Rational orthonormal bases, state transformations, and dissipativity

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***Abstract*—We establish formulas relating the state of a continuous-time system with that of the transformed one in a Takenaka-Malmquist-Kautz orthogonal basis. We use such relation to give a simple proof that if the original system is dissipative, then the transformed one is also dissipative with the same storage function. We apply our results to continuous-time subspace system identification.**

I. INTRODUCTION

Rational orthogonal basis functions have been used for decades to decompose continuous time signals in a series representation, using recursively-generated functions orthogonal to each other in the $\mathcal{L}_2$-sense. When the signals being represented are the input and the output of a linear continuous-time system, a discrete *transformed system*, formally introduced in sect. II-B, can be defined that maps the basis representation of the one signal into the other. Such transformation has been used effectively in (approximate) modelling, system identification, control theory, robust control, and signal processing (see [4], [6], [11]–[14], [16], [17], [22]).

Past investigations in this area have adopted a purely input-output point of view, in which the state of the transformed system is only instrumental to the computation of the input-output map in the orthogonal basis, and to the study of its properties. To the best of the authors' knowledge, no attention has been given to the actual relation of the state of the original continuous-time system with that of the transformed one. One of the contributions of this paper consists in formulas (see sect. III-A) relating such state variables to each other: we show that the state of the transformed system is a weighted average of the coefficients of the orthonormal basis representations of the continuous-time system state and of its derivative.

In sect. III-B, we use such expression to establish that the orthonormal basis representation of the state of the continuous-time system satisfies a difference equation of first order in the state and in the input coefficient sequences. In sect. IV, we exploit our formulas to give a simple proof of the preservation of dissipativity of the continuous-time system to that of the transformed system, generalizing our previous results for the Hambo basis case (see [18]) to the more general case of bases obtained through a Takenaka-Malmquist-Kautz construction (see [3], [9]). We also show that our formulas provide a "variational" interpretation of the transformed system state.

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A mathematical model that does not mirror the physical properties of a system is useless for practical purposes. Consequently, in system identification and model order reduction it is particularly relevant to devise algorithms ensuring that structural properties of the original system (e.g. passivity) are preserved in the identified or reduced-order model (see e.g. [20], [21]). In sect. V we illustrate an application of our results to continuous-time system identification, where we present some preliminary results refining existing procedures, showing the potential of our approach to contribute to the development of new algorithms that preserve dissipation. Another potential area of application is robust control, where basis functions are used for the approximation of multipliers in the IQC approach (see [23], in particular remark 5 p. 6).

II. BACKGROUND MATERIAL AND NOTATION

A. Orthonormal bases

The space of real-valued functions square integrable on $\mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geq 0\}$ is denoted as $L^2(0,\infty)$, with the norm of $u \in L^2(0,\infty)$ defined as $\|u\| := \sqrt{\int_0^\infty |u(t)|^2 dt}$. The Hardy space H^2 is the space of functions from $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \Re(z) \geq 0\}$ to $\mathbb{C}$ analytic on the right half plane, with norm $\|\widehat{u}\| := \sup_{\sigma>0} \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |\widehat{u}(\sigma+j\omega)|^2 d\omega}$. The spaces $L^2(0,\infty)$ and H^2 can be identified with each other via the Fourier or Laplace transform. If $u : \mathbb{R} \to \mathbb{R}$, then $\widehat{u}$ denotes its transform.

These definitions are generalized in the usual way to vector-valued functions. To avoid cumbersome notation, when it is clear from the context what the dimension of the vectors is, we use the same symbols to denote the set of square integrable vector-valued functions and that of their transforms.

To define an orthonormal basis for H^2, we follow the Takenaka-Malmquist-Kautz construction for the case of a bounded sequence $\lambda_k \in \mathbb{R}_+$, $k = 0, \ldots$, that is also bounded away from zero. We consider the case of complex λ_k in Rem. 6. For each $k \in \mathbb{N}$, we define the inner functions

$$\widehat{\phi}_k(s) := \frac{s-\lambda_k}{s+\lambda_k}\ ,\ k = 0, \ldots\ , \tag{1}$$

and from it we construct the sequence defined by

$$\begin{aligned} \widehat{b}_0(s) &:= \frac{\sqrt{2\lambda_0}}{s+\lambda_0} \\ \widehat{b}_k(s) &:= \left(\prod_{i=0}^{k-1} \widehat{\phi}_i(s)\right) \frac{\sqrt{\lambda_k}}{\lambda_k+s}\ ,\ k \geq 1\ . \end{aligned} \tag{2}$$

Special cases of this construction are $\lambda_k = \lambda$ (Laguerre basis) and $\lambda_{k+p} = \lambda_k$, where $p > 1$ is the fixed period of the sequence $\{\lambda_k\}$ (Hambo or generalized orthonormal basis).

Given the identification of H^2 with $L^2(0,\infty)$, if $\{\widehat{b}_k\}_{k=0,\ldots}$ is an orthonormal basis for H^2 then the sequence of real-valued functions $\{b_k\}_{k=0,\ldots}$ is an orthonormal basis for $L^2(0,\infty)$. It follows that every $f \in L^2(0,\infty)$ can be written in terms of an orthonormal basis $\{b_k\}_{k=0,\ldots}$ as $f = \sum_{k=0}^{\infty} \widetilde{f}_k b_k$, for suitable coefficients $\widetilde{f}_k \in \mathbb{R}$, $k = 0, 1, \ldots$. When f is a vector-valued function, the orthonormal basis representation is component-wise and the coefficients are real vectors.

B. The transformed system

We study state space systems described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \, , \end{aligned} \tag{3}$$

with A Hurwitz and $x(0) = 0$. In the rest of this paper, the state-space dimension is denoted by n, the number of input variables by m, and the number of outputs by p.

If $u \in L^2(0,\infty)$ in (3), with orthonormal basis representation $u = \sum_{k=0}^{+\infty} \widetilde{u}_k b_k$, then also $y \in L^2(0,\infty)$ and can be represented by $y = \sum_{k=0}^{+\infty} \widetilde{y}_k b_k$ for some sequence $\{\widetilde{y}_k\}_{k=0,\ldots}$ of real vector coefficients. The coefficients $\{\widetilde{u}_k\}$ and $\{\widetilde{y}_k\}$ of the series expansion of the input, respectively output trajectory of (3) are related to each other through the dynamics of the time-varying transformed system

$$\begin{aligned} \widetilde{\xi}_{k+1} &= \widetilde{A}_k \widetilde{\xi}_k + \widetilde{B}_k \widetilde{u}_k \\ \widetilde{y}_k &= \widetilde{C}_k \widetilde{\xi}_k + \widetilde{D}_k \widetilde{u}_k \, , \end{aligned} \tag{4}$$

where

$$\begin{aligned} \widetilde{A}_k &:= (\lambda_k I - A)^{-1} (\lambda_k I + A) \\ \widetilde{B}_k &:= \sqrt{2\lambda_k} (\lambda_k I - A)^{-1} B \\ \widetilde{C}_k &:= \sqrt{2\lambda_k} C (\lambda_k I - A)^{-1} \\ \widetilde{D}_k &:= C (\lambda_k I - A)^{-1} B + D \, . \end{aligned} \tag{5}$$

The expressions for $\tilde{A}_k$ and $\tilde{D}_k$ in (5) are the same as those in Prop. 1 of [16]. The expressions for $\tilde{B}_k$ and $\tilde{C}_k$ in (5) follow from evaluating the integrals for $\mathbf{B}$ and $\mathbf{C}$ given therein. See also Lemma 4 p. 370 of [14].

