



## Toric Objects Associated with the Dodecahedron

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**Abstract.** In this paper we illustrate a tight interplay between homotopy theory and combinatorics within toric topology by explicitly calculating homotopy and combinatorial invariants of toric objects associated with the dodecahedron. In particular, we calculate the cohomology ring of the (complex and real) moment-angle manifolds over the dodecahedron, and of a certain quasitoric manifold and of a related small cover. We finish by studying Massey products in the cohomology ring of moment-angle manifolds over the dodecahedron and how the existence of nontrivial Massey products influences the behaviour of the Poincaré series of the corresponding Pontryagin algebra.

### 1. Introduction

Toric topology studies topological properties of spaces that are equipped with well-behaved toric symmetries, and extend its applications to related areas of topology, geometry, combinatorics, and mathematical physics. The study of these objects focuses on the rich and varied interactions between the combinatorial structure of the orbit spaces and the equivariant topology of the actions. In this paper we survey some homotopy theoretical constructions and methods coming from toric topology and illustrate them on toric objects over the dodecahedron.

We start by recalling a number of key definitions and results from toric topology concerned with the notion of a moment-angle-complex. For an exhaustive source on ideas and methods of toric topology we refer the reader to the fundamental monograph [22].

#### 1.1. Moment-Angle-Complexes and Manifolds

An *abstract simplicial complex*  $K$  on a vertex set  $[m] = \{1, 2, \dots, m\}$  is a collection of subsets of  $[m]$  such that,

- (i) for each  $i \in [m]$ ,  $\{i\} \in [m]$ ,
- (ii) for every  $\sigma \in K$ , if  $\tau \subset \sigma$  then  $\tau \in K$ .

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We assume that  $\emptyset \in K$ . The elements of  $K$  are called *faces*. The maximal faces (under inclusion) are called *facets*. The dimension of a face  $\sigma$  of a simplicial complex  $K$  is defined as  $\dim \sigma = |\sigma| - 1$ , where  $|\sigma|$  denotes the cardinality of  $\sigma$ . The dimension of  $K$ ,  $\dim K$ , is defined as the maximum dimension of the faces of  $K$ . A complex  $K$  is *pure* if all of its facets are of the same dimension. The *i-skeleton*  $sk^i(K)$  of a simplicial complex  $K$  is the collection of all faces of  $K$  of dimension not greater than  $i$ .

For a subset  $J \in [m]$  we define a *full subcomplex*  $K_J$  of the simplicial complex  $K$  as the subcomplex of  $K$  containing all simplices of  $K$  whose vertices lie in  $J$

$$K_J := \{\sigma \in K \mid \sigma \subset J\}.$$

Let  $K$  be a simplicial complex on  $[m]$  and  $\dim K = n - 1$ . Let us denote by  $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$  a collection of topological pairs of CW-complexes. The *polyhedral product* of  $(\underline{X}, \underline{A})$  over  $K$  is a topological space  $\mathcal{Z}_K(\underline{X}, \underline{A}) = (\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} D(\sigma)$  where

$$D(\sigma) = \prod_{i=1}^m Y_i, \quad \text{and} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma \end{cases}$$

By definition,  $D(\emptyset) = A_1 \times \dots \times A_m$ .

In the case  $(\underline{X}, \underline{A}) = \{(D^2, S^1)\}_{i=1}^m$ ,  $\mathcal{Z}_K(\underline{X}, \underline{A})$  is denoted by  $\mathcal{Z}_K$  and it is called *the moment-angle-complex* of  $K$  and in the case  $(\underline{X}, \underline{A}) = \{(D^1, S^0)\}_{i=1}^m$ ,  $\mathcal{Z}_K(\underline{X}, \underline{A})$  is denoted by  $\mathcal{R}_K$  and it is called *the real moment-angle-complex* of  $K$ . The spaces  $\mathcal{Z}_K$  and  $\mathcal{R}_K$  are intensively studied in toric topology, see [22, Chapter 4].

One of the most important classes of simplicial complexes  $K$  is that of triangulated spheres. If  $K$  is a triangulation of a sphere,  $\mathcal{Z}_K$  is a topological  $(m + n)$ -dimensional manifold. In this case much more can be said about topology and geometry of a (real) moment-angle manifold.

In what follows we shall be concerned by a subclass of triangulated spheres that is called *polytopal spheres*. Suppose a simplicial complex  $K$  on the vertex set  $[m]$  is a *polytopal* triangulated sphere, that is,  $K$  is the boundary of a simplicial polytope  $P^*$ , combinatorially dual to a convex simple  $n$ -dimensional polytope  $P$  with  $m$  facets  $F_1, \dots, F_m$ . Thus,  $K$  is a pure simplicial complex of dimension  $\dim K = n - 1$ . Moreover,  $K = \partial P^*$  is simplicially isomorphic (combinatorially equivalent) to the nerve complex  $K_P$  of  $P$  with respect to the closed covering of the boundary of  $P$  by its facets. Thus, combinatorial type of  $K_P$  determines the face poset structure of  $P$ , and vice versa.

For the sake of completeness we recall the classical definition of a simple convex polytope.

**Definition 1.1.** A simple convex  $n$ -dimensional polytope  $P$  in the Euclidean space  $\mathbb{R}^n$  with scalar product  $\langle \cdot, \cdot \rangle$  is a bounded intersection of  $m$  halfspaces

$$P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geq 0 \quad \text{for } i = 1, \dots, m\} \tag{1}$$

where  $\mathbf{a}_i \in \mathbb{R}^n, b_i \in \mathbb{R}$ . We assume that its *facets*

$$F_i = \{\mathbf{x} \in P : \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i = 0\} \quad \text{for } i = 1, \dots, m.$$

are in general position, that is, exactly  $n$  of them meet at a single point. Furthermore, we assume that there are no redundant inequalities in (1), no inequality can be removed from (1) without changing the set  $P$ .

The following construction first appeared in the seminal paper of Davis and Januszkiewicz [29].

**Definition 1.2.** Suppose  $P^n$  is a simple convex polytope with the set of facets  $\mathcal{F} = \{F_1, \dots, F_m\}$ . Denote by  $T^{F_i}$  a 1-dimensional coordinate subgroup in  $T^{\mathcal{F}} \cong T^m$  for each  $1 \leq i \leq m$  and  $T^G = \prod T^{F_i} \subset T^{\mathcal{F}}$  for a face  $G = \cap F_i$  of a polytope  $P^n$ . Then the *moment-angle manifold* over  $P$  is defined as a quotient space

$$\mathcal{Z}_P = T^{\mathcal{F}} \times P^n / \sim,$$

where  $(t_1, p) \sim (t_2, q)$  if and only if  $p = q \in P$  and  $t_1 t_2^{-1} \in T^{G(p)}$ ,  $G(p)$  is a minimal face of  $P$  which contains  $p = q$ .

If in the above settings we replace  $T^{\mathcal{F}} \cong T^m$  by  $\mathbb{Z}_2^{\mathcal{F}} = \mathbb{Z}_2^m$  and  $t_1 t_2^{-1}$  is replaced by  $t_1 - t_2$ , we get a *real moment-angle manifold* over  $P$  as a quotient space:

$$\mathcal{R}_P = \mathbb{Z}_2^{\mathcal{F}} \times P^n / \sim .$$

**Remark 1.3.** (1) It can be deduced from Construction 1.2 that if  $P_1$  and  $P_2$  are *combinatorially equivalent*, that is, their nerve complexes are combinatorially equivalent, then  $\mathcal{Z}_{P_1} \cong \mathcal{Z}_{P_2}$  and  $\mathcal{R}_{P_1} \cong \mathcal{R}_{P_2}$ . The opposite statement is *not* true, in general.

(2) For any  $n$ -dimensional convex simple polytope  $P$ , Buchstaber and Panov proved the following facts: the moment-angle manifold  $\mathcal{Z}_P$  is a smooth  $(m + n)$ -dimensional closed 2-connected manifold which can be embedded in  $\mathbb{C}^m$  as a nondegenerate intersection of Hermitian quadrics; the real moment-angle manifold  $\mathcal{R}_P$  is a smooth  $n$ -dimensional closed non simply connected orientable manifold which can be embedded in  $\mathbb{R}^m$  as a nondegenerate intersection of real quadrics, see [18] and [22, Chapter 6].

(3) For any  $n$ -dimensional convex simple polytope  $P$ , it was also proved in [18] that:  $\mathcal{Z}_P$  is equivariantly homeomorphic to the moment-angle-complex  $\mathcal{Z}_{K_P}(D^2, S^1)$ ;  $\mathcal{R}_P$  is equivariantly homeomorphic to the real moment-angle-complex  $\mathcal{Z}_{K_P}(D^1, S^0)$ , see [22, Chapter 6] for more details.

It is easy to see that, using the face poset structure of a simple polytope  $P$  and Buchstaber-Panov theorem, one can obtain the following well known description of the (singular) cohomology groups of a (real) moment-angle manifold over  $P$  as a reformulation of the Hochster’s formula.

For a simple polytope  $P$  with the set of facets  $\mathcal{F} = \{F_1, \dots, F_m\}$  and a subset  $J \subseteq [m]$ , denote  $P_J = \cup_{j \in J} F_j$ . Then

$$H^l(\mathcal{Z}_P) = \sum_{J \subseteq [m]} \tilde{H}^{l-|J|-1}(P_J),$$

$$H^l(\mathcal{R}_P) = \sum_{J \subseteq [m]} \tilde{H}^{l-1}(P_J).$$

### 1.2. Quasitoric Manifolds and Small Covers

Quasitoric manifolds and small covers are extensively studied in toric topology in the last twenty years. A detailed exposition on them can be found in Buchstaber and Panov’s monographs [20] and [22]. Here we briefly review the main definition and results about them.

Let

$$G_d = \begin{cases} S^0 & \text{if } d = 1 \\ S^1 & \text{if } d = 2 \end{cases}, \mathbb{R}_d = \begin{cases} \mathbb{Z}_2 & \text{if } d = 1 \\ \mathbb{Z} & \text{if } d = 2 \end{cases} \quad \text{and} \quad \mathbb{K}_d = \begin{cases} \mathbb{R} & \text{if } d = 1 \\ \mathbb{C} & \text{if } d = 2 \end{cases}$$

where  $S^0 = \{-1, 1\}$  and  $S^1 = \{z \mid |z| = 1\}$  are multiplicative subgroups of real and complex numbers, respectively. The standard action of  $G_d^n$  on  $\mathbb{K}_d^n$  is given as

$$G_d^n \times \mathbb{K}_d^n \rightarrow \mathbb{K}_d^n : (t_1, \dots, t_n) \cdot (x_1, \dots, x_n) \mapsto (t_1 x_1, \dots, t_n x_n).$$

A  $G_d^n$ -manifold is a differentiable manifold with a smooth action of  $G_d^n$ .

**Definition 1.4.** A map  $f : X \rightarrow Y$  between two  $G$ -spaces  $X$  and  $Y$  is called *weekly equivariant* if for any  $x \in X$  and  $g \in G$  it holds

$$f(g \cdot x) = \psi(g) \cdot f(x)$$

where  $\psi : G \rightarrow G$  is some automorphism of group  $G$ .

Let  $M^{dn}$  be a  $dn$ -dimensional  $G_d^n$ -manifold. A *standard chart* on  $M^{dn}$  is a pair  $(U, f)$ , where  $U$  is a  $G_d^n$ -stable open subset of  $M^{dn}$  and  $f$  is a weekly equivariant diffeomorphism from  $U$  onto some  $G_d^n$ -stable open subset of  $\mathbb{K}_d^n$ . A *standard atlas* is an atlas which consists of standard charts. A  $G_d^n$  action on a  $G_d^n$ -manifold  $M^{dn}$  is called *locally standard* if manifold  $M^{dn}$  has a standard atlas. The orbit space for a locally standard action is naturally regarded as a manifold with corners.

**Definition 1.5.** A  $G_d^n$ -manifold  $\pi_d : M^{dn} \rightarrow P^n$ ,  $d = 1, 2$  is a smooth closed  $dn$ -dimensional  $G_d^n$ -manifold admitting a locally standard  $G_d^n$ -action such that its orbit space is a simple convex  $n$ -polytope  $P^n$  regarded as a manifold with corners. If  $d = 1$  such a  $G_d^n$ -manifold is called a *small cover* and if  $d = 2$  a *quasitoric manifold*.

1.3. Finite Group Actions and Equivariant Cohomology of Polyhedral Products

Alongside with the action of a compact torus  $T^m$  and its subgroups on  $\mathcal{Z}_K$ , or on any polyhedral product  $(X, A)^K$ , where  $K$  is a simplicial complex on the vertex set  $[m]$ , one can consider the following class of naturally arising (finite) group actions on polyhedral products.

Suppose  $K$  is a  $G$ -simplicial complex on  $[m] = \{1, 2, \dots, m\}$ , that is a finite group  $G \subseteq \text{Aut}(K) \subseteq \Sigma_m$  acts on  $[m]$  in such a way that each simplex  $\sigma = (i_1, \dots, i_k) \in K$  is mapped to a simplex  $g(\sigma) = (g(i_1), \dots, g(i_k)) \in K$  for any  $g \in G$ . Note that  $G$  always acts on  $2^{[m]}$  and the orbit of  $J \in 2^{[m]}$  is  $G(J) = \{g(J) \mid g \in G\} \subset 2^{[m]}$ . This action of  $G$  on  $2^{[m]}$  restricts to the initial  $G$ -action on  $K \subset 2^{[m]}$  if and only if  $G$  acts simplicially on  $K$ .

**Remark 1.6.** Note that,  $G$  takes simplices of  $K$  to simplices of the same dimension (every  $g \in G$  is 1-1 on the set of simplices  $K$ ), therefore,  $G$  takes missing faces of  $K$  to missing faces of  $K$ . Thus, we can define  $G$ -action on  $K$  by either images of the simplices of  $K$  or images of the missing faces of  $K$ .

