {} & N_{1|2,3,4,5}(\ell) I^{a_1,\ldots,a_5}_{1,2,3,4,5}(\ell) \\ & + \frac{1}{2s_{12}} N_{12|3,4,5}(\ell) I^{a_1,\ldots,a_5}_{12,3,4,5}(\ell) + \frac{1}{2s_{23}} N_{1|23,4,5}(\ell) I^{a_1,\ldots,a_5}_{1,23,4,5}(\ell) \\ & + \frac{1}{2s_{34}} N_{1|2,34,5}(\ell) I^{a_1,\ldots,a_5}_{1,2,34,5}(\ell) + \frac{1}{2s_{45}} N_{1|2,3,45}(\ell) I^{a_1,\ldots,a_5}_{1,2,3,45}(\ell) \\ & + \frac{1}{2s_{51}} N'_{51|2,3,4}(\ell) I^{a_1,\ldots,a_5}_{1,2,3,4}(\ell) \,. \end{aligned} \tag{2.30}$$

Since the field-theory limit rules behave differently for labels at the extremities of the color ordering, the 51-pentagon numerator is denoted $N'_{51|2,3,4}(\ell)$. Using the field-theory limit (2.10) and comparing the outcome with (2.30) we can read off the box numerators. They are independent of the loop momentum and are uniformly described by

$$N_{A|B,C,D} = V_A T_{B,C,D} \,. \tag{2.31}$$

In particular, $N'_{51|2,3,4} = N_{51|2,3,4} = V_{51} T_{2,3,4}$. This result agrees with the analysis of [1].

The pentagon $I^{a_1,\ldots,a_5}_{1,2,3,4,5}(\ell)$ arises from the pieces with no kinematic poles in (2.10) and collecting its associated superfields yields the numerator

$$\begin{aligned} N^{a_1,\ldots,a_5}_{1|2,3,4,5}(\ell) =& V_1 T^m_{2,3,4,5}\ell^m + \Big[V_{12}T_{3,4,5}\big(a_2 - a_1 + \frac{1}{2}\big) + (2 \leftrightarrow 3,4,5)\Big] \\ &+ \Big[V_1 T_{23,4,5}\big(a_3 - a_2 + \frac{1}{2}\big) + (2,3|2,3,4,5)\Big]. \end{aligned} \tag{2.32}$$

A straightforward but tedious calculation shows that

$$\begin{aligned} QN^{a_1,\ldots,a_5}_{1|2,3,4,5}(\ell) =& \frac{1}{2}V_1V_2T_{3,4,5}((\ell + f_{a_1\ldots a_5} - k_{12})^2 - (\ell + f_{a_1\ldots a_5} - k_1)^2) \\ &+ \frac{1}{2}V_1V_3T_{2,4,5}((\ell + f_{a_1\ldots a_5} - k_{123})^2 - (\ell + f_{a_1\ldots a_5} - k_{12})^2) \\ &+ \frac{1}{2}V_1V_4T_{2,3,5}((\ell + f_{a_1\ldots a_5} - k_{1234})^2 - (\ell + f_{a_1\ldots a_5} - k_{123})^2) \\ &+ \frac{1}{2}V_1V_5T_{2,3,4}((\ell + f_{a_1\ldots a_5} - k_{12345})^2 - (\ell + f_{a_1\ldots a_5} - k_{1234})^2) \end{aligned} \tag{2.33}$$

with the $f_{a_1\ldots a_5}$ defined as in (2.7). It is then not hard to check that the above cancels the BRST variation of the box terms. For example, the terms proportional to $(\ell+f_{a_1\ldots a_5}-k_{123})^2$ are given by

$$\frac{1}{2}(V_1V_3T_{2,4,5} - V_1V_4T_{2,3,5}) = -\frac{1}{2s_{34}}QV_1T_{2,34,5} \tag{2.34}$$

and cancel the BRST variation of the 34-box in (2.30) since

$$(\ell + f_{a_1\ldots a_5} - k_{123})^2 I^{a_1,\ldots,a_5}_{1,2,3,4,5}(\ell) = I^{a_1,\ldots,a_5}_{1,2,34,5}(\ell)\,. \tag{2.35}$$

Similar calculations show that $QN^{a_1,\ldots,a_5}_{1|2,3,4,5}(\ell)I^{a_1,\ldots,a_5}_{1,2,3,4,5} = -QA_{\rm box}(1,2,3,4,5)$ and therefore the five-point SYM integrand (2.30) is BRST invariant.

The BRST cohomology identities [22]

$$\begin{aligned} \langle V_1 k^1_m T^m_{2,3,4,5}\rangle &= \langle -V_{12}T_{3,4,5} + (2 \leftrightarrow 3,4,5)\rangle \\ \langle V_1 k^2_m T^m_{2,3,4,5}\rangle &= \langle V_{12}T_{3,4,5} + \big[-V_1T_{23,4,5} + (3 \leftrightarrow 4,5)\big]\rangle \end{aligned} \tag{2.36}$$

can be used to show that

$$\langle N^{(5)}_{1|2,3,4,5}(\ell + a^i k_i)\rangle = \langle N^{a_1,\ldots,a_5}_{1|2,3,4,5}(\ell)\rangle \tag{2.37}$$

where $N^{(5)}_{1|2,3,4,5}(\ell)$ is given by (1.15) and $I^{a_1,\ldots,a_5}_{1,2,3,4,5}(\ell) = I_{1,2,3,4,5}(\ell + a^i k_i)$. This is an important consistency check on the field-theory rules spelled out in section 2.3.

All color ordering permutations of the five-point SYM integrand is available to download from [28].

2.4.4. Six points

The six-point genus-one superstring correlator is given by [11]

$$\begin{aligned}\mathcal{K}_6(\ell) = \frac{1}{2} V_{A_1} T^{mn}_{A_2,\ldots,A_6} \mathcal{Z}^{mn}_{A_1,\ldots,A_6} + \big[123456|A_1,\ldots,A_6\big] \\ + V_{A_1} T^{m}_{A_2,\ldots,A_5} \mathcal{Z}^{m}_{A_1,\ldots,A_5} + \big[123456|A_1,\ldots,A_5\big] \\ + V_{A_1} T_{A_2,\ldots,A_4} \mathcal{Z}_{A_1,\ldots,A_4} + \big[123456|A_1,\ldots,A_4\big]\,,\end{aligned} \tag{2.38}$$

with the worldsheet functions [10],

$$\begin{aligned}\mathcal{Z}_{123,4,5,6} &= g^{(1)}_{12} g^{(1)}_{23} + g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13}\,,\\ \mathcal{Z}_{12,34,5,6} &= g^{(1)}_{12} g^{(1)}_{34} + g^{(2)}_{13} + g^{(2)}_{24} - g^{(2)}_{14} - g^{(2)}_{23}\,,\\ \mathcal{Z}^m_{12,3,4,5,6} &= \ell^m g^{(1)}_{12} + (k_2^m - k_1^m) g^{(2)}_{12} + \big[k_3^m (g^{(2)}_{13} - g^{(2)}_{23}) + (3 \leftrightarrow 4,5,6)\big]\,,\\ \mathcal{Z}^{mn}_{1,2,3,4,5,6} &= \ell^m \ell^n + \big[(k_1^m k_2^n + k_1^n k_2^m) g^{(2)}_{12} + (1,2|1,2,3,4,5,6)\big]\,.\end{aligned} \tag{2.39}$$

To illustrate the field-theory rules of the previous section we will derive the SYM integrand $A(2,3,4,5,6,1;\ell) = A(1,2,3,4,5,6;\ell+k_1)$. We begin with the field theory limit rules given by (2.10) and (2.11)

$$\begin{aligned}g^{(1)}_{ij} &\to \frac{1}{2}\mathrm{sgn}^{234561}_{ij} I^{234561} + \frac{1}{2}\phi_{ij|\mathrm{Ord}_{234561}(ij)} I^{234561}_{ij}\\ g^{(2)}_{ij} &\to \frac{1}{12} I^{234561} + \frac{1}{2s_{12}}(\delta_{1i}\delta_{2j} + \delta_{1j}\delta_{2i}) I^{234561}_{12}\\ g^{(1)}_{ij} g^{(1)}_{kl} &\to \frac{1}{4}\mathrm{sgn}^{234561}_{ij}\mathrm{sgn}^{234561}_{kl} I^{234561} + \frac{1}{4}\mathrm{sgn}^{234561}_{kl}\phi_{ij|\mathrm{Ord}_{234561}(ij)} I^{234561}_{ij}\\ &\quad + \frac{1}{4}\mathrm{sgn}^{234561}_{ij}\phi_{kl|\mathrm{Ord}_{234561}(kl)} I^{234561}_{kl} + \frac{1}{4}P(ij,kl)\end{aligned} \tag{2.40}$$

where

$$P(ij,kl) = \begin{cases} \phi_{ijl|Ord_{234561}(ijl)} I^{234561}_{ijl} & \text{if } j=k\\ -\phi_{ijk|Ord_{234561}(ijk)} I^{234561}_{ijk} & \text{if } j=l\\ -\phi_{jil|Ord_{234561}(ijl)} I^{234561}_{jil} & \text{if } i=k\\ \phi_{kij|Ord_{234561}(kij)} I^{234561}_{kij} & \text{if } i=l\\ \phi_{ij|Ord_{234561}(ij)}\phi_{kl|Ord_{234561}(kl)} I^{234561}_{ij,kl} & \text{else}\end{cases} \tag{2.41}$$

Extracting the terms proportional to $I^{234561} = I_{2,3,4,5,6,1}$, we find the hexagon numerator

$$N_{2|3,4,5,6,1}(\ell) = \frac{1}{2} V_1 T^{mn}_{2,3,4,5,6}\big(\ell^m\ell^n - \frac{1}{12}[k^1_m k^1_n + (1 \leftrightarrow 2,3,4,5,6)]\big) \tag{2.42}$$

$$
\begin{aligned}
&+\frac{1}{2}(V_1 T^m_{23,4,5,6}(\ell^m - \frac{1}{6}k_2^m + \frac{1}{6}k_3^m) + (2,3|2,3,4,5,6))\\
&-\frac{1}{2}(V_{12} T^m_{3,4,5,6}(\ell^m + \frac{1}{6}k_1^m - \frac{1}{6}k_2^m) + (2 \leftrightarrow 3,4,5,6))\\
&+\frac{1}{6}V_1(T_{[[2,3],4],5,6} + T_{[2,[3,4]],5,6} + (2,3,4|2,3,4,5,6))\\
&+\frac{1}{4}(V_1 T_{23,45,6} + (2,3|4,5|2,3,4,5,6))\\
&-\frac{1}{4}(V_{12} T_{34,5,6} + (2|3,4|2,3,4,5,6))\\
&-\frac{1}{6}((V_{123} - 2V_{132})T_{4,5,6} + (2,3|2,3,4,5,6))\,.
\end{aligned}
$$

We then identify the pentagon numerators, which in all but one case are given by a generalization of the formulae from [1],

$$
\begin{aligned}
N_{A|B,C,D,E1} = V_{E1}T^m_{A,B,C,D}\ell^m &+ \frac{1}{2}(V_{[A,E1]}T_{B,C,D} + (A \leftrightarrow B,C,D))\\
&+ \frac{1}{2}(V_{E1}T_{[A,B],C,D} + (A,B|A,B,C,D))
\end{aligned}
\tag{2.43}
$$

The exception to the above is the 12-pentagon, which differs as it has a contribution from the $g^{(2)}$ terms due to the color ordering 234561. It is given by the coefficient of $1/2s_{12}I^{a_1=1}_{2,3,4,5,6}$ (note the absence of the label 1 from the ordering in $I^{a_1=1}_{2,3,4,5,6}$)

$$
\begin{aligned}
N'_{21|3,4,5,6}(\ell) = &- V_1 T^{mn}_{2,3,4,5,6}k_2^m k_1^n\\
&- (V_1 T^m_{23,4,5,6}k_1^m + (3 \leftrightarrow 4,5,6))\\
&+ V_{12}T^m_{3,4,5,6}(\ell^m + k_1^m - k_m^2)\\
&- (V_{13}T^m_{2,4,5,6}k_2^m + (3 \leftrightarrow 4,5,6))\\
&+ \frac{1}{2}(V_{12}T_{34,5,6} + (3,4|3,4,5,6))\\
&- (V_{13}T_{24,5,6} + (3|4|3,4,5,6))\\
&+ \frac{1}{2}((2V_{132} - V_{123})T_{4,5,6} + (3 \leftrightarrow 4,5,6))
\end{aligned}
\tag{2.44}
$$

The box numerators have the standard form, with the word containing the label 1 assigned to the V superfield, and the other blocks of indices assigned to the T

$$
N_{A|B,C,D1E} = V_{D1E}T_{A,B,C}\,, \qquad N_{E1A|B,C,D} = V_{E1A}T_{B,C,D}\,. \tag{2.45}
$$

A long calculation shows that the BRST variation of the above integrand is purely anomalous and given by[7]

$$QA^{a_1=1}(1,2,3,4,5,6) = \frac{1}{2} V_1 Y_{2,3,4,5,6}(I_{2,3,4,5,6} - \ell^2 I_{2,3,4,5,6,1}) \tag{2.46}$$

This is then of a similar form to the $a_1 = ... = a_6 = 0$ result found in [1], and by an analogous argument to the one presented there one finds the same result for the integrated anomaly

$$\int d^{10}\ell QA^{a_1=1}(1,2,3,4,5,6) = -\frac{\pi^5}{240} V_1 Y_{2,3,4,5,6} \,. \tag{2.47}$$

Of course the type I superstring theory with gauge group $SO(32)$ is free of gauge anomalies [30], but this property does not survive the field-theory limit of its planar sector and the six-point one-loop SYM amplitude in ten dimensions is anomalous [31]. The result (2.47) written in terms of the anomalous building block $Y_{2,3,4,5,6}$ [22] is the pure spinor superspace encoding of the field-theory anomaly [6,7].

