University of Southampton

Faculty of Social, Human and Mathematics Sciences

Southampton Education School

Investigating Pedagogical Content Knowledge for Teaching Calculus at University Level: A Framework and Analysis of Four Cases

by

Ibrahim Abdah Alzubaidi

Thesis for the degree of Doctor of Philosophy

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Abstract

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Within higher education mathematics, the focus on the teaching of calculus is continuing to be highlighted. Researchers have long sought to enhance the quality of calculus teaching at the university level. Even so, there is little clarity regarding the Pedagogical Content Knowledge (PCK) of calculus teachers in higher education. This study seeks to go some way to closing this identified gap by investigating the PCK of calculus teachers.

This study proposes a model of PCK for calculus teaching and uses this model to identify how calculus teachers articulate and demonstrate their PCK to achieve their teaching goals, to deliver the building blocks to construct and enable their students' mathematical understanding, to apply instructional strategies, and to utilise calculus connections with other academic subjects and wider applications.

In order to understand the PCK of calculus teachers, this study is situated in higher education in Saudi Arabia. The sample group comprises calculus teachers of first-year university students. This study uses multiple cases and qualitative and quantitative data collected through a triangulated approach using survey, semi-structured interview and observation of teaching. The analysis of the data employs a specially developed analytical framework for PCK for teaching calculus. Cross-case analysis identified, in detail, how these teachers articulate and demonstrate their PCK to develop learners' cognition of calculus; address the developmental aspects of the curriculum, apply instructional strategies to deliver their teaching aims and objectives, and to utilise calculus connections.

This study's findings are steeped in fine detail and have appropriately addressed the research questions. It is significant in conceptualising, and analysing empirically, the PCK of calculus teachers. The findings identify that all the teachers showed their PCK in relation to how they taught calculus, it was also clear that not all aspects of PCK were equally evident among them. Some focused on specific instructional strategies to target learners' needs, others highlighted students' misconceptions about calculus in different ways. Knowledge of students' thinking about calculus concepts was narrow, while little effort about knowledge of calculus connections was identified. Although the teachers attempted to highlight real-world applications of calculus, identifying real-world connections that the students could understand was lacking. Significantly, calculus relating to other academic subjects was least identified.

The findings pave the way for future developments of university calculus teaching and provides a model that can be developed and used widely within the field of calculus teaching in higher education. It is anticipated that this model will support the development of mathematics teaching in higher education in Saudi Arabia and elsewhere.

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Research Thesis: Declaration of Authorship

Print name:	Ibrahim Alzubaidi

Title of thesis:	Investigating Pedagogical Content Knowledge for Teaching Calculus at University Level: A Framework and Analysis of Four Cases

I declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Part of this work have been published as:

   Alzubaidi, I., & Jones, K. (2018). A case study of a university teacher of Calculus 1. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of INDRUM 2018: the Second Conference of the International Network for Didactic Research in University Mathematics* (pp. 452-453). (Proceedings of the International Network for Didactic Research in University Mathematics; Vol. 2). Kristiansand, Norway: University of Agder and INDRUM.

Signature:		Date:	

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The Dedication

To my dear father and mother for all that they have given.

To my wife and my children for their patience and their sacrifices to enable me to complete this thesis, and for their continuing support over the years of my study.

Definitions and Abbreviations

Abbreviation	The term
CK	Content Knowledge
CKTM	Content knowledge for teaching mathematics
COACTIV	Cognitive Activation in The Mathematics Classroom: the Orchestration of Learning Opportunities for the Enhancement of Insightful Learning in Mathematics
DACaCu	Developmental Aspects of The Calculus Curriculum
GOTVT	General Organization for Technical and Vocational Training
GPK	General Pedagogical Knowledge
IBL	Inquiry-Based Learning
ISs	Instructional Strategies
LCCa	Learners' Cognition of Calculus
KSA	Kingdom of Saudi Arabia
KCaCos	Knowledge of Calculus Connections
MC	Mathematical Challenge
MCK	Mathematical Content Knowledge
MKT	Mathematical Knowledge for Teaching
ML	Management of Learning
MOE	Ministry of Education
NCAAA	National Commission for Academic Accreditation and Assessment
PCK	Pedagogical content knowledge
SS	Sensitivity to Students
TA	Teaching Assistants
TEDS-M	Teacher Education and Development Study in Mathematics
TPACK	Technological, Pedagogical and Content Knowledge
TVTC	Technical Vocational Training Corporation

Chapter 1 Introduction

1.1 Overview

Currently, parts of the Middle East are largely under-represented when it comes to educational quality initiatives, but this perspective is slowly changing. Examples include the implementation of educational policies, such as Vision 2030 in the Kingdom of Saudi Arabia (KSA) and Education 2020 in the United Arab Emirates (UAE). The governments of many Middle Eastern countries are seeking to improve the quality of education within higher education (Raven, 2011). For Middle Eastern students to demonstrate enhanced competence in the global marketplace, and for international students considering studying in the Middle East, the quality and type of instruction needs to be appropriate.

As the KSA attempts strategically to place its tertiary institutions on the world stage, teacher quality is of significant importance. Teachers need to be able to understand the impact of their teaching and have a strong pedagogical background in addition to their research responsibilities. While research quality can be, to some extent, measured through numbers of publications, conference proceedings, keynote speeches, and other academic appearances, capturing the nature of the quality of teacher knowledge in the classroom is considerably more difficult (Mansour et al., 2013). While research suggests that one way this can be achieved is through focusing on specific aspects of teacher knowledge, Khakbaz (2016, p. 185) argues that "there is little information about the teaching knowledge of mathematics university teachers". As calculus is fundamental for students pursuing business, commerce, economics, management, as well as many of the natural sciences, it is essential that the teachers working on first-year programmes are able to provide the foundation for concepts that require scaffolding during a student's university journey.

Studies have been conducted on teachers' pedagogic content knowledge (PCK). Such research is scattered among quantitative and qualitative approaches, with multiple different focuses (e.g. Aydin el at., 2015; Fan, 2014; Krauss et al., 2008; Petrou & Goulding, 2011; Rollnick, 2016). Currently, none of this research has specifically targeted calculus teachers within the context of the Middle East. Understanding how teachers are applying their PCK to their calculus classroom context is necessary, as the Middle East attempts to compete in the global market. In consideration of these points, this study sets out to examine the extent to which university teachers of mathematics, and calculus in particular, are able to demonstrate an understanding of the PCK that is associated with teaching first year university students' calculus. As a teacher, finding a balance between the theoretical underpinnings of pedagogy, and pairing this with the conceptual knowledge of the subject matter to be taught can be particularly interesting. While calculus is necessarily something the researcher is acutely familiar with, the education associated with teaching such a crucial and important concept is very intriguing. Through a careful analysis of this topic, and carefully chosen methods, the researcher seeks to comprehend the overall field of study. The reasons why it is necessary to undertake this type of research, surrounding the identified topic and context, include there being 1) no studies on teachers' PCK of calculus 1 at university level in the KSA to date, 2) closing the gap in the literature related to PCK among calculus teachers in the KSA, and 3) making contributions to research that reflect the globalisation of higher education. Biza et al. (2016) reviewed studies published after 2014 and summarized theoretical and methodological perspectives. They were looking for the link between theories and university mathematics teachers' practices and posed the question "how can knowledge and competence developments be described and analysed effectively and validly?" (Biza et al., 2016, p.24).

As a Saudi national, the researcher has witnessed the shift in the educational system in the KSA from one of being unorganised, and not well established, to the current model. For more detail on the educational system in the KSA, see Chapter 3 and Appendix A. While this model still has issues, which the government along with educational institutions are attempting to improve, it is still considerably better than the system existing a decade ago. Furthermore, the field of tertiary education has been one of the major places where education has undergone expansion (Al-Aqeel, 2016; Yamani et al., 2000). Along with this expansion, the 'growing pains' corresponding to the implementation of new programmes, departments, and institutions, raises the need for faculties to demonstrate excellent quality. Initially, this was challenging because the demand for teachers exceeded the supply (Borg & Alshumaimeri, 2012; Sabah et al., 2014).

This researcher's personal interest and professional experience in university level mathematics education can be considered as the personal motivation for undertaking this study. O'Leary (2017) considers that it is important, when conducting research, to select a topic that evokes passion and interest in the researcher. The researcher has a personal interest in both the fields of education and mathematics. While the researcher's academic strength is in the field of education, both fields have played a role in the researcher's education experiences. The researcher is a lecturer in mathematics education and obtained an MSc in Curricula and Teaching Methods in Mathematics with a GPA 3.95 out of 4 and overall grade Excellent with First Class Honours, and a BSc with Excellent with Second Class Honours from the Department of Mathematics. Not only does the researcher have subject content knowledge, he has also has pedagogic experience obtained in his role as a mathematics teacher at intermediate and secondary schools for many years and as a teaching assistant in the Mathematics Department and for one year when teaching calculus to first year students at university, and then teaching subjects related to mathematics education at university level for five years. To stimulate his wider experience and motivation the researcher has attended courses in designing educational podcasts, designing interactive lessons using Course LAB, conferences on scientific research, its skills, and statistical applications in the field of curriculum and educational supervision. Combining the importance of personal interest and research motivation and focussing on a topic that links mathematics with education is the choice that this researcher has made in order to benefit the academic community and potentially policymakers, but also to feed the researcher's passion in the field of calculus education.

1.2 The Research Gap

Numerous studies (Hill et al., 2008, Khakbaz 2016; Krauss et al., 2008; Lesseig, 2016; Fan, 2014; Marks, 1990; Petrou & Goulding, 2011; Shulman 1986, 1987; Sowder et al., 1998) have indicated that teachers require different types of knowledge in facilitating learning in the classroom, yet researchers often disagree about what types of knowledge to include. Pedagogical content knowledge (PCK) was identified by Shulman (1987) as an essential component of teacher knowledge and defined as a “special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (p.8). More recently Miller (2006) argued that “PCK provides a framework that can be used to describe the origin of this critical teacher knowledge; i.e., that PCK represents an epistemological approach to constructing teaching knowledge” (p.91). In Shulman’s (1987) opinion, it was the interconnectedness of content and pedagogy that was vital. He considered that it was necessary for the teacher to go beyond the subject and interpret the subject matter and how it is linked to their role in facilitating learning that contributes to their effectiveness.

Research on PCK has been expanding and developing. In recent years, PCK has been examined through the lens of technology (Tamir, 1988; Mishra & Koehler, 2006). Though this is not necessarily fully relevant to the field of mathematics, where much of the material continues to be taught in the classroom on a whiteboard (i.e. not online or through technology-enhanced methods), this does not mean to say that PCK itself is outdated, the outcome is quite to the contrary. Despite a gap in the research, any previous research that has been conducted on PCK has quickly become outdated (i.e. less explicit) as a theoretical framework. In particular, using the notions and assumptions of the framework, it is important, for methodological reasons, to make the theoretical framework as explicit as possible (Depaepa et al., 2013). This field has not been examined in detail in the Saudi context and therefore leaves a research gap that requires filling. The researcher conducted a review of literature to identify similar work conducted in the Middle East and, to the best of the researcher’s knowledge, no study to date has investigated the PCK of mathematics teachers at university level in general and calculus teaching in particular. Moreover, in research reviews of the literature related to the research problem such as those of Biza et al., 2016; Bressoud et al., 2016; Larsen et al., 2017; Petropoulou et al., 2016; Rasmussen et al., 2014, there was no research that had investigated the PCK of calculus teachers at university level. This study therefore aims to bridge this gap by conducting research in the context of university education in the KSA and to provide a platform for future research in the field of PCK of not only calculus teachers, but also mathematics teachers at university level.

1.3 Research Aims

The aim of this study is to analyse the PCK demonstrated by a sample of university level teachers teaching a first-year calculus course. This research focuses on both the knowledge of content and students seen as essential (Ball et al., 2008; Lesseig, 2016) to this group of teachers, as well as the knowledge of content and teaching calculus (Lesseig, 2016; Khakbaz, 2016; COACTIV, 2004 (source: Baumert & Kunter, 2013); TEDS-M, 2008 (source: Tatto et al., 2008, p5)). Within this focus, the aims are:

- to verify the extent to, and the circumstances under, which PCK is used;
- to highlight any issues related to either content and students or content and teaching (Ball et al., 2008; Lesseig, 2016);
- identify areas where PCK can be improved within an analytical framework.

The types of instructional strategies for teaching calculus and the calculus connections that exist within the classroom are also considered especially in areas of demonstrated knowledge of content and teaching calculus and learners' cognition of calculus and developmental aspects of the calculus curriculum. These subcategories of PCK are seen as essential components for calculus teachers to achieve and could contribute to better overall performance at the university level. This is particularly important within the context of the KSA as it attempts to demonstrate exceptional quality on a worldwide stage.

It is also essential to consider the limitations of the research aims and objectives, as this is a small-scale research project and there is the issue of generalizable results. Hence, one of the aims of this research is to expand the aims and objectives of future research projects that seek to gain a better understanding of what could be done to further improve university mathematics teachers' practice through a broader context.

1.4 Objectives and Research Questions

The focus for this study is to investigate teachers' calculus teaching with an overarching goal to detail the PCK for teaching calculus and to analyse their PCK for teaching calculus. This will be achieved through the development of a model of PCK for teaching calculus that represents this area. As such, this research project has two overarching objectives:

OB1. To propose a model of PCK for teaching calculus.

OB2. To explore calculus teachers' PCK.

In line with the objectives, the RQs for the study are:

1. What would be a model of PCK for teaching calculus?

2. Using this model of PCK, how do calculus teachers articulate and demonstrate their PCK?

1.5 Pedagogy and Teacher Knowledge

This section begins with the term 'pedagogy', leading to the assumption that teachers require certain competencies to be successful or effective in the classroom. Overall, these competencies are multidimensional and generally consist of three overlying components - content knowledge (CK), pedagogical content knowledge (PCK) and general pedagogical knowledge (GPK). While the focus of this research is primarily concerned with PCK, it is important to understand the components of all three competencies to define the threshold encompassed by PCK.

In order to clarify the terminology to be used within this thesis, the researcher reviewed literature in relation to other research projects and found three terminologies to describe the method and practice of teaching: pedagogy, andragogy, and heutagogy. According to Palaiologos (2011), the learning method of pedagogy is 'teacher-driven' while andragogy is 'learner-driven' and heutagogy is 'self-determined'. While andragogy or heutagogy may be better 'literal' terms given the context of this study, they are not best suited for the context of this study. Nevertheless, the definitions provide clarity to the framework and scope of this thesis, while providing indications about what is, and is not, achievable in relation to the research questions.

According to Palaiologos (2011), the learning method of pedagogy is 'teacher-driven' and the main focus in this study is on university teachers and their knowledge. Pedagogy, in its direct translation, seems to relate to the teaching of children, yet the understanding of how it is applied goes well beyond this literal translation into how it is employed in the literature. Pedagogy is "the methods and principles of teaching" (MacMillan English dictionary, 2007, p.1101), so “widely assumed to be self-evident” (Adams, 2011, p.467). Bernstein in his grand theory of social structure and reproduction claimed pedagogy as a "cultural relay" (1990, p.191), but Alexander (2004, p.9) argued that “the spectrum of available definitions ranges from the societally broad to the procedurally narrow". In this study, the researcher adopts Alexander's (2004, p.8) definition of pedagogy, which is "conceived as encompassing both act, and thought, about teaching". In addition, under the definition of pedagogy, learning is a process of acquiring subject matter where content is sequenced according to the subject matter (Darder & Baltodano, 2003). All of these elements can be combined to not only explain pedagogy, but to explain the way that teaching, and learning occurs within the context of a Saudi university. As a result, the term pedagogy is used throughout this thesis.

1.5.1 Pedagogic Content Knowledge (PCK)

Shulman (1987), defined PCK as “a special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding” (p.8). Subsequently, Ball et al. (2008, p. 399) divided PCK into “knowledge of content and students”, “knowledge of content and teaching”, and “knowledge of content and curriculum”. For the purpose of this thesis, PCK is defined as a combination of subject content knowledge and general pedagogical knowledge, while taking into account both learners’ conceptions and learning difficulties. Investigating teachers' PCK is a useful starting point in determining their classroom practice and professional knowledge; it offers insights about them and their pedagogical strategies within a certain context (in this case, university calculus teachers in the KSA).

The overlap between pedagogy and content may differ among teachers, with some more heavily influenced by the pedagogical aspect and others by the content. Ultimately, since the purpose of this study is to determine the PCK of calculus teachers working in first year calculus courses at university level, it is assumed that each teacher, in this study, has the ‘competencies’ to teach the first-year calculus course, and would therefore have at least some level of both content and pedagogical knowledge that could be investigated. While the definition offered for PCK has some limitations, it allows for the scope of this study to be bounded by the definition provided in order to provide some sort of analysis. This definition is also consistent with the literature on the subject of PCK, and this consistency provides further justification for the definition (see Petrou & Goulding, 2011; Miller, 2006; Ball et al., 2008; Grossman, 1990; Shulman, 1987).

1.5.2 Content Knowledge (CK)

Shulman (1987) defined CK as that which "refers to the amount and organization of knowledge per se in the mind of teacher" (p.8). Ball et al. (2008, p.391) subsequently took CK to include "knowledge of the subject and its organizing structures". CK can be divided into two smaller components: syntactic content knowledge and substantive content knowledge (Barnes, 2007). Syntactic content knowledge is a set of strategies that can be employed to establish truth, validity, invalidity or falsehood (Shulman, 1986), while substantive content knowledge comprises the concepts, models, laws and principles associated with a particular discipline (Barnes, 2007). More is said about syntactic content knowledge and substantive content knowledge in Section 2.4.

1.5.3 General Pedagogical Knowledge (GPK)

Shulman (1987, p. 8) considered that GPK is a "special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter". Later, Grossman and Richert (1988) stated that GPK "includes knowledge of theories of learning and general principles of instruction, an understanding of the various philosophies of education, general knowledge about learners, and knowledge of the principles and techniques of classroom management" (p. 54). Subsequently, König et al. (2011, p. 189) defined GPK as "Generic theories and methods of instruction and learning, as well as of classroom management" (see Section 2.3).

1.6 The Research Background

The KSA, located on the Arabian Peninsula, was founded in 1932. It has borders with the Red Sea to the west, with Yemen to the south, with Iraq, Jordan and Kuwait to the north, and with the Arabian Gulf, Qatar, and the UAE to the east. It is an Islamic state, which means that the Shari'ah (Islamic Holy Law) acts as both the legal framework for the country and as its constitution (Yamani, 2000). Saudi Arabia enjoys a high GDP, which is a result of the discovery of oil in the region paired with the increasing worldwide consumption of oil that exists today. Petroleum exports account for a significant portion of the country's financial ties, on which the economy is heavily dependent. It should be noted that while the country has enjoyed significant benefits from the worldwide demand for petroleum products, experts in the country are attempting to also ensure that the country has alternative sources of revenue and a strategy if requests for petroleum, from foreign nations, decline (Al-Amri, 2011).

One strategy for achieving diversification within the country has been through advancements in education, and more specifically, advancement in education that strategically aligns the KSA, through the implementation of educational policies such as Vision 2030, with other major worldwide experts (MOE, 2017). The government of the KSA has, as a result, spent substantial financial resources developing a higher education system (Al-Amri, 2011), expanding its university infrastructure (i.e. building more universities) and encouraging more participation in higher education.

Higher education, and more specifically undergraduate education, has become much more commonplace in the KSA as the numbers of students enrolled in various programmes continues to rise annually. According to recent figures, in 2015 the KSA enrolled 1,454,692 students in higher education; of these 1,240,117 were studying at the undergraduate level (Central Department of Statistics and Information, 2015). Much of this increase is due to policy, funding, administration and regulation of the university environment, with the Ministry of Higher Education (MOE, 2017) and the Technical Vocational Training Corporation (TVTC) offering highly subsidised packages for students wishing to pursue this path of education. As this growth continues, the Saudi government consistently allocates significant resources to developing an education programme that competes on an international level (MOE, 2017).

In 2016, the Saudi government spent in excess of KSA Riyal 80 billion on improvements to the education sector, in addition to the allocation of KSA Riyal 184 billion from the national budget (which was also the largest single item of the national budget) (MOE, 2017). This money has been earmarked for educational developments relating to infrastructure and the development of new programmes and courses. The intention of this funding is to develop a Saudi population that can sustain the country, so as to diversify the region thus reducing the dependence on oil and oil products (Al-Aqeel, 2016). While some steps have been taken to increase participation through the creation of new disciplines for study, there has also been a need to refine the current areas of study, specifically in STEM subjects (science, technology, engineering and mathematics). One of the main focuses is on the development of mathematics and sciences at the undergraduate level.

While the focus on university-level mathematics may be documented among multiple sections in Saudi education policy, how this is to be achieved is much less clear. There is the burden of providing an excellent standard of higher education, but this assumes that there are university teachers available to teach to such standards. As the KSA moves into a position where it seeks to be competitive within the global university rankings, it requires educators with high levels of professional knowledge (Krieger, 2007). Teachers at the tertiary level must continue to draw on inspiration and innovation to teach students in a variety of different styles and context (Sabah et al., 2014). This personal innovation is often stimulated by new and conceptually unique approaches to understanding, pedagogy and assessment. Teachers must navigate the field of higher education to provide students with a learning experience that not only meets specific learning objectives, but also one that offers students the facility to scaffold and develop their understanding as they move through university (Borg & Alshumaimeri, 2012).

1.7 The Research Context

The gap in the research identified for this study applies to the university sector not only in the KSA but also internationally. This study examines university calculus teachers' PCK, in particular those who teach first year calculus programmes, at a reputable Saudi institution. The rationale of choosing calculus for this study is that calculus is important at the university level. The course is the combination of several subjects in mathematics that include numerical calculation, graphical representation, and symbolic manipulation (Alcock, 2014; Tall & Ramos, 2004). For good learning, teachers need the professional knowledge to present the course and should attempt to attract students' interest in the learning of calculus by, for example, relating the mathematics to real-life applications. It is also important to explore teachers' knowledge of calculus. Specifically, this research aims to investigate the PCK of calculus teachers, focusing on proposing a model of PCK for calculus teaching and using this model to identify how calculus teachers articulate and demonstrate their PCK to achieve their teaching goals, to deliver the building blocks to construct and enable their students' mathematical understanding, to apply instructional strategies, and to utilise calculus connections with other academic subjects and wider applications.

1.7.1 The Saudi Arabian University Education Context

The focus of the Saudi education system has four overarching principles. There must be a focus on a centralised system of control, educational support, state funding (i.e. all education is free in the KSA), and a general policy of gender segregation (Smith & Abouammoh, 2013). While there are multiple bodies that oversee these principles, the ones relating to higher education generally include the General Presidency of Girls' Education, which is responsible for the segregated education of girls and women (Hamdan, 2005). The MOE, which is responsible for aspects of university involvement, and the General Organization for Technical and Vocational Training (GOTVT), which is responsible for technical colleges and trade training. At the university level, men and women are segregated in different campuses and most universities in the KSA provide opportunities for both genders, though in separate locations (exceptions include King Fahd University, which is male-only and Princess Nora University, which is female-only) (Al-Aqeel, 2016). There are instances where both genders are taught together in higher education; for example, in some medical schools (Al-Amri, 2011). While all of these universities offer different benefits, this research is solely focused on the male population at one university (that offers female schooling on a different campus).

1.7.2 University Level Calculus

University level calculus is a challenging area regardless of which educational institution is offering the instruction (Kidron, 2014). Calculus is essentially the study surrounding how things change. It is a comprehensive framework for modelling systems, and by utilising such systems predictions can be made to deduce consequences (Barrett & Suli, 2012). Calculus requires that a systematic view be taken on phenomena involving change, especially along dimensions such as time, force, mass, length and temperature (Back & Wright, 2012). Calculus is a method that uses science to provide outcomes for quantifiable real-world situations (Doorman & Van Maanen, 2008) and can be viewed as a type of language used by mathematicians to communicate clearly as they explain how objects behave in nature (Biza et al., 2016; Bressoud et al., 2016; Larsen et al., 2017; Petropoulou et al., 2016; Rasmussen et al., 2014).

For the average person, the above definition may not be particularly helpful, especially at the university level. Many university students view calculus as just another class where memorisation of equations is required in order to pass (Bresoud et al., 2013). As such, students can have the impression that these equations are far removed from the ‘real world’ situations. Without a concrete demonstration of how calculus is relevant to life beyond the university classroom, it can be difficult to convince first-year university students of its benefits. Calculus is important, specifically at the university level, because understanding calculus is essentially the first step in understanding how the world works. It is a foundation on which other skills can be built. Through learning calculus, students become masters of a mathematical language that is essential in many real-world applications of mathematics.

The importance of calculus lies within its application within other disciplines studied at the university level. Calculus does not solely relate to the field of mathematics. It is used within engineering, physical, business, and economics to make accurate predictions about systems that are constantly in adaptive and fluctuating circumstances, allowing for profit maximisation and wise decision making. It is used in engineering to predict how certain elements (e.g. force) might affect structures and other infrastructure. It is used in computer science for applications, such as with large scale data-analysis problems using large clusters of computers or for machine learning. It is also used in the sciences for research purposes. In addition, it is often a requirement or prerequisite for many additional programmes (e.g. for engineering chemistry, physics, biology, mathematics, computer science, etc.).

Many of the mathematics theorems taught in first year calculus classes are then applied in upper years, suggesting an essential need for students to be fluent in the language of calculus after their first semester taking this subject. The influential nature of calculus suggests that studying the teachers who are required to teach the material to students seems beneficial, especially when considering the Saudi context and the desire of the KSA to be competitive in worldwide university rankings.

1.7.3 Content Knowledge of Calculus 1 Teachers

Calculus 1 is a first-level course of calculus at university presenting fundamental topics (functions, limits and continuity, the derivatives, and integrals) which are of use to all undergraduate mathematics students and is a pre-requisite for many other programmes, including but not limited to, Engineering, Chemistry, Physics, Biology, Computer Science, and Environmental Sciences (Alamolhodaei, 1996; Alcock, 2014). In the case of teachers' calculus 1, their CK would typically have first been acquired at the undergraduate university level. Calculus is not a subject that is typically or consistently taught at the high school level in the KSA, though students are required to take some aspects of mathematics that act as a prerequisite to beginning a calculus programme. Calculus teachers in the KSA are unlikely to solely have a calculus background; it is much more likely that they would have undertaken a degree in mathematics with a calculus focus. As a result, calculus teachers at university level demonstrate their knowledge for teaching in this field in accordance with the rules and procedures associated with this branch of mathematics.

1.8 The Sample Location

The selected research context is a comprehensive public university in the KSA, referred to as University X in this thesis. University X supports both graduate and undergraduate programmes for both male and female students. There are approximately 100,000 students and associated with this population are approximately 5,000 faculty members who are responsible for teaching courses and often for conducting the associated research projects typically linked with higher education (MOE, 2017). University X has a fairly large mathematics department (in comparison with other universities of similar size) and this department supports students either as part of a degree programme in mathematics or as prerequisites for other subjects (MOE, 2017).

University X is unique and appropriate for this study for a variety of reasons. University X has one of the KSA's top mathematics populations, it is also above the national average on mathematics scores (MOE, 2016). The average proportion of males to females is approximately 46% to 54%. The mathematics school is known for its positive learning culture, which promotes academic excellence and a highly supportive system for mathematics students, which ensures they achieve success. For more detail on the sample location, see Appendix A.

Being a university that is heavily influenced by the expansion of the higher education system in the KSA, the mathematics school has embraced the use of Technology Enhanced Learning (TEL) in some of its lessons and is investing in infrastructure, hardware, software, teaching and learning resources, and research in a variety of different fields. In 2013, at the onset of the pilot study for their foundational calculus programme, the researcher was involved in some of the preliminary discussions surrounding the implementation, which has provided a good grounding and positive impetus for a study on calculus teaching in the classroom. The selected university may therefore provide examples of good practice. While a variety of research has been conducted on university education in mathematics and other research has been completed on topics focusing on the KSA, the two have never been paired.

1.9 Theoretical Framework

Key studies were drawn on in terms of their findings regarding PCK, both generically and specifically. This has informed, enhanced and further developed decisions made in the context of this study. One key finding is that whilst there is a proliferation of research in the area of the theory of PCK, not all of the research is as firmly rooted in practice as might be expected as "research on collegiate teachers' actual classroom teaching practice is virtually non-existent" (Speer et al., 2010, p. 99). Being tested in practical surroundings, more often than not meant in professional development scenarios of mathematics teachers in schools and universities, became a key criterion for inclusion in this research. This is relevant not only for the theoretical underpinnings of this study but also in terms of the motivation for, and practice-informed nature of, the research presented here. Smith's (2014) articulation of the importance of a tool acting as a "bridge between research and practice" (p.3) has pivotally informed this study. The theoretical underpinning of the proposed model, itself informed by numerous researchers engaged in the field, is critical in providing order and the opportunity for categorisation within a contested field that, as is shown in Chapter 4, is characterised by as much disagreement as it is by agreement amongst scholars. The analytical framework provides a construct that is firmly based on the conceptual framework but has allowed for the systemisation of historical data gathered by other scholars whose work has informed this study, together with the empirical research conducted for this study. In their interaction, the conceptual and analytical frameworks ensure focus is retained on the subject at the heart of this study. It also ensures that the findings from this research can be re-tested in different contexts, given that the conceptual framework and the analytical framework provide a roadmap for future research.

1.10 Proposed Model

This thesis is influenced by Khakbaz's (2016) suggestion that "Pedagogical Content Knowledge (PCK) provides a suitable framework to study knowledge of teachers'' (p. 185). By using a relevant and an up-to-date theory that combines pedagogical knowledge with content knowledge, this thesis is not only able to demonstrate what teachers know, but how the results link to previous research on the topic. The proposed model is developed from the work of Khakbaz (2016); COACTIV, 2004 (source: Baumert & Kunter, 2013); TEDS-M, 2008 (source: Tatto et al., 2008, p.5); Lesseig, (2016); (see Chapter 4). This study set out to develop a framework of PCK, to be tested by investigating how calculus teachers articulate and demonstrate their PCK and from which a model of PCK has been developed (see Figure 4-6, p.71).

1.11 Contribution to Knowledge

This study is highly relevant to teaching and learning and has direct implications for how teachers at university level perceive aspects of PCK. It highlights the issues surrounding the form of PCK that university teachers and professors have. The concept of PCK is prevalent, not just in the KSA, but around the world, as studies have been conducted on PCK in many countries (e.g. Grossman & Yerian, 1992; Niess, 2005; Hill et al., 2008; Watson el at., 2008; Toerien, 2011; Khakbaz, 2016; Akerson el at., 2017). To the best of the researcher's knowledge and experience there has been no other study on teachers' PCK of calculus 1 at university level.

This study focuses on first year calculus teachers at university level. It addresses the recurring themes and theories corresponding to PCK literature, namely the issues of pedagogy and how this affects the relationship and teaching experiences in a calculus classroom. The uniqueness of the calculus classroom in the KSA, at the university level, lies in some interesting and interconnected factors, which include knowledge of content and teaching calculus, but also knowledge of content and students when teaching calculus.

The university calculus teacher is explored in this study through the lens of interpretive implications of PCK which may be employed and/or developed though a strategic approach, specifically to enhance achievement and provide a better overall experience for students. The study moves beyond a 'deficit model' and what the teachers might not express in their current state of PCK, and it provides a way forward that is consistent with the growth and expansion of the Saudi model of higher education growth. It is acknowledged that the teachers participating in this study may be likely to demonstrate a balance of both positive and negative traits, though these traits may likely differ between participants. If the KSA seeks to demonstrate excellence in higher education on the worldwide stage, then the PCK of teachers, especially teachers in core subject areas, must also demonstrate excellence.

The findings from this research provide a platform for future research in the field of PCK of not only calculus teachers, but also mathematics teachers at university level. The contribution this research makes, paves the way for the future development of calculus teachers and students and provides a model that can be developed and used widely within the field. Although this research was situated within the Saudi university system, which is therefore its priority, it also makes a global contribution to the knowledge and understanding of calculus teaching in universities. In addition, it is demonstrated throughout the literature review of this study that there is a scarcity of research examining the field of PCK within calculus at university level (e.g. Khakbaz, 2016). This is particularly problematic because of the way that calculus integrates into so many other subjects within the university context (i.e. students must take calculus for entrance into the sciences, engineering, finance, business, etc.).

While diverse studies have been carried out in PCK, none has been carried out in the Saudi higher education mathematics context. This research has investigated teachers' perceptions about PCK, in addition to its demonstration, by them, in the classroom. This methodological approach differs somewhat from the way data are presented in previous studies. Justification for the approach is outlined in the Methodology chapter.

1.12 Thesis Organisation

This study comprehensively details the issues surrounding PCK within four main areas: (1) Learners' Cognition of Calculus, (2) Developmental Aspects of the Calculus Curriculum, (3) Instructional Strategies of Teaching Calculus and (4) Knowledge Calculus Connections, and how do calculus teachers articulate and demonstrate them within the context of the KSA, with initial research questions being posed. The thesis structure and outline are presented in Figure 1-1. Chapter 2, the Literature Review shares with Chapter 3 Background to University Level Education in the KSA, where past and present research in the current field is paired with key concepts that shape and inform the study. The theoretical aspect is elaborated in Chapter 4, the theory chapter of the study, in order to ensure that a foundation is established from which this research project can be built. Chapter 5 then outlines the methodology and explains the methods applied in the study in addition to the selection and analysis processes. Chapters 6 presents the data analysis and findings. Chapter 7 discusses the findings and Chapter 8 sums up the whole study, draws conclusions and identifies further research.

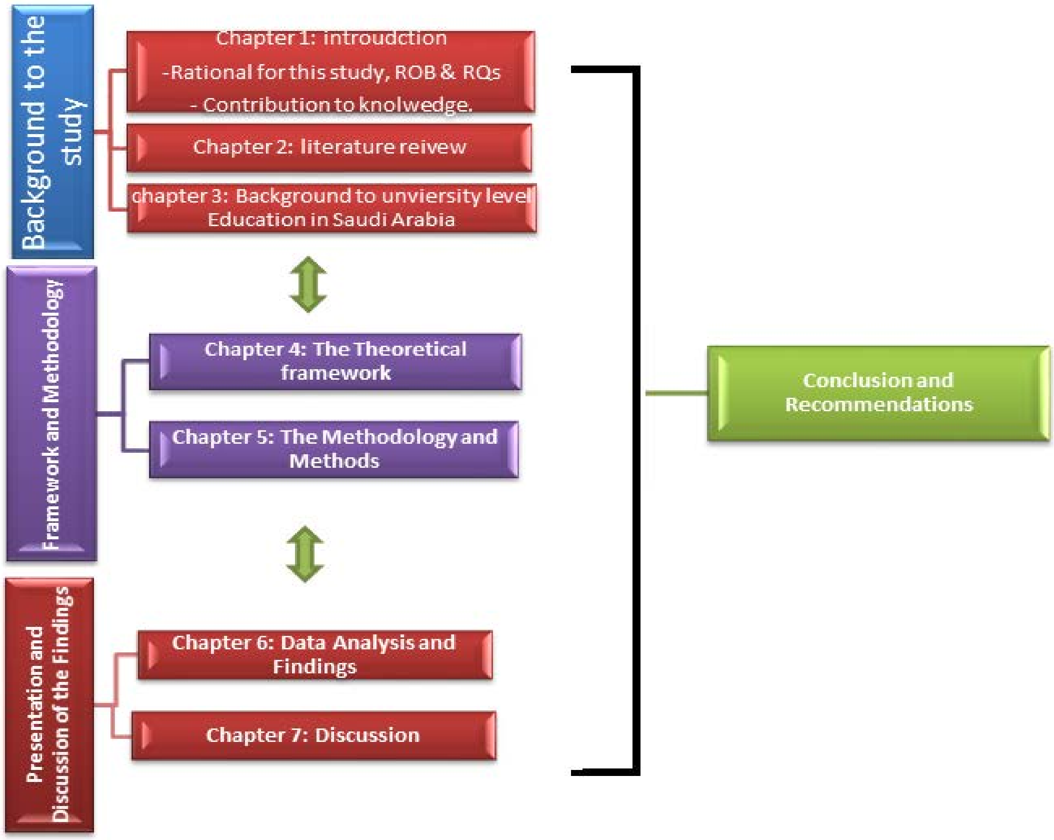

Figure 1-1: The Thesis Structure and Outline.

1.13 Chapter Summary

In the introduction, the principle objectives and the rationale for this study have been outlined together with the background information and context of where the research is situated. The research questions have been specified (and are addressed in more detail throughout the study), and the theoretical framework that informs the study has been established. The potential contributions to knowledge within the fields of education, mathematics, calculus, and pedagogy have been communicated and the case for research into the PCK of calculus teachers in the KSA made. Considerations have been given as to how teachers (and potentially their students) could benefit from a better understanding about their own PCK and how it might develop over time in different ways. The unique university system in the KSA has been highlighted and this has assisted in laying the foundations for this study. In the next chapter – the literature review, the literature relevant to this thesis is critically examined and analysed, in order for the reader to gain enough background knowledge to fully understand the components of this research project.

Chapter 2 Literature Review

2.1 Introduction

This chapter is concerned with the identification and analysis of existing literature in the field of study. The purpose of this literature review is to delineate some of the key concepts in relation to the subject under investigation, as well as how they relate to what other authors have presented. To establish the focus for this study, an analysis of the existing literature is undertaken in order to reveal the gap in the literature within the field of study.

The starting point for the literature review is to identify and describe key concepts and constructs that relate to the complexities of knowledge. These are articulated and contrasted in terms of their relation to the teaching and learning of mathematics, specifically calculus. Different types of knowledge are identified with relevance to mathematical and subject content knowledge, as well as the knowledge required to teach calculus effectively.

The chapter then goes on to identify difficulties students have with calculus, and effective teaching strategies at under-graduate level, that aim to address those challenges.

2.2 Teachers' Knowledge

Over the past thirty years teachers' knowledge is one of the most important fields which researchers are continuing to investigate, including types of knowledge and understanding of teachers' knowledge and approaches to knowledge (Barnett & Hodson, 2001; Cochran et al., 1993; Fennema & Franke, 1992; Grossman, 1990; Halim & Meerah, 2002; Jong, 2003; Marks, 1990; Shulman, 1986, 1987; Van Driel et al., 1998). Teachers' knowledge is "conceived as all profession-related insights, which are potentially relevant to a teacher's activities" (Verloop et al., 2001, p 24). In order to instruct students effectively in mathematics, at whatever level, it is often said that teachers need to know more than content or, as Chapman (2017, p. 304) states, "teachers need to hold knowledge of their students beyond the content that provides appropriate context to engage them meaningfully in the mathematics classroom and the learning of mathematics". Consequently, knowledge required by teachers, encompasses a variety of constructs. Chapman's description relates to culture, as one challenge for researchers is defining teachers' knowledge in that knowledge is difficult to define with precision. Approaches to knowledge need to take into account varying perspectives on the nature of knowledge such as rational and instrumental constructs, declarative versus procedural knowledge, moral determinants, subject matter background, professional knowledge, affective-motivational characteristics, and willingness to teach (Blömeke & Delaney 2014; Grossman, 1990; Loughran, 2008; Shulman, 1987). Shulman (1987) introduced the characterisation of teacher knowledge, making seven distinct categorisations: 'pedagogical content knowledge', 'curriculum knowledge', 'subject matter knowledge', 'general pedagogical knowledge', 'knowledge of learners' characteristics', 'knowledge of educational context', and 'knowledge of educational purposes and values'. Despite the fact that these categorisations are general and not specific to mathematics, a number of researchers have nevertheless utilised them within their educational research. The diversity of constructs evident in contemporary research in mathematical education can be traced back to a fundamental divergence of what Dossey (1992) calls external and internal (to the mathematical community) expectations and interpretations of the nature of mathematics and the knowledge and beliefs of mathematics teachers. By way of delineation and definition, the subsequent sections deal with a number of these constructs, such as GPK, CK, PCK, and mathematical knowledge for teaching (MKT).

2.3 General Pedagogical Knowledge (or Educating Strategies) (GPK)

Some studies simply refer to GPK as knowledge about teaching (Cochran et al., 1993; Grossman, 1990) or of the knowledge used for teaching (Vistro-Yu, 2005). In this sense GPK can include aspects such as techniques for teaching, processes for teaching, psychological principles and other classroom management strategies. Essentially, any GPK encompasses general types of knowledge that the teacher needs in order to teach the students in their classroom. As such GPK fuses range of knowledge including reasonable frameworks for organising, classroom plans, conducting organisational systems, providing various classroom-levelled strategies, and through implementing motivational strategies (Stylianides & Stylianides, 2013).

Other literature suggests that GPK largely encompasses two components, instruction and classroom management, though it was a challenge to find consistency within the literature (Konig et al., 2011). In the USA, for example, GPK encompasses educational foundations and teaching methods, but these cannot be justified in all contexts, as culture seems to play a pivotal role in what actually constitutes ‘general pedagogy’ (Konig et al., 2011). In Germany, as another example, ‘general pedagogy’ is less about teaching methods and more about the underlying theories derived from educational psychology, the sociology of education, and educational histories. As such underlying ideas can be cultural, teachers are taught to uphold certain roles and identifying characteristics related to GPK within their own teaching context. Therefore, applying either the US or German examples to the case of KSA is challenging. Nevertheless, despite any differences that are in part rooted in cultural traditions, there seems to be consensus on the importance of a positive correlation between GPK and educational outcomes because it includes all principles of teaching, such as a body of general knowledge skills (Grossman, 1990).

In a study by Rollnick et al. (2008), mathematics teachers who were competent in GPK were able to manage classrooms effectively, thus leading to more positive learning experiences for the students. It is acknowledged that much of the previous work in this area has been conducted at lower levels of education (e.g. elementary and secondary), and it is a possibility that aspects like classroom management may be more of an issue at lower levels than at the university level being examined in this study (Jong et al., 2005; Vistro-Yu, 2003).

While classroom management may not end up being a focus at the university level, there are still aspects of GPK that apply to the university context. Both classroom management and the component of psychological processes require a bigger picture approach, much of which falls outside the scope of this study.

2.4 Content Knowledge/Subject Matter Knowledge (CK)

2.4.1 Overview

Grossman (1990, p.6) states that CK refers to “knowledge of the major facts and concepts within a field and the relationship among them”, while Petrou and Goulding (2011, p.11) define CK as including “knowledge of the subject and its organising structures”. Ball et al. (2008, p.403) expand on this by defining “subject matter knowledge” as being “composed of three key elements: 'Common content knowledge' that any well-educated adult should have, Horizon content knowledge, and mathematical knowledge that is 'Specialized content knowledge' to the work of teaching and that only teachers need know”. Content knowledge is generally obtained during disciplinary education (Jong, 2003).

As noted in Chapter 1, content knowledge can be considered as having two components: syntactic content knowledge and substantive content knowledge (Barnes, 2007) and this is considered in the next section.

2.4.2 Syntactic and Substantive Structures/Content Knowledge

Syntactic content knowledge is a set of strategies that can be employed to establish truth, validity, invalidity or falsehood (Shulman, 1986). This contrasts with substantive content knowledge, which is defined as the concepts, models, laws and principles associated with a particular discipline (Barnes, 2007). In this case, by linking the substantive with the syntactic, teachers should be able to define the concept for the learner, explain the concept in relation to theory and to practice, but also be able to relate that concept to external situations as well as to situations within the mathematical discipline.

Both syntactic and substantive content knowledge are essential for calculus teachers when relating this to PCK. This is because teachers need to be able to have both an understanding of the material being taught in first year calculus, but also should be able to think about how it should be taught. Wu (2005) argues that teachers with strong PCK generally have a sound understanding of CK and are often able to design instructional strategies that accurately allow learners to best understand the concept or material. Content knowledge is also linked to experience. There have been numerous studies that have examined the CK of teachers at different stages of their careers (i.e. pre-service, novice, intermediate, expert) (see Aydin et al., 2015; Carpenter et al., 1988; Even, 1993; Hill et al,, 2008; Krauss et al., 2008; Manoucherhri, 1997; Muijs & Reynolds, 2000; Rollnick, 2016). These studies all have determined that PCK is strongly affected by good CK. As such, it is essential to both to have a clear definition of CK and also to judge the influence of teachers' CK in relation to first year calculus. For the purpose of this study, CK is a component of knowledge of content and teaching and is considered to influence the way the calculus content is taught.

2.4.3 Mathematical Content Knowledge (MCK)

French (2005) distinguishes between subject knowledge and mathematical content knowledge (MCK) inasmuch as that the former is more all-encompassing in terms of the knowledge of the subject in contrast to content knowledge which, he argues, is defined by curricular and assessment requirements. Plotz (2007) has a somewhat diverging view, as he references general subject matter content knowledge as MCK without, for example, the link being drawn to a curricular specification. Plotz (2007) particularly highlights the fact that MCK stems from primary and secondary schooling. Other academics, notably Van Driel, et al., (1998), Jong (2003), Jong et al. (2005); Khakbaz (2016) and Tamir (1988) reference subject matter knowledge as largely acquired through formal tertiary education and training.

Knowing that much of the literature that exists generally relates to the western context, many of the policymakers in these western countries indicate concern when students do not demonstrate proficiency on standardised testing in mathematics. Their conclusions, when they occur, generally seem to indicate that students would do better if their teachers knew more about mathematics (Kahan et al., 2003). For example, in the US context, the US Department of Education suggests that teachers should have a 'deep knowledge' of the subject matter (US Department of Education, 1998, p.22). In their conclusions about the lack of proficient subject matter knowledge, these policymakers indicate that teachers teach in a more authoritarian way (Thompson, 1992). Still considering the US context, this 'authoritarian' manner is seen as a bad thing because under this teaching style, students may be less likely to ask questions. A lack of questions may lead to misunderstandings or misconceptions about the topic. As mathematics is very much a scaffolded learning process, a lack of understanding at a lower level has the potential to lead to larger problems in the future. This poses multiple issues in the Saudi context. First, this outcome may suggest that when considering the university context, students previously taught by teachers at the high school level may not have the same level of mathematics understanding as those who have been taught by a content-expert teacher. This perhaps is not surprising, given the fact that it is inevitable that students experience both good and bad teachers. What is, perhaps, more interesting are the cultural implications associated with the questioning process. While in the US context, the idea of questioning the teacher in order to facilitate better overall understanding of the classroom material is seen as a very positive interaction, it appears that, in much of the literature, a relationship between teacher and students is formed and that interaction during class is encouraged. This is not necessarily the case in the Saudi context. Classes can be quite rigid and the idea of an engaging question and answer session, or of the notion of a student disrupting the classroom with a question, might typically be discouraged.