**Remark 1.7.** Note that, due to the above remark,  $G$  acts on each skeleton  $sk^i(K)$  of  $K$  for  $0 \leq i \leq \dim(K)$ , as well as on barycentric subdivision  $bs(K)$  of  $K$ , cones and suspensions over  $K$  (if  $I \subset J$  then  $g(I) \subset g(J)$  for any  $g \in G$ , the vertices of cones and suspensions are fixed points of  $G$ -action).

**Proposition 1.8.** For a full subcomplex  $K_J \subset K$  on  $J \subseteq [m]$  and any  $g \in G$ ,  $g(K_J) = K_{g(J)}$ . Moreover, both the image  $G(K_J) = \cup_{g \in G} K_{g(J)}$  and the full subcomplex  $K_{G(J)}$  are  $G$ -simplicial complexes on the same vertex sets and  $G(K_J) \subset K_{G(J)}$ .

*Proof.* If  $\sigma \in K_J = K \cap 2^J$ , then  $g(\sigma) \in g(K_J)$  by definition of  $G$ -action on  $K$ , and  $g(K_J) \subset K_{g(J)}$ . On the other hand, if  $\tau \in K_{g(J)}$  then  $\tau = g(\omega)$ , where  $\omega \subset J$  as  $g$  is a 1-1 map on  $[m]$ , thus  $K_{g(J)} \subset g(K_J)$ . As an element of  $K_{G(J)}$  is of the form  $\sigma = (h_1(i_1), \dots, h_k(i_k))$ , where  $h_j \in G, i_j \in J$  and  $h_s \neq h_t$  for  $s \neq t$ , proving that  $G(K_J) \subset K_{G(J)}$ .  $\square$

**Definition 1.9.** Let  $G$  be a topological group. Suppose  $X$  is a  $G$ -space and consider the principal  $G$ -bundle  $EG \rightarrow BG$ . The orbit space  $X_G = (EG \times X)/G$  of the free action  $g(e, x) = (eg^{-1}, gx)$  of  $G$  on  $EG \times X$  is called the *Borel construction* of  $X$  with respect to the  $G$  action. *Equivariant cohomology* of  $X$  with respect to the  $G$  action is ordinary cohomology of the Borel construction,

$$H_G^*(X; \mathbb{k}) = H^*(X_G; \mathbb{k}).$$

**Example 1.10.** Suppose  $X = *$ . Then  $H_G^*(X; \mathbb{k}) = H^*(EG/G; \mathbb{k}) = H^*(BG; \mathbb{k})$ . Thus, the projection  $X_G \rightarrow BG$  yields the  $H^*(BG; \mathbb{k})$ -module structure on  $H_G^*(X; \mathbb{k})$ .

Suppose  $X = \mathcal{Z}_K$  and  $G = T^m$ . Then the Borel construction  $X_G$  is homotopy equivalent to the Davis-Januszkiewicz space of  $K$ ,  $(\mathbb{C}P^\infty, *)^K$ , and the equivariant cohomology of a moment-angle-complex

$$H_G^*(\mathcal{Z}_K) = H^*((\mathbb{C}P^\infty, *)^K) \cong \mathbb{Z}[K]$$

is isomorphic to the Stanley-Reisner algebra of  $K$  (see [21, Theorem 7.30]). Moreover, for the canonical action of  $T^n$  on a quasitoric manifold  $M^{2n} = M(P, \Lambda)$  one has a homotopy equivalence

$$M_{T^n}^{2n} \simeq (\mathcal{Z}_P)_{T^n}.$$

Therefore, the  $T^n$ -equivariant cohomology ring of  $M^{2n}$  is also isomorphic to the face ring  $\mathbb{Z}[P]$  of  $P$ , see [21, Proposition 7.38, Corollary 7.39].

We consider the case of a moment-angle-complex  $\mathcal{Z}_K$  and a simplicial action of  $G$  on  $K$ . The  $\mathbb{Z}[G]$ -module structure on  $H^*(\mathcal{Z}_K)$  was studied in [1, 36]. We are interested in the  $H^*(BG)$ -module structure on  $H_G^*(\mathcal{Z}_K)$ . Note that  $H_{\{e\}}^*(\mathcal{Z}_K) = H^*(\mathcal{Z}_K)$  for the trivial group action on  $\mathcal{Z}_K$ .

**Proposition 1.11.** *Let  $K$  be a simplicial complex on  $[m]$  and let  $G \subseteq \text{Aut}(K) \subset \Sigma_m$ . Then  $\mathcal{Z}_K$  is a  $G$ -space.*

*Proof.* Consider the cellular decomposition of  $\mathcal{Z}_K$  based on the decomposition of the unit disk  $D^2$  into 3 cells: one 2-dimensional  $D$ , one 1-dimensional  $T$  and one 0-dimensional  $\{1\}$ , see [21, Section 7.3]. Then the cells of  $\mathcal{Z}_K$  are enumerated by vectors  $C \in \{D, T, 1\}^m$  and  $\mathcal{Z}_K$  becomes a cellular subcomplex in  $(D^2)^m$ ; a cell  $C$  of  $(D^2)^m$  is in  $\mathcal{Z}_K$  if and only if the set of indices  $C_D$  corresponding to  $D$  coordinates of  $C$  forms a simplex in  $K$ . Since  $G \subseteq \text{Aut}(K)$  acts simplicially on  $K$ , it acts by permuting cells in the above cell decomposition of  $\mathcal{Z}_K$ .  $\square$

**Problem 1.12.** *Determine the  $H^*(BG)$ -module structure and the  $\mathbb{Z}[G]$ -algebra structure of the equivariant cohomology  $H_G^*(\mathcal{Z}_K)$  when  $G \subseteq \text{Aut}(K)$ . In particular, find the widest possible class of  $K$  and  $G$  such that the Serre spectral sequence of the fibration*

$$\mathcal{Z}_K \rightarrow (\mathcal{Z}_K)_G \rightarrow BG$$

*collapses in the  $E_2$  term.*

**Problem 1.13.** *Compute the cohomology ring and determine the homotopy type of the fixed point set  $F(G, \mathcal{Z}_K)$  of a  $G$ -action on  $\mathcal{Z}_K$ .*

## 2. The Cohomology Ring of $\mathcal{Z}_I$ and $\mathcal{R}_I$

In this section we study combinatorics of the simplicial complex  $I$  given as the boundary of the icosahedron and generators of cohomology ring of its moment-angle complex  $\mathcal{Z}_I$  and real moment-angle-complex  $\mathcal{R}_I$ . Recall that the icosahedron is the regular polyhedron and Platonic solid having 12 vertices, 30 edges and 20 equivalent equilateral triangle faces. Any polyhedron can be associated with a combinatorial dual figure, where the vertices of one correspond to the faces of the other and the edges between pairs of vertices of one correspond to the edges between pairs of faces of the other. However, not all such duals are geometric polyhedra. The dual of the icosahedron is the dodecahedron, another regular polyhedron and Platonic solid having 20 vertices, 30 edges and 12 equivalent regular pentagon faces. In the rest of the paper we assume that the vertices of the icosahedron are labelled with numbers  $1, \dots, 12$  as in Figure 1.

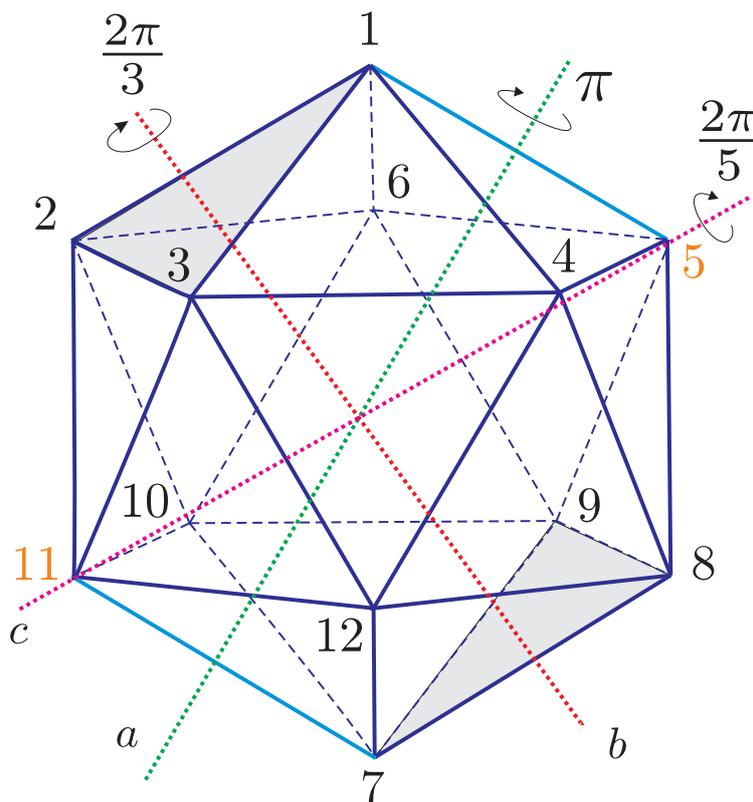


Figure 1: The icosahedron

2.1. Cohomology of Moment-Angle Manifolds

Let  $\mathbb{k}$  be a field or the ring of integers  $\mathbb{Z}$  and  $\mathbb{k}[v_1, \dots, v_m]$  be the polynomial algebra where  $\deg v_i = 2$ . Let  $\Lambda[u_1, \dots, u_m]$  be the exterior algebra where  $\deg u_i = 1$ . Given a subset  $I = \{i_1, \dots, i_k\} \subset [m]$  let  $v_I$  denote the square-free monomial  $v_{i_1} \cdots v_{i_k}$  in  $\mathbb{k}[v_1, \dots, v_m]$ .

The Stanley-Reisner ring or the face ring of a simplicial complex  $K$  on  $m$  vertices is the quotient graded ring

$$\mathbb{k}[K] := \mathbb{k}[v_1, \dots, v_m] / \mathcal{I}_K,$$

where  $\mathcal{I}_K = (v_I \mid I \notin K)$  is the ideal generated by those monomials  $v_I$  for which  $I$  is not a simplex of  $K$ . The ideal  $\mathcal{I}_K$  is known as the Stanley-Reisner ideal of  $K$ .

The cohomology ring of  $\mathcal{Z}_K$  over  $\mathbb{k}$  was obtained by Buchstaber, Panov, and Baskakov.

**Theorem 2.1.** ([22, Buchstaber-Panov Theorem 4.5.]) *There are isomorphisms, functorial in  $K$ , of bigraded algebras*

$$H^{*,*}(\mathcal{Z}_K; \mathbb{k}) \cong \text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}(\mathbb{k}[K], \mathbb{k}) \cong H^{*,*}[\Lambda[u_1, \dots, u_m] \otimes \mathbb{k}[K], d]$$

where the bigrading and the differential on the right hand side are defined by

$$\text{bideg } u_i = (-1, 2), \quad \text{bideg } v_i = (0, 2), \quad du_i = v_i, \quad dv_i = 0.$$

A module structure of  $H^*(\mathcal{Z}_K; \mathbb{k})$  can thus be obtained using a well-known result from combinatorial commutative algebra, the Hochster’s formula, which represents the above Tor-algebra as a direct sum of reduced simplicial cohomology groups of all full subcomplexes in  $K$ . Multiplication in  $H^*(\mathcal{Z}_K; \mathbb{k})$  was firstly described by Baskakov.

**Theorem 2.2.** ([7, Baskakov Theorem 1], [22, Hochster Theorem 4.5.8]) *There are isomorphisms of  $\mathbb{k}$ -modules*

$$\mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i, 2j}(\mathbb{k}[K]; \mathbb{k}) \cong \bigoplus_{J \subset [m], |J|=j} \widetilde{H}^{|J|-i-1}(K_J; \mathbb{k}) H^l(\mathcal{Z}_K; \mathbb{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}^{l-|J|-1}(K_J; \mathbb{k})$$

*These isomorphisms sum up into a ring isomorphism  $H^*(\mathcal{Z}_K; \mathbb{k}) \cong \bigoplus_{J \subset [m]} \widetilde{H}^*(K_J; \mathbb{k})$  where the ring structure on the right hand side is given by the canonical maps*

$$H^{k-|I|-1}(K_I; \mathbb{k}) \otimes H^{l-|J|-1}(K_J; \mathbb{k}) \rightarrow H^{k+l-|I|-|J|-1}(K_{I \cup J}; \mathbb{k})$$

*which are induced by simplicial maps  $K_{I \cup J} \rightarrow K_I * K_J$  for  $I \cap J = \emptyset$  and zero otherwise.* □

Due to [22, Construction 3.2.8, Theorem 3.2.9] the Tor-algebra of  $K$  acquires a multigraded refinement and the multigraded components can be calculated in terms of full subcomplexes. Namely, for any simplicial complex  $K$  on  $[m]$  we have

$$\mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i, 2J}(\mathbb{k}[K], \mathbb{k}) \cong \widetilde{H}^{|J|-i-1}(K_J; \mathbb{k})$$

where  $J \subset [m]$  and  $\mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i, 2\mathbf{a}}(\mathbb{k}[K], \mathbb{k}) = 0$ , if  $\mathbf{a}$  is not a  $(0, 1)$ -vector.

Moreover, if we denote by  $R(K) = \Lambda[u_1, \dots, u_m] \otimes \mathbb{k}[K]/(v_i^2 = u_i v_i = 0, 1 \leq i \leq m)$  a graded algebra with the differential  $d$  as in Theorem 2.1, then  $R(K)$  also acquires multigrading and the following  $\mathbb{k}$ -module isomorphism holds

$$\mathrm{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{-i, 2\mathbf{a}}(\mathbb{k}[K], \mathbb{k}) \cong H^{-i, 2\mathbf{a}}[R(K), d]$$

for any simplicial complex  $K$ .