2.4.5. Seven Points

At seven points, the numerators become far too complex to state here. One example can be found in the appendix D. The derivation of these numerators has one additional complication; as was discussed in [11] the refined worldsheet functions are given by

$$\mathcal{Z}_{12|3,4,5,6,7} = \partial g_{12}^{(2)} + s_{12} g_{12}^{(1)} g_{12}^{(2)} - 3 s_{12} g_{12}^{(3)} \,. \tag{2.48}$$

The derivative and the double pole are then removed by using partial integration with the Koba-Nielsen factor $\mathcal{I}_7(\ell)$

$$\begin{aligned}(\partial_1 g_{12}^{(2)})\mathcal{I}_7(\ell) &= \partial_1(g_{12}^{(2)}\mathcal{I}_7(\ell)) + g_{12}^{(2)}\partial_2\mathcal{I}_7(\ell) \\ &= \partial_1(g_{12}^{(2)}\mathcal{I}_7(\ell)) + g_{12}^{(2)}\Big((\ell\cdot k_2) + s_{21}g_{21}^{(1)} + s_{23}g_{23}^{(1)} + ... + s_{27}g_{27}^{(1)}\Big)\mathcal{I}_7(\ell),\end{aligned} \tag{2.49}$$

which gives the reformulated expression for (2.48)

$$\mathcal{Z}_{12|3,4,5,6,7} = -3s_{12}g_{12}^{(3)} + g_{12}^{(2)}(\ell\cdot k_2 + s_{23}g_{23}^{(1)} + s_{24}g_{24}^{(1)} + ... + s_{27}g_{27}^{(1)}) \tag{2.50}$$

This is the form of the refined worldsheet function we use to extract the numerators and the computation proceeds analogously as before. And we have verified the vanishing of the BRST variation of the resulting general expression.

[7] See the discussion of [29] as summarized in section 4.5 of [1] to understand why (2.46) is not trivially zero due to the cancellation of propagators in the integrand.

3. Local BCJ-satisfying numerators

In this section we will obtain the kinematic numerators associated to various one-loop cubic graphs using the field-theory limit rules of section 2.3 applied to the superstring correlators for six external states as well as some seven-point numerators. The results of this section resolve a puzzle in the analysis of [1]. Namely, the representation in [1] of the six-point integrand did not satisfy the color-kinematics duality by terms which suspiciously were related to the gauge anomaly. We now show that the six-point integrand representation arising from the field-theory limit of the string correlator satisfies all the color-kinematic Jacobi dual relations of Bern-Carrasco-Johansson.

3.1. Color-kinematics duality

The color factors of amplitudes in gauge theory depend on the structure constants of some gauge group, f^{abc}, that satisfy the Jacobi identity,

$$f^{abe}f^{cde} + f^{bce}f^{ade} + f^{cae}f^{bde} = 0\,. \tag{3.1}$$

The color-kinematics duality conjecture posed by Bern, Carrasco and Johansson (BCJ) states that the kinematic numerators of cubic-graph diagrams can be chosen to satisfy the same Jacobi identity relating their color factors [16]. That is, if a triplet of diagrams i, j, k whose color factors c_i, c_j, c_k vanish due to the Jacobi identity (3.1), $c_i + c_j + c_k = 0$, the corresponding numerators N_i, N_j, N_k of the diagrams satisfy $N_i(\ell) + N_j(\ell) + N_k(\ell) = 0$ as well. Stated originally at tree-level [16] (and proven by the field-theory limit of string theory tree amplitudes [32,33]) the duality was conjectured at loop-level in [34], where the kinematic numerators also depend on loop momenta ℓ parameterizing various n-gon cubic graphs. Through this approach, properties of $4 \leq \mathcal{N} \leq 8$ supergravity up to four loops have been made manifest [35] (for the five-loop extension see [36,37]).

As part of the color-kinematics duality, once the gauge-theory amplitude is written down using kinematic numerators that satisfy all the kinematic Jacobi identities and automorphism symmetries of the cubic graphs, the gauge amplitude can be used to construct a gravity amplitude by replacing the color factors by a second copy of numerators $c_i \to \tilde{N}_i(\ell)$ [16,34]. For more details see the review [2].

We will now show that the numerators extracted from the one-loop string correlators using the field-theory rules of section 2.3 satisfy all the color-kinematics relations. However, starting at six points the numerators do not satisfy the required symmetries under shifts of the loop momentum required by the automorphism symmetries of the cubic graphs (see [24]), leading to subtleties in the construction of the gravity amplitudes which we defer to future work.

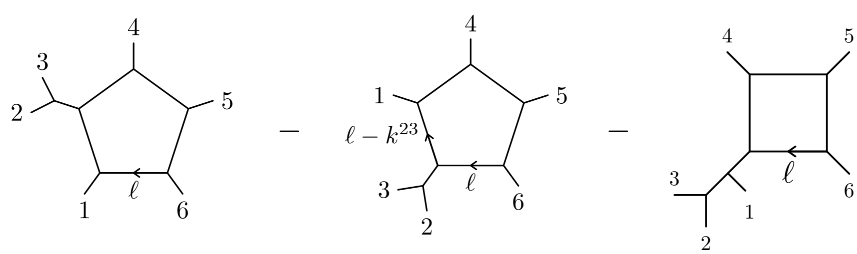
The one-loop five-point integrand of SYM in ten dimensions was already discussed in section 1.3.2 so we will focus on the six-point SYM integrand and briefly outline the discussion of the seven-point numerators.

3.2. Six points

The color-kinematics relations are manifestly satisfied within external tree graphs due to the BCJ gauge used in the multiparticle superfields [38,15]. Therefore we will discuss the kinematic Jacobi identities among p-gons with different values of p.

3.2.1. Kinematic Jacobi between pentagons and a box

The pure spinor superspace expressions of the numerators associated to the graphs in the following linear combination



constitute a good example of how our methods give rise to a BCJ-satisfying parameterization of the six-point integrand.

Two of the above graphs are part of the integrand in the canonical color ordering $A(1,2,3,4,5,6;\ell)$. According to the color-kinematics identity that we are seeking to show, the loop momentum parameterization of the graphs must have the same momentum flowing in the edges that are kept the same for all graphs. Therefore the middle graph must have loop momentum ℓ flowing from leg 6 to the fork 23. According to the convention shown in fig. 2 this is the 23-pentagon $N_{23|1,4,5,6}(\ell)$ from the integrand $A(2,3,1,4,5,6;\ell)$ whose expression can be read off from the field-theory limit rules for this particular ordering.

However the assumption used in the parameterization of [1] was that this pentagon is obtained in a crossing symmetric way as $N_{1|4,5,6,23}(\ell - k_{23})$. As shown in [1], using these kinematic numerators the algebraic translation of the BCJ triplet linear combination above becomes

$$
\begin{aligned}
N_{1|23,4,5,6}(\ell) - N_{1|4,5,6,23}(\ell - k_{23}) - N_{[1,23]|4,5,6}(\ell) = \\
k^{23}_m V_1 T^m_{23,4,5,6} + V_{231}T_{4,5,6} + \big[V_1 T_{234,5,6} + (4 \leftrightarrow 5, 6)\big]
\end{aligned}
\tag{3.2}
$$

which is not in the cohomology of the BRST charge and therefore is not vanishing. In other words, the representation of the six-point integrand chosen in [1] does not satisfy the color-kinematics duality.

In contrast, using the field-theory limit rules of this work the cubic graphs above can be derived in their native color ordering and they satisfy the BCJ triplet numerator identity:

$$
N_{1|23,4,5,6}(\ell) - N_{23|1,4,5,6}(\ell) - N_{[1,23]|4,5,6}(\ell) = 0\,. \tag{3.3}
$$

To see this vanishing we begin with the box numerator $N_{[1,23]|4,5,6}(\ell)$, the coefficient of $\frac{1}{4s_{23}s_{123}} I_{123,4,5,6}$ in the integrand $A(1,2,3,4,5,6;\ell)$. According to (2.11) and (2.13) the only functions that can generate such a factor are $g^{(1)}_{12} g^{(1)}_{23}$ and $g^{(1)}_{13} g^{(1)}_{23}$ via $P(12,23)$ and $P(13,23)$. There are only two terms featuring these functions in the six-point string correlator (1.3),

$$
V_{123}T_{4,5,6} g^{(1)}_{12} g^{(1)}_{23} + V_{132}T_{4,5,6} g^{(1)}_{13} g^{(1)}_{32}\,. \tag{3.4}
$$

The field-theory limit of $g^{(1)}_{12} g^{(1)}_{23}$ and $g^{(1)}_{13} g^{(1)}_{23}$ gives rise to the box integrand through the $P(ij,jk)$ terms in

$$
\begin{aligned}
\frac{1}{4}V_{123}T_{4,5,6}P(12,23) + \frac{1}{4}V_{132}T_{4,5,6}P(13,32) &= \frac{1}{4}V_{123}T_{4,5,6}\phi_{123|123}I_{123} + \frac{1}{4}V_{132}T_{4,5,6}\phi_{132|123}I_{123} \\
&= \frac{1}{4}V_{123}T_{4,5,6}\left(\frac{1}{s_{12}s_{123}} + \frac{1}{s_{23}s_{123}}\right) I_{123} + \frac{1}{4}V_{132}T_{4,5,6}\left(-\frac{1}{s_{23}s_{123}}\right) I_{123}\,.
\end{aligned}
\tag{3.5}
$$

The box numerator $N_{[1,23]|4,5,6}(\ell)$ is given by the coefficient of $\frac{1}{4}\frac{1}{s_{23}s_{123}} I_{123}$,

$$
N_{[1,23]|4,5,6} = V_{123}T_{4,5,6} - V_{132}T_{4,5,6} = V_{[1,23]}T_{4,5,6} \tag{3.6}
$$

The pentagon $N_{1|23,4,5,6}(\ell)$ is given by the coefficient of $\frac{1}{2s_{23}} I_{23}$ in the field theory limit of the correlator $\mathcal{K}_6(\ell)$ for the color ordering $A(1,2,3,4,5,6;\ell)$. Such factors arise from any appearance of $g^{(1)}_{23}$ in (1.3),

$$
\begin{aligned}
&V_1 T^m_{23,4,5,6}\ell^m g^{(1)}_{23} + \Big[V_{123}T_{4,5,6} g^{(1)}_{12} g^{(1)}_{23} + (2 \leftrightarrow 3)\Big] + \Big[V_1 T_{234,5,6} g^{(1)}_{23} g^{(1)}_{34} + (4 \leftrightarrow 5, 6)\Big] \\
&\quad + \Big[V_{14}T_{23,5,6} g^{(1)}_{14} g^{(1)}_{23} + (4 \leftrightarrow 5, 6)\Big] + \Big[V_1 T_{23,45,6} g^{(1)}_{23} g^{(1)}_{45} + (4,5|4,5,6)\Big]
\end{aligned}
\tag{3.7}
$$

Taking the limits and collecting terms proportional to $\frac{1}{2s_{23}} I_{23}$ we get

$$N_{1|23,4,5,6}(\ell) = V_1 T^m_{23,4,5,6}\ell^m + \frac{1}{2}\big[V_{[1,23]}T_{4,5,6} + (23 \leftrightarrow 4,5,6)\big] \\ + \frac{1}{2}\big[V_1 T_{[23,4],5,6} + (23,4|23,4,5,6)\big]\,. \tag{3.8}$$

The numerators (3.6) and (3.8) agree with the numerators obtained in [1].

The middle pentagon in the above figure is the 23-pentagon in the integrand of $A(2,3,1,4,5,6;\ell)$ since the loop momentum is running from leg 6 to 2. Alternatively, a cyclic rotation as in (2.25) yields the integrand $A(1,4,5,6,2,3;\ell-k_{23})$ whose field-theory limit is computed using the rules of section 2.3 with with $a_2 = a_3 = -1$, $a_i = 0$ for all other i. The calculation proceeds similarly to the above. The relevant terms are now[8]

$$V_1 T^m_{4,5,6,23}\ell^m g^{(1)}_{23} + \Big[V_{123}T_{4,5,6}g^{(1)}_{12}g^{(1)}_{23} + (2\leftrightarrow 3)\Big] \\ + \Big[V_1 T_{423,5,6}g^{(1)}_{42}g^{(1)}_{23} + V_1 T_{432,5,6}g^{(1)}_{43}g^{(1)}_{32} + (4\leftrightarrow 5,6)\Big] \\ + \Big[V_{14}T_{5,6,23}g^{(1)}_{14}g^{(1)}_{23} + (4\leftrightarrow 5,6)\Big] + \frac{1}{2}\Big[V_1 T_{45,6,23}g^{(1)}_{45}g^{(1)}_{23} + (4,5|4,5,6)\Big]. \tag{3.9}$$

Taking the field theory limits and restricting ourselves to the s_{23} single poles, we see that the numerator is given by

$$N_{23|1,4,5,6}(\ell) = V_1 T^m_{4,5,6,23}\ell^m - \frac{1}{2}V_{[1,23]}T_{4,5,6} + \frac{1}{2}(V_{[1,4]}T_{5,6,23} + (4\leftrightarrow 5,6)) \\ + \frac{1}{2}(V_1 T_{[23,4],5,6} + (23,4|23,4,5,6)) \tag{3.10}$$

This differs from the parameterization of this graph used in [1], namely $N^{(5)}_{1|4,5,6,23}(\ell-k_{23})$ with the expression for $N^{(5)}_{A|B,C,D,E}(\ell)$ given in (1.17). While the representation of [1] fails to satisfy the color Jacobi identity, the new representation derived here obeys the color-kinematics duality. To see this we plug the superfield expressions of the new field-theory representations of the box (3.6) and pentagons (3.8), (3.10) in the kinematic Jacobi relation (3.3) to obtain

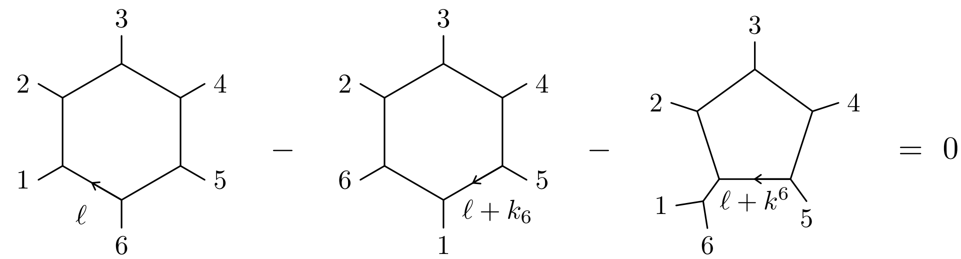
$$N_{1|23,4,5,6}(\ell) - N_{23|1,4,5,6}(\ell) - N_{[1,23]|4,5,6}(\ell) = 0\,. \tag{3.11}$$

And we note that the BCJ relation is identically satisfied at the superfield level (i.e. no BRST cohomology identity is needed). This trivial vanishing for the BCJ triplet at one loop parallels the superfield vanishing of the BCJ triplet of tree-level numerators obtained from the field-theory of the disk correlators as seen in (1.14).