Based on the above, there is some difficulty in determining the nature of MCK if using US style approaches, as the definition of teaching may require modification. In the Saudi context MCK, at the university level as it relates to calculus, generally includes instructional skills of calculus using axiom, definition, relating a definition to an example, theorem, proof, example, diagrams, and generality (MOE, 2017). Also included are concepts of function, limits and continuity, differentiation rules, application to graphing, rates, and approximations, definite and indefinite integration, the fundamental theorem of calculus, applications to geometry (area, volume, and arc length), applications to science (average values, work, and probability), techniques of integration, approximation of definite integrals, and improper integrals (Neill & Shuard, 1982). It should be noted that these topics do vary by programme, department and university.

How teachers might acquire competencies in these areas relates to their MCK. According to Kahan et al. (2003), there are three features to MCK: a deep understanding of factual knowledge, the understanding of how this understanding fits within a larger conceptual framework, and an organisation of knowledge where retrieval and application are possible. This is contrasted by Kaput et al. (1998) who suggest that MCK has only two components: knowledge inventory and organisation. They suggest that knowledge inventory is likely to include what one knows, while organisation relates to how this knowledge is accessed.

While it is interesting to think of MCK as a specific component of a teacher's knowledge, figuring out how to analyse it poses some considerable challenges. Within those challenges, there are suggestions by previous researchers that a quantitative approach might be optimal, as sometimes CK is analysed through the number of courses a teacher has completed or by their scores in university. These quantitative findings can be challenged on a number of different grounds. First, there is not necessarily a relationship between how a teacher scored in a university mathematics course and what they know about calculus - even if the subject matter is related. Other researchers (e.g. Ball, 1990; Even, 1990; Leinhardt & Smith, 1985; Shulman, 1986; Wilson et al., 1987) have suggested a more qualitative approach, which tends to focus more on how knowledge is organised and whether knowledge and understanding of facts can relate to what the teacher knows about the discipline. This is also problematic, as without at least some sort of quantitative component, comparisons are inevitably difficult.

2.5 Pedagogical Content Knowledge (PCK)

While research on PCK has been wide ranging, the focus for this thesis relates to mathematics education, and therefore it is essential to outline what has previously been studied regarding PCK within the field of mathematics and in demarcation to constructs referenced in previous sections of this chapter.

The idea of PCK was initially developed by Shulman (1987) and colleagues in the 'Knowledge Growth in Teaching' project as a broader perspective model for understanding teaching and learning (e.g. Shulman & Grossman, 1988). The focus of their project was on how novice teachers acquired new understandings of their content, and how these new understandings influenced their teaching. What emerged from this project was the concept of PCK, which was formed by the synthesis of three knowledge bases: subject matter knowledge, pedagogical knowledge, and knowledge of context. Ultimately, under Shulman (1987, p.8), PCK was defined as a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding". In other words, PCK can be considered unique to teachers and separates, for example, a science teacher from a scientist or a mathematics teacher from a mathematician. Along the same lines, Cochran et al. (1991, p.6) differentiated between a teacher and a content specialist in the following manner:

> *Teachers differ from biologists, historians, writers, or educational researchers, not necessarily in the quality or quantity of their subject matter knowledge, but in how that knowledge is organized and used. For example, experienced science teachers' knowledge of science is structured from a teaching perspective and is used as a basis for helping*

> *students to understand specific concepts. A scientist's knowledge, on the other hand, is structured from a research perspective and is used as a basis for the construction of new knowledge in the field.*

PCK was acknowledged by Shulman (1987) as an essential component to consider because, up until that point, research in areas of teaching and teacher education had not necessarily focussed on key aspects - such as lesson content, combined with questions posed to students or explanations offered by teachers to students' questions. In Shulman's opinion, it was the interconnectedness of content and pedagogy that was lacking in novices, and that it was necessary to go beyond the subject and find out how the teacher interpreted the subject matter and how this linked to their role in facilitating learning. In his work, Shulman (1987) acknowledges that there is a knowledge base for teachers, and that this knowledge base is going to be a key component in the way that the subject matter is presented (e.g. a teacher who has a strong interest in Calculus might present it in a clearer way than a teacher who lacks interest). One final key aspect of Shulman's (1987) theory on PCK is that regardless of whether the teacher's previous knowledge came from teaching or from research, the teacher must also be able to navigate the preconceptions and misconceptions of students in order to ensure that the teaching is completed successfully.

PCK may consist of multiple elements, making the analysis of these elements challenging without an overarching definition. To further complicate matters, there is a divide between what constitutes knowledge and what constitutes a belief. In an attempt to clarify this issue, Phillip et al. (2007) suggest that beliefs are an inherent part of knowledge. They suggest that beliefs generally involve a level of certainty as perceived by the believer. In this way, beliefs are not consistent among individuals but rather that they differ depending on the conceptions of the believer. In this way, knowledge can be considered a set of beliefs and more specifically can be a set of beliefs that are typically justified in the mind of the teacher (Eichler & Erens, 2014). This relates to PCK because it is something that encompasses different types of knowledge, which are significantly influenced by the beliefs a teacher has about pedagogy or about learning mathematics.

PCK is a key construct that sits alongside other constructs such as GPK or specific CK for example. It denotes a teacher's knowledge of ways of helping students to understand specific concepts, as well as "relational understanding and adaptive reasoning of the subject matter" (Kathirveloo & Puteh, 2014, p.1). PCK could thus be interpreted as bridging subject content on the one hand and pedagogy on the other. Without a solid understanding of how to teach particular mathematical content, as well as a robust understanding of that mathematical matter, teachers may not be able to teach well, and students would not learn as well. From the perspective of teaching calculus, this has a number of specific implications, not least an understanding of the constituent components of teaching and learning calculus as a cognitive development. Bressoud et al. (2016), for example, reference particularly students' difficulty regarding the conception of the limit process, rooted in prior experiences that "can block understanding" (p. 6); therefore PCK denotes a teacher's understanding of the phenomenon, as well as knowledge about pedagogical principles enabling the effective teaching of the content matter at heart.

Some studies (e.g. Ball et al., 2008; Bromme, 1995) have argued that there is a lack of clarity of empirical grounding and theoretical explanation for the existence of PCK. While it is acknowledged that several studies have included ideas about how PCK can be conceptualised, according to some researchers (e.g. Cochran, 1993; Fernandez, 2014: Jong, 2003) the development of a teacher's PCK can be 'integrative' or 'transformative'. To aid clarification, Gess-Newsome (1999) and Jong (2003) explain that in the integrative model the types of knowledge become integrated as PCK while in the transformative model they are transformed into PCK. The two models are illustrated in Figure 2-1 (Gess-Newsome, 1999, p. 12).

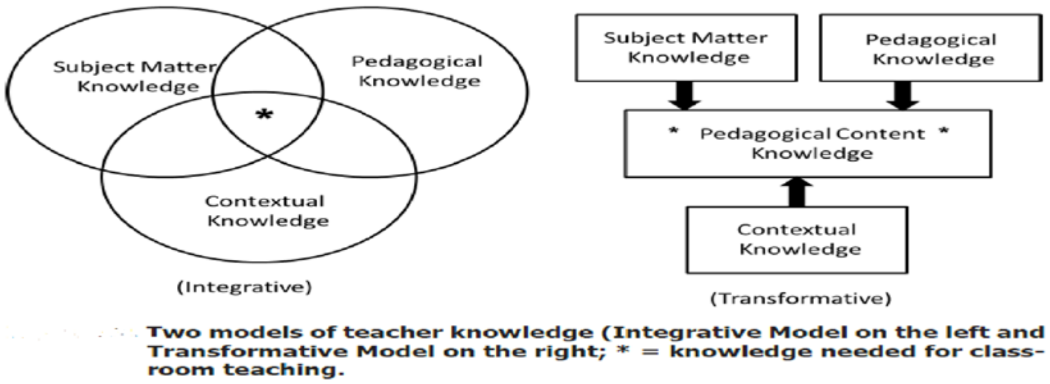

Figure 2-1: Integrative Model and Transformative Model (Gess-Newsome, 1999, p. 12)

Fernandez (2014, p.94) explains the differences between the two models as follows: "In the integrative model, PCK does not exist as a domain of knowledge, and knowledge of teachers would be explained by the intersection of three constructs – subject matter, pedagogy and context" while in the transformative model "PCK would be the synthesis of all the knowledge necessary for the teacher's education". As an illustrative analogy used by the author, the difference is likened to the formation of a mixture vs. a chemical transformation that results when two chemical substances react; in the former the substances remain chemically distinct while in the latter the initial substances cannot be separated, and the initial properties cease to exist. While these models are interesting, the *development* of PCK is not the focus of this thesis. Both models result in PCK and it is that result that is PCK that is the focus of the present study.

As Depaepe et al. (2013) confirm, “PCK was - and still is - very influential in research on teaching and teacher education” (p.12). Depaepe et al. conducted a systematic review of PCK, and how researchers in teaching mathematics conceptualised PCK. The findings of Depaepe et al. (2013) appear to correspond to the claims of Jong (2003) that PCK can be conceptualised in different ways. Depaepe et al. (2013) indicate that PCK is influenced by the methods used. Similarly, Jong et al. (2005) say that PCK is heavily influenced by the topic, context, content, and teachers’ feelings on a particular day, among other influences.

Other researchers (e.g. Bednarz & Proulx, 2009; Hodgen, 2011; Mason, 2008; Petrou & Goulding, 2011) consider PCK as ‘knowing-to-act’, a more dynamic view on PCK that is situated in the act of teaching within a particular context. Still others (e.g. Baumert et al., 2010; Bednarz & Proulx, 2009; Huillet, 2009; Marks, 1990) ponder the distinction between CK and PCK theoretically and empirically. They argue that multiple dimensions affect the act of teaching, for example pedagogical and mathematical dimensions. In their review, Depaepe et al. (2013) stated that many researchers (e.g. Ball et al., 2008; Cochran, DeRuiter, & King, 1993; Grossman, 1990; Hill et al., 2008; Hill et al., 2004; Marks, 1990) have adopted Shulman's model and developed it in several subjects, especially in mathematics and language, and expanded and refined PCK's definition. Grossman (1990), for instance, argued that there are four components central to teachers’ PCK: (1) knowledge of students’ understanding, (2) knowledge of curriculum, (3) knowledge of instructional strategies, and (4) knowledge of purposes for teaching, while Marks (1990) supported the following structure of PCK: (1) knowledge of students’ understanding, (2) knowledge of media for instruction, (3) knowledge of subject matter, and (4) knowledge of instructional processes (Depaepe et al., 2013, p.13).

To sum up, while the findings of the discussed empirical research on PCK can be used for exploring PCK, it can also provide a platform for future researchers to analyse and explore PCK and develop new PCK frameworks. The researcher's experience and knowledge of the situation in the KSA suggests that teachers in the KSA would lean towards the integrative end of the spectrum because of the lack of teacher education or training in any of the transformative practices. Despite this situation, there is a push for teachers to be more innovative in the classroom, so this situation may change in the near future (or may have already changed for some university teachers in the KSA), though more research on this topic is required.

2.5.1 PCK Within the Field of Mathematics

The four-point structure of PCK proposed by Marks (1990) (see above) was specifically about mathematics teachers’ PCK and this indicates that mathematics teachers’ PCK may be different to teachers of other disciplines. Nevertheless, Marks' model does not differ much from Shulman's original conception of PCK, despite the claim that Marks' model offers unique insights into mathematics teachers' PCK. Hill et al. (2008) suggest that in the field of mathematics, PCK comprises knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum.

What can be seen from the various models of PCK that have developed over time is that the field of mathematics may have aspects of instruction (teaching and learning) that do not necessarily coincide with other disciplines. In previous studies there is a lack of identified agreement among researchers (e.g. Khakbaz 2016; Krauss et al., 2008; Lesseig, 2016; Fan, 2014; Ijeh, 2012; Petrou & Goulding, 2011) regarding, for example, students' perceptions and students' learning outcomes. Much of the literature seems to focus on instructional models rather than also considering perceptions of students' difficulties, as previously outlined by Shulman (1987). This seems problematic, as mathematics is, by its very nature, a scaffolded learning experience where students must master more basic concepts before moving onto higher levels (Krauss et al., 2008). Saudi Arabia has a clear curriculum in its high school mathematics and foundation year programmes that attempt to ensure that students are prepared for university, should they choose to pursue it. There are a number of higher order mathematical functions that are essential, should students wish to begin a programme in calculus. By thinking about the context of the KSA, it is essential that conceptions of students are considered because such conceptions influence how teachers 'teach' in the classroom.

Depaepe et al. (2013) point out that some researchers (e.g. Ball et al., 2008; Hill et al., 2004; 2005; 2008) have worked to reconceptualise mathematics teachers' PCK and have used MKT or content knowledge for teaching mathematics (CKTM) as overall terms while still working with PCK and CK. The next section provides more detail about this area.

2.6 Mathematical Knowledge for Teaching (MKT)

"What do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill?" (Ball et al., 2005, p.17). This question started Ball et al.'s research about MKT in several studies (e.g. Ball et al., 2005; 2008). Their answer to the question was that MKT is knowledge of subject matter (or CK) *and* PCK, and that PCK has three main categories: knowledge of content and students, knowledge of content and teaching, and knowledge of content and the curriculum.

Ball and Sleep (2007) and Ball et al. (2005) suggest that, under their framework, the central tasks of teaching mathematics include the following:

- mathematical knowledge
- unpacking and decompressing mathematical ideas
- sequencing ideas
- choosing and using representations and examples
- explaining and guiding explanation
- using mathematical language and notation
- analysing errors
- interpreting and evaluating alternative solutions and thinking
- analysing mathematical treatments in textbooks
- making mathematical practices explicit
- attending to issues of equity (e.g. language, contexts, mathematical practices)

(Ball & Sleep, 2007; Ball et al., 2008; Hill et al.,2005)

These researchers (Ball & Bass, 2003; Hill et al., 2005; Ball & Sleep, 2007) also note that teachers may rely on their own past experiences when attempting to teach mathematics lessons in the classroom. This, they say, is typically referred to as 'practical knowledge', and according to Nisbett and Ross (1980) this typically refers to instances where a teacher's beliefs are derived from personal incidents related to their personal experiences (which are typically established early in life) and are resistant to change, even when contradictory evidence is supplied. Additionally, Borg (2003) notes that these personal beliefs and experiences are how teachers "learn a lot about teaching" (p.86).

One of the themes of PCK in the MKT literature is that knowledge of content and students often occurs as long as prior learning experiences are considered. For calculus, the teacher may expect students to employ their critical thinking skills in a step-by-step process and through re-evaluation of their prior knowledge of calculus. According to Bressoud et al. (2016), it is imperative for teachers to question their own assumptions about students' knowledge and to understand the gap between students' existing knowledge and the mathematical foundation of calculus concepts. If pedagogy is based on cognition and prior knowledge, mathematics teachers may not have the PCK they require to be effective in the classroom. This current study aims to identify the PCK of calculus teachers.

The MKT model is considered in more depth, and alongside other models of teacher knowledge, in Chapter 4.

2.7 Calculus at the University Level

This study focuses on calculus as "At university level, calculus is among the more challenging topics faced by new undergraduates" (Kidron, 2014; p 69). Calculus is a first-year compulsory course because of its importance as a platform for many other courses at university level (Artigue, 2001; Gueudet, 2008; Nardi, 2008; Maat et al., 2011; Petropoulou et al., 2016). University education involves the teaching of large numbers of students, with university lecturers usually assuming the role of teacher within lectures, which is the assumed teaching method (Petropoulou et al., 2016). Within a systematic literature review conducted by Speer et al. (2010), teaching was seen to be “an unexamined practice” (p.99), and they, and other recent researchers (e.g. Biza et al., 2016; Bressoud et al., 2016; Larsen et al., 2017; Petropoulou et al., 2016; Rasmussen et al., 2014), highlighted the need for future in-depth empirical research studies to evaluate teaching at the university level. Speer et al. (2010) noted that whilst most research had been undertaken in the compulsory sector of education, only some had examined mathematics teaching at the university level. A case in point is the work of Weber (2004) who analysed lecturing and explained how lecturers do their lectures. Weber’s (2004) analysis of lecturing and teaching highlighted tensions experienced by lecturers. These tensions generally included higher levels of work-related stress due to the wide variety of background mathematical knowledge in the classroom and the challenges associated with teaching/designing classes that needed to target a multitude of skill levels. Speer el al.'s (2010) review of research on teaching mathematics at university level used a frame to distinguish between teaching practice and instructional activities. Their findings identified "six different instructional activities used in the course—lecture, reflective teacher presentations, student presentations, small group work, whole-class discussions, and individual student work" (p 106). Speer el al. argue that empirical studies are one of the best research designs that can reveal the main differences between teachers’ thinking and their practice. Wagner et al. (2007) referred to the dearth of research into post-compulsory education and stated that “post-compulsory mathematics teachers’ ‘pedagogical content knowledge […] is closely tied to the nature of instructional experiences they have” (p. 251).

With particular reference to calculus teaching and learning at under-graduate level, Sofronas et al. (2011) investigated what a selection of eminent mathematicians, from a variety of backgrounds (including calculus textbook authors, calculus committee members, national teaching/scholarship award recipients [p. 133]), thought under-graduate students should know about calculus by way of defining a common understanding. The researchers reported that the fundamental theorem of calculus was cited by all of the participants in their research as a ‘unifying idea’ (p. 142). In addition, they all identified ‘the derivative and the integral as concepts foundational to the first-year calculus’ (p. 142).

In calculus courses, some researchers have attempted to assess new methods in teaching calculus in the classroom. For example, Kashefi et al. (2012) suggest an integrated environment of calculus learning based on both classroom face-to-face learning and e-learning concepts and added that assessments, IT and web-based assistance, and blended learning activities in mathematics would effectively facilitate students' learning. Kashefi et al.'s empirical study used multiple methods, including semi-structured interviews, in the data collection of students' written solutions for eight problems of difficult topics which students were faced with. Skjelstad (2009) used a quantitative study to investigate two correlations of students' perceptions in calculus about their teacher's teaching and strategies used, this was between teachers' 'immediacy behaviors' and teaching strategies and students' motivation, and another correlation was studied between 'immediacy behaviors' and teaching strategies with student effort attributions. Skjelstad found a significant positive correlation between calculus teaching strategies and student motivation and teacher behaviour. Furthermore, Barclay (2012), in a qualitative study, identified that in-depth understanding and analytical thinking are required during the application of problems, which call for an above-average level of mathematics skills. Therefore, investigation of more comprehensive abilities of mathematics in students is important, as the in-depth understanding of concepts is important to resolve the practical problems in the classroom (e.g. having classes where the students have a similar level of attainment) (Barclay, 2012).

The teaching of particular calculus topics, and the means of doing so, has become the focus of contemporary analyses (Rasmussen et al., 2014). One of the most frequent attributes of recent research is the focus on a particular aspect of calculus teaching and learning. For instance, Kabael (2010) investigated students' comprehension of the chain rule, while Sealey (2014) proposed a framework for characterising student's comprehension of 'definite integrals' and 'Riemann Sum'. As a means of perceiving the entire calculus educational context, such targeted research is useful, especially its significant contribution to students being able to comprehend aspects of a particular calculus topic. Yet Rasmussen et al. (2014, p.508) state that: "the studies leave the field with a hit-or-miss map of the terrain in calculus learning, teaching, and understanding".

This section has shed light on central aspects of calculus and also the relationship between background, research practice and knowledge; the importance of teachers' research practices in their teaching, and resources and preparation for teaching. Some wider aspects of calculus are considered in the next section.

2.7.1 Wider Aspect of Calculus

There exist issues with the effective teaching and learning of calculus, especially if the theoretical and pragmatic issues do not converge (Robert & Speer, 2001). Mathematics is usually regarded as a subject of great precision, of which calculus is often seen as a refined subdivision (Tall & Vinner, 1981). Yet, this refinement is countered by the fact that future teachers, engineers, doctors, economists, scientists, and mathematicians require calculus (Rasmussen, Marrongelle & Borba, 2014). Despite the fact that many major areas of study require calculus as a prerequisite, many students struggle and may ultimately fail at their calculus attempts.

There has been extensive research on calculus learning and teaching, much of which has followed a pattern of (1) identifying the difficulties of students, (2) investigating how students learn a particular concept, (3) evolving the classroom to address this concept, and (4) research on the teacher, instructor, teaching assistant, or graduate student (Rasmussen, Marrongelle & Borba, 2014). This is supported by Hitt and Gonzalez-Martin (2016) who suggest that "theoretical frameworks only really targeted advanced mathematical thinking through a cognitive approach" (p. 4). Yet they note that in the last decade, research into calculus has undergone significant development, where communication in the classroom has become a key component of much more interesting research outcomes. According to Boaler (2002), research into the wider aspects of calculus goes beyond the theoretical and that assessments can be made about the learning opportunities provided. Boaler (2002) supports the notion that knowledge development is a complex process where working through exercises and discussing mathematical ideas were only pieces in what was ultimately a much larger and immersive experience necessary for success. In determining what opportunities are available for students, Boaler (2002) highlights the benefits between the application of classroom knowledge and the real-world perspective. Khakbaz (2016, p.190) suggests "that application has more than one meaning (...) application in the real world, application in other disciplines and application in mathematics".

2.7.2 Influence of Teachers' Background, Research Practices on Their Teaching

In an attempt to reflect in terms of university mathematics teachers' background in pedagogy and their preparation for teaching, Petropoulou et al. (2011) demonstrated that lecturers' research practices influence their approaches to teaching. Petropoulou et al. examined two researchers of mathematics education and research of mathematicians. They chose a first-year calculus course as the context for the study, with the focus being the lecturer's teaching decisions, reflections and actions and looking for the link between the lecturer's teaching experiences and research, by using counterexamples for refuting invalid claims. Petropoulou et al. (2011) attempted to determine the resources that the lecturer used to make teaching decisions, they argued that the practice was influenced by experiences as a research mathematician and involvement in mathematics education research. It is noteworthy to mention this, because these experiences seemed to be blended and reflected in teaching decisions and actions. Similarly, Mali et al. (2014) conducted their study in small group tutorials for first year mathematics students. Mali et al.'s study was part of a PhD project and they chose one of three tutors, an experienced lecturer, who holds a doctorate in mathematics. Through the lecturer's teaching and using generic examples, they pointed out that their teaching practices were influenced by their background as a research mathematician, which revealed that "aspects of a mathematical concept and links with the tutor’s particular research practice, didactics and pedagogy emerge" (p.161).

2.7.3 Resources and Preparation for Teaching

Resources and preparations for teaching are considered one of several sources which can influence the teaching process in the classroom. Gueudet (2015) conducted a study, using interview and material resources, on six university teachers in France about their “resources and documentation work”. The author was looking for teachers' interactions with resources for preparing and delivering their teaching and suggested that using resources and preparations for teaching can be addressed through professional development or training. Biza et al., (2016) point out that "there is an increasing interest by tertiary teachers in non-lecture pedagogies" (p.5) and Hayward et al. (2015) argue that "relatively little is known about the impact of professional development on teaching practice in higher education" (p.59). Hayward et al. focussed on participant outcomes from a series of annual, week-long professional development workshops for college mathematics instructors about Inquiry-Based Learning (IBL) in undergraduate mathematics. Hayward et al. found that around 60 % of teachers used IBL strategies in the year following the workshop they had attended.

From the literature reviewed on the influence of teachers’ backgrounds, perspectives, and research practices on their teaching, it can be considered that the practices of teachers in the KSA’s universities can be influenced by a number of sources including background and research practice, and pedagogic training.

2.8 Teachers’ Practice

According to Nardi et al. (2014), the past three decades have seen significant improvements in calculus education across the entire academic curriculum (Nardi et al., 2014). For many universities, this now means there are multiple programmes within each topic area; therefore, it is appropriate for calculus teachers to have a good understanding of these programmes.

Weber (2004) argues that students may learn about calculus concepts in distinctly different ways. These include the "the natural learning approach, the formal learning approach and the rote/procedural learning approach" (pp. 129-130). Weber considers that students who learn best through natural learning are most likely to use their pre-existing intuitive understanding of the concept to derive meaning and thus understand the definition. Natural learners may benefit from teaching practices that offer a pseudo-structural approach where the meaning of the concept is discussed but formal definitions are not offered (Habre & Abboud, 2005). In this way, the teacher is encouraging students to use their intuition in an attempt to explain 'why'. Weber (2004) also suggested that students that employ the formal learning approach are unlike natural learners in that they may not have an intuitive understanding of the concept. Students who employ formal learning strategies use a logical or sequential approach to justify why their proofs are valid.

Contrastive to both the natural and formal approaches to learning, students that rely on rote/procedural approach take what their professors have taught them and then apply it to their own examples (Weber, 2004). Students who employ rote/procedural learning are generally unlikely to be able to link formal theory to the solution to their problem (Sofronas & DeFranco, 2010). According to Johnson et al. (2016), the lecture format is still the most predominant teaching strategy employed in calculus courses. According to these researchers, one of the reasons why teachers suggest that this type of teaching practice is so common relates to the sheer amount of material that needs to be covered in each lesson.

In terms of teachers' practices, a number of researchers (Hawkins et al., 2012; Jaworkski et al., 2017; Petropoulou et al., 2016) characterize mathematics teaching as consisting of three inter-related elements (1) management of learning (ML), (2) sensitivity to students (SS), and (3) mathematical challenge (MC). In this 'teacher triad', the construct of ML can include the planning of classroom tasks, use of textbooks, and setting of norms. The construct of SS describes the teachers' understanding of the students and attention to their cognitive, social, and affective needs. Finally, MC describes the challenges experienced by students that affect their mathematical thinking and activity, including the questions posed and the emphasis on metacognitive processing.

Knowledge of instructional strategies can be considered in terms of two categories: "knowledge of subject-specific strategies, and knowledge of topic-specific strategies" (Magnusson et al., 1999, p 109-110). The application of each is highly variable; for example, strategies that are subject-specific are likely to be much more widely applicable than those that are topic-specific. Topic-specific strategies can only be applied to the specific topic for which they have been developed; as a result, subject-specific strategies coincide with "orientations to teaching calculus" within PCK components

(Kashefi et al., 2012). They enhance the teaching of calculus in a manner that is consistent with their mandated goals.

There are many subject-specific strategies that can assist in the teaching of calculus topics. Many of these strategies include an instructional sequence, for example, lecture, problem solving, reflective teacher presentations, inquiry learning, student presentations, small group work, whole-class discussions, and individual student work (Doorman el at., 2008; Handelsman et al., 2004; Kashefi et al., 2012a; Lawson et al., 1989; Speer et al., 2010). When teaching a topic though discussion, or small group work, the learning is informative. Fraser's study (2016) identified that participants use strategies that are "often aligned with the text used in the unit (subject) and usual discipline approaches across the sector and were somewhat dependent upon the lecturers' expertise in the content'' (p. 152).

Other teaching strategies include "active exploration that uses critical, logical, and creative thinking skills to answer questions by teacher guidance" (Aulia et al., 2018, p.1), drawing out any pre-instructional conceptions students may have (Doorman el at., 2008), and inciting cognitive conflicts in students by highlighting anomalies (Bode et al., 2009). Still more strategies can help students identify patterns occurring in the world that they can then "discover" as well as consider when explanations must be devised (Kashefi et al., 2012). Many such teaching techniques are referred to as 'scaffolding', a set of strategies for helping students develop their own thoughts through exploring and challenging the validity of their own opinions (Doorman el at., 2008).

A study by Siyepu (2009) explored the effects of self-study activities on students' understanding of differential calculus, using mixed methods and multiple methods. The study found that "students improved their study skills; understanding; positive self-esteem; confidence and lack of insight in aspects of mathematics such as substitution, simplification, trigonometry identities, algebraic identities, conceptualisation, and derivatives of algebraic expressions and trigonometry functions" (Siyepu, 2009, p.136). The study motivated the students to learn challenging problems and to enjoy the learning of 'differential calculus'.

Another teacher practice is highlighting the difference between pivotal example and counter-example (Klymchuk, 2005, 2014). Klymchuk (2014) reported how that usage of counter-examples significantly improved students' performance. Klymchuk (2016) described his experiences in both teaching with, and research on, counterexamples in calculus as a pedagogical strategy and he discussed his findings of several experimental studies with students and lecturers of calculus. The study found that lecturers and students' attitudes were very positive about using counter-examples and this strategy is effective (a form of PCK). Klymchuk (2016) upheld the results of other studies that had examined counter-examples in calculus (Coupland et al., 2016; Klymchuk, 2005; 2014).

This suggests that using pivotal examples and counterexamples is important for calculus teachers to provide better opportunities for students to learn and deepen conceptual understanding and eliminate common misconceptions (Klymchuk, 2016).

To formulate a framework surrounding both the classification and order of the building blocks of mathematical theories, it is also important to clarify how these building blocks might be interpreted by teachers. As such, there are several terms used in mathematical theory. Hence, "the main components of a mathematical theory like analysis are axioms, definitions, theorems, and proofs" (Alcock, 2014, p. 8). In addition, there is the use of examples and diagrams to explain mathematical concepts.

In the majority of the mentioned research studies, including more recent studies (e.g. Biza et al., 2016; Bressoud et al., 2016; Petropoulou et al., 2016; Rasmussen et al., 2014), the aim of the analysis has been to analyse data from real-life educational practice. It appears from the teacher's understanding, that the knowledge that is needed to teach students is founded upon the mathematical requirements of educational lecturing or teaching itself, and this is different to the understanding a teacher acquires in their own education (Ball & Bass, 2003; Cooney & Wiegel, 2003). This aspect of knowledge also stands for the capacity of teachers in calculus teaching to derive connectivity in mathematical principles and link this to their application in practical fields (Biza et al., 2016; Bressoud et al., 2016). The extension of calculus concepts to other disciplines of engineering and physical sciences is also an attribute of teachers' knowledge (Maciejewski & Star, 2016); therefore, a holistic approach is required for teachers to develop an understanding of, and to employ, the application techniques of calculus in classes. Apart from subject knowledge, the understanding of teaching strategies to convey the material effectively is also an important consideration in the development of teachers' knowledge (Cooney & Wiegel, 2003). Although numerous studies have investigated the methods and forms of mathematics knowledge among teachers in elementary and secondary schools, there remains a scarcity of research on university calculus teacher knowledge (Biza et al., 2016; Bressoud et al., 2016; Maciejewski & Star 2016; Potari et al., 2007; Tall, 2010). Therefore, further research is required to better understand the discrepancy between acquiring specialised calculus knowledge and the traditional methods and implications this has for teachers' own formal education. The need for training and development is quite high for university teachers because of the versatile challenges of calculus courses and their application in other science disciplines.

The questions this research study asks sits within the aims for teaching calculus, from the practical to the theoretic as discussed by Martinez-Luaces and Noh (2015). In particular, they stress learning calculus for its historical-sociological significance, inasmuch as its study exposes the student to a major shift in mathematical thinking, which influences our scientific worldview today. Such an approach clearly influences, not only the content of calculus lessons – by focusing on scientific discoveries directly resulting from this shifting world view –, but a pedagogical approach that incorporates a sense of wonder and discovery. However, in the context of teaching calculus in the KSA, given the MOE's emphasis on practical application, the calculus teacher should focus more on the necessity of calculus in such fields as physics, and engineering (Mesa & Burn, 2015). As far as students' comprehension about the content of this topic, Nardi et al. (2014) discuss the idea that learning mathematics in general, and calculus in particular (as the first higher-level mathematics students' encounter in university), often entails substantial discursive shifts for learners. Weller et al. (2004) use the term "resolutions of cognitive issues" to describe the mental blocks and preconceptions that prevent students from grasping the difficult topic at hand. This was demonstrated in Ferrini-Mundy and Graham's (1991) study of first semester calculus students, who were unable to provide a general definition of function, despite their ability to write formulae (Buck, 1970; Nardi, 2008; Seldon et al., 1989; Viirman, 2014).

Knowledge of students' inherent preconceptions and conceptual blocks should also influence the sequence in which the subject is imparted – at least in its initial stages. Gyöngyösi et al. (2011), in Denmark, studied the transition from concrete to abstract perspectives, which mark the transition from the secondary school to the university study of calculus, with the former being focused on practical-theoretical blocks of concrete analysis, and the latter on more complex praxeologies (where praxeologies constitute the basic units into which one can analyse "human action at large", Chevallard, 2005, p. 23). In terms of teaching tactics when teaching content, Seldon et al. (1989) discussed the use of introducing calculus at the university level through practical problem solving. Such an approach would work even at the university level in the KSA, due to the Saudi government's focus, as mentioned above.

Teaching mathematics requires teachers to demonstrate learning outcomes through more than just words, formulae and equations. There are certain picture-related items and strategies that are employed in the mathematics classroom that may differ from other subjects (e.g. the humanities or social sciences). One of the ways this is highlighted is through knowledge of mathematics procedures. This is explained as a symbolic representation system (Star, 2002) and generally includes aspects such as algorithms or rules that are specifically employed to complete a mathematical task. In the calculus classroom this might include bar graphs, scatter diagrams, histograms, axiom, definition, example, theorem, proof, etc. where decisions need to be made on aspects such as selecting the scale, drawing the axes, plotting the points, and joining the line of best fit (Neill & Shuard, 1982; Leinhardt, 1990). The key difference of knowledge of mathematics procedures is that teachers may use it without providing an explanation of the procedures used to complete the task. Knowledge of mathematics procedures can be linked to aspects of misconception, as teachers who employ the knowledge of mathematics procedures approach may assume students are able to understand and complete tasks related to symbolic representation, though this may not necessarily be the case.

While knowledge of mathematics procedures is an important component in mathematics teaching, it is not solely the focus of this research. As it is an intrinsic component of many mathematics lessons, it may be considered in the observation component of this research and may be further considered briefly in the discussion chapter.

The focus of teaching calculus has four overarching principles. There must be a focus on the content, a pedagogy, students' understanding, and the purpose; which is the framework of PCK (Grossman, 1990; Khakbaz, 2016). Lachner and Nuckles (2015) investigated the instructional explanations given by university mathematicians, who they considered to have less PCK, with those of school mathematics teachers and found that deep content knowledge helped instructors generate explanations. Similarly, Hill et al. (2008) have pointed to strong relations between levels of teachers' PCK and the mathematical quality of their instruction in secondary school. All teachers who want to be good teachers need deep PCK in order to provide high-quality teaching.

To sum up, one of the main reasons to choose university calculus teaching as the topic of study is the identified the gap in the literature which clearly indicates there is little empirical research that focuses on mathematics teacher's practice. A number of researchers (e.g. Biza et al., 2016; Bergsten, 2012; Bressoud et al., 2016; Jaworski el at., 2016; Khakbaz, 2016; Speer el at., 2010; Petropoulou et al., 2016) suggest more research on teaching practice, teacher knowledge, and how teachers do and think in their teaching.

2.9 Purposes for Teaching Subject Matter (Calculus)

An important aspect of PCK is an awareness of goals and objectives. This requires a teacher to understand exactly what his or her students are required to understand about the given subject (Wilson et al., 1987). McCallum (2000) investigated the goals of a calculus curriculum, that "had a variety of goals, which may be grouped into three broad areas: conceptual understanding, realistic problems, and use of technology" (p 14). This knowledge is not time specific to the current academic year; a teacher must consider the wider curriculum, understanding that students may have already gained some knowledge of the topic in previous years and will continue to build on their knowledge base in the future (Grossman, 1990). A teacher can enhance their understanding of the goals and objectives outlined for their curriculum by reading documents at either national or state level that articulate frameworks for curricular decision-making. The idea that learning objectives may be shared with students in the lesson is an important one (Hannah et al., 2011) and has been shown to be beneficial in the mathematics classroom (Jaworski at el., 2009; Tall, 2004) and furthermore, obvious objectives make students actively want to participate to gain concepts (Hannah et al., 2011). Stating learning objectives also makes more sense for the teacher's actions and that students retain more knowledge (Morgan, 1998; Petropoulou et al., 2016). For a calculus curriculum, many universities will have their own documentation that outlines specific concepts for individual courses, and it is advantageous for calculus teachers to examine these and achieve the objectives. McCallum (2000) summarised the goals of first year calculus course in five points: "make calculations with agility, accuracy, intelligence and flexibility, explain the basic concepts of calculus clearly and reason mathematically with them, solve extended problems with good judgment, and make connection between different incarnations of the same idea, and use calculus to model realistic situations from engineering, physical, life, and social sciences" (p.17). In relation to this, Wagner et al. (2007), by focussing on PCK in the context of teaching undergraduate students, acknowledging the vital role of PCK in the process of achieving instructional goals.

The goals and purposes of teaching calculus include reference to calculus goals and general institution goals. The main aim is that students acquire cognitive skills through thinking and problem solving. In the KSA, the 'purpose' for the calculus course is outlined by the MOE in the syllabus, the:

- Student should mature in their understanding of calculus through the study of limits, derivatives, and integrals and their applications.
- Student acquires knowledge by learning derivatives and integrals of the logarithmic, exponential, inverse trigonometric, hyperbolic functions.
- Student studies the techniques of derivation, tangent line, rate of changes, fundamental theorem of calculus, integration, finding the area between two curves, volumes of revolution, and volumes of a solid with known cross sections and find the length of a curve.
- Student knows the limit of sequences, sum of infinite series and finding Maclaurin, Taylor expansion of functions in one variable.
- Student acquires cognitive skills through thinking and problem solving.
- Student acquires cognitive skills by building blocks of mathematical theories.
- Student becomes responsible for their own learning through solutions of assignments and time management (MOE, 2017).

When considering PCK it is advantageous to consider two distinct aspects. The first is goals and objectives that are specifically outlined by governing authorities, and, more often than not, are enshrined in subject specifications and assessment objectives, while the second is understanding the curricula and materials. Interestingly, some researchers have argued for a clear distinction between the knowledge required to be a teacher (pedagogical knowledge) and the knowledge base required for a particular curriculum (content knowledge) (Wilson et al., 1987).

2.10 Students' Difficulties with Calculus

Calculus is one of the most complex fields in mathematics for both teachers and students to understand, but the prequalification of this subject is necessary to develop the basis of engineering and physical sciences (Kashefi et al., 2012). As a result, the level of calculus taught at college and university levels differs and reflects variations in standards because not all universities require the same level of calculus in order to obtain an undergraduate degree. The understanding of this subject is considered a challenging task and is something not made simpler by students' varying conceptual mathematics understandings prior to and during the university experience (DeGeorge & Santoro, 2004). One key issue Sofronas et al. (2011) identified additionally, is the fragmented nature of instruction at college or university level which results in the students losing oversight and context, and subsequently, their grasp on calculus concepts.

While the learner is not the focus of this study, how the teacher interprets the learner is a key component in PCK. Conceptions, as defined for this study, include both preconceptions and misconceptions. To this end, a misconception within the field of calculus might include an idea or belief that is founded on incorrect of erroneous information about some aspect or detail relating to calculus theory (Olivier, 1989; Robert & Speer, 2001; Jones & Alcock, 2014). This notion of misconception often arises because pre-existing concepts must exist for students to function in first year calculus (i.e. students must have a certain level of understanding about mathematics in order to be successful in calculus). Challenges arise, however, when teachers' preconceptions about students' knowledge differ from the actual competencies. According to Jones and Alcock (2014), preconceptions are pivotal in the link between pre-calculus knowledge and new knowledge. This idea relates to teaching and to PCK because it is the teacher's responsibility to address and to resolve mathematical misconceptions through the development of a learning approach. The choice a teacher makes could potentially be wide ranging, as it is impossible to predict the level of knowledge each cohort of students, or even each student, brings to a first-year calculus class. Ultimately, what much of the literature does agree upon is that calculus teachers must address these misconceptions before moving on to higher levels of knowledge; failure to do this would demonstrate poor PCK (Penso, 2002; Cazorla, 2006). Cazorla (2006) indicates that a failure to address misconceptions along with the structure of mathematics-based coursework often leads to learning difficulties among students. Knowing that students' learning difficulties often arise from the way lessons are taught (Penso, 2002) allows researchers to delve deeper into the reasoning behind this. According to Penso (2002) learning difficulties are often the result of lesson preparation, the learning atmosphere, lesson content, and how the lesson is implemented in the classroom setting (e.g. the lecture approach). In addition to these, learning difficulties can also be classified as including misconceptions that learners have, leading to problems later in existing courses as well as affective characteristics of the learner (e.g. whether they like calculus, whether they are motivated to study, the goal-oriented nature of the student, etc). While learners' motivations are interesting, and certainly worthy of study, their own perceptions are beyond the scope of this study. What needs to be considered in this study is the teachers' interpretations of students' challenges and difficulties. First, what needs to be considered is how these are identified. This needs to be followed with a discussion on the strategies employed to appropriately deal with student misconceptions and with students' learning difficulties.

Bressoud et al. (2016) argue that the transition from secondary level mathematical education, and content requirements, to that at post-compulsory level is inhomogeneous. As a result, students lack foundational concepts and knowledge for the effective transition and thus experience difficulties in grasping calculus concepts. Gruenwald and Klymchuk (2003) reference the fact that "misconceptions or unsuitable preconceptions cause many difficulties" (p. 2). According to Sonnert et al. (2015), students who are taking calculus courses at college demonstrate a stymied motivation with regard to mathematical courses, and this is something that can have an impact on students' aims, goals and motivation to continue their mathematical learning. It is possible that this is due to higher levels of rigour when weighted against coursework completed below the college-level. The challenges faced by students, with regard to calculus, include manipulation of algebraic concepts and a poor understanding of such concepts, which are, according to Kashefi et al. (2012), two major barriers for student education. In addition, Rasmussen (2012) states that students face difficulties as a result of the issues in developing concepts. Additionally, with regard to the study conducted by Tall (2010), the author stressed the significance and necessity of educational stratagems and tools used for learning, in the context of a wider educational application and setting with regard to classroom learning and students' understanding of the subject.

Calculus is generally the initial occasion where a student is faced with a concept where calculations are unlikely to be solved using algebraic or arithmetical tasks or by infinite processes, which may be solved or tackled with the use of indirect argument (Tall, 1993). Indeed, teachers generally endeavour to overlook students lack of background knowledge or inability to apply certain basic mathematical concepts by applying an 'informal' method, which requires teachers to provide background information (unrelated to calculus) in order to overcome the fact that many of their students are unprepared to learn (Lachner & Nuckles, 2015). Nevertheless, despite the methods used, an overall unhappiness with the 'calculus course', as a subject in university, has been increasingly evident across a number of universities worldwide (Törner et al., 2014).

Tall (2010) and Thompson et al. (2010) conducted studies that concluded that learners are not succeeding in relating an understanding of the symbol and the terminology utilised within their calculus courses and within calculus itself. Thus, conceptualisation is not complete, and the significant ideas tend to be neglected, with incorrect meanings being attributed to each and every one of them. Consequently, conceptualisation is not realised without relating the correct symbol-word pairing, and the significance of this problem is generally under-stressed, according to Adams (2008). Thompson et al. (2010) consider that calculus teachers and lecturers need to increase the extent and level of their students' understanding, particularly with regard to the more challenging concepts within the classroom. Thus, the challenge lies in relating applications to procedures with regard to calculus concepts learned within the classroom (Hiebert & Grouws, 2007; Schoenfeld, 2006). Tall (2010) concludes that writing the ideas and application of calculus principles is more helpful than writing only the definitions and summaries of topics. Tall further discusses the significance of prior mathematics knowledge and experience among undergraduate university students.

According to Tall (1993), there seems to be a number of fundamental differences that universally pose challenges for students in the area of calculus. The first of these is language, especially as certain terms such as 'limit' have a different colloquial meaning and conceptual meaning in calculus. On top of this, particular challenges for students include:

- Language terminology – terms such as 'limit,' 'tends to,' and 'approach' have both implicit and explicit meanings and cause confusion.
- The notion that students experience difficulties determining whether the 'limit' can actually be reached.
- Confusion surrounding the progression from finite to infinite, in an attempt to understand what happens at infinity (Broussard et al., 2016; Tall, 1993).

Kymnchuk et al. (2010) further elaborate on this by suggesting that the process of translation is not only verbal but is also embedded in the difficulty between application of practical situations and mathematical notation form. They offer suggestions that translation of language includes symbols, because the mental processes that guide students in the construction of equations may utilise unfamiliar symbols; the processes used to create such symbols offers challenges in comprehension that are sometimes difficult to overcome.

Muzangwa and Chifamba (2012) consider that students' difficulties are largely associated with errors. They classify errors as structural, meaning that students have difficulty grasping a principle that is essential to the solution or that they cannot appreciate one of the relationships essential for a solution. This may be, according to Kymnchuk et al. (2010), that students have challenges with the mathematical modelling process, which is something that typically affects first year students more than it does for more advanced calculus students. This disconnect between the 'real world' and the mathematical world requires some navigation and offers a particularly unique challenge for calculus students that may not be found in other subjects.

In addition to the above challenges, Tarmizi (2010) suggests that students who experience greater difficulties with calculus tend to focus on different targets than students who are more successful. In the Tarmizi study, students who experienced difficulties tended to seek a step-by-step methodical method to reach a conclusion, whereas students with greater ability were much more flexible in their approach to problem solving. Tarmizi equates the ability to understand calculus as synonymous with the mathematical thinking process. Tarmizi suggests that the first component of this process is building upon real world actions and linking this to conceptual embodiment, whereas the second process is much more procedural, which he suggests is perceptual symbolism. Students who lack these processes are likely to experience more difficulties.