Now we turn to the real case. By [20, Theorem 8.9] one has  $\mathbb{k}$ -module isomorphisms

$$H^p(\mathcal{R}_K; \mathbb{k}) \cong \bigoplus_{J \subset [m]} H^{p-1}(K_J; \mathbb{k}). \tag{1}$$

The multiplicative structure was given firstly by Cai [23]. Consider a differential graded algebra  $r(K)$  which is a quotient algebra of a free graded algebra on  $2m$  variables  $u_i, t_j$ , where  $\deg(u_i) = 1, \deg(t_j) = 0$ , by the Stanley-Reisner ideal of  $K$  in variables  $\{u_i\}$  and the following relations:

$$u_i t_i = u_i, t_i u_i = 0, u_i t_j = t_j u_i, t_i t_i = t_i, t_i t_j = t_j t_i, u_i u_i = 0, u_i u_j = -u_j u_i.$$

**Theorem 2.3.** *There is a graded ring isomorphism*

$$H^*(\mathcal{R}_K; \mathbb{k}) \cong H^*[r(K), d]$$

*where  $d(t_i) = u_i$  and  $d(u_i) = 0$ .* □

### 2.2. Cohomology Generators of $\mathcal{Z}_I$

We explicitly list the generators of the cohomology ring of the moment-angle complex of the boundary of the icosahedron. In our description, we appeal on the action of the group  $A_5$  on the icosahedron. Denote by  $\mathcal{Z}_I$  the moment-angle manifold of the boundary of the icosahedron, which is equivariantly homeomorphic to the moment-angle manifold  $\mathcal{Z}_D$  of the dodecahedron  $\mathcal{D}$ .

By Hochster theorem, nontrivial cohomological classes arise in noncontractible full subcomplexes  $I_J$ .

**Proposition 2.4.**  $b^3(\mathcal{Z}_I) = b^{-1,4}(\mathcal{Z}_I) = 36$ .

*Proof.* Noncontractible simplicial complex on two vertices is  $S^0$ . We consider the orbits of subcomplexes on two vertices with respect to the symmetry group  $A_5$  of the icosahedron. There are two types of full subcomplexes on two vertices of  $\mathcal{Z}_I$  homotopy equivalent to  $S^0$ , as illustrated in Figure 2.

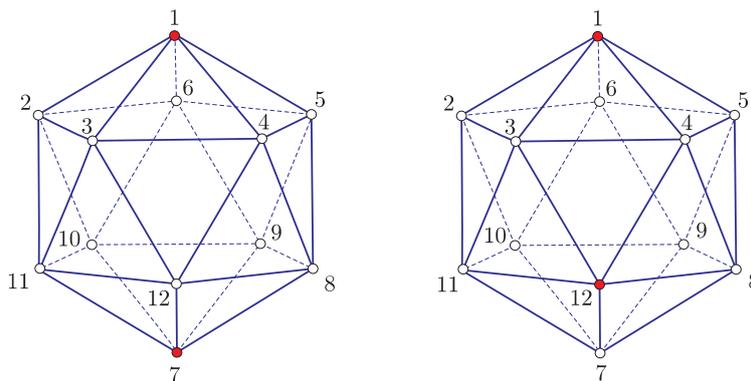


Figure 2: The case  $|J| = 2$

There are six subcomplexes in the orbit of the full subcomplex on  $J = \{1, 7\}$  and 30 subcomplexes in the orbit of the full subcomplex on  $J = \{1, 12\}$ , thus

$$b^3(\mathcal{Z}_I) = b^{-1,4}(\mathcal{Z}_I) = 6 + 30 = 36.$$

□

**Proposition 2.5.**  $b^4(\mathcal{Z}_I) = b^{-2,6}(\mathcal{Z}_I) = 160.$

*Proof.* Noncontractible simplicial complexes on three vertices are either homotopy equivalent to  $S^1$ ,  $S^0$  or  $S^0 \vee S^0$ . In the boundary of the icosahedron any noncontractible full subcomplex on three vertices lies in the orbit of one of the following full subcomplex on the sets  $\{1, 8, 11\}$ ,  $\{1, 11, 12\}$  or  $\{1, 7, 12\}$  as illustrated in Figure 3.

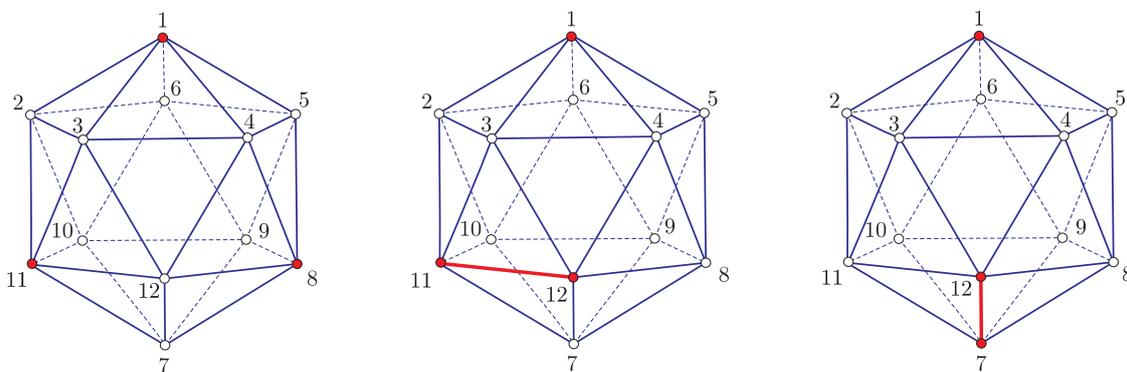


Figure 3: The case  $|J| = 3$

There are 20 full subcomplexes in the orbit of the full subcomplex on  $\{1, 8, 11\}$  and they are homotopy equivalent to  $S^0 \vee S^0$ . In each of the orbits of the full subcomplexes on  $\{1, 11, 12\}$  and  $\{1, 7, 12\}$  there are 60 full subcomplexes and they are homotopic to  $S^0$ . They contribute to the cohomology of  $\mathcal{Z}_I$  with

$$b^4(\mathcal{Z}_I) = b^{-2,6}(\mathcal{Z}_I) = 20 \cdot 2 + 60 + 60 = 160.$$

□

**Proposition 2.6.**  $b^5(\mathcal{Z}_I) = b^{-3,8}(\mathcal{Z}_I) = 315.$

*Proof.* A noncontractible flag 2-dimensional simplicial complex on four vertices is homotopy equivalent to  $S^1$ ,  $S^0$ ,  $S^0 \vee S^0$  or  $S^0 \vee S^0 \vee S^0$ . All noncontractible full subcomplexes on four vertices lie in some of the orbits of the full subcomplex on the sets  $\{1, 7, 9, 10\}$ ,  $\{1, 8, 9, 11\}$ ,  $\{1, 5, 7, 11\}$ ,  $\{1, 5, 11, 12\}$ ,  $\{1, 2, 5, 12\}$  or  $\{1, 2, 5, 7\}$  as in Figure 4.

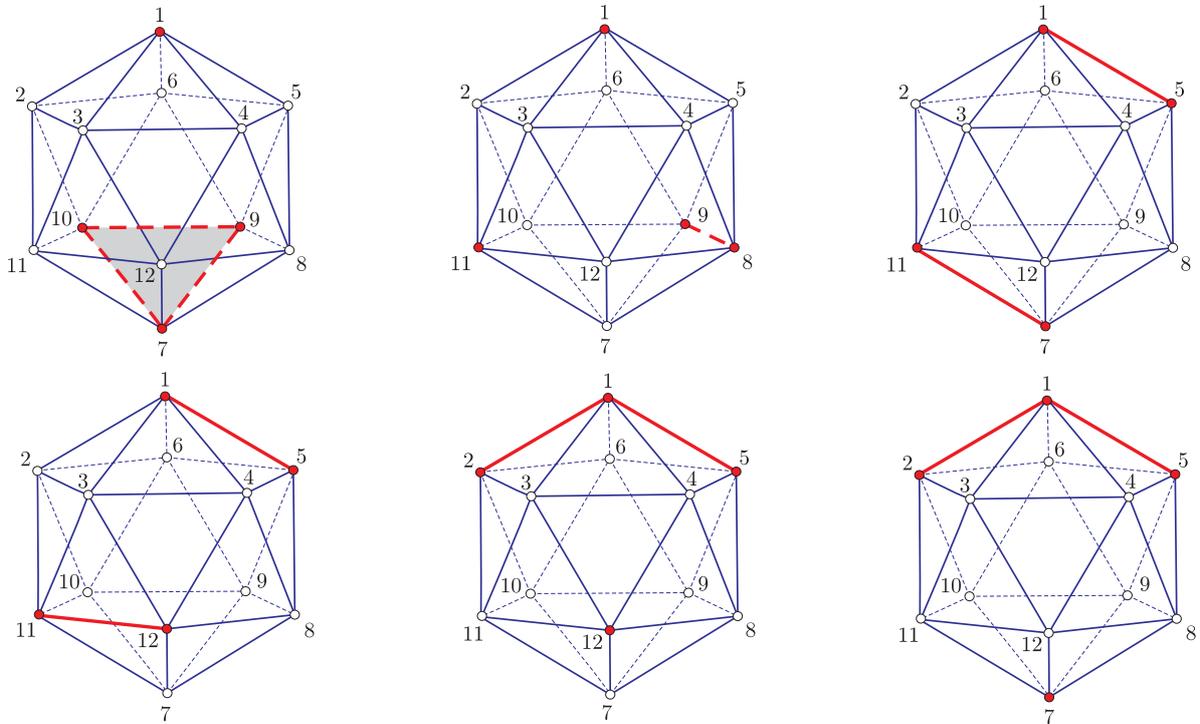


Figure 4: The case  $|J| = 4$

Full subcomplexes  $\mathcal{I}_{\{1,7,9,10\}}$ ,  $\mathcal{I}_{\{1,5,7,11\}}$ ,  $\mathcal{I}_{\{1,5,11,12\}}$ ,  $\mathcal{I}_{\{1,2,5,12\}}$  and  $\mathcal{I}_{\{1,2,5,7\}}$  are homotopy equivalent to  $S^0$ , while  $\mathcal{I}_{\{1,8,9,11\}}$  is homotopy equivalent to  $S^0 \vee S^0$ . In the orbits of full subcomplexes  $\mathcal{I}_{\{1,7,9,10\}}$ ,  $\mathcal{I}_{\{1,5,11,12\}}$ ,  $\mathcal{I}_{\{1,2,5,12\}}$  and  $\mathcal{I}_{\{1,2,5,7\}}$  there are 60 full subcomplexes, in the orbit of  $\mathcal{I}_{\{1,5,7,11\}}$  there are 15 full subcomplexes and 30 full subcomplexes in the orbit of  $\mathcal{I}_{\{1,5,7,11\}}$ . They contribute to the cohomology of  $\mathcal{Z}_{\mathcal{I}}$  with

$$b^5(\mathcal{Z}_{\mathcal{I}}) = b^{-3,8}(\mathcal{Z}_{\mathcal{I}}) = 60 + 60 + 60 + 60 + 15 + 30 \cdot 2 = 315.$$

□

**Lemma 2.7.**  $b^{-3,10}(\mathcal{Z}_{\mathcal{I}}) = 12$ .

*Proof.* There are six types of noncontractible full subcomplexes on five vertices as in Figure 5, but by the Hochster formula only the pentagons have a nontrivial class in the first cohomology group  $\tilde{H}^1(\mathcal{I}_J)$  and give generators of  $H^{-3,10}(\mathcal{Z}_{\mathcal{I}})$ . It is obvious that all 12 pentagons lie in the orbit of the full subcomplex  $\mathcal{I}_{\{2,3,4,5,6\}}$ .

□

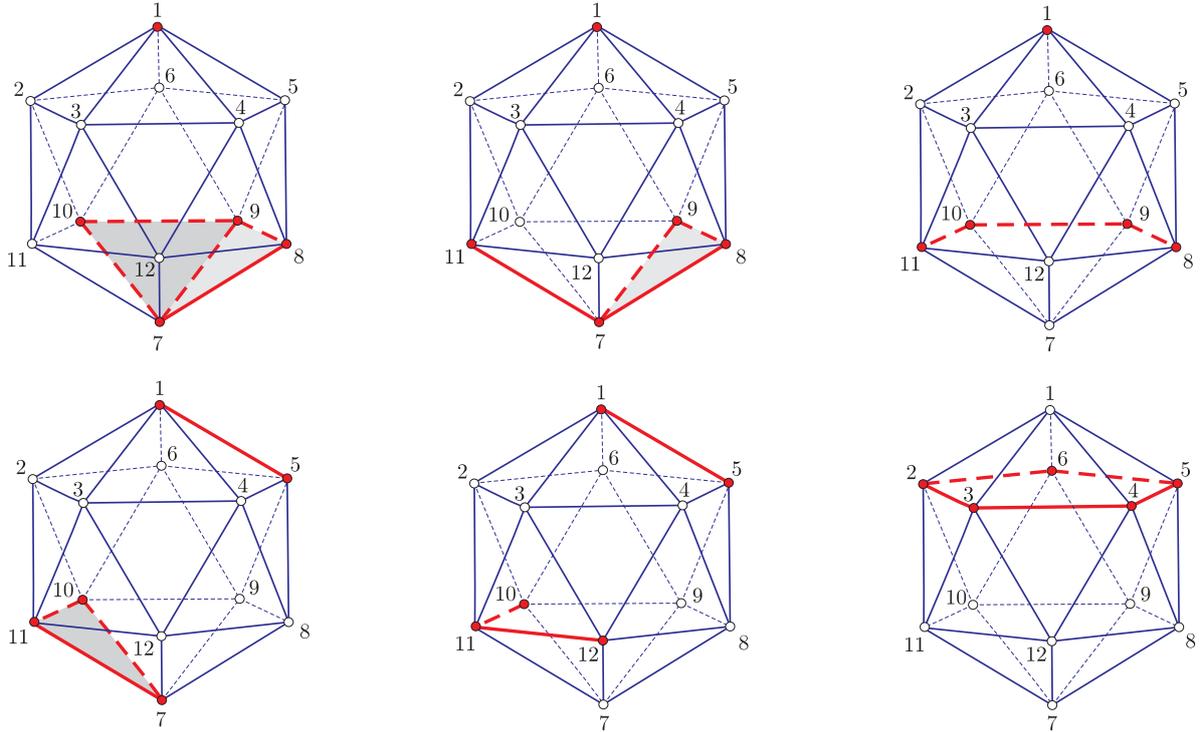


Figure 5: The case  $|J| = 5$

**Proposition 2.8.**  $b^6(\mathcal{Z}_I) = b^{-4,10}(\mathcal{Z}_I) = 300$ .

*Proof.* The full subcomplexes which give generators of  $H^{-4,10}(\mathcal{Z}_I)$  are those of five types homotopy equivalent to the union of two points, see Figure 5. In the orbit of each of the full subcomplexes  $\mathcal{I}_{\{1,7,8,9,10\}}$ ,  $\mathcal{I}_{\{1,7,8,9,11\}}$ ,  $\mathcal{I}_{\{1,8,9,10,11\}}$ ,  $\mathcal{I}_{\{1,5,7,10,11\}}$  and  $\mathcal{I}_{\{1,5,10,11,12\}}$  lie exactly 60 full subcomplexes. Thus,