[8] We exploit the total symmetry of the six-point correlator (1.3) in $2,3,4,5,6$

3.2.2. Kinematic Jacobi between hexagons and a pentagon

In a given color ordering, all of the pentagons have a similar structure apart from the ij-pentagon whose labels are cyclically split at the extremities $A(i,\ldots,j;\ell)$. In this subsection we will demonstrate the validity of a BCJ relation involving such a numerator. The relation we will show is



which corresponds to

$$N_{1|2,3,4,5,6}(\ell) - N^{a_6=1}_{1|6,2,3,4,5}(\ell) - N_{61|2,3,4,5}(\ell) = 0\,. \tag{3.12}$$

To find the hexagon numerators, we look at the piece of the field theory limits proportional to $P = I$. In the first case, this means making the substitution

$$g^{(1)}_{ij} \to \frac{1}{2}\mathrm{sgn}^{123456}_{ij} I\,, \qquad g^{(1)}_{ij} g^{(1)}_{kl} \to \frac{1}{4}\mathrm{sgn}^{123456}_{ij}\mathrm{sgn}^{123456}_{kl} I\,, \qquad g^{(2)}_{ij} \to \frac{1}{12} I\,. \tag{3.13}$$

This then gives the value of the first hexagon numerator as

$$\begin{aligned} N_{1|2,3,4,5,6}(\ell) = &+ \frac{1}{6}((V_{[[1,2],3]} + V_{[1,[2,3]]})T_{4,5,6} + (2,3|2,3,4,5,6)) \\ &+ \frac{1}{6}V_1(T_{[[2,3],4],5,6} + T_{[2,[3,4]],5,6} + (2,3,4|2,3,4,5,6)) \\ &+ \frac{1}{4}V_{[1,2]}T_{[3,4],5,6} + (2|3,4|2,3,4,5,6)) \\ &+ \frac{1}{4}V_1 T_{[2,3],[4,5],6} + (2,3|4,5|2,3,4,5,6)) \\ &+ \frac{1}{2}(V_{[1,2]}T^m_{3,4,5,6}(\ell^m - \frac{1}{6}k^m_1 + \frac{1}{6}k^m_2) + (2 \leftrightarrow 3,4,5,6)) \\ &+ \frac{1}{2}(V_1 T^m_{[2,3],4,5,6}(\ell^m - \frac{1}{6}k^m_2 + \frac{1}{6}k^m_3) + (2,3|2,3,4,5,6)) \\ &+ \frac{1}{2}V_1 T^{mn}_{2,3,4,5,6}(\ell^m\ell^n - \frac{1}{12}k^m_1 k^n_1 - \frac{1}{12}k^m_2 k^n_2 - \cdots - \frac{1}{12}k^m_6 k^n_6)\,. \end{aligned} \tag{3.14}$$

For the second hexagon, we consider the field-theory limit of the correlator with the color ordering $A(1,6,2,3,4,5;\ell+k_1)$. The limits needed now have the form

$$\begin{aligned} g^{(1)}_{ij} &\to \frac{1}{2}\mathrm{sgn}^{162345}_{ij} + \delta_{j6} - \delta_{i6}\,, \\ g^{(1)}_{ij} g^{(1)}_{kl} &\to \big(\frac{1}{2}\mathrm{sgn}^{162345}_{ij} + \delta_{j6} - \delta_{i6}\big)\big(\frac{1}{2}\mathrm{sgn}^{162345}_{kl} + \delta_{l6} - \delta_{k6}\big)\,, \\ g^{(2)}_{ij} &\to \frac{1}{12} + \frac{1}{2}\big(\delta_{i6}(1-\mathrm{sgn}^{162345}_{ij}) + \delta_{j6}(1+\mathrm{sgn}^{162345}_{ij})\big)\,. \end{aligned} \tag{3.15}$$

Using these, the numerator is identified as

$$
\begin{aligned}
N^{a_6=1}_{1|6,2,3,4,5}(\ell) = &+ \frac{1}{2} V_1 T^{mn}_{2,3,4,5,6}(\ell^m \ell^n + 2k_1^m k_6^n - \frac{1}{12}(k^1_m k^1_n + k^2_m k^2_n + \cdots k^6_m k^6_n)) \\
&+ \frac{1}{2}(V_1 T^m_{[2,3],4,5,6}(\ell^m - \frac{1}{6}k_2^m + \frac{1}{6}k_3^m) + (2,3|2,3,4,5,6)) \\
&- (V_1 T^m_{[2,6],3,4,5} k_1^m + (2 \leftrightarrow 3,4,5)) \\
&+ \frac{1}{2}(V_{[1,2]} T^m_{3,4,5,6}(\ell^m - \frac{1}{6}k_1^m + \frac{1}{6}k_2^m + 2k_6^m) + (2 \leftrightarrow 3,4,5)) \\
&+ V_{[1,6]} T^m_{2,3,4,5}(\frac{3}{2}\ell^m - \frac{13}{12}k_1^m + \frac{13}{12}k_6^m) \\
&+ \frac{1}{6} V_1 (T_{[[2,3],4],5,6} + T_{[2,[3,4]],5,6} + (2,3,4|2,3,4,5,6)) \\
&+ \frac{1}{6}((V_{[[1,2],3]} + V_{[1,[2,3]]}) T_{4,5,6} + (2,3|2,3,4,5)) \\
&- \frac{1}{3}((V_{[[1,2],6]} + V_{[1,[2,6]]}) T_{4,5,6} + (2 \leftrightarrow 3,4,5)) \\
&+ \frac{1}{4}(V_1 T_{[2,3],[4,5],6} + (2,3|4,5|2,3,4,5,6)) \\
&+ \frac{1}{4}(V_{[1,2]} T_{[3,4],5,6} + (2|3,4|2,3,4,5)) \\
&- \frac{3}{4}(V_{[1,2]} T_{[3,6],4,5} + (2|3|2,3,4,5)) \\
&+ \frac{3}{4}(V_{[1,6]} T_{[2,3],4,5} + (2,3|2,3,4,5))
\end{aligned} \tag{3.16}
$$

Finally we have the pentagon term, the superfield coefficient of $\frac{1}{2s_{16}} I_{61,2,3,4,5}$ from the integrand $A(1,2,3,4,5,6;\ell)$. This can be found to be

$$
\begin{aligned}
N'_{61|2,3,4,5}(\ell) = &+ \frac{1}{2}\big[(V_{[[1,2],6]} + V_{[1,[2,6]]}) T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big] \\
&+ \big[V_{[1,2]} T_{[3,6],4,5} + (2|3|2,3,4,5)\big] \\
&- \frac{1}{2}\big[V_{[1,6]} T_{[2,3],4,5} + (2,3|2,3,4,5)\big] \\
&- \big[V_{[1,2]} T^m_{3,4,5,6} k_6^m + (2 \leftrightarrow 3,4,5)\big] \\
&+ \big[V_1 T^m_{[2,6],3,4,5} k_1^m + (2 \leftrightarrow 3,4,5)\big] \\
&- V_{[1,6]} T^m_{2,3,4,5}(\ell^m + k_6^m - k_1^m) \\
&- V_1 T^{mn}_{2,3,4,5,6} k_1^m k_6^n
\end{aligned} \tag{3.17}
$$

It is then simply a matter of plugging the numerators into the identity (3.12) to verify its validity.

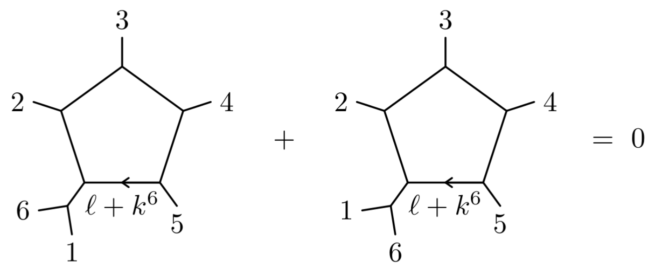


Fig. 3 The antisymmetry of the 61-pentagon from the integrand $A(1,2,3,4,5,6;\ell)$. The momentum running into the 61 external tree in the graph on the right is $\ell + k_6$ because in the color ordering $1,2,3,4,5,6$ a momentum ℓ must run between 6 and 1. Therefore in order to preserve the momentum assignment in the edges between the two cubic graphs, the pentagon on the left is part of the integrand $A(1,6,2,3,4,5;\ell+k_6)$ with momentum $\ell + k_6$ running between legs 5 and 1 as dictated by the convention (2.24). Therefore to extract this pentagon the field-theory rules of section 2.3 must be used with $a_6 = 1$.

3.2.3. Antisymmetry of ij-pentagons from $A(i,P,j;\ell)$ in $i \leftrightarrow j$

As mentioned above, the color-kinematics duality relations within external tree diagrams is manifestly satisfied due to the usage of multiparticle superfields in the BCJ gauge. For instance, all the boxes and all but one pentagon for an integrand of arbitrary color ordering $A(P;\ell)$ can be described by

$$N_{A|B,C,D}(\ell) = V_A T_{B,C,D}(\ell) + (A \leftrightarrow B,C,D) \tag{3.18}$$

$$\begin{aligned} N_{A|B,C,D,E}(\ell) &= \big[V_A T^m_{B,C,D,E}\ell_m + (A \leftrightarrow B,C,D,E)\big] \\ &+ \frac{1}{2}\big[V_A T_{[B,C],D,E} + (A|B,C|A,B,C,D,E)\big] \\ &+ \frac{1}{2}\big[V_{[A,B]} T_{C,D,E} + (A,B|A,B,C,D,E)\big] \end{aligned} \tag{3.19}$$

with the additional constraint that $T^{\cdots}_{\ldots,A1B,\ldots} = 0$ (i.e., setting to zero all terms in which the label 1 is not assigned to a multiparticle vertex V_P). For example, using (3.19) we recover the 23-pentagon (3.10)

$$\begin{aligned} N_{23|1,4,5,6}(\ell) = V_1 T^m_{4,5,6,23}\ell^m - \frac{1}{2}V_{[1,23]}T_{4,5,6} + \frac{1}{2}(V_{[1,4]}T_{5,6,23} + (4 \leftrightarrow 5,6)) \\ + \frac{1}{2}(V_1 T_{[23,4],5,6} + (23,4|23,4,5,6)) \end{aligned} \tag{3.20}$$

where we used (1.17) and the constraint $T^{\cdots}_{\ldots,A1B,\ldots} = 0$. Since in the BCJ gauge [38,15] the multiparticle labels (words) in (3.18) and (3.19) satisfy generalized Jacobi identities, the color-kinematics duality are manifest within those words, with a notable exception.

The exception arises for the ij-pentagon when the labels i, j are adjacent up to a cyclic rotation, e.g. the 61-pentagon in $A(1,2,3,4,5,6;\ell)$ or the 12-pentagon in $A(2,3,4,5,6,1;\ell)$ do not follow the general formula (3.19), as can be seen for example in (2.44). The reason this happens is due to a clash between the ij pentagon labels in $A(j,P,i;\ell)$ and the convention that the loop momentum ℓ runs between i and j. So to verify the antisymmetry of the 61-pentagon from $A(1,2,3,4,5,6;\ell)$ one needs to compare it to the 16-pentagon from $A(1,6,2,3,4,5;\ell+k_6)$ using the field-theory rules from section 2.3. The result is

$$\begin{aligned} N^{a_6=1}_{16|2,3,4,5}(\ell) = &-\frac{1}{2}\big[(V_{[[1,2],6]} + V_{[1,[2,6]]})T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big] \\ &- \big[V_{[1,2]}T_{[3,6],4,5} + (2|3|2,3,4,5)\big] \\ &+ \frac{1}{2}\big[V_{[1,6]}T_{[2,3],4,5} + (2,3|2,3,4,5)\big] \\ &+ \big[V_{[1,2]}T^m_{3,4,5,6}k^m_6 - V_1 T^m_{[2,6],3,4,5}k^m_1 + (2 \leftrightarrow 3,4,5)\big] \\ &+ V_{[1,6]}T^m_{2,3,4,5}(\ell^m + k^m_6 - k^m_1) \\ &+ V_1 T^{mn}_{2,3,4,5,6}k^m_1 k^n_6 \,. \end{aligned} \tag{3.21}$$

Comparing (3.21) with (3.17) one immediately verifies the color-kinematics identity depicted in fig. 3

$$N^{a_6=1}_{16|2,3,4,5}(\ell) + N_{61|2,3,4,5}(\ell) = 0 \,. \tag{3.22}$$

It is interesting to observe that the field-theory limit rules yield a different 16-pentagon in the in color ordering without a shift in the loop momentum $A(1,6,2,3,4,5;\ell)$, namely

$$N_{16|2,3,4,5}(\ell) = V_{16}T^m_{2,3,4,5}\ell_m + \frac{1}{2}\big[V_{16}T_{23,4,5} + (2,3|2,3,4,5)\big] + \frac{1}{2}V_{162}T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big] \,. \tag{3.23}$$