C. Continuous- and discrete-time dissipativity

Let $\Sigma = \Sigma^\top \in \mathbb{R}^{(m+p)\times(m+p)}$; the system (3) is *dissipative* with respect to the *supply rate*

$$Q_\Sigma(u, y) := \begin{bmatrix} u^\top & y^\top \end{bmatrix} \Sigma \begin{bmatrix} u \\ y \end{bmatrix}$$

if there exists $P = P^\top \in \mathbb{R}^{n\times n}$ such that $\frac{d}{dt}\left(x^\top P x\right) \leq Q_\Sigma(u, y)$ for all trajectories (x, u, y) satisfying (3). The quadratic form $Q_P(x) := x^\top P x$ is called a *storage function*. Dissipativity is equivalent to the existence of $\Phi = \Phi^\top \in \mathbb{R}^{n+m}$, $\Phi \geq 0$, such that for all (x, y, u) satisfying (3) the *dissipation equality*

$$Q_\Sigma(u, y) = \frac{d}{dt}\left(x^\top P x\right) + Q_\Phi(x, u) \tag{6}$$

is satisfied. Such Q_Φ is called the *dissipation rate*. Using the second equality in (3) to rewrite the supply rate as a quadratic function $Q_{\Sigma'}$ of (x, u), the dissipation equality is shown to be equivalent with the *linear matrix inequality*

$$\Sigma' - \begin{bmatrix} A^\top P + PA & PB \\ B^\top P & 0_{m\times m} \end{bmatrix} \geq 0 \, . \tag{7}$$

The definition of dissipativity for a discrete time-invariant system $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}\right)$ is based on the dissipation equality

$$Q_\Sigma(u_k, y_k) = x_{k+1}^\top P x_{k+1} - x_k^\top P x_k + Q_\Phi(x_k, u_k) \, . \tag{8}$$

Rewriting Q_Σ as a function $Q_{\Sigma'}$ of (x_k, u_k), (8) is equivalent with the discrete-time linear matrix inequality

$$\Sigma' - \begin{bmatrix} \widetilde{A}^\top P \widetilde{A} - P & \widetilde{A}^\top P \widetilde{B} \\ \widetilde{B}^\top P \widetilde{A} & \widetilde{B}^\top P \widetilde{B} \end{bmatrix} \geq 0 \, . \tag{9}$$

Adapting the definition of [5] for discrete *time-varying* systems and general (not necessarily positive-semidefinite) storage functions, we call a system (4) dissipative if there exists $P = P^\top \in \mathbb{R}^{n\times n}$ such that for all $k_0 \leq k_1$ it holds that

$$\sum_{k=k_0}^{k_1} Q_\Sigma(\widetilde{u}_k, \widetilde{y}_k) \geq \widetilde{\xi}_{k_1}^\top P x_{k_1} - \widetilde{\xi}_{k_0}^\top P x_{k_0} \, .$$

III. Continuous- and transformed system state

If the input trajectory u is square-integrable, then the corresponding continuous-time state trajectory x and its derivative $\dot{x}$ satisfying (3) are also square-integrable, and consequently they have orthonormal basis representations

$$x = \sum_{k=0}^{+\infty} \widetilde{x}_k b_k \quad \text{and} \quad \dot{x} = \sum_{k=0}^{+\infty} \widetilde{\dot{x}}_k b_k \, . \tag{10}$$

In the first part of this section we establish a relation between the real vector-valued sequences $\{\widetilde{x}_k\}$ and $\{\widetilde{\dot{x}}_k\}$ and two sequences computed from $\widetilde{\xi}_k$ in (4). In the second part, we compute dynamical equations for the coefficients $\{\widetilde{x}_k\}_{k=0,\ldots}$.

A. Weighted average and increment of $\{\widetilde{\xi}_k\}_{k=0,\ldots}$

In the literature about orthonormal bases and dynamical systems the state $\widetilde{\xi}$ of the transformed system is introduced as an *auxiliary* variable needed to link the sequences $\{\widetilde{u}_k\}_{k=0,\ldots}$ and $\{\widetilde{y}_k\}_{k=0,\ldots}$ via (4). We show that an *instrinsic* relation exists between the state variables of the original continuous-time and of the transformed system.

Theorem 1. *Given the orthonormal basis (2), consider the representations (10) of the state trajectory and its derivative of the continuous-time system (3) in the orthonormal basis. Then*

$$\widetilde{x}_k = \frac{1}{\sqrt{\lambda_k}} \frac{\widetilde{\xi}_k + \widetilde{\xi}_{k+1}}{\sqrt{2}} \text{ and } \widetilde{\dot{x}}_k = \sqrt{\lambda_k} \frac{\widetilde{\xi}_{k+1} - \widetilde{\xi}_k}{\sqrt{2}} \, . \tag{11}$$

Proof. From the first equation in (3) it follows that $\widetilde{\dot{x}} = A\widetilde{x} + B\widetilde{u}$. Define a new continuous-time system defined by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= I_n x \, , \end{aligned} \tag{12}$$

whose transfer function is $(sI-A)^{-1}B$. From the definition of transformed system (4) and the state-space description (12), it follows that the sequence $\{\widetilde{x}_k\}$ is the output of the transformed system

$$\begin{aligned} \widetilde{\xi}_{k+1} &= \widetilde{A}_k\widetilde{\xi}_k + \widetilde{B}_k\widetilde{u}_k \qquad (13)\\ \widetilde{x}_k &= \underbrace{\sqrt{2\lambda_k}\left(\lambda_k I - A\right)^{-1}}_{=:\widetilde{G}_k}\widetilde{\xi}_k + \underbrace{\left(\lambda_k I - A\right)^{-1}B}_{=:\widetilde{H}_k}\widetilde{u}_k \ . \end{aligned}$$

From the second equation of (13) and $\dot{x} = Ax + Bu$ we also obtain the following equation expressing $\dot{\widetilde{x}}_k$ as a linear function of $\widetilde{\xi}_k$ and $\widetilde{u}_k$:

$$\begin{aligned} \dot{\widetilde{x}}_k &= \underbrace{\sqrt{2\lambda_k}A\left(\lambda_k I - A\right)^{-1}}_{=:\widetilde{G}'_k = A\widetilde{G}_k}\widetilde{\xi}_k \\ &\quad + \underbrace{\left(A\left(\lambda_k I - A\right)^{-1} + I\right)B}_{=:\widetilde{H}'_k = A\widetilde{H}_k + B}\widetilde{u}_k \ . \qquad (14) \end{aligned}$$

Since $(\lambda_k - s)^{-1}(\lambda_k + s) + 1 = \frac{2\lambda_k}{\lambda_k - s}$, for every k it holds that

$$\widetilde{A}_k + I = (\lambda_k I - A)^{-1}(\lambda_k I + A) + I = 2\lambda_k(\lambda_k I - A)^{-1} \ ,$$

and consequently

$$\begin{aligned} \widetilde{\xi}_{k+1} + \widetilde{\xi}_k &= \left(\widetilde{A}_k + I\right)\widetilde{\xi}_k + \widetilde{B}_k\widetilde{u}_k = 2\lambda_k\left(\lambda_k I - A\right)^{-1}\widetilde{\xi}_k \\ &\quad + \widetilde{B}_k\widetilde{u}_k \\ &= \sqrt{2\lambda_k}\widetilde{G}_k\widetilde{\xi}_k + \sqrt{2\lambda_k}\widetilde{H}_k\widetilde{u}_k = \sqrt{2\lambda_k}\widetilde{x}_k \ , \end{aligned}$$

from which the first equation (11) follows.