$$b^{-4,10}(\mathcal{Z}_I) = 60 + 60 + 60 + 60 + 60 = 300.$$

□

**Lemma 2.9.**  $b^{-4,12}(\mathcal{Z}_I) = 112$ .

*Proof.* There are seven types of noncontractible full subcomplexes on six vertices, see Figure 6. Three of them are homotopy equivalent to  $S^1$ , one to the disjoint union of the point and  $S^1$  and three to  $S^0$ . Therefore, only  $\mathcal{I}_{\{1,3,5,8,11,12\}}$ ,  $\mathcal{I}_{\{1,2,4,7,8,11\}}$ ,  $\mathcal{I}_{\{1,2,5,7,8,11\}}$  and  $\mathcal{I}_{\{1,8,9,10,11,12\}}$  have nontrivial  $\tilde{H}^1(\mathcal{I}_J)$ . In the orbit of  $\mathcal{I}_{\{1,3,5,8,11,12\}}$  and  $\mathcal{I}_{\{1,2,4,7,8,11\}}$  there is exactly 60 full subcomplexes, while in the orbits of  $\mathcal{I}_{\{1,2,4,7,8,11\}}$ ,  $\mathcal{I}_{\{1,2,5,7,8,11\}}$  and  $\mathcal{I}_{\{1,8,9,10,11,12\}}$  there are thirty, ten and twelve full subcomplexes, respectively.

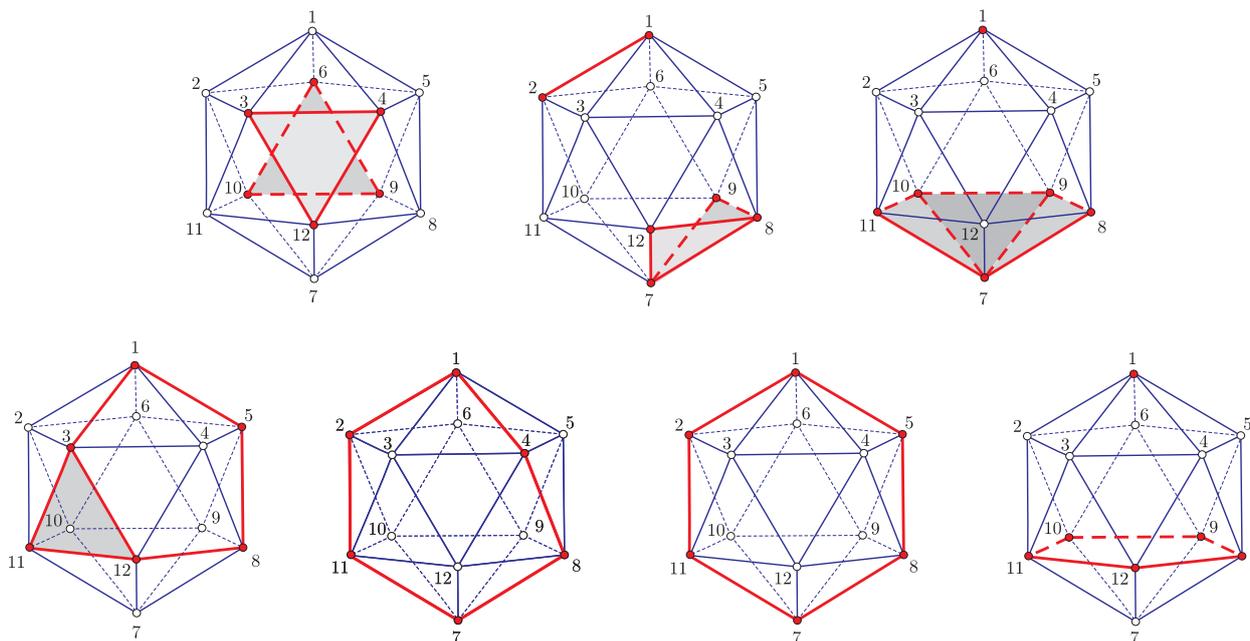


Figure 6: The case  $|J| = 6$

Thus,

$$b^{-4,12}(\mathcal{Z}_I) = 60 + 30 + 10 + 12 = 112.$$

□

**Lemma 2.10.**  $b^{-5,12}(\mathcal{Z}_I) = 112$ .

*Proof.* Among seven types of noncontractible full subcomplexes on six vertices shown at Figure 6 there are four homotopy equivalent to the disjoint union of two points and one to the the disjoint union of the point and  $S^1$  and they only have nontrivial  $\tilde{H}^0(I_J)$ . In the orbits of  $\mathcal{I}_{\{3,4,6,9,10,12\}}$ ,  $\mathcal{I}_{\{1,2,7,8,9,12\}}$ ,  $\mathcal{I}_{\{1,7,8,9,10,11\}}$  and  $\mathcal{I}_{\{1,8,9,10,11,12\}}$  there are 10, 30, 60 and 12 full subcomplexes.

Thus,

$$b^{-5,12}(\mathcal{Z}_I) = 10 + 30 + 60 + 12 = 112.$$

□

Using Lemma 2.7 and Proposition 2.10, the seventh Betty number is calculated

**Proposition 2.11.**  $b^7(\mathcal{Z}_I) = 124$ .

**Proposition 2.12.**  $b^9 = b^{-5,14}(\mathcal{Z}_I) = 300$ .

*Proof.* The full subcomplexes giving generators of  $H^{-6,16}(\mathcal{Z}_I)$  are those of five types homotopy equivalent to  $S^1$ , see Figure 7. Thus,

$$b^9 = b^{-6,16}(\mathcal{Z}_I) = 60 + 60 + 60 + 60 + 60 = 300.$$

□

**Lemma 2.13.**  $b^{-6,14}(\mathcal{Z}_I) = 12$ .

*Proof.* There are six types of noncontractible full subcomplexes on seven vertices as in Figure 7, but only complements of the pentagons give generators of  $H^{-6,14}(\mathcal{Z}_I)$  since  $\tilde{H}^0(I_J)$  for  $J = \{1, 7, 8, 9, 10, 11, 12\}$ . □

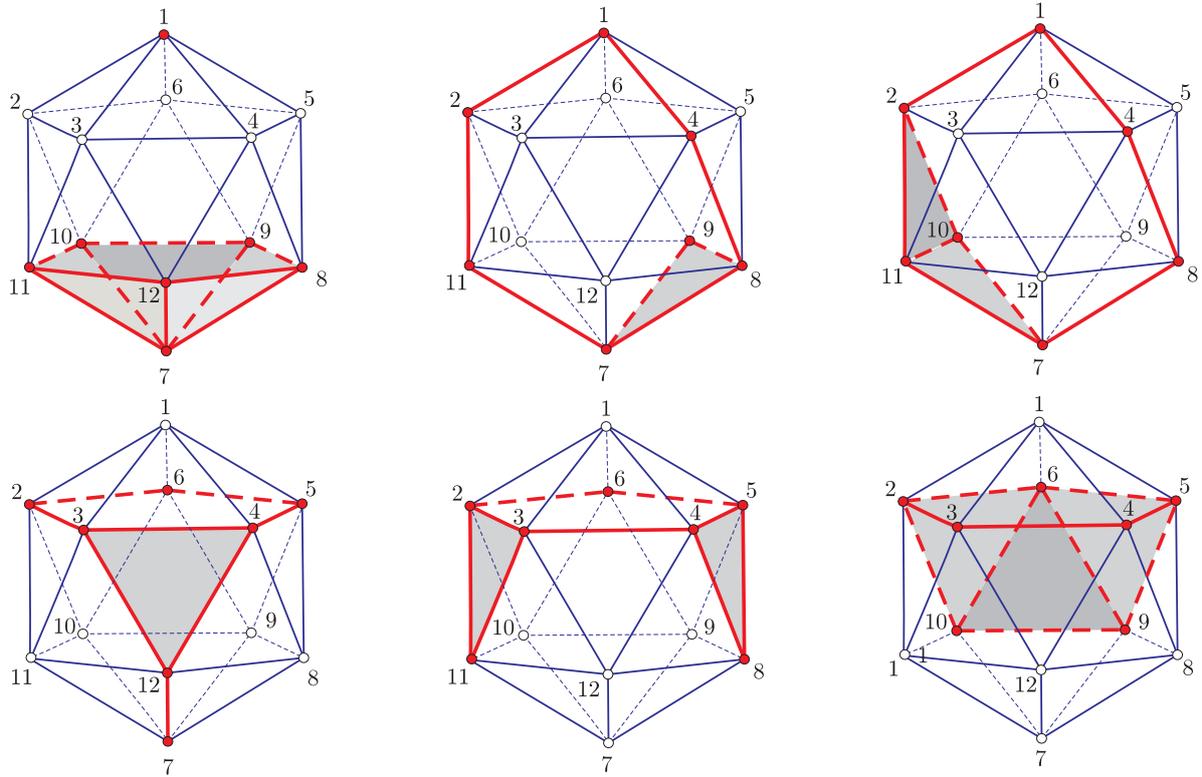


Figure 7: The case  $|J| = 7$

By Lemmas 2.9 and 2.13, the eighth Betty number can be calculated.

**Proposition 2.14.**  $b^8(\mathcal{Z}_I) = 124$ .

**Proposition 2.15.**  $b^{10}(\mathcal{Z}_I) = b^{-6,16}(\mathcal{Z}_I) = 315$ .

*Proof.* There are six types of noncontractible full subcomplexes on eight vertices as in Figure 8. Seven of them have homotopy type of  $S^1$ , while one is homotopy equivalent to  $S^1 \vee S^1$ .

The full subcomplexes in the orbits of  $\mathcal{I}_{\{2,3,4,5,6,7,9,10\}}$ ,  $\mathcal{I}_{\{1,3,5,6,8,9,11,12\}}$ ,  $\mathcal{I}_{\{1,3,5,8,9,10,11,12\}}$ ,  $\mathcal{I}_{\{1,2,3,5,7,8,9,11\}}$ ,  $\mathcal{I}_{\{1,2,5,7,8,9,10,11\}}$  and  $\mathcal{I}_{\{2,3,4,5,6,8,11,12\}}$  contribute to the cohomology of  $\mathcal{Z}_I$  with

$$b^{10}(\mathcal{Z}_I) = b^{-6,16}(\mathcal{Z}_I) = 60 + 60 + 30 \cdot 2 + 15 + 60 + 60 = 315.$$

□

**Proposition 2.16.**  $b^{11}(\mathcal{Z}_I) = b^{-7,18}(\mathcal{Z}_I) = 160$ .

*Proof.* There are three types of noncontractible full subcomplexes on nine vertices as in Figure 9.

They contribute to the cohomology of  $\mathcal{Z}_I$  with

$$b^{11}(\mathcal{Z}_I) = b^{-7,18}(\mathcal{Z}_I) = 60 + 60 + 20 \cdot 2 + 60 = 160.$$