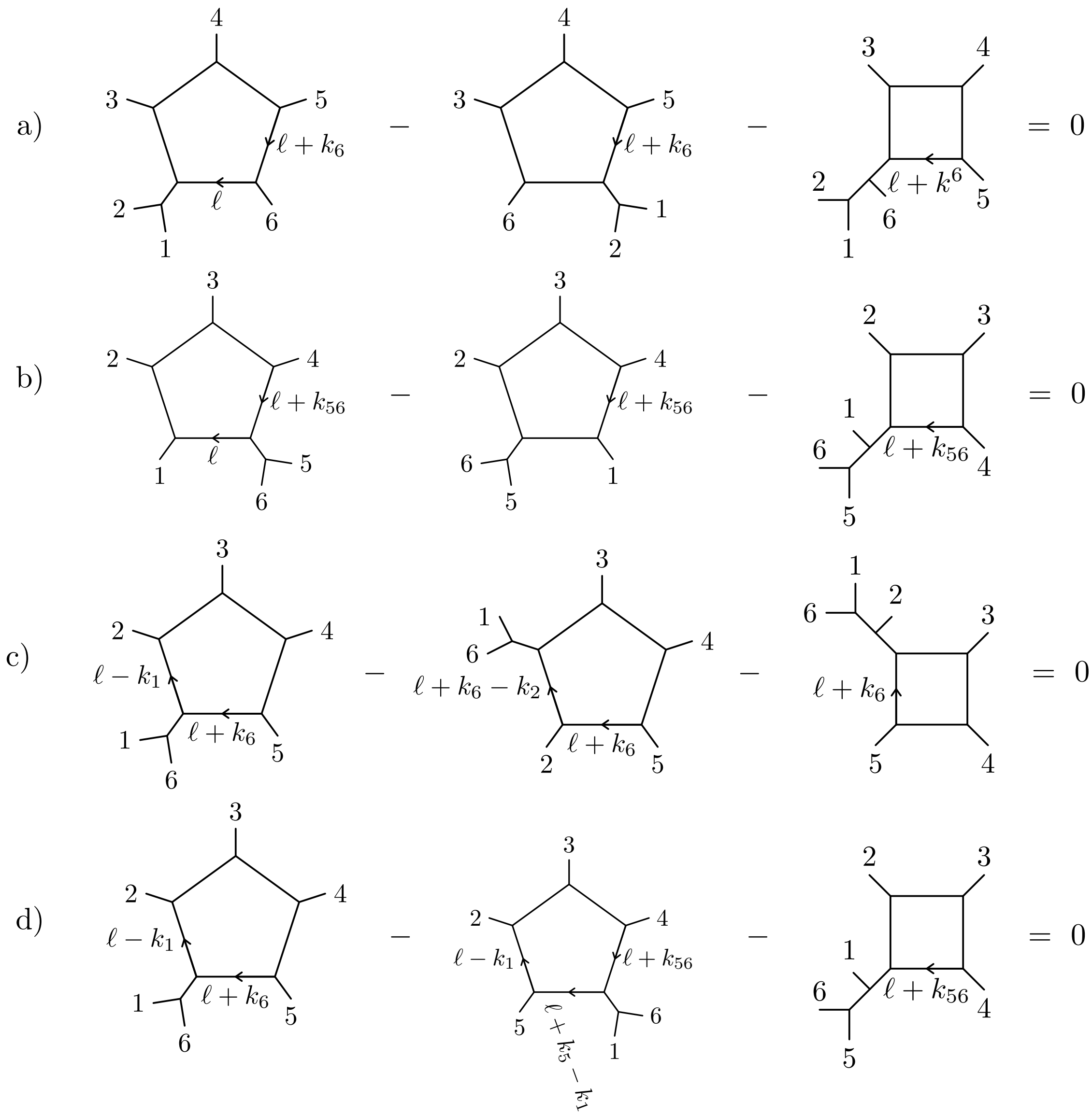
If we now perform a manual shift $\ell \to \ell + k_6$ in the 16-pentagon numerator (3.23) and compare it with the 16-pentagon from the shifted amplitude $A(1,6,2,3,4,5;\ell+k_6)$ we find that they are not BRST equivalent,

$$Q\big(N^{a_6=1}_{16|2,3,4,5}(\ell) - N_{16|2,3,4,5}(\ell+k_6)\big) = Q(s_{16}V_1 J_{6|2,3,4,5}) \,. \tag{3.24}$$

This shows that the field-theory rules of section 2.3 capture the shifts in the loop momentum parameterization in a non trivial way, as the limit for $A(1,6,2,3,4,5;\ell+k_6)$ does not follow from naively shifting $\ell \to \ell + k_6$ in $A(1,6,2,3,4,5;\ell)$.

3.2.4. Remaining BCJ triplets

There are then a number of relations between pentagons and boxes left to show in order to see that we have a BCJ representation of $A(1,2,3,4,5,6)$, and these can be seen in the cases a) to d) in the next figure. For each of these in turn we just follow the rules (2.10) for the following amplitudes with the following assignments of values for the a_i

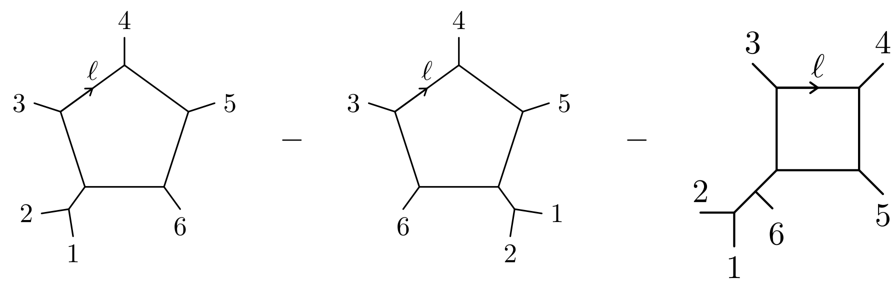


$$
\begin{aligned}
&A(1,2,6,3,4,5;\ell+k_6), && a_1=a_2=a_3=a_4=a_5=0,\ a_6=1\\
&A(1,6,5,2,3,4;\ell+k_{56}), && a_1=a_2=a_3=a_4=0,\ a_5=a_6=1\\
&A(1,3,4,5,2,6;\ell-k_2), && a_1=a_3=a_4=a_5=a_6=0,\ a_2=-1\\
&A(1,5,2,3,4,6;\ell+k_5), && a_1=a_2=a_3=a_4=a_6=0,\ a_5=1
\end{aligned}
\tag{3.25}
$$

These have been verified to give amplitudes which are BRST invariant and satisfy the relations a) to d) in the figure above. We will not detail their construction any further, as they can be obtained by analogous manipulations as discussed above.

3.2.5. Other parameterization of cubic graphs

Note that the choice of loop momentum to parameterize the cubic graphs plays an important role due to the inherent asymmetry of the numerators with respect to the label 1 (which must always be associated with V_P). The cases considered above are the ones which maximize the chances of failure. For instance, if we choose to position ℓ in the edge between 3 and 4 in the graphs depicted in a) in the previous figure the resulting triplet of numerators



is easily seen to satisfy the color-kinematics identity. In this case we get,

$$N_{4|5,6,12,3}(\ell) - N_{4|5,12,6,3}(\ell) - N_{4|5,[6,12],3}(\ell) = 0\,. \tag{3.26}$$

To see this it is enough to use the pentagon (1.17) to obtain

$$\begin{aligned} N_{4|5,6,12,3}(\ell) = V_{12}T^m_{3,4,5,6}\ell_m - \frac{1}{2}\Big[&V_{12}T_{34,5,6} + V_{12}T_{35,4,6} + V_{12}T_{36,4,5} \\ &- V_{12}T_{45,3,6} - V_{12}T_{46,3,5} - V_{12}T_{56,3,4} - V_{123}T_{4,5,6} + V_{124}T_{3,5,6} \\ &+ V_{125}T_{3,4,6} + V_{126}T_{3,4,5}\Big] \\ N_{4|5,12,6,3}(\ell) = V_{12}T^m_{3,4,5,6}\ell_m - \frac{1}{2}\Big[&V_{12}T_{34,5,6} + V_{12}T_{35,4,6} + V_{12}T_{36,4,5} \\ &- V_{12}T_{45,3,6} - V_{12}T_{46,3,5} - V_{12}T_{56,3,4} - V_{123}T_{4,5,6} + V_{124}T_{3,5,6} \\ &+ V_{125}T_{3,4,6} - V_{126}T_{3,4,5}\Big]\,, \end{aligned} \tag{3.27}$$

from which we get $N_{4|5,6,12,3}(\ell) - N_{4|5,12,6,3}(\ell) = -V_{126}T_{3,4,5}$ and (3.26) is satisfied since $N_{4|5,[6,12],3} = V_{[6,12]}T_{3,4,5} = -V_{126}T_{3,4,5}$.

Thus we conclude that the field-theory limit of the genus-one six-point string correlator (1.3) for various color orderings as dictated by the ordering of vertex operators on the boundary of the Riemann surface satisfies all the color-kinematics identities.

3.3. Seven points

At seven points, BCJ relations are analogously satisfied. Given their significantly more complex structure, we will not demonstrate these explicitly here and we will only outline their construction below.

As alluded to earlier, at seven points there is an extra complication that must be dealt with: the refined superfields. To find the field theory limits of the refined terms, we have to use an alternative method and partially integrate the worldsheet functions against the Koba-Nielsen factor. This then means that, when we want to verify BCJ relations, we must rearrange these refined terms. For relations in which the loop momentum structure is unchanged between terms (that is, BCJ relations in which there is always momentum ℓ going into leg 1), this amounts to canceling all $(\ell \cdot k)$ terms. Take for instance the relation

$$N_{1|2,3,4,5,6,7}(\ell) - N_{1|2,4,3,5,6,7}(\ell) - N_{1|2,34,5,6,7}(\ell) = 0\,, \tag{3.28}$$

and consider the refined terms $V_1 J_{34|2,5,6,7}$ within it. In the standard ordering correlator, these terms are associated with the worldsheet function $\mathcal{Z}_{34|1,2,5,6,7}$ and we would therefore expect the heptagon numerator $N_{1|2,3,4,5,6,7}(\ell)$ to contain the terms

$$-\frac{1}{12} V_1 J_{34|2,5,6,7}\Big(\ell \cdot k_4 - \frac{1}{2} k^{12} \cdot k^4 + \frac{1}{2} k^4 \cdot k^{567}\Big)\,. \tag{3.29}$$

Likewise, the other numerators we would expect to contain the terms

$$\begin{aligned} N_{1|2,4,3,5,6,7}(\ell) &\leftrightarrow -\frac{1}{12} V_1 J_{43|2,5,6,7}\Big(\ell \cdot k_3 - \frac{1}{2} k^{12} \cdot k^3 + \frac{1}{2} k^3 \cdot k^{567}\Big) \\ N_{1|2,34,5,6,7}(\ell) &\leftrightarrow 0\,. \end{aligned} \tag{3.30}$$

The relation (3.28) is clearly not satisfied with these values.

Instead, we cancel the $\ell \cdot k$ terms. For example, we rewrite (3.29) as

$$\begin{aligned} &-\frac{1}{12} V_1 J_{34|2,5,6,7}\Big(\frac{1}{2}(\ell - k^{123})^2 - \frac{1}{2}(\ell - k^{1234})^2 + k^{123} \cdot k^4 - \frac{1}{2} k^{12} \cdot k^4 + \frac{1}{2} k^4 \cdot k^{567}\Big) \\ =&-\frac{1}{12} V_1 J_{34|2,5,6,7}\Big(\frac{1}{2}(\ell - k^{123})^2 - \frac{1}{2}(\ell - k^{1234})^2 + \frac{1}{2} k^3 \cdot k^4\Big)\,. \end{aligned} \tag{3.31}$$

We then cancel the $(\ell - k)^2$ terms with the denominator of the Feynman loop integrand $I_{1,2,3,4,5,6,7}(\ell)$ associated with this term, and so they contribute to hexagons instead. Hence there is only one term of this form associated with the heptagon,

$$N_{1|2,3,4,5,6,7}(\ell) \leftrightarrow -\frac{1}{24} s_{34} V_1 J_{34|2,5,6,7}\,. \tag{3.32}$$

Similarly, the other heptagon numerator undergoes this procedure and is associated with

$$N_{1|2,4,3,5,6,7}(\ell) \leftrightarrow \frac{1}{24} s_{34} V_1 J_{34|2,5,6,7} \,. \tag{3.33}$$

There are then extra terms in the hexagons arising from the canceled portion of the terms from the heptagons. The 34-hexagon we are interested in inherits a term from the cancellation (3.31). Hence we now have

$$N_{1|2,34,5,6,7}(\ell) \leftrightarrow -\frac{1}{12} s_{34} V_1 J_{34|2,5,6,7} \,. \tag{3.34}$$

Note this differs from what may be naively expected from (3.31) due to the hexagon containing an extra $2s_{34}$ in its denominator compared with the heptagon. Now plugging (3.32), (3.33), (3.34) into the relation (3.28) we see it is now satisfied

$$-\frac{1}{24} s_{34} V_1 J_{34|2,5,6,7} - \frac{1}{24} s_{34} V_1 J_{34|2,5,6,7} - \Big(-\frac{1}{12} s_{34} V_1 J_{34|2,5,6,7} \Big) = 0 \,. \tag{3.35}$$

Similar manipulations hold for other BCJ relations of this sort. Additional complications arise when the BCJ relations involve terms of different loop momentum structure, and we have yet to identify a general algorithm for these situations. However, by explicitly rearranging amplitudes term by term, we have been able to structure them so that they satisfy all of the BCJ relations we have tested. Namely, we have been able to simultaneously satisfy the following more complex relations

$$\begin{aligned}
N_{1|2,3,4,5,6,7}(\ell) - N^{a_7=1}_{1|7,2,3,4,5,6}(\ell) - N_{[7,1]|2,3,4,5,6}(\ell) &= 0 \,, \\
N_{1|2,3,4,5,7,6}(\ell) - N^{a_6=1}_{1|6,2,3,4,5,7}(\ell) - N_{[1,6]|2,3,4,5,7}(\ell) &= 0 \,, \\
N_{[7,1]|2,3,4,5,6}(\ell) - N^{a_6=1}_{[7,1]|6,2,3,4,5}(\ell) - N_{[6,[7,1]]|2,3,4,5} &= 0 \,, \\
N_{[6,[7,1]]|2,3,4,5}(\ell) - N^{a_5=1}_{[6,[7,1]]|5,2,3,4}(\ell) - N_{[5,[6,[7,1]]]|2,3,4}(\ell) &= 0 \,, \\
N^{a_6=1}_{[1,6]|2,3,4,5,7}(\ell) + N_{[6,1]|2,3,4,5,7} &= 0 \,, \\
N_{[7,1]|2,3,4,5,6}(\ell) + N^{a_7=1}_{[1,7]|2,3,4,5,6} &= 0 \,.
\end{aligned} \tag{3.36}$$

Though this is not an exhaustive test, we hope that it is sufficient to serve as a proof of concept that this method work, and that it should always be possible to rearrange the refined terms to satisfy the color-kinematics duality.