To prove the second equation (11), write $(\lambda_k - s)^{-1}(\lambda_k + s) - 1 = \frac{2s}{\lambda_k - s}$; then

$$\widetilde{A}_k - I = (\lambda_k I - A)^{-1}(\lambda_k I + A) - I = 2A(\lambda_k I - A)^{-1} \ .$$

Consequently

$$\begin{aligned} \widetilde{\xi}_{k+1} - \widetilde{\xi}_k &= \left(\widetilde{A}_k - I\right)\widetilde{\xi}_k + \widetilde{B}_k\widetilde{u}_k \\ &= 2A\left(\lambda_k I - A\right)^{-1}\widetilde{\xi}_k + \widetilde{B}_k\widetilde{u}_k \\ &= \frac{\sqrt{2}}{\sqrt{\lambda_k}}A\widetilde{G}_k\widetilde{\xi}_k + \widetilde{G}_k B\widetilde{u}_k \ . \end{aligned}$$

Since

$$\begin{aligned} A\widetilde{H}_k + B &= A\left(\lambda_k I - A\right)^{-1}B + B = \frac{\widetilde{A} - I}{2}B + B \\ &= \frac{\widetilde{A} + I}{2}B = \frac{\sqrt{\lambda_k}}{\sqrt{2}}\widetilde{G}_k \ , \end{aligned}$$

it follows that

$$\begin{aligned} \dot{\widetilde{x}}_k &= A\widetilde{x}_k + B\widetilde{u}_k = A\left(\widetilde{G}_k\widetilde{\xi}_k + \widetilde{H}_k\widetilde{u}_k\right) + B\widetilde{u}_k \\ &= A\widetilde{G}_k\widetilde{\xi}_k + \left(A\widetilde{H}_k + B\right)\widetilde{u}_k \ . \end{aligned}$$

Consequently, $\dot{\widetilde{x}}_k = A\widetilde{G}_k\widetilde{\xi}_k + \frac{\sqrt{\lambda_k}}{\sqrt{2}}\widetilde{G}_k B\widetilde{u}_k$, from which the second equation in (11) follows. □

We call the expressions on the right-hand side of equations (11) respectively the (time-dependent) weighted *average* and the (time-dependent) weighted *increment* of the sequence $\{\widetilde{\xi}_k\}$. We now establish equivalent expressions for the state variables of the transformed and of the original system.

Corollary 1. *The following statements are equivalent:*

1) *Equations (11) hold for every* $k \in \mathbb{N}$*;*
2) *The following equations hold for every* $k \in \mathbb{N}$*:*

$$\begin{aligned} \widetilde{\xi}_k &= \sqrt{\frac{\lambda_k}{2}}\widetilde{x}_k - \frac{1}{\sqrt{2\lambda_k}}\dot{\widetilde{x}}_k \\ \widetilde{\xi}_{k+1} &= \sqrt{\frac{\lambda_k}{2}}\widetilde{x}_k + \frac{1}{\sqrt{2\lambda_k}}\dot{\widetilde{x}}_k \ . \end{aligned}$$

3) *The following equations hold for every* $k \in \mathbb{N}$*:*

$$\begin{aligned} \widetilde{\xi}_k &= \left(\sqrt{\frac{\lambda_k}{2}}I_n - \frac{1}{\sqrt{2\lambda_k}}A\right)\widetilde{x}_k - \frac{1}{\sqrt{2\lambda_k}}B\widetilde{u}_k \\ \widetilde{\xi}_{k+1} &= \left(\sqrt{\frac{\lambda_k}{2}}I_n + \frac{1}{\sqrt{2\lambda_k}}A\right)\widetilde{x}_k + \frac{1}{\sqrt{2\lambda_k}}B\widetilde{u}_k \ . \end{aligned}$$

Proof. The equivalence between statements 1 and 2 follows from (11) and

$$\begin{bmatrix} \frac{1}{\sqrt{2\lambda_k}}I_n & \frac{1}{\sqrt{2\lambda_k}}I_n \\ -\sqrt{\frac{\lambda_k}{2}}I_n & \sqrt{\frac{\lambda_k}{2}}I_n \end{bmatrix}^{-1} = \begin{bmatrix} \sqrt{\frac{\lambda_k}{2}}I_n & -\frac{1}{\sqrt{2\lambda_k}}I_n \\ \sqrt{\frac{\lambda_k}{2}}I_n & \frac{1}{\sqrt{2\lambda_k}}I_n \end{bmatrix} \ .$$

Equivalence $(2) \Leftrightarrow (3)$ follows from $\dot{\widetilde{x}}_k = A\widetilde{x}_k + B\widetilde{u}_k$. □

Remark 1. The dynamics of the transformed system state given by the first equation in (4) can also be deduced from the equalities established in statement 3 of Cor. 1. To show this, we first prove that the matrix $\sqrt{\frac{\lambda_{k+1}}{2}}I_n - \frac{1}{\sqrt{2\lambda_{k+1}}}A$ is nonsingular. By contradiction, if there exists $v \in \mathbb{C}^n$, $v \neq 0$ such that $\sqrt{\frac{\lambda_{k+1}}{2}}v = \frac{1}{\sqrt{2\lambda_{k+1}}}Av$, then A has an eigenvalue at $\lambda_{k+1} \in \mathbb{R}_+$; however, A is Hurwitz by assumption.

From the first equation in statement 3 of Cor. 1 conclude that

$$\begin{aligned} \widetilde{x}_k &= \left(\sqrt{\frac{\lambda_k}{2}}I_n - \frac{1}{\sqrt{2\lambda_k}}A\right)^{-1}\widetilde{\xi}_k \\ &\quad + \frac{1}{\sqrt{2\lambda_k}}\left(\sqrt{\frac{\lambda_k}{2}}I_n - \frac{1}{\sqrt{2\lambda_k}}A\right)^{-1}B\widetilde{u}_k \ . \end{aligned}$$

Substituting this expression in the second equation of statement 3, one obtains the first equation in (4). The second equation in (4) follows from $\widetilde{y}_k = C\widetilde{\xi}_k + D\widetilde{u}_k$. □

B. Dynamics of $\{\widetilde{x}_k\}_{k=0,\dots}$

From the dynamics of the transformed system and from the relations (13) and (14) one can obtain first-order time-varying equations describing the dynamics of the coefficients $\widetilde{x}_k$ of the orthonormal basis representations of the continuous-time state x.

Theorem 2. *Define*

$$\begin{aligned} A_{k,-} &:= \sqrt{\frac{\lambda_{k+1}}{2}}I_n - \frac{1}{\sqrt{2\lambda_{k+1}}}A \\ A_{k,+} &:= \sqrt{\frac{\lambda_k}{2}}I_n + \frac{1}{\sqrt{2\lambda_k}}A \ . \end{aligned}$$

The sequence $\{\widetilde{x}_k\}$ of orthonormal basis coefficients of x in the representation (10) satisfies the time-varying difference equation:

$$\widetilde{x}_{k+1} = A_k\widetilde{x}_k + B_k\left(\frac{1}{\sqrt{2\lambda_k}}\widetilde{u}_k + \frac{1}{\sqrt{2\lambda_{k+1}}}\widetilde{u}_{k+1}\right) \quad (15)$$

where $A_k := A_{k,-}^{-1}A_{k,+}$ and $B_k := A_{k,-}^{-1}B$.