□

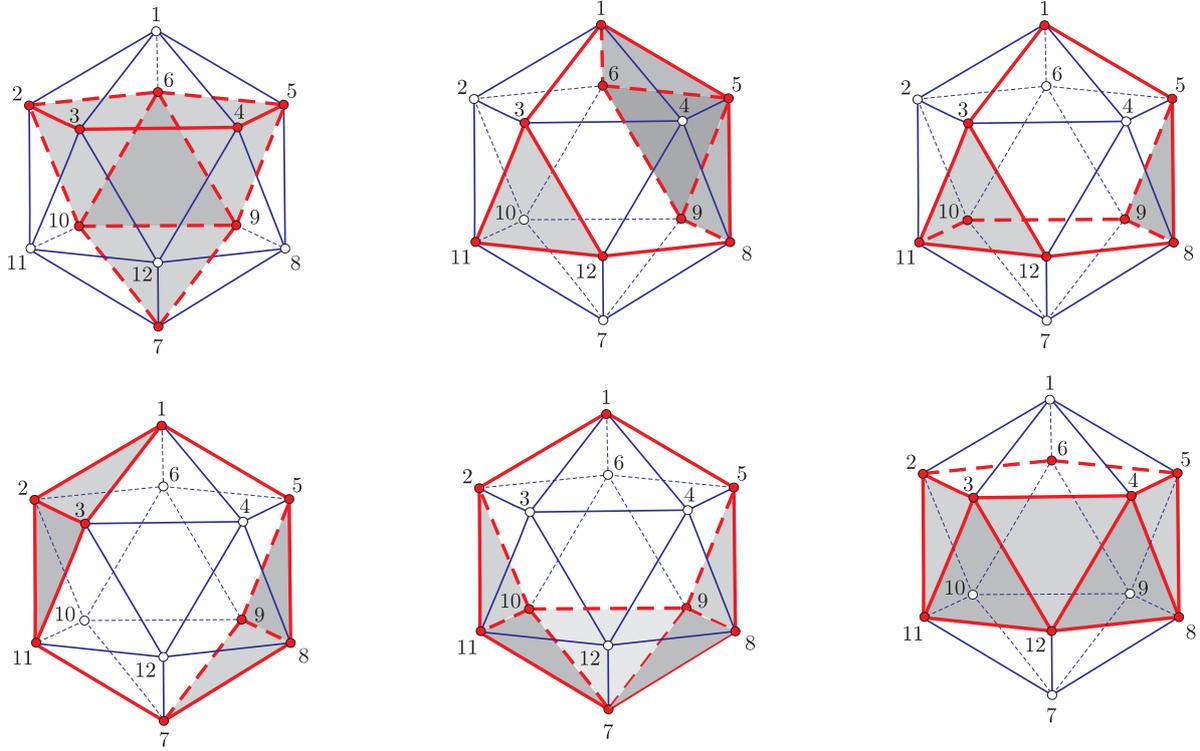


Figure 8: The case  $|J| = 8$

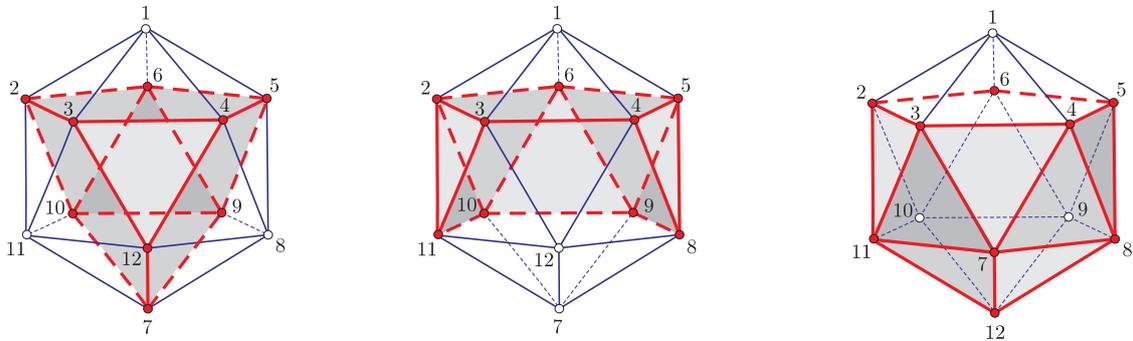


Figure 9: The case  $|J| = 9$

**Proposition 2.17.**  $b^{12}(\mathcal{Z}_I) = b^{-8,20}(\mathcal{Z}_I) = 36$ .

*Proof.* There are two types of noncontractible full subcomplexes on ten vertices as in Figure 10.

By the symmetry,

$$b^3(\mathcal{Z}_I) = b^{-1,4}(\mathcal{Z}_I) = 6 + 30 = 36.$$

□



**Theorem 2.18.** (a)

$$H^i(\mathcal{Z}_I, \mathbb{k}) \cong \begin{cases} \mathbb{k} & \text{if } i = 0 \text{ or } 15, \\ \mathbb{k}^{36} & \text{if } i = 3 \text{ or } 12, \\ \mathbb{k}^{160} & \text{if } i = 4 \text{ or } 11, \\ \mathbb{k}^{315} & \text{if } i = 5 \text{ or } 10, \\ \mathbb{k}^{300} & \text{if } i = 6 \text{ or } 9, \\ \mathbb{k}^{124} & \text{if } i = 7 \text{ or } 8, \\ 0 & \text{elsewhere.} \end{cases}$$

(b)  $H^*(\mathcal{Z}_I, \mathbb{Z})$  is torsion free.

Next we briefly discuss the real moment-angle manifold  $\mathcal{R}_D$  over the dodecahedron  $\mathcal{D}$ . Real moment-angle manifolds (also known as universal abelian covers over polytopes, see [29, §4.1]) and real moment-angle-complexes have recently attracted attention in geometric group theory and combinatorics due to their relations to the study of the right-angled Coxeter group. Recall that for a simplicial complex  $K$  on the set of vertices  $[m]$ , its right-angled Coxeter group is the quotient of the free group on  $m$  generators  $g_1, \dots, g_m$  modulo relations  $g_i g_j = g_j g_i$  for all  $\{i, j\} \in \text{sk}^1(K)$ .

The right-angled Coxeter group  $RC_P$  of a simple polytope  $P$  is the right-angled Coxeter group of the simplicial complex  $\partial P^*$ . Thus the right-angled Coxeter group of the dodecahedron  $\mathcal{D}$  is given by

$$RC_{\mathcal{D}} = \frac{F\langle g_1, \dots, g_{12} \rangle}{(g_i^2 = 1, g_i g_j = g_j g_i \mid 1 \leq i \leq 12, \{i, j\} \in \text{sk}^1(\mathcal{K}))}$$

**Proposition 2.19.** The following statements hold:

(a)  $\mathcal{R}_D$  is a 3-dimensional closed orientable manifold and a space of a regular 512-fold covering over the small cover  $M_{\mathbb{R}}(\mathcal{D})$ .  $\mathcal{R}_D \simeq K(G, 1)$ , where  $G = [RC_{\mathcal{D}}, RC_{\mathcal{D}}]$  is a commutator subgroup of  $RC_{\mathcal{D}}$ .

(b)  $H^*(\mathcal{R}_D)$  is torsion free. The Betti numbers of  $\mathcal{R}_D$  are given by:

$$b^0(\mathcal{R}_D) = b^3(\mathcal{R}_D) = 1, b^1(\mathcal{R}_D) = b^2(\mathcal{R}_D) = 935.$$

*Proof.* To prove statement (a) first note that  $\mathcal{D}$  is of dimension  $n = 3$  and has  $m = 12$  vertices. Therefore,  $\mathcal{R}_D$  is a smooth 3-dimensional closed orientable manifold and a covering space of a regular  $2^{m-n} = 512$ -fold covering over a small cover  $M_{\mathbb{R}}(\mathcal{D})$  (see [37]).

There is a homotopy fibration

$$\mathcal{R}_D \rightarrow (\mathbb{R}P^\infty, *)^I \rightarrow B\mathbb{Z}_2^m \tag{2}$$

and a short exact sequence of groups

$$1 \rightarrow [RC_{\mathcal{D}}, RC_{\mathcal{D}}] \rightarrow RC_{\mathcal{D}} \rightarrow \mathbb{Z}_2^m \rightarrow 1. \tag{3}$$

It follows from the works of Davis [28] and Davis-Januszkiewicz [29] that  $(\mathbb{R}P^\infty, *)^I \simeq K(RC_{\mathcal{D}}, 1)$ . Thus, the exact homotopy sequence of the fibration (2) shows that  $\mathcal{R}_D$  is aspherical and  $\pi_1(\mathcal{R}_D) = [RC_{\mathcal{D}}, RC_{\mathcal{D}}]$ .  $\mathbb{Z}_2^m$  is abelian, so  $[RC_{\mathcal{D}}, RC_{\mathcal{D}}] \subseteq \pi_1(\mathcal{R}_D)$  and from the exact sequence (3) the index  $(\pi_1(\mathcal{R}_D) : [RC_{\mathcal{D}}, RC_{\mathcal{D}}]) = 1$ . This finishes the proof of statement (a).

To prove statement (b) observe that by Alexander duality and the universal coefficient theorem, the integral cohomology  $H^*(\mathcal{R}_D)$  is torsion free. Now, using (1)

$$b^1(\mathcal{R}_D) = \sum_{J \subseteq [m]} \text{rk } \tilde{H}^0(\mathcal{D}_J).$$

By Theorem 2.2, the sum in the right side of the above equality and the calculation shown in Table 1, it follows that

$$\sum_{|J|=1}^m \beta^{-(|J|-1), 2|J|}(J) = 935.$$

The rest follows from statement (a) and Poincaré duality for  $\mathcal{R}_{\mathcal{D}}$ .  $\square$

### 3. Quasitoric Manifolds and Small Covers over Dodecahedron

A *fullerene*  $P$  is a 3-dimensional polytope with only 5- and 6-gonal faces. Grbić and Beben [8] recently proved that a moment-angle manifold  $\mathcal{Z}_P$  over a fullerene  $P$  has LS-category equal to 3. Moreover, they proved that all its triple and higher Massey products of decomposable elements vanish. Fullerenes are originally considered in quantum physics and chemistry of carbon allotropes. This class of polytopes were brought into algebraic topology in the series of recent works by Buchstaber and Erokhovets [10–13]. In those papers they established a number of combinatorial properties of fullerenes and topological properties of their toric spaces. Furthermore, they constructed all fullerenes from a dodecahedron via several polytopal truncation operations. In the recent paper with Masuda, Panov, and Park [14] they solved positively the cohomological rigidity problem for quasitoric manifolds over *Pogorelov class* polytopes. The Pogorelov class polytopes are 3-dimensional polytopes with no 3- and 4-belts of facets and it contains fullerenes.

It was proved in [30] that if there exists a smooth projective toric variety over a simple 3-dimensional polytope  $P$ , then  $P$  has at least one triangular or quadrangular face. Therefore, it follows from combinatorial structure of fullerenes and, more generally, Pogorelov class polytopes, that they support quasitoric manifolds ([29]) but do not support smooth projective toric varieties.

In particular, a dodecahedron is a flag 3-polytope which can not have a Delzant realization in the ambient Euclidean space.

The Four Color Theorem was used in [29, Example 1.21] to prove that there is a quasitoric manifold (and small cover) over any simple 3-polytope. The argument is simple: assume that a simple 3-polytope is colored regularly by four colors  $a, b, c$  and  $d$ , then assign the columns  $(1, 0, 0)^t$ ,  $(0, 1, 0)^t$  and  $(0, 0, 1)^t$  to the facets colored by color 1, 2 and 3, respectively. To the facets colored in 4, assign  $(-1, -1, -1)^t$  and in this way, form the characteristic matrix that satisfies the non-singular condition. However, there are quasitoric manifolds over a 4-colorable simple polytope that are not arising in this way. For example, Garrison and Scott [37] using computer search found 25 small covers up to homeomorphism over the dodecahedron.

The quasitoric manifolds and small covers discussed above are examples of a general class of small covers and quasitoric manifolds in toric topology, see [29, Example 1.15]. The orbit polytopes for these manifolds are simple  $n$ -polytopes which admit regular coloring in  $n$  or  $n + 1$  colors while manifolds are called canonical. The class of canonical small covers and quasitoric manifolds contains pullbacks from the linear models which have various nice properties, for example they are all stable parallelizable [29, Corollary 6.10]. Baralić and Živaljević [5] found an application of canonical quasitoric manifolds and used it to prove results of a Knaster-Kuratowski-Mazurkiewicz type.

We will describe quasitoric manifolds over the dodecahedron arising from 4-coloring. In general, it is possible to have non-isomorphic regular 4-coloring over a simple 3-polytope whose characteristic matrices may produce non-homeomorphic quasitoric manifolds. In [58] it is stated that there exist exactly two nonisomorphic 5-colorings of the dodecahedron with four faces of each color and that they are symmetric to each other up to permutation of colors. Two colorings are considered to be isomorphic if they differ by permutation of colors and direct isometric transformation. We prove this results in the next lemma.

**Lemma 3.1.** *There are exactly two nonisomorphic regular colorings of the dodecahedron in 4 colors and they are plane symmetric.*

*Proof.* Regular coloring of the faces of the dodecahedron corresponds dually to the coloring of the vertices of the icosahedron. First, observe that in a regular 4-coloring it is not possible to color four vertices of the icosahedron in the same color since it would result in having two neighboring vertices colored in the same color. Thus, every color is used exactly three times.

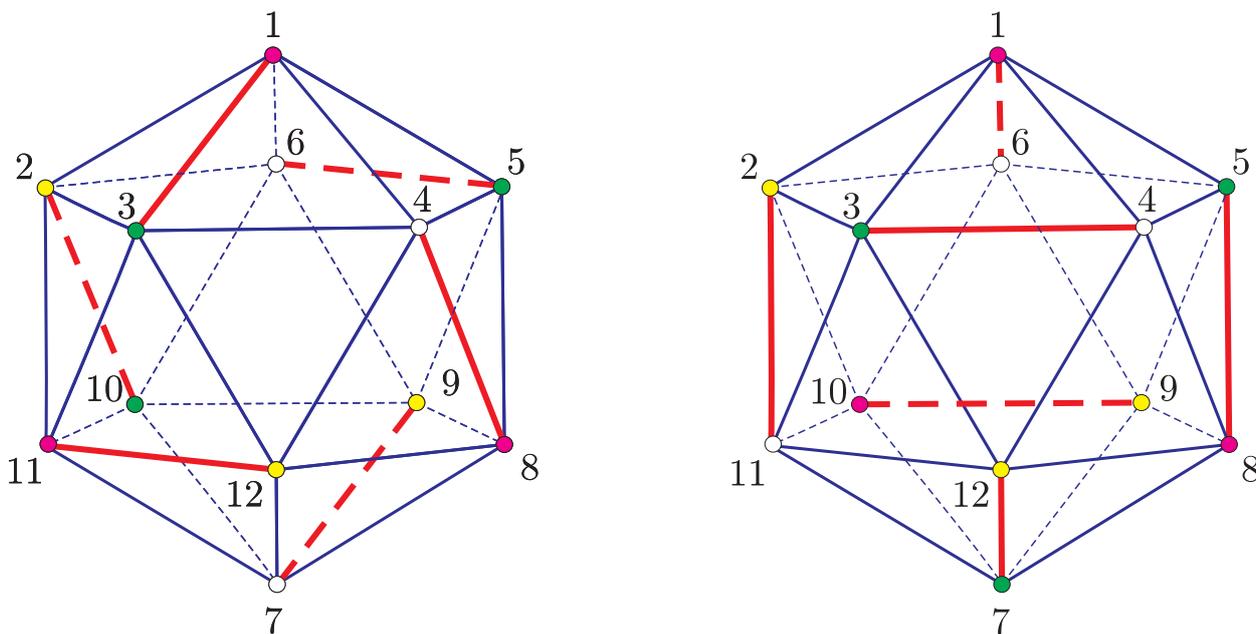


Figure 11: Regular 4-coloring of the icosahedron

Without loss of generality we assume that the vertex labeled by 1 and the vertices in its link, labeled by 2, 3, 4, 5 and 6 are colored as in Figure 11. The vertex 8 can be colored either in yellow or in pink. Assume that it is yellow. Then the remaining five vertices of the icosahedron lie in the link of the vertex 6 or in the link of vertex 8 and they cannot be yellow, contradicting the fact that there must be three yellow vertices in regular 4-coloring. Thus, the vertex 8 is pink, and consequently the vertices 9 and 12 are yellow. For the remaining three vertices there are only two options as shown in Figure 11.