Research (such as Carlson, 1997; Clement, 2001; Sierpinska,1992; Thompson,1994) suggests that there are specific examples from calculus concepts and procedures that students typically find difficult. For example, functions are considered to be the major aspect of calculus formation, but students face some difficulty with this concept. The literature indicates that most students have a perception that function is a mathematical statement with an equal sign. In addition, some studies (e.g. Dubinsky & Wilson, 2013; Clement, 2001; Sierpinska, 1992) have argued that students think functions must be continuous, and they cannot imagine that a function can be defined over split domains or constant. Viirman (2014) reviewed many studies and found that students' conceptions of the function concept show "inconsistencies both within conceptions and between conceptions and definitions" (p.17). The definition of concept of function and its representation and determining its co-domain are considered to be difficulties faced by students.

Another specific example from calculus concepts and procedures that students typically find difficult is the idea of limit. Tall and Vinner (1981) provided evidence that the difficulty for students is to conceive the limit process as a number. Kidron (2014) emphasises that "Conceptual problems in learning calculus are also related to infinite processes. Research demonstrates that some of the cognitive difficulties that accompany the understanding of the concept of limit might be a consequence of the learners' intuition of infinity"(p.70). In addition, Todorov, (2001) found that students have difficulty when they apply the definition of limit. Williams (1991) conducted his study on 10 students with concept images of limit by using the formal definition. His data showed that "it is not surprising that the students in this study failed to adopt a more formal view of limit after only five sessions". (p.235). Therefore, it can be concluded that the limit concept and applying its definition are difficult and could affect students' understanding of other concepts based on them.

In addition, some literature shows that students can find it difficult to understand the derivative. Baker et al. (2000) studied student difficulty in incorporating knowledge about the second derivative into sketching the graph and reported how hard it is for many students to utilise all of the data that may be available. The authors argued that students have difficulty in understanding the derivative as a function. Also, an interesting study was conducted by Zandieh (2000), who provided a framework for understanding student difficulties with the concept of derivative and shows that multiple representations, or contexts, is important to help students to understand the derivative. Moreover, Zandieh (2000) also identified that students can use the derivative to find speed without understanding the limit process of this. However, the author argued that this lack of understanding can lead to misapplication of the derivative. Zandieh highlighted that another difficulty of derivative is moving from the notion of derivative at a point to derivative as a function, which presents as an example of understanding the relationship between a derivative as a function to the derivative at a point.

For this present study, it is recognised that students' conceptions exist. These include preconceptions and misconceptions that will differ among students. As a result of these conceptions, students may experience learning difficulties. This is taken as a fact, and it is necessary to approach it this way in order to interpret teacher's perspectives on these issues.

2.11 Calculus Reform

Research has highlighted ongoing 'calculus reform' which is considered to be crucial for improving calculus teaching (Hurley et al., 1999). Hurley et al. (1999) stated in their article that:

> *The instructional practices of calculus-reform programs differ markedly from those that had persisted for decades (some would say, centuries). It is only natural for faculty to question whether the new modes really improve the approach that in their own education worked successfully. Some observe little if any improvement in conceptual understanding among students from reform courses* (p 800).

Another of the most profound areas of calculus reform has been in the area of technology (Schoenfeld 1995). For learners of all disciplines, this has been a fairly substantial modification to the teaching of subject matter, but in the case of calculus the implications have affected both teaching and learning significantly. Technology in calculus was seen as a way that calculus could become more meaningful for students. It was noted that students' difficulties often stemmed from a lack of understanding about the mathematical processes, which were aspects of thought that accrued over time. When considering the way technology could be used to reform the way that calculus was taught, some saw it as a way to improve on the typical lecture style method that existed previously (Schoenfeld, 1995).

Technology offered a slow change in calculus reform; it began with the introduction of calculators that could offer much quicker and detailed responses than the paper-and-pencil option. While calculators were expensive at the time, they were seen as a fundamental advancement, because they were ‘computers’ that were both portable and accessible. Once the calculator became a staple in the calculus classroom, the expansion of technology to include computers was seen as particularly helpful in facilitating understanding (Vincent et al., 2015). From this viewpoint, visual representations of graphs were deemed to be helpful in assisting students to see the modelling of functions. The initial programs that were offered were broadly seen as offering some real-world insight into the theoretical models.

Yet as time has passed, technology continues to flourish and gives students the opportunity to develop their skills in calculus, both in and outside the classroom (Kumsa, Pettersson & Andrews, 2017). Students who are experiencing difficulties with a particular concept in calculus now have access to a wide range of videos and tutorials online, which can offer support and explanations in a way that differs from what they have experienced in lectures. This reform has allowed for underperforming students to gain extra help in areas of weakness, thus possibly contributing to a better overall classroom environment. Furthermore, teachers are also using technology more in the classroom, which facilitates opportunities for a more active and engaged lesson, thus moving away from the more traditional model of instruction. Schoenfeld (1995) identified this shift in technology "as one that is more engaging, and activity based, indicating this as an advantage to calculus reform, thus assisting students in making the mental constructions necessary for the understanding of calculus" (p. 3). Additionally, regardless of the assistance provided to researchers, with respect to calculus reform, the problem requires training in order to help the teachers and to facilitate their understanding and their use of educational methods (Thompson et al., 2010).

2.12 Calculus Curriculum

There has never been one universal calculus curriculum, but Ferrini-Mundy and Graham (1991) summarized the area of fundamental calculus concepts as falling into four areas: functions, limits and continuity, the derivative, and integral. However, with the advancement of technology, what can be accomplished within a single course has expanded (Schoenfeld 1995). Despite the diversity of the topics being selected by each institution, calculus is a subject that is taken by hundreds of thousands of students annually (Ferrini-Mundy & Graham, 1991). The outcome of asking this many students to think about higher level processes is that many students either fail their courses or struggle to achieve the grades necessary to demonstrate complete understanding (Ferrini-Mundy & Graham, 1991). The reasoning for why this occurs seems to relate to both the teaching styles in the classrooms (and the tutorials) and the students' learning processes.

In terms of teaching, Ferrini-Mundy and Graham (1991) suggested that curriculum development has focused on making the instruction more 'lean and lively' (p. 628) as opposed to the operational plan of content that is often distributed in a more traditional curriculum. Curriculum development in this area may benefit from a number of considerations (1) decisions about content, (2) information about student's previous knowledge, (3) teachers' perspectives on the process of learning, and (4) teachers' practical knowledge (Ferrini-Mundy & Graham, 1991). These considerations may also be considered in conjunction with reflection, evaluation, and redevelopment, to ensure that students are gaining the most benefit from the revised curriculum. In addition, Ferrini-Mundy and Graham (1991) indicated that viewing the calculus curriculum as one that is a constructive process can be particularly valuable for teachers. This is because by reconstructing the classroom so that the learner can attempt to make sense of the information through evaluation, connection, and organisation they are more likely to be able to work with the problems, in an attempt to solve them. This is contrastive to the more traditional approach where students would be passive recipients of knowledge and where an explanation of a concept would be repressed by a perfect explanation.

2.13 Chapter Summary

This chapter has provided an overall view of teachers' knowledge. It has also given an overview of types of teacher knowledge, and then discussed faculty and learning and teaching practices of calculus and mentioned knowledge of MCK. In addition, the researcher has reviewed the nature of calculus teachers' knowledge and PCK. Finally, the researcher has been motivated to look for what influences calculus teachers in implementing their PCK in the classroom setting.

Chapter 3 details the background to university level education in the KSA.

Chapter 3 Background to University Level Education in Saudi Arabia

3.1 Introduction

The MOE in the KSA is the primary body responsible for making decisions at the university level; this responsibility is for planning, coordinating and supervising the universities. It was initially established in 1975, though with the recent push by the government on aspects of education, several academically focused centres have been established to support the overarching ministry (Rugh, 2002). Most notably, the National Centre for Assessment in Higher Education oversees the entry tests for students wishing to pursue higher education and the National Commission for Academic Accreditation and Assessment, which is responsible for achievement of quality standards among Saudi universities (Alamri, 2011). These entities, among others, are particularly important for my study because they are responsible for changes made to the universities themselves (i.e. through the implementation of a foundation year which includes compulsory calculus programmes), that place the universities in direct competition with those from other nations.

In January 2015, the Saudi government approved expenditure of approximately KSA Riyal 217 billion to be allocated to higher education (MOE, 2017). This push, to be more focused on education, came to the forefront in the Reform Agenda. Over a five-year period, the funds would be used to support various aspects of teaching and learning, and include teacher training and professional development, curriculum and textbook reviews and the adaptation of certain programmes to include electronic components (e.g. e-learning, online courses, etc.) together with the creation of programmes that incorporate innovative practice (e.g. active learning). The Reform Agenda followed the previous project, the King Abdullah Project, and has been nicknamed the Horizon Project because of its mission to span all universities across the Kingdom. This project is essential in understanding the nature of the Saudi university context (MOE, 2017). The project typically makes the assumption that all universities can operate under the same set of strategies, and that consistency and equality among schools is achievable. This set of underlying assumptions is based upon the view of cultural association with its sense of compliance and central control (Abdullah, 2006; Al-Aqeel, 2016). For institutions, this becomes challenging as there seems to be a desire to implement innovative pedagogical strategies, but the lack of experience of institutional autonomy has hindered these attempts. Educational leadership is evolving in the KSA, but it is a slow process, perhaps a slower process than either the government or the universities have anticipated.

In its attempt to compete on the world stage, the KSA requires a focused, clear and detailed plan that moves beyond the Horizon Project. As a result, two additional strategies were suggested and included:

1. A collaborative higher education system to include all major stakeholders including the government, individual universities, industry, and community representatives. This system requires:
    - A clear and widely communicated vision requiring 10 and 20 years forward thinking initiatives that need flexibility, especially at critical times.
    - Well defined objectives that outline what needs to occur to achieve the future components of the vision.
    - Processes that define how each objective will be met.
    - A plan detailed enough to ensure the appropriate allocation of resources. In this instance, resources are defined as more than just financial but also include equipment and infrastructure.
    - A feedback process that is both rigorous and constructive.
2. A comprehensive and compatible system that allows (and continues to allow) for the collection, analysis, and reporting of progress and performance. This being required at both the system and at the institutional levels (Abdullah et al., 2006; Smith & Abouammoh, 2013).

These strategies are still underway, as the KSA attempts to abide by its strategic Vision 2030, and it can be considered that, through these strategies, the KSA will be able to compete in the field of higher education. These strategies have also paved the way for mathematics education in specific universities within the Saudi context.

3.2 Current Teaching Practices at University Level in Saudi Arabia

One of the main arguments surrounding the need for research into improving the education system in the KSA is that the current system is broken, or at the very least inefficient (Al-Husain & Hammo, 2015; Asiri, 2012). This is not a concept that is unique to the KSA. Many countries have experienced issues in demonstrating competencies in mathematics and in literacy (De Lange, 2003; Cai et al., 2016), but not all countries experience the exact same issues. In the case of the KSA, the education system has been repeatedly examined over the last decade. Globally, the country has been significantly influenced by several major historical events, the most important being the end of the 'oil boom years', which in turn have influenced the education system. During this volatile period criticism arose with respect to the education system, however instead of listening to the criticism (which came from both inside and outside the county's borders), the government took a firm stance. While this stance first suggested that imposing changes on the educational curricula was not appropriate (Prokop, 2003), teaching practices at the university level were later identified as a point where the KSA could improve (Alshahrani & Ally, 2016). The strong stance against change was a significant eye-opener to the people of the KSA, and to university teachers working in the field.

In the last decades, rote learning has played a role in nearly every part of the educational system within the KSA, as well as the wider Gulf Region (Alshahrani & Ally, 2016). Prokop (2003), points out the typical teaching method in the Saudi school system includes very repetitive activities and rote learning. In addition, "Saudi Arabia has received sustained international criticism over many years about the quality of its education system" (Smith & Abouammoh, 2013, p. 6). Furthermore, the Saudi population is growing, and with the 'oil boom years' receding into the past, there is now fierce competition for jobs (Prokop, 2003). Yet with this competition "there is a clear divide between the output of the education system and the requirements of the domestic labour market" (p.87).

Mathematics, in leading up to university, in most courses can be overly complex in relation to the level of the students. While this may be problematic in the sciences or some of the arts subjects, it is a particularly problematic issue for university level teaching, especially in mathematics (Abu Asaad, 2010). This is because most university tasks cannot be effectively taught through rote learning and simple repetition. Such strategies can be considered to be archaic in teaching (though they are still widely used) and reflect back to the question about why the higher education context in the KSA works in the way that it does (Alamri, 2011; Prokop, 2003). There is concern that where students are expected to have 'unquestioning' attitudes, these do not provide opportunities for students to ask questions or to truly engage in the material. While this type of approach may work well where memorisation is the key testing mechanism, for interactive material, such as the application of theory to context (i.e. mathematics), there is little benefit from such an approach (Al-Khateeb, 2011). This brings the thought process back to the questions surrounding the delivery of education, specifically mathematics education in schools.

Considerable research has been conducted on some strategies and delivery of material in the mathematics classroom (e.g. Finelli et al., 2001). Finelli et al.'s study was inspired by one of the technical sessions of the 29th Annual IEEE/ASEE Frontiers in Education Conference and describes some strategies which help teachers to improve their teaching and facilitate students' learning. This conference gave researchers the opportunity to discuss the strategies to improve learning style and teaching. They summarised their ideas into: Planning the Course, Conducting the Course, Active Learning, Learning Styles and Class Participation, Face-to Face Interaction, Individual Accountability, Interpersonal and Small Group Skills, and Group Processing. The Saudi education system has chosen to focus on thought process development, underpinned by suitable teaching. The dominant current pedagogy is based on lectures and there are no other strategies and delivery materials used in classrooms.

There is also the question about the influence of non-Saudis on teaching mathematics in the KSA and how they influence the quality of education. The need to enhance the quality of teachers at the university level has left the MOE with no choice but to recruit teachers from other countries to meet the increasing demand for teachers (Borg & Alshumaimeri, 2012; Smith & Abouammoh, 2013). These mathematics teachers, currently working in the public system in the KSA, have good subject knowledge, mathematics proficiency and competence in mathematics teaching methodology (Borg & Alshumaimeri, 2012; Sabah et al., 2014). The choice of teachers is based on their high qualifications and considers the strength of their influence on those around them because one of the areas of most concern is the pre-service teacher programmes currently available in the KSA (Borg & Alshumaimeri, 2012; Smith & Abouammoh, 2013).

3.3 Statements on University Mathematics by the Saudi Government

The Saudi MOHE has provided universities with information that is discipline specific as a way to maintain consistency, despite offering universities a certain level of independence. These discipline specific statements outline both discipline and societal expectations of mathematics. Among these statements, the MOE contends that mathematics students must be made aware of problems in the physical and social world (within the context of mathematics) and that this should be done though aspects of creative and logical reasoning (MOE, 2017). Mathematics is thus regarded as a field of study whose primary focus is on problem solving through logical thinking. Through various types of exercises, societal patterns can be discerned, providing students with an opportunity to not only understand the world, but to use their understanding to improve the world.

In statements, the MOH has acknowledged the role that mathematics plays in various programme streams, in addition to the more quotidian contexts of daily life. Because of mathematics' inherent ability to wed theory to reality, students must also be able to think in both the abstract and the practical. This can be achieved by discovering relationships and patterns through descriptive, numerical and systematic ways of thinking. Learners engage in problem solving exercises, collect, organise, interpret and analyse data, and establish abstract models based upon current mathematical theories. In a statement issued by the MOE, mathematics generally:

1. Allows students to analyse situations [in reality or in the abstract] and to justify the decisions they have made. Students should be encouraged to seek empowerment to work towards a critical decision-making process.

2. Provides for equal opportunities and a variety of choices towards many different aspects of society.
3. Contributes to developing the Saudi culture.
4. Encourages the pursuit of rigorous and elegant patterns and relationships, from which pleasure and satisfaction can be attained.
5. Engages with other political, socio-economic, and organizational bodies to foster critical reasoning within a broad range of disciplines (MOE, 2017).

The overarching framework is provided to all teachers on an annual basis. As the MOE focuses on broader overarching concepts rather than individual programmes, it is somewhat difficult to see how calculus fits within this framework. Calculus is very much an abstract component of mathematics that does not necessarily fit within some of the points outlined in the above statement. As such, it is possible that calculus teachers, within the Saudi context, may have difficulty in applying the above statements to their teaching in the classroom. Therefore, it is important to note that while calculus is part of the mathematics department, it has some unique challenges that may not align with the overarching philosophy. Hence, these challenges lead to the question as to why calculus is important.

3.4 Mathematics and Calculus in Saudi Universities

Students who are accepted into university in the KSA, especially those who wish to pursue mathematics, computer science, biology, physics, business, engineering and chemistry are required to take some form of calculus as a pre-requisite to their future course work. While the calculus courses are offered as half year or full year courses (depending on the university, the programme, and the nature of the pre-requisite), students generally opt to take calculus in their first year at university (Yushau, 2006). The demographic profile of Saudi university students, (i.e. the majority of students are Saudi nationals who have completed high school within the Saudi school system), shows that they all come from similar mathematical backgrounds (Al-Aqeel, 2016; Sabah et al., 2014). In high schools, students are required to take a certain level of mathematics in order to qualify for entry into university. While some students may opt to take higher-level mathematics classes (e.g. International Baccalaureate (IB) mathematics), the uptake on these courses is low. Students are also required to sit standardised final exams in mathematics, contributing to the idea that many students come with approximately the same level of background knowledge in mathematics. As such, the teachers who work with first year calculus students (should) have a fairly strong understanding of what the students were taught in high school mathematics, as the curriculum is standardised across the country (Al-Aqeel, 2016). While teachers may have knowledge on what the students have learned, this may lead to underlying assumptions as to what the students actually know. This may be problematic, as students' misconceptions can lead to teachers' preconceptions about their level of knowledge, further complicating the issue (Bressoud et al., 2016; Nardi, 2011).

Compounding the above issue is the actual curriculum taught in high school classrooms. Students who complete the Saudi Grade 12 high school mathematics programme generally are taught a simple form of calculus, though it is acknowledged that they are taught many of the theories and principles on which calculus is built (Alamri, 2011). Therefore, first year university is the first time that students are exposed to the full range of knowledge about calculus and teachers need to make decisions about how to proceed through the material. A balance is required because, as a pre-requisite, a certain level of understanding is required in order to complete upper year courses, but these outcomes must be achievable, or students will become de-motivated and burn out before completing the course. These are essential aspects that teachers must consider in the first-year model (Alamri, 2011).

In addition to considering past experiences, there is a need to consider the present, and how calculus is taught in the university classroom. While other countries (France, Germany, USA, United Kingdom, Hong Kong, Singapore, and South Korea) have in some instances adopted active learning strategies to encourage problem solving within the classroom context (Bressoud et al., 2016), the KSA still focuses on a lecture style format. A typical first year calculus lecture would comprise approximately 24 hours of taught lecture time fronted by a professor, with certain additional time scheduled in tutorials (Al-Aqeel, 2016). Students sit, facing the front of the classroom, where the professor typically uses a board and marker (i.e. a whiteboard or chalkboard) to demonstrate mathematical equations and formulae related to the course. Multimedia slides may also be employed (e.g. PowerPoint slides) and other more basic educational tools often appear in classrooms (e.g. overhead projectors and/or cameras). There has been very little push to incorporate other forms of technology into the first-year calculus programmes, though the textbooks may offer an online lab component where students can complete problem sets.

This type of teaching style appears in contrast to the Horizon Project vision and the subsequent steps that have been employed to encourage innovation (Hamdan, 2005). While it is not for certain, one of the reasons for the lack of innovation in teaching these courses could relate to its lack of influence on university rankings. If the KSA wants to compete at a world-class level, it must focus on aspects that affect the rankings (e.g. research, infrastructure, programming, etc.). Teaching is difficult to measure and so is frequently absent from university ranking. Measuring teaching is difficult because it is both difficult to quantify, and because it is impossible to compare academic teaching across countries (Barnes, 2007). In more recent years, some rankings have attempted to target teaching (e.g. The Times Higher Education [THE] rankings), though these are largely based on teacher-student ratios, number of PhDs awarded per faculty member, and other aspects that generally are unrelated to first year calculus teaching (Barnes, 2007).

Because rankings tend to focus on research-oriented processes, much of the ‘innovation’ that the KSA tries to achieve is centred around research projects (Al-Aqeel, 2016). Expenditure by the Saudi government on supporting university research is increasing, especially in the areas of science (MOE, 2017). In addition, postgraduate research also has acquired multiple sources of funding; this is because the demographics of the university (being primarily Saudi students) means that these postgraduate students may go on to work at the university level as faculty, and therefore funding their research has more potential for benefit in the future (MOE, 2017). However, teachers must take steps to improve their own professional development, this can be challenging because of the workload requirements placed on these teachers. It is typically, though not always the case, that first year courses are taught by more novice faculty, as experienced faculty tend to gravitate to smaller upper year courses that focus more specifically on their area of study, allowing more time for research (Borg & Alshumaimeri, 2012; Sabah et al., 2014).

In summary, many students enrolled in undergraduate programmes must take first year calculus as a requirement for their programme of study, even if this programme of study is not solely mathematics focused. First year classes are typically large and taught in a lecture style format, which in most instances dictates that students are passive learners during these lectures. Teachers are often novice faculty members, as first year classes require a significant time commitment. Teachers are provided with standard materials in the classroom, including chalkboard, OHP, overhead camera, and projection screen. The MOE is focused on improving the rankings of Saudi universities through considerable investment, though first year classes are largely overlooked because they have very little impact on the rankings.

3.5 University Faculty Development in Saudi Arabia

As previously identified, as the KSA has only recently taken steps to be competitive on the world stage, there are questions surrounding the quality of the current faculty population. Students' experiences in the classroom have the potential to impact on how a university is perceived, and while teaching may not be measured in the rankings of universities, student performance and perceptions are. As such, it is necessary to continue to improve skills development among faculty in order to improve the quality of their teaching.

Currently, in the KSA the faculty development process consists of faculty evaluation, which is generally conducted within the institution. It is touted as a way for faculty members to develop their teaching, improve the quality of their instruction, and ensure that they are meeting their responsibilities to the institution (Alebaikan & Troudi, 2010). The central premise, in the KSA, of this evaluation process is to provide faculty members with some indication of their performance.

It is argued in the literature (Borg & Alshumaimeri, 2012; Smith & Abouammoh, 2013) that in the Saudi context faculty evaluation is essential in the improvement of the institution in addition to faculty development. It offers the opportunity to raise academic standards and is considered an essential factor in the overall effectiveness of an institution (Sabah et al., 2014). While the literature provided in the KSA on faculty evaluations generally suggests that a faculty should use these evaluations at a formative level, in reality administrators in the KSA may use these evaluations for quality assurance and to inform decisions related to renewal of contracts and promotions (Al-Aqeel, 2016; Borg & Alshumaimeri, 2012). This researcher considers that the strengths of faculty members should be shared, while the weaknesses should be addressed.

Literature (e.g. Darandari et al., 2009; Sabah et al., 2014; Smith & Abouammoh, 2013) has identified four factors relating to the concerns of higher education, these include:

- the establishment of the National Commission for Academic Accreditation and Assessment (NCAAA);
- the open acknowledgement that the pedagogical techniques employed by teachers are inefficient;
- the rapid increase in the actual number of universities;
- the competition for students to enrol in particular universities (Darandari et al., 2009).

The NCAAA is responsible for overseeing the improvement of programmes and institutions. Faculty evaluations often are used as a means to show that the university is meeting the NCAAA quality standards in terms of performance. The issue with this connection is that it is in the best interest of the university to provide favourable faculty reviews in order to satisfy this requirement. Satisfactory reviews may lead universities to obtain additional government funding and an increase in student enrolment (Al-Dakhil., 2011). In addition, because faculty evaluations are generally university crafted instruments that are given to all departments to complete, the questions and reflections that faculty members are asked to generate may not truly represent the role of the faculty member or what they have accomplished (Borg & Alshumaimeri, 2012; Smith & Abouammoh, 2013).

As such, in the KSA, many faculty members are resistant to these faculty evaluations. This is the case because the faculty member feels that they are not being assessed on their own merits but for the purpose of either institutional requirements or to satisfy the NCAAA standards. Within this framework, faculty that fall outside the more typical teaching strategies may be in jeopardy of receiving negative performance responses. The process of evaluation is conducted by other teachers or administrators that typically expect to see certain outcomes in the classroom. In a study by Campos and Pinto (2016), in the Brazilian context, considerable challenges were associated with the shift between expectations and reality, as internal tensions were highlighted. This is particularly influential to the current research, as it highlights the difficulties that teachers face when attempting to utilise teaching methods that fall outside the typical classroom expectations.

The second factor that commonly arises as an issue in the KSA is teaching standards. Saudi mathematics teachers generally begin their teaching careers without receiving any formal training or pedagogical preparation and, thus not surprisingly, they often lack the effective teaching skills they require to teach in a lecture style format (or any format for that matter) (Goldhaber, 2002). The literature suggests that while teachers may have good levels of CK in mathematics and may be well suited and equipped to conduct research in their discipline, they may not necessarily be able to communicate their knowledge effectively to students (Handal, 2003). There is often a desire among these teachers (and with other teachers across the world) to teach in the way that they have been taught. As such, traditional lecture approaches, which are familiar to the teachers, are typically employed in mathematics teaching. In the same way, assessment strategies (i.e. testing in a similar way to the way these teachers were tested) also follow a traditional format (Handal, 2003).

While some universities in the KSA have developed programmes to enhance teaching quality, this has not been a governmental initiative and so programmes differ in scope, delivery and content. There has also been a push at some universities to encourage professional development for teaching assistants (TA). At the current time, TAs in the KSA are typically postgraduate students (either Master’s or PhD Candidates). Their selection for a TA position largely relates to their scores at the undergraduate level, assuming that CK is the most important point for them to display. This is not unlike other universities across various different countries; TAs are in many instances responsible for overseeing the tutorials. These tutorials are in addition to the lectures given by the professor and the purpose is to provide students with opportunities where they can get additional support (Al-Dossary, 2008). It would seem from this description of the tutorial process that teaching pedagogy should be very important in such settings, yet training in pedagogy is not a pre-requisite, although some institutions are working on a model to correct this discrepancy.

In the current system, there are opportunities for faculty development but many of the reviews that are conducted in the university system relate to aspects of promotion and are not particularly useful in assessing teaching improvement. Teachers are hired because they have completed PhDs and demonstrate the ability to conduct research in their field. Many teachers have not been exposed to teaching pedagogy, and because much of the promotion process is tied to research, many lack the desire to achieve a better understanding of teaching pedagogy. This can be frustrating for students, because not only are the professors unenthusiastic about teaching pedagogy, the teaching assistants who guide students through tutorials are also typically operating with more content knowledge than pedagogical knowledge. For more details see Appendix A.

3.6 Chapter Summary

Conclusions can be drawn that indicate that the university education system in the KSA is moving from a centralised system of control to one where institutions are obtaining more power to run programmes and to make decisions that best fit their student populations. This is a shift from the way education has been run previously and offers a good context for research, as these changes could mean differences to multiple aspects within higher education. In order to work as a faculty member in mathematics, teachers require a PhD in mathematics, though they are not required to have any formal pedagogical training (i.e. they do not need a teaching degree).

Teaching mathematics, in general, is regarded highly in the KSA and the MOE has gradually introduced higher order thinking into the schooling system in earlier years of schooling. Based on this, there is backing to suggest that mathematics (and the teaching of calculus) is a useful and beneficial skill for university students to learn. Therefore, there are still questions surrounding the reasoning why research suggests there is little knowledge about mathematics university teachers and their teaching knowledge (Khakbaz, 2016). With many jobs requiring mathematical ability, the deciding factor (or one of the requirements) is proficiency in calculus because it is "the combination of several strands in mathematics that include numerical calculation, graphical representation, and symbolic manipulation which contribute to the development in technology" (Maat et al., 2011, p.26). This researcher considers that this situation is directly relevant to the mathematics programmes at the university level in the KSA, and this current study seeks to analyse the PCK of teachers who teach calculus at the university level.

Chapter 4 The Theoretical Framework

4.1 Introduction

Trochim (2006) suggests that there are two domains in research—theory and observation. A suitable theory can guide every aspect of the empirical component of a study from developing the research questions and problem statement, analysing the data through to discussing the findings and finally drawing and writing conclusions (Trochim, 2006).

A theory, according to Kerlinger (1986), is "a set of interrelated constructs, definitions, and propositions that present a systematic view of phenomena by specifying relations among variables with the purpose of explaining and predicting phenomena" (p. 9). The phenomena, in this present study, is the PCK of university calculus teachers. A framework is *"a set of ideas that you use when you are forming your decisions and judgements"* (MacMillan English dictionary, 2007, p.561), providing the structure within which the relationships between variables of a phenomenon function. Kerlinger (1986), also considers that "a theory can be used to successfully make predictions and this predictive power of the theory can help guide researchers to ask appropriate research questions" (p.9). In other words, to base research on a theory allows the researcher to design their study and ask appropriate questions.

When 'a theory' and 'a framework' are considered together to create a theoretical framework, this can provide well-supported justification to conduct a study and can help the reader understand the researcher's perspective. In addition, a suitable theoretical framework indicates that the investigation proposed by the researcher is not based entirely as a result of their own instincts or guesses, but rather is informed by theoretical and empirical facts obtained from trustworthy and verifiable research studies. Since Shulman (1987) introduced the PCK notion several decades ago, PCK has been seen as a suitable framework through which to research teachers' knowledge and many studies have been conducted using PCK as a theoretical framework (e.g. Khakbaz, 2016; Lesseig, 2016; Rollnick, 2016; Aydin el at., 2015; Fan, 2014; Nordin et al., 2013; Ijeh, 2012; Petrou & Goulding, 2011; Krauss et al., 2008; Baker & Chick, 2006; Miller, 2006; Duling, 1992; Grossman, 1990; Tamir, 1988).

This current research makes the assumption that teachers require certain knowledge in the classroom teaching situation. This knowledge is multidimensional and generally consists of three overlapping components - content knowledge (CK), pedagogical content knowledge (PCK) and general pedagogical knowledge (GPK). This research focuses on the concept of PCK. Therefore understanding, contextualising and investigating the competencies within these three components is important to define the threshold encompassed by PCK.

Shulman's (1987) ideas have become a part of educational research tradition and as such many scholars have offered elaborations on Shulman's ideas, or perhaps different conceptualisations of PCK (see Barnett & Hodson, 2001; Cochran et al., 1993; Grossman, 1990; Halim & Meerah, 2002; Jong, 2003; Marks, 1990; Van Driel et al., 1998). The use of PCK, as a theoretical framework, has presented researchers with a new perspective for collecting and analysing data about teachers' knowledge and cognition (Jong, 2003; Rollnick et al., 2008; Toerien, 2011). In order to understand how calculus teachers are informed by their pedagogic understanding and skills and their subject content knowledge in their teaching, this chapter describes a conceptual framework for PCK, based on research carried out in practical teacher development contexts (Lesseig, 2016; Khakbaz, 2016; Baumert & Kunter, 2008; Senk et al., 2008).

4.2 Approaches to Conceptualising Teachers' PCK

The concept of PCK, as developed by Shulman (1986), has undergone significant re-interpretations and redefinitions over the years. Scholars have used Shulman's original framework (1987) and adapted it in the light of further research and developments, but this does not mean the discrediting of one version in favour of another. Rather, it shows how a theory, rooted in the practice of teaching, needs to evolve to respond to priorities being revised and being subject to different weightings. Hu (2014) concludes that whilst PCK is considered as a holistic conceptual framework, its components are nonetheless re-examined over time, not least as it is of "practical significance to clarify its components" (p. 411). Referring to Shulman's original framework (1987) and subsequent iterations by other researchers, Hu (2014) refers to clarifications of the components when analysing similarities and differences between them. One of the key distinguishing factors is whether the components can be described as generic or specific, i.e. applicable only within a (specific) certain subject or more widely (generic).

A great deal of research has been conducted in an attempt to identify and characterise PCK during classroom practice, but research communities continue to call for studies to devise methods of conceptualising PCK (Miller, 2006). PCK is founded on the interpretivist process and is constantly in a state of exploration. This state of exploration can be applied at several levels (i.e. teacher, context, institution, etc.) and this requires insight when attempting to measure it. However, knowledge construction is collaborative and based upon social negotiation (Vygotsky, 1978) and requires thoughtful reflection on experiences and this reflection then needs to be integrated into a larger knowledge community. It is assumed that knowledge construction, under PCK, is based on personal experiences and of the continuous testing of hypotheses. Yet PCK also falls within social constructivism, a branch of constructivist theory that considers that culture plays a significant role in pedagogy, as does the social context (Miller, 2006).

For the purpose of this research, this study focuses on components that have been identified from existing literature in the field. The resulting summary tables show two different approaches: Table 4.1 based on Van Driel et al. (1998), Park and Oliver (2008) and Ball et al. (2008) and Table 4-2 based on Park and Oliver (2008) and Ball et al. (2008). In Table 4-1, looking at columns 1,2 and 4 suggests that there is major support for these components, also early focus appeared to be on ‘purpose’, ‘student understanding’ and ‘instructional strategies’ for teaching. It can be seen that ‘student understanding’ and ‘instructional strategies’ are in every conceptualisation of PCK. The purpose of Table 4-1 is to demonstrate that PCK has existed in the literature for several decades and has consistently been mentioned, researched, and analysed in a wide range of disciplines. This contributes to the overall justification for PCK research to continue in different subject areas, such as in mathematics. For example, Table 4-1 indicates that 75% of the identified authors consider knowledge of the purpose of teaching a subject as one of component of PCK.

While the summary in Table 4-1 illustrates the components agreed upon by most authors, the summary uses general terms and may lead to misunderstandings of PCK components. The summary also has the disadvantage that it does not include sub-components of each knowledge component, so the summary may not be sufficiently specific. Therefore, re-summarizing the components, in some commonly-referred-to conceptualization of PCK, is shown in Table 4-2. In Table 4-2, one position is taken for one knowledge component and different conceptualizations of PCK are analysed respectively. Moreover, the general knowledge components are divided into sub-components to make the summary more specific. In Table 4-2 ‘representations’ in knowledge of instructional and knowledge of students' understanding of the subject with different sub-components are in every conceptualisation of PCK. In this table, previous authors’ strategies for navigating the topic of PCK are demonstrated through the creation of consistent sub-categories that have already been well-established.

Both summary Tables (4-1 and 4-2) are clearly related, focusing on the components of PCK that exist in the literature. However, as is shown in Table 4-2, authors differ in how they conceptualize approaches in terms of the definition of, and interaction between, the components. Thus, to avoid misunderstandings, the researcher created Table 4-3 to provide more detail about the components agreed upon by most authors and distinguished general knowledge components from the amalgam of subject matter knowledge and mathematical knowledge in knowledge base for teaching.

Authors	Knowledge of							
	Purposes for Teaching a Subject	Student Understanding	Curriculum	Instructional Strategies Representation	Assessment	Subject Matter	Content	Pedagogy
Shulman 1987	*	PCK	*	PCK		*	*	*
Tamir 1988	PCK	PCK	PCK	PCK	PCK	*		*
Grossman 1990	PCK	PCK	PCK	PCK		*		
Marks 1990	PCK	PCK		PCK		PCK		
Smith & Neale 1989	PCK	PCK		PCK		*		
Cochran et al. 1993	PCK	PCK		**		PCK	PCK	PCK
Geddis et al. 1993	PCK	PCK	PCK	PCK				
Femandezbalboa & Stiehl 1995	PCK	PCK		PCK		PCK	PCK	
Magnasson et al. 1999	PCK	PCK	PCK	PCK	PCK			
Hasweh 2005	PCK	PCK	PCK	PCK	PCK	PCK	PCK	PCK
Ball et al., 2008	**	PCK	PCK	PCK	**	*	PCK	PCK
Loughran et al. 2009	PCK	PCK		PCK		PCK	PCK	PCK

*separate category in the knowledge base for teaching **Not discussed explicitly

Table 4-1: Summary of Components in Different Conceptualisations of PCK (using Van Driel et al., 1998; Park & Oliver, 2008; Ball et al., 2008)

Components & sub-components \ Authors		Shulman 1987	Gudmundsdotti & Shulman 1987	Grossman 1990	Tamir 1998	Magnusson et al. 1999	Ball et al. 2008
Knowledge of the goals for teaching a subject		**	**	PCK	**	PCK	**
Knowledge of students' understanding of the subject	Students' conceptions of learning	PCK	PCK	PCK	PCK	**	**
	Students' learning interest in the subject area	**	PCK	**	**	PCK	PCK
	Students' learning approaches	**	**	**	**	PCK	PCK
	Students' difficulties in learning	PCK	PCK	**	**	PCK	PCK
Knowledge of curriculum in specific subject area	Selection of content	PCK	PCK	**	**	**	PCK
	Teaching materials	**	**	PCK	**	**	PCK
	Organization of content	**	**	PCK	PCK	**	PCK
Knowledge of instructional and strategies	Representations	PCK	PCK	PCK	PCK	PCK	PCK
	Activities	**	**	**	PCK	PCK	**
Knowledge of assessment of students' learning of the subject matter		**	**	**	PCK	PCK	**
General knowledge of curriculum		*	*	*	**	**	PCK
Subject matter knowledge		*	*	*	*	*	PCK
Knowledge of context		**	*	*	**	*	PCK
Knowledge of students		**	*	*	**	*	PCK
General pedagogical knowledge		**	*	*	*	*	**

*separate category in the knowledge base for teaching **Not discussed explicitly

Table 4-2 Summary of Components in Different Conceptualisations of PCK (using Park & Oliver, 2008; Ball et al., 2008)

4.3 Developing the Framework

4.3.1 Overview

Table 4-3 identifies the stages in development of the PCK framework starting with Shulman (1986), its originator, through to Lesseig (2016) and Khakbaz (2016). Table 4-3 was created based on the results documented in Table 4-1. and 4-2. It presents PCK components based on researchers' clarifications that were reviewed above as the generic and specific nature of PCK components and explains the trend of clarifying PCK components for teaching mathematics. Using the coding structures and the interpretation of PCK in the literature, it was possible to demonstrate the stages of development that have been modified over time. This is important because when considering the conceptualisation of PCK in the modern period, it is imperative to consider its evolution.

Originator	Concept developed
Shulman (1986/7)	Pedagogic knowledge (what teachers know about teaching) + Subject matter knowledge (what teachers know about what they teach) = Pedagogic Content Knowledge (PCK)
Tamir (1988)	Considered teacher's knowledge consisting of six major categories, namely: "general liberal education, personal performance, subject matter, general pedagogical, Pedagogical content knowledge, and foundations of the teaching profession" (p.99), and proposed a fourth component of PCK consisting of two categories: - knowledge of the dimensions of learning that are important to assess; - knowledge of the methods by which learning can be assessed.
Grossman (1990)	outlined PCK as having four main elements: - conceptions of purposes for teaching subject matter; - knowledge of students' understanding; - curricular knowledge; - knowledge of instructional strategies.
The COACTIV Project (2004)	Approaches PCK generically with the research focusing on a theoretical model which was tested empirically. Their conclusion is that professional competence can be seen 'as a Multidimensional construct' (p. 17). The dimensions are: - Professional knowledge (= content knowledge, pedagogical content knowledge, pedagogical/psychological knowledge, organisational knowledge, counselling knowledge). - Values and beliefs. - Motivational orientations and self-regulation.

Ball et al. (2008)	The development is based on two categories: - Subject matter knowledge (or content knowledge) which is divided into three types of mathematical subject matter (or content) knowledge, two of which - specialised and common content knowledge. - PCK includes three main categories: knowledge of content and students, knowledge of content and teaching, and knowledge of content and the curriculum. The basis of this work is the design of tools for the measurement of teachers' content knowledge in the context of teaching elementary school level mathematics. The tools were empirically tested. The resulting framework was organised in mathematical topics (such as algebra, number and operations) and domains (e.g. knowledge of content, knowledge of students and content).
Teacher Education and Development Study (TEDs-M) (2008)	Presents three theoretical sub-domains (elements) of mathematics as a specific PCK - mathematical curricular knowledge; - knowledge of planning for mathematics teaching and learning; - enacting mathematics for teaching and learning.
Khakbaz (2016)	a phenomenological study of 10 university mathematics teachers at Bu Ali Sina University (Islamic Republic of Iran) resulted in a model comprising four cognitive themes: - mathematics syntactic knowledge; - knowledge about mathematics curriculum planning; - knowledge about students' mathematics learning; - knowledge about creating an influential mathematics teaching – learning environments (p. 1). In addition, Khakbaz (2016) identified three contextual themes: - the nature of mathematics subjects; - university teachers' features; - terms of learning environment. (p. 1)
Lesseig (2016)	based on data from 35 teacher-leaders from three US school districts, investigated PCK on the specific example of mathematical concept of proof by establishing two two-pronged frameworks consisting of - *'knowledge of content and students'* - encompassed explicit knowledge of student proof schemes and developmental aspects of proof; - *'knowledge of content and teaching'* - relationship between instruction and proof schemes, questioning strategies and knowledge of proof connections.

Table 4-3: Developments in PCK

4.3.2 Shulman's Model of Teacher Knowledge

PCK is a type of knowledge exclusively used by teachers (Shulman, 1986; 1987) and to bring about effective teaching, teachers need to combine the subject and pedagogy so that they demonstrate "an understanding of how particular topics, problems, or issues are organized, represented, and adapted to diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). Shulman considered that PCK was largely about understanding specific teacher CK and how to modify or transform that knowledge into an accessible version for students through the use of specific pedagogical strategies. The overall model of teacher knowledge is illustrated in Figure 4-

1, thus, PCK, in its most basic form, can be described as the blending of aspects of content and of pedagogy. However, this blended nature of PCK is not quite as straightforward as simply a combination of two components.

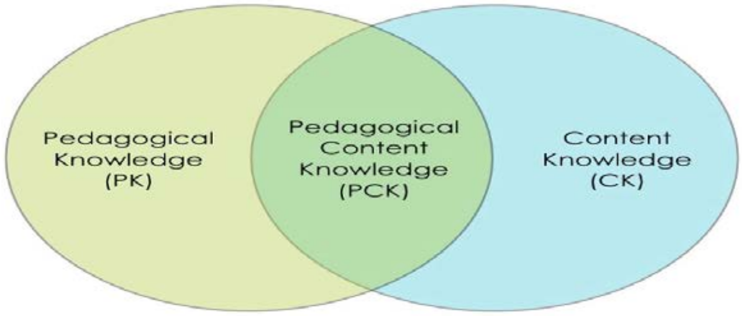

Figure 4-1: Shulman's Model of Teacher Knowledge (Shulman, 1987)

Shulman's (1987) work provides the foundation in that his original features of PCK still remain consistent in much of the subsequent research undertaken (e.g. Khakbaz 2016; Lesseig, 2016; Rollnick, 2016; Aydin el at., 2015; Fan, 2014; Nordin et al., 2013; Ijeh, 2012; Petrou & Goulding, 2011; Krauss et al., 2008; Baker & Chick, 2006; Miller, 2006; Duling, 1992; Grossman, 1990; Tamir, 1988). PCK is commonly taken to be the transformation of at least two constituent knowledge domains: general pedagogical knowledge and subject matter knowledge (Gess-Newsome, 1999 cited in Hadiyanti et al., 2014).

4.3.3 Tamir's Model for Teachers' Knowledge

The model by Tamir (1988) is significant as it appeared the year after Shulman (1986/87) and could possibly be considered as an early response to Shulman's work. Tamir attempted to develop and extend the categories which were suggested by Shulman and Sykes (1986) and the framework was suggested as a basis for teacher education. Tamir posed the question "What kinds of knowledge do teachers need in order to be effective in their classrooms?"(p.99) and attempted to answer this question through his model (see Figure 4-2) which clarifies a framework for teachers' knowledge, consisting of six major categories, namely: "general liberal education, personal performance, subject matter, general pedagogical, subject matter specific pedagogical, and foundations of the teaching profession" (p.99).

1. GENERAL LIBERAL EDUCATION
2. PERSONAL PERFORMANCE How do I look, speak, listen, move in class?
3. SUBJECT MATTER
   - 3.1 Knowledge: Major ideas and theories of a particular discipline
   - 3.2 Skills: How to use a microscope
4. GENERAL PEDAGOGICAL
   - 4.1 Student
     - 4.1.a Knowledge: Piaget's development levels
     - 4.1.b Skills: How to deal with hyperactive students
   - 4.2 Curriculum
     - 4.2.a Knowledge: The nature, structure, and rationale of Bloom's Taxonomy
     - 4.2.b Skills: How to prepare a learning unit
   - 4.3 Instruction (Teaching and management)
     - 4.3.a Knowledge: Different ways of assigning turns to students in class discussion
     - 4.3.b Skills: How to formulate a high level question
   - 4.4 Evaluation
     - 4.4.a Knowledge: Different types of tests
     - 4.4.b Skills: How to design a multiple choice item
5. SUBJECT MATTER SPECIFIC PEDAGOGICAL
   - 5.1 Student
     - 5.1.a Knowlege: Specific common conceptions and misconceptions in a given topic
     - 5.1.b Skills: How to diagnose a student conceptual difficulty in a given topic
   - 5.2 Curriculum
     - 5.2.a Knowledge: The pre–requisite concepts needed for understanding photosynthesis
     - 5.2.b Skills: How to design an inquiry oriented laboratory lesson
   - 5.3 Instruction (Teaching and management)
     - 5.3.a Knowledge: A laboratory lesson consists of three phases: pre–lab discussion, performance, and post–laboratory discussion.
     - 5.3.b Skills: How to teach students to use a microscope
   - 5.4 Evaluation
     - 5.4.a Knowledge: The nature and composition of the Practical Tests Assessment Inventory
     - 5.4.b Skills: How to evaluate manipulation laboratory skills
6. FOUNDATIONS OF THE TEACHING PROFESSION

Figure 4-2: Tamir's Model for Teacher Knowledge (Tamir, 1988, p.100)

4.3.4 Grossman's Model of Teacher Knowledge

In the book 'The Making of a Teacher' (1990), Grossman challenged the assumption that anyone who has command of the subject matter can teach well and that experience is the best 'teacher'. Grossman conducted case studies of six novice teachers and focused on pedagogical understanding of subject matter that distinguishes between subject expertise and experience. Grossman (1990) developed and gave more detail to Shulman's (1987) ideas and outlined PCK as having four main components. Figure 4-3 provides an overview of Grossman's model of teacher knowledge, which captures the inter-relation between PCK and the other components of teacher knowledge.