3.4. Supergravity amplitudes and the double copy

One of the goals in obtaining a parameterization of gauge theory 1-loop integrands that satisfies the color-kinematics duality is to construct corresponding supergravity integrands via the double-copy construction [2]. For five points this construction was carried out explicitly in four dimensions in [24] while the ten-dimensional analysis using pure spinor superspace was done in [1]. In the pure spinor superspace setup, the supergravity integrand obtained via the double copy must be checked to be BRST invariant, as that guarantees gauge and supersymmetry invariance of its component expression in terms of polarizations and momenta [5].

We will now repeat the five-point supergravity construction of [1] to highlight that it is BRST invariant but that it is so only because the numerators satisfy the dihedral symmetries of the cubic graphs in the cohomology of pure spinor superspace (see [24] for a discussion of these symmetries). While at five points our numerators satisfy these symmetries in addition to the color Jacobi identities, the corresponding symmetries at six points are not satisfied by our BCJ-satisfying six-point numerators and will prevent the double-copy construction of a BRST-closed supergravity integrand. Applying the double-copy procedure at six points will be left for a future work.

3.4.1. The five-point supergravity integrand

Let us construct the five-point supergravity integrand using the double-copy procedure to highlight the existence of a subtlety: the consistency of the double-copy construction requires the five-point numerators not only to satisfy the kinematic Jacobi identities but also the dihedral symmetries of the cubic graphs. We will see that these symmetries, unlike the kinematic Jacobi identities, are satisfied in the cohomology rather than identically.

Starting with the color-dressed integrand (E.1) we replace the color factors by an extra copy of duality-satisfying kinematic numerators. This yields

$$\begin{aligned} M_5(\ell) = \Big(&\frac{1}{2}\mathcal{N}_{1|2,3,45}I_{1,2,3,45}\tilde{N}_{1|2,3,45} + \frac{1}{2}\mathcal{N}_{1|2,34,5}I_{1,2,34,5}\tilde{N}_{1|2,34,5} \\ &+ \frac{1}{2}\mathcal{N}_{1|23,4,5}I_{1,23,4,5}\tilde{N}_{1|23,4,5} + \frac{1}{2}\mathcal{N}_{12|3,4,5}I_{12,3,4,5}\tilde{N}_{12|3,4,5} \\ &+ \frac{1}{2}\mathcal{N}_{51|2,3,4}I_{51,2,3,4}\tilde{N}_{51|2,3,4} + \mathcal{N}_{1|2,3,4,5}(\ell)I_{1,2,3,4,5}\tilde{N}_{1|2,3,4,5}(\ell) + \mathrm{perm}(2,3,4,5)\Big) \end{aligned} \tag{3.37}$$

Note that the kinematic numerators on the left are written in terms of Berends-Giele numerators $\mathcal{N}$ of the appendix A while those on the right are the local numerators N.

After setting up the double-copy supergravity integrand (3.37) we must check its BRST variation. Since (3.37) is left/right symmetric[9] it is enough to consider the left-moving BRST variation, which we will see vanishes only if the right-movers are in the cohomology of the right-moving pure spinor superspace. To see this surprising fact, consider the variation of the left-moving pentagon $N_{1|2,3,4,5}(\ell)$ multiplied by the loop-momentum integrand $I_{1,2,3,4,5}$:

$$\begin{aligned} QN_{1|2,3,4,5}(\ell)I_{1,2,3,4,5} &= \frac{1}{2}V_1V_2T_{3,4,5}\big[(\ell-k_{12})^2-(\ell-k_1)^2\big]I_{1,2,3,4,5} \\ &+ \frac{1}{2}V_1V_3T_{2,4,5}\big[(\ell-k_{123})^2-(\ell-k_{12})^2\big]I_{1,2,3,4,5} \\ &+ \frac{1}{2}V_1V_4T_{2,3,5}\big[(\ell-k_{1234})^2-(\ell-k_{123})^2\big]I_{1,2,3,4,5} \\ &+ \frac{1}{2}V_1V_5T_{2,3,4}\big[\ell^2-(\ell-k_{1234})^2\big]I_{1,2,3,4,5} \\ &= \frac{1}{2}V_1V_2T_{3,4,5}\big[I_{1,23,4,5}-I_{12,3,4,5}\big]+\frac{1}{2}V_1V_3T_{2,4,5}\big[I_{1,2,34,5}-I_{1,23,4,5}\big] \\ &+ \frac{1}{2}V_1V_4T_{2,3,5}\big[I_{1,2,3,45}-I_{1,2,34,5}\big]+\frac{1}{2}V_1V_5T_{2,3,4}\big[I_{1,2,3,4}-I_{1,2,3,45}\big] \end{aligned} \tag{3.38}$$

where we used identities such as $(\ell-k_1)^2I_{1,2,3,4,5} = I_{12,3,4,5}$ that follow from (2.6). These loop-momentum identities are trivial but one of them on the last line, namely $\ell^2I_{1,2,3,4,5} = I_{1,2,3,4}$, has a peculiar behavior: the right-hand side has no label 5. This seemingly innocuous fact will have a surprising implication in the double-copy construction of the five-point supergravity integrand when (3.38) appears multiplied by a right-moving factor $\tilde{N}_{1|2,3,4,5}(\ell)$.

The reason is that the right-moving pentagon $\tilde{N}_{1|2,3,4,5}(\ell)$ depends on the loop momentum and picks up the shift[10] $\ell\to\ell-k_5$ needed when rewriting $I_{1,2,3,4}\to I_{51,2,3,4}$. More explicitly, one can show that the BRST variation of (3.37) contains

$$\begin{aligned} QM_5(\ell) &= \ldots+\frac{1}{2}V_1V_5T_{2,3,4}\big[I_{1,2,3,4}\tilde{N}_{1|2,3,4,5}(\ell)+I_{51,2,3,4}(\tilde{N}_{15|2,3,4}-\tilde{N}_{1|5,2,3,4}(\ell))\big] \\ &= \ldots+\frac{1}{2}V_1V_5T_{2,3,4}\,I_{51,2,3,4}\big[\tilde{N}_{15|2,3,4}+\tilde{N}_{1|2,3,4,5}(\ell-k_5)-\tilde{N}_{1|5,2,3,4}(\ell)\big]. \end{aligned} \tag{3.39}$$

---

[9] The left- or right-moving terminology refers to the two sides of the double-copy kinematic factors, distinguished by the tildes.

[10] In the gauge-theory integrand the term $V_1V_5T_{2,3,4}I_{1,2,3,4}$ from the last line of (3.38) can be trivially rewritten as $V_1V_5T_{2,3,4}I_{51,2,3,4}$ since its kinematic factor is invariant under the shift $\ell\to\ell-k_5$.

On the one hand we know from section 3 that the kinematic Jacobi identity

$$\tilde{N}_{5|1,2,3,4}(\ell) - \tilde{N}_{1|5,2,3,4}(\ell) + \tilde{N}_{15|2,3,4} = 0\,. \tag{3.40}$$

is satisfied[11]. Therefore the vanishing of the left-moving BRST variation (3.39) hinges on the dihedral symmetry of the pentagon $\tilde{N}_{1|2,3,4,5}(\ell - k_5) = \tilde{N}_{5|1,2,3,4}(\ell)$. One can show that this symmetry is satisfied *in the cohomology* of the right-moving pure spinor superspace given by the pure spinor bracket

$$\langle \tilde{N}_{1|2,3,4,5}(\ell - k_5)\rangle = \langle \tilde{N}_{5|1,2,3,4}(\ell)\rangle\,, \tag{3.41}$$

where we emphasize that the above would not be true in terms of superfields, i.e. without the pure spinor brackets. To see this we use the numerators obtained from the field-theory limits to get that $\langle \tilde{N}_{5|1,2,3,4}(\ell) - \tilde{N}_{1|2,3,4,5}(\ell - k_5)\rangle$ is given by

$$\langle \tilde{V}_1 \tilde{T}^m_{2,3,4,5} k^m_5 + \tilde{V}_{51}\tilde{T}_{2,3,4} + \big[\tilde{V}_1 \tilde{T}_{52,3,4} + (2 \leftrightarrow 3,4,5)\big]\rangle = 0\,, \tag{3.42}$$

as can be seen using the cohomology identity (1.19).

To summarize, the five-point supergravity integrand is BRST invariant. But there is a subtlety: the double-copy construction seems to require more than just the kinematic Jacobi identities, the numerators must also satisfy the dihedral symmetries of the cubic graphs[12] (which are satisfied in the cohomology of the right-movers).

3.4.2. Six-point double copy and automorphism symmetries

At six points a naive application of the double-copy procedure with BCJ-satisfying numerators obtained in the previous sections does not produce a consistent supergravity integrand: it fails to be BRST invariant in pure spinor superspace. This happens because the numerators, even though they satisfy the color-kinematics duality they do not satisfy the automorphism symmetries of their associated cubic graphs.

To see this it is enough to use the BCJ-satisfying six-point numerators in a tentative double-copy construction to obtain, among many others, the following terms under a left-moving BRST variation $QM_6(\ell)$,

$$\begin{aligned}
&-\frac{1}{4s_{23}} V_1 V_{23} T_{4,5,6}\Big(I_{123,4,5,6}\tilde{N}_{1|23,4,5,6}(\ell) - I_{1,4,5,6}\tilde{N}_{1|4,5,6,23}(\ell) - I_{123,4,5,6}\tilde{N}_{[1,23]|4,5,6}\Big)\\
&= -\frac{1}{4s_{23}} V_1 V_{23} T_{4,5,6} I_{123,4,5,6}\Big(\tilde{N}_{1|23,4,5,6}(\ell) - \tilde{N}_{1|4,5,6,23}(\ell - k_{23}) + \tilde{N}_{231|4,5,6}\Big)\,.
\end{aligned} \tag{3.43}$$

[11] These numerators are readily available to download from [28].

[12] At tree level for the double copy construction of supergravity amplitudes to be BRST invariant it is enough for the numerators to satisfy the kinematic Jacobi identities

Similarly as described in (3.38) at five points, the missing labels in $I_{1,4,5,5}$ arise from loop-momentum cancellations in $QN_{1|4,5,6,23}(\ell)I_{1,4,5,6,23}$. This is compensated by the shift $\ell \to \ell - k_{23}$ which is picked up by the right-moving pentagon in the second line. If the condition $\tilde{N}_{1|4,5,6,23}(\ell - k_{23}) = \tilde{N}_{23|1,4,5,6}(\ell)$ for the automorphism symmetry of the pentagon was satisfied then the terms (3.43) would vanish identically since

$$\tilde{N}_{1|23,4,5,6}(\ell) - \tilde{N}_{23|1,4,5,6}(\ell) + \tilde{N}_{231|4,5,6} = 0\,, \tag{3.44}$$

as can be verified using the numerators available to download from [28]. Unfortunately it is not true that $\tilde{N}_{1|4,5,6,23}(\ell - k_{23}) = \tilde{N}_{23|1,4,5,6}(\ell)$ and, unlike the case at five points, this is not true even in the cohomology[13],

$$\langle \tilde{N}_{1|4,5,6,23}(\ell - k_{23})\rangle \neq \langle \tilde{N}_{23|1,4,5,6}(\ell)\rangle\,. \tag{3.45}$$

Therefore the naive application of the double-copy construction at six points is not consistent even though the numerators satisfy the color-kinematics duality.

It is interesting to observe that the automorphism symmetries of the graphs encoded in the loop momentum shifts $\ell + a_i k_i$ are satisfied by the numerators from the integrands with shifted loop momentum $A(\sigma; \ell + a_i k_i)$. In the case of (3.45) we have the identity (valid at the superfield level)

$$N^{a_2=-1,a_3=-1}_{1|4,5,6,23}(\ell) = N_{23|1,4,5,6}(\ell), \tag{3.46}$$

where the numerator on the left-hand side is the 23-pentagon from the amplitude with shifted loop momentum, $A(1,4,5,6,2,3;\ell - k_{23})$. This integrand is computed with the field-theory limits of section 2.3 with $a_2 = a_3 = -1$ corresponding to the shifted loop momentum $\ell - k_2 - k_3$. Unfortunately it is not clear how to use these numerators directly as functions of ℓ rather than as functions of the shift parameters a_i.

[13] Note that the last line of (3.43) is identical (apart from the left/right-moving nature of the numerators) to the BCJ-triplet failure in the representation of [1], given in equation (6.12) of that reference. Unlike the representation of [1], the six-point integrand of gauge theory found here satisfies all BCJ relations for the left- and right-moving numerators. However, once terms in the left-moving BRST variation are collected we see that the BCJ failure of [1] in the left-moving sector appears here as a failure in the right-moving sector due to a shift of the loop momentum.

3.4.3. Comments on the double-copy construction in pure spinor superspace

The failure of the automorphism symmetry (3.45) for the 23-pentagon is a contact term in s_{23} after its component expansion is evaluated through the pure spinor bracket, that is $\langle \tilde{N}_{1|4,5,6,23}(\ell - k_{23})\rangle - \tilde{N}_{23|1,4,5,6}(\ell)\rangle \sim s_{23}(\ldots)$. In pure spinor superspace we have

$$N_{23|1,4,5,6}(\ell) - N_{1|4,5,6,23}(\ell - k_{23}) = k_{23}^m V_1 T^m_{23,4,5,6} + V_{231} T_{4,5,6} + \big[V_1 T_{234,5,6} + 4 \leftrightarrow 5, 6\big] \tag{3.47}$$

which represents the same failure to satisfy the color-kinematics duality as pointed out in equation (6.12) of [1].