Next we prove that these two colorings are not isomorphic, that is, one coloring cannot be obtained from the other using symmetries of the icosahedron and permutations of colors. Assume that this can be achieved. Consider six red edges in Figure 11. They all have property that the opposite edge has different colored vertices from the vertices of the edge, so the icosahedron on the right in Figure 11 should be put in position that red edges are as in the icosahedron on the left. There are exactly 12 positions that fix red edges and they are obtained using rotations through pairs of opposite red edges or using rotations around axes through the shaded triangles, see Figure 12. Clearly, all are distinct from the coloring of the left icosahedron in Figure 11, while the icosahedrons in second and third row of Figure 12 depict that two distinct colors have chiral distribution of colors.  $\square$

There is a procedure to construct a small cover and a quasitoric manifold over a given polytope and we will illustrate it on a dodecahedron.

Let  $P^n$  be a simple polytope with  $m$  facets  $F_1, \dots, F_m$ . By Definition 1.5, every point in  $\pi^{-1}(\text{int}(F_i))$  has the same isotropy group which is one-dimensional subgroup of  $G_d^n$ . We denote it by  $G_d(F_i)$ .

**Definition 3.2.** Let  $P^n$  be a combinatorial simple polytope and  $l$  is a map from facets of  $P^n$  to one-dimensional subgroups of  $T^n$ . Then  $(P^n; l)$  is called a *characteristic pair* and  $l$  is called a *characteristic map* if

$$l(F_{i_1}) \times \dots \times l(F_{i_k}) \longrightarrow T^n$$

is injective whenever  $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$ .

Each  $G_d^n$ -manifold  $\pi_d : M^{dn} \rightarrow P^n$  determines a characteristic map  $l_d$  on  $P^n$

$$l_d : \{F_1, \dots, F_m\} \rightarrow \mathbb{K}_d^n$$

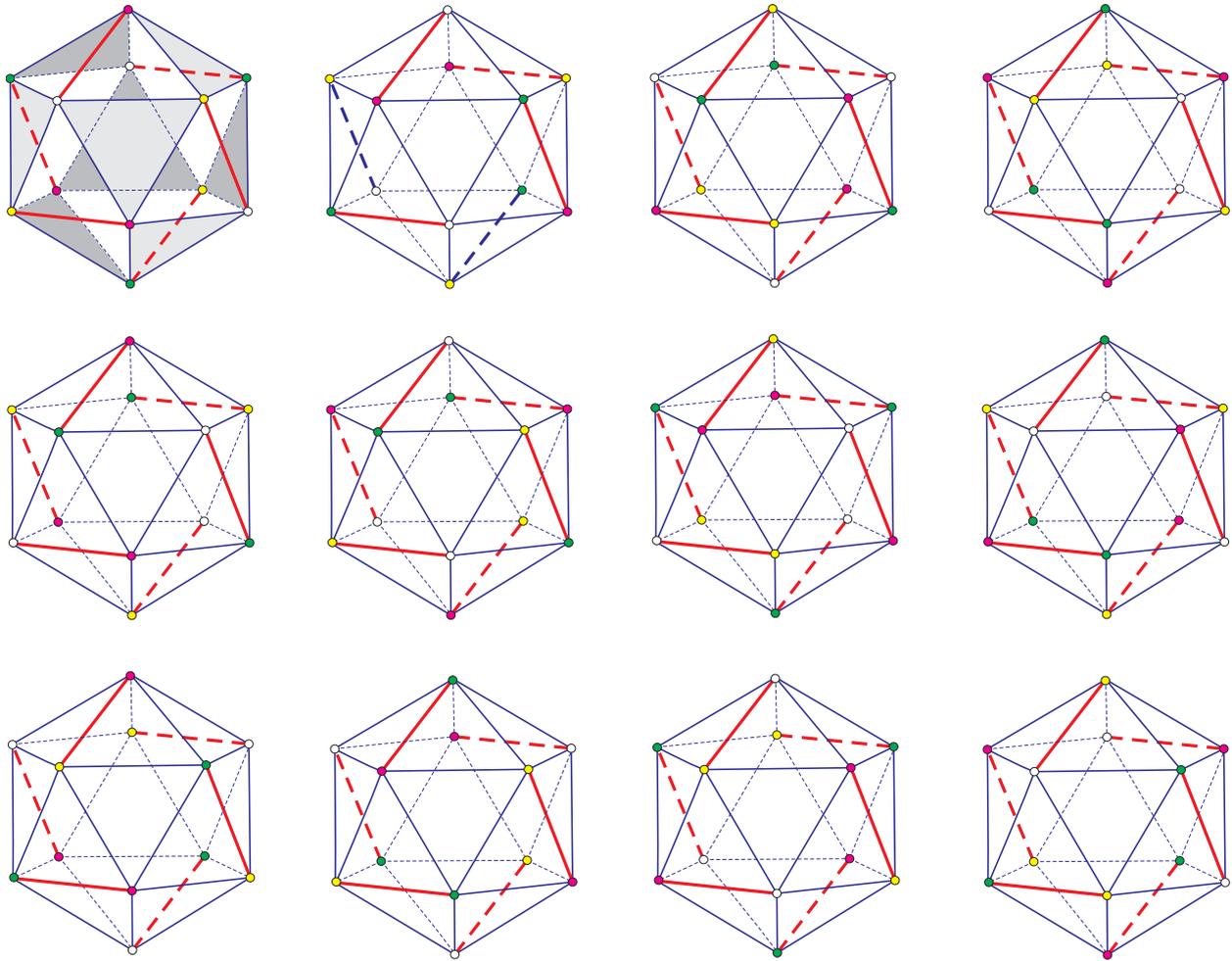


Figure 12: Symmetries of the icosahedron fixing six red edges

defined by mapping each facet of  $P^n$  to nonzero elements of  $\mathbb{K}_d^n$  (for notation see subsection 1.2) such that  $l_d(F_i) = \lambda_i = (\lambda_{1,i}, \dots, \lambda_{n,i})^t \in \mathbb{K}_d^n$ , where  $\lambda_i$  is primitive vector such that

$$G_d(F_i) = \{ (t^{\lambda_{1,i}}, \dots, t^{\lambda_{n,i}}) \mid t \in \mathbb{K}_d, |t| = 1 \}.$$

From the characteristic map we obtain an integer  $(n \times m)$ -matrix  $\Lambda_{\mathbb{K}_d}(M^{dn}) := (\lambda_{i,j})$  which is called *the characteristic matrix of  $M^{dn}$* . For  $d = 2$  each  $\lambda_i$  is determined up to sign. Since the  $G_d^n$ -action on  $M^{dn}$  is locally standard, the characteristic matrix  $\Lambda_{\mathbb{K}_d}(M^{dn})$  satisfies the nonsingular condition for  $P^n$ , i. e. if  $n$  facets  $F_{i_1}, \dots, F_{i_n}$  of  $P^n$  meet at vertex, then  $|\det \Lambda_{\mathbb{K}_d}^{(i_1, \dots, i_n)}(M^{dn})| = 1$ , where  $\Lambda_{\mathbb{K}_d}^{(i_1, \dots, i_n)}(M^{dn}) := (\lambda_{i_1}, \dots, \lambda_{i_n})$ . Any integer  $(n \times m)$ -matrix satisfying the non-singular condition for  $P^n$  is also called a *characteristic matrix on  $P^n$* .

Vice versa, we can construct a small cover and a quasitoric manifold over a polytope  $P$ , from the characteristic pair  $(P^n, \Lambda_{\mathbb{K}_d})$  as described in [20, Construction 5.12].

The cohomology ring of the  $G_d^n$ -manifold corresponding to the characteristic pair is known and could be described in the following way. There are two ideals naturally assigned to  $P^n$  and the characteristic matrix  $\Lambda_d$ . Let  $F_1, \dots, F_m$  be the facets of  $P^n$ . Let  $\mathbb{K}_d[v_1, \dots, v_m]$  be the polynomial algebra over  $\mathbb{K}_d$  on  $m$  generators with  $\deg(v_i) = d$ . The Stanley-Reisner ideal  $\mathcal{S}_P$  is the ideal generated by all square-free monomials  $v_{i_1} v_{i_2} \cdots v_{i_s}$

such that  $F_{i_1} \cap \dots \cap F_{i_s} = \emptyset$ . Let  $\Lambda_d = (\lambda_{ij})$  be a characteristic  $n \times m$  matrix over  $P^n$ . We define linear forms

$$\theta_i := \sum_{j=1}^m \lambda_{ij} v_j$$

and define  $\mathcal{J}$  to be the ideal in  $\mathbb{K}_d[v_1, \dots, v_m]$  generated by  $\theta_i$  for all  $i = 1, \dots, n$ . Let  $M^{dn}$  be a  $G_d^n$  manifold corresponding to the characteristic pair  $(P^n, \Lambda_d)$  and  $\pi : M^{dn} \rightarrow P^n$  be the orbit map. From Definition 1.5 we obtain that each  $\pi^{-1}(F_i)$  is a closed submanifold of dimension  $d(n - 1)$  which is itself a  $G_d^{n-1}$ -manifold over  $F_i$ . Let  $v_i \in H^d(M^{dn}; \mathbb{K}_d)$  denote its Poincaré dual. The ordinary cohomology of small covers and quasitoric manifolds (see [29]) is given by

$$H^*(M^{dn}) \simeq \mathbb{K}_d[v_1, \dots, v_m]/(\mathcal{S}_P + \mathcal{J}). \tag{4}$$

Let  $\mathcal{D}$  denote the dodecahedron and denote by  $\mathcal{I}$  the boundary of its dual, the icosahedron. Enumerate the facets of  $\mathcal{D}$  by numbers  $1, 2, \dots, 12$  using enumeration of the vertices of  $\mathcal{I}$ , as shown in the Figure 1. Divide all facets of  $\mathcal{D}$  in four groups, denoted  $a, b, c$  and  $d$  of three facets, so that within each group all facets are mutually disjoint, that is, there are no edges that they span. Fix the following division:  $\{1, 8, 10\}$ ,  $\{2, 9, 12\}$ ,  $\{3, 5, 7\}$  and  $\{4, 6, 11\}$ . We denote facets from the first set by  $a_1, a_2, a_3$ , from the second set by  $b_1, b_2, b_3$ , etc. and order them as they are written.

Consider the matrix

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix} \tag{5}$$

which defines a map  $l$  from the facets of  $\mathcal{D}$  to one-dimensional subgroups of  $T^3$ . That map  $l$  satisfies the condition from Definition 3.2, it is enough to check it at each vertex of  $\mathcal{D}$  separately. Therefore, the pair  $(\mathcal{D}, l)$  (or  $(\mathcal{D}, \Lambda)$ ) is a characteristic pair. Denote by  $M^6$  the quasitoric manifold over  $\mathcal{D}$  induced by this pair.

By [29, Theorem 3.1], the Betti numbers of  $M^6$  are equal to

$$\beta_2 = \beta_4 = 9, \beta_0 = \beta_6 = 1$$

while all other Betti numbers are zero.

By (4),

$$H^*(M^6; \mathbb{Z}) \simeq \mathbb{Z}[a_1, a_2, a_3, b_1, \dots, d_3]/(\mathcal{S}_{\mathcal{D}} + \mathcal{J})$$

where  $\mathcal{S}_{\mathcal{D}}$  is the Stanley-Reisner ideal of  $\mathcal{D}$  and  $\mathcal{J}$  is generated by  $\theta_1, \theta_2$  and  $\theta_3$  defined by rows of  $\Lambda$ . By abuse of notation, we denote the generators by the facets they represent. More precisely,

$$\theta_1 = a_1 + a_2 + a_3 - d_1 - d_2 - d_3$$

$$\theta_2 = b_1 + b_2 + b_3 - d_1 - d_2 - d_3$$

$$\theta_3 = c_1 + c_2 + c_3 - d_1 - d_2 - d_3.$$

Using relations obtained from  $\mathcal{J}$ , express three generators using other nine of them. For example,  $a_3 = d_1 + d_2 + d_3 - a_1 - a_2$ , likewise  $b_1$  and  $c_3$ .

To summarize, we get

$$H^*(M^6; \mathbb{Z}) = \mathbb{Z}[a_1, a_2, b_2, b_3, c_1, c_2, d_1, d_2, d_3]/\mathcal{S}_{\mathcal{D}}$$

where all generators are of degree 2.

By direct calculation we obtained the following:

- a) the generators of  $H^2$  are  $a_1, a_2, b_2, b_3, c_1, c_2, d_1, d_2, d_3$ ;
- b) the generators of  $H^4$  are all the squares of generators of  $H^2$ :  $a_1^2, \dots, d_3^2$ ;
- c) the generator of  $H^6$  is the fundamental class of  $Z_{\mathcal{D}}$  and can be represented by  $a_1^3$ ;
- d) multiplication of 2-dimensional generators is given in the Table 2. Numbers in brackets after generators represent the number of the facet corresponding to that generator. For example,  $b_2(9)$  means that generator  $b_2$  corresponds to the facet number 9 of the dodecahedron, or equivalently, to the vertex number 9 of the icosahedron in the Figure 1. Since all generators are of even degree, the multiplication is commutative.
- e) multiplication of 2-dimensional and 4-dimensional generators is given in Table 3, where  $X$  represents the fundamental class  $[M^6] \in H^6(M^6; \mathbb{Z})$ .

For the small cover  $M^3$  over dodecahedron  $\mathcal{D}$  induced by the characteristic pair  $(\mathcal{D}, \Lambda)$ , there is the same calculation. The difference is in dimensions of the generators and here we use  $\mathbb{Z}/2$  instead of  $\mathbb{Z}$  as coefficients. More precisely,

$$H^*(M^3; \mathbb{Z}/2) = \mathbb{Z}/2[a_1, a_2, b_2, b_3, c_1, c_2, d_1, d_2, d_3]/\mathcal{S}_{\mathcal{D}}$$

where all generators are of degree 1.