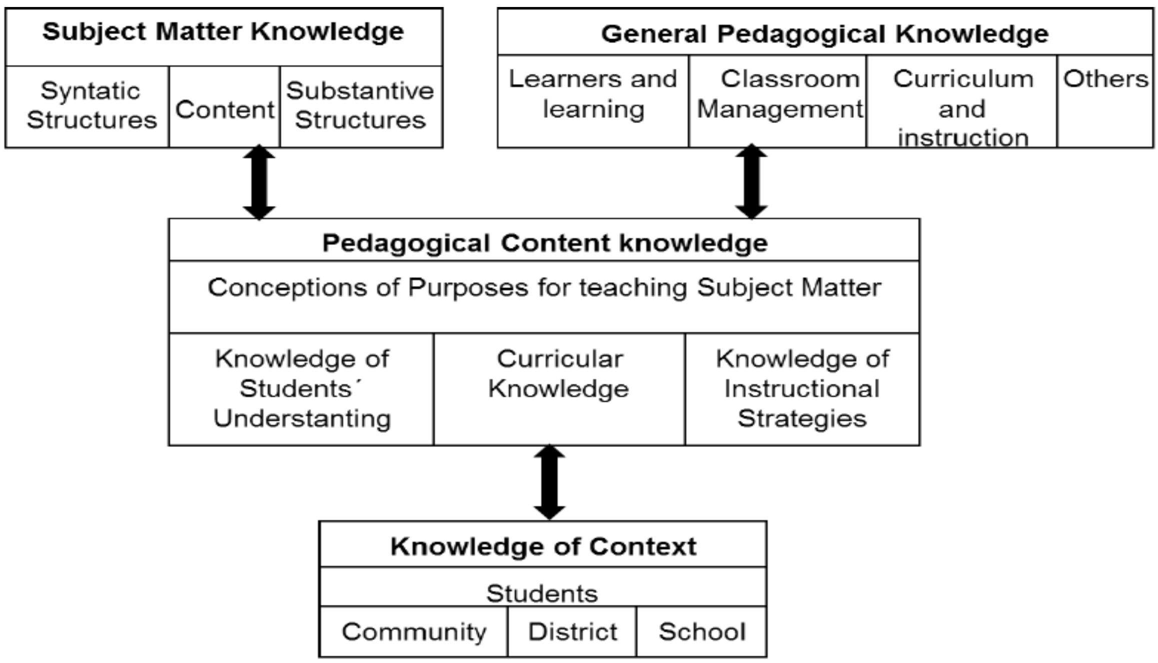

Figure 4-3: Grossman's Model of Teacher Knowledge (Grossman, 1990, p.5)

4.3.5 The COACTIV Project Model of Teacher Knowledge and Competency

The COACTIV (2004) (source: Baumert & Kunter, 2013) project provided one of the models used to develop the proposed framework of this study. It was a mathematics education research project funded by the German Research Foundation and linked to the 2003/04 testing cycle of the Programme for International Student Assessment (PISA). The aim of the research project was twofold, inasmuch as a theoretical model was to be developed for teacher competence and tested empirically. The starting point was that while it was acknowledged that many models have been developed regarding teacher competence and knowledge, very few were based on empirical research in practice.

The COACTIV model encompasses professional knowledge, values, beliefs, motivational orientations, and self-regulation. It builds on the work of other scholars, notably Ball et al. (2001) and Senk et al. (2008) but what differentiates their model is that it is based on the fact that “four forms of mathematical knowledge are theoretically distinguished” (Baumert & Kunter, 2004, p. 9). They divided professional knowledge to five areas of competence, which are CK, PCK, PK, organisational, and counselling knowledge. They outlined PCK as having three main elements: explanatory knowledge, knowledge of students' mathematical thinking, and knowledge of student assessment. The model is shown in Figure 4-4.

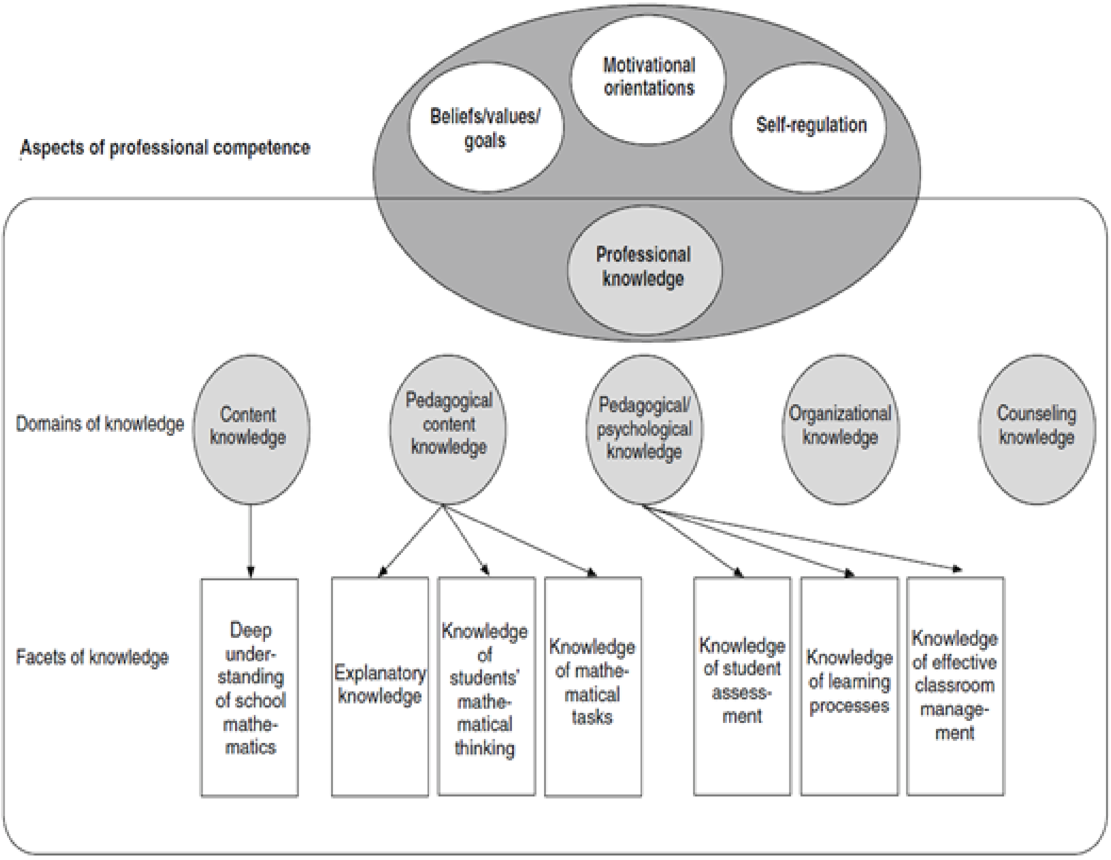

Figure 4-4: Overarching Framework of the COACTIV Project

(source: Baumert & Kunter, 2013, p.29)

4.3.6 The Model by Ball et al.

The model by Ball et al. (2008) advanced a domain map for mathematical knowledge (Figure 4-5) for teaching and explaining the relationship between their model and two of Shulman's (1987) initial categories: subject matter knowledge and PCK and they moved Shulman's third category, curriculum knowledge from a main category to a sub-category of PCK. This is consistent with Grossman's work (1990) who was one of Shulman's research team (Grossman, 1990). They developed measures for mathematical knowledge for teaching and their project indicated that there are empirically discernible sub-domains into PCK (knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum). Moreover, there is an important sub-domain of 'pure' content knowledge unique to the work of teaching, specialized content knowledge, which is distinct from the common content knowledge.

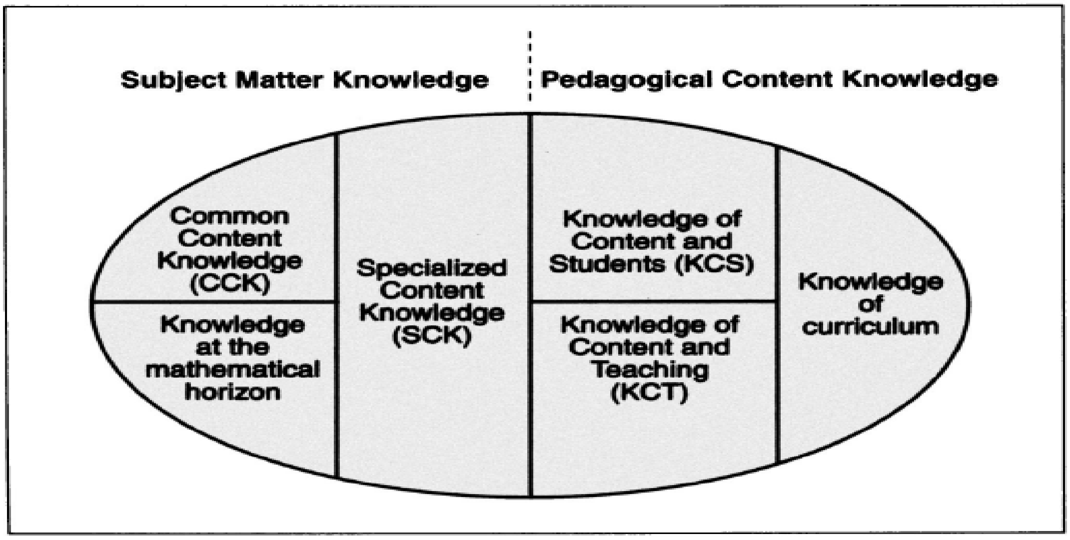

Figure 4-5: Domain Map for Mathematical Knowledge for Teaching (Ball et al., 2008, p.403)

4.3.7 Teacher Education and Development Study (TEDS-M)

The Teacher Education and Development Study (TEDS-M) (Tatto et al., 2008) is another theoretical model used to develop the proposed framework for this current study. Where Baumert and Kunter's research (2004) is rooted just in the PISA cycle of that year, Senk et al. (2008) linked their research to the Trends in International Mathematics and Science Study (TIMSS), inasmuch as they aligned their frameworks to the TIMSS cognitive parameters, i.e. "knowing, applying, and reasoning" (Mullis & Martin, 2017, p.13). Senk et al. (2008) presented the findings of their multi-national, longitudinal study about mathematical knowledge for teaching and introduced a framework based on two dimensions differentiated as 'mathematics content knowledge' and 'mathematics pedagogical content knowledge'. Each of these dimensions is underpinned by a number of sub-domains. The authors went on to reference three sub-domains which underpin the dimension of mathematics pedagogical content knowledge as "mathematics curricular knowledge; knowledge of planning for mathematics teaching and learning; knowledge of enacting mathematics" (Senk et al., 2008, p. 4). The model is shown in Table 4.4.

Sub-domains and aspects of the sub-domain of mathematics pedagogical content knowledge used in TEDS-M.

Mathematical curricular knowledge	• Establishing appropriate learning goals • Knowing different assessment formats • Selecting possible pathways and seeing connections within the curriculum • Identifying the key ideas in learning programs • Knowledge of mathematics curriculum
Knowledge of planning for mathematics teaching and learning	• Planning or selecting appropriate activities • Choosing assessment formats • Predicting typical students' responses, including misconceptions • Planning appropriate methods for representing mathematical ideas • Linking didactical methods and instructional designs • Identifying different approaches for solving mathematical problems • Planning mathematical lessons
Enacting mathematics for teaching and learning	• Analyzing or evaluating students' mathematical solutions or arguments • Analyzing the content of students' questions • Diagnosing typical students' responses, including misconceptions • Explaining or representing mathematical concepts or procedures • Generating fruitful questions • Responding to unexpected mathematical issues • Providing appropriate feedback

Table 4-4: Mathematical Pedagogical Content Knowledge as Used in the TEDS-M Framework (source: Tatto et al., 2008, p.5)

4.3.8 Khakbaz's Model of Mathematics University Teachers' Perception of PCK

Khakbaz's (2016) theoretical model also informs the proposed framework for this study. Khakbaz undertook research with a group of university teachers who were asked to reflect both on their experience of students and their experience of teachers. Based on this research, Khakbaz (2016) developed a two-pronged framework for PCK which differentiated between cognitive on the one hand and contextual on the other. The model is illustrated in Table 4-5. In order to arrive at a definition for contextual PCK, Khakbaz tested the participants' understanding of 'application' for context. As the participants' conceptualisation of application varied between participants, Khakbaz (2016) concluded that teachers do not only need subject specific CK but also need an awareness of "applications of mathematical concepts and the main ideas behind them" (p. 5).

Categories	Themes	Sub-themes
Cognitive themes	Mathematics syntactic knowledge	Application of the mathematics concept Main idea behind a mathematics problem
	Knowledge about students	Major and grade of students Students' misconceptions and learning difficulties
	Knowledge about mathematics curriculum planning	Knowledge about mathematics problems Make a coherent and meaningful content To make the content appropriate with students
	Knowledge about creating an influential teaching–learning environment	Knowledge about different representation approaches Knowledge about how to say and how to write Knowledge about how to engage students in teaching–learning mathematics Knowledge about giving feedback to students Knowledge about using information communication technology (ICT) in teaching Knowledge about using aesthetic sense in teaching Classroom management
Contextual themes	Nature of subject	
	Professor's features	
	Terms of learning atmosphere	Physical aspect Sociocultural aspect

Table 4-5: Khakbaz's Model of PCK for Teaching Mathematics in Higher Education. (Khakbaz, 2016, p.191)

4.3.9 Lesseig's Model

Lesseig advanced a model for the Mathematical Knowledge for Teaching Proof (MKT for Proof), which details required knowledge *"across subject matter and pedagogical domains"* (Lesseig, 2016, p. 253). From this model, this researcher adopted and developed Lesseig's framework for the present study, as Lesseig's framework uses knowledge about proof, and proof is considered as one of the components of calculus.

Lesseig's data was gathered at four different professional development events across a variety of school settings. Lesseig (2016) referred to Smith (2014) who argued that tools employed in the elicitation of data can be utilised to articulate "a standard or shared understanding of practice" (p. 265). Lesseig aimed to develop a framework based on teachers' responses to a variety of assumptions and posed scenarios around the teaching of proof, and then to test the suitability of the framework for professional development purposes for teachers of mathematics. Lesseig (2016) developed this concept further by stating that the framework presented can be both a data gathering tool and a tool for the development of teachers. In terms of PCK for teaching proof, Lesseig's framework differentiates between knowledge of content and students on the one hand and knowledge of content and teaching on the other. The model is shown in Table 4-6.

Pedagogical content knowledge components of the MKT for proof framework

Pedagogical Content Knowledge for Teaching Proof	
Knowledge of Content and Students	**Knowledge of Content and Teaching**
Explicit knowledge of student proof schemes • Characteristics of external, empirical and deductive proof schemes • Students' tendency to rely on authority or empirical examples • Typical progression from inductive to deductive proof	*Relationship between instruction and proof schemes* • Methods of answering questions, responding to student ideas, using examples and lecturing that either promote or diminish authoritarian or empirical proof schemes
Developmental aspects of proof • Definitions & statements available to students • Representations within students' conceptual reach • Forms of argumentation appropriate for students' level • Relationship between mathematical and everyday use of terms	*Questioning strategies* • To elicit justification beyond procedures • To encourage thinking about the general case *Use of pivotal examples or counter-examples* • To extend, bridge or scaffold thinking • To focus on key proof ideas *Knowledge of proof connections* • How to link visual, symbolic & verbal proofs • How argument structure depends on accepted definitions • How to produce a general argument from a numerical example or specific diagram

Table 4-6: The PCK Components of Lesseig's MKT for Proof Framework (Lesseig, 2016, p. 257)

4.4 The Proposed Model of PCK for Teaching Calculus

As the aim of this study is to propose a model of PCK for teaching calculus and then use this model to explore calculus teachers' PCK, this section presents the proposed model for the study and provides justification for choosing the components and codes and why other elements were not used. In the thesis, the model is systematically analysed and refined in the light of the data collected for this study.

In exploring potential models for this research, the model mainly considered was the framework for teachers' knowledge of teaching proof used by Lesseig (2016). Lesseig's model is suitable for this study as it can be adapted for use in analysing the PCK of teachers of calculus in higher education, primarily to uncover how such calculus teachers articulate and demonstrate their PCK. The framework builds on Ball's and colleagues construct of PCK to include knowledge of content and students, and knowledge of content and teaching, Lesseig's model has been chosen and integrated with the Khakbaz (2016); COACTIV (2004); TEDS-M (2008) models. The development of PCK is critical to effective teaching mathematics, and the codes have been chosen which support sub-categories in Lesseig's model. As this current research focuses primarily on teachers' teaching, and how the teachers articulate their teaching, it was decided not to use codes relating to mathematics curriculum planning, knowledge of planning for mathematics teaching and learning, and summative assessment of learning.

Where Lesseig (2016) has two categories: knowledge of content and students, and knowledge of content and teaching, the framework proposed for this study uses the categories of ***knowledge of content and students when teaching calculus***, and ***knowledge of content and teaching calculus***. These categories are underpinned by a number of first-level and second-level sub-categories as shown in Figure 4-6. This addresses:

RQ1: What would be a model of PCK for teaching calculus?

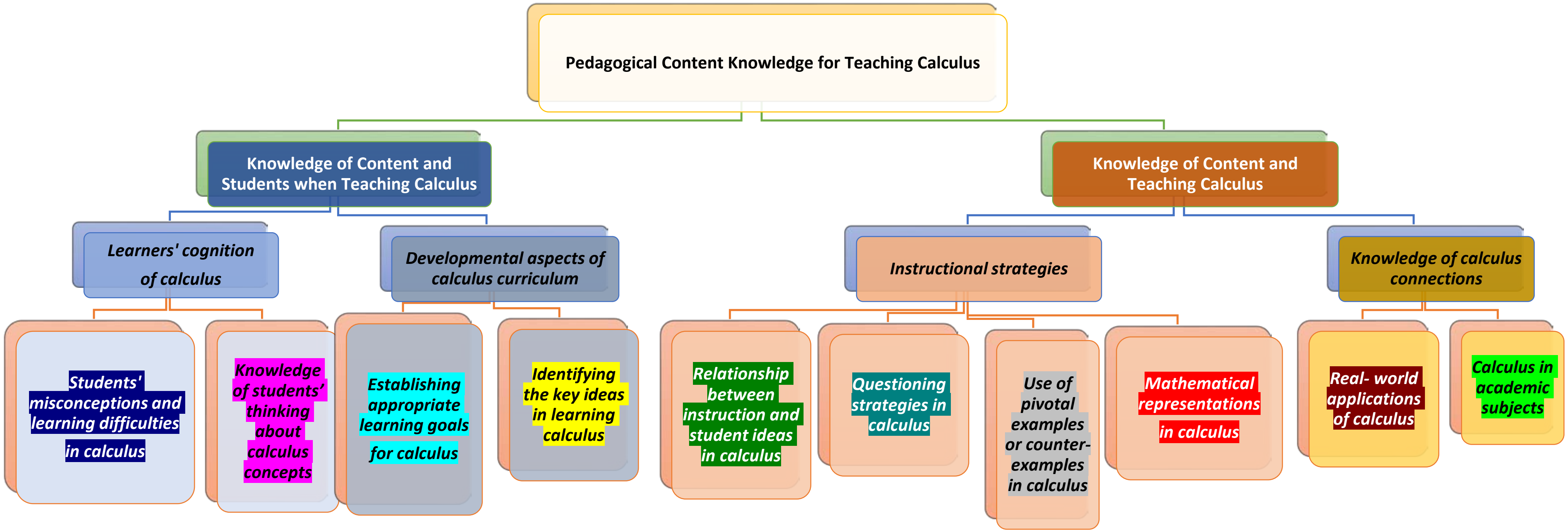

Figure 4-6: The Proposed Model of PCK for Teaching Calculus

The first-level sub-categories for the category of ***knowledge of content and students when teaching calculus*** are ***learners' cognition of calculus*** and ***developmental aspects of the calculus curriculum***. The former is informed by the systemisation of Baumert and Kunter (2004) who, in turn, aligned their analysis to the previous frameworks and models. This led them to include a category in their PCK model of student cognition "including misconceptions and strategies" (Baumert & Kunter, 2004, p. 32). Khakbaz (2016) echoes the notion of the significance of teacher knowledge of ***students' misconceptions and learning difficulties in calculus***. In the proposed model, therefore, students' misconceptions and error production have fed into the first of two second-level sub-categories for learners' cognition, with ***knowledge of students' thinking about calculus*** concepts (Baumert & Kunter, 2004; Lesseig, 2016) representing the second. The latter first-level sub-category presented here, ***developmental aspects of calculus curriculum***, is informed by Lesseig's work (2016) who argues that teachers' understanding of application (of proof) changes according to their context and students, and notions of curriculum thus evolve as well. Based on this, the sub-category has been differentiated further into two second-level sub-categories: ***identifying the key ideas in learning calculus*** and ***the establishing of appropriate learning goals for calculus***.

The second category ***knowledge of content and teaching calculus*** is underpinned by two first-level sub-categories which are ***instructional strategies*** and ***knowledge of calculus connections***. Instructional strategies are further underpinned by four second-level sub-categories and knowledge of calculus connections by two. The second-level sub-categories underpinning instructional strategies have been informed by the work of Lesseig (2016) which is testimony to the viability of her framework which was developed and tested through a variety of professional development settings. These four second-level sub-categories include: ***relationship between instructions and student ideas in calculus; mathematical representation in calculus; questioning strategies in calculus*** and ***use of pivotal examples or counter-examples in calculus.*** Similarly, knowledge of calculus connections is underpinned by two second-level sub-categories which draw on Lesseig (2016) and Khakbaz (2016) in contextualising calculus in ***real world application of calculus*** in everyday life and ***calculus in other academic areas.***

In summary, the components listed below and the text within brackets explains this study's interpretation of the four sub-elements of the proposed model of PCK for calculus teaching.

- learners' cognition of calculus (knowledge of what level students are functioning at i.e. what they know, what they think they know, what they think they do not know);
- developmental aspects of the calculus curriculum (how and what for);

- instructional strategies (how best to teach the subject matter to meet all students' needs and curricular requirements in the context of the students' levels; knowledge of instructional strategies relevant to the context);
- knowledge of calculus connections (what needs to be indicated, so that people benefit from the applications of calculus every day and linking between calculus concepts and application of calculus in everyday use).

For more detail on the sources of the model, see Appendix B.

4.5 Chapter Summary

This chapter has outlined the theoretical basis for this research. It is acknowledged that, when detailing definitions for a study, past researchers did not always agree on the underlying uses of a word or a phrase. Further, the use of words in different contexts have been exemplified throughout the different examples of research that have been conducted. This chapter has outlined the theories relating to teacher knowledge that have provided the foundations for the development of the theoretical framework devised and proposed for use in this study. Furthermore, working on the chapter has contributed towards the researcher’s thinking about the research questions for this current study.

Chapter 5 Methodology

5.1 Introduction

This chapter accounts for the methodology and methods used to investigate calculus teachers' PCK. Initially, the research objectives and questions are re-stated. This is followed by the methodology, which includes the philosophical approaches associated with this research. The focus on several 'cases' is justified as an appropriate means of research. The methods used, including survey, observation, and interview are detailed, as is the pilot study. As part of this, the justification for selecting the calculus course within mathematics and the participant sample is explained. The data analysis procedures are described followed by considerations regarding validity and reliability. Finally, the ethical considerations related to this research study are explained. The chapter concludes with a summary.

5.2 Research Objectives and Research Questions

The focus for this study is to investigate teachers' calculus teaching. The overarching goal is to detail the PCK for teaching calculus and to analyse the teachers' PCK for teaching calculus. This was achieved through the development of a model of PCK for teaching calculus and as such, this research project has two overarching objectives:

OB1. To propose a model of PCK for teaching calculus.
OB2. To explore calculus teachers' PCK.

In line with the objectives, the research questions for the study are:
RQ1. What would be a model of PCK for teaching calculus?
RQ2. Using this model of PCK, how do calculus teachers articulate and demonstrate their PCK?

5.3 Methodology

One of the main components of this research that needed to be addressed was the perceived gap in the literature. A case in point stated by Khakbaz (2016, p. 185) is that:

> *Teaching mathematics in university levels is one of the most important fields of research in the area of mathematics education. Nevertheless, there is little information about teaching knowledge of mathematics university teachers. PCK provides a suitable framework to study knowledge of teachers.*

One of the gaps identified includes the lack of qualitative research findings that relate to PCK within the field of calculus teaching, but more specifically within the given context of this study. The researcher sought to ensure that this gap is addressed through the formation of the research instruments. Through careful discussions with the thesis supervisor, an extensive review of the literature, and by attempting to match the methods to the research questions posed, the intention was to ensure that the methodology would be appropriate and deliverable.

Methodology can be described as the lens through which the researcher views and makes decisions about the study (Mills, 2013), while Shank and Brown (2007) consider that methodology can be the philosophical framework within which the research is conducted. Together these can be interpreted as being the philosophical assumptions underpinning the selected research methods, including why qualitative or quantitative methods, or a mixture of both, are selected. If inappropriate methodology is used, or if appropriate methodology is used poorly, the results of a study could be misleading. In this study, the researcher drew on comprehensive paradigms, research approaches, research designs, and research methods to identify the most suitable for this research. The research process for this study has been developed in order systematically to achieve the goal of this research, which is to examine particular levels of sub-sets of the PCK of calculus teachers.

5.3.1 Philosophical Assumptions Underlying the Research

Creswell (2013) identifies that research paradigms can affect every level of study. A 'paradigm' is depicted as a full structure or system that influences and directs both research and practice (Willis el al., 2007). The assessment of a level suggests the level is measurable, consequently placing research into a positivist paradigm. Positivism, while useful for measurement of levels of knowledge cannot fully explain all aspects of this study. The research questions needed data that went beyond the observable and quantifiable. Consequently, the empirical research within this study needed to be based upon an interpretivist research philosophy to interpret the data through a qualitative lens to reveal the human-interest perspective. The next sections outline the approaches associated with the positivist and interpretivist paradigms adopted.

5.3.2 Interpretivist Paradigm

There are certain philosophical assumptions underpinning this research and its methodology. In particular, it follows an interpretivist paradigm that recognises that the individual and society are inseparable and assumes that the process of research can be used to uncover a person's understanding of a particular phenomenon (O'Donoghue, 2006). It also recognises that everyday life is based in the social and is constructed through "people employed within the system acting together and producing their own roles and patterns of action" (Blackledge & Hunt, 2018, p.235). This approach follows a long-standing tradition rooted in the social sciences that enables understanding of people's views through interaction with them in their own language and on their own terms (Kirk & Miller, 1986). It also recognises that an individual's sense of self is created through their interaction with others (Mead, 1934) and so this must be considered during the research process. The philosophical underpinnings of an interpretivist approach also considers that conclusions drawn from the research cannot be separated from the participants' own experiences and that 'meaning' may shift and change rather than remain static and fixed.

5.3.3 Ontological Reasoning

Ontology is based upon the actuality of being and of reality of a phenomenon. It is depicted in this light as the 'study of being'; ontology is a systematic account of existence (Crotty, 1998). For knowledge-based systems, what 'exists' is exactly that which can be represented. In other words, things that exist have a reality that can be described and have a relationship with each other. Crotty (1998) considers that the language, or vocabulary, used in these descriptions represents knowledge of the reality.

Within the realm of social science, ontology focuses more upon social reality. This notion is considered objective in the sense that social reality is an exterior concept, which is what the individual sees. However, because it forms and grows within ones' consciousness it can be considered in a subjective sense (Cohen et al., 2013). Thus, the focus of a research project can be approached objectively to seek to understand the effect different variables can have on something in terms of governing, growing, or altering it. In other words, the reality identifies how human actions are controlled and overseen by rules that are all encompassing with a constant nature as their foundation (Cohen et al., 2013). In the context of this study, these rules assist with the interpretation of culture, not just culture as is associated with a national framework such as the Middle East, but the examination of the culture within the educational context, allowing for opportunities for theoretical clarity to emerge.

Social reality is considered a consequence of events, since social individuals construe particular connotations from events and situations; they construct or interpret a theoretical structure (Crotty, 1998). The ontology, or foundation on which interpretivist research is based, accepts that the social world faces ongoing and continual development via the exchanges and relations between people. This study, within qualitative aspects, aims to create an understanding of calculus teachers' actual implementation of their PCK within their teaching practices. Thus, their social reality can be explained and recognised by seeking to identify their viewpoints, as they are the individuals who are immersed in the development of their sense and comprehension of their reality.

When seeking to comprehend social phenomena, the perspectives that different people have are critical. To truly understand the knowledge teachers, hold and how they impart this knowledge necessitates an ontological belief, which incorporates the specificity of circumstances and the individuality of different people. This amounts to an interpretive notion. Adopting this perspective enabled the researcher to identify how teachers understand and abstractly identify with the world they live in together with any subjective interpretations they allocate to internal rather than external exchanges (Cohen et al., 2013).

As part of its interpretivist paradigm this research used inductive reasoning through the researcher's engagement with the data uncovered through its research methods. This approach to its adopted paradigm follows Merriam's (2009) view that a focus on 'cases' is an appropriate process for carrying out inductive reasoning. The aim is not to test a particular hypothesis, but instead generate findings (Kohlbacher, 2006). This form of research is 'grounded' in the data collected through the study and is particularly appropriate to use in research that focuses its attention on everyday situations, such as in educational contexts (Hancock & Algozzine, 2006). Eisenhardt (1989) states that analysing data drawn from cases can be challenging; however, it is also acknowledged that the process of interpreting data can be illuminating through an inductive approach.

5.4 Studying 'Cases'

Philosophically, the study of 'cases' recognises that there are multiple meanings and realities, which are, in part, produced by the researcher (Lincoln et al., 2011; Yin, 2014). In studying cases, the researcher's positionality must be considered to ensure the reliability and validity of findings. In particular, the researcher's philosophical position must be acknowledged to understand what informs the approach they have used (Harrison et al., 2017). It must also be recognised that the researcher must use their own previous knowledge and judgement when interpreting data (Babbie, 2001). This means that one researcher may interpret the same data differently from another researcher and this must be acknowledged, and steps must be put in place to ensure that the findings of a study are reliable and valid. Consequently, the positionality of the researcher must be acknowledged as potentially influencing the findings of the research. This research closely follows the view of Stake (1995; 2006) that meaning must be uncovered through experience in context.

The studying of 'cases' has undergone shifts over recent decades resulting in a flexible approach to research (Harrison et al., 2017). This makes it most appropriate to this study as it provides a flexible approach to looking at an issue from multiple perspectives in a real-world setting. To understand this fully, it is important to look to the history of studying 'cases' with its roots in qualitative research in anthropology, history, psychology, and sociology (Merriam, 1998; Simons, 2009; Stewart, 2014) and in education as a way to evaluate curriculum design (Merriam, 2009; Simons, 2009; Stake, 1995). However, despite the great potential of this approach, it is important to understand how its philosophy can be interpreted in a practical sense and how this can be influenced by the presence of the researcher.

Yin's (1998) definition of a case is that it "investigates a contemporary phenomenon within its real-life context; when the boundaries between phenomenon and context are not clearly evident and where multiple sources of evidence are used" (p.23). In addition, Bassey (1999) suggests that a case is "a study of a singularity conducted in depth in natural settings" (p.47). According to Stake (2006), studying cases enables a researcher to explore complex phenomena. Similarly, Yin (2003) suggests that studying cases is useful for examining simple and complex phenomena and further suggests that a multiple methods approach is particularly useful (see Section 5.7). In addition to Yin's (2003) interpretation of studying cases, Kennedy et al. (2006) suggest that the study of cases tends to employ qualitative strategies for data analysis because one key component of such research is its illustrative nature. Berg (2007) elaborates on this by suggesting that by being illustrative, the study of cases allows for actualised elements, patterns and nuances to be highlighted. Researchers, such as Miles et al. (2014) and Stake (2000) argue that multiple cases are more effective than single cases, as they offer more in-depth understanding of the phenomenon. Furthermore, Merriam (2009) considers that the use of multiple cases adds to the study's validity. Acknowledging that the findings are not be generalisable across the wider field, instrumental cases can provide a foundation on which further research could be built. Researchers, according to Stake (1995), have different motivations for selecting cases to research (i.e. not all case research is conducted with the same purpose). Stake (1995) identified three types of such research: collective, instrumental and intrinsic. For collective research, there are a number of individual case studies (or each one is treated separately) and these works together towards a final outcome or research question (Hancock & Algozzine, 2017). In an instrumental case, the issue or factor is the focus of the study rather than the case (Stake, 1995). Finally, in intrinsic research, the researcher has a personal and specific interest in some case, thus making it an 'intrinsic interest' (Stake, 1995).

In this research study, the researcher is particularly interested in understanding teachers' PCK in depth. This research study utilised an instrumental approach, in line with Stake's (2000) explanation that instrumental cases "concentrate on phenomena instead of the case itself. The case is of secondary interest; it plays a supportive role and facilitates our understandings of something else"

(p. 437). The phenomenon examined in this study is calculus teachers' PCK in teaching Calculus 1, within the context of university level teaching.

5.4.1 Justification for the Study of Cases

This research is justified in selecting the study of cases for several reasons, the first being that this approach is commonly used as a method for educational research. More importantly, however, it has been used by other researchers in closely related studies (e.g. Sowder et al., 1998; Holton, 2001). The study of cases of mathematics teachers is not a new phenomenon. Sowder et al. (1998), when conducting a two-year research project with mathematics teachers, considered that studying cases was the most appropriate method to obtain and analyse data. Sowder et al. (1998) suggest that by studying cases there are opportunities for both explanation of individual case narratives and for cross-case analysis. Their view supports the strategies adopted for this current research, which examined the personal narratives of teachers, and observations of teaching and brought these together through the use of cross-case analysis.

Flyvbjerg's (2006) argument that studying cases suits research involving practical and professional knowledge ultimately supports the purpose of this thesis. Flyvbjerg suggests that theoretical knowledge is derived from "rule-based learning" (p.223), whereas practical knowledge is more about experiences or reflections that teachers have had. He continues by arguing that practical knowledge is in fact context-dependent and that the study of cases is "well suited" (p.223) to provide this type of knowledge. Flyvbjerg's (2006) differentiation between theoretical and practical knowledge is acknowledged, as is his view of the case being context dependent.

5.4.2 Generalisation and the Contribution to Research

Generalisation, according to Richards (2003), contributes to understanding the nature of the problem, "which is different from attempting to classify or to justify the outcomes" (p. 216). While Flyvbjerg (2006) indicates that research that uses cases is not generalizable, Eisenhardt (1989) argues that "a good cross-case analysis is counteracting these tendencies by looking at the data in many divergent ways, one tactic is to select categories or dimensions, and then to look for within-group similarities coupled with intergroup differences. Dimensions can be suggested by the research problem or by existing literature, or the researcher can simply choose some dimensions" (p.541). In this current study, the goal was to reveal the PCK of the university calculus teacher participants in the selected cases, by analysing the categories and dimensions of PCK that were suggested by the research problem. The goal was to examine existing literature, develop a theoretical framework, and provide a study of cases.

5.4.3 Verification and its Association with Researcher Bias in Studying Cases

According to Van Wynsberghe and Khan (2007) researchers are generally biased, based upon their own preconceived notions towards confirmation rather than falsification of the case. Van Wynsberghe and Khan's argument comes down to the nature of selectivity. They suggest that researchers interpret data (specifically qualitative data) in a particular way, and that researchers also tend to recall certain themes and justifications from memory which may not accurately portray the overall findings of the research.

What Flyvbjerg (2006) and Van Wynsberghe and Khan (2007) are highlighting is, ultimately the notion of researcher bias. This is not something unique to the study of cases, though both articles seem to highlight this as a particular weakness. Researcher bias happens at all levels, it occurs as a result of the types of questions asked in a quantitative questionnaire and can continue into qualitative aspects of the study through the interpretations of observations. Steps need to be taken to minimise this bias. In this study, this occurred through the use of a pilot study, (see Section 5.10) through careful consideration of the instruments (along with my PhD supervisor) (see Section 5.9) and through methodical analysis of the data collected (see Section 5.12).

5.4.4 Limitations of the Study of Cases

The study of cases is not without limitations. The purpose in the identification of such limitations is that by addressing the issues, justification for this type of approach can be made. According to Flyvbjerg (2006), there are five topics worthy of identification and debate in examining the study of cases. It is essential, considering all of these elements, that these limitations are addressed within this study. While Table 5-1 below outlines the limitations, it is clear that for many of those identified, a solution, or at least a mitigation of the limitation, needed to be achieved.

Limitations	Mitigations applied in this study
(Flyvbjerg, 2006, p.219).	
Theoretical knowledge can be more valuable for teachers to possess than practical knowledge. Cases typically focus on practical knowledge.	In this study, the researcher is particularly interested with how practical knowledge and content knowledge are intertwined, and so, this limitation can only be further justification that the study of cases is suitable for this current study.
It is impossible to generalise from a single case. As a result, making a contribution to research can be challenging.	The use of several cases, alongside a cross-case analysis, is a suitable means of ensuring the study makes a contribution to research.
The case study approach is not valuable when attempting to test and build a theory; it is much more appropriate for generating hypotheses.	This research does not intend to test or build theory in the 'grand' sense but rather to propose and use a theoretical framework for the specific topic.
There is bias in the case approach, especially towards verification.	By using methodological triangulation, this study is able to balance the weaknesses of one research method with the strengths of another. Theory triangulation is also employed.
The ability for a researcher to accurately summarise a case can be particularly challenging.	By developing and using a theoretical framework, data collected within it allows for accurate analysis and summarisation of the cases.

Table 5-1: Limitations and Mitigations of Studying Cases

Considering all of these limitations, but also the positive aspects of studying cases, the choice to employ this approach was warranted and appropriate within the context of this current research study. Through the use of this approach, the researcher has conducted research that adheres to the research questions provided at the beginning of this chapter and to look at the intrinsic nature of the case under examination (Richards, 2003).

5.5 Section Summary

A number of researchers (e.g. Sowder et al., 1998; Holton, 2001) have used the study of cases with PCK in different subjects in higher education. In line with this tradition, the study of cases was considered suitable, with the primary influence being that of Sowder et al. (1998) who studied cases in their mathematics research study. It is acknowledged that bias exists, as it would for any study, and that limitations are present. These limitations have been addressed, and while it is not possible to entirely mitigate all of these limitations, the researcher has provided an overview of how the limitations were addressed within this context. This research answers the research questions while still falling within time and budgetary constraints held by the researcher.

5.6 Triangulation

By recognising that the data collection process is often ‘messy’ and complex, Hartley (2004) suggests that researchers studying cases employ triangulation in order to better facilitate research validation. Triangulation can help to avoid errors and bring findings closer to the ‘truth’ (Lincoln et al., 2011). This is particularly important in this research study as it involves subjective views as well as observations. Greene et al. (1989) provide a detailed justification for the use of triangulation in research that combines qualitative and quantitative research methods. For them, the process of triangulation is one of "convergence, corroboration, correspondence" and during the coding of data, there is a strong emphasis on identifying points of corroboration between qualitative and quantitative data (p.105). Carvalho and White (1997) propose four reasons for undertaking triangulation:

- *Enriching:* the outputs of different informal and formal instruments add value to each other by explaining different aspects of an issue.
- *Refuting:* where one set of options disproves a hypothesis generated by another set of options.
- *Confirming:* where one set of options confirms a hypothesis generated by another set of options.
- *Explaining:* where one set of options sheds light on unexpected findings derived from another set of options.

Denzin (1973, p.301) proposes four basic types of triangulation:

- *Data triangulation:* involves time, space, and persons
- *Investigator triangulation:* involves multiple researchers in an investigation
- *Theory triangulation:* involves using more than one theoretical scheme in the interpretation of the phenomenon
- *Methodological triangulation:* involves using more than one option to gather data, such as interviews, observations, questionnaires, and documents.

Triangulation is, essentially, the “combination of two or more data sources, investigators, methodological approaches, theoretical perspectives or analytical methods” (Thurmond, 2001, p. 253). Triangulation offers researchers the opportunity to approach the data from multidimensional perspectives, which in turn increases external and internal validity, as well as reliability within the project (Boyd, 2000). Within this research, methodological triangulation was employed as an essential component. Under this assumption, researchers that employ triangulation, through the use of multiple methods (e.g. surveys, interviews, observations) that produce similar results, would likely be able to demonstrate that appropriate measures have been implemented (Moran-Ellis et al., 2006). This notion has also been referred to in the literature as ‘convergence’ (Matheson, 1988); but along the same lines, if convergence fails to appear the results may end up being contradictory or inconsistent, suggesting flaws in the research instruments (Matheson, 1988).

Methodological triangulation, also known within the literature as method triangulation, mixed method triangulation, or multimethod triangulation is used to limit possible biases generated by only using a single method (Barbour, 1998; Greene & Caracelli, 1997). It is preferred in the field of education because it allows for the cross-verification of data (Lincoln & Guba, 2000). By using methodological triangulation, this study was able to balance the weaknesses of one research method with the strengths of another. Furthermore, this research used theory triangulation, which involved more than one theoretical scheme in the interpretation of the phenomenon (see Chapter 3 where a number of theory approaches were combined to develop the theoretical framework used in this study). Table 5-2 below shows this study's research questions and how they relate to the methodological and theoretical framework.

Research Questions	Theoretical framework	Data Collection	Methods
RQ1: What would be a model of PCK for teaching calculus?	Model of teacher knowledge (Lesseig, 2016; Khakbaz, 2016; COACTIV, 2004; TEDS-M, 2008)	Theory approaches are combined to develop the theoretical framework	Theory triangulation, which involved more than one theoretical scheme in the interpretation of the phenomenon
RQ2: Using this model of PCK, how do calculus teachers articulate and demonstrate their PCK?	The proposed model of PCK for teaching calculus	Quantitative Qualitative	Survey Interview Observation

Table 5-2: Research Questions and Relationship to the Methodological and Theoretical Framework

When planning a piece of multi-stage research, several questions must be asked (Bryman, 2006). For example, an early question asked in this research was whether quantitative and qualitative data should be collected simultaneously or sequentially and whether qualitative or qualitative data has priority (Morgan, 1998; Morse, 1991). As a research study using both qualitative and quantitative data collection, no one form of data was more important than the other and triangulation was ensured at an early stage to ensure valid and reliable results (Tashakkori & Teddlie, 1998; Creswell, 2003; Creswell et al., 2003; Greene et al., 1989). This approach to multi-stage research occurred at the research question stage and was embedded throughout the research process.

5.7 Multiple Methods

In his definition, Creswell (2003) talks about underlying philosophical assumptions that guide the collection and analysis of data and the "collection, analysis and mixing of data", the central premise being "... that the use of quantitative and qualitative approaches in combination provides a better understanding of research problems than either approach alone" (p. 5).

5.7.1 Rationale for Multiple Methods of Quantitative and Qualitative Research

Teachers' PCK is a challenging and multifaceted field (Magnusson et al., 1999) and it is, therefore, difficult to capture it by using one method alone. It was decided that using a single method of data collection would not meet the aims and objectives of this research. Many studies have used multiple approaches together to investigate teachers' PCK (e.g. Duling, 1992; Baker & Chick, 2006; Ijeh, 2012; Nordin et al., 2013; Gall et al., 2003). The use of multiple approaches, over many years, suggests that using multiple approaches is consistently considered to be appropriate within the field of education.

This study used both qualitative and quantitative methods as the researcher acknowledges that "all forms of measurement can be imperfect" (Harrison et al., 2017, p.5) and, therefore prone to errors, which may invalidate the findings of a piece of research. To address this, this study used triangulation in the form of observations, interviews and survey, to avoid any potential errors in the aim of understanding what is happening and reach as close to the 'truth' as possible (Lincoln, Lynham & Guba, 2011).

In order to give a clear perspective to the overall research, using quantitative and qualitative methods need to be considered together. This comparison of data, as a whole, offers a more detailed approach than analysing individual components. In this study, this was achieved through survey (quantitative), observation (qualitative) and interview (qualitative) design. While quantitative data offers the 'hard' data, according to multiple researchers (e.g. Denzin & Lincoln, 1994; Domegan & Flemming, 2007; Myers, 2009), the use of qualitative methods helps to explore and discover issues about the problem at hand and is designed to help researchers understand people, and the social and cultural contexts within which they live and/or work. This is particularly pertinent to this study, as the examination of calculus teachers, within the context of the university setting, required both an understanding of the culture surrounding pedagogy as well as aspects of pedagogical knowledge. This group of teachers falls within a specific community of practice, where their knowledge, experience, and training all contribute to their setting. As such, a multiple methods approach was deemed to be most suitable for this research study.

This study aims to identify calculus teachers' PCK and how they use their knowledge of PCK in practice, together with the factors that influence their practical decisions in the classroom, which influenced the decision to use multiple methods within a studying cases methodology. Punch (1998) suggests that the researcher "... should choose a method that is appropriate to what you are trying to find out" (p. 244). The research questions, the focus of the phenomenon being studied, the philosophical approach or paradigm were all involved in directing the research methods (Punch, 1998). Table 5-3 below presents, in detail, the connection between the theoretical framework and the data collecting methods. Each selected method of data collection is explained in detail in the following sections.