Since the issue with missing labels in the loop momentum integral as a result of a BRST variation will always be present for the BCJ-satisfying numerators obtained in this work, solving this problem seems to require a different approach to the double-copy construction in the pure spinor superspace context. Given that the failures are purely contact terms, the generalized double-copy prescription of [36] may be applicable[14] and it will be interesting to see how BRST invariance is restored. It is reasonable to speculate that the deformations of the right-moving BCJ triplets by contact terms as a result of loop momentum shifts due to canceled loop propagators in the left-moving BRST variation may be a generic feature of the double copy in pure spinor superspace. This characteristic may be especially important at higher loops. We plan to investigate this problem in future work.

We note that supergravity integrands have been constructed using BCJ numerators in four dimensions for up to seven points in [39] and to all multiplicity in [20] using spinor helicity in the MHV sector. Supergravity amplitudes were also constructed in [40] but using a partial-fraction representation of the loop momentum integrands.

4. Conclusion

In this work we obtained a set of field-theory limit rules for the Kronecker-Eisenstein coefficient functions present in the genus-one superstring correlators derived in [9,10,11]. Using these rules we found local numerators for ten-dimensional SYM integrands at one loop for five, six and seven points that satisfy the BCJ color-kinematics duality. These results resolve the difficulties in an earlier analysis of the six-point SYM integrands which did not satisfy the color-kinematics duality [1].

[14] We thank Oliver Schlotterer for discussions on this point.

These field-theory limits have an special affinity with the pure spinor superspace representation of the superstring correlators. They take into account arbitrary choices in the parameterization of the loop momentum integrands, shuffling terms among various numerators preserving BRST invariance of the SYM one-loop integrands while changing the BRST properties of individual numerators in a non-trivial way, see the discussion around (3.24). The prescription to find the field-theory limit of the correlator whose parameterization contains shifts of the loop momentum by arbitrary linear combinations of external particle momenta is crucial in demonstrating all the BCJ color-kinematic identities of our ten-dimensional SYM representation.

However, in attempting to use the BCJ-satisfying six-point numerators in a double-copy construction of the supergravity integrand we learned that the numerators must satisfy, in addition to the kinematic Jacobi identities, also the various graph automorphism symmetries in order for the supergravity integrand to be BRST invariant. Unfortunately our six-point numerators viewed as functions of the loop momentum (rather than as the numerators from integrands with general loop momentum as described at the end of section 3.4.2) do not satisfy these symmetries and the double-copy construction initiated here remains incomplete. However, the contact-term nature of the automorphism symmetry failure indicates that the generalized double-copy prescription of [36] may resolve this. We defer the full analysis of this problem to future work.

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Appendix A. Conventions

In this appendix we briefly summarize some of the conventions used in the main text.

Sums over deconcatenations are denoted by $\sum_{A_1 \ldots A_n = a_1 \ldots a_m}$ They represent the sum over all possible ways of generating n words from $a_1 \ldots a_m$, while maintaining the order. These words may be empty, but often when they are the terms being summed over will be zero. So, to give an example, the sum $\sum_{ABC=12}$ denotes the sum over six cases; three of them are where two of A, B and C are empty and the third is 12, and the other three are where $A = 1, B = 2, C = \emptyset$, $A = 1, B = \emptyset, C = 2$, and $A = \emptyset, B = 1, C = 2$.

Another notation commonly used is

$$(terms) + (a_1, ..., a_m|N_1, ..., N_n), \qquad m \leq n. \tag{A.1}$$

This notation works means a sum over all possible ways of replacing $a_1, ..., a_n$ in the terms with n terms from the ordered list $N_1, ..., N_n$. Further generalizations of this follow naturally, with (terms) + $(a_1, ..., a_{m_1}|b_1, ..., b_{m_2}|N_1, ..., N_n)$ meaning sum over all ways of generating two ordered lists from $N_1, ..., N_n$, one of length m_1, one of length m_2, and substituting them in for $a_1, ..., a_{m_1}$ and $b_1, ..., b_{m_2}$. For example, in $V_{[1,23]}T_{[4,56],7,8} +$ $(23|4, 56|23, 4, 56, 7, 8)$ possible terms are $V_{[1,4]}T_{[23,56],7,8}$ and $V_{[1,23]}T_{[7,8],4,56}$, but not $V_{[1,23]}T_{[8,7],4,56}$ as the latter would violate the ordering constraint.

Another summation notation to note is

$$(terms) + [1...n|A_1, ..., A_m], \qquad m \leq n. \tag{A.2}$$

This denotes the sum over A_1, ..., A_m all possible Stirling cycle permutations constructed from $1, ..., n$ [11]. This means that you take the set of numbers 1, ..., n, and construct all possible permutation cycles from it, select those involving m brackets, and canonicalise by having the first term in each cycle be its lowest element, and the cycles ordered by their lowest elements. Each cycle is then substituted in for an A. For example, consider the sum $+[1234567|A_1, ..., A_4]$. One possible permutation of $1, \ldots, 5$ involving 4 brackets would be (12)(64)(3)(57), which swaps 1 with 2, 6 with 4, and 5 with 7. We then begin canonicalising by using that permutation cycles have cyclic symmetry to rewrite this as (12)(46)(3)(57), and then order the cycles by their lowest values, (12)(3)(46)(57). Hence, one term in this sum would set $A_1 = 12$, $A_2 = 3$, $A_3 = 46$, $A_4 = 57$. These sums may be thought of as being $A_1 = 1$ followed by any terms from $2...n$ in any order, then A_2 is the next lowest value left followed by any possible set of values in any order from the numbers left, and so on. So in the above example, $A_1 = 15$ would be a possible term, which would mean A_2 starting with a 2 and so it could be $A_2 = 23$, then A_3 starts with a 4 and so we could have $A_3 = 4$, and then finally A_4 follows the same rules and uses up all remaining letters, so $A_4 = 67$.

A.0.1. Lie algebra notation and Berends-Giele currents

We frequently use the notation of words and Lie brackets, especially when indexing SYM multiparticle superfields, see the discussion on section 3 of [9]. In any situation where a Lie bracket would be expected but a word A is found instead, this should be regarded as being the left-to-right Dynkin bracket $\ell(A)$ [41],

$$\ell(a_1 ... a_n) \equiv [[...[[a_1, a_2], a_3]...], a_n] \,. \tag{A.3}$$

For example, $[[[1, 23], 45], 678]$ is interpreted as $[[[1, [2, 3]], [4, 5]], [[6, 7], 8]]$ and vice-versa.

A mapping from words to Lie brackets which will be particularly useful is the b-map defined by [42][15]

$$b(i) = i, \qquad b(P) = \frac{1}{2s_P} \sum_{XY=P} [b(X), b(Y)]. \tag{A.4}$$

For example, $b(12) = \frac{1}{2s_{12}}[1, 2]$, and $b(123) = \frac{1}{4s_{12}s_{123}}[[1, 2], 3] + \frac{1}{4s_{23}s_{123}}[1, [2, 3]]$.

Superfields are described in terms of two broad classes of objects. The first are local and denoted by V, T, J, and N. The composition of the first three of these objects can be found in more detail in [3,22]. The fourth will be used to refer to amplitude numerators and are detailed on a case by case basis. These objects have a number of slots for indices labelling their superfield contents, and all such indices will be Lie brackets. The second class of objects are Berends-Giele (BG) currents. These are related to the local objects previously described through the use of the b-map on each of their blocks of indices. The BG current of particular use to us is denoted by $\mathcal{N}$, defined in terms of local objects N as

$$\mathcal{N}_{A_1|A_2,\ldots,A_m}(\ell) \equiv N^{(m)}_{b(A_1)|b(A_2),\ldots,b(A_m)}(\ell) \tag{A.5}$$

For example, a seven-point box Berends-Giele numerator is expanded as

$$\begin{aligned}\mathcal{N}_{1|23,456,7}(\ell) &= N_{b(1)|b(23),b(456),b(7)}(\ell) \\ &= \frac{1}{s_{23}s_{456}} \left(\frac{1}{s_{45}} N_{1|[2,3],[[4,5],6],7}(\ell) + \frac{1}{s_{56}} N_{1|[2,3],[4,[5,6]],7}(\ell) \right).\end{aligned}$$

It should be noted that generalized Mandelstam invariants are defined with a $\frac{1}{2}$ factor,

$$s_{i_1 \ldots i_n} \equiv \frac{1}{2}(k^m_{i_1} + \ldots + k^m_{i_n})^2 = \sum_{1 \le a < b \le n} k_{i_a} \cdot k_{i_b} \tag{A.7}$$

[15] Note the extra factor of $\frac{1}{2}$ in (A.4) compared to the definition in [42]. This convention leads to local BCJ numerators which are correctly normalized.

Appendix B. Cyclic symmetry of the field-theory limit rules

In this appendix we will show that the definitions for the field theory limits we have given yield the cyclic symmetry relations seen in (2.25)

$$A(1,2,...,n;\ell+\Sigma_i a_i k_i) = A(2,3,...,n,1;\ell-k_1+\Sigma_i a_i k_i) \tag{B.1}$$

We refer to terms from $A(1,2,...,n;\ell+\Sigma_i a_i k_i)$ with a (I), and $A(2,3,...,n,1;\ell-k_1+\Sigma_i a_i k_i)$ with a (II).

First, we compare their $b_{ij}^{(p)}$ terms. We restrict ourselves to the limit of a single Kronecker-Eisenstein coefficient function, as the limits of their products are the natural generalization of this and will follow accordingly. Referring to (2.14), and using the notation $a_{ji} := a_j - a_i$, we see that they differ by

$$\begin{aligned} b_{ij}^{I\ (p)} - b_{ij}^{II\ (p)} &= \sum_{m=0}^{p} \Big(\left(\text{sgn}_{ij}^{12\ldots n}\right)^m \frac{B_m a_{ji}^{p-m}}{m!(p-m)!} - \left(\text{sgn}_{ij}^{23\ldots n1}\right)^m \frac{B_m (a_{ji}+\delta_{j1}-\delta_{i1})^{p-m}}{m!(p-m)!} \Big) \\ &= \sum_{m=0}^{p} \frac{B_m}{m!(p-m)!} \Big(\left(\text{sgn}_{ij}^{12\ldots n}\right)^m a_{ji}^{p-m} - \left(\text{sgn}_{ij}^{23\ldots n1}\right)^m (a_{ji}-\delta_{j1}+\delta_{i1})^{p-m} \Big) \end{aligned} \tag{B.2}$$

Clearly in all cases where neither of i or j is 1 this vanishes. If we suppose $i=1$, the first *sgn* function is 1, and the second is -1. Hence this difference becomes

$$b_{ij}^{I\ (p)} - b_{ij}^{II\ (p)} = \sum_{m=0}^{p} \frac{B_m}{m!(p-m)!} \Big(a_{j1}^{p-m} - (-1)^m (a_{j1}+1)^{p-m} \Big) \tag{B.3}$$

This can be verified to vanish on a case by case basis with relative ease. Taking for instance the $p=3$ case, we have

$$\begin{aligned} b_{ij}^{I\ (3)} - b_{ij}^{II\ (3)} &= \frac{B_0}{6}\Big(a_{j1}^3 - (-1)^0 (a_{j1}+1)^3\Big) + \frac{B_1}{2}\Big(a_{j1}^2 - (-1)^1 (a_{j1}+1)^2\Big) \\ &+ \frac{B_2}{2}\Big(a_{j1}^1 - (-1)^2 (a_{j1}+1)^1\Big) + \frac{B_3}{6}\Big(a_{j1}^0 - (-1)^3 (a_{j1}+1)^0\Big) \\ &= \frac{1}{6}\Big(a_{j1}^3 - a_{j1}^3 - 3a_{j1}^2 - 3a_{j1} - 1\Big) + \frac{1}{4}\Big(a_{j1}^2 + a_{j1}^2 + 2a_{j1} + 1\Big) \\ &+ \frac{1}{12}\Big(a_{j1} - a_{j1} - 1\Big) + 0 = 0 \end{aligned} \tag{B.4}$$

With the aid of FORM [43] we have verified that this vanishes in at least the first 700 cases. Similar will hold if we instead take $j=1$. Hence, the b part of the field theory limits matches in both representations.

Then, we move onto the c piece. This difference is given by

$$c_{ij}^{I\,(p)} - c_{ij}^{II\,(p)} = \frac{1}{2(p-1)!}\Big(\big(a_{ji} + \mathrm{sgn}_{ij}^{12\ldots n}\mathrm{dist}_4^{12\ldots n}(i,j)\big)^{p-1} - \big(a_{ji} - \delta_{j1} + \delta_{i1} + \mathrm{sgn}_{ij}^{23\ldots n1}\mathrm{dist}_4^{23\ldots n1}(i,j)\big)^{p-1}\Big) \tag{B.5}$$

Again, we need only consider the cases where one of i and j is 1. If we take $i = 1$ we get

$$c_{ij}^{I\,(p)} - c_{ij}^{II\,(p)} = \frac{\big(a_{j1} + \mathrm{dist}_4^{12\ldots n}(1,j)\big)^{p-1} - \big(a_{j1} + 1 - \mathrm{dist}_4^{23\ldots n1}(1,j)\big)^{p-1}}{2(p-1)!} \tag{B.6}$$

We now consider the two pieces of the numerator, and see that these are given by

$$(a_{j1} + \mathrm{dist}_4^{12\ldots n}(1,j))^{p-1} = \begin{cases} a_{j1}^{p-1} & j \leq 4 \\ (a_{j1}+1)^{p-1} & j > 4 \end{cases}, \tag{B.7}$$
$$(a_{j1} + 1 - \mathrm{dist}_4^{23\ldots n1}(1,j))^{p-1} = \begin{cases} a_{j1}^{p-1} & j \leq n-2 \\ (a_{j1}+1)^{p-1} & j > n-2 \end{cases}.$$

When $n = 4, 5$, the only Kronecker-Eisenstein functions in amplitudes is $g_{ij}^{(1)}$, and we see that setting $p = 1$ in the above gives equivalence. When $n = 6$, these coincide in that $n - 2 = 4$. When $n = 7$ and $p > 1$, they differ when $j = 5$. However, this disagreement will not matter. At 7 points a term $g_{15}^{(2+)}$ is multiplied by at most one other $g_{ij}^{(q)}$ function, but we need at least two Kronecker-Eisenstein coefficient functions in order to make the corresponding P function non-zero. That is, for example,

$$g_{15}^{(2)}g_{56}^{(1)} \Rightarrow P(15,56) = \phi_{156|5671}I_{156} = 0\,, \tag{B.8}$$
$$g_{15}^{(2)}g_{56}^{(1)}g_{67}^{(1)} \Rightarrow P(15,56,67) = \phi_{1567|5671}I_{5671} \neq 0\,.$$

At 8 points, this will of course become an issue. However, the description of the *dist* function was chosen purely for simplicity. If we instead think of this function as asking whether the pole being approached crosses the boundary between particles n and 1, then consistency should be maintained to higher points.