Proof. Write (11) for two consecutive indices k, $k+1$, obtaining two expressions for $\widetilde{\xi}_{k+1}$:

$$\begin{aligned}\widetilde{\xi}_{k+1} &= \sqrt{\frac{\lambda_k}{2}}\widetilde{x}_k + \frac{1}{\sqrt{2\lambda_k}}\widetilde{\dot{x}}_k \\ &= \left(\sqrt{\frac{\lambda_k}{2}}I_n + \frac{1}{\sqrt{2\lambda_k}}A\right)\widetilde{x}_k + \frac{1}{\sqrt{2\lambda_k}}B\widetilde{u}_k\end{aligned}$$

and

$$\begin{aligned}\widetilde{\xi}_{k+1} &= \sqrt{\frac{\lambda_{k+1}}{2}}\widetilde{x}_{k+1} - \frac{1}{\sqrt{2\lambda_{k+1}}}\widetilde{\dot{x}}_{k+1} \\ &= \left(\sqrt{\frac{\lambda_{k+1}}{2}}I_n - \frac{1}{\sqrt{2\lambda_{k+1}}}A\right)\widetilde{x}_{k+1} \\ &\quad -\frac{1}{\sqrt{2\lambda_{k+1}}}B\widetilde{u}_{k+1} \ .\end{aligned}$$

To complete the proof recall from Rem. 1 that $\left(\sqrt{\frac{\lambda_{k+1}}{2}}I_n - \frac{1}{\sqrt{2\lambda_{k+1}}}A\right)$ is nonsingular; now equate the two expressions obtained for $\widetilde{\xi}_{k+1}$, and multiply on the left both sides of the resulting equation by $\left(\sqrt{\frac{\lambda_{k+1}}{2}}I_n - \frac{1}{\sqrt{2\lambda_{k+1}}}A\right)^{-1}$. □

Remark 2. The proof of Th. 2 shows that equation (15) can be interpreted as a *consistency condition* obtained by equating the two expressions for $\widetilde{\xi}_{k+1}$ obtained from (11). It is a matter for future research to investigate whether such consistency equation uniquely identifies the orthonormal basis. □

Remark 3 (The Laguerre basis)**.** When the sequence $\{\lambda_k\}$ is constant, $\lambda_k = \lambda \in \mathbb{R}_+$ for all k, then the orthonormal basis (2) is called the Laguerre basis. In this case the equalities established in Theorem 1 read $\widetilde{x} = \frac{\widetilde{\xi}_k+\widetilde{\xi}_{k+1}}{\sqrt{2\lambda}}$ and $\widetilde{\dot{x}} = \sqrt{\lambda}\frac{\widetilde{\xi}_{k+1}-\widetilde{\xi}_k}{\sqrt{2}}$, and the dynamics of $\widetilde{x}$ are time-invariant:

$$\begin{aligned}\widetilde{x}_{k+1} &= \left(\sqrt{\frac{\lambda}{2}}I_n - \frac{1}{\sqrt{2\lambda}}A\right)^{-1}\left(\sqrt{\frac{\lambda}{2}}I_n + \frac{1}{\sqrt{2\lambda}}A\right)\widetilde{x}_k \\ &+ \frac{1}{\sqrt{2\lambda}}\left(\sqrt{\frac{\lambda}{2}}I_n - \frac{1}{\sqrt{2\lambda}}A\right)^{-1}B\left(\widetilde{u}_k + \widetilde{u}_{k+1}\right) \ .\end{aligned}$$

□

Remark 4 (Generalized orthonormal basis)**.** In the case of a generalized orthonormal basis the sequence $\{\lambda_k\}_{k=0,\ldots}$ is p-periodic: $\{\lambda_k\} = \{\lambda_0, \ldots, \lambda_{p-1}, \lambda_0, \ldots\}$. Applying the equalities established in Theorem 1 leads to the equations

$$\widetilde{x} = \frac{\widetilde{\xi}_k + \widetilde{\xi}_{k+1}}{\sqrt{2\lambda_{k(\text{mod }p)}}} \quad \text{and} \quad \widetilde{\dot{x}} = \sqrt{\lambda_{k(\text{mod }p)}}\frac{\widetilde{\xi}_{k+1} - \widetilde{\xi}_k}{\sqrt{2}} \ .$$

From Th. 2 it follows that $\widetilde{x}$ satisfies the periodic dynamics

$$\begin{aligned}\widetilde{x}_{k+1} &= A_k\widetilde{x}_k \\ &\quad +B_k\left(\frac{1}{\sqrt{2\lambda_{k(\text{mod}p)}}}\widetilde{u}_k + \frac{1}{\sqrt{2\lambda_{k+1(\text{mod }p)}}}\widetilde{u}_{k+1}\right) \ ,\end{aligned}$$

where $A_k := A_{k,-}^{-1}A_{k,+}$ and $B_k := A_{k,-}^{-1}B$, with $A_{k,-}$ and $A_{k,+}$ defined by

$$\begin{aligned}A_{k,-} &:= \sqrt{\frac{\lambda_{k+1(\text{mod }p)}}{2}}I_n - \frac{1}{\sqrt{2\lambda_{k+1(\text{mod }p)}}}A \\ A_{k,+} &:= \sqrt{\frac{\lambda_{k(\text{mod }p)}}{2}}I_n + \frac{1}{2\sqrt{\lambda_{k(\text{mod }p)}}}A \ .\end{aligned}$$

□

IV. Dissipation and system transformation

We now prove that if the continuous-time system (3) is dissipative, then the transformed system (4) is also dissipative. The following theorem is the main result of this section.

Theorem 3. *Let $\Sigma = \Sigma^\top \in \mathbb{R}^{(m+p)\times(m+p)}$, and assume that the system (3) is dissipative with respect to the supply rate $Q_\Sigma(u, y)$; then the transformed system is also dissipative with respect to the supply rate*

$$Q_\Sigma(\widetilde{u}_k, \widetilde{y}_k) := \begin{bmatrix}\widetilde{u}_k^\top & \widetilde{y}_k^\top\end{bmatrix}\Sigma\begin{bmatrix}\widetilde{u}_k \\ \widetilde{y}_k\end{bmatrix} \ .$$

If $P = P^\top \in \mathbb{R}^{n\times n}$ induces a storage function for (3), then it also induces a storage function for the transformed system.

Proof. Let (x, u, y) satisfy (3), with orthonormal basis representation $(\widetilde{x}_k, \widetilde{u}_k, \widetilde{y}_k)$ and associated transformed system trajectory $(\widetilde{\xi}_k, \widetilde{u}_k, \widetilde{y}_k)$, $k = 0, \ldots$. Consider a fixed, but otherwise arbitrary, $k \in \mathbb{N}$, and define $\left(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k\right)$ by (5). We now prove that a linear matrix inequality holds for such arbitrary k; this will be instrumental to prove the claim of the Theorem.

From statement (2) in Cor. 1, for every k it holds that $\widetilde{\xi}_k = \sqrt{\frac{\lambda_k}{2}}\widetilde{x}_k - \frac{1}{\sqrt{2\lambda_k}}\widetilde{\dot{x}}_k$. It follows that

$$\begin{aligned}&\sqrt{2\lambda_k}\left(\lambda_k I - A\right)^{-1}\widetilde{\xi}_k + \left(\lambda_k I - A\right)^{-1}B\widetilde{u}_k \\ &= \lambda_k\left(\lambda_k I - A\right)^{-1}\widetilde{x}_k - \left(\lambda_k I - A\right)^{-1}\widetilde{\dot{x}}_k \\ &+ \left(\lambda_k I - A\right)^{-1}B\widetilde{u}_k = \widetilde{x}_x \ .\end{aligned}$$

Consequently, for a fixed k it holds that

$$\underbrace{\begin{bmatrix}\sqrt{2\lambda_k}\left(\lambda_k I - A\right)^{-1} & \sqrt{2\lambda_k}\left(\lambda_k I - A\right)^{-1} \\ 0_{m\times n} & I_m\end{bmatrix}}_{=:T_k}\begin{bmatrix}\widetilde{\xi}_k \\ \widetilde{u}_k\end{bmatrix} = \begin{bmatrix}\widetilde{x}_k \\ \widetilde{u}_k\end{bmatrix} \ .$$

Denote by $M(A, B, C, D)$ the left-hand side of the matrix inequality (7), and by $\widetilde{M}(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ the matrix on the left-hand side of (9). Using the definitions (5), straightforward calculations show that

$$\widetilde{M}(\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k) = T_k^\top M(A, B, C, D)T_k \ . \quad (16)$$

Since (3) is dissipative, $M(A,B,C,D) \geq 0$ and consequently $\widetilde{M}(\widetilde{A}_k,\widetilde{B}_k,\widetilde{C}_k,\widetilde{D}_k) \geq 0$: for a fixed k, the time-invariant discrete-time system associated with $(\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D})$ is Σ-dissipative.