First Level categories	**Second Level Categories (Lesseig, 2016; Khakbaz, 2016; COACTIV, 2004; TEDS-M, 2008)**	**Data collecting methods**
PCK - Learners' cognition of calculus	Students' misconceptions and learning difficulties in calculus.	Survey, Interview, Observation
	Knowledge of students' thinking about calculus concepts.	Survey, Interview, Observation
PCK- Developmental aspects of the calculus curriculum.	Knowledge of aims for teaching calculus.	Survey, Interview, Observation
	Knowledge of sequencing of building blocks of mathematical theories.	Survey, Interview, Observation
PCK - Knowledge of instructional strategies	Relationship between instruction and students' ideas in calculus.	Survey, Interview, Observation
	Questioning strategies in calculus.	
	Use of pivotal examples or Counter-examples in calculus.	
	Mathematical representation in calculus.	
PCK - Knowledge of calculus connections.	Real- world applications of calculus.	Survey, Interview, Observation
	Calculus in academic subjects.	

Table 5-3: Connecting the Theoretical Framework and the Data Collecting Methods

5.8 Selection of the University Course

As the researcher had a prior relationship with the mathematics faculty in the college at University X, so the location of the study was confirmed, and the researcher decided to choose the calculus course. Since this was the only first year calculus course offered at University X, it was deemed to be the only option available to the researcher. Yet, despite being the only option, it provided clear and consistent messaging through the syllabus to both the teachers and the students. It also allowed for data to be compared across the four participants, who teach mathematics students only. The researcher's relationship was specifically with the administrator overseeing programme development. Based on previous communications with him, along with the researcher’s own investigation into the syllabi of other first-year calculus courses across different universities in the KSA, these topics seemed to generally be consistent with those which are taught at other institutions. This was not necessarily important at this stage of the research process, as with a small-scale study of multiple cases generalisability is not the aim. However, in future research projects, this information may have the potential to be useful. Table 5-4 shows the topics of the calculus lectures selected for observation.

Proposed Model / Lecture	Learners' cognition of calculus	Developmental aspects of calculus curriculum	instructional strategies	Knowledge of calculus connections
Functions	√	√	√	
Limit	√	√	√	
Continuity	√	√	√	
Derivatives	√	√	√	
Following the Differentiation Rules	√	√	√	
Applications of Differentiation	√	√	√	√
Integrals	√	√	√	
Applications of Integration	√	√	√	√

Table 5-4: Topics of the Calculus Lectures Selected for Observation

5.9 Methods

One of the main aspects of data collection that is an essential component of any research study are ethical issues and considerations associated with each type of approach. Within this current research, both the pilot study and the main study required the researcher to have informal conversations with both administrators (i.e. through the organisation of the observations) and with the participants (i.e. for scheduling the observations, interviews, and for completion of the survey). The researcher applied for ethical approval for his research via the University of Southampton Ethics and Research Governance Online (ERGO), see Appendix D and F, and section 5.15 outlines the ethical considerations associated with this research.

All three data collecting instruments were piloted, prior to their final use. Conducting a pilot study is an essential component of the research process as it provides further justification for the instruments used and ensures that they meet the needs of the research process (Hazzi & Maldaon, 2015). By involving both the Research Supervisor and PhD peers within this process, the survey, observation, and interview were evaluated to a higher level of expectation. In terms of the survey, the question types were modified to include questions that were easy to understand and appropriate to both the study and the level of the teachers. It was also ensured that ambiguous questions were avoided, and that Arabic was seen as the best language to conduct the study in. Further, for the interview piloting assured that the grammar, question type, and word choice was most appropriate for the situation and that the researcher was well prepared to answer the questions that might be posed by the teachers. The observation schedule was revised to better suit what the researcher was able to do within the classroom and these results were further discussed with the PhD supervisor. It is recognised that no instrument will ever end up being perfect and that limitations will always need to be considered. However, through the use of a pilot study, these challenges were able to be minimized and addressed prior to the main study. As a result of this pilot study, the researcher was able to proceed with the final study from April to July 2017. A thorough and detailed report of the pilot study can be found in Appendix E.

5.9.1 Overview of Survey

As there may not be a relationship between what a teacher knows and what they score in a test, quantitative findings may not present a complete picture. Researchers (e.g. Ball, 1990; Even, 1990; Leinhardt & Smith, 1985; Shulman, 1986; Wilson et al., 1987) suggest that a more qualitative approach, focusing more on how knowledge is organised and whether it can relate to what the teacher knows about the discipline may be beneficial. Without some sort of quantitative component, however, comparisons are inevitably difficult. As this research only comprised four cases, devising an instrument that would allow for the delivery of a level of feelings from the participants was considered necessary in order to support the rich data obtained from the interviews and observations conducted in this study.

Survey is a widely used instrument for researchers when collecting data (McNeill, 1985). It is prepared by developing a form of statements, which may relate to attitudes, facts or beliefs (Ary et al., 1979) and presented to a target group or a sample of the population. Kumar (2011) defines a questionnaire as “a written list of questions, the answers to which are recorded by respondents” (p.145). Using a survey technique in this research study, allowed the researcher to complete a form of triangulation. Using observations provided the actuality of what the teacher did, interviews allowed the teacher to say what they did and the survey provided the teacher the opportunity to indicate what they thought they did.

One of the major components of survey design is the actual phrasing of the statements. Decisions on language can have a significant impact on the data that ends up being collected (Fowler & Cosenza, 2009). It is essential to review and evaluate the design of survey research in order to assure that the statements being created actually relate to the topic being discussed and the research questions posed. Fowler and Cosenza (2009, p.376) identify what constitutes a “good” question and outlines four characteristics, which can equally be applied to survey statements as they:

1. must be clearly understood consistently (i.e. by all participants);
2. must allow for answering/reporting in an appropriate way;
3. be such that the participants have the information needed and/or required to actually be able to respond;
4. be such that the participants are willing to respond.

5.9.1.1 Discussion of Survey Procedure

As a first step to answering the research questions, and based on the research objectives, the researcher decided to choose a survey-based method for exploring the teachers’ background. This included many steps, including starting to review literature (e.g. Cohen et al., 2013) on designing the survey. Furthermore, the researcher reviewed many studies (e.g. Eley, 2006; Ijeh, 2012; Melibari, 2015; Ng, 1995; Sulaiman, 2011; Thomas, 2012) that employed questionnaire to investigate PCK, teaching mathematics, and the nature of mathematics, and used them to inform the survey statement development for the present study.

The survey form was designed to obtain certain demographic information, and to obtain the teachers' judgements regarding PCK in relation to their classroom practices within calculus teaching, focusing on both their own understanding as well as how they felt about the challenges faced by their students. The researcher attempted to target the planning stage (the aims of teaching), the implementation stage, and the reflection stage through the survey statements, while also addressing the relationship between this information and students' prior knowledge. The survey targeted their knowledge of calculus teaching, instructional skills/strategies, curriculum and, finally, student difficulties and assessment, using a five-point attitude scale (see Appendix I).

In terms of demographic questions, the researcher was primarily interested in the pedagogical and professional backgrounds of each teacher, along with their world experiences. This section began with simple questions about the participants' university experience and qualifications. This simple beginning served two purposes: first, it was designed to provide the researcher with a baseline for the explanation/results phase of the project. Second, it offered an easy transition into other instruments, while attempting to build rapport and trust, something McNamara (2009) highlights as essential. After establishing this baseline, questions in the demographic section covered teaching or travelling abroad. While not necessarily relevant for this project, these questions provided insights into the line of thinking for each teacher and benefitted the researcher moving forward. The researcher met the participants and gave them the information sheet and consent form for the study (see Appendix I {I.2, I.3, I.4, I.5}) and explained the stages of the data collection, then distributed the survey.

5.9.2 Overview of Observation

Observation allows the researcher to obtain most of the information from their surrounding through looking. Simpson and Tuson (1995) argue that observation is to be the main method and most versatile way of gathering data in a small-scale study on the one hand, on the other hand it can be complex and difficult to undertake. In this study, observations of teaching were used as the main research tool to gain insight into aspects of the PCK of university teachers, in order to examine the extent to which they demonstrate an understanding of the pedagogy that is associated with teaching a particular group of students within a particular set of circumstances.

5.9.2.1 Justification for Using Observation

Ritchie's et al., (2013, p.245) four main components were used to justify the use of observation in this research study. In Table 5-5, Column 1 presents the four components of Ritchie et al., and Column 2 the applicability of these components in this study.

The justification for the use of observations relates to four main components identified by Ritchie et al., (2013, p.245)	**The applicability of Ritchie's et al., components (2013) in this study.**
1. Studies that examine the way people interact with an environment or other physical context.	The participants should be able to demonstrate their content knowledge and, to some extent, describe the pedagogical approaches underpinning their work; however, the ability to link the two together is generally classified as a complex process.
2. Studies that have complex processes or interactions that are difficult to describe accurately.	When addressing complex mathematical issues and student responses to these issues, participants are likely to undertake instinctive actions in the classroom that they are not aware of; the process of classroom observation may help to account for these phenomena.
3. Studies that need the interpretation of instinctive actions or subconscious behaviours, where participants might not be aware of their own behaviours.	To appropriately assess both the pedagogical and content components of teachers, observing the interaction between teachers and students in the lecture environment will be essential.
4. Studies that focus on social norms or pressures to conform to certain behaviours (for example, situations in which participants cannot or may not be willing to verbalise an accurate description of their behaviours)	With pedagogical understanding, there is often a need to be seen as "innovative" or to conduct a calculus lesson in a certain way. When expressing their own interpretations, teachers may feel pressured to suggest that they act or teach in a certain way, though this may not be the case in practice; therefore, observing actual lessons is an appropriate way to address this.

Table 5-5: Justification for Using Observation (source: Ritchie et al., 2013)

5.9.2.2 Discussion of Observation Procedure

A researcher who chooses to study cases must be sensitive to both opportunistic and planned data collection (Hartley, 2004). This is particularly true within the context of this research, as the observations allowed for unplanned situations which could produce results not previously discussed in the literature. This research project used longitudinal data collecting within the context of each case. This choice was made in order to ensure that when the observations were conducted, the teachers were given multiple opportunities to demonstrate their PCK in the classroom, as it was recognised that some concepts and/or lessons are easier to teach than others, and the researcher wanted to provide the teachers with multiple opportunities to demonstrate the relationship between knowledge and practice.

Four participants were observed in this study, with each being observed for one lesson per week over the course of eight weeks, totalling 32 total observations. Participants were observed over eight separate sessions in order to ensure that an accurate representation of their teaching methods and practices were obtained. Teachers have certain strengths in different areas; by using a classroom observation sheet and video-recording the calculus lessons over eight weeks, participants were given the opportunities to demonstrate PCK over a wider range of topics. This also mitigated the notion of 'having a bad day', as the participants had multiple opportunities to teach.

In the KSA at university level, first-year calculus classes are rather large, and auditoriums generally hold up to 100 students. Prior to the observations taking place, the researcher obtained consent from each teacher to record the classes. After approval has been obtained from both the head of the maths department and participant teachers, the students were notified by email, prior to the commencement of the study, about the use of video-recording in four of their calculus classes.

Teachers were notified in advance which classes were to be recorded, and it was the same class each week. The researcher relied mainly on the video recordings and the classroom observation sheets (see Appendix J). As the students were not part of this research project, the video camera was not directed towards the students. In using both recording methods, the researcher used different techniques for capturing what happened in each case and what each case was about. Given that each session lasted two hours, the important consideration at this stage of the project was that when all these data were obtained the researcher could concentrate on those points that would help answer the research questions. The main reason for using classroom observation was that it would allow the researcher to keep thinking in terms of, and looking through, the lens of PCK. In this way it was considered that other interesting data could be revealed.

The researcher chose to video-record the lectures for several reasons. First, due to the nature of the subject material, the participants would often work on blackboards/whiteboards. Second, the researcher wanted to consider aspects of instructional strategies (for example, interactive instruction, gestures, movement around the classroom, etc.) as aspects of a pedagogical theory that suggests that non-verbal elements also have an effect on delivery and comprehension (Power, 1998). In addition, the researcher focused on the building blocks of mathematical theories, such as axioms, definitions, theorems, proofs and diagrams.

5.9.2.3 Using an Observation Schedule

While the researcher video recorded the classroom lessons, there was still the need to be methodical in what aspects of the lecture contributed to PCK, especially when considering the different teaching styles, the teachers may present and the way that information may be communicated. As the researcher lacked experience of using observation within qualitative research, using a structured observation schedule offered an added level of support to ensure that the research questions were addressed, and the data collected remained focused.

Four areas were identified as being particularly important to the field of PCK (see proposed model Figure: 4-6, p.71). Once the overall themes were identified, the researcher utilised aspects of previous observation methods designed by a number of researchers (e.g. Wragg, 1999; Ijeh, 2012; Henze & Van Driel, 2015) to fill in the gaps. Then, after adding calculus concepts (Alcock, 2014) to develop the observation schedule, the researcher investigated different versions by applying three different observation schedules to existing video recordings of the teaching of calculus 1 that are available on YouTube. This was done so that the analysis would best address the research questions posed. As a result of this pilot study, the researcher produced a new observation schedule and outline for the lecture (Appendix J, J.1). The researcher met the participants and gave them the information sheet and consent form for observation (see Appendix, J.1, J.2, J.3, J.4) and explained the stages of the data collection, then selected the lectures which the researcher would attend. In some instances, aspects of the observation were followed up with a brief interview, for example, with the participating teacher (see Appendix K.7).

5.9.2.4 Sampling Strategy for the Observations

The sampling strategy was purposive, and the type being critical case sampling, the definition of which is the process of selecting a small number of important cases - cases that are likely to "yield the most information and have the greatest impact on the development of knowledge" (Patton, 2002, p. 236). Also, critical case sampling is:

> *Extremely popular in the initial stages of research to determine whether or not a more in-depth study is warranted or where funds are limited, critical case sampling is a method where a select number of important or "critical" cases are selected and then examined. The criterion for deciding whether or not an example is "critical" is generally decided using the following statements: "If it happens there, will it happen anywhere?" or "if that group is having problems, then can we be sure all the groups are having problems?* (Etikan et al., 2016, p.3).

The reason for using critical case sampling is that, although sampling for one or more critical cases may not yield findings that are broadly generalisable, they may allow researchers to develop logical generalisations from the rich evidence produced when studying a few cases in depth. To identify critical cases, the researcher needed to be able to identify the dimensions that make the cases critical. In this study, each case was a university calculus teacher. In this study the PCK of teachers teaching calculus is the tightly framed framework by which this research is bounded i.e. university level, particular branch (mathematics department) of a particular subject (calculus).

5.9.2.5 Reasons for Selecting Three Observations for Each Participant

Observing teachers can be seen as a sensitive process. Conducting eight lesson observations for each participant had a number of effects, including:

- the presence of the observer became normalised to the participant and the students;

- the participants did not know which observation was to be analysed;
- the participant's behaviour in their teaching practice was not atypical of their normal classroom practice i.e. they did not do a 'special lesson'. There was no guarantee that the teacher would not make a special effort during each observation, but it reduced this possibility;
- the observer, as well the camera, became a familiar sight in the classroom.

The researcher took the list of lectures and divided the lectures according to the elements of the proposed model of this study into two groups that contained similar characteristics. The researcher followed the process of systematic sampling, which according to Cohen and Manion (1994) is “a modified form of simple random sampling” (p. 87). The researcher applied this process to the observed lectures as a population, rather than the participants. All the lectures were listed in their order of 1 to 8 on pieces of paper. Then these were divided into two groups for systematic sampling, 1,2,3,4,5,7 in bowl 1 and 6 and 8 in bowl 2. Two numbers were picked out of bowl 1 and one out of bowl 2. Figure 5-1 illustrates the method of selecting three observations for each participant. It is acknowledged that there are advantages and disadvantages to this approach. The advantages were that the observations were not subject to additional researcher bias and provided a fair representation of the material. The disadvantages were that it was possible that some lessons would not necessarily be representative of what actually happened across different lectures by a participant. As this disadvantage could not be prevented, even if lessons were specifically selected, the method employed for selection was seen as the most appropriate approach.

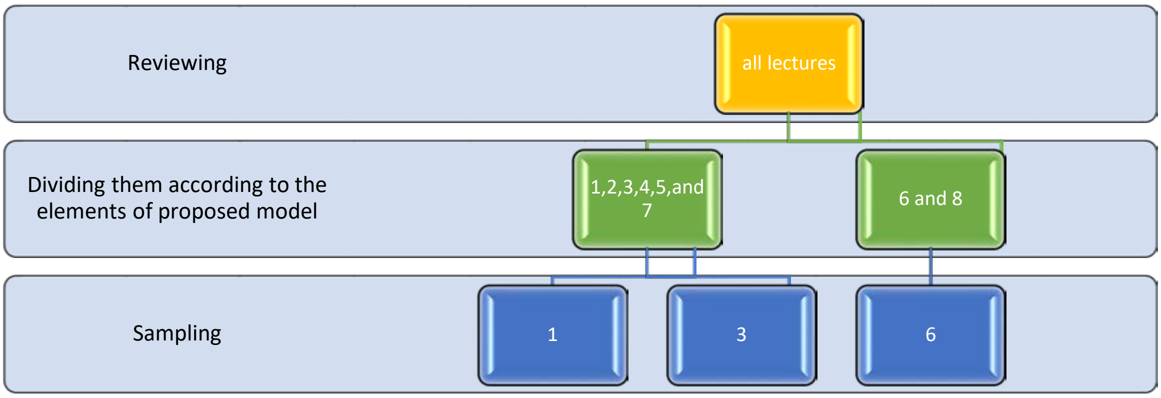

Figure 5-1: The Process for Selecting Observed Lectures for Analysis

5.9.3 Overview of Interview

An interview is a conversation that has a structure and/or a purpose and is led by one party: the interviewer. In the current research, to meet the research objectives, it was necessary to interview participants to gain their insights and perspectives (Cohen et al., 2013). Mouly (1978) defines an interview as "a conversation carried out with the definite purpose of obtaining certain information" (p.201). Asking about the interviewee's everyday life can reveal powerful insights into the life and understanding of the participant (Kvale, 2008). Nevertheless, despite their powerful nature, interviews require various other sources of data to reinforce their accuracy (Silverman, 2015), such as observation.

An interview provides an opportunity for the researcher to gain insights into their participants' perspectives and experiences (Cohen et al., 2011). Interviews are also often used for follow-up data collection, should a topic of interest arise that requires greater illumination, or triangulation with other methodologies to improve reliability (Denscombe, 2003). Therefore, the calculus teachers' PCK was explored using interviews to provide qualitative data. Triandis and Berry (1980) suggest that "a research interview is a two-person conversation; it is initiated by the interviewer for the purpose of obtaining research-relevant information and focused by him on the content specified by research objectives of systematic description, prediction, or explanation" (p.142). Using interviews in this study was considered advantageous, as they would generate comparative data while permitting greater freedom than the use of a survey.

Face-to-face individual interviews were used, as they typically result in responses that are more accurate than if an interview is conducted by another means (for example, a telephone interview) (Silverman, 2010). Furthermore, due to the detailed nature of the topic being discussed, an individual interview was deemed appropriate as opposed to, for example, a group interview. Codo (2008) suggests that individual interviews provide a more personal environment and are generally linked to obtaining more accurate responses. While it is acknowledged that group interviews are also useful for the exchange of ideas (Codo, 2008), in this research, the detailed nature of the topic, as well as the busy schedule of the participants, meant that face-to-face interviews were considered the most appropriate.

Face-to-face interviews are particularly useful because they allow the researcher to gain access to an interviewee's insights into human affairs and behavioural events, which are essential to studying case evidence (Yin, 2003). Furthermore, non-verbal communication can be more easily observed during face-to-face interviews. In order to ensure the participants provide honest, in-depth responses, the interviewer is required to quickly build a good rapport with the interviewee (Robson, 2007) as "People tend to enjoy the rather rare chance to talk about their ideas at length to a person whose purpose is to listen and note the ideas without being critical" (Denscombe, 2003, p. 190). Finally, non-verbal communication can also be observed during a face-to-face interview.

Despite the many positives associated with interviews, there are also negatives. Yin (2003) suggests that interviews should be taken as verbal reports only and that, when reporting such events, it is essential to note that interviews are subject to the "common problems of bias, poor recall and poor or inaccurate articulation" (p.109). In addition, the interviewer can be biased with their questions or the topics they choose to pursue (Mouley, 1978; Cohen et al., 2011). It is also unlikely that two interviewers obtain the same findings, as "interviewers are human beings and not machines, and their manner may have an effect on respondents" (Bell, 2005, p. 166). Researchers typically have a research question and will be more likely to pursue topics that validate this question, as opposed to topics that may call it into question (Borg, 1981). Interestingly, participants can also be biased in their responses; rather than being honest, they may try to answer in a manner they think the interviewer desires. A further limitation of interviews is the fact that they are highly time-consuming; they take time to arrange and conduct, and post-interview transcription and analysis is a slow process (Drever, 2003). Interview participants also need to be well informed of the ethical implications of the interview, including informed consent, confidentiality and the intention to do no harm. It was necessary to record the interviews, because taking notes while participants are speaking is inherently subjective and it is impossible to record every word, phrase and gesture participants produce, thus making a non-recorded interview less reliable. However, in this case, participants were encouraged to confirm their interview notes after the event, thus minimising this limitation.

A semi-structured interview was deemed appropriate for this study because the advantages are greater than the limitations. This enabled the researcher to delve deeper and obtain greater insights from the participants, thus obtaining richer data.

5.9.3.1 Selecting the Interview Type

A semi-structured interview is created in such a way that flexibility is allowed within the interview process (Silverman, 2015). In a semi-structured interview, a few essential questions are designed for the researcher to follow, although leniency is permitted when steering the conversation toward topics that arise as a result of the conversation with the interviewee. A semi-structured approach was selected for this research study for a variety of different reasons. First, as this study is exploratory in nature, semi-structured interviews allowed for topics to be discussed before further exploration of topics that were not initially identified by the survey. Second, semi-structured interviews allowed the researcher to prepare some key questions allowing the interview to flow, but still maintain a focus on the topic of PCK in university calculus. If an unstructured interview had been selected, the topic or research questions may not have been fully addressed. The semi-structured interview format selected allowed for greater flexibility than a structured interview (Burns, 2000). However, flexibility can also present limitations, as participants may spend time discussing something tangential and unrelated to the interview topic (Burns, 2000). The interviewer will initially have a set of open-ended questions to form some basic structure; however, they are then free to ask supplementary questions according to participant responses (Burns, 2000; Descombe, 2003; Bryman, 2004). If required, the initial question structure can therefore be reordered during the interview, or questions can be deleted if needed (Bryman, 2004; Robson, 2002).

Semi-structured interviews were used to explore how the participating teachers articulated their PCK on the particular topics they had taught. The interviews also focused on how teachers' educational background facilitated their teaching of calculus (Jong et al., 2005; Rollnick et al., 2008). This interview schedule was initially piloted (see Appendix E.2) before the final version was used. A mathematics expert and two education specialists evaluated the stipulated questions to identify whether sufficient information could be obtained to comprehensively understand the extent to which the participants' PCK background has facilitated teaching calculus. These experts concluded that the interview schedule was appropriate. While interviews are semi-structured, meaning they contain specific questions for all participants, they also enabled the interviewer to pursue topics of interest in greater depth (Bell, 2005) and thus obtain explanations that could not be gathered during an observation.

5.9.3.2 Designing the Interview Schedule

In designing the interview schedule, the researcher reviewed many studies (e.g. Sowder et al.,1998; Sulaiman, 2011; Ijeh, 2012; Henze & Van Driel, 2015) which were also based on theoretical frameworks and employed interviews to investigate PCK and teaching mathematics.

Three main areas required consideration, these included asking questions on knowledge of teaching calculus, on knowledge of student understanding within calculus, and the sequencing of building blocks of mathematical theories in calculus. These areas were identified as essential in relation to the literature and theoretical framework, but also in support of the surveys conducted prior to the interviews.

The remaining parts of the interview included asking participants about their own teaching experiences and a variety of questions surrounding pedagogy, content and administrative matters. While the purpose of this study is to examine PCK, it was important for the researcher to gain a thorough understanding of the participants as people (Kvale, 2008), as people are multifaceted. For example, it was important to determine whether teachers were selecting their own textbooks and course materials since a lack of consistency in this area may indicate that while some teachers are using what they feel to be the best possible resources for their students, other teachers may be hindered by administrative bodies (i.e., their topics, texts and assignments may be imposed from above). These factors may play a role in the pedagogical development of the participants. In addition, the interview method was designed to further clarify the pedagogical knowledge of the participants by investigating their knowledge of the "typical" learner in their university classes. This section refers to students' levels of success and areas of difficulty. The interview also posed questions about the understanding of assessment strategies, which are an essential component of PCK (see Appendix K.1 for the interview schedule and K.2 for this in Arabic).

5.9.3.3 Conducting the Interviews

Much like other research methods, interviews require significant preparation before their actual delivery. McNamara (2009) mentions eight principles in the preparation stage of an interview: selecting a suitable environment, explaining aim, format and confidentiality, pointing to the length of time, asking for questions before starting, giving the researcher's contact details and utilising some kind of video/audio-recording equipment. In this study, all the participants were calculus teachers from University X. The researcher provided the Dean of Faculty of University X, and the head of the mathematics department with an information sheet and/or letter of research introduction with details of the study and what involvement in the study requires from the participants. The researcher met the participants and gave them the information sheet and consent form for the study (see Appendix, K.3, K.4, K.5, K.6) and explained the stages of the data collection, then booked a room for the interviews.

5.10 Pilot Study

In order to answer the research questions, it is essential to examine some of the main components of this research project that may influence the findings. Within research, some aspects are pre-testable. For example, a pilot study can test survey instruments, interview schedules and observation protocols, with the aim of striving for a high level of accuracy, especially in the terms of clarity (Creswell, 2014). Creating and implementing valuable instruments in order to obtain useful and helpful data requires a multiple step process which requires several attempts in order to provide the clearest instruments possible. This section outlines the benefits of a pilot study before discussing the specifics of the data collection and the changes made.

Pilot studies, also known as feasibility studies, exist for the primary purpose of pre-testing a particular instrument which will be used in the research. It is typically a common step in the research process when using surveys, observations and interviews (Van Teijlingen & Hundley, 2002). The advantages of conducting a pilot study include knowing the weaknesses of a current instrument and where it might fail, whether or not research protocols might be followed, and to determine the appropriateness of the instruments (De Vaus, 1993). Pilot studies can be used in both quantitative and qualitative research and should be completed enough in advance of the main study so that changes can be made to the instruments in a timely but organised manner.

The pilot study for this research was completed in December 2016 through to January 2017 and spanned 23 days. During this period the researcher returned to KSA in order to use face-to-face meetings in order to ensure that each step in the pilot process was clear and that follow up questions could be asked, if necessary. Before leaving to conduct the pilot study, the researcher had already completed a working draft of the instruments, which had been seen by the PhD supervisor. Once discussions around these instruments had occurred, the researcher contacted two participants and asked them to participate in the pilot study process. The participants are calculus teachers at the university level, but they teach the foundation year for engineering students. The research project was explained to the participants prior to the researcher's arrival in the KSA (this was completed by email) and they were given copies of the Participant Information Sheet (see Appendix I.2, I.4, J.1, J.3, K.3, K.5) as a general guideline.

With respect to the participants, both were seasoned mathematics teachers. Pilot Participant 1 is someone who I had observed in the past. He had previously taught calculus and spoke passionately about the students in his course. He was considered to be a mid-career mathematician, being between 36-50 in age and with 8 years of experience in mathematics teaching. He had completed undergraduate, graduate and PhD level study in mathematics and was currently employed as an assistant professor. Pilot Participant 2 is also a mid-career professor with 13 years of experience in mathematics teaching - 6 of those years as a teacher in the KSA. He is between 36-50 years of age and currently employed as an associate professor having completed undergraduate, postgraduate, and PhD level study in mathematics. I selected Pilot Participant 2 for the pilot study because I was familiar with his teaching philosophies and I had previously observed his classes. For more detail on the pilot study (see Appendix E), which includes any changes that were made to the data collecting instruments.

5.10.1 Amendments and Considerations Following the Pilot Study

It is noteworthy that the pilot study gave the instruments of this study more accuracy and clarity. Summary of amendments:

In the survey, adding the demographic information as the first part; there were several changes for example, some statements were unclear in meaning; changing the survey questions in more than two directions to discriminate significantly between subjects, and to elicit valuable information about PCK.

In the observation schedule, the researcher began with three different observation schedules, as it was unclear which one would be most appropriate, and the researcher aimed to use different observation schedules in order to design a final appropriate observation schedule. It was hoped that by collecting the data and then analysing it according to the three different schedules, that the outcomes would best represent the research questions posed as a result of the pilot stud. The researcher produced a new observation schedule and outline for the lectures (Appendix J), informed a little by version 2 and 3 and much by version 1.

The interview was very long, and many of the interesting questions could lead to lengthy responses. Therefore, it was be better to record the demographic information beforehand, by a questionnaire rather than as part of the interview and reduce the final interview schedule from five parts to three parts.

5.11 Initial Participant Identification and Sampling

Participants were selected through purposive sampling. Purposive sampling occurs when the participants are selected based upon the judgement of the researcher, and this method is best employed when undertaking small-scale research projects (Creswell, 2013), as was the case in this study. In addition, it offers the researcher the opportunity to target specific individuals that meet the study requirements within bounded criteria set by the researcher. Purposive sampling, which is a kind of non-probability sampling, generally offers an appealing solution for a researcher looking to study a specific construct (Tongco, 2007; Palys, 2008; Oliver & Jupp, 2006) and can be particularly beneficial if the researcher has limited time constraints (Castillo, 2009).

The research criteria for this study required participants to currently be employed (as either a lecturer or professor, or some combination of the two) and to be teaching students enrolled in the first-year calculus course at University X. In addition, the researcher selected participants with diverse backgrounds (e.g. education level, teaching experience, student class size, etc.) in an attempt to maximise the perspectives provided, though this was limited by the number of teachers working within the first-year calculus programme. Participants were required to be the main contact for the course (i.e. teaching assistants on the course were not considered in the initial participant pool). In addition, while knowledge surrounding the implementation of the pre-calculus course was beneficial, it was not a requirement for participation in this study.

In total, there were seven potential teachers in the first-year calculus courses on the male campus of University X. I sent requests out to all seven, four responded that they would be happy to participate, two responded with a refusal, and the last one responded that he would only participate if I could not find four other participants to do so. As a result, the decision was made not to include this final participant, consequently the researcher was limited to four participants.

The selected participants had different backgrounds, and different levels of experience, but are all were teaching first-year calculus using the same syllabus. The demographic questions asked enabled the researcher to establish a profile of the sample group (see Table 5-6). All four participants fell into the 36-50 age group and held PhDs in Mathematics obtained from non-Saudi universities. Their experience of teaching calculus at university level ranged from 4 to 8 years. Out of the four participants, only one had experience of prior teaching and the other three were students prior to teaching calculus at university level. None of these participants had attended academic conferences about the teaching and learning of calculus. These teachers worked individually, but with the same preliminary material. They did not have many opportunities to work or collaborate because of other demands, such as research, taking up their time.

Teacher	Qualification	Research activity	Teaching calculus experience (in years)	Background pedagogy
John	BSc in mathematics education, MSc and PhD in Applied Mathematics	Yes	4	Yes
Alex	BSc in mathematics education, MSc and PhD in Applied Mathematics	Yes	4	Yes
Sam	BSc, MSc, and PhD in Applied Mathematics	Yes	6	No
Tom	BSc, MSc, and PhD in Applied Mathematics	Yes	8	No

Table 5-6: Information About the Participants

5.12 The Process of Data Collection

Ethical approval for conducting this research from the University of Southampton (Appendix F) was obtained and the data collection process began by sending a letter regarding data collection (Appendix G) in order to obtain the consent of University X; the gatekeeper for the selected case setting (Appendix H). The researcher provided the Dean of Faculty of University X, and the head of the mathematics department, with an information sheet and letter of data collection with details of the study and what involvement in the study requires from the participants. Then, the researcher sought the consent of the participants after the research project was explained to them. The participants they were given copies of the consent forms (see Appendix I.3, I.5, J.2, J.5, K.3, K.5) as a general guideline. Creswell (2007) advises that participants should be made aware of all the process of data collection and they should be made aware of objective of the research, procedure for collecting data, participant withdrawal right, researcher's plan to protect the confidentiality of the participants, a statement about the risks participation in the study could entail, and the possible benefits to the participants. This was achieved through the participant information sheets (see Appendix I.2, I.4, J.1, J.3, K.4, K.6) and the consent forms the participants were requested to sign.

This study adopted a multiple methods research design. Data were collected from calculus teachers through the use of questionnaires, interviews and observations to investigate PCK for teaching calculus at university level. While each method was very important, and no one form was more important than other, each method was designed look at a specific aspect of PCK (e.g. considering how the teaching practices of the four teachers, and what they say about this, provides evidence of their PCK of teaching calculus) and also aimed at triangulating and supplementing each other method.

The questionnaire was distributed to the four teachers in the first week of data collection. For the interviews, the researcher, guided by the interview schedules, recorded the data mainly using audio recording. The researcher followed the recommendations of Simons (2009) in being mindful of certain factors: for example, the necessity of establishing rapport; using focused questions to fill in certain gaps in the data; maintaining an open questioning style and being a good and careful listener. The interviews lasted between thirty to forty and minutes and took place in the college in a meeting room. All interviews were in Arabic and translated and transcribed into English. The researcher interviewed John and Alex in week two of data collection and observation, Tom in week three and finally Sam in week four. Some aspects of the observation were followed up with a brief interview, this was with John only, on one occasion.

For the observations, the researcher agreed with each teacher that he would attend eight lectures, two hours for each teacher per week. The researcher provided an overview about the research in general and about the observation in particular. In the overview, the researcher emphasised that his attendance, in each lecture, would be as an observer. Observation schedules were used to take notes, make outlines for each lecture, which were video-recorded to enable further reference. Figure 5-2 shows the setting of lectures. The number of students in the lectures were between 56 to 75. The classroom had a whiteboard and data show projector. A benefit from the pilot study was that the researcher knew to choose a suitable position in classroom for him, and the camera, for successful observation. The data collection process took roughly three months. After finishing the data collection process, the researcher transformed the interview data and observation data into textual data for analysis. In addition, the researcher made sure all the data collected in the different forms were all saved in retrievable formats for any future reference.

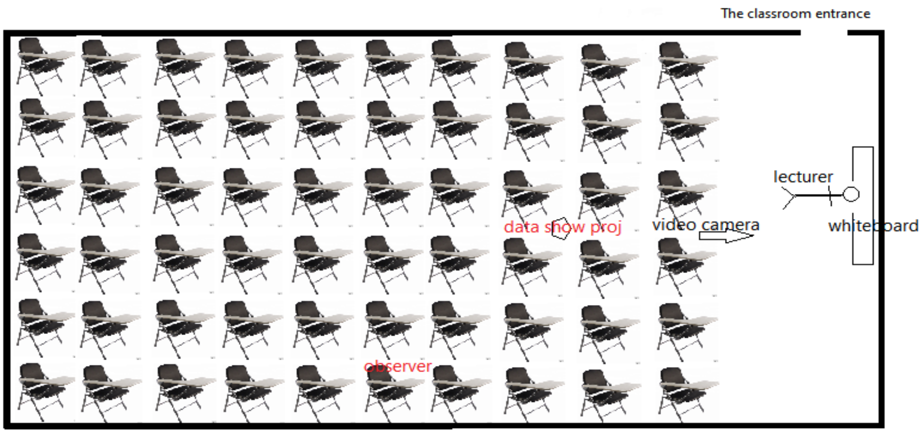

Figure 5-2: The Setting of Lectures

5.12.1 Summary of Information on Students' Background

The focus of this research is not the students; however, a general overview can be presented. All students were from KSA and there are no non-Saudis students, and all the students were in the same age band. They were a mixture of city and rural origin and they all had the same curriculum input regardless of where they came from in KSA.

In high school, students are required to take a certain level of mathematics in order to qualify for entry into university, contributing to the idea that many students come with approximately the same level of background knowledge in mathematics. Students who complete the KSA Grade 12 high school mathematics programme generally are taught a simple form of calculus, though it is acknowledged that they are taught many of the theories and principles on which calculus is built (Alamri, 2011). Therefore, first year university is the first time that students are exposed to the full range of knowledge about calculus and teachers need to make decisions about how to proceed through the material, because students' background influence their teaching of calculus, and students come with some misconceptions. A balance is required because, as a pre-requisite, a certain level of understanding is required in order to complete upper year courses, but these outcomes must be achievable, or students can become de-motivated and burn out before completing the course. These are essential aspects that teachers must consider in the first-year model (Alamri, 2011).

Moreover, while all students need to learn the fundamentals of calculus, classes are focused in different ways (i.e. students majoring in mathematics to become teachers focus on slightly different lessons than engineering students (MOH)). This may mean that the lecturers focus on different pedagogical strategies based on the audience - because the students will be mathematics teachers in the future - but the concepts being taught (especially in the first few weeks) are quite similar.

Typically, students were asked for responses in a lecturer-to-student dialogue, and while there were some instances where students were permitted to work in groups, and the lecturer moved around these were not always common and the strategies employed varied between teachers. In the KSA, it is still very common for the lecturer to stand at the front of the class with chalk and solve problems in front of the students. While the students are permitted to ask questions during this process, oftentimes they remain silent and do not engage with the teacher.

The following sections present the data analysis procedures.

5.13 Data Analysis Procedures

5.13.1 Overview

The method for devising and testing frameworks, both conceptual and analytical, is well-tested in educational research and social science research more widely (Ritchie et al., 2013). However, managing qualitative research is not without challenges, particularly when it comes to the presentation of findings and the management of data. This was addressed in this study by referencing both the conceptual framework and the analytical framework for the purpose of focus and clarity. Drawing on previous research, focused through the two frameworks, provided origin and context, whereas the original empirical research, based on the data gathered for this study, tested the conceptual framework in practice.

Identifying and analysing PCK constructs can be achieved by using methods designed for that purpose. In this study, the components of PCK were analysed using multiple assessment strategies. These multiple assessment strategies were informed by other researches, an example of which is presented in Appendix C. This ensured that the conceptual framework would be tested both in the context of other researchers' work and the original empirical data gathered in this study (see Appendix L for examples).

The proposed model of PCK for teaching calculus with characteristics (See Table 5-7) presents the categories of *knowledge of content and students when teaching calculus* and *knowledge of content and teaching calculus,* each underpinned by a number of first and second level sub-categories. These sub-categories were informed, created and categorised from previous research in the literature.

Categories	First-Level Subcategories	Second-Level Subcategories (elements)	Characteristics
Knowledge of content and students when teaching calculus	- *Learners' cognition of calculus.*	- *Students' misconceptions and learning difficulties in calculus.*	- identifying students' difficulties with both constructing and evaluating calculus concepts; - identifying students' difficulties, including their inability to state definitions, not knowing how a proof should begin, inadequate concept images, and an inability or disinclination to generate and use examples; - Using knowledge of learners' cognitions to address anticipated questions and students' misconceptions.
		Knowledge of students' thinking about calculus concepts.	- identifying the characteristics of external, empirical and deductive concepts of calculus; - Identifying students' formation of mathematical concepts in calculus. - Identifying students' progression in understanding typical calculus concepts.
	- *Developmental aspects of the calculus curriculum.*	*Establishing appropriate learning goals in calculus*	- demonstrating ways to define and explain the lesson's goal and objectives to the students.
		- *Identifying the key ideas in learning calculus.*	- providing and making available definitions, theorems and proofs to students; - providing forms of argumentation appropriate for students' levels; - identifying relationships between mathematical and everyday use of terms; - demonstrating "routes" to explain calculus ideas, examples, or proofs.
Knowledge of content and teaching calculus	- *Instructional strategies.*	*Relationship between instruction and students' ideas in calculus.*	- teaching calculus ideas using a systematic approach based on a solid grounding in logic and its associated linguistic expressions; - presenting and sequencing problems that can lead students to more easily see the structure of certain calculus concepts; - demonstrating methods of answering questions, responding to students' ideas, using examples; - Demonstrating use of appropriate instructional methods.
		Questioning strategies in calculus.	- obtaining justification beyond just procedures; - encouraging thinking about the general case; - actively involving students in the lesson through questioning; - develop critical thinking skills, by asking why; - Checking understanding on completion of work.
		Use of pivotal examples or counter-examples in calculus.	- *Using examples or counter-examples to focus on key ideas in calculus*
		- *Mathematical Representation in calculus.*	- demonstrating the role of representations as recognised in manipulating mathematical objects, communicating ideas, and assisting in problem solving; - drawing strong connections between the representation's students use and their understanding; - using tasks that require a flexible use of different representations; - linking visual, symbolic, and verbal ideas in calculus
	- *Knowledge of calculus connections.*	- *Real- world applications of calculus.*	- emphasising mathematics as a way of interpreting experience or as a human activity; - indicating that people benefit from the applications of calculus every day; - Linking between calculus concepts and application of calculus in everyday use.
		- *Calculus in academic subjects.*	Demonstrating calculus in various academic subjects.

Table 5-7: The Proposed Model of PCK for Teaching Calculus with Characteristics

The data analysis in this research took an approach influenced by Hancock and Algozzine (2006) (see Figure 5-3) however, in this research Step 6 was omitted, as statistical data was not needed for analysis. For the purposes of analysis, each lecture was summarised using the field notes and video footage and divided each lecture into several episodes, each episode containing one point of the lesson. The researcher utilised all episodes in the selected lectures to analyse the PCK of the teacher. The approach was manual and involved sorting through the data and coding the data in a systematic way. The process was one of categorising data, which relied on secondary data gathered through a literature review as well as through primary research methods. It was vitally important to ensure that this process of categorisation was rigorous (Harrison et al., 2017). When categories had been established, patterns could then be identified and examined. This follows Yin's (2014) view that data analysis involves "examining, categorizing, tabulating, testing, or otherwise recombining both quantitative and qualitative evidence to address the initial propositions of a study" (p.109) and also supports Neuman's (1997) view that "data analysis means a search for patterns in data" (p.426). The patterns revealed, through the data analysis process, then formed the basis of the research findings.

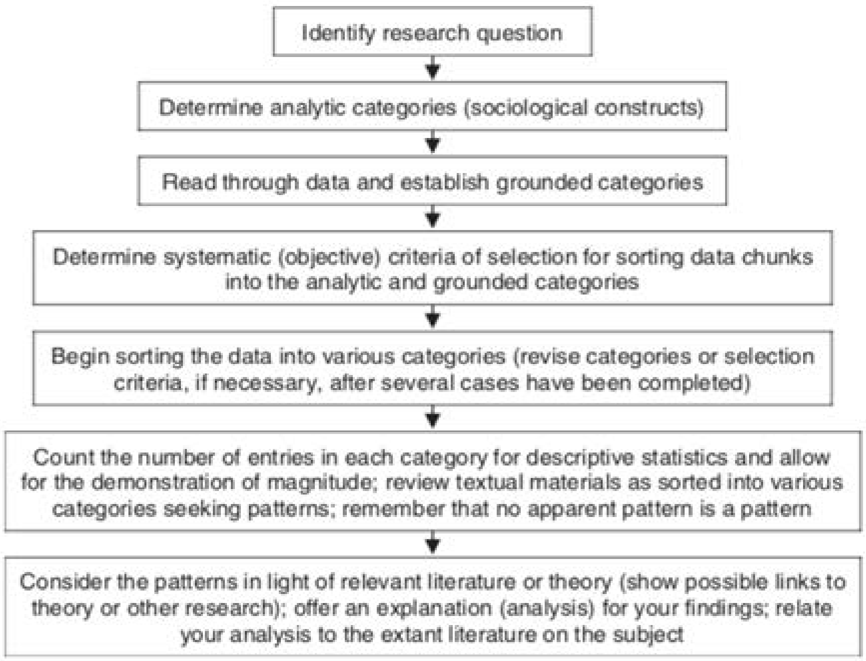

Figure 5-3: Stage Model of Qualitative Content Analysis (Hancock & Algozzine, 2006)

5.13.2 Cross-Case Analysis

As a form of analysis that examines themes across several case studies, cross-case analysis is suited to this research, in line with Eisenhardt (1989) view that studying cases is "particularly well suited to new research areas or research areas for which existing theory seems inadequate" (pp.549-549) and extends this by considering multiple case studies, gathered both through secondary research and primary research, using qualitative and quantitative research methods. This approach has been termed by Stake (2000, p.437) as the 'collective case' and its value can be recognised as drawing data from a range of contexts together for analysis as a means to explore new avenues of inquiry. This builds a stronger evidence base to answer a research question. This approach has been identified as particularly suitable for this research as it seeks to understand a range of experiences and perspectives and considers a new area of underexplored research.

5.14 Research Validity and Reliability

Reliability can be defined as "the stability or consistency with which we measure something" (Robson, 2002, p. 101) and the "degree to which an assessment or instrument consistently measures an attribute" (Pellissier, 2008, p.6). A study can be considered reliable if, given either the same or similar conditions, there is stability in the results (Lincoln & Guba, 1985). Validity can be understood as "the extent to which a measure accurately reflects the concept that it is intended to measure" (Pellissier, 2008, p. 6). In this study, reliability and validity have been maintained through the process by which the sample was chosen and through rigorous methodological design.

There are many ways to measure validity and reliability. For the evaluation of instruments that will be collecting the data, the researcher tests their validity and reliability to affirm that the control measures are correct and will provide solid results. Validity and reliability are utilised to evaluate the nature and quality of all "pre-established measures for qualitative and quantitative" methods (Lodico et al., 2010). When considering research credibility, validity and reliability are two factors that should be carefully considered (Patton, 2002). Whilst the definition of validity is widely disputed, it is often considered as "An account is valid or true if it represents accurately those features of the phenomena that it is intended to describe, explain or theorise" (Hammersley, 1987, p.69). Validity is concerned with whether the measure is actually measuring what it claims to and whether it does so accurately (Winter, 2002). Validity is incredibly difficult to ascertain as it "is part of a dynamic process that grows by accumulating evidence over time. Without it, all measurement becomes meaningless" (Neuman, 2007, p.69).