Appendix C. The field-theory limit at higher points

We anticipate that the field theory limit rules for an arbitrary product of $g_{ij}^{(n)}$ functions should generalize in the natural way

$$\prod_{a=1}^{n} g_{i_a j_a}^{(p_a)} \to \sum_{A\in\mathcal{P}(12\ldots n)}\Bigg(\Big(\prod_{a\in A} b_{i_a j_a}^{(p_a)}\Big)\Big(\prod_{b\in A^c} c_{i_b j_b}^{(p_b)}\Big)P(i_{B_1}j_{B_1},\ldots,i_{B_{|B|}}j_{B_{|B|}})\Bigg) \tag{C.1}$$

where $\mathcal{P}(12...n)$ denotes the power set of $12...n$, A is an element of this, and A^c its complement. We stress that the indices of the $c^{(p)}$ and those in the P function are identical.

The general P functions will be as in (2.13), with $P(i_1j_1,...,i_nj_n)$ chaining together i_mj_m pairs as much as possible, and then using these as indices for ϕ and I functions. So, for instance, we would expect

$$\begin{aligned} P(12,23,34,45,56,67) &\leftrightarrow \hat{\phi}(\sigma|1234567)I_{1234567} \\ P(15,32,56,24) &\leftrightarrow \hat{\phi}(\sigma|156)\hat{\phi}(\sigma|324)I_{156,324} \end{aligned} \tag{C.2}$$

As for the limits of $b^{(p)}$ and $c^{(p)}$ at higher points, these we expect will generalize from (2.10) in the natural way. As evidence of this, we look to the Fay identity for $g_{12}^{(n)}g_{23}^{(1)}$

$$g_{12}^{(n)}g_{23}^{(1)} = -g_{13}^{(n+1)} + g_{13}^{(1)}g_{12}^{(n)} - ng_{12}^{(n+1)} + \sum_{j=0}^{n}(-1)^j g_{13}^{(n-j)}g_{23}^{(1+j)} \,. \tag{C.3}$$

We begin by looking at $b^{(n)}$, and restrict ourselves to the case $a_i = 0 \;\forall\, i$ initially. In these circumstances we know that $b_{ij}^{(1)} = \frac{1}{2}\mathrm{sgn}_{ij}^{12\cdots n}$, and we would expect the general order $b_{ij}^{(n)}$ to depend only upon the order of i and j with respect to the color ordering. Hence, we substitute into (C.3) the values

$$g_{13}^{(1)}, g_{23}^{(1)} \to \frac{1}{2}\,, \qquad g_{12}^{(n)}, g_{13}^{(n)}, g_{23}^{(n)} \to b^{(n)}\,. \tag{C.4}$$

Upon rearranging this gives us the recursion relation

$$b^{(n+1)} = -\frac{1}{n+1-(-1)^n}\sum_{j=1}^{n}(-1)^j b^{(n-j+1)}b^{(j)}\,. \tag{C.5}$$

This can be seen to vanish for n even, $n > 0$, by virtue of the symmetry in the gg terms and the antisymmetry of the $(-1)^j$. For n odd, it simplifies to

$$b^{(2n)} = -\frac{1}{2n+1}\sum_{j=1}^{2n-1}(-1)^j b^{(2n-j)}b^{(j)} = -\frac{1}{2n+1}\sum_{j=1}^{n-1} b^{(2n-2j)}b^{(2j)}\,, \tag{C.6}$$

where the second equality follows from the vanishing of the b with odd indices. It may then be proved by induction that this is solved by

$$b^{(n)} = \frac{B_n}{n!}\,, \tag{C.7}$$

where B_n is the n^{th} Bernoulli number. Showing this requires an identity due to Euler [44],

$$\sum_{k=1}^{n-1} \binom{2n}{2k} B_{2k} B_{2n-2k} = -(2n+1) B_{2n}\,, \quad n \geq 2\,. \tag{C.8}$$

Hence, we speculate that when $a_i = 0\ \forall i$, the field theory limit of a general term from the Kronecker-Eisenstein series away from poles is given by (C.7). The first few (non-zero) values are

$$\begin{aligned} &b^{(0)} = 1\,, \quad b^{(1)} = \frac{1}{2}\,, \quad b^{(2)} = \frac{1}{12}\,, \quad b^{(4)} = -\frac{1}{720}\,, \quad b^{(6)} = \frac{1}{30240}\,, \\ &b^{(8)} = -\frac{1}{1209600}\,, \quad b^{(10)} = \frac{1}{47900160}\,, \quad b^{(12)} = -\frac{691}{1307674368000}\,. \end{aligned} \tag{C.9}$$

We can then extend this to the general a_i case, though with less elegance. If instead of making the substitution (C.4) into (C.3), we instead use the general a_i values of the $b^{(1)}$ terms, we find the relation

$$\begin{aligned} \left(\frac{1}{2} + a_3 - a_2\right) b_{12}^{(n)} = -b_{13}^{(n+1)} &+ \left(\frac{1}{2} + a_3 - a_1\right) b_{12}^{(n)} - n b_{12}^{(n+1)} \\ &+ \left(\frac{1}{2} + a_3 - a_2\right) b_{13}^{(n)} + \sum_{j=1}^{n} (-1)^j b_{13}^{(n-j)} b_{23}^{(1+j)}\,. \end{aligned} \tag{C.10}$$

This cannot be as easily rearranged into a recursion relation. However, if we assume that $b_{ij}^{(n)}$ is a polynomial in $a_j - a_i$ up to order n, we may use the above to identify the polynomial coefficients. Doing this reveals the value of $b_{ij}^{(4)}$ as would be expected from (2.10) as the unique solution. And then we have verified that the relation above is satisfied in a number of further cases if we assume this general form of $b^{(n)}$.

We can perform a similar exercise for the $c_{ij}^{(n)}$ pole terms. In its current form (C.3) is not the most useful for this, as we would like the *dist* functions to be non-zero. Instead, we suppose the amplitude we are considering is $A(1, 2, ..., m)$ for convenience, and look at an alternative Fay identity,

$$g_{1m}^{(n)} g_{m(m-1)}^{(1)} = -g_{1(m-1)}^{(n+1)} + g_{1(m-1)}^{(1)} g_{1m}^{(n)} - n g_{1m}^{(n+1)} + \sum_{j=0}^{n} (-1)^j g_{1(m-1)}^{(n-j)} g_{m(m-1)}^{(1+j)}\,. \tag{C.11}$$

We need not restrict ourselves to the $a_i = 0\ \forall i$ case here, as the computation is simpler. Looking at the s_{1m} single poles leads us to the relation

$$\begin{aligned} c_{1m}^{(n)} \left(-\frac{1}{2} + a_{m-1} - a_m\right) &= \left(\frac{1}{2} + a_{m-1} - a_1\right) c_{1m}^{(n)} - n c_{1m}^{(n+1)} \\ \Rightarrow \quad c_{1m}^{(n+1)} &= \frac{1}{n} c_{1m}^{(n)} (1 + a_m - a_1) \end{aligned} \tag{C.12}$$

Using that we know $c_{1m}^{(1)} = \frac{1}{2}$, this becomes

$$c_{1m}^{(n)} = \frac{1}{2(n-1)!} \left(1 + a_m - a_1\right)^{n-1} \tag{C.13}$$

This agrees with the known values of $c_{17}^{(2)}$ and $c_{17}^{(3)}$ also. We can also repeat this calculation for poles of $g_{12}^{(n)}$ to see what would happen if the *dist* function were not triggered, and find the similar relation

$$c_{12}^{(n)} = \frac{1}{2(n-1)!} \left(a_2 - a_1\right)^{n-1} \tag{C.14}$$

Hence the form of $c_{ij}^{(n)}$ presented in (2.10) is the natural generalization, and we expect (2.10) to hold to higher points.

We end this discussion though by stressing that this approach is highly speculative, and we have not tested these values produced in any way beyond the aforementioned discussion. They are however a strong candidate for what they are attempting to describe.

Appendix D. The BRST analysis of a seven-point numerator

In this appendix we identify the full expression for the $[5,[6,7]]$-pentagon in the amplitude $A(1,2,3,4,5,6,7;\ell+4k_4-6k_5)$, and confirm that its variation has the desired form. We begin by finding the coefficient of one term contributing to the numerator in detail, namely $V_1T_{2576,3,4}$. Within the string correlator this is associated with the worldsheet function

$$\begin{aligned}\mathcal{Z}_{1,2576,3,4} =& g_{25}^{(1)} g_{57}^{(1)} g_{76}^{(1)} + g_{25}^{(3)} + g_{57}^{(3)} + g_{76}^{(3)} - 2g_{62}^{(3)} + g_{25}^{(1)}(g_{57}^{(2)} + g_{76}^{(2)} - g_{62}^{(2)}) \\ &+ g_{57}^{(1)}(g_{25}^{(2)} + g_{76}^{(2)} - g_{62}^{(2)}) + g_{76}^{(1)}(g_{25}^{(2)} + g_{57}^{(2)} - g_{62}^{(2)})\,. \end{aligned} \tag{D.1}$$

Only two of these terms contain the $s_{67}s_{567}$ pole structure, $g_{25}^{(1)} g_{57}^{(1)} g_{76}^{(1)}$ and $g_{76}^{(1)} g_{57}^{(2)}$. The contribution of the former was identified in (2.23), and the latter follows from (2.11),

$$c_{76}^{(1)} c_{57}^{(2)} = \frac{1}{2} \cdot \frac{6}{2} = \frac{3}{2}\,. \tag{D.2}$$

Summing these together, the $V_1T_{2576,3,4}$ contribution to the $[5,[6,7]]$-pentagon is

$$\left(-\frac{11}{8} + \frac{3}{2}\right) V_1 T_{2576,3,4} \hat{\phi}(1234567|576) I_{567} = -\frac{1}{8 s_{67} s_{567}} V_1 T_{2576,3,4} I_{1,2,3,4,567} \tag{D.3}$$

Similar calculations for all other terms in the correlator yield the numerator expression

$$
\begin{aligned}
N^{a_4=4,a_5=-6}_{1|2,3,4,[5,[6,7]]}(\ell) &= 6V_1T^{mn}_{2,3,4,5,67}k^m_5k^n_{67} + V_1T^m_{2,3,4,[5,67]}(\ell^m - 6k^m_5 + 6k^m_{67}) \\
&- 6\left((V_1T^m_{25,3,4,67}k^m_{67} + (2 \leftrightarrow 3,4)) + V_{15}T^m_{2,3,4,67}k^m_{67} + (5 \leftrightarrow [6,7])\right) \\
&+ \frac{1}{2}\left(V_{12}T_{3,4,[5,67]} + (2 \leftrightarrow 3,4,[5,67])\right) \\
&+ \frac{1}{2}\left(V_1T_{23,4,[5,67]} + (2,3|2,3,4,[5,67])\right) \\
&+ 6\left((V_1T_{25,[3,67],4} + (2,3|2,3,4)) + (2 \leftrightarrow 3)\right) \\
&+ 6\left((V_{15}T_{[2,67],3,4} + (2 \leftrightarrow 3,4)) + (5 \leftrightarrow [6,7])\right) \\
&+ 6\left((V_1T_{2675,3,4} + (2 \leftrightarrow 3,4)) - (6 \leftrightarrow 7)\right) \\
&+ 6\left(V_{1675}T_{2,3,4} - (6 \leftrightarrow 7)\right) + 4\left(V_1T_{24,3,[5,67]} + (2 \leftrightarrow 3)\right) \\
&+ 4V_{14}T_{2,3,[5,67]} - 4V_1T_{2,3,[4,[5,67]]} + 6V_1J^m_{5|2,3,4,6,7}(k^m_6 - k^m_7) \\
&+ 6s_{67}\left((V_1J_{5|27,3,4,6} + (2 \leftrightarrow 3,4,6)) + V_{17}J_{5|2,3,4,5,6} - (6 \leftrightarrow 7)\right)
\end{aligned}
\tag{D.4}
$$