Now consider $k_0, k_1 \in \mathbb{N}$, $k_0 \leq k_1$. From the inequality $\widetilde{M}(\widetilde{A}_k,\widetilde{B}_k,\widetilde{C}_k,\widetilde{D}_k) \geq 0$ for $k \in \mathbb{N}$ conclude that for every k it holds that $Q_\Sigma(u_k, y_k) \geq \widetilde{\xi}_{k+1}^\top P \widetilde{\xi}_{k+1} - \widetilde{\xi}_k^\top P \widetilde{\xi}_k$, and consequently

$$\begin{aligned}\sum_{k=k_0}^{k_1} Q_\Sigma(u_k, y_k) &\geq \sum_{k=k_0}^{k_1} \widetilde{\xi}_{k+1}^\top P \widetilde{\xi}_{k+1} - \widetilde{\xi}_k^\top P \widetilde{\xi}_k \\ &= \widetilde{\xi}_{k_1}^\top P \widetilde{\xi}_{k_1} - \widetilde{\xi}_{k_0}^\top P \widetilde{\xi}_{k_0} ,\end{aligned}$$

which proves the dissipativity of the time-varying system (4), (5). ☐

Remark 5. For the case of the generalized (Hambo) orthonormal basis, a result analogous to that of Th. 3 was established in [18] with a less straightforward argument.

Example 1. Consider the matrices $A = \begin{bmatrix} -2 & -\frac{3}{4} \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $D = 0$, corresponding to the positive-real transfer function $G(s) = \frac{s+1}{(s+\frac{1}{2})(s+\frac{3}{2})}$. The continuous-time positive-real lemma LMI has a solution $P := \begin{bmatrix} 0.2500 & 0.2500 \\ 0.2500 & 0.6875 \end{bmatrix} \succ 0$. It can be verified that P also solves the discrete positive-real lemma LMIs for every transformed system $(\widetilde{A}_k,\widetilde{B}_k,\widetilde{C}_k,\widetilde{D}_k)$, due to the congruence relation (16).

We now prove an orthonormal basis version of the dissipation equality for the continuous-time system (3).

Proposition 1. *Let $\Sigma = \Sigma^\top \in \mathbb{R}^{(m+p)\times(m+p)}$; assume that (3) is dissipative with respect to the supply rate $Q_\Sigma(u,y) = \begin{bmatrix} y^\top & u^\top \end{bmatrix} \Sigma \begin{bmatrix} y \\ u \end{bmatrix}$. Let $P = P^\top \in \mathbb{R}^{n\times n}$ induce a storage function, with corresponding dissipation rate $Q_\Phi(x,u) = \begin{bmatrix} x^\top & u^\top \end{bmatrix} \Phi \begin{bmatrix} x \\ u \end{bmatrix}$. The following equalities holds for the orthonormal basis coefficients $\widetilde{x}$, $\widetilde{\dot{x}}$, $\widetilde{y}$, $\widetilde{u}$ and transformed system state variable $\widetilde{x}$:*

$$\begin{aligned}&\sum_{k=0}^{\infty} \begin{bmatrix} \widetilde{u}_k^\top & \widetilde{y}_k^\top \end{bmatrix} \Sigma \begin{bmatrix} \widetilde{u}_k \\ \widetilde{y}_k \end{bmatrix} - \begin{bmatrix} \widetilde{u}_k^\top & \widetilde{x}_k^\top \end{bmatrix} \Phi \begin{bmatrix} \widetilde{u}_k \\ \widetilde{x}_k \end{bmatrix} \\ &= \sum_{k=0}^{\infty} \frac{1}{2}\widetilde{\dot{x}}_k^\top P \widetilde{x}_k + \frac{1}{2}\widetilde{x}_k^\top P \widetilde{\dot{x}}_k \sum_{k=0}^{\infty} \left(\widetilde{\xi}_{k+1}^\top P \widetilde{\xi}_{k+1} - \widetilde{\xi}_k^\top P \widetilde{\xi}_k \right)\end{aligned} \tag{17}$$

Proof. To prove the first equality, integrate both sides of the dissipation equality and use the orthonormality of the basis:

$$\begin{aligned}&\int_0^{+\infty} Q_\Sigma(u,y) - Q_\Phi(u,y)\ dt \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} \widetilde{u}_k^\top & \widetilde{y}_k^\top \end{bmatrix} \Sigma \begin{bmatrix} \widetilde{u}_k \\ \widetilde{y}_k \end{bmatrix} - \begin{bmatrix} \widetilde{u}_k^\top & \widetilde{x}_k^\top \end{bmatrix} \Phi \begin{bmatrix} \widetilde{u}_k \\ \widetilde{x}_k \end{bmatrix} \\ &= \int_0^{+\infty} \frac{1}{2}\dot{x}^\top P x + \frac{1}{2} x^\top P \dot{x}\ dt \\ &= \sum_{k=0}^{\infty} \frac{1}{2}\widetilde{\dot{x}}_k^\top P \widetilde{x}_k + \frac{1}{2}\widetilde{x}_k^\top P \widetilde{\dot{x}}_k .\end{aligned}$$

To prove the second equality, use (11) to write $\frac{1}{2}\widetilde{\dot{x}}_k^\top P \widetilde{x}_k + \frac{1}{2}\widetilde{x}_k^\top P \widetilde{\dot{x}}_k = \widetilde{\xi}_{k+1}^\top P \widetilde{\xi}_{k+1} - \widetilde{\xi}_k^\top P \widetilde{\xi}_k$. ☐

We now illustrate a "variational" interpretation of the state variable $\widetilde{\xi}$ of the transformed system.

Consider first the following linear algebraic result.

Proposition 2. *Let $\Pi = \Pi^\top \in \mathbb{R}^{n\times n}$, $\{\theta_k\}$, $\{v_k\}$, $\{w_k\}$ be n-dimensional real sequences, and let $\{\alpha_k\}$ be a sequence of nonzero real numbers. The following statements are equivalent:*

1) *The sequences $\{\theta_k\}$, $\{v_k\}$, $\{w_k\}$ satisfy*

$$\begin{aligned}\theta_k &= \frac{1}{\sqrt{2}}\left(\frac{1}{\alpha_k}v_k - \alpha_k w_k\right) \\ \theta_{k+1} &= \frac{1}{\sqrt{2}}\left(\frac{1}{\alpha_k}v_k + \frac{\alpha_k}{\sqrt{2}} w_k\right) ;\end{aligned} \tag{18}$$

2) *The sequences $\{\theta_k\}$, $\{v_k\}$, $\{w_k\}$ satisfy*

$$\begin{aligned}v_k &= \frac{\alpha_k}{\sqrt{2}}(\theta_k + \theta_{k+1}) \\ w_k &= \frac{1}{\sqrt{2}\alpha_k}(\theta_{k+1} - \theta_k)\end{aligned} \tag{19}$$

Moreover, if statement 1 or statement 2 holds, then for every $k \in \mathbb{N}$ it holds that $\frac{1}{2}\left(v_k^\top \Pi w_k + w_k^\top \Pi v_k\right) = \theta_{k+1}^\top \Pi \theta_{k+1} - \theta_k^\top \Pi \theta_k$.