Reliability, on the other hand, is "the consistency of a measure of a concept" (Bryman, 2008, p.140). Reliability can be measured in terms of stability, internal reliability and inter-observer consistency. Stability is the measure's ability to obtain similar test-retest data under the circumstance of no intervention. Internal reliability concerns the consistency or relatedness of the questions or indicators included in the scale. Finally, two markers using the same measure should draw the same conclusion or result if the measure has good inter-observer consistency.

This study utilised semi-structured interviews, structured observations in classrooms providing qualitative data, and survey. The validity and reliability were measured for each of these instruments, requiring two different approaches.

Several basic strategies help to enhance internal validity, including triangulation, requesting that participants verify the accuracy of collected data, observing the same scenarios over a longer period, and requesting peer opinions or validations on findings (Merriam, 1998). Sapsford and Jupp (2006) state that: "One form of replication involves examining the extent of agreement between two observers of the same behaviour. This technique is more often used in more structured observation" (p.88). Structured observations are highly advantageous. It is possible to argue greater reliability from this method, as it is possible to obtain information that is more precise in terms of time of occurrence, duration and frequency. Moreover, the ordering of variable occurrences and reconstructions are more accurate (McCall, 1984).

Multiple data sources can often contribute to an increase in validity and reliability, as these are fundamental in ensuring the results obtained are of scientific use and highly valid and reliable in the research process. In this study it was deemed advantageous to triangulate several methodological approaches. Triangulation is highly beneficial in ensuring validity; if one of the methods is weak, the remaining methodological approaches will maintain strength in the results (Cunningham, 1997). Different calculus 1 classes were observed in this study, and to ensure that the structured observations were valid method triangulation was used. The two other methods were semi-structured interviews with the four participants and survey. In addition, the researcher used video recording to allow for re-examination of any uncertain or unclear interpretation by the researcher. Video recordings also allowed the researcher to go back to the participants for clarifications.

5.15 Ethical Considerations

5.15.1 Overview

One of the main aspects of data collection, that is an essential component of any research study, are the ethical issues and considerations associated with each type of approach. It is initially acknowledged that ethics, generally, contain 'grey areas' and it is the responsibility of the researcher to interpret these areas in a methodological way. This can be particularly challenging for novice researchers, as while ethics are essentially regulations, the interpretation of these regulations can vary (Felicio & Pienidiaz, 1999). It is suggested that this research study falls within the boundaries of 'standard ethics' being a low risk project. Despite this low risk, some interpretation of research ethics was inherently necessary for the completion of this study. The sections below highlight the current ethical practices accepted in research practices and to justify how this research project falls within acceptable boundaries. Creswell (2013) acknowledges that there are essential steps in the research process that ensure that a researcher maintains transparency and honesty within their field of research. Using Creswell's (2013) guidelines and assisted by the accepted standards of the Economic and Social Research Council (ESRC) and the British Education Research Association (BERA), the researcher, as noted above, applied for ethical approval for the research via the University of Southampton Ethics and Research Governance Online (ERGO) (see Appendix F). The following sections outline the ethical considerations associated with this research. Saudi letters of consent (see Appendix G and H).

5.15.2 Creating a Beneficial Research Problem

According to Creswell (2013), it is essential that the research question(s) identify a problem and then provide insight into the benefits of addressing this problem. He argues that this benefit needs to go beyond the researchers own interests (e.g. for the purpose of completing a PhD) and ascertains that it must also provide benefit to the participants. One of the ways this 'benefit' can be achieved is by ensuring that participants find their own involvement in the research meaningful.

Within this current research, both the pilot study and the main study required the researcher to have informal conversations with administrators (i.e. for the organisation of the observations) and with the participants (i.e. for scheduling the observation, interviews, and for completion of the survey). During these conversations, participants (and administrators) were provided with an information sheet and consent letter (Appendix I.2, I.3, 1.4, 1.5, J.1, J.2, J.3, J.4, K.3, K.4, K.5, K.6) outlining the nature of the research and how they would be affected. They were also given access to the research questions. By providing a letter to participants, and through the informal conversations conducted with the participants, this researcher considers that the provision of beneficial research for both has been met.

5.15.3 Ethics and Participant Selection

It was necessary for participants within this research to be carefully selected, yet this needed to be paired with what would be achievable for the researcher during the set period of study (i.e. time). According to Polkinghorne (2005), it is essential that the researcher selects participants who can provide full and saturated descriptions of their experience (in this case PCK - calculus) being investigated. Additionally, Polkinghorne (2005) suggests that, at least in the field of purposive sampling, exemplars need to be determined so that the research can learn substantive information by conducting the research. In this research, this meant selecting participants where the most information could be obtained.

Polkinghorne (2005) suggests that researchers have considerable freedom (especially in qualitative aspects of research) when purposively selecting participants. This freedom expands when studying a number of cases. According to Gregory (2003), the researcher will already have a set of morally guided principles that will often lead the researcher to select participants that relate to the same set of principles. This can be identified as a limitation. Both Gregory (2003) and Polkinghorne (2005) suggest that integrity and honesty are essential for the trustworthiness of the data, and that it is essential for the researcher to be transparent with all steps taken in the selection of participants.

In relating the ideas of Polkinghorne (2005) and Gregory (2003) to this study context, both participants and administrators were made aware of the purpose of this study and how the data collected would be utilised. Participant selection (see Section 5.10) required the researcher to balance time constraints with purposive selection of participants. As the research methods and participant sampling strategies have been outlined above, this ethical consideration has been met.

5.15.4 Ethics Involving Consent

Before beginning the data collection component of a research project two elements of consent need to be achieved with the participants. First, they must agree to participate and second, they must fully understand what they are participating in. Gregory (2003) highlights that this type of consent must be voluntary and that the participants must be ‘fully informed.’ He suggests that ‘fully informed’ essentially constitutes that participants should be “free of unwarranted pressures upon … arriving at the decision” (p.38). Rosnow and Rosenthal (2011) suggest that this needs to include a fair agreement between the researcher and participant. The agreement, as it relates to ethics, requires the researcher to avoid deception whenever possible. Creswell (2013) suggests that deception occurs when the researcher and the participants have different motivations, when there is a lack of trust, or when participants feel an obligation to participate. In order to avoid deception in my research, I offered the participants the opportunity to review the information and consent letter (see Appendix I.2, I.3, 1.4, 1.5, J.1, J.2, J.3, J.4, K.3, K.4, K.5, K.6) prior to asking for official acceptance. This gave each of them the opportunity to think about the topic before agreeing to participate.

5.15.5 Ethics Involving Confidentiality

While Gregory (2003) indicates that participants may be unlikely to express their views, attitudes and beliefs without the assurance of confidentiality, there are certainly difficulties associated with actually implementing confidentiality in practice. With the increasing availability of published research accessible through online portals, there is the possibility that participants could be identified, despite the best efforts of the researcher. This can be particularly true for qualitative research, as the participants may use unique phrasing that makes them more identifiable to people who already know their backgrounds. Noting that it is unlikely that total confidentiality can be achieved, there are steps that can be taken by the researcher to minimise this risk.

Crow et al. (2006) explain that confidentiality has, essentially, three main components. First, they indicate that data must be separated from identifiable individuals. This was achieved in this study by removing observation and survey data from University X's premises immediately after data collection, as well as the anonymising of data with participant pseudonyms. Secondly, Crow et al. suggest that those who have access to the data maintain confidentiality. This was achieved by ensuring that only the researcher had access to the hard copies of the data and that these were securely stored. Finally, Crow et al. (2006) indicate that confidentiality can be assured through the anonymising of individuals. In this study, this was done at the beginning of the data collection process and the pseudonyms being added before the data transcription process began. Participants were assured of confidentiality, and anonymity was explained as was the use of pseudonyms.

5.16 Researcher Bias and Limitations

The ESRC (2015) suggests that researcher bias can occur for a variety of reasons; one of the ways that this can ethically affect data is based on pre-conceived notions that the researcher has on the answers to the research questions. Often at times, researchers want to be ‘proven’ correct, and as such may even have unconscious bias within the research process.

The observation method allowed the researcher to indicate what happened at what time. As observation was the primary method of data collecting, the researcher utilised themes and sub-themes to code the transcription, in order to provide a level of consistency across the observations. While some level of bias might have existed within this interpretation, this was considered the most controlled and methodical way of approaching the qualitative data. In addition, with the use of a pilot study, aspects of the observation method were streamlined to maintain a workable framework.

While researcher bias is recognised as a limitation, this research study attempted to minimise bias by undertaking a pilot study and by having the research instruments reviewed by other researchers in the field, for feedback. While it is recognised that this does not fully provide justification for researcher independence, it meets the ethical requirements outlined within Southampton University (ERGO).

5.17 Section Summary

Ethics are an essential and comprehensive component of any research project. This section has simply addressed the most relevant ethical issues related to this topic, though further considerations were addressed with the research supervisor and through the ethics ERGO form submitted prior to this study's commencement. While it is acknowledged that there are some limitations to ethical considerations (e.g. maintaining confidentiality), the researcher has attempted to ensure that all these considerations have been addressed as fully as possible.

5.18 Chapter Summary

This chapter has outlined the methodological framework underlying this research study. Because the use of both qualitative and quantitative methods required the need for multiple methods research, the associated methods of the observation, interview and survey instrument and schedule have been addressed. This information comes in addition to the theoretical framework, which was presented in support of minimising the gap in current research. Ethics were also a crucial element within the context of this research study, and these have been presented with perspectives from both the researcher’s university framework, but also in relation to the larger ethical context identified within the literature. The next chapter presents the data analysis and discusses the findings.

Chapter 6 Data Analysis and Findings

6.1 Introduction

This chapter presents the findings of the cases examining the PCK of four calculus teachers at university level. In order to maintain anonymity, each has been provided with a pseudonym so that their identity is not revealed. For the purpose of this study, they are referred to as John, Alex, Sam, and Tom. The research used three different data collection methods: detailed observations of multiple class sessions delivered by each teacher, survey, semi-structured interview, and some follow-up questions for clarification if there was any ambiguity regarding the data collected. The goal of these findings is not to indicate ‘good’ or ‘bad’ teaching, but rather to consider how the teaching practices of the four teachers, and what they say about this, provides evidence of their PCK of teaching calculus.

In this study calculus teachers' knowledge is investigated within a PCK theoretical framework for teaching calculus composed of four main components which are:

1. **Learners' cognition of calculus** (LCCa) **-** involves students' misconceptions and learning difficulties in calculus and knowledge of their thinking about calculus concepts.

2. **Developmental aspects of the calculus curriculum** (DACaCu) **-** establishes appropriate learning goals and identifies the key ideas in learning calculus. The establishment of learning goals is closely tied to the identification of key concepts in learning calculus. This not only requires the teacher to provide the fundamental proofs, theorems and definitions to students, but also includes targeting forms of argumentation that are appropriate for the level of the students.

The building blocks of mathematical theories within the calculus classroom start from identifying learning goals to identifying the key ideas in learning calculus and covers multiple categories. These categories include students' misconceptions and learning difficulties in calculus, mathematical representations, use of pivotal examples or counterexamples in calculus and knowledge of calculus connections.

3. **Instructional strategies (**ISs) **-** focuses on the relationship between instruction and students' ideas, mathematical representations, questioning strategies, and use of pivotal examples or counterexamples in calculus. In order to effectively demonstrate PCK, both in personal understanding of the topic and in practice in lectures, teachers need to be able to demonstrate certain types of instructional strategies. These ISs require calculus teachers to

use a systematic approach to instruction, for example a solid grounding in logic and the corresponding linguistic expressions.

4. **Knowledge of calculus connections (**KCaCos) **-** identifies recurring characteristics of KCaCos being highlighted, and codes established to identify patterns from which calculus connections could be generated.

The interview questions are divided into three parts; part one about knowledge of teaching calculus (13 questions), part two about knowledge of student understanding within calculus (8 questions), and final part about the sequencing of building blocks of mathematical theories in calculus (7 questions). Also, the questionnaire is divided into six parts; part one about demographic information, part two about knowledge of aims for teaching calculus and included six statements, part three and five about knowledge of instructional strategies for calculus and knowledge of curriculum for calculus, each one contained seven statements, part four about knowledge of student understanding within calculus and included eight statements, part six about knowledge of assessment for mathematics and contained four statements. The data collected were analysed in respect to the proposed model of PCK (see Chapter 4) and analytical approach (see Chapter 5). The data were then re-examined within the four main components to ensure that nothing significant was missed.

6.2 Case Teacher John

6.2.1 John's PCK of LCCa

John displayed his knowledge of learners' cognition of calculus by identifying his students' difficulties with concepts of both constructing and evaluating calculus. At the beginning of the first lecture (Lecture 1, Episode 1 (00m18s-13m42s)) John told his students: *"Now I am writing a diagnostic test on the whiteboard ... think about it until I finish the writing*". The students were expected to complete 5 questions in 7 minutes. By composing and administering the diagnostic test, John showed that he had some understanding of the fact that students could have misconceptions and he needed to understand these.

In the interview, John suggested that students have "… *certain difficulties in the basics of mathematics ….*" (Interview, Part 2, Q1a) identifying mathematical terms, types of functions, domain and range as areas of particular focus. He showed consistency in both his interview and survey responses. For example, he suggested that logarithmic, exponential and rational functions are calculus concepts that are challenging to students. He also identified, in the survey, that he modifies his approach, or the content of each lecture, to accommodate these misconceptions. This is demonstrated through his disagreement with the survey statement: "I never adjust my progress through the calculus syllabus to take account of students' understanding and misconceptions." (Part 4, Statement 2)

In Lecture 3, Episode 2 (12m01s-22m50s) John spent a considerable amount of time discussing limits and promised to include more exercises on this idea. In the next lecture, however, I observed that the students were not interacting with John when he started to prove using the precise definition of limits. The students had difficulties with the formal epsilon delta definition of limits and John reassured them by saying "*… we will make some exercises in the next lecture …*". In addition, John showed consistency in his teaching practice and in his thinking in the interview when he said:

> "*I feel that students have difficulties to find a technique that helps them evaluate limits, also I feel that the student needs a lot of exercises to absorb this idea. Now I tell student that ∞/∞ the student feels perplexed - but I give several examples, and then I explain in more than one method for them to understand the idea … .*" (Part 2, Q1a)

This approach was evidenced in Lecture 3, Episode 3 (23m01s-32m17s) when he told the students:

> *"… what are the indeterminate forms? There are seven of them I am going to talk about what it means to be indeterminate and explain why they are indeterminate ... I am going to list them* $\frac{0}{0}$*;* $\frac{\infty}{\infty}$*;* 1^{∞}*;* 0^{0}*;* ∞^{0} *;* $\infty - \infty$ *;* $0 \cdot \infty$*". "… It will be clear in the following examples … we will see a technique that helps us evaluate limits that we otherwise cannot evaluate or ... to use more complicated techniques ... ."*

(with the whiteboard work showing some techniques to evaluate limits in Figure 6-1).

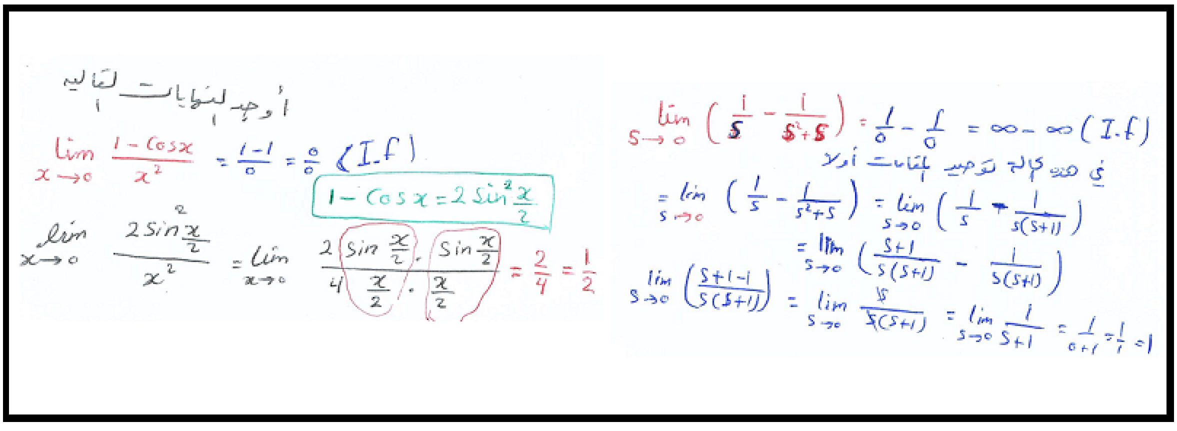

Figure 6-1: Example of Case John in Lecture 3 - Technique to Evaluate Limits

At this point John asked the students if they understood and the students responded "*No*". John then re-explained to them. A second time, with another method, students engaged in the discussion and they were interacted with teacher, so it seemed that they understood the examples. This approach aligns with his disagreement with the survey statement: "I never adjust my progress through the calculus syllabus to take account of common student misunderstandings and misconceptions." (Part 4, Statement 2) When he re-explained the examples to his students, he was being consistent with his response to the survey statement: "I evaluate my students' understanding of the calculus topic that I am teaching." (Part 4, Statement 6) Likewise, skipping the proofs of some theorems by saying these are difficult to prove, showed that John was ensuring that the students knew about which theorems they should prove in this course. In Lecture 6, Episode 3 (23m34s-31m16s) John talked about the extreme value theorem and mentioned that:

> "*… it is difficult to prove but we will explain it by examples, but we should know the meaning of these terms, endpoints, stationary points, singular points and as we discussed critical points ... .*"

This corresponds with his interview answer when he explained that:

> "*… it depends on the type of theorem. There are difficult theorems for students to understand, but if these theorems will be taught, for example at higher levels, we in calculus 1 do not give great importance to them. I try to give them a brief introduction, which means the basic things so that when studying calculus 2, 3 or real analysis, they have the basis of this theorem in a simple way and we note that they will study it more widely in the advanced levels … .*" (Part 3, Q4)

In other cases, John mentioned to the students that typical misconceptions existed about a certain topic. This was evidenced in Lecture 3, Episode 5 (44m00s-50m11s), Example 1: $f(x) = x^2$? And $f(x) = \begin{cases} 2, & x \in \mathbb{Z} \\ 3, & x \notin \mathbb{Z} \end{cases}$. In this example, John asked his students: *"When we see this function in the general $f(x) = x^2$ what are the domain and range of this function?"*

One of the students said:"$\mathbb{R} \to \mathbb{R}$*".* John asked: "*Is* $\mathbb{R}$ *the range*?" He then addressed this misconception by saying that: "*... the range is* $[0, \infty)$*… but we can say for your answer as co-domain … .*" He had noted that students had misconceptions with the definition of the term 'range'. Overall, it was in Lectures 1, 3 and 6 where students' misconceptions were explicitly addressed in the class.

Another area that links PCK with LCCa is in the typical progression that students follow when learning new calculus concepts. In his interview, John's perspective on this matter suggested that he considered that learners' progression is best achieved through the use of examples. He acknowledged that he attempted to use several examples for each concept to ensure that the progression in learners' cognition continued (Part 2, Q1a). Moreover, in the interview, John seemed to recognise that learners process calculus concepts in different ways. John tried to assist this progression through the use of examples that demonstrate a multifaceted explanation of the concepts. In terms of his PCK, observation of his lectures supported the implementation of this knowledge in the classroom setting. For instance, in Lecture 3, Episode 2 (12m01s-22m50s) John gave the following example (with the whiteboard work shown in Figure 6-2)

> Example 1: "*Evaluate the limit:* $\lim_{x \to 3}(x^2 + x - 5)$. *What is the difference between this example and when we say prove this* $\lim_{x \to 3}(x^2 + x - 5) = 7$?".

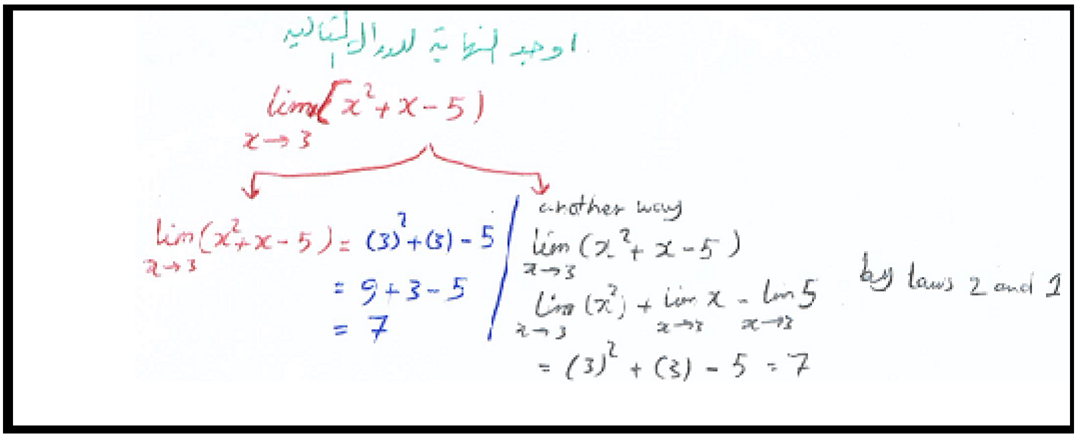

Figure 6-2: Example of a Multifaceted Explanation of the Concepts

The students gave many answers and John explained the difference between them:

> "*... when we see evaluate the limit in the question and do not give limit of computation, we do direct substitution, while if we see in the question prove we should use this the precise definition of limit ... .*"

When he realised that the students' understanding of this topic was flawed, John chose to offer an explanation of limits of computation and direct substitution and proof by using the precise definition of limit. Despite this explanation, students still demonstrated having challenges with the concept, at which point John moved onto further examples. As the weeks progressed John continued to use examples, but there were certainly instances where his patience with the class thinned, for example, in a somewhat weary tone, he said:

> "*... I think you learnt this term in the secondary school?*" Lecture 6, Episode 2 (12m09s-23m00s).

In his interview, John suggested that students' formation of concepts and progression through varying concepts was best supported by both collaborative and cooperative learning. In Part 1 of his interview (Q4, Q10 and Q11) he highlighted these strategies as a means to final solutions. There was some evidence that John applied these concepts in practice; for example, in Lecture 3, Episode 4 (32m25s-43m40s) John encouraged the students to work in groups of 5 or 6 to solve problems. In these cases, John allowed the students to form their own groups.

6.2.2 John's PCK in the DACaCu

While John did not make his teaching aims explicit in all of his lectures, there was evidence that he employed this aspect of PCK in Lectures 1 and 6. In Lecture 1, Episode 1 (00m18s-13m42s) he stated the aim:

> *''In this semester, we will study functions which will help us to understand the rest of this course and other courses in upper level, limits ... applications of integration, which will help us to understand other subjects as physics, engineering etc."*

John outlined the goals of understanding, which is a basic strategy and likely very appropriate for a preliminary lecture.

He stated in his interview that: "*In the mathematics department, we have an existing plan for each subject with objectives - general objectives and specific objectives for each lesson …"*. (Part 1, Q5) He also stated that: "... *I give a simple introduction about the lesson and I explain the objectives."* (Part 1, Q1b) In Lecture 6, Episode 1 (01m01s-11m02s) John acted in accordance with this interview comment, when I observed him tell the class that:

> *"The topic which we will learn today can help us solve many types of real-world problems. For example, we will see the maximum and minimum values of particular functions how can help us to find for example amount of material used in a building, cost, loss, profit, strength … lots engineering and science problems which we can solve by using derivatives … we will focus on the maximum and minimum values of functions, increasing and decreasing."*

As the weeks progressed, John's establishment of learning goals became much more specific, as demonstrated in Lecture1, Episode 7 (70m14s-88m07s):

> "*We will see if the function's graph is symmetric with respect to the y-axis, $origin$ or no … the aim of this idea that leads us to know if this function is even, odd or not and vice versa ... ."*

In another example, in Lecture 6, Episode 2 (12m09s-23m00s) John stated:

> "*I am going to set the derivative equal to zero, what is the aim of that? The aim of finding where the slope is equal to zero ... the critical point what do I mean by critical point?"*

In these two examples John was highlighting very particular and specific aims for what he was teaching; asking the students what they felt the aims were before providing his own interpretation. What was observed in these instances was consistent with John's interview comment that strongly indicated that he is focused on helping his students to understand the new concepts. He commented in his interview (Part 1, Q1a) that dealing with problems, understanding the function, and deriving the function were all essential goals in many of the lessons, which is supported by his strong agreement with the survey statement: "I always know how to organise the main aims of each calculus lesson that I teach." (Part 2, Statement 1) Furthermore, his agreement with the statement: "I always select lesson objectives for each calculus lesson by considering suitable methods for teaching." (Part 2, Statement 3) shows that John is linking his lesson teaching aims with teaching methods. His disagreement with this statement: "At the start of each calculus lesson I never define the aims of the lesson to students." (Part 2, Statement 5) indicated a consistent attitude.

John frequently mentioned key ideas prominently in almost every lecture episode. As such, it was evident that he was trying to analyse each calculus topic using definitions, theorems, proofs and examples. This is demonstrated through his agreement with the survey statement: "I analyse each calculus topic by building blocks of mathematical theories using axioms, definitions, theorem, proof." (Part 4, Statement 2) and in his interview (Part 1, Q1b) he stated that he analyses each topic of calculus:

> "... *then I give a simple introduction about the lesson and I explain the objectives. I usually start with definitions ... then we care for the sequence of the ideas starting from the definitions, theorems, proofs ... and then examples* ... ."

This approach was evidenced in Lecture 3, Episode 9 (79m19s-87m38s) when John stated: "*I will show you some properties of continuity before we talk about them what do you think they are?*" One student responded: "*I think they look like the properties of limits*". John completed his talk: "... *we will see, let's say if we have two functions f and g… the following function are also continuous at …* ." John asked a question: "*What do I mean by the function is continuous?".* The students gave answers and John mentioned this definition and then kept talking: *"… theorem 1. $f + g$… Let's take the first theorem $f + g$ please read the proof in the page 121 …* ." John asked his students to read the proof before explaining it, in doing so he was using two first-level subcategories of PCK, which are DACaCu and ISs. After two minutes, John asked: "*How many students understand this proof? ...*

One; three; seven just …". John then explained the proof and identified the students' difficulties in understanding the proof. I observed the next lecture given by John where he asked the students:

> "*Do you remember in the second lecture where we studied a catalogue of essential functions, who can remember that? … There are lots and lots of kinds of functions like ... are polynomials continuous? Why? Let's prove that … in the proof we will use limits laws and obtain justification beyond just procedures ... .*" (Lecture 3, Episode 10 (87m50s-95m24s)).

In Lecture 6, Episode 4 (32m00s-42m55s) John talked about Rolle's Theorem and explained the proof and then talked about the geometrical interpretation of it and then gave the students an example. In addition, in Lecture 6, Episode 5 (43m28s-55m59s) John used the same structure with the Mean Value Theorem and talked about it and then asked the students to read the proof in the book. There then followed an oral explanation and John talked about the geometrical interpretation of it and gave the students an example. John stated in his interview (Part 3, Q3):

> *"We have a sequence in lesson plans and a sequence in the same lesson. For the plan of lessons, I rely on the textbook and the main reference. I do not try to get out of them because they are connected, and I choose the things that fit my students' abilities. They are the basis for them and help them understand the higher levels in the future. For the lesson … during the lecture, I always have an introduction and an explanation of the idea and then the application. The lesson depends on the main idea, definitions, theorems, proof and examples ... which are sequenced according to the written book."*

In the survey John agreed with the statement: "I only use examples and diagrams after having introduced the formal calculus theory." (Part 3, Statement 4) Phrases used by John were specific to the calculus context, for example *"maximum and minimum value"* (Lecture 6, Episode 3 (12m09s-23m00s)), *"extreme value theorem"* (Lecture 6, Episode 3 (23m34s-31m16s)), and *"applications of differentiation and applications of integration"* (Lecture 1, Episode 1 ((00m18s-13m42s)). In addition to the availability of these definitions, John also demonstrated routes to explain the calculus ideas; this was achieved through examples, as stated above, but also through explanations and proofs: "*… theorem 1.* $f + g$ *… Let's take the first theorem* $f + g$ *please read the proof in page 121…* " (e.g. in Lecture 3, Episode 9 (79m19s-87m38s)).

6.2.3 John's PCK of ISs

John consistently used both examples and calculus-based definitions in his lectures. In addition to these specific strategies, John demonstrated the ability to imagine the bigger strategic picture. In his interview, he suggested that he used a scaffolding approach to systematically build students' knowledge by starting with a simple introduction, moving through to the aims, and then beginning to ‘sequence’ the ideas related to the topic:

> “*Then I give a simple introduction about the lesson and I explain the objectives. I usually start with definitions ... then we take care of the sequence of the ideas starting from the definition, theorems … and then examples.”* (Part 1, Q1b)

This strategy consisted of a series of specific moves that utilised what the students already knew and then built on their previous knowledge with corresponding topics in a scaffolding approach; for example, in Lecture 3, Episode 10 (87m50s-95m24s) (see Section 6.2.2). While John’s understanding of scaffolding and the bigger picture represents one component of PCK, he also identified and discussed other ISs in his teaching. John stated in his interview that:

> “*I sometimes use a wide range of teaching approaches, yes, I often use lecture because it helps me to provide as much content as possible as well as the strategy of collaborative learning, and discussion that I use in all lessons …."* (Part 1, Q11)

Moreover, elsewhere in the interview he said:

> "*I sometimes use cooperative learning which means that during the lesson after explaining the main idea, I give an example or .... We distribute the worksheets to the students then I give them time to work together … but we are trying to use cooperative learning by encouraging students to try to solve and explore information by cooperation among them ...."* (Part 1, Q4)

This approach was clear, for example in Lecture 3, Episode 4 (32m25s-43m40s), when John told his students to: “… *please make small groups where each group contains 5 to 6 students ….”* He then distributed worksheets and asked his students to answer the questions together and gave them 10 minutes answering time. John walked around amongst them and discussed their thoughts (another strategy - discussion methods). In taking these approaches, John's actions supported his disagreement with the survey statement: "I avoid using a wide range of teaching approaches in a classroom setting.” (Part 3, Statement 5) but agreement with the statement: "I am aware of using a wide range of knowledge in planning my calculus lessons." (Part 2, Statement 4)

John would start with a problem and/or a definition that he believed his students would understand. This was evident in Lecture 1, Episode 2 (14m03s-26m30s) when he used the example and diagram as a tool for introducing the new definition: “... *from the final question in this test and our discussion, we commence the new lesson .... This is an example of function what is the function?*”. (see Figure 6-3).

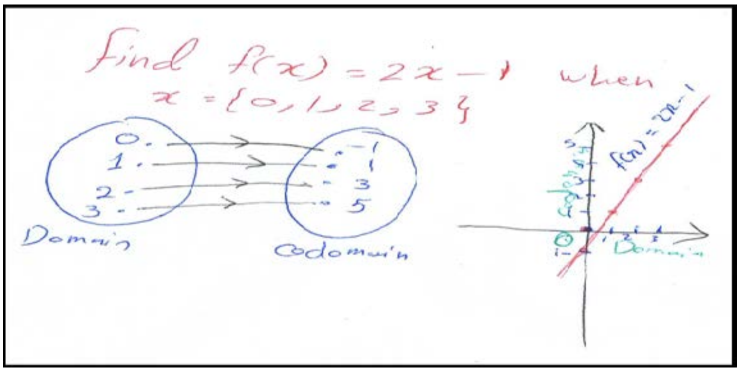

Figure 6-3: Example of Function in Lecture 1

In addition, in Lecture 3, Episode 3 (23m01s-32m17s) when he used the problem in an example from a previous episode (12m1s-22m50s) as an introduction for a new idea he made the link between two different ideas. He explained three examples by using direct substitution $\lim_{x\to 2}(\frac{3+x^2}{x+1})$; $\lim_{\theta\to\frac{\pi}{2}}(\sin\theta + 2\theta)$ and $\lim_{x\to 1}(\frac{2x^2-3x+1}{1-x^2})$? And a final example was a problem when they applied the direct substitution. John wanted to use it to introduce indeterminate forms which he explained in the follow up clarification:

> "*… you can see in the short video which you sent, I mentioned in the beginning 'but it won't always work' this as counter example and would like to use it as an introduction for the next idea …"* (follow up clarification regarding Lecture 3, Episode 2 (12m1s-22m50s)).

So, John's claim demonstrated his use of appropriate instructional methods.

For the most part John used questioning strategies. For example, in Lecture 1, Episode 2 (14m03s-26m30s) he was observed to do so when he told the students: "*… this is an example of function, what is the function?*". If the students were successful in answering the problem, John moved on to a new question that either introduced a new concept or modified the current one in a different way. This was clear in Lecture 3, Episode 6 (51m15s-61m27s) when John stated:

> *"We will know the qualifications of continuity, that there are three things that should be if f is continuous at a: 1) $f(a)$ is defined, 2) $\lim_{x\to a} f(x)$ exists and 3) $\lim_{x\to a} f(x) = f(a)$… ."*

John then requested his students to " … *explain how $f(a)$ is defined?"*

John gave three questions where each example was not met by at least one of the previous qualifications and "… *each example clarified one kind of function discontinuity* …". Another example occurred in Lecture 3, Episode 10 (87m50s-95m24s see p130). When the students responded *"Yes"*

John wanted to obtain justification beyond just procedures, so he asked: "*Why?... Let's prove that*". As long as the students continued to be correct, the process continued. If the students were incorrect or did not answer, John continued to use questioning strategies to determine where the fault in the students' logic occurred and then redirected them, either back to the original problem or to another one that addressed the fault in the logic. A case of point being presented in Lecture 3, Episode 2 (12m01s-22m50s): Example 1: *Evaluate the limit:* $\lim_{x\to 3}(x^2 + x - 5)$ … . John asked the students: *"What is the difference between this example and when we say prove this* $\lim_{x\to 3}(x^2 + x - 5) = 7$*?"*. Another example of his questioning strategies occurred in Lecture 3, Episode 5 (44m00s-50m11s) when John tried encouraging thinking about the general case. Example 1: $f(x) = x^2$? And $f(x) = \begin{cases} 2, & x \in \mathbb{Z} \\ 3, & x \notin \mathbb{Z} \end{cases}$? John asked his students: "*I would like to ask you when we see this function in the general* $f(x) = x^2$ *what are the domain and range of this function?".* One of the students said: "$\mathbb{R} \to \mathbb{R}$*". "Is* $\mathbb{R}$ *the range*?" John responded. He waited for students to think and one of the students gave the answer. While John did not explicitly state this process in his interview, he hinted in Part 1, Q9 that: *"I encourage the students to try to get the idea and they try to answer some questions ….*" He further alluded to instances where he had discussed questioning strategies and his use of examples in the classroom. From my observations this was evident in Lectures 1, 3, and 6, especially in Lecture 3, Episodes 2 and 3, as John moved through the lesson. John was observed using different types of sequencing in an attempt to lead the students to understand some of the more challenging calculus concepts. John's use of questioning strategies was corroborated by his responses in the survey where he agreed with two statements: "I always ask questions to evaluate my students' understanding of calculus topic that I am teaching." (Part 4, Statement 5) and: "I always use a variety of ways and strategies to develop students' understanding of calculus." (Part 3, Statement 7)

One of the elements of PCK where John showed particularly strong connections was in his use of questioning strategies. For John, questioning was not only about the students' understanding, as highlighted above, but was also used to involve the students in the lesson (i.e. to maintain focus). For example, in Lecture 1, Episode 2 (14m03s-26m30s) John said: "*The function f is a rule that assigns to ... who knows what set X is called?*", thus encouraging critical thinking while working on a particular case. For instance, in Lecture 3, Episode 7 (61m49s-72m00s) he asked:

> "*Is the function* $f(x) =$ *1-* $\sqrt{1 - x^2}$ *continuous on the interval*$[\![-1,1]\!]$*? Before giving me, your answer who knows the domain and co-domain in general of this function?*"

John employed significant questioning strategies to evoke active participation. He was clear in his interview that:

> "*At the end of each lesson, I give some exercises. I ask the students in the lecture to solve one or two exercises. Through the answers, I discussed their answer, and know how the students' progress and achieve the objectives and I give them 2 or 3 exercises as homework ….*" (Part 2, Q1a)

Furthermore, he agreed with the survey statement: "I always ask questions to evaluate my students' understanding of calculus topic that I am teaching." (Part 4, Statement 6) Sometimes, John was observed to ask rhetorical questions and did not expect an answer from the students. This was particularly evident in Lecture 1, Episode 5 (48m40s-65m00s) when he asked: "W*ho can remember the definition of absolute value?"* When using the word "remember" when discussing the absolute value, this was closely linked to John's desire to explain things in a variety of different ways. The use of the word "remember" was meant as a trigger, suggesting to the students that they had learned this term previously. Another example, in the same episode, occurred when he asked the students to "… *find the domain and co-domain of function* $y = \frac{|x|}{x}$ *and sketch that. What do you think?"*

John's students were observed to willingly participate in the lecture and to engage with the material, even when they were not entirely sure of the answer. This was demonstrated when John posed a question and received multiple different answers to the same problem. In Lecture 3, Episode 2 (12m01s-22m50s) Example 1: "*Evaluate the limit:* $\lim_{x \to 3}(x^2 + x - 5)$ *What is the difference between this example and when we say prove this* $\lim_{x \to 3}(x^2 + x - 5) = 7$*?"* The students gave dozens of answers. The fact that the students were able to move from a series of incorrect answers to ones that were correct demonstrated that the students' thinking, generally, was changing over the course of the lecture, seemingly as a response to both the questioning strategies and the use of examples (Episode 8 (72m14s-79m05s)). When John sketched four graphs of functions, a discussion about them followed (see Figure 6-4). He said: *"We will see three different types of discontinuities those are a removable discontinuity, jump discontinuities, and infinite discontinuity"*. John then asked: *"Who can imagine what the kind of functions in those graphs is?"* There was no response from the students. He continued: *"… the first and third graphs look like rational functions and the second graph seems a piecewise function … we will see that later when we take examples".* For those who were still unable to grasp the concept by the end of the lesson, homework was often assigned so that these concepts could be reviewed at a later date.

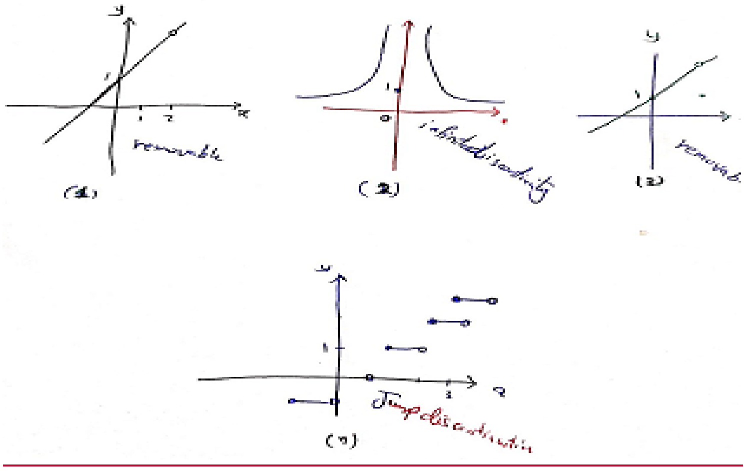

Figure 6-4: Sketching of kinds of discontinuities

John was able to pair different skill areas in his explanations and in the problems that he chose, so that the students were getting different stimuli in simultaneous instances. For example, in Lecture 1, where John discussed functions (Episode 4 (37m10s-47m30s)) he said: "*One-to-one functions if it never takes on the same value twice … we will see three different examples*". There were some obvious uses of pivotal examples and counter-examples (See Figure 6-5).

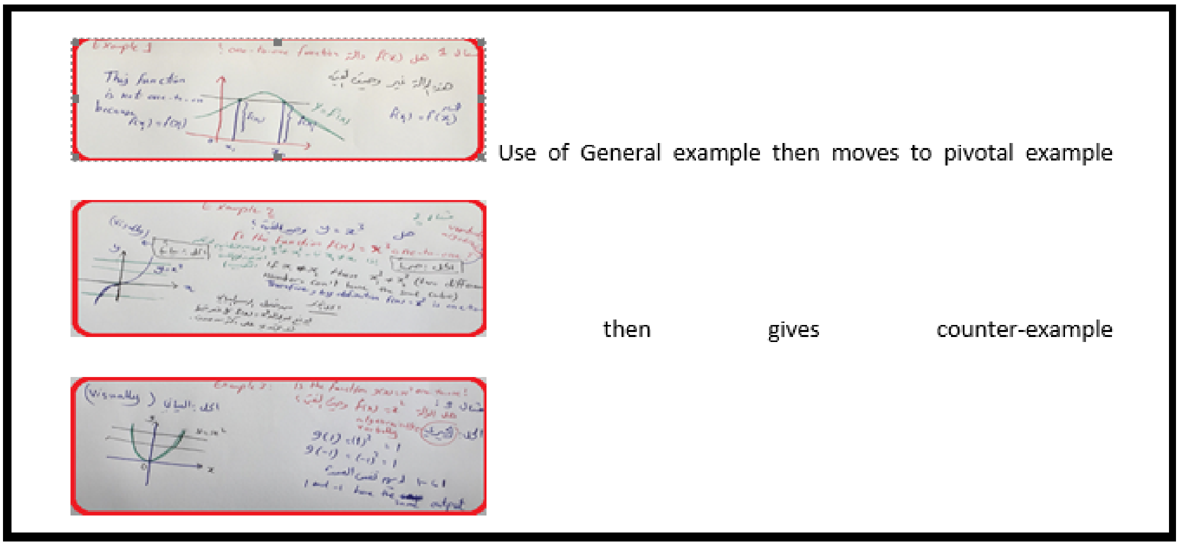

Figure 6-5: A Pivotal Example and Counter-Example with Representations

In another example, John used two of the second-level subcategories of ISs (relationship between instruction and students' ideas in calculus and use of counter-examples). In his interview (Part 1, Q2) John explained that: "*Sometimes I use examples to introduce the new idea".* This was observed in Lecture 3, Episode 2 (12m1s-22m50s) where he explained three examples by using direct substitution $\lim_{x\to 2}(\frac{3+x^2}{x+1})$; $\lim_{\theta\to\frac{\pi}{2}}(\sin\theta+2\theta)$ and $\lim_{x\to 1}(\frac{2x^2-3x+1}{1-x^2})$?. A final example was a problem when the students applied direct substitution. John wanted to give his students a counter-example, the use of which he justified by explaining that he wanted to use it: *"as a counter-example and* *would like to use it as an introduction for the next idea"* (follow up clarification regarding Lecture 3, Episode 2 (12m1s-22m50s)). This was also mentioned in his interview (Part 1, Q3):

> "*I emphasise that planning is always important ... I am looking for pivotal examples and counter examples to show them to students and help me to explain calculus ideas.*"

It was evident that John was using various instructional methods. In these scenarios, John was not only offering a verbal explanation of the concept, but also employing other strategies as well including numeric and algebraic presentations on a whiteboard. For example, in Lecture 1, Episode 4 (37m10s-47m30s) when he paired visual representation with the spoken explanation see Figure 6-5. A case in point, in Lecture 6, Episode 4 (32m00s-42m55s) and Episode 5 (43m28s-55m59s) when John explained Rolle's Theorem and the Mean Value Theorem, he talked about the geometrical interpretation of them and used the graphs (see Figure 6-6).

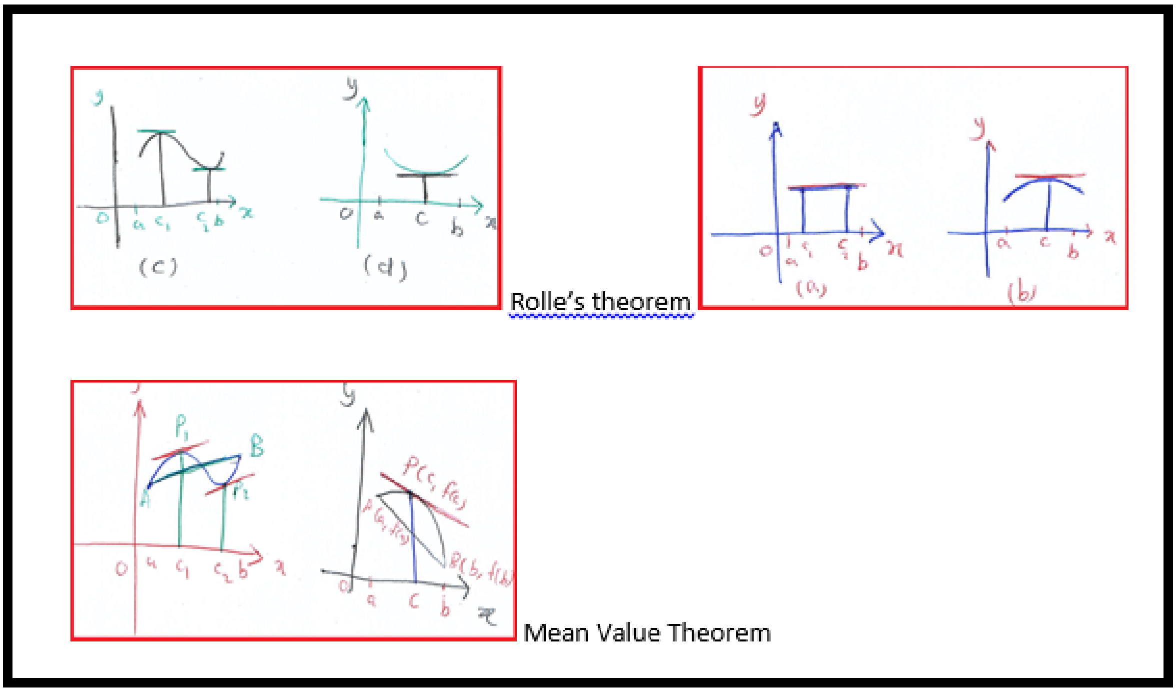

Figure 6-6: Example of a Visual Representation in Lecture 6

Such choices attempt to get students to really understand the model and to avoid misrepresentation of symbols (Berry & Nyman, 2003). In Lecture 3, Episode 8 (72m14s-79m05s) John sketched three graphs of functions and a discussion followed and John explained that: "*... we will see three different types of discontinuities those are a removable discontinuity, jump discontinuity, and infinite discontinuity".* Then he asked: *"Who can imagine what is the kind of functions in those graphs?"* There was no response from the students. John continued: *"The first and third graphs look like rational functions and the second graph seems a piecewise function … we will see that later when we take examples."* By doing so John was linking visual, symbolic, and verbal ideas in calculus.