All other conclusions including multiplication tables are the same, as for quasitoric manifold. One should notice that since we are dealing with  $\mathbb{Z}/2$  coefficients, there are no signs in the Table 2.

	$a_1(1)$	$a_2(8)$	$b_2(9)$	$b_3(12)$	$c_1(3)$	$c_2(5)$	$d_1(4)$	$d_2(6)$	$d_3(11)$
$a_1$	$a_1^2$	0	0	0	$c_1^2$	$a_1^2 - c_1^2$	$d_1^2 - a_2^2$	$a_1^2 + a_2^2 - d_1^2$	0
$a_2$	0	$a_2^2$	$a_2^2 - b_3^2$	$b_3^2$	0	$c_1^2 + c_2^2 - a_1^2$	$a_2^2$	0	0
$b_2$	0	$a_2^2 - b_3^2$	$b_2^2$	0	0	$c_2^2$	0	$b_2^2$	0
$b_3$	0	$b_3^2$	0	$b_3^2$	*	0	$d_1^2$	0	$b_3^2 - d_1^2$
$c_1$	$c_1^2$	0	0	*	$c_1^2$	0	$d_1^2 + d_2^2 - c_2^2$	0	$c_1^2 + c_2^2 - d_1^2 - d_2^2$
$c_2$	$a_1^2 - c_1^2$	$c_1^2 + c_2^2 - a_1^2$	$c_2^2$	0	0	$c_2^2$	$c_2^2 - d_2^2$	$d_2^2$	0
$d_1$	$d_1^2 - a_2^2$	$a_2^2$	0	$d_1^2$	$d_1^2 + d_2^2 - c_2^2$	$c_2^2 - d_2^2$	$d_1^2$	0	0
$d_2$	$a_1^2 + a_2^2 - d_1^2$	0	$b_2^2$	0	0	$d_2^2$	0	$d_2^2$	0
$d_3$	0	0	0	$b_3^2 - d_1^2$	$c_1^2 + c_2^2 - d_1^2 - d_2^2$	0	0	0	$d_3^2$

Table 2: Multiplication in  $H^2(M^6; \mathbb{Z})$

where  $*$  =  $b_3c_1 = c_1b_3 = b_2^2 + b_3^2 + c_1^2 - d_1^2 - d_2^2 - d_3^2$ .

	$a_1^2$	$a_2^2$	$b_2^2$	$b_3^2$	$c_1^2$	$c_2^2$	$d_1^2$	$d_2^2$	$d_3^2$
$a_1$	X	0	0	0	X	0	0	X	0
$a_2$	0	X	0	X	0	X	X	0	0
$b_2$	0	0	X	0	0	X	0	X	0
$b_3$	0	X	0	X	0	0	X	0	0
$c_1$	X	0	0	0	X	0	X	0	0
$c_2$	0	X	X	0	0	X	0	X	0
$d_1$	0	X	0	X	X	0	X	0	0
$d_2$	X	0	X	0	0	X	0	X	0
$d_3$	0	0	0	0	0	0	0	0	X

Table 3:  $H^2(M^6; \mathbb{Z}) \cup H^4(M^6; \mathbb{Z})$

#### 4. Massey Products in $H^*(\mathcal{Z}_{\mathcal{I}})$ and Poincaré Series for $\mathbb{k}[\mathcal{I}]$

We start this section with a definition of a Massey product in the cohomology of a differential graded algebra. First, we need the notion of a defining system for an ordered set of cohomology classes. Our presentation follows that in [45], [21, Appendix  $\Gamma$ ] and [17].

**Definition 4.1.** Suppose  $(A, d)$  is a differential graded algebra,  $\alpha_i = [a_i] \in H^*[A, d]$  and  $a_i \in A^{n_i}$  for  $1 \leq i \leq k$ . Then a *defining system* for  $(\alpha_1, \dots, \alpha_k)$  is a  $(k+1) \times (k+1)$ -matrix  $C$  satisfying the following conditions:

- (1)  $c_{i,j} = 0$  if  $i \geq j$ ,
- (2)  $c_{i,i+1} = a_i$ ,
- (3)  $a \cdot E_{1,k+1} = dC - \bar{C} \cdot C$  for some  $a = a(C) \in A$ , where  $\bar{c}_{i,j} = (-1)^{\deg(c_{i,j})} \cdot c_{i,j}$  and  $E_{1,k+1}$  is a  $(k+1) \times (k+1)$ -matrix with '1' in the position  $(1, k+1)$  and all other entries equal to zero.

One readily checks that the above conditions imply that  $d(a) = 0$  and  $a \in A^m$ ,  $m = n_1 + \dots + n_k - k + 2$ . Therefore, the cohomology class  $\alpha = [a(C)] \in H^m[A, d]$  of  $a = a(C)$  is defined.

**Definition 4.2.** A *Massey product*  $\langle \alpha_1, \dots, \alpha_k \rangle$  is the set of all cohomology classes of the type  $\alpha = [a(C)]$ , where  $C$  is a defining system for  $(\alpha_1, \dots, \alpha_k)$ . The  $k$ -fold Massey product  $\langle \alpha_1, \dots, \alpha_k \rangle$  is said to be *defined* if  $\langle \alpha_1, \dots, \alpha_k \rangle \neq \emptyset$ , that is, if there exists a defining system  $C$  for it. A defined Massey product is called:

- (i) *trivial* if  $[a(C)] = 0$  for some  $C$ ;
- (ii) *decomposable* if  $[a(C)] \in H^+(A) \cdot H^+(A)$  for some  $C$ ;
- (iii) *strictly defined* if  $\langle \alpha_1, \dots, \alpha_k \rangle = \{[a(C)]\}$  for some  $C$ .

It can also be verified that a Massey product  $\langle \alpha_1, \dots, \alpha_k \rangle$  is determined by the cohomology classes  $\{\alpha_i, 1 \leq i \leq k\}$  of particularly chosen representing cocycles  $\{a_i, 1 \leq i \leq k\}$ , which we used in the above definition, see [45].

In what follows we call a  $k$ -fold Massey product for  $k = 3$  a *triple* Massey product and we refer to a  $k$ -fold Massey product for  $k > 3$  as a *higher* Massey product.

**Example 4.3.** Suppose in the above definition  $k = 4$ . If  $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$  is defined, then we have:

$$a = d(c_{1,5}) - \bar{a}_1 \cdot c_{2,5} - \bar{c}_{1,3} \cdot c_{3,5} - \bar{c}_{1,4} \cdot a_4,$$

$$d(c_{1,3}) = \bar{a}_1 \cdot a_2, d(c_{1,4}) = \bar{a}_1 \cdot c_{2,4} + \bar{c}_{1,3} \cdot a_3, d(c_{2,4}) = \bar{a}_2 \cdot a_3, d(c_{2,5}) = \bar{a}_2 \cdot c_{3,5} + \bar{c}_{2,4} \cdot a_4, d(c_{3,5}) = \bar{a}_3 \cdot a_4.$$

These identities show that for a higher Massey product  $\langle \alpha_1, \dots, \alpha_k \rangle$  to be defined it is necessary that all the lower order Massey products of consecutive elements are defined and *trivial simultaneously*. If all the lower order Massey products of consecutive elements vanish, but not simultaneously, then the whole product may not exist, see [54, Example I].

May [52] introduced a generalization of the ordinary Massey product, the *matrix Massey product*, and studied its basic properties. He also showed that differentials in the Eilenberg–Moore spectral sequence of the path loop fibration for any path connected simply connected space can be described in terms of (matrix) Massey products. However, Massey products are not completely determined by those differentials, see [54, Example II]. For the discussion of ordinary and matrix Massey products and their applications in rational homotopy theory and symplectic geometry see [3, 52].

From now on let  $\mathcal{I}$  denote the boundary of icosahedron, see Figure 1.

**Proposition 4.4.** Suppose  $a_i = v_i u_{i+6}$  for  $i = 2, \dots, 6$  are 3-dimensional cocycles in  $R^{-1A}(\mathcal{I})$  representing the classes  $\alpha_{i-1} = [a_i]$ . Then the 5-fold Massey product  $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle \subset H^*(\mathcal{Z}_{\mathcal{I}})$  is defined and trivial.

*Proof.* Consider the following defining system C:

$$c_{1,2} = v_2u_8, c_{2,3} = -v_9u_3, c_{3,4} = v_4u_{10}, c_{4,5} = -v_{11}u_5, c_{5,6} = v_6u_{12},$$

and all other  $c_{i,j}$  being zero. Then  $0 \in \langle \alpha_1, \dots, \alpha_5 \rangle$  and this Massey product is defined and vanishes.  $\square$

Note, that any of the triple products of consecutive elements in  $\langle \alpha_1, \dots, \alpha_5 \rangle$  is trivial by [31, Theorem 6.1.1], see Figure 13. Furthermore, observe that a nonzero cohomology class is contained in such a product alongside with zero, for example

$$[v_4v_8u_2u_3u_9u_{10}] \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle.$$

**Remark 4.5.** It follows from the proof of [31, Theorem 6.1.1] that a nontrivial triple Massey product of 3-dimensional classes in  $H^*(\mathcal{Z}_K)$  is given by a unique and decomposable cohomology class.

**Theorem 4.6.** (a) *There are no nontrivial Massey products  $\langle \alpha_1, \dots, \alpha_k \rangle \subset H^*(\mathcal{Z}_I)$  with  $\dim \alpha_1 = \dots = \dim \alpha_k = 3$  for  $k = 3$ , or  $k \geq 6$ ;*

(b) *Consider the following classes in  $H^3(\mathcal{Z}_I)$  (see Figure 14)*

$$\alpha_1 = [v_2u_9], \alpha_2 = [v_3u_{10}], \alpha_3 = [v_4u_{11}], \alpha_4 = [v_5u_{12}].$$

*Then the Massey product  $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$  is strictly defined and nontrivial.*

*Proof.* Recall that for an element  $\alpha \in \langle \alpha_1, \dots, \alpha_k \rangle \subset H^*(\mathcal{Z}_I)$ ,  $\dim \alpha = 2k + 2$ . The moment-angle manifold  $\mathcal{Z}_I$  is 2-connected, thus  $H^p(\mathcal{Z}_I) = 0$  for all  $p \geq m + n - 2 = 13, p \neq 15$ . Observe that all the defined  $k$ -fold Massey products of 3-dimensional classes in  $H^*(\mathcal{Z}_I)$  for  $k \geq 6$  vanish for dimensional reasons.

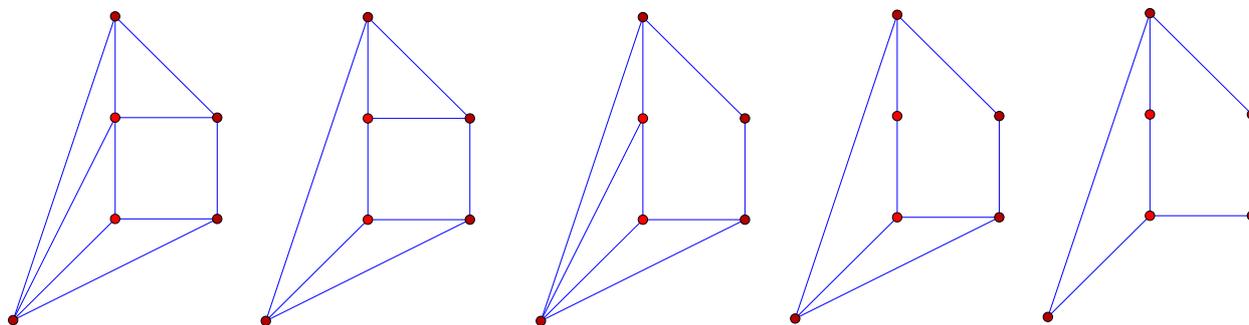


Figure 13: Five obstruction graphs for nonexistence of a triple Massey product of 3-dimensional classes in  $H^*(\mathcal{Z}_K)$

To prove the rest of statement (a) assume the converse is true. Then, by [31, Theorem 6.1.1], there exists an induced subgraph in  $sk^1(I)$  of one of the five types shown in Figure 13. This implies that among the flagizations of those five obstruction graphs there exists at least one which is isomorphic to a full subcomplex in  $I$  on 6 vertices. Furthermore, each of these flagizations is easily seen to be homotopy nontrivial. However, none of these five flagizations appears in Figure 6 above. We get a contradiction which finishes the proof of the first statement.