Proof. The equivalence of statements 1 and 2 follows from

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\alpha_k} I_n & -\alpha_k I_n \\ \frac{1}{\alpha_k} I_n & \alpha_k I_n \end{bmatrix}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} \alpha_k I_n & \alpha_k I_n \\ -\frac{1}{\alpha_k} I_n & \frac{1}{\alpha_k} I_n \end{bmatrix}$$

The final claim follows from the fact that $\begin{bmatrix} 0 & \Pi \\ \Pi & 0 \end{bmatrix}$ equals

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\alpha_k} I_n & \frac{1}{\alpha_k} I_n \\ -\alpha_k I_n & \alpha_k I_n \end{bmatrix}\begin{bmatrix} -\Pi & 0 \\ 0 & \Pi \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\alpha_k} I_n & -\alpha_k I_n \\ \frac{1}{\alpha_k} I_n & \alpha_k I_n \end{bmatrix} .$$

☐

The last claim of Prop. 2 shows that any bilinear form on a pair of sequences $\{v_k\}$, $\{w_k\}$ can be rewritten as the increment of a quadratic function of a sequence of auxiliary variables $\{\theta_k\}$ defined from $\{v_k\}$, $\{w_k\}$ by the equations (18). Equivalently, see equation (19), the sequences $\{v_k\}$ and $\{w_k\}$ are the (time-dependent) weighted average and increment, respectively, of the sequence $\{\theta_k\}$, and the parameter α_k determines how much the contribution of v_k and w_k, respectively, weighs in the definition of θ_k.

Assume that the sequence $\{\alpha_k\}$ is bounded, and bounded away from zero. Define λ_k by $\lambda_k := \frac{1}{\alpha_k^2}$; then the equations (19) coincide with those relating $\widetilde{\xi}$ with $\widetilde{x}$ and $\widetilde{\dot{x}}$, see Th. 1. The state variable of the transformed system can thus be interpreted as also an *auxiliary* variable defined to rewrite the orthonormal basis version of the dissipation equality (17) as an increment of a quadratic form.

A consequence of Cor. 1 and Rem. 1 is that *any* variable ξ defined as a weighted average and increment of *some* orthonormal basis representation of the trajectories x and $\dot{x}$ of a continuous-time system satisfies the dynamic equations (4). As mentioned in Rem. 2, the dynamics of $\widetilde{x}$ is described by the consistency condition (15); it is a matter for future research to investigate whether such consistency equation uniquely identifies the orthonormal basis.

We conclude this section considering the case of complex poles in the construction of the orthonormal basis.

Remark 6. We consider here the case when complex poles are used in the construction of orthonormal bases; we follow the approach of [1], [11]. We assume that a finite sequence $\lambda_0, \ldots, \lambda_{n-1}$ of real numbers have been chosen to build an orthonormal basis $\{\widehat{b}_k(s)\}$ according to the formulas (1), which we rewrite for for the complex case, denoting the real part of the complex number λ_k by $\Re(\lambda_k)$:

$$\widehat{\phi}_k(s) := \frac{s - \lambda_k^*}{s + \lambda_k}\ ,\ k = 0, \ldots\ , \tag{20}$$

and

$$\begin{aligned} \widehat{b}_0(s) &:= \frac{\sqrt{2\Re(\lambda_0)}}{s + \lambda_0} \qquad (21)\\ \widehat{b}_k(s) &:= \left(\prod_{i=0}^{k-1} \widehat{\phi}_i(s)\right) \frac{\sqrt{2\Re(\lambda_k)}}{\lambda_k + s}\ ,\ k \geq 1\ . \end{aligned}$$

Assume that $\lambda_n \in \mathbb{C}$ with nonzero imaginary part is chosen for the construction of the next basis element; then necessarily $\lambda_{n+1} := \lambda_n^*$. To guarantee that real signals correspond to real orthonormal basis representations, the transformation

$$\begin{bmatrix} \widehat{b}'_n(s) \\ \widehat{b}'_{n+1}(s) \end{bmatrix} = \underbrace{\begin{bmatrix} c_0 & c_1 \\ c'_0 & c'_1 \end{bmatrix}}_{=:C} \begin{bmatrix} \widehat{b}_n(s) \\ \widehat{b}_{n+1}(s) \end{bmatrix} \tag{22}$$

is defined, where the coefficients c_i, c'_i, $i = 0, 1$ are chosen such that the orthonormality condition for the basis elements

$$\widehat{b}_1(s), \ldots, \widehat{b}_{n-1}(s), \widehat{b}'_n(s), \widehat{b}'_{n+1}(s) \tag{23}$$

is satisfied. This is equivalent to requiring that the matrix C defined in (22) is orthonormal: $CC^* = C^*C = I_2$. Such condition can be shown to be equivalent (see pp. 267-268 of [1]) to the parametrization

$$\begin{aligned} c_0(\theta) &:= \frac{\lambda_n^* \sqrt{2} \cos\theta + |\lambda_n| \sqrt{2} \sin\theta}{2\lambda_n^*} \\ c_1(\theta) &:= \frac{\lambda_n^* \sqrt{2} \cos\theta - |\lambda_n| \sqrt{2} \sin\theta}{2\lambda_n^*} \\ c'_0(\theta) &:= \frac{-\lambda_n^* \sqrt{2} \sin\theta + |\lambda_n| \sqrt{2} \cos\theta}{2\lambda_n^*} \\ c'_1(\theta) &:= \frac{-\lambda_n^* \sqrt{2} \sin\theta - |\lambda_n| \sqrt{2} \cos\theta}{2\lambda_n^*}\ , \end{aligned}$$

where $0 \leq \theta < 2\pi$.

With the change of basis to (23), if $\widetilde{f}_n$, $\widetilde{f}_{n+1}$ denote the n-th and $(n+1)$-th coefficient of the orthonormal basis representation of a function f in the basis

$$\widehat{b}_1(s), \ldots, \widehat{b}_{n-1}(s), \widehat{b}_n(s), \widehat{b}_{n+1}(s)\ ,$$

and $\widetilde{f}'_n$, $\widetilde{f}'_{n+1}$ denote the coefficients of the representation of f in the basis $\widehat{b}_1(s), \ldots, \widehat{b}_{n-1}(s), \widehat{b}'_n(s), \widehat{b}'_{n+1}(s)$, the relation between the coefficients is

$$\begin{aligned} \begin{bmatrix} \widetilde{f}'_n \\ \widetilde{f}'_{n+1} \end{bmatrix} &= C \begin{bmatrix} \widetilde{f}_n \\ \widetilde{f}_{n+1} \end{bmatrix} \\ \begin{bmatrix} \widetilde{f}_n \\ \widetilde{f}_{n+1} \end{bmatrix} &= C^* \begin{bmatrix} \widetilde{f}'_n \\ \widetilde{f}'_{n+1} \end{bmatrix}\ . \end{aligned}$$