6.2.4 John's PCK in the KCaCos

Khakbaz (2016, p.190) suggests:

> *" ... that application has more than one meaning (...): application in the real world, application in other disciplines and application in mathematics".*

In this study, after revisiting the collected data and field notes, two subcategories (real-world applications of calculus and calculus in academic subjects) were derived inductively. Here the researcher used specific extracts from the observation, interview and survey to illustrate these subcategories and how teachers *"make connections between students' knowledge and mathematical content through focusing on the main idea behind a mathematics problem" (*Khakbaz, 2016, p.190).

In the interview John explained that: *"We use real-life examples ... I always focus the derivative applications ... and I often mention there is a clear connection between calculus and academic subjects such as engineering, physics, chemistry."* (Part 1, Q10c) It was somewhat evident, however, that he had difficulty in identifying real-world connections that the students would understand. John highlighted real-world problems and discussed the application of differentiation using examples relating to building, cost, loss, profit and strength. While these real-world examples suggest that John was applying the theoretical framework to his lessons, it was unclear how relevant these examples of stocks and buildings were to the students' own experiences. Instead, it appeared much more as an attempt to justify to the students why their instructor stated that these will happen. An observed example was when he discussed the price calculation in supermarkets paired with examples of the stock market:

> *"When I explain something, which can be used in another subject ... it is difficult to determine one topic because each one has many applications, especially derivatives … real examples and physical models on the limits which mean the value of a function at a certain point. We can calculate the price calculation in the supermarket stock price."* (Part 1, Q10c)

John applied the real-world application of the calculus aspect of PCK into his limits lesson, Lecture 3, Episode 1 (00m45s-12m11s)), where he used examples from simple stock market trading limits to best explain his point. He again made use of real-world experiences in Lecture 6, Episode 1 (01m01s-11m02s) when he explained that:

> "*There are a lot of different ways in which we can apply doing derivatives and utilizing derivatives and differentiation … in this topic we will see how it is very important with specific topics we can see derivatives and the concept in engineering, economics a lot of*

good applications the engineering and economics with derivatives. The topic, which we will learn today that can help us solve many types of real-world problems. For example, we will see the maximum and minimum values of particular functions how can help us to find for example amount of material used in a building, cost, loss, profit, strength, … lots of engineering and science problems which can be solved by using derivatives."

And in Episode 2 (12m09s-23m00s):

"As we had in the previous lecture rates of changes in the natural and social sciences, we remembered that $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ *we have seen the interpretations in physics, chemistry, biology, and economics … we're going to learn about some applications of the derivative one of the applications is to find the maximum and minimum values on a function and before we define the absolute maximum and minimum value of f and local maximum and minimum value of f ."*

In the interview John went on to highlight that while calculus can be applied broadly, its application to these subjects may be somewhat vague because the application can vary depending on the topic. Despite this challenge, John was able to suggest these applications in his lectures, and most specifically in Lecture 6, Episode 1 (01m01s-11m02s) where he highlighted engineering, economics, and science, specifically focusing on the use of derivatives. Nevertheless, the examples he provided were in the form of generalisations. However, his agreement with the survey statement: "I am not interested in how calculus is taught at other (similar) university institutions in other parts of the world." (Part 5, Statement 7) suggests a somewhat closed approach to his teaching. Statements in the lecture such as: *"We can see derivatives in subjects like engineering"* and *"Economics has a lot of good applications here"* (Lecture 6, Episode 1 (01m01s-11m02s)), do not really tell the students much about how the derivatives might be used in other contexts, but simply that they do.

6.3 Case Teacher Alex

6.3.1 Alex's PCK of LCCa

At the beginning of the first lecture (Lecture 1, Episode 2 (06m03s-25m50s)) Alex administered a 10-question diagnostic test to the students and informed them that: *"The aim of this diagnostic test is to identify your background on the foundations of calculus."* Composing and using a diagnostic test to some extent indicated that Alex understood something about students' conceptions. During his interview, he stated:

> *"Honestly, I start with the students from the basics because if there is a misconception and misunderstanding so I have been with them from scratch and most students currently have a problem in secondary education. They do not come with basic information … in the first lecture I always give students a diagnostic test for 10-15 minutes.*" (Part 2, Q2)

Alex indicated in the survey response that he felt that students' performance could be reliably assessed in the classroom. (Part 6, Statement 1) As the students were completing the test, Alex walked between the rows making notes about areas of concern. He then began the lecture with an overview of the identified concerns. As a result of this test, Alex was able to offer some suggestions for weaker students about the resources they could utilise to ensure that they had enough background knowledge in calculus to be successful in the course. This approach was supported by his strong disagreement with the survey statement: "I never adjust my progress through the calculus syllabus to take account of students' understanding and misconceptions." (Part 4, Statement 2) and his agreement with the statement: "I anticipate my students' prior calculus knowledge before the lesson." (Part 4, Statement 4).

Another area that Alex was most conscious about was the level of complexity of definitions. He indicated in his interview that it was his role, as the teacher, to ensure that the students were relying on the essential information because, in the case of calculus, there are a lot of concepts and complicated language associated with the field, and his students were just starting out. In order to ensure that the learning curve was not too steep for his students, he suggested that it was his goal to provide guidance on what information was important for retention and to correct the students' misconceptions surrounding the complexities of certain topics:

> "*I am with my students to correct their wrong information and misconceptions and tell them how to get the information. There are too many sources of information and the role of the teacher is to direct the student through proper guidance in obtaining information and correcting misconceptions*." (Part 1, Q4b)

In practice, Alex highlighted this in Lecture 1 when he directed the students to deal with particularly challenging definitions. He asked his students to define the function and they said: *"It is a class of ordered pairs*". Alex stated that: "*This is a simple definition, and I consider that as a poor definition; we need a definition that provides a rich preparation for our study of functions."* (Lecture 1, Episode 3 (26m22s-34m15s)). This suggests that Alex was aware that the definition was probably more difficult than the students could handle, or that a simpler explanation would reduce the number of difficulties that were being experienced by them. By highlighting, for the students, that the definition was poor, Alex was demonstrating PCK in the classroom because he had anticipated an area of difficulty and had attempted to mitigate the challenges associated with a particular concept.

Another example was observed in Lecture 3, as Alex identified students' difficulties in adequate concept images, saying:

> *"... power rule. I want you to analyse this power by taking a very basic binomial of $(a + b)$ and we will raise it to different powers of n and analyze what is going on in this pattern so that when I get to the power rule ... I will give you method ... take $(a + b)$ and raise it to the zero power ... anything to the zero power what does it equal?"*

One of his students answered: *"1".* Alex replied:

> *"... $(a + b)$ to the first power is just itself, and $(a + b)^2$ I know many of you have difficulty ... so $(a + b)$ times $(a + b)$ is $a^2 + 2ab + b^2$ ... then if we would like to get this trinomial and wrap it in parenthesis and multiply by $(a + b)$ again is $(a + b)^3$ we will get $a^3 + 3a^2b + 3ab^2 + b^3$ ... I will give you rule ... the first term starts with the highest power n and then counts down, whereas the second term exponents are counting up .... So, when we see $(a + b)^3$, the first term being a and its exponent is counting down in value ...."* (Lecture 3, Episode 4 (35m16s-46m49s)).

The third example for addressing students' misconceptions came in Lecture 6, Episode 6 (70m12s-81m08s) when Alex talked about the definition of the number e. My notes recorded that Alex asked: *" What is the derivative of $y = e^x$?".* One student applied the rule of x^n, where n is a real number. In this case Alex noted a misconception and emphasised the difference among the three terms e^x, x^n and a^x.

Another instance where Alex was particularly cognisant of the students' misconceptions, and how to address them, was in the area of homework. This does not necessarily relate to the implementation of homework into the students' course requirements, but rather relates to how Alex used the results from the homework as a foundation for addressing the students' misconceptions. Alex suggested in his interview that he used homework to address students' difficulties by bringing up common errors within the larger lecture setting. He suggested that this particularly addressed their misconceptions as *"I use the homework to address these difficulties … then use their answers to show them their misconceptions….*" (Part 2, Q5) This was demonstrated through his strong agreement with the survey statement: "I always provide constructive formative feedback to calculus students." (Part 6, Statement 4) and in his practice in Lecture 3, where he began the lecture by commenting on learning difficulties. He suggested that: "*When I marked your homework, I saw common learning difficulties when you wanted to find the limits of rational functions as X approaches infinity*" (Episode 1 (01m02s-12m55s)). He then went on to explain, on the whiteboard, about the easiest way to find limits at infinity, using different coloured markers to highlight the different steps in the process (see Figure 6-7).

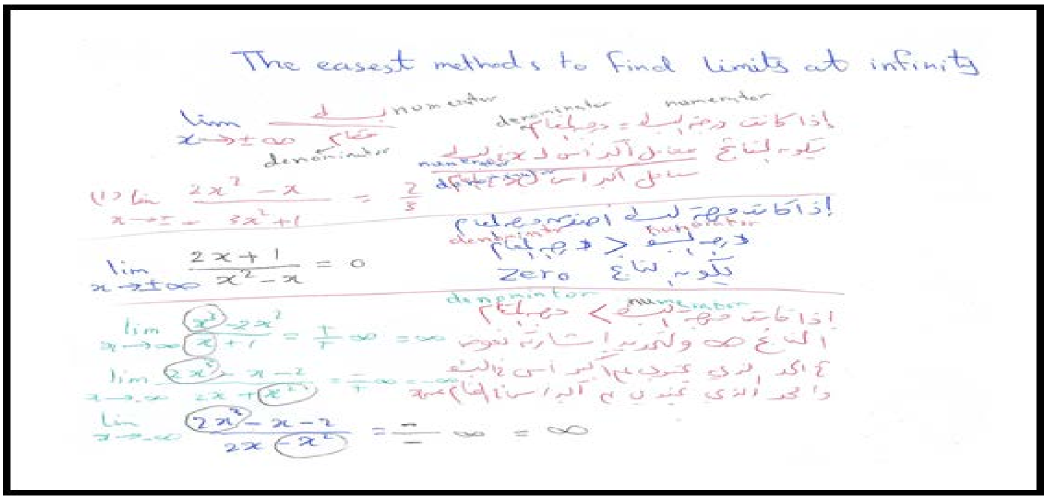

Figure 6-7: Finding Limits at Infinity

In fact, the strategy of using homework in multiple different ways suggests that Alex was confident that he would be able to use it to identify students' misconceptions and challenges.

In terms of knowledge of students' thinking about calculus, Alex was observed to consistently ask *"Why?"* rather than *"Do you understand?"*. For example, in Lecture 3, Episode 8 (95m27s-108m55s) he said: *"I have a question that I wanted to ask you in the first lecture, but now is a better time, why are we studying this kind of function - the trigonometric functions*.*"* Another example, in Lecture 3, Episode 3 (25m23s-34m51s) was observed. Alex mentioned a theorem: "*If f is differentiable at c, then f is continuous at c... who can interpret this theorem?"* He gave his students the opportunity to express their thoughts.

Student 1: "... *repeating the same statement."*

Student 2: "*f should meet all the qualifications of continuity."*

Alex: *"What are the qualifications of continuity*?"

Student: " $\lim_{x \to a} f(x)$ *exists and* $\lim_{x \to a} f(x) = f(a)$"

Alex: "*Just?"*

No answer.

Alex: "*Do not forget that f is defined."*

Alex*: "We will see the interpretation of this theorem when we prove it ... do you prove that, or do we want to prove that together*?"

Students: “*No, we prefer to prove that together.”*

Alex: *“What you think about the inverse of this theorem and why?*”

Most of students answered: “*Yes, if f is continuous ….”*

Alex: *“Why?”*

Alex: “*Let’s see that, we will take this example* $f(x) = |x|$*….”*

Alex*: “What do you think now? And why?”*

Students*: “No, it is wrong.”*

Alex then sketched a graph (see Figure 6-8) and showed the students where a function can fail to be differentiable, when it is discontinuity, corner, or vertical tangent.

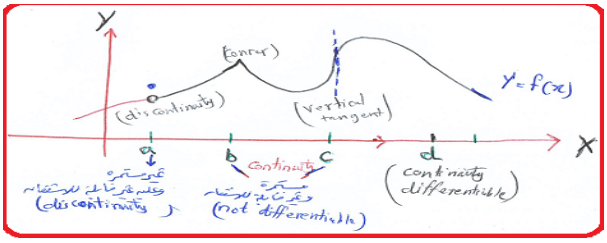

Figure 6-8: Types of Discontinuity

This required the students to employ their critical thinking skills, but it also provided Alex with some understanding of how much information the students actually understood. With this information, he could both identify the students’ formation of mathematical concepts and determine their progression. In his survey response (Part 4, Statement 6) Alex indicated that he always asked questions to evaluate his students’ understanding of the calculus topic he was teaching, demonstrating his use of this strategy.

Finally, in linking Alex’s knowledge of students’ thinking to LCCa, Alex was able to, in some instances, identify the characteristics of the empirical concepts of calculus. Alex highlighted, in his interview, that he valued and always used real-life examples. He mentioned that: *"I always look for real life example rather than just examples ... if you would like to attract your students to the lesson ... make linking between that and their life*." (Part 1, Q10) In this way, he was suggesting that it attracted his students to the topic and made the empirical relevant to the classroom. This was, in some instances, demonstrated in the classroom, such as in Lecture 6, when Alex highlighted real-world problems when referring to the applications of differentiation and other examples in other lectures (see Section 6.3.4 on calculus connections).

6.3.2 Alex's PCK in the DACaCu

Alex consistently indicated the aims of the lesson at the beginning of the lecture and used certain types of scaffolding to ensure that the students were notified of the aims. For example, in Lecture 1, Alex not only stated the aim of the lecture, but also the wider aim of the course by saying that: "*The aim for this course is for you to be able to know the functions domain, limits of function, and other applications* ...." (Episode1 (00m03s-6m11s)). Alex identified these basic, but fundamental components, as essential for students' learning development. He commented in the interview that: "*The most important objectives are that the student has the basic ability or basic concepts that he can understand the subject*" (Part 1, Q1a) and he disagreed with the survey statement: "At the start of each calculus lesson I never define the aims of the lesson to students." (Part 2, Statement 5) Another example, in Lecture 3, Episode1 (01m02s-12m55s) was observed when Alex said that: "*We will be able to know a derivative presented graphically, numerically … and we will interpret the derivative ... then we will find the derivative of the function using the definition …*". In this lecture, Alex then focused on the specific aim, saying:

> "*We will study how to find the derivative using the definition of the derivative formula so basically, we need to have the derivative of a function using the limit process and f'(x) as "F dash x" and we can say F prime of X ….*" (Episode 2 (13m20s-25m00s))

Moreover, Alex was observed doing this again in Lecture 6, when he said: *"As we saw the benefits of the first derivative of a function, we now need to find out the benefits of second derivative of a function ... that will help us determine the intervals of concavity*." (Episode 7 (68m19s-80m47s)). This corresponds to his answer in the interview where he stated that:

> *"It starts from the process of preparation for these objectives. Frankly, I know the lesson and have a look at it. Do we take advantage of it in future lessons? I take into account the lessons in the future in the subject, which can depend on those lessons or build on it and then I start explaining the objectives. Then I choose the suitable opener (introduction) for the lesson and then I explain the lesson."* (Part1, Q2)

In addition, Alex stated: "*I always set the learning objectives in mind and then I build the whole lesson of planning and choosing the method of explanation … etc. and the goals are actually the basics.*" (Part1, Q3)

In the survey, Alex strongly agreed with the two following statements: "I always know how to organise the main aims of each calculus lesson that I teach." (Part 2, Statement 1), and "I always select lesson objectives for each calculus lesson by considering suitable methods for teaching." (Part 2, Statement 3)

Alex consistently highlighted ways in which the students could apply the outcomes from the lesson to the wider context; this included instances where the students might use the information in the real world. For example, in Lecture 6, Episode1 (01m51s-08m15s) Alex was observed to tell the students that: "*The main aim is to apply our knowledge in this course for solving real-world problems ... how we find maximum and minimum values*...." In order to make this connection, he often focused on the basics in the belief that if students could successfully understand and utilise basic elements of calculus, they would be able to successfully build on their understanding. For this to occur, Alex considered that methodical explanation of the learning goals was necessary.

In terms of his ability to identify key ideas in the learning of calculus Alex indicated, on several occasions, that he had a clear understanding of how to choose both the calculus topics for instruction and the teaching strategies to implement them. This was demonstrated through his interview response:

> "*I focus on concepts ... must have a starting point from which to focus on definitions and explain them in detail and give my students what other books say about the concept, maybe in the symbols ... open their minds as I can … because if the student does not understand the definition, it is difficult to understand what follows and then sequentially according to the textbook, theorem, result, proof and diagram. Nevertheless, the usual sequence begins from the definition and then an example or theorem and proof according to the importance. My point of view is for them to understand the definition and examples with applications then I focus on sketching, if there is the possibility of a graph.*" (Part 3, Q3)

These views were supported by Alex's responses in the survey (Part 3, Statement 2) where he indicated his strong knowledge between teaching strategies and topic selection. Alex demonstrated this in practice where he was observed, in Lecture 1, to highlight the available definitions to students when discussing even and odd functions from their curves and was able to explain the definition and to support it with a graph and example of each case:

> *"The even function curve can be identical on the axis Y and the curve of the odd function can be identical on the point of origin … from a function's curve we can know if that function is an odd or even nor not …."* (Episode 4 (35m00s-46m10s)

In this way, Alex was essentially providing the students with a 'route' to explain calculus ideas, and was observed to use a discussion approach in this situation when one of his students asked: "*How do we know a function is not even nor odd from their curve?*" Alex gave his students the opportunity to explain their thoughts on this question and one answered: “*Maybe apply or use their definition*". Alex mentioned the key ideas prominently in almost every lecture episode. As such, it was evident that he was trying to analyse each calculus topic using definitions, theorems, proofs and examples. For example, when he touched upon the rules of finding derivatives in Lecture 3, he was observed telling the students that:

> "*We will take the function then will define it then will see its theorem and prove it and take an example .... In the first and second lecture we had some commonly used functions ... constants function do you remember the definition of that? What is its domain? ...the theorem 1 (constants function rule) ... I will prove this theorem ... to help you to understand that let's take this example ...what is identity function? ... theorem 2 (identity function rule) ... the proof of this theorem is ... take this example for understanding that ... power function is ...theorem 3 (Power Rule) ... in your notebook try to prove that ....*" (Episode 4 (35m16s-46m49s))

These particular examples demonstrate the link between the theoretical framework surrounding DACaCu and Alex’s actions.

Some of Alex's observed lectures provided clear indications that he was able to identify the key learning ideas in calculus. For example, he referred to important issues at the beginning of Lecture 3 and used them as a tool for introducing the derivatives which are the tangent line - he mentioned two axioms:

> *“As we had in the second lecture that tangent lines were used as an introduction to the concept of limit … today we use it also as an introduction to the concept of derivative, and we will see in the future. How we use the concept of tangent line to solve several problems such as distance optimisation problems … how many straight lines can they pass through a single point?”* (Episode 1 (01m02s-12m55s))

When a student commented that "*Infinite lines can pass*" Alex explained that: “*This is one of the axioms of Euclidean geometry, and how many straight lines can touch the curve at one point only? ... and the second issue which is instantaneous velocity … that is related to today's lesson."* In addition, Alex tried to provide some common alternative notation when he saw this as necessary, such as in Lecture 3 where he was observed to ask: "*I want you to know some common alternative notation for the derivative such as that symbols D, d/dx, Df, f... .*” (Episode 4 (35m16s-46m49s)

Alex's teaching practice was not always consistent. For example, in Lecture 6 he was observed to go straight into the concept being presented, attempting to get the students to define the local minimums and maximums in order to view the function as increasing or decreasing:

> "*In this graph I want you to define a local minimums and local maximums and we can see the basic idea of what it means for a function to be increasing and decreasing ... then we can get their definitions through your understanding."* (Episode 2 (09m00s-22m11s))

He initially asked the students to move forward with the task but did not provide a lot of detail on the background of such an exercise. For Alex, this specific example was an anomaly to his general practice and the assumption here is that he anticipated either that the students would already have this knowledge, because his action supported his agreement with the survey statement: "I anticipate my students' prior calculus knowledge before the lesson." (Part 4, Statement 4) or that the students would likely not experience instances of misconceptions associated with this task, where Alex's action was supported by his strong disagreement with the survey statement: "I do not know where to direct the students if they need assistance with a particular mathematical concept." (Part 5, Statement 4)

One area pertaining to the DACaCu that Alex clearly demonstrated was in regard to the relationship between mathematics and the everyday use of terms. In his interview he stated that: "*I explain the lesson and I rely on the application dramatically ... because it makes the concept clear by intensifying examples from our life and exercises within the classroom."* (Part 1, Q2) This was evident in Lecture 6, Episode 8 (82m19s-95m31s), when Alex was observed to emphasise mathematics as a way of interpreting experience or as a human activity, indicating that people benefit from the applications of the first and second derivative of a function every day, and linking between calculus concepts and application of calculus in everyday use. He asked the students: "*Let f(x) be the temperature at time t where you live and suppose that time t = 3 you feel uncomfortably hot. How do we feel?*"

In summary, Alex was observed to frequently make the definitions he provided in lectures simpler than what was outlined in the students' textbooks. He indicated that he found visual representations easier for the students to understand and this was effective in avoiding misunderstanding, as he explained in the interview: *"My point of view to understand the definition and examples and applications of concepts, I should focus on sketches of the possibility of the graph because it is a way to show the concept idea to the students.*" (Part 3, Q3) These demonstrate a link between theory and practice as it relates to delivering the building blocks required to construct and enable Alex's students' mathematical understanding.

6.3.3 Alex's PCK of ISs

Alex was observed to be quite methodical in his approach to teaching calculus, but what was most apparent was that he required all the students, from the inception of the course, to have the same fundamental concepts. This was identified through the diagnostic readiness assessment that he provided in Lecture 1, Episode 2 (06m03s-25m50s) but what was apparent was that he expected students to self-direct their learning if they felt they did not have sufficient previous knowledge. Alex, in this case, suggested supplementary resources, including a particular book, which would help to facilitate understanding. This was demonstrated through his strongly disagree response to the survey statement: "I am not interested in how calculus is taught at other (similar) university institutions in other parts of the world." (Part 5, Statement 7) and when he told his students that:

> *"We will take this book 'calculus early transcendentals' as a main reference and we will use "التعامل مع التفاضل والتكامل" as the Arabic reference, I will give you some YouTube links to see more than one way to see how the concept of calculus is taught in other universities"* (Lecture 1, Episode1 (00m03s-6m11s)).

Seemingly realising that not all the students complete the readings or that they all learn through visual text, Alex offered multiple options to supplement his teaching outside of the classroom. In Lecture 1, he suggested a list of YouTube links so that students could not only refresh their own knowledge but also so that they could see how calculus is being taught at other universities. His approach links to the theoretical model of PCK, as Alex demonstrated the use of appropriate instructional methods in his teaching. This was exemplified by the sharing of resources, but also by assuring students that there were many styles of teaching and that his style embodied resources from others that might prove to enhance understanding.

This approach assumes that students take the initiative to achieve the baseline knowledge required. This can be challenging for some students because not only are they trying to learn the concepts of the course, but they may also be trying to catch up on previous knowledge. Alex stated in his interview that:

> *"Getting the information or knowledge is not only from the teacher ... I think that the student can get the information from Google or YouTube or he can get it from his classmates. He can get it from the program Maplesoft ... I'm with the student to correct wrong information and to tell them how to get the information. There are too many sources of information and the role of the teacher is to direct the student through proper guidance in obtaining information and correcting misconceptions."* (Part 1, Q4a, b)

Alex was also cognisant of his instructional methods when teaching. He indicated that selecting the appropriate teaching methods was essential. He indicated in his interview that teachers generally have the content knowledge for calculus teaching and that knowing the references and a selection of materials, exercises, or activities are valuable, but if not delivered through an appropriate method, these can be less effective:

> *"… because they are the basics in this subject and they are the starting point. It means selecting the appropriate teaching methods as well as knowing more than a reference and selection of the material of exercises and examples to suit the potential of students. Achieving the goal comes with the concerted efforts of the teacher by preparing the lesson and choosing the appropriate teaching methods for the lesson. For the students, by reading the lesson before entering the room and attention to the lecturer's explanations."* (Part 1, Q1b)

Interestingly, in addition to commenting on his own teaching methods, Alex placed a great deal of responsibility on the students, suggesting that they should be prepared for the lesson by reading and maintaining attention to the lecturer's explanations. In his interview, Alex suggested that he used many teaching methods:

> "*I often use the deductive method - sometimes it is how student deduces the solution. Sometimes I use the method of inductive.... There is the participation method ... also students work in a group which is called cooperative education. I also use another way to ask them that they are going to prepare the lesson and explain to their colleagues. The students prepare the lesson and in ten minutes they explain to their classmates and here the students also need to try to understand the lesson very well in this way.*" (Part 1, Q4c)

Alex's comment indicated this aspect of his pedagogical knowledge. This was supported by observation of his use of an inductive method in Lecture 3, Episode 4 (35m16s-46m49s)) when he talked about: "*Power rule. I want you to analyse this power by taking a very basic binomial of $(a+b)$ ... the first term starts with the highest power n and then counts down, whereas the second term exponents are counting up."* Alex applied the deductive method in Lecture1, when he talked about linear models:

> *"... use the slope-intercept form of the equation of a line to write a formula for the functions as $y = mx + b$, m is the slope of the line and b is the y-intercept try to solve this problem ... dry air moves upwards ... if the ground temperature is 20c ... and the temperature at ... 1 km is 10c ... draw the graph and what is the temperature ..."* (Episode 5 (47m03s-59m50s)).

Also in Lecture 1, Episode 10 (104m30s-113m11s) Alex demonstrated use of the instructional method, when he asked his students to prepare the lesson and explain it to their classmates: "*The next lecture will cover some examples and the increase and decrease of functions … please all of you prepare these topics and I will choose three of you to explain each topic for ten minutes.”* Another example occurred in Lecture 3, Episode 9 (102m04s-118m42s) when he was observed telling the students that:

> *"We will stop there, but please you think and take a look about how to differentiate a composite function and the chain rule theorem ... the next lecture I will give some of you ten minutes when you can explain these calculus ideas."*

In Lecture 6, Alex used both the instructional and cooperative learning methods when he directed his students to: " ... *stop here and make groups ... this for some exercises, I will see how you can answer them … then I will choose each group to stand in front of the board to explain one of these exercises to us"* (Episode 9 (98m04s-118m42s).

Another area where Alex attempted teaching calculus ideas using a systematic approach, based on a solid grounding in logic and its associated linguistic expressions, was observed when he explained theorem 4 (constant multiple rule) and explained the proof of it, then explained theorem 5 (sum rule) and discussed its proof, then he gave an example of each theorem. In theorem 6 (difference rule) he explained that: "*In this theorem we can use the previous theorems (4 and 5) to prove it. How can we do that?"* There was no answer and Alex asked the students to: *"... think about $f - g$ as $f + (-1)g$ does it not look like theorem 4 and 5*” (Lecture 3, Episode 5 (47m00s-69m35s)).

These approaches align with his agreement with the survey statement: "I have experienced and investigated different ways of teaching calculus." and his strong level of disagreement with the statements: "I do not know how to choose the teaching strategies to achieve the aims of the calculus topics that I teach." and "I avoid using a wide range of teaching approaches in a classroom setting." (Part 3, Statements 1, 2, 5)

Another aspect related to ISs and PCK highlights the benefits of questioning strategies. Under this subcategory, teachers are encouraged to actively involve students in the lesson through the use of *‘why’* questions and by obtaining justification beyond the procedural elements of the task. Alex attempted to use questioning strategies, for example in Lecture 1, Episode 7, (73m37s-80m49s) he was observed to seek answers to a question about an explicit algebraic function. His class was unusually quiet, his students, however, were either not willing or not able to provide a response. This lack of response to questions was consistent throughout all the observed lessons. Yet while questioning strategies in Alex’s class were not always interactive, he was able to engage the students in other ways. This was demonstrated in Lecture 3, Episode 3, (25m23s-34m51s) where Alex was observed attempting to explain a particular theorem: "*If f is differentiable at c, then f is continuous at c ... who can interpret this theorem?"*. He first gave the students the opportunity to work in pairs before asking for comments. Alex was able to elicit responses from the students and allow them to express their thoughts. In this example, students were verbally working together to solve the problem that Alex posed. This group work was much more collaborative than the class questioning strategies and allowed Alex to ultimately explain a misconception. The students could then describe where they were going wrong and come to a logical conclusion. This links well to PCK, as while Alex was not seeking answers to specific questions, the discussions elicited critical thinking skills and offered the students the opportunity to reflect on a specific case. Alex was able to consistently ask the question *'why'* of the students. In taking these approaches, Alex's actions supported his response in his interview when he said: "*I use questions and discussion throughout the lesson and at the end of each idea, also from the questions and feedback I know how my students think. Also, I always encourage my students to ask questions.*" (Part 1, Q7) Furthermore, Alex also agreed with the survey statement: "I always ask questions to evaluate my students' understanding of the calculus topic that I am teaching." (Part 4, Statement 6)

Alex's lessons were indicative of his typical teaching style, although in the later lessons he used classroom activities in many ways, including the use of technology. In his interview he explained that: "*I use both YouTube and Maplesoft personally. I try to use it in my lectures and send links to the students that help them understand calculus.*" (Part 1, Q13). In the survey he strongly agreed with the statement: "I always use a variety of ways and strategies to develop students' understanding of calculus." (Part 3, Statement 7) This was evidenced in a number of lectures including Lecture 1, Episode 6 (60m35s-73m19s) where he explained trigonometric functions and used Maplesoft to show the students graphs, and in Lecture 3, Episode 8 (95m27s-108m55s) Alex mentioned that people benefit from the applications of trigonometric functions every day:

> "*These functions are used in modelling real-world phenomena, such as waves, vibrations also, in an elastic motion. I remember when I was studying in Australia how the professor taught us to use Maplesoft in these functions. I will show you that.*"

Another area that links PCK with ISs is the use of pivotal examples and counterexamples. Alex always used pivotal examples but did not mention counterexamples in his practice and his responses in the survey and interview. He moved to high level examples and made links between mathematical and everyday use of terms through examples. From his interview, Alex's perspective on this matter suggests that he considered that learners' progression is best achieved through the use of real-life examples, as he explained that: *"I always use real-life examples, I always look for* *real life example rather than just examples. if you would like to attract your students to the lesson use examples to make linking between that and their life."* (Part 1, Q2) He further explained that: *"I explain the lesson and I rely on the application dramatically because it makes the calculus ideas clear by intensifying examples and exercises within the classroom."* (Part 1, Q10).

In terms of using real-life examples to focus on key ideas in calculus, observation during lectures identified Alex implementing this practice in the classroom setting, saying:

> *"I will leave these simple examples to you to answer them as homework, and we move to higher level examples ... let's take one of using of trigonometric functions quantities that vary a periodic manner... simple harmonic motion ... an object at the end ... vertical spring is ...s=f(t)=4 cos t ... ."* (Lecture 3, Episode 8 (95m27s-108m55s)

Other examples occurred in Lecture 1, Episode 3 (26m22s-34m15s) "... *dry air moves upwards if the ground temperature is 20c ...and the temperature at* ...", in Lecture 6, Episode 3 (23m09s-31m15s) "... *Hubble space telescope was deployed on* ... " and Episode 8 (82m19s-95m31s) "… *let f(x) be the temperature at time t where you live and suppose that time t = 3 you feel uncomfortable."* Interestingly, Alex used examples as a tool for introducing formal calculus theory such as in Lecture 3, Episode 3 (25m23s-34m51s) where he mentioned a theorem: *"If f is differentiable at c, then f is continuous at c ... let's see that we will take this example f(x) =|x|… ."* He confirmed this approach in his survey response when he agreed with the statement: "I often use examples and diagrams as a tool for introducing formal calculus theory." (Part 3, Statement 6)

Alex frequently used mathematical representation in calculus in almost every lecture episode. As such, it was evident that he was trying to show all possible ways to represent the calculus concepts as revealed in his interview:

> *"I always use the representations of functions to show all possible ways to my students. As you know there are three or four possible ways to represent a function by formula, graph, or description in words ... there is one, but I cannot remember that."* (Part 1, Q10)

In Lecture 1, Episode 3 (26m22s-34m15s) Alex was observed to ask his students:

> *"How can we sketch the graph of function? We should look for some of this function notation (Cartesian coordinates) then we draw coordinate axes then sketch the graph by using Cartesian coordinates. We can see that in these three examples...."*

On another occasion in Lecture 3, Episode 6 (70m12s-81m08s) he asked: *"Who can sketch the graph of $f(x) = e^x$?"* In another example, Alex used two of the second-level subcategories of PCK for teaching calculus (establishing appropriate learning goals in calculus and mathematical representation in calculus). In Lecture 3, Episode 1 (01m02s-12m55s) he pointed out the general purpose and three specific aims for this lecture: *"*We will be able to know a derivative presented graphically, numerically … and we will interpret the derivative … ." and in Lecture 6, Episode 2 (09m00s-22m11s) he said: *"In this graph I want you to define local minimums and local maximums and we can see the basic idea of what it means for a function to be increasing and decreasing*." In his interview Alex explained that: *"I always set the learning objectives in my mind and then I build the whole lesson of planning and choosing the mathematical representation in calculus etc."* (Part 1*,* Q2)

*T*he observation and interview responses align with Alex's agreement with the survey statement: *"I always select teaching approaches that build on student thinking and learning in calculus." (Part 4, Statement 3*) In these scenarios, Alex is not only offering a verbal explanation of the concept, but also employing other strategies including numeric, graphic, and algebraic presentations.

6.3.4 Alex's PCK in the KCaCos

In order to consider the KCaCos, two second-level subcategories need to be identified: (1) the real-world applications of calculus and (2) calculus in academic subjects. This is a challenging subcategory because the emphasis of mathematics, and particularly calculus to real-world settings is not always apparent. In Lectures 1, 3, and 6 for example, Alex was able to demonstrate this applicability. When discussing limits and derivatives with the students, Alex mentioned that people benefit from the application of calculus every day. He suggested that: "*These functions are used in modelling real-world phenomena, such as waves, vibrations, and in elastic motions*" (Lecture 3, Episode 8 (95m27s-108m55s)). In the same lesson, he also talked about simple harmonic motion and how this is related to the use of trigonometric function quantities. For students, this link to real-world applicability was beneficial because it offered an opportunity for them to see the applied value, rather than thinking of calculus in the abstract. It also put the entirety of the lesson into perspective for the students through the use of visualisation. Some students may have benefitted from visually linking the examples Alex was presenting using actual real-world entities, such as waves and vibrations. Alex's use of these real-world applications was emphasised in his interview, suggesting a good connection between his opinion and his implementation of the concept in the classroom. Alex suggested that: "*I always use real-life examples and I always look for real life example rather than just example. If you would like to attract your students to the lesson, make the link between it and their life."* (Part 1, Q10) While it is recognised that Alex, as a teacher, is more likely to have a good knowledge, everyday use of calculus, not only did he attempt to link the concepts to real-life examples such as waves and vibrations, but he also attempted to explain, and to demonstrate, to students how calculus fits within everyday usage.

In terms of the application of calculus in other academic subjects, a subset of the demonstration of PCK, Alex's choice of examples related well to other subjects. The links made by Alex, in this sense, however, were not entirely explicit, as he did not often mention other subjects by name. It was not clear if the students were actually aware of the link between calculus and other subjects, as while the links existed implicitly, there could have been more explicit connections made. For example, in Lecture 6, Episode 9 (102m04s-118m42s) Alex explained some of applications of differentiation in health science, he told the students that: "*the blood vascular system consists of blood vessels … the resistance R of the blood as* $R = C\frac{L}{r^4}$, *as example*…." The strategies that Alex used and the demonstration of calculus as a real world human activity is a good demonstration of PCK. Alex's indication that he commonly used YouTube and Maplesoft in his own work and offering students similar resources was particularly valuable in making him relatable to the students.

6.4 Case Teacher Sam

6.4.1 Sam's PCK of LCCa

In his interview, Sam suggested that:

> "*In calculus 1, the student may come without the basic information. As a teacher, in order to motivate the students to attend this course and to pay attention, he must start from the basics… you cannot present difficult things in calculus while the student does not know what x and y are.*" (Part 2, Q4)

This response by Sam not only addresses students' misconceptions but the misconceptions of teachers as well. In addressing his students' cognition of calculus, Sam's interview response indicated that his focus is on his students' preconceived knowledge, or their lack of knowledge, prior to entrance into his class.

The data collected from the observation of Sam's teaching indicated that he used certain strategies to interpret the students' misconceptions in his lectures. Sam started out with some basic concepts of calculus by asking: "*Can we discuss these term sets, the real numbers, absolute values, square roots, and the square, the inequalities involving absolute ... what is the slope of the line?*" (Lecture 1, Episode 2, (08m00s-22m13s)). When he did not receive a response, Sam reviewed the concept and determined whether the students understood by asking for their explanation after he had finished. His methodical and systematic approach generally meant that each concept was presented with an introduction of the most salient points, followed by more detailed explanations on the topic.

When Sam was teaching the students in Lecture 3 (Derivatives), there were instances in the lecture where he anticipated students' misconceptions. This was particularly evident in Episode 2 (12m18s-25m40s) where he created a linear stepped progression from one concept to another. This progression was indicated through signposting behaviour, where Sam used words such as *'first', 'then'*, and *'next'* to direct the students' attention:

> *"Let's start with definition of a tangent ... then numerical exploration of gradients of chords we can find the gradient of function ... this leads us to study derivatives we can define that ... and this will be easy when we take this example ... next I want you to think about the relationship between the tangent line and the derivative ... the tangent line to y=f(x) at … ."*

In this case, Sam was using learners' cognition to address misconceptions, thus indicating aspects of PCK in his teaching delivery. This was consistent with his responses to the interview questions where Sam highlighted the benefits of sequential teaching as essential for the building of students' understanding. He suggested that: *"The professor sets several considerations in the order of the subject so that it is easy for the students to understand the lesson sequentially."* (Part 1, Q8) Furthermore, this approach was confirmed when he stated that: "*We follow step 1, step 2, and step 3 in a systematic way to help the students to face up to calculus difficulties.*" (Part 3, Q2). Additionally, in the survey, Sam indicated that he anticipated his students' prior calculus knowledge before the lesson when he agreed with the statement: "I anticipate my students' prior calculus knowledge before the lesson." (Part 4, statement 4) indicating that his personal reflections accurately matched his teaching practices in this instance. This use of sequential teaching was consistent in all of his observed lessons and in most instances linked to the identification of students' misconceptions.

Sam used language to determine that a concept existed. He presented difficult concepts in an obvious way with examples and included phrases such as: "*Let's take power functions with negative integer exponents … what about n if it is a fraction ... if n is any real number ...*" (Lecture 3, Episode 4 (39m52s-51m21s)). He attempted to make a concept understandable and easy to grasp through starting from integer numbers to rational then real numbers. While this was a demonstration of PCK, as it relates to students' cognition of calculus, Sam presented the same logical approach to definitions as he did with moving through the concepts of the lecture. Sam presented definitions to students, such as when he discussed concavity later in Lecture 6 by simply saying: "*The concave upward is … and concave downward is … and the concavity test is …*" (Episode 7 (69m02s-83m27s)). For each of these terms, Sam provided a definition, but from the perspective of the observer, and possibly from the students' perspective, the fact that definitions were being provided was not always evident. In terms of how this relates to PCK, it is evident that Sam understood that the students required these definitions for success in the course and he wanted to make linking between these definitions.

In addition to knowledge of misconceptions, Sam also provided evidence in Lecture 3, Episode 5 (51m40s-63m30s) that he could relate to students' knowledge of the proof by presenting and sequencing the problem of proof. He provided forms of argumentation appropriate for the students' levels, this was in the theorem of product rule, when he started explaining this issue: "2^2=$2 + 2 = 2 \times 2$ *then* $3^2 = 3 + 3 + 3 = 3 \times 3$ *then* $5^2 = 5 + 5 + 5 + 5 + 5$ *then* $x^2 = x + x + x + x + x \ldots + x$ $(x\ times)$ *we will take the derivative of each side of final equation* $2x = 1 + 1 + 1 + 1 \ldots + 1(x\ times)$ $\rightarrow 2 = 1.$" Sam asked the students if this was true, however, the students did not reply. Sam counted and said: *"This is a wrong derivative of x added to itself x times."* He then asked: *"Do you think this is a simple linear equation?"* The students replied: *"No".* Sam then continued: *"Think about the right side we have $x \times x$ this means $x^2 = x \times x$. We should apply the product rule … ."* He highlighted different ways that the problems provided would be of help.

Beginning with a discussion about what the students actually knew, demonstrated that Sam had knowledge of his students' thinking and the calculus concepts that they needed to know. The class began with a discussion about what the students knew and this moved onto the theorems that the students were required to grasp. This seemed like a logical transition, as Sam was able to discern what the students knew, thus minimising a redundancy when presenting information. In his interview, Sam commented on how explanations functioned in his class:

> *"In each lecture, it is necessary to bring your pedagogical knowledge and to use this knowledge to facilitate the learning process. I use these skills so that the students can benefit from the explanation and to receive all the information. The information is firmly kept in their minds through using a pedagogical knowledge that helps me to communicate the information in the right way."* (Part 1, Q4)

Sam was observed to encourage his students to seek support when needed and continued to revise the material regularly. This behaviour suggested that Sam had a good understanding about how students learn and retain information. His incorporation of study skills and useful practices for learning calculus demonstrated an awareness of pedagogy rather than just of calculus concepts. Sam's responses in the interview indicated, however, that his desire for the students to employ concepts did necessarily transcend problems of examples: "*I present an example. In another example, I give enough time for the students to discuss this example, then I try to correct their mistakes and write the correct things on the board ….*" (Part 1, Q2b)

Sam indicated in the survey that he had a 'neutral' view regarding the statement: "I never adjust my progress through the calculus syllabus to take account of common student misunderstandings and misconceptions." (Part 4, Statement 2) This does not seem to support his practice in the classroom and his response in the interview:

> *"Most students have some misconceptions they have had from secondary or pre-secondary. These misconceptions or misunderstandings have influenced their learning ... in calculus 1 ... we must revise some definitions and concepts again in order to establish the student's information in an excellent way. This helps the continuity of the study in clear ways and the concepts are correct in college. When the concepts are right, we form the student correctly ... we must correct these things of course. There are some students who have right concepts, and some have semi-correct concepts. There are others without information, and we must re-establish them so that all students have the same level or a convergent level."* (Part 2, Q2)

He took account of misconceptions, as they could be concepts applicable across a range of subjects. Despite the discrepancy between his teaching practice and questionnaire response, Sam showed evidence that his teaching practice aligned with this subcategory of PCK.

6.4.2 Sam's PCK in the DACaCu

In terms of teaching aims and learning goals, Sam was quite clear about both the learning goals of the course and the teaching aims of each lesson, and he consistently shared this information with the students at the beginning of each lecture. In his interview, Sam explained that:

> "*At the beginning of each term, I must set the objectives of the course completely. At the beginning of each lesson, I must set the objectives of the lecture. I should set the main objective and get there by the end of the lecture."* (Part 1, Q5)

Sam employed this logical and sequential process of instruction. This was clear from the observations when he defined the learning goals at the beginning of each lesson. For example, in Lecture 1, Episode1 (02m07s-07m41s) he told the students that: "*The main aims of this course and our aims in this lecture we know the functions, their domain ..."* and also, in Lecture 3, Episode 1 (02m17s-12m15s) he told them: " *... today we aim to understand a derivative ... we are looking to know how ... why is it important? And why do we study it?"* This approach by Sam is supported by his strong agreement with the survey statements: "I always know how to organise the main aims of each calculus lesson that I teach." (Part 2, Statement 1) and "I always select lesson objectives for each calculus lesson by considering suitable methods for teaching." (Part 2, Statement 3) Sam's strong agreement with these two statements and his explanations of the aims of his lectures showed that he was employing aspects of PCK consistent with the theoretical model. His insights seemed to be based on outcomes of learning, as supported by a response in his interview:

> *“Every lesson has its objectives. Through the objectives you expect to achieve you know if the lesson is effective or not? For example, you set three objectives in the beginning of the lecture and you see if you have achieved them then by the end of the lecture. If you do not achieve an objective, you will feel that this lesson is ineffective and that you presented it in the wrong way. But if you achieved all the objectives and you have benefitted the students and they have developed their skill, I consider that the lesson is effective. Achieving the objectives is done by choosing the right way.”* (Part 1, Q12)

In continuing with DACaCu, Sam was also able to demonstrate ‘routes’ in the classroom by explaining calculus ideas, definitions, examples, theorems or proofs. In Lecture 3, Sam used the Theorem of the Quotient Rule prior to beginning an example with his students (Lecture 3, Episode 5 (51m40s-63m30s)). In this lecture, Sam explained to his students that: *“From this example, we want to deduce the quotient rule …”.* He indicated in his interview that:

> *"Each university professor sets several considerations in the order of the subject so that it is easy for the student to understand the lesson in a sequential way. The students can build their information from this course. In calculus, there are certain ideas in the order of the subject. We start with definition, then an example or a definition then a theorem and proof or an example and a theorem, then I ask for the definition according to the lesson and the background of the student for this lesson."* (Part 1, Q8)

This approach was supported by his agreement with the survey statement: “I often use examples and diagrams as a tool for introducing formal calculus theory." (Part 3, Statement 6) This suggests that Sam attempted to make a connection between the theoretical calculus ideas and the examples that were used to demonstrate these theories. Through this method, Sam was identifying the key ideas in learning calculus, consistent with the PCK framework.