The VJ terms above are those which arise naively by looking to the $s_{67}s_{567}$ poles in the correlator. As discussed previously it may be that they require some rearrangement to be in a BCJ representation, but for illustrating the field theory limit methods we give the numerator in the above form. A lengthy calculation yields the variation

$$
\begin{aligned}
QN^{a_4=4,a_5=-6}_{1|2,3,4,[5,[6,7]]}(\ell) &= \frac{1}{2}V_1V_2T_{3,4,[5,67]}\left((\ell - k_{12} + 4k_4 - 6k_5)^2 - (\ell - k_1 + 4k_4 - 6k_5)^2\right) \\
&+ \frac{1}{2}V_1V_3T_{2,4,[5,67]}\left((\ell - k_{123} + 4k_4 - 6k_5)^2 - (\ell - k_{12} + 4k_4 - 6k_5)^2\right) \\
&+ \frac{1}{2}V_1V_4T_{2,3,[5,67]}\left((\ell - k_{1234} + 4k_4 - 6k_5)^2 - (\ell - k_{123} + 4k_4 - 6k_5)^2\right) \\
&+ \frac{1}{2}V_1V_{[5,67]}T_{2,3,4}\left((\ell - k_{1234567} + 4k_4 - 6k_5)^2 - (\ell - k_{1234} + 4k_4 - 6k_5)^2\right) \\
+(k^6 \cdot k^7)\Bigg(&\left(6V_1V_{26}T^m_{3,4,5,7}k^m_5 + (2 \leftrightarrow 3,4,5)\right) + V_1V_{57}T^m_{2,3,4,6}\left(\ell^m + 6k^m_{57}\right) \\
&+ 6V_1V_7T^{mn}_{2,3,4,5,6}k^m_5k^n_{67} + 6k^m_5(V_1V_6T^m_{27,3,4,5} + (2 \leftrightarrow 3,4,5)) \\
&+ \left(\ell^m + 6k^m_{67}\right)V_1V_7T^m_{2,3,4,56} + 6(V_1V_6T^m_{25,3,4,7}k^m_{67} + (2 \leftrightarrow 3,4)) \\
&+ 6V_{15}V_6T^m_{2,3,4,7}k^m_{67} + 6V_{17}V_6T^m_{2,3,4,5}k^m_5 \\
&+ 6V_1V_5T^{mn}_{2,3,4,6,7}k^m_5k^n_7 + V_{16}V_5T^m_{2,3,4,7}k^m_5 \\
&+ (6V_1V_{25}T^m_{3,4,6,7}k^m_6 + (2 \leftrightarrow 3,4)) + (6V_1V_5T^m_{26,3,4,7}k^m_5 + (2 \leftrightarrow 3,4,7)) \\
&+ \frac{1}{2}(V_1V_{[2,57]}T_{3,4,6} + (2 \leftrightarrow 3,4))
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
&- \frac{1}{2}(V_1V_{26}T_{3,4,57} + (2 \leftrightarrow 3,4)) - \frac{1}{2}(V_1V_{56}T_{23,4,7} + (2,3|2,3,4,7)) \\
&+ \frac{1}{2}(V_1V_7T_{[2,3],4,56} + (2,3|2,3,4,56)) + \frac{1}{2}(V_{12}V_{57}T_{3,4,6} + (2 \leftrightarrow 3,4)) \\
&+ \frac{1}{2}(V_{12}V_7T_{3,4,56} + (2 \leftrightarrow 3,4,56)) - \frac{1}{2}V_{17}V_{56}T_{2,3,4} + \frac{1}{2}V_{175}V_6T_{2,3,4} \\
&+ 6\big((V_1V_{27}T_{[3,5],4,6} + (3 \leftrightarrow 4,6)) + (2 \leftrightarrow 3,4)\big) + 6(V_1V_7T_{[26,5],3,4} + (2 \leftrightarrow 3,4)) \\
&+ 6(V_1V_7T_{25,36,4} + V_1V_7T_{26,35,4} + (2,3|2,3,4)) + 6(V_{15}V_{27}T_{3,4,6} + (2 \leftrightarrow 3,4)) \\
&+ 6(V_{15}V_7T_{26,3,4} + (2 \leftrightarrow 3,4)) + 6(V_{16}V_7T_{25,3,4} + (2 \leftrightarrow 3,4)) \\
&+ 6\big((V_1V_{25}T_{37,4,6} + (3 \leftrightarrow 4,6)) + V_{17}V_{25}T_{3,4,6} + (2 \leftrightarrow 3,4)\big) \\
&+ 6V_{165}V_7T_{2,3,4} + 6(V_1(V_{257} + V_{275})T_{3,4,6} + (2 \leftrightarrow 3)) + 6V_1V_{576}T_{2,3,4} \\
&+ 4(V_1V_{57}T_{24,3,6} + (2 \leftrightarrow 3,6)) + 4V_{14}V_{57}T_{2,3,6} + 4(V_1V_7T_{24,3,56} + (2 \leftrightarrow 3,56)) \\
&+ 4V_{14}V_7T_{2,3,56} + 4V_1V_{46}T_{2,3,57} + 2V_1V_{457}T_{2,3,6} + 20V_1V_{475}T_{2,3,6} \\
&+ 6V_1Y^m_{2,3,4,5,6,7}k^m_7 + 6(V_1Y_{26,3,4,5,7} + (2 \leftrightarrow 3,4,5,7)) + 6V_{16}Y_{2,3,4,5,7} \\
&- (6 \leftrightarrow 7)\Bigg) \\
+(k^5 \cdot k^{67})\Bigg(&\Big(\frac{1}{2}(V_1V_{[2,67]}T_{3,4,5} + V_{12}V_{67}T_{3,4,5} + (2 \leftrightarrow 3,4)) + 4V_1V_5T_{2,3,67} + 4V_{14}V_{67}T_{2,3,5} \\
&+ \frac{1}{2}(V_1V_{67}T_{23,4,5} + (2,3|2,3,4,5)) + 4(V_1V_{67}T_{24,3,5} + (2 \leftrightarrow 3,5)) - (5 \leftrightarrow 67)\Big) \\
&- \frac{1}{2}(V_{15}V_{67}T_{2,3,4} + (5 \leftrightarrow 6,7)) + 6\big((V_1V_25T_{3,4,67} - (25 \leftrightarrow 67)) + (2 \leftrightarrow 3,4)\big) \\
&- 6V_{15}V_{67}T_{2,3,4} - 6V_1Y_{2,3,4,5,67}\Bigg) \\
+6(k^6 \cdot k^7)&(k^5 \cdot k^6)V_1V_5(J_{7|2,3,4,6} + J_{6|2,3,4,7}) - 6(k^5 \cdot k^{67})(k^6 \cdot k^7)V_1V_5J_{7|2,33,4,6}
\end{aligned}
$$

This has intentionally been expressed with factors $(\ell \cdot k)$ reformulated in terms of propagators. For an n-point amplitude in the canonical ordering with arbitrary loop momentum structure, this is done with

$$
\begin{aligned}
(\ell \cdot k_{i(i+1)\ldots j}) = -\frac{1}{2}(\ell + \sum_{m=1}^{n} a_m k_m - k_{12\ldots.j})^2 + \frac{1}{2}(\ell + \sum_{m=1}^{n} a_m k_m - k_{12\ldots.(i-1)})^2 \\
- k_{i(i+1)\ldots j} \cdot \left(\sum_{m=1}^{n} a_m k_m - \frac{1}{2} k_{i(i+1)\ldots j}\right) .
\end{aligned} \tag{D.6}
$$

We may then be reassured of the validity of this numerator expression, as those terms in the variation proportional to propagators cancel terms from other box numerators. For example, one such set of terms is

$$
\begin{aligned}
&V_1 V_3 T_{2,4,[5,67]} \left((\ell - k_{123} + 4k_4 - 6k_5)^2 - (\ell - k_{12} + 4k_4 - 6k_5)^2 \right) I^{a_4=4,a_5=-6}_{1,2,3,4,567} \\
&\qquad = V_1 V_3 T_{2,4,[5,67]} \left(I^{a_4=4,a_5=-6}_{1,2,34,567} - I^{a_4=4,a_5=-6}_{1,23,4,567} \right)
\end{aligned} \tag{D.7}
$$

This then cancels one term in the variation of the $[3,4],[5,[6,7]]$-box, and one from the $[2,3],[5,[6,7]]$ box. Similar holds true for all other terms in the variation, and the remaining terms in (D.5) are canceled themselves by analogous results in the variation of hexagons.

Appendix E. The five-point color-dressed integrand

In this appendix the five-point color-dressed integrand will be written down after the application of the color decomposition techniques of [45].

The five-point color-dressed one-loop integrand can be written as

$$
\begin{aligned}
M_5(\ell) = \Big(&\frac{1}{2} \mathcal{N}_{1|2,3,45} I_{1,2,3,45} B_{1,2,3,45} + \frac{1}{2} \mathcal{N}_{1|2,34,5} I_{1,2,34,5} B_{1,2,34,5} \\
&+ \frac{1}{2} \mathcal{N}_{1|23,4,5} I_{1,23,4,5} B_{1,23,4,5} + \frac{1}{2} \mathcal{N}_{12|3,4,5} I_{12,3,4,5} B_{12,3,4,5} \\
&+ \frac{1}{2} \mathcal{N}_{51|2,3,4} I_{51,2,3,4} B_{51,2,3,4} + \mathcal{N}_{1|2,3,4,5}(\ell) I_{1,2,3,4,5} P_{1,2,3,4,5} + \text{perm}(2,3,4,5) \Big)
\end{aligned} \tag{E.1}
$$

where $\mathcal{N}$ denotes the Berends-Giele counterpart of the n-gon numerator as described in the appendix A while the color factors of the box and pentagon cubic graphs are

$$
B_{12,3,4,5} = f^{a12} f^{eab} f^{b3c} f^{c4d} f^{d5e}, \qquad P_{1,2,3,4,5} = f^{a1b} f^{b2c} f^{c3d} f^{d4e} f^{e5a}\,. \tag{E.2}
$$

The factor of $\frac{1}{2}$ in (E.1) compensates the overcounting of graphs due to symmetries. Note that the box numerators do not depend on the loop momentum.

The color-dressed integrand (E.1) is BRST closed. To see this we expand all color factors in terms of their pentagon constituents using the Jacobi identity as $B_{12,3,4,5} = P_{1,2,3,4,5} - P_{2,1,3,4,5}$ [45] and consider the terms proportional to $P_{1,2,3,4,5}$. Using the five-point numerators of section 2.4.3 these are

$$
\begin{aligned}
M_5(\ell)\Big|_{P_{1,2,3,4,5}} &= \mathcal{N}_{1|2,3,4,5}(\ell) I_{1,2,3,4,5} + \frac{1}{2} \Big(N_{12|3,4,5} I_{12,3,4,5} - \mathcal{N}_{21|3,4,5} I_{1,3,4,5} \\
&+ \big[\mathcal{N}_{1|23,4,5} - \mathcal{N}_{1|32,4,5} \big] I_{1,23,4,5} + \big[\mathcal{N}_{1|2,34,5} - \mathcal{N}_{1|2,43,5} \big] I_{1,2,34,5} \\
&+ \big[\mathcal{N}_{1|2,3,45} - \mathcal{N}_{1|2,3,54} \big] I_{1,2,3,45} + \mathcal{N}_{51|2,3,4} I_{1,2,3,4} - \mathcal{N}_{15|2,3,4} I_{15,2,3,4} \Big)\,.
\end{aligned} \tag{E.3}
$$

After using $N_{ij|k,l,m} = -N_{ji|k,l,m}$ by (2.31) and performing the loop momentum shifts $\ell' = \ell - k_2$ in $I_{1,3,4,5}$ and $\ell' = \ell + k_5$ in $I_{15,2,3,4}$ these terms become the integrand $A(1,2,3,4,5;\ell)$ of (2.30),

$$\begin{aligned} M_5(\ell)\Big|_{P_{1,2,3,4,5}} &= \mathcal{N}_{1|2,3,4,5}(\ell) I_{1,2,3,4,5} + \mathcal{N}_{1|23,4,5} I_{1,23,4,5} + \mathcal{N}_{1|2,34,5} I_{1,2,34,5} \\ &+ \mathcal{N}_{1|2,3,45} I_{1,2,3,45} + \mathcal{N}_{12|3,4,5} I_{12,3,4,5} + \mathcal{N}_{51|3,4,5} I_{1,2,3,4} \,. \end{aligned} \tag{E.4}$$

Hence, after considering all the permutations the color-dressed integrand (E.1) becomes

$$M_5(\ell) = A(1,2,3,4,5;\ell)\, P_{1,2,3,4,5} + \text{perm}(2,3,4,5) \tag{E.5}$$

and it is manifestly BRST closed. The rewriting (E.5) agrees with the general result of [45] (see e.g. equation (3.4) of [46]).

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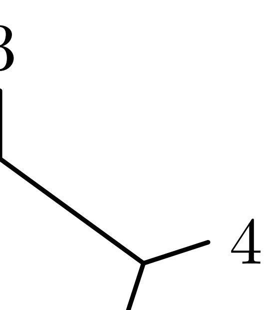
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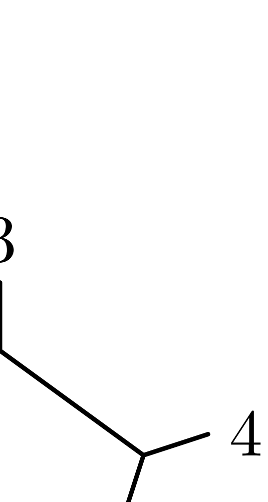
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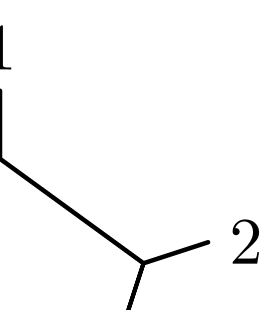
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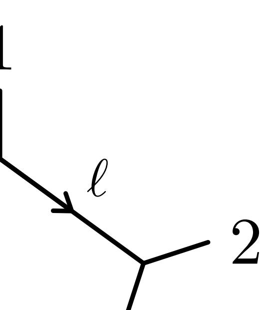

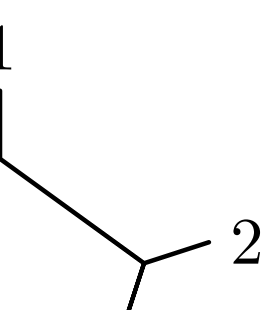

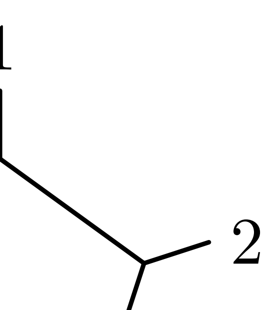
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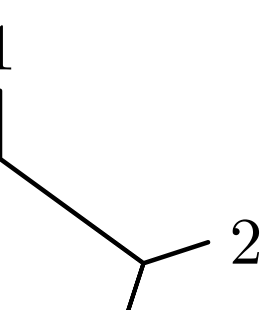
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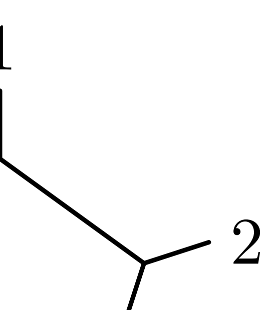

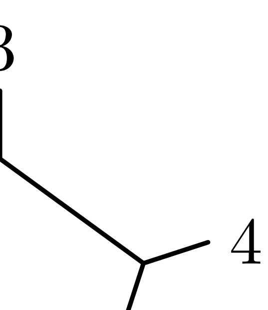
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