Using these equations component-wise and defining

$$C_e := \begin{bmatrix} c_0 I_n & c_1 I_n \\ c'_0 I_n & c'_1 I_n \end{bmatrix}\ ,$$

we obtain the representation $\begin{bmatrix} \widetilde{\xi}'_k \\ \widetilde{\xi}'_{k+1} \end{bmatrix}$ of $\begin{bmatrix} \widetilde{\xi}_k \\ \widetilde{\xi}_{k+1} \end{bmatrix}$ in the new basis by

$$\begin{bmatrix} \widetilde{\xi}'_k \\ \widetilde{\xi}'_{k+1} \end{bmatrix} = C_e \begin{bmatrix} \widetilde{\xi}_k \\ \widetilde{\xi}_{k+1} \end{bmatrix}\ . \tag{24}$$

From Cor. 1 it follows that $\begin{bmatrix} \widetilde{\xi}_k \\ \widetilde{\xi}_{k+1} \end{bmatrix} = T(\lambda_k) \begin{bmatrix} \widetilde{x}_k \\ \dot{\widetilde{x}}_k \end{bmatrix}$, where

$$T(\lambda_k) := \begin{bmatrix} \sqrt{\frac{\lambda_k}{2}} I_n & -\frac{1}{\sqrt{2\lambda_k}} I_n \\ \sqrt{\frac{\lambda_k}{2}} I_n & \frac{1}{\sqrt{2\lambda_k}} I_n \end{bmatrix}\ ,$$

and from (24) we conclude that $\begin{bmatrix} \widetilde{\xi}'_k \\ \widetilde{\xi}'_{k+1} \end{bmatrix} = C_e T(\lambda_k) \begin{bmatrix} \widetilde{x}_k \\ \dot{\widetilde{x}}_k \end{bmatrix}$, the counterpart to statement 2) of Cor. 1.

Now let $P = P^\top \in \mathbb{R}^{n \times n}$ and define $P_e := \begin{bmatrix} 0_{n\times n} & P \\ P & 0_{n\times n} \end{bmatrix}$; since $T(\lambda_k)^{-1} C_e^* \begin{bmatrix} \widetilde{\xi}'_k \\ \widetilde{\xi}'_{k+1} \end{bmatrix} = \begin{bmatrix} \widetilde{x}_k \\ \dot{\widetilde{x}}_k \end{bmatrix}$, then

$$\begin{aligned} \widetilde{x}_k^\top P \dot{\widetilde{x}}_k + \dot{\widetilde{x}}_k^\top P \widetilde{x}_k &= \begin{bmatrix} \widetilde{x}_k^\top & \dot{\widetilde{x}}_k^\top \end{bmatrix} P_e \begin{bmatrix} \widetilde{x}_k \\ \dot{\widetilde{x}}_k \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{\xi}_k'^\top & \widetilde{\xi}_{k+1}'^\top \end{bmatrix} C_e T(\lambda_k)^{-\top} P T(\lambda_k)^{-1} C_e^* \begin{bmatrix} \widetilde{\xi}'_k \\ \widetilde{\xi}'_{k+1} \end{bmatrix}\ ; \end{aligned}$$

it follows that the last equality in (17) can expressed in terms of the auxiliary variables $\widetilde{\xi}'_k$, $\widetilde{\xi}'_{k+1}$.

V. Application: subspace identification of continuous-time systems

We apply our results to system identification of deterministic continuous-time systems. Given space limitations, we use the Laguerre basis; the case of a generalized orthonormal basis will be illustrated elsewhere. Given an input-output pair (u, y) of $\mathcal{L}_2$-trajectory, we compute its representation $u = \sum_{k=0}^{+\infty} \widetilde{u}_k b_k$ and $y = \sum_{k=0}^{+\infty} \widetilde{y}_k b_k$ in the Laguerre basis.

Using $p + 2q + N$ consecutive terms

$$\{\widetilde{u}_k\}_{k=p,\ldots,p+2q+N} \text{ and } \{\widetilde{y}_k\}_{k=p,\ldots,p+2q+N} \tag{25}$$

from these representations, we construct for fixed integers p and q the "past" Hankel matrix

$$\mathcal{H}^- \quad := \quad \begin{bmatrix} \widetilde{u}_p & \widetilde{u}_{p+1} & \cdots & \widetilde{u}_{p+N} \\ \widetilde{u}_{p+1} & \widetilde{u}_{p+2} & \cdots & \widetilde{u}_{p+N+1} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{u}_{p+q-1} & \widetilde{u}_{p+q,1} & \cdots & \widetilde{u}_{p+q+N-1} \\ \widetilde{y}_p & \widetilde{y}_{p+1} & \cdots & \widetilde{y}_{p+N} \\ \widetilde{y}_{p+1} & \widetilde{y}_{p+1,1} & \cdots & \widetilde{y}_{p+N+1} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{y}_{p+q-1} & \widetilde{y}_{p+q,1} & \cdots & \widetilde{y}_{p+q+N-1} \end{bmatrix} \tag{26}$$

and the "future" Hankel matrix

$$\mathcal{H}^+ \quad := \quad \begin{bmatrix} \widetilde{u}_{p+q} & \widetilde{u}_{p+q+1} & \cdots & \widetilde{u}_{p+q+N} \\ \widetilde{u}_{p+q+1} & \widetilde{u}_{p+q+2} & \cdots & \widetilde{u}_{p+q+N+1} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{u}_{p+2q} & \widetilde{u}_{p+2q+1} & \cdots & \widetilde{u}_{p+2q+N} \\ \widetilde{y}_{p+q} & \widetilde{y}_{p+q+1} & \cdots & \widetilde{y}_{p+q+N} \\ \widetilde{y}_{p+q+1} & \widetilde{y}_{p+q+2} & \cdots & \widetilde{y}_{p+q+N+1} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{y}_{p+2q} & \widetilde{y}_{p+2q+1} & \cdots & \widetilde{y}_{p+2q+N} \end{bmatrix} . \tag{27}$$

It follows from the state equation (4) for the transformed system (see also sect. 2.3.1 of [19]) that the k-th column of any basis matrix for the intersection of the row spaces of $\mathcal{H}^-$ and $\mathcal{H}^+$, is the state value at k of the system (4) corresponding to the input-output trajectory $\{(\widetilde{u}_k, \widetilde{y}_k)\}_{k=0,\dots,N}$:

$$\begin{aligned} &\text{row span}\left(\mathcal{H}^+\right) \cap \left(\mathcal{H}^-\right) \\ &= \text{row span}\left(\begin{bmatrix} \widetilde{\xi}_{p+q} & \widetilde{\xi}_{p+q+1} & \dots & \widetilde{\xi}_{p+q+N} \end{bmatrix}\right) . \end{aligned} \tag{28}$$

Applying the formula in Remark 3, one can compute from such a basis matrix for row span $(\mathcal{H}^+) \cap (\mathcal{H}^-)$ N consecutive values $\{\widetilde{x}_k\}_{k=p+q,\dots p+N}$ and $\left\{\widetilde{\dot{x}}_k\right\}_{k=p+q,\dots p+N}$ of the orthonormal basis representation of the state trajectory x and its derivative $\frac{d}{dt}x$ corresponding to (u, y).