While Sam demonstrated PCK in relation to DACaCu, there were areas where his abilities to determine students' learning may have been challenged by some of the strategies he chose to implement. In terms of his ability to identify key ideas in the learning of calculus Sam indicated, on several occasions, that he had a clear understanding of how to choose both the calculus topics for instruction and the teaching strategies to implement them. This was demonstrated through his interview response:

> *"There is a logical sequence of ideas. This sequence is logical .... we follow step 1, step 2, and step 3 in a systematic way. There is a sequence within the lesson which depends on the analysis of the lesson in terms of theories and definitions, examples and evidence. I use definitions and I sometimes explain examples. Then I move to theorem, proof and then an example ...."* (Part 3, Q2)

In addition, Sam often linked his organisational structure of the topic with the course syllabus and the textbook and following the textbook. This corresponds to his answer in the interview when he explained that:

> *"I use the sequence of the textbook which is acceptable. Every professor tries to strive and develop according to the quality of the students. Sometimes you enter the lecture room and you find students with no background in this area. It means you have to do a review for the basics, for example, to review many things in order to benefit the students to understand the remaining part...."* (Part 3, Q3)

Sam was recorded in Lecture 3, Episode 4 (39m52s-51m21s) and Episode 5 (51m40s-63m30s) to provide available definitions, theorems and proofs to students:

> *"We will use our knowledge of derivatives definition to find the rules of finding Derivatives ... we will have definitions, theorems and proofs, we will start with constants function ... ."*

Furthermore, Sam was able to use the sequence of calculus ideas, definitions, theorems and proofs with pivotal examples together in his lessons to give the students a sequence of calculus ideas and as a way to link theory to more practical applications. This was particularly noted in Lecture 3 (mentioned above) and Lecture 6 where Sam was explaining the maximum and minimum values, as in Episode 2 (11m43s-25m32s) and used the sequence of calculus ideas when he explained Rolle's Theorem in Episode 5 (45m41s-56m06s) and the Mean Value Theorem in the same lecture.

While Sam demonstrated PCK in relation to DACaCu, there was a logical sequence of ideas, he used the sequence of the textbook and argued this sequence is logical and he considered that most professors use it.

6.4.3 Sam's PCK of ISs

In the case of Sam, nowhere was his demonstration of PCK more profound than in his systematic approach to teaching and he was very clear that this was his foundation and justification for the way that he approached his teaching. Sam, in his interview, commented on this approach:

> *"There is a logical sequence of ideas... You cannot know integration without knowing the derivation .... All things are related; of course, we follow step 1, step 2 and step 3 in a systematic way. There is a sequence within the lesson which depends on the analysis of the lesson in terms of the theorems and definitions, examples and evidence."* (Part 3, Q2)

This systematic sequencing was demonstrated often in his lectures, with a useful example of this being observed in Lecture 1, Episode 8 (85m10s-99m10s) where Sam presented sequencing problems leading students to determine the structure of trigonometric functions. He told his students: "*Let's take the super hexagon that will help us to remember all the trigonometric formulae and identities and can help us in the future to remember them and we can find their derivative.*" (see Figure 6-9). He highlighted the difficulties with both constructing and evaluating calculus concepts in order to ensure that the students were aware of areas that were particularly challenging.

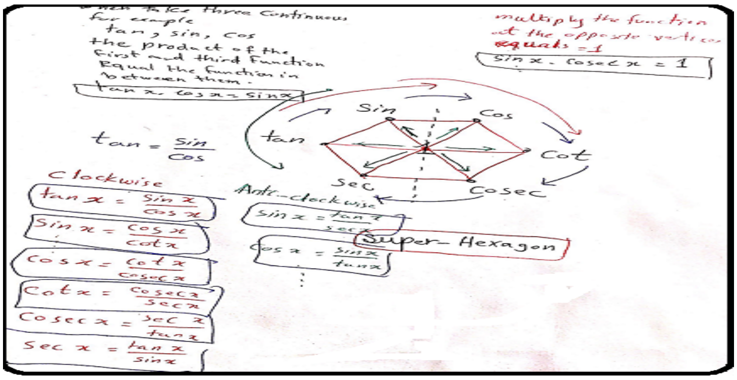

Figure 6-9: Super Hexagon for Trigonometric Functions

Another example was observed in Lecture 3, where Sam presented sequencing problems that could lead the students to more easily see the structure of certain calculus concepts:

> *"If f a differentiable function, then its derivative f' is also function we can see its domain... so f' might have a derivative of its own is called the second derivative we can use f'' ...and we can see the third derivative... we can have this example $f(x) = 5x^6 + 2x^2$...this is called higher derivatives ...*" (Episode 3 (26m03s-37m45s)).

This approach was corroborated through his strong agreement with the survey statement: "I always use a very mathematical way of teaching calculus." (Part 3, Statement 3)

While Sam's systematic approach to instruction meant that his students were well aware of how the lesson would proceed, I noted that there was an abundance of teacher talk within his lessons. In his interview Sam indicated that he valued group work and collaboration as he mentioned that: "*I use cooperative learning, homework and discussion ...* " (Part 1, Q4a), but in many of the lectures that I observed, he did most of the speaking and he did not ask for the involvement of students.

While there were instances when Sam asked the students for an answer (e.g. when he asked the students to deduce the rule from the example in Lecture 3, Episode 5(51m40s-63m30s)), more often he was telling them to remember certain mathematical concepts, as demonstrated in Lecture 3. In this lecture he reminded the students to: "... *remember all of the laws to make the table of differentiation formulas"* (Episode 6 (64m28s-70m05s)). Part of the challenges associated with Sam's linear progression, through the objectives of the lessons, seemed to be the cause for the lack of questioning. In Sam's case, from his interview response it appeared that his ultimate goal was to teach the students the objective of the day, as he explained: "... *we reach the ideas and conclusions then write them on the blackboard after all the discussions with the students*" (Part 1, Q4b), rather than to facilitate or check for the students' understanding of the concepts. Sam's linear approach to meeting course and lesson objectives somewhat deviated from the questioning strategies subcategory, as outlined in the theoretical model.

Despite Sam's lack of questioning strategies, he did use a significant number of examples to help him to explain the concepts that he was trying to teach. In Lecture 6, Episode 6 Sam utilised multiple examples as he encouraged the students to think about the general case: "*Now we will take some examples to explain these ideas... the final example I will change the idea of question ... suppose the first derivative of a function g is g'(x)=* $(x+1)^2(x-3)^5(x-8)^4$*..."* (57m43s-68m12s)). In this case, he did utilise some questioning but was primarily focused on using the pre-prepared examples to help focus and explain the ideas of the lesson.

Sam was more forthright in his responses to the survey and interview components, suggesting that he frequently used questioning strategies as a means to encourage thinking about the general case. During his interview, he stated:

> "*The discussion is the most important thing to achieve objectives. This gives the students freedom so they can speak and discuss and that there is no fear of the professor.We motivate all students to work in groups. We give them worksheets to work on outside of class. They find solutions as assignments. The last example of this is the homework that was presented in groups yesterday. Formation of the groups is free among students. Each student chooses his colleague as he wishes… I always ask students questions and use discussion on a permanent basis.*" (Part 1, Q9)

Sam indicated in his survey response that he agreed with the statement: "I always ask questions to evaluate my students' understanding of the calculus topic that I am teaching." (Part 4, Statement 5) However, this was not as evident in practice, suggesting some discrepancy between his pedagogical understanding and his practical application.

In terms of relating this to the PCK model, there are questions surrounding what constitutes a 'pivotal' example, as described by the theoretical framework. The examples Sam presented were useful, but because they were pre-prepared, they often relied on assumptions about what the students would find valuable. Sam highlighted in his interview that he valued and always used examples:

> *"... because all the definitions and theorems are the same in all the resources and only the examples differ. What examples can you provide for the benefit of the student? The students benefit from the examples not only the definition or the theorem ...."* (Part 1, Q6).

Correspondingly, in the survey, Sam agreed with the statement: "I often use examples and diagrams as a tool for introducing formal calculus theory." (Part 3, Statement 6) and I noted that Sam used examples in almost every lecture episode. As such, it was evident that he was trying to explain calculus ideas through examples. A case in point was noted in Lecture 1, Episode 3 (22m16s-31m25s) where he told his students: "*Let's take these examples find the domain and range and sketch the graph of these functions ....*" Sam always attempted to present his examples in different ways, for example in Lecture 3, Episode 8 (83m00s-104m00s) (See Figure 6-10). This corresponds to the answer he gave in the interview when he explained that: *"I present an example then another example, I give enough time for the students to discuss this example ... I teach the students different approaches and methods in solving examples."* (Part 1, Q2a) Moreover, Sam used the counterexample once in Lecture 1, Episode 7 (72m00s-84m17s) when he explained to the students that: "*We will take these examples on the even, odd functions and counter-example to show you the non-odd and non-even function ....*" Therefore, while they were likely very important, Sam's direct line of thinking may have influenced which examples could be seen as pivotal for the students and which ones were simply helpful within the context of the lesson.

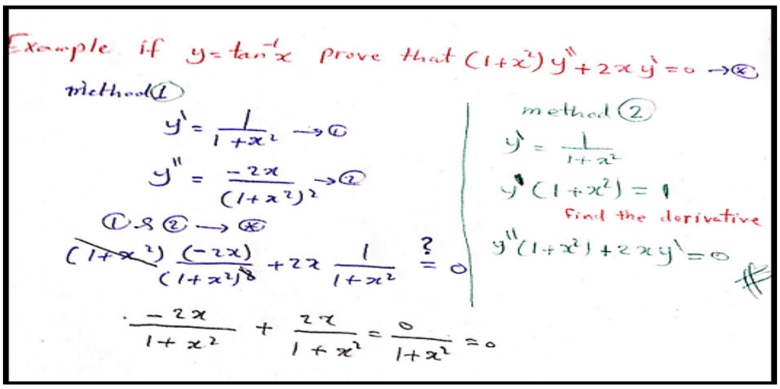

Figure 6-10: Example of Presenting an Example in Different Ways

An area that was particularly prominent in the lessons taught by Sam related to the mathematical representation in calculus, as described in the theoretical framework. In his lectures, Sam was observed to be a good communicator when it came to the topic being discussed. As everything had clearly been planned out in advance, he was able to draw on the strong connection between the representations that students used and their understanding. For example in Lecture 1, Episode 8 (85m10s-99m10s) he explained that: "... *these functions' examples, we will try to sketch their graphs we are going to start with the trigonometric functions* ... ." Sam highlighted that there were instances where students could make good connections if they were able to use diagrams and explanations as tools for understanding theory (in line with his response to the survey statement Part 3, Q6). In his interview, Sam's perspective on this matter suggested that he considered that mathematical representations are best achieved through the use of many ways. He acknowledged that: *"These things, of course, are algebraic symbols and we can express them with graphics. We can represent function, derivative and limit ... graphically with real examples. We can express many images in these things … ."* (Part 1, Q10)

Furthermore, Sam was able to use many different representations, such as visual, symbolic and verbal ideas together in his lessons to give the students a wide range of perspectives and as a way to link theory to more practical applications. This was particularly noted in Lectures 1 and 3 where Sam was discussing the basic properties of trigonometric functions, in Lecture 6, when he talked about maximum and minimum values such as in Episode 2 (11m43s-25m32s) and use the role of representations as recognised in communicating ideas when he explained Rolle's Theorem in Episode 5 (45m41s-56m06s) and the Mean Value Theorem in the same lecture.

6.4.4 Sam's PCK in the KCaCos

In Sam's classroom, there were certain instances where he made reference to ideas that went beyond the actual learning of the materials associated with calculus. For example, in Lecture 3, Episode 9 (102m04s-118m42s) Sam discussed James Gregory, giving a historical overview of this person and how he was associated with mathematics:

> *" I will stop as a lecturer of calculus and will talk as a history teacher about the chain rule ... this is what we will have next lecture, but I would like to give you a summary about the history of that and help you to prepare ... the first person to formulate that was James Gregory ... he was a Scottish mathematician .... then Andrews ...."*

This historical piece was, from Sam's perspective, fundamental to the students understanding of calculus because knowing the origin could provide context for future learning. In addition to linking calculus to a historical component, Sam was very focused on linking calculus to real-world applications as well. One of the ways that Sam was able to demonstrate the connections to the real world occurred in Lecture 6 where he was discussing the applications of differentiation. In this scenario, Sam was highlighting not only the real-world components but how enjoyable this lesson was because of its practicality. In this lecture the researcher recorded that Sam stating that: *"I really like these lectures, which are enjoyable ... today we will apply what we have learned ... we will combine functions with (domain, range, interval), limits, continuity, and derivatives in real-life examples ..."* (Episode1 (04m09s-11m22s)).

Sam indicated in his interview that he felt that representations and images were of particular importance when teaching. He suggested that examples gave context to students and this assisted in the avoidance of misconceptions. He explained that: *"We can represent function, derivative and limit ... graphically with real examples. We can express many images in these things…."* (Part 1, Q10). While a suggestion of real-world examples is necessarily a component of the PCK theoretical model that has been identified, it is generally posited that if a teacher has more interest in a specific area of study, they may be more inclined to show real-world applications.

Additionally, in terms of Sam's demonstration of the real-world application of calculus, there were instances where he emphasised the benefits of calculus and linked it to specific applications, as described in the theoretical model. In Lecture 6, Episode 3 (25m59s-34m15s) Sam spent some time talking about the Hubble telescope and used a pivotal example for identifying relationships between mathematics and application by telling the students that: "... *the Hubble space telescope was deployed on ... is given by ... estimate the absolute max and min values of the* ..." where he focused on minimum and maximum values. By putting this example in the perspective of a calculus model, Sam was able to hold the students' interest and express value in the application of the topic. This teaching practice, and others, generally demonstrated that in certain instances, Sam was able to demonstrate his knowledge of the calculus connections in wider real-world applications.

6.5 Case Teacher Tom

6.5.1 Tom's PCK of LCCa

In the case of Tom, this teacher seemed to be drawn to the weaker students and making sure that they were being supported. However, he seemed to find it difficult to find a balanced approach in teaching the students that were weaker and those that excelled in his course. For example, he was observed to indicate that he knew his students were going to experience difficulty with functions when he asked them what *"… exponential functions are... if x and y are positive numbers and n and m are any real numbers, can you find $x^{n+m} =?$ and $x^{n-m}?$"* (Lecture 1, Episode 7 (72m13s-84m44s)) There was no response from the students so Tom continued "… *we will review the laws of exponents ... .*" As a result, he suggested to the students that this would be an area to focus on.

These types of recommendations were consistently deployed by Tom in his lectures. For example, they were noted in Lecture 3, Episode 5 (42m15s-55m44s) where he discussed the challenges behind proofs:

> " ... *the limit of a sum is the sum of the limits ... I will prove the first law and the third ... then we will take the law and give an example ... I will leave the proofs of the rest of theorems to you to read them at home ... I know you have difficulties how a proof should begin ... these proofs will help you to know how you can use other laws in manipulating mathematical ideas."*

And when he explained the difficulties students might have with constructing and evaluating limits at infinity, as demonstrated by his comment: "*We have some laws which will help you to know evaluating limits at infinity …."* (Episode 5 (61m00s-68m22s)) (See Figure 6-11)

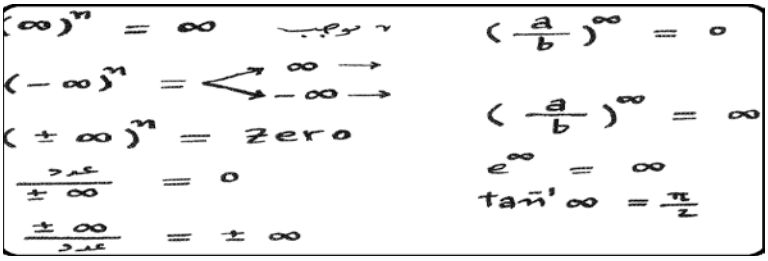

Figure 6-11: Laws of Infinity

Tom spoke directly to the students and forthrightly highlighted areas where the students tended to struggle. This type of approach is a type of signposting of students' misconceptions and learning difficulties. Tom was ensuring that essential concepts that were fundamental to success throughout the course were flagged up for the students. There are many benefits to this type of approach that relate to supporting the student. In the case of Tom, this type of behaviour indicated that, from a PCK perspective, he was aware of the misconceptions and able to address them in the classroom. Should weaker students in a class be struggling with the comprehension of certain topics, their stress may be reduced simply by knowing that these topics are likely to be common challenges among students overall.

The way that Tom addressed the students' misconceptions was paralleled by his own knowledge about students' thinking about calculus concepts. Tom attempted to anticipate areas of particular concern and to make these important aspects in the course structure. In the very first lesson, Tom was observed to suggest to his students that they: *"you will be mathematics teachers ... must have a background in the functions and derivatives ... and limits and a background in differentiation and integration."* (Lecture 1, Episode1 (02m02s-09m532s)). He also indicated that this was an essential component of calculus when he spoke in his interview, where he said:

> “*There is little difference in objectives, for example, between teaching calculus for mathematics students and chemistry, physics students or any other subjects. For example, in mathematics, the focus of our conversation is mathematics students, who will be maths teachers ... my goal for students to have a better understanding of calculus at the end of the course, they must have some knowledge prior to attending and my role is to correct their misconceptions and I help them overcome difficulties.*” (Part 1, Q1a)

While Tom seemed to inherently know the fundamental challenges that his students were likely to have in the classroom, there were some deviations from this way of thinking in his responses in the survey. For example, Tom indicated ‘neutral’ to the statement: ‘I anticipate my students’ prior calculus knowledge before the lesson.’ (Part 4, Statement 4) This response is surprising, as it was clear that Tom both understood the fact that some students were weak at calculus and that there were many common areas where students tended to struggle. Therefore, it was evidenced in his teaching practice that he did anticipate students’ prior knowledge, though the self-reflection on this aspect of PCK was not as clear.

Tom also provided evidence in his teaching that he could identify students’ progression in understanding typical calculus concepts. This was demonstrated in Lecture 1, Episode 6 (61m05s-70m40s) when Tom explained to the students that there were foundational requirements of algebraic and trigonometric functions that were paramount to moving forward in the course. He asked them: "... *what does it mean algebraic functions? ... classify the following functions as one of the types of functions that we have learned ...*". By establishing some of the mathematical concepts that were necessary, but that were identified as previous mathematical knowledge, such as in the end of this episode, Tom sought to obtain justification beyond just procedures. He asked the students: “W*hat is the difference between* $f(x) = 5^x$ *and* $g(x) = x^5$?" and was indicating to them the logical sequencing of mathematics, to ensure that they had the building blocks needed to be successful in the course. This was a demonstration of PCK, in relation to the theoretical framework, as Tom's approach was highlighting the learners’ cognition; furthermore, he was indicating that students learning follows a scaffolding strategy.

Other than the slight anomaly with the survey, Tom seemed to be cognisant of both the challenges that the students have in calculus and how he, as a teacher, can most effectively address these challenges. In his interview, Tom was forthcoming about the types of students that enrolled in his class. He suggested:

> “*When the students come to the mathematics department, they vary at different levels: distinguished students, modest students and weak students. So, what do I do in this case? Of course, if you follow every weak student, this will hurt the distinguished; and the opposite*

is true. There are people who are not good, because they are not bad, because they have not worked on themselves, I have to find a balance." (Part 2, Q2)

This was demonstrated through his strong disagreement with the survey statement: "I never consider individual differences among students when planning my calculus lessons." (Part 2, Statement 2) When considering his response to this statement, it is clear that Tom is demonstrating aspects of PCK as they relate to the theoretical model because he is demonstrating an ability to address misconceptions and knowledge of students' thinking in his explanations and in the classroom setting.

6.5.2 Tom's PCK in the DACaCu

When considering the developmental aspects of calculus, one area where Tom was particularly vocal about a particular concept was in the area of learning goals. In the first lecture, Tom set out the learning goals for the students by explaining ways of thinking: "*You will be maths teachers ... must have a background in the functions and derivatives ... and limits and a background in differentiation and integration, you should know the best methods to understand them ....*" (Episode1 (02m02s-09m532s)). Tom suggested to the students that they should think like future maths teachers, indicating that they should able to fully explain a concept to someone else and have them be able to understand the material. In terms of written material, Tom pointed to the syllabus as a particularly useful resource for students. He suggested in the interview that important objectives were set in the course syllabus and these assisted the students in knowing the objectives of the lesson. He suggested that: *"… through the course syllabus I set the important objectives and I often explain them to the students. Sometimes through the lesson, the student knows the objectives of the lesson….*" (Part 1, Q5) Tom's explanation of the goals of the lessons continued throughout the observation period. For example, at the beginning of Lecture 3 (Episode1 (05m20s-07m58s) he told the students that: *"Our topic today is limits, and this lecture is considered as an introduction to limits and the most important lessons in calculus 1 ... by the end of this topic of limits, you will be able to find the limit of the function ... ."* Tom again highlighted the goals of the lesson and what the students should be able to accomplish by the end of the lesson. The same was true in Lecture 6 where he stated that: "... *derivatives which can be applied to solving problems in several subjects such as engineering, physics...etc ...."* (Episode1 (03m01s-10m48s)). He outlined the aim of the lesson along with a definition and a practical example of derivatives to demonstrate to the students what they would be discussing in the class. This was supported through his strong agreement with the survey statements: "I always know how to organise the main aims of each calculus lesson that I teach." and "I always select lesson objectives for each calculus lesson by considering suitable methods for teaching." (Part 2, Statements 1, 3) All of the above examples indicate that Tom was establishing learning goals in his calculus lessons. As the students were able to progress through the course, it can only be assumed that these learning goals were 'appropriate' as outlined in the theoretical model of PCK.

In addition to his continuous description of learning goals, Tom also prominently discussed definitions, theorems, and proofs that the students required for success in his course. For example, at a critical moment in Lecture 1 (Episode 2 (10m01s-22m58s)) Tom provided a definition of one-to-one functions to his students when he told them that: *" ... we will talk about functions definitions ... focus on this example on the whiteboard ... look at representations of functions by a graph ... we define one-to-one functions ... and we have piecewise defined functions ... ."* Tom felt that this was a pivotal moment for his students to have the definition because they would be working with it throughout the lecture and it would be something that was required knowledge for further concepts later in the course. He suggested that definitions were essential as they related to formal calculus theory (survey, Part 3, Statement 3) and that examples and diagrams were generally better used after this definition was presented. Tom's response to this survey statement was generally supported in his teaching as he went on to use the familiar structure of (1) providing a definition, (2) providing an example/graphic, and (3) explaining the value of the concept throughout the lectures observed. Tom showed consistency in his teaching practice and in his thinking in the interview also, as exemplified when he said:

> "*I start with the definition and give examples to simplify this definition. Then after giving these examples, I give theorems, and the basic properties with proofs. If the characteristics of the theorem and proofs are facilitated and simplified, I try to demonstrate simplified ... .*" (Part 1, Q2a)

In terms of the building blocks of mathematical theories, Tom often linked his organisational structure of the topic with the course syllabus and the textbook and followed the textbook. He was recorded in Lecture 3, Episode 5 (42m15s-55m44s) to provide available definitions, theorems and proofs to students:

> " *... the limit of a sum is the sum of the limits ... I will prove the first law and the third ... then we will take the law and give an example ... I will leave the proofs of the rest of theorems to you to read them at home ... I know you have difficulties how a proof should begin ... these proofs will help you to know how you can use other laws in manipulating mathematical ideas ....*"

In addition, this approach was obvious in Lecture 6, Episode 3 (30m00s-40m10s) when he told the students to: *"Look to your textbook we will have the Fermat's theorem and explain the proof"* and

Episode 7 (81m24s-96m11s) when Tom provided available definitions, theorems and proofs to the students when he explained Rolle's Theorem with three cases and did the same structure with the Mean Value Theorem in Episode 8 (97m18s-113m07s). In all these instances Tom was essentially providing a route for the students to follow in order to achieve success in this topic and provide a valuable overview of the material for the students. This corresponds to his answer in the interview when he explained that:

> "*I try to adhere to the Arabic reference "التعامل مع التفاضل والتكامل" in the sequence so that I do not confuse the students. Teaching topics based on their sequence in references and syllabus is better. I don't use other references except in rare cases."* (Part 3, Q2)

Moreover, Tom justified his actions in his answer to another question in the interview when he explained that:

> "*I respect the plan, the programme and the existing approved course elements. If there is an addition, there is no objection to this so as not to change the order. I adhere to the reference that I use in the lecture through the arrangement of ideas and lessons so as not to make a difference for the students between what that takes in the lecture and the approved reference*." (Part 1, Q8)

Tom's response to the survey statements: "I am aware of how the calculus material I teach fits within the bigger university context." and "I am not interested in how calculus is taught at other (similar) university institutions in other parts of the world." (Part 5, Statements 5, 6) was 'neutral'. This does seem to support both his practice in the classroom and his response in the interview, when he stated that he was interested in following the textbook. From these data, it can be assumed that Tom is less concerned about where the material appears in the lecture, but rather that theorems are appropriately explained to students in a way that is clear and logical for the particular situation. Again, while this seems like a valuable use of building blocks to construct, it was not always demonstrated in practice. Tom did not introduce any theorem in Lecture 1, and in Lecture 3, he did not focus on the theorem, but it was presented as any information in the lecture. This also occurred in Lecture 6, though theorems were heavily targeted in this lecture right at the beginning. In both of these instances, Tom highlighted the value of the inclusion of theorems into his teaching, though less was provided on the building blocks of mathematical theories that enable students' mathematical understanding.

6.5.3 Tom's PCK of ISs

Tom, in his responses in the interview, generally suggested that he did not have particular ISs or teaching methods that he specifically employed. He suggested that this was because he did not have enough experience in this field:

> *"I do not claim knowledge of the full teaching methods, but I consider myself that I have little experience that allows me to evaluate the level of the student in front of me. I do not have a particular teaching method, or I'm not interested in the ways of presenting a lesson which I consider to be more formal than useful. I do not have enough experience in this field, as I told you. I do not prepare teaching methods, but I try to make links between visual, symbolic, and verbal ideas in my lesson ...."* (Part 1, Q4)

This was supported by his responses in the survey, where he indicated that he did not have a wide range of knowledge in planning calculus lessons and had not experienced or investigated different ways of teaching calculus (Parts 2 and 3, Statements 4,1). In terms of his instructional methods, Tom tended to follow the same strategy throughout. He began with the lesson overview and an evaluation of the homework. Definitions and examples were then provided for the students with strategies for continued study appearing at the end. He stated in his interview that: *"I do not like to use different teaching methods ..."* (Part 1, Q9) indicating that he preferred the lecture style when he said: *"... the lecture and urge the students to work on their own through doing a lot of exercises."* (Part 1, Q11)

In considering the link to PCK, one of the characteristics that falls into the category describing the relationship between instruction and students' ideas suggests that teachers should demonstrate appropriate instructional methods. While Tom may not have necessarily acknowledged that the methods he utilised were thought out in detail, from the lessons observed they appeared to be appropriate to meet the needs of the students. This was discerned from the observation in Lecture 6, Episode 2 (11m09s-29m15s) where Tom explained to his students that:

> *"We face a lot in our daily lives the problem of finding the best way for doing something, sometimes the problem turns into a matter of searching for maximum and minimum values ... can you give me examples of that .... "*

In the middle of this episode he said: *"I want to mention the issue of existence case ... this includes type of set and also, type of function ... who can talk to us about that?"* Finally, at the end of this episode, Tom actively involved the students in the lesson through questioning, as demonstrated when he asked his students: *"… where do extreme values occur?"* Here, the students were able to participate in the lesson on the applications of differentiation. In this lecture, the students were required to discuss and talk and because of this interaction there was some evidence of Tom using ISs, however, in his interview he did not recognise that he had done so.

Another aspect of PCK relates to questioning strategies, which Tom did not employ consistently. Areas where he did use these strategies appeared most prominently in Lecture 6, Episode 2 (11m09s-29m15s). In this episode, he initially asked the students some questions about the applications of differentiation. This relates to the point above about appropriate ISs. Tom then continued the lesson and used questioning techniques to ensure that the students understood the various points of the lesson. This also gave the students the opportunity to participate in the lecture and to be able to demonstrate their knowledge. This also allowed the students to employ their critical thinking skills, as they looked for answers to the questions posed. Throughout this process, Tom was employing PCK when he encouraged students to think about the general case and when he checked their understanding in this lecture only, the students were interacting with him. Tom indicated a 'neutral' response to the survey statement "I always ask questions to evaluate my students' understanding of the calculus topic that I am teaching." (Part 4, Statement 6). He stated in his interview that:

> *"I sometimes ask in the lesson "do you understand?" As I told you before, there are some students who do not answer if they understand or not. But when I expect the majority to understand, I move to the second part because I am obliged to a particular curriculum ... ."* (Part 2, Q6)

In the observed lessons, Tom used many pivotal examples, moreover there were many obvious situations where he demonstrated knowledge of using examples to focus on key ideas in calculus in practice. It is acknowledged that Tom certainly used example strategies to determine explanations of calculus ideas. These examples, however, were largely directed at students who were weaker, thus not necessarily addressing the distinguished students in the class that Tom highlighted were a subset of his student population. He tried to provide some examples for the distinguished students, such as in Lecture 1 when he provided an example of linear function: "*This example for distinguished students ... dry air moves upwards ... if the ground temperature is 20c ... and the temperature at ... 1km is 10c ... draw the graph and what is the temperature ...*" (Episode 5 (46m01s-60m12s)). Another example occurred in Lecture 3, Episode 8 (85m20s-108m56s) when he told the students that: *"The signum function denoted by sgn is defined by ...."* Tom told the students to copy some examples from the whiteboard and practice the questions for homework. In his interview Tom suggested that examples are very important, as indicated by his comment: *"I start by definition and giving examples to simplify this definition. Then after giving these examples, I give theorems, and the basic properties with proofs ...."* (Part 1, Q2a) He suggested that the examples were essential as they related to explaining and clarifying calculus concepts (survey, Part 3, Statement 3) and that examples and diagrams are generally better used after the definition has been presented.

One of the ways that teachers can move beyond CK to aspects of PCK is through the enhancement of linking visual, symbolic, and verbal ideas in calculus. This approach was evidenced in Lecture 1, Episode 2 (10m01s-22m58s), where Tom drew strong connections between the representations use and the students' understanding when he told them that:

> *" ... we will talk about functions definition ... focus on this example on the whiteboard .... look at representations of functions by a graph ... we have three item domains? Range? Co-domain? the domain ...."*

Another example observed in Lecture 6, Episode 3 (30m00s-40m10s) was given by Tom: *"How to find maximum value or minimum value? Let's have these examples to focus on max-min values, and I am sketching their graphs ...."* A teacher with strong PCK in calculus is likely to ensure that multiple mathematical representations are employed in order to ensure individual comprehension. According to Tom:

> "*I give them a way to understand and do not memorise. You know that mathematics does not encourage or advise to memorise, but [for the students to] understand and know how to retrieve information ... graphs. Sometimes, they play this role, may be one of the means by which we remember certain laws and certain relations or specific theories ... I use the diagrams in my teaching."* (Interview, Part 1, Q10)

Tom was more forthright in his response to the interview questions, suggesting that he frequently used mathematical representations as a means to encourage thinking about the general case (see interview Part 1, Q4, p 171).

6.5.4 Tom's PCK in the KCaCos

In presenting the KCaCos to the students, Tom completed this undertaking as it relates to the theoretical model but generally did so in a rather vague way. For example, in Lecture 6, Episode 2 (11m09s-29m15s) Tom stated to the students: *"We face a lot in our daily lives, the problem of finding the best way for doing something, sometimes the problem turns into a matter of searching for maximum and minimum values …."* Yet despite this general interpretation of how calculus might fit into the larger picture, Tom was quite aware of the challenges that students could face trying to link the abstractness of some of the calculus concepts with everyday use. In his interview, he commented on that see (interview part 1, Q10 above).

This demonstrates the applicability of Tom's knowledge of the real-world applications of calculus despite not fully realising this vision in the classroom. In addition, when Tom participated in the interview on this subject, he highlighted the link between "... *calculus and physics and how derivatives were a useful example of how these two academic subjects could coexist."* (Part 1, Q10) He suggested that the mathematical diagrams and the associated laws may help students to consider which application to use when attempting to solve a problem. In considering this connection, Tom is suggesting links to other academic subjects, which is a product of the theoretical model of PCK. Yet despite Tom's conscious understanding of these connections, his ability to relay this information to the students was lacking. He briefly mentioned physics in Lecture 6, Episode 1 (03m01s-10m48s) but did not go into any detail about the connection that students should form. Instead, he required the students to make the connection between the two implicitly, but with very little emphasis on this component, it is difficult to determine whether students were actually able to draw the conclusions Tom expected of them.

6.6 Cross-Case Analysis

This cross-case analysis, related to the PCK elements, is built on the findings from the four specific cases. This cross-case analysis is presented in the sequence of the research question.

> **RQ2:** Using this model of PCK, how do calculus teachers articulate and demonstrate their PCK?

6.6.1 Learners' Cognition of Calculus

Learners' cognition of calculus comprises two second-level subcategories; students' misconceptions and learning difficulties in calculus, and knowledge of students' thinking about calculus concepts.

6.6.1.1 Students' Misconceptions and Learning Difficulties in Calculus

In terms of students' misconceptions, the calculus teachers in this study addressed these aspects in different ways. All of the teachers, however, presented a clear understanding of what students had learned at the secondary school level and all indicated that the students did not have much prior calculus knowledge upon entering university. For these teachers, the outcome was a need to have the students learn the material of the course, but also to really understand the 'why' associated with the fundamental concepts that exist for each calculus topic taught. For John, this came primarily from explanations, whereas Alex suggested that misconceptions were best addressed through cooperative education. John's methodical approach ensured that every concept received diligent teacher-talk associated with it; however, Alex provided opportunities for students' participation. Both John and Alex used their knowledge of learners' cognitions to address anticipated questions and students' misconceptions and identified the students' formation of mathematical concepts in calculus. Furthermore, John always asked his students if they understood and if they answered no, he attempted to re-explain the examples using more than one method. John's challenges to engage students in the classroom was not unique, as other teachers experienced difficulties with interaction. What made John's case unique was how he demonstrated consistency, suggesting that he knew what the students' misconceptions and difficulty were and did his best to address them. While Teacher Alex tended to use a more implied approach by using other examples and sometimes tended to use cooperative learning methods in this situation. In the instances where the proofs were considerably difficult, the teacher allowed the students to work in peer groups prior to a general group discussion for comprehension, as occurred in one class conducted by Teacher Alex. While Tom did not re-explain any example, but Sam used the discussion approach when he has to re-explain the example.

Both John and Alex used diagnostic tests in their first lecture with their new students, devising suitable diagnostic tests, which posed appropriate questions. Alex, while having a clear understanding of what challenges the students faced was willing, or proficient, in finding strategies that would effectively address these misconceptions in the calculus classroom. He was able to offer some suggestions for weaker students about the resources they could utilise to ensure that they had enough background knowledge in calculus to be successful in the course. In contrast, Sam did not use a diagnostic test, but rather started his first lecture with some basic concepts of calculus through discussion. Sam's interview response not only showed his awareness the students' misconceptions but the misconceptions of teachers as well. Sam's comment suggests that teachers also come to a class with a certain set of preconceived notions about what students are expected to know. If students' knowledge is very basic, then the construction of calculus concepts, necessary for the class, may not be fully understood, leading to significantly more difficulties. In addressing his students' cognition of calculus, Sam's interview response indicated that his focus on his students' preconceived knowledge, or their lack of knowledge, and his methodical and systematic approach generally meant that each concept was presented with an introduction of the most salient points, followed by more detailed explanations on the topic. It seemed different in the case of Tom, who did not start by reviewing his students' knowledge, but appeared to be generally very supportive of his students. He chose another way to address his students' misconceptions. While Tom seemed inherently to know the fundamental challenges that his students were likely to have, there were some deviations from his way of thinking in his responses in the survey. For example, Tom indicated 'neutral' to the statement: 'I anticipate my students' prior calculus knowledge before the lesson.' This response is surprising, as it was clear that Tom understood both the fact that some students were inherently weaker at calculus and that there were many common areas where students tended to struggle.

In the survey, both John and Alex showed that they were willing to modify their approach, or the syllabus, to accommodate the students' misconceptions. It was clear that they both understood that some students were inherently weaker at calculus and that there were many common areas where students tended to struggle. On the other hand, Sam indicated in the survey that he had a 'neutral' view regarding the statement: "I never adjust my progress through the calculus syllabus to take account of common student misunderstandings and misconceptions." This, however, was not his practice in the classroom nor his response in the interview. He recognised that not all students came to his calculus classroom with the same level of knowledge and therefore he made the homework difficult to ensure that students were both challenged and sought out collaboration to complete the activities to address their misconceptions. To address the students' misconceptions in practice, Alex challenged the students in another way by directing them to deal with particularly challenging definitions. This suggested that Alex was aware that the definition was probably more difficult than the students could handle, or that a simpler explanation would reduce the number of difficulties that were being experienced by the students in order to ensure that the students were only receiving material on calculus functions that they could reasonably evaluate.

Both John and Tom closely adhered to the textbook without reference to correcting the previous concepts of definitions, whereas Sam attempted to make a concept understandable and easy to grasp. Both Alex and Tom were particularly cognisant of the students' misconceptions and how to address them using homework as a foundation to build on. This approach does not relate to the implementation of homework in the students' course requirements. Furthermore, rather than assuming that all calculus students would have the same misconceptions as past and present students, Alex used the identified learning difficulties of his students, which goes beyond the notion of general misconceptions. Alex used real-world challenges to address the students' difficulties. On the other hand, Sam selectively chose structures, (see Lecture 3), where he felt that his choices would alleviate some of the misconceptions that the students experienced. John encouraged the students to work in self-determined groups of 5 or 6 to solve problems, however, by putting the students in pre-designed groups may have been preferable because students would then be able to use their differing knowledge and skills levels to correct each other's misconceptions.

6.6.1.2 Knowledge of Students' Thinking About Calculus

The second aspect of the theoretical model that addresses learners' cognition of calculus relates to the knowledge of students' thinking about calculus. Sam began with a class discussion about what the students knew, and this moved onto the theorems that the students were required to grasp. On the other hand, Alex seemed to assess whether the students were able to employ the thinking skills that would allow them to move to the next level of calculus understanding.

6.6.2 Developmental Aspects of the Calculus Curriculum

In considering the developmental aspects of the calculus curriculum, two second level subcategories require consideration; the establishment of appropriate learning goals, and the identification of key ideas.

6.6.2.1 Establishment of Appropriate Learning Goals for Calculus

All four teachers demonstrated the use of PCK in identifying their teaching aims. They demonstrated strategies for presenting the learning aims, objectives, and/or learning goals for the lessons in different ways and supported the observed strategies by their responses in the interviews and questionnaires.

Sam was the most linear of the four participants as his lectures followed a similar format. Not only did he explicitly present the teaching aims of each lesson at the beginning, but he also stated the overall objectives for the course. For Alex, the learning goals seemed to be more flexible, depending upon the students’ understanding. He often focused on the basics in the belief that if students could successfully understand and utilise basic elements of calculus, they would be able to successfully build on their understanding. Alex made many more links between the learning goals and the topics and the way that students progressed through the course. He consistently highlighted ways in which the students could apply the outcomes from the lesson to the wider context; using instances where the students might use the information in the real world. For this to occur, Alex considered that a step-by-step methodical explanation of the learning goals, based on the level of students' understanding, was necessary. Similarly, John's use of learning goals to facilitate students' understanding and comprehension mirrored this strategy. John highlighted very particular and specific aims for what he was teaching, asking the students what they felt the aims were before providing his own interpretation. In contrast, Tom was particularly vocal about what a particular concept was in the area of learning goals. In the first lecture, Tom set out the learning goals for the students by explaining ways of thinking and suggested to the students that they should think like future maths teachers, indicating that they should be able to fully explain a concept to someone else and have them be able to understand the material. In terms of written material, Tom pointed to the syllabus as a particularly useful resource for students. He suggested in the interview that important objectives were set in the course syllabus and these assisted the students in knowing the objectives of the lesson.

When establishing the appropriate learning goals, the different strategies used by the four teachers ranged from using the objectives set in the course syllabus to step-by-step explanation of the learning goals, based on the level of students' understanding. The definition of key ideas to show how teachers use their PCK include:

- focusing on the basics to enable students to successfully understand and utilise basic elements of calculus to build their understanding;
- applying the outcomes from the lesson to the wider context; use the information in the real world;
- using learning goals to facilitate students' understanding;
- enabling students to fully explain a concept to others and have them understand it.

6.6.2.2 Identifying the Key Ideas in Learning Calculus

The PCK theoretical framework requires calculus teachers to provide, and make available, definitions, theorems and proofs to students as well as to provide them in a sequence appropriate for their levels of understanding. The teachers in the current study used different methods to demonstrate these key ideas. John, Alex and Sam frequently mentioned key ideas prominently in almost every lecture episode. As such, it was evident that they were trying to analyse each calculus topic using the definitions, theorems, proofs and examples. Sam, in addition, used his background in teaching and experience which enabled him to give the students a sequence of calculus ideas and as a way to link theory to more practical applications. In contrast Alex indicated, on several occasions, that he had a clear understanding of how to choose both the calculus topics for instruction and the teaching strategies to implement them. Alex, for example, when describing theorems, used a ‘think, pair, share’ tactic that allowed students to have a bit of time to discuss their thoughts before actually having to produce output in front of the entire class. Tom often linked his organisational structure of the topic with the course syllabus, set by the MOE, and followed the textbook. Tom began with a definition and specifically highlighted ideas that students tended to struggle with in relation to the concept. In this way he was demonstrating signposting, which can assist in creating the building blocks (definition, theorem, and proof) for students' learning. This outcome not only assisted students in identifying key ideas, but it flagged specific areas where weaker students could focus their attention. In contrast, but still pedagogically relevant, Sam, John, and Alex also used an example as a foundation to present the theorem and went on to explain the definition, theorem and proof, and the relevance to the aims of the lesson. These three teachers used the example and diagram as a tool for introducing the new definition, theorem, and then proof. John, however, differed in this aspect of demonstrating PCK when he asked his students to read the theorem and proof before explaining it. In doing so he was using three second-level subcategories of this study's PCK framework, which are identifying the key ideas in learning calculus, relationship between instruction and students' ideas in calculus and students' learning difficulties in calculus. John then explained the proof and identified the students' difficulties in understanding the proof. All the teachers were able to explain the definition and to support it with a graph and example of each case, in this way they were essentially providing the students with a 'route' to understanding calculus ideas. Alex and John were observed to use a discussion approach in this situation and only Alex referred to important issues at the beginning of Lecture 3 and used them as a tool for introducing the derivatives and mentioned two axioms. Alex was observed to frequently make the definitions he provided in lectures simpler than what was outlined in the students' textbooks. He indicated that he found visual representations easier for students to understand and this was effective in avoiding misunderstanding. These demonstrate a link between theory and practice as it relates to delivering the building blocks required to construct and enable Alex's students' mathematical understanding.

Overall, all the teachers suggested that definitions were essential as they related to formal calculus theory. The lectures observed showed that they used the similar structure of (1) providing a definition, (2) providing an example/graphic, and (3) explaining the value of the concept throughout. To deliver the building blocks to construct and enable students' mathematical understanding, calculus teachers use their PCK in a number of ways.

6.6.3 Instructional Strategies

In terms of instructional strategies, there are four second level subcategories identified in the theoretical framework; relationship between instruction and student ideas in calculus, questioning strategies, use of pivotal examples or counterexamples, and mathematical representation.

6.6.3.1 Relationship Between Instruction and Student Ideas in Calculus

The teachers in the current study had different ways of demonstrating these instructional strategies. For Alex, in general, all four elements were prominent in both his teaching practice and his responses in the questionnaire/interview. While John utilised counterexamples and questioning strategies, which were prominent in both his teaching practice and his responses in the survey and interview.

In terms of the teachers demonstrating the relationship between instruction and student ideas in calculus, John suggested that he used a scaffolding approach to build students' knowledge by starting with a simple introduction, moving through to the aims, and then beginning to 'sequence' the ideas related to the topic. This strategy consists of a series of specific moves that utilise what the students already know and then to build on previous knowledge with corresponding topics in a scaffolding approach. While John's understanding of scaffolding and the bigger picture represents one component of instructional strategies, he also identified and discussed in his interview other instructional strategies in his teaching, such as lecture, collaborative learning and discussion and he gave reasons to use cooperative learning to encourage the students to try to solve and explore information by cooperating among themselves. In an attempt to lead the students to understand some of the more challenging calculus concepts, John used different types of sequencing such as starting with a problem and/or a definition that he believed his students would understand.

While Alex was observed to be quite methodical in his approach to teaching calculus but what was most apparent was that he required all the students, from the inception of the course, to have the same fundamental concepts. He expected the students to self-direct their learning if they felt they did not have sufficient previous knowledge. Alex, in this case, suggested supplementary resources, including a particular book, which would help to facilitate understanding. This approach assumes that students will take the initiative to achieve the baseline knowledge required, which can challenge some students while trying to learn the concepts of the course and concurrently to catch up on previous knowledge.

Alex was also cognisant of his instructional methods when teaching. He indicated that selecting the