To prove statement (b), let us denote

$$I_1 = \{2, 9\}, I_2 = \{3, 10\}, I_3 = \{4, 11\}, I_4 = \{5, 12\}.$$

Figure 14 shows that

$$\tilde{H}^0(I_{I_1 \sqcup I_2}) = \tilde{H}^0(I_{I_2 \sqcup I_3}) = \tilde{H}^0(I_{I_3 \sqcup I_4}) = 0$$

and

$$\tilde{H}^1(I_{I_1 \sqcup I_2}) = \tilde{H}^1(I_{I_2 \sqcup I_3}) = \tilde{H}^1(I_{I_3 \sqcup I_4}) = 0.$$

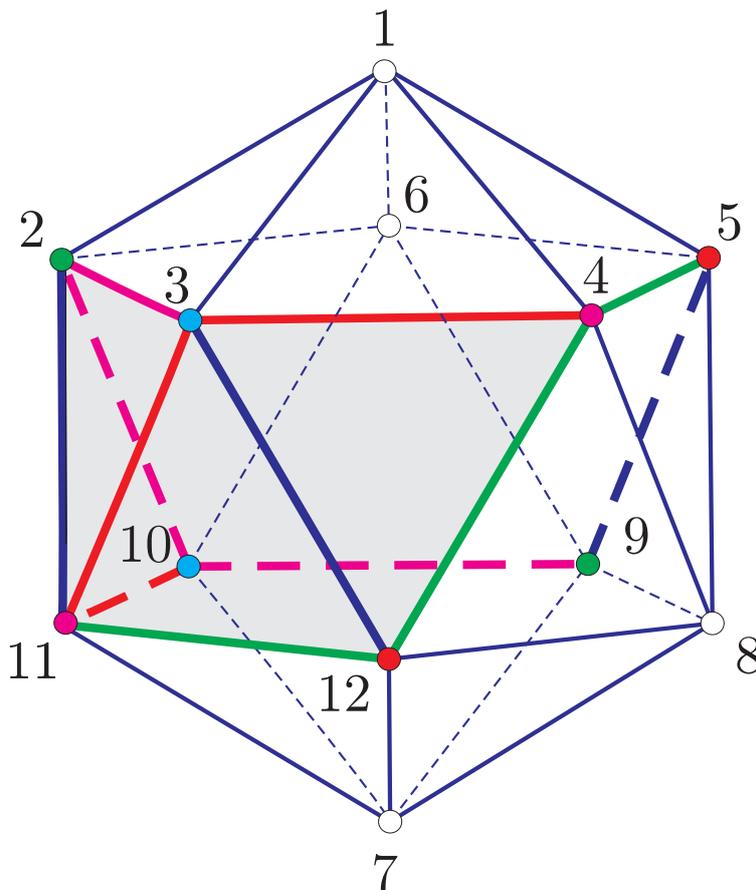


Figure 14: Classes  $\alpha_1, \alpha_2, \alpha_3$ , and  $\alpha_4$  in  $H^*(\mathcal{Z}_I)$

Furthermore, Figure 14 shows that

$$\tilde{H}^0(\mathcal{I}_{I_1 \sqcup I_2 \sqcup I_3}) = \tilde{H}^0(\mathcal{I}_{I_2 \sqcup I_3 \sqcup I_4}) = 0$$

and

$$\tilde{H}^1(\mathcal{I}_{I_1 \sqcup I_2 \sqcup I_3}) = \tilde{H}^1(\mathcal{I}_{I_2 \sqcup I_3 \sqcup I_4}) = 0.$$

Due to [46, Lemma 3.3], the 4-fold Massey product  $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$  is strictly defined. A straightforward calculation shows that (up to sign)

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \{[v_5 v_9 u_2 u_3 u_4 u_{10} u_{11} u_{12}]\}.$$

The cocycle

$$v_5 v_9 u_2 u_3 u_4 u_{10} u_{11} u_{12} \in R^{-6, 2(0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1)}(\mathcal{I})$$

is not a coboundary, since if in its multigraded component

$$v_5 v_9 u_2 u_3 u_4 u_{10} u_{11} u_{12} = d\left(\sum_{j=2}^{12} a_j v_j u_2 \dots \hat{u}_j \dots u_{12}\right)$$

then

$$a_2 = a_4 = a_{11} = a_9 = -a_3 = -a_{10} = -a_{12} = -a_5$$

and  $-a_5 - a_9 = 1$ , which is a contradiction.  $\square$

Let  $P = P^4$  be the 120-cell, that is, a convex regular simple polytope all of whose 120 facets are dodecahedra  $\mathcal{D}$ , see [37, §3.4].

**Proposition 4.7.** (a) *There exists a strictly defined nontrivial Massey product of order 4 in  $H^*(\mathcal{Z}_P)$ . In particular,  $\mathcal{Z}_P$  is a nonformal manifold.*

(b) *The hyperbolic 3-manifold  $\mathcal{R}_{\mathcal{D}}$  is a submanifold and a retract of the hyperbolic 4-manifold  $\mathcal{R}_P$ . In particular,  $\pi_1(\mathcal{R}_P)$  is a semi-direct product of  $[RC_{\mathcal{D}}, RC_{\mathcal{D}}]$  with  $\text{Ker } r_*$ , where  $r : \mathcal{R}_P \rightarrow \mathcal{R}_{\mathcal{D}}$  is the retraction. There is a short exact sequence*

$$1 \rightarrow \text{Ker } r_* \rightarrow \pi_1(\mathcal{R}_P) \rightarrow [RC_{\mathcal{D}}, RC_{\mathcal{D}}] \rightarrow 1.$$

*Proof.* Since  $P$  has no 3- or 4-belts of facets, it is a flag polytope. Due to [15, Theorem 2.16],  $\mathcal{Z}_{\mathcal{D}} \hookrightarrow \mathcal{Z}_P$  is a submanifold and a retract of  $\mathcal{Z}_P$ . Statement (a) now follows by applying Theorem 4.6 (b).

For statement (b),  $\mathcal{R}_{\mathcal{D}}$  is a submanifold of  $\mathcal{R}_P$ , since  $\mathcal{D} \subset P$  is a face, and the existence of a retraction follows from [56, Proposition 2.2] and [15, Lemma 2.13]. By Proposition 2.19 (a), fundamental group of  $\mathcal{R}_{\mathcal{D}}$  equals  $[RC_{\mathcal{D}}, RC_{\mathcal{D}}]$ , which finishes the proof.  $\square$

**Remark 4.8.** (1) For more on the combinatorics of  $\mathcal{D}$  and  $P^4$ , and the hyperbolic structures on small covers and real moment-angle manifolds over dodecahedron and 120-cell, see [6, 14, 37]. (2) Zhuravleva [59] constructed a nontrivial triple Massey product in  $H^*(\mathcal{Z}_K)$  when  $K = K_P$  is a nerve complex of an arbitrary Pogorelov polytope  $P$ . The latter class includes all fullerenes and, in particular, the dodecahedron.

Now we turn to a discussion of Poincaré series of face rings and related topics.

In homological theory of local rings nontrivial Massey products in Tor-algebras  $\text{Tor}_*^S(R, \mathbb{k})$  play an important role. Here,  $S$  is a polynomial ring,  $R = S/I$  is a monomial ring, and  $\text{Tor}_*^S(R, \mathbb{k})$  is the so called Koszul homology of the local ring  $R$ , see [42].

**Definition 4.9.** Suppose  $\mathbb{k}$  is a field. A face ring  $\mathbb{k}[K]$  is called *Golod (over  $\mathbb{k}$ )* if multiplication and all (triple and higher) Massey products in  $\text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^+(\mathbb{k}[K], \mathbb{k})$  vanish. If this holds over every field  $\mathbb{k}$ , the complex  $K$  is also called *Golod*.

This area of homological algebra dates back to the pioneering work of Golod [39], in which it was proved that a local Noetherian commutative ring  $R$  is *Golod*, that is, the product and all Massey products in its Koszul homology are trivial ( $\mathbb{k} = R/I$ ,  $I$  is the unique maximal ideal in  $R$ ), if and only if the Poincaré series of  $R$  is given by a rational function of a certain type. For a face ring  $\mathbb{k}[K]$  and its Tor-algebra  $\text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{*,*}(\mathbb{k}[K], \mathbb{k})$ , an analogous result was proved by Grbić and Theriault [41, Theorem 11.1].

Namely, the previous definition is equivalent to the following property of a Poincaré series of a face ring.

**Definition 4.10.** A face ring  $\mathbb{k}[K]$  is called *Golod (over  $\mathbb{k}$ )* if the following identity for the Poincaré series of its Yoneda algebra holds

$$P(\text{Ext}_{\mathbb{k}[K]}(\mathbb{k}, \mathbb{k}); t) = \frac{(1+t)^m}{1 - \sum_{i,j>0} \beta^{-i,2j}(\mathbb{k}[K])t^{-i+2j-1}}.$$

We denote the left hand side by  $P(\mathbb{k}[K]; t)$ , or simply,  $P(K; t)$ , and call it the *Poincaré series* of  $\mathbb{k}[K]$ , or simply, of  $K$ .

**Remark 4.11.** Note that the Poincaré series of the Yoneda algebra above (or, equivalently, the Poincaré series of a face ring) acquires a topological interpretation in toric topology. Namely, there is an isomorphism of graded (noncommutative) algebras if  $\mathbb{k}$  is a field, see [18, Theorem 5.3.4]

$$\text{Ext}_{\mathbb{k}[K]}(\mathbb{k}, \mathbb{k}) \cong H_*(\Omega(\mathbb{C}P^\infty, *)^K; \mathbb{k})$$

and thus

$$P(\mathbb{k}[K]; t) = P(H_*(\Omega(\mathbb{C}P^\infty, *)^K; \mathbb{k}); t).$$

Moreover, the following inequality holds.

**Theorem 4.12.** ([41, Theorem 11.1]) *For any simplicial complex  $K$  there is an inequality*

$$P(\mathbb{k}[K]; t) \leq \frac{t(1+t)^m}{t - P(\tilde{H}^*(\mathcal{Z}_K; \mathbb{k}); t)}.$$

The equality is obtained if and only if  $\mathbb{k}[K]$  is Golod.

As a counterexample to an earlier claim by Berglund and Jöllenbeck, Katthän [44] constructed a simplicial complex  $K$  such that all products of elements of positive degrees are trivial in  $\text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{*,*}(\mathbb{k}[K], \mathbb{k})$ , but  $\mathbb{k}[K]$  is not Golod having a nontrivial triple Massey product in its Koszul homology  $\text{Tor}_{\mathbb{k}[v_1, \dots, v_m]}^{*,*}(\mathbb{k}[K], \mathbb{k})$ . Furthermore, it was also shown in [44, Theorem 4.1] that a monomial ring  $R$  is Golod if all  $r$ -fold Massey products vanish for all  $r \leq \max(2, \text{reg}(R) - 2)$ .

On the other hand, finding a widest possible class of simplicial complexes  $K$  for which Golodness of  $\mathbb{k}[K]$  is equivalent to vanishing of the cup product in  $H^*(\mathcal{Z}_K; \mathbb{k})$  remains a good question to look at. It follows from [40, Theorem 4.6] that this property holds for all flag simplicial complexes  $K$ . Frankhuizen [34] has recently proved that for a monomial ring  $R$  whose minimal free resolution is rooted,  $R$  is Golod if and only if the product in  $\text{Tor}_r^{\mathbb{k}}(R, \mathbb{k})$  is trivial.

We finish this section with a discussion on the Poincaré series of the Stanley-Reisner ring  $\mathbb{k}[I]$  of the boundary of icosahedron  $I$ . Here, we also compute the Poincaré series of the Pontryagin algebra  $H_*(\Omega\mathcal{Z}_I; \mathbb{k})$ . In what follows  $\mathbb{k}$  denotes either a field, or the ring of integers  $\mathbb{Z}$ .

**Proposition 4.13.** *For the Poincaré series of  $\mathbb{k}[I]$ ,*

$$P(I; t) = \frac{(1+t)^3}{1-9t+9t^2-t^3} < \frac{t(1+t)^m}{t - P(\tilde{H}^*(\mathcal{Z}_I; \mathbb{k}); t)} = \frac{(1+t)^{12}}{1-Q(t)}$$

where  $Q(t) = 36t^2 + 160t^3 + 315t^4 + 300t^5 + 124t^6 + 124t^7 + 300t^8 + 315t^9 + 160t^{10} + 36t^{11} + t^{14}$ . Moreover, the Pontryagin algebra  $H_*(\Omega\mathcal{Z}_I; \mathbb{Z})$  is torsion free and the following equality holds for its Poincaré series

$$P(H_*(\Omega\mathcal{Z}_I; \mathbb{k}); t) = \frac{1}{(1+t)^9(1-9t+9t^2-t^3)}.$$

*Proof.* Since  $\mathcal{D}$  is a simple 3-dimensional polytope, using the Euler formula, one gets

$$f(\mathcal{D}) = (f_0(\mathcal{D}), 3f_0(\mathcal{D}) - 6, 2f_0(\mathcal{D}) - 4) = (12, 30, 20)$$

By definition,

$$h_0t^3 + h_1t^2 + h_2t + h_3 = (t-1)^3 + f_0(t-1)^2 + f_1(t-1) + f_2$$

which implies that  $\mathbf{h}(\mathcal{D}) = (1, 9, 9, 1)$ . Applying [55, Proposition 9.5], the Poincaré series  $P(H_*(\Omega\mathcal{Z}_I; \mathbb{k}); t)$  is given by

$$\frac{1}{(1+t)^{m-n}(1-h_1t + \dots + (-1)^n h_n t^n)} = \frac{1}{(1+t)^9(1-9t+9t^2-t^3)}$$

since  $I = \partial\mathcal{D}^*$  is a flag simplicial complex.

Furthermore, by [55, (8.2)] there is an exact sequence of noncommutative algebras

$$1 \rightarrow H_*(\Omega\mathcal{Z}_K; \mathbb{k}) \rightarrow H_*(\Omega(\mathbb{C}P^\infty, *)^K; \mathbb{k}) \rightarrow \Lambda[u_1, \dots, u_m] \rightarrow 1$$

for any simplicial complex  $K$ , where  $\mathbb{k}$  is a field or the ring of integers  $\mathbb{Z}$ , and  $\Lambda[u_1, \dots, u_m]$  is the exterior algebra on  $m$  generators of degree one. It follows that

$$P(I; t) = P(H_*(\Omega(\mathbb{C}P^\infty, *)^I; \mathbb{k})) = P(H_*(\mathbb{T}^m; \mathbb{k}); t) \cdot P(H_*(\Omega\mathcal{Z}_I; \mathbb{k}); t)$$

which gives the desired value of the Poincaré series of  $\mathcal{I}$ .

The strong inequality in the statement follows from Theorem 4.12, since  $\mathbb{k}[\mathcal{I}]$  is not a Golod ring by Theorem 4.6 (b). Explicit calculation shows that the Betti numbers vector of  $\mathcal{Z}_{\mathcal{I}}$  is given by

$$\beta(\mathcal{Z}_{\mathcal{I}}) = (1, 0, 0, 36, 160, 315, 300, 124, 124, 300, 315, 160, 36, 0, 0, 1).$$

It follows that  $P(\tilde{H}^*(\mathcal{Z}_{\mathcal{I}}; \mathbb{k}); t) = tQ(t)$  which implies the right hand side of the inequality in the statement.

Finally, the Pontryagin algebra  $H_*(\Omega\mathcal{Z}_{\mathcal{I}}; \mathbb{Z})$  is torsion free by [40, Corollary 5.2]. This finishes the proof.  $\square$

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