Now define the matrices

$$\begin{aligned} X_N &:= \begin{bmatrix} \widetilde{x}_{p+q} & \widetilde{x}_{p+q+1} & \dots & \widetilde{x}_{p+q+N} \end{bmatrix} \\ \overset{\bullet}{X}_N &:= \begin{bmatrix} \widetilde{\dot{x}}_{p+q} & \widetilde{\dot{x}}_{p+q+1} & \dots & \widetilde{\dot{x}}_{p+q+N} \end{bmatrix} , \end{aligned}$$

computed from the data, and the matrices

$$\begin{aligned} U_N &:= \begin{bmatrix} \widetilde{u}_{p+q} & \widetilde{x}_{p+q+1} & \dots & \widetilde{u}_{p+q+N} \end{bmatrix} \\ Y_N &:= \begin{bmatrix} \widetilde{x}_{p+q} & \widetilde{y}_{p+q+1} & \dots & \widetilde{y}_{p+q+N} \end{bmatrix} \end{aligned}$$

and assume that u and its transformation $\{u_k\}_{k=0,\dots}$ are persistently exciting for the continuous and transformed system, respectively. The matrices of the continuous state model can be computed solving, for example using least-squares methods, the system of equations

$$\begin{bmatrix} \overset{\bullet}{X}_N \\ Y_N \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_N \\ U_N \end{bmatrix} . \tag{29}$$

This procedure computes the continuous-time model *directly* from the data (25); compare it with the standard approach of first identifying the transformed model and subsequently computing from it the continuous-time model (see [7], [10], [15]. Moreover, the last two steps of our procedure constitute a *data-driven* inversion of the Laguerre transform. An analogous procedure can be stated for data-driven inversion of the Hambo transform (see Prop. 6 p. 660 of [7] and Th. 6 of [16]); this will be presented elsewhere.

Since different choices of the basis for row span $(\mathcal{H}^+) \cap (\mathcal{H}^-)$ in (28) correspond to different choices of the state-space basis for the transformed model and consequently for the continuous-time model associated with it, with our apprach it may be possible to compute "special" representations (e.g. input-output balanced, Riccati-balanced, and so forth) directly from data, a feature especially interesting for data-driven model-order reduction of continuous-time systems. This is ongoing research that will be presented elsewhere.

We apply our procedure to data generated by the positive-real transfer function $G(s) = \frac{s+1}{(s+\frac{1}{2})(s+\frac{3}{2})}$ excited by the input signal

$$u(t) = e^{-\frac{3t}{4}} + e^{-\frac{2t}{3}} + e^{-\frac{3t}{5}} + e^{-\frac{2t}{5}} + e^{-\frac{t}{3}} + e^{-\frac{t}{4}} + e^{-\frac{t}{5}} + e^{-\frac{t}{6}} .$$

The corresponding output is

$$\begin{aligned} y(t) \quad = \quad & -\frac{278555}{72072}e^{-\frac{3t}{2}} - \frac{4}{3}e^{-\frac{3t}{4}} - \frac{12}{5}e^{-\frac{2t}{3}} - \frac{40}{9}e^{-\frac{3t}{5}} \\ & -\frac{19}{6}e^{-\frac{t}{2}} + \frac{60}{11}e^{-\frac{2t}{5}} + \frac{24}{7}e^{-\frac{t}{3}} + \frac{12}{5}e^{-\frac{t}{4}} \\ & +\frac{80}{39}e^{-\frac{t}{5}} + \frac{15}{8}e^{-\frac{t}{6}} . \end{aligned}$$

We choose a Laguerre orthonormal basis associated with $\lambda = 1$. We compute the coefficients $\widetilde{y}_k$ and $\widetilde{u}_k$, $k = 0, \dots, 22$, and we choose $p = 0$, $q = 4$, $N = 18$ in (26).

It can be verified that the Hankel matrix with 8 rows and 18 columns constructed from the coefficients $\widetilde{u}_k$, $k = 0, \dots, 22$, has full row rank and consequently $\widetilde{u}$ is persistently exciting. The 16×18 matrix $\begin{bmatrix} \mathcal{H}^- \\ \mathcal{H}^+ \end{bmatrix}$ has rank 10, equal to the sum of the degree of excitation of $\widetilde{u}$ and the dimension of the state space of the transformed system.

To compute a basis for row span $(\mathcal{H}^+) \cap (\mathcal{H}^-)$, we compute a SVD of $\begin{bmatrix} \mathcal{H}^- \\ \mathcal{H}^+ \end{bmatrix}$, obtaining row span $(\mathcal{H}^+) \cap (\mathcal{H}^-) =$ row span (Ξ), where

$$\Xi := \begin{bmatrix} -0.916817 & -0.1372 \\ -0.580368 & -0.0866933 \\ -0.370617 & -0.0553367 \\ -0.240547 & -0.0358771 \\ -0.158203 & -0.0235766 \\ -0.105281 & -0.0156783 \\ -0.0707379 & -0.0105281 \\ -0.0479052 & -0.00712646 \\ -0.0326521 & -0.00485551 \\ -0.0223741 & -0.00332608 \\ -0.0153994 & -0.00228863 \\ -0.0106385 & -0.00158072 \\ -0.00737286 & -0.00109529 \\ -0.00512374 & -0.000761045 \\ -0.00356927 & -0.00053008 \\ -0.00249167 & -0.000369996 \end{bmatrix}^\top .$$

Since we work with the Laguerre basis, the transformation (11) can be performed multiplying Ξ on the left by $S := \frac{1}{\sqrt{2}}\begin{bmatrix} -I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$, obtaining

$$S\Xi = \begin{bmatrix} \tilde{\dot{x}}_4 & \tilde{\dot{x}}_5 & \dots & \tilde{\dot{x}}_{18} \\ \tilde{x}_4 & \tilde{x}_5 & \dots & \tilde{x}_{18} \end{bmatrix} = \begin{bmatrix} \overset{\bullet}{X}_N \\ X_N \end{bmatrix} .$$

Solving the system (29) with least squares methods yields the identified continuous-time model

$$\begin{aligned} \widehat{A} &= \begin{bmatrix} -2.74109 & 14.2345 \\ -0.195398 & 0.74108 \end{bmatrix} , \ \widehat{B} = \begin{bmatrix} -0.512773 \\ -0.0672392 \end{bmatrix} \\ \widehat{C} &= \begin{bmatrix} -3.89906 & 14.8625 \end{bmatrix} , \ \widehat{D} = -0.0000190403 . \end{aligned}$$

Note that small numerical errors are present: $\widehat{D} < 0$. It can be verified that the poles of the identified model are -1.50001, -0.500007, very close to those of the original system. The transfer functions $G(s)$ and $\widehat{G}(s) := \widehat{C}\left(sI_2 - \widehat{A}\right)^{-1}\widehat{B} + \widehat{D}$ are close in the H_∞-sense: $\|G(s) - \widehat{G}(s)\|_\infty = 2.5904 \times 10^{-5}$. Most importantly, the transfer function of the identified system is also positive-real, since the inequality

$$\begin{aligned} &\widehat{G}(j\omega) + \widehat{G}(-j\omega) \\ &= \frac{-0.0000380806\omega^4 + 1.99983\omega^2 + 1.50005}{\omega^4 + 2.50003\omega^2 + 0.562521} > 0 \end{aligned}$$

holds for all $\omega \in \mathbb{R}$. Thus the transformation preserves passivity, but is not immune from numerical errors, whether because of a numerically unsound implementation, or the low number of expansion coefficients considered in the example (22 in total).

VI. Conclusions

We have investigated some properties of the orthonormal basis representation $\{\tilde{x}_k\}$ of the continuous-time state. We showed the relation between the state trajectories of the transformed system, and the sequences $\{\tilde{x}_k\}$ and $\{\tilde{\dot{x}}_k\}$ of the coefficients of the orthonormal basis representation of the continuous-time state and its derivative, and we illustrated the dynamics of $\tilde{x}_k$ in Th. 2. We have used these results to give a self-contained proof of the preservation of dissipativity through transformation in Th. 3, and illustrated in sect. V some preliminary results in their application to system identification. We believe that these results have also the potential to be relevant in (linear and non-linear) model-order reduction, where the preservation of physical properties such as dissipativity in an identified or approximate model (e.g. that of a nonlinearity) is essential (see [2], [8]). This is ongoing research that will be presented elsewhere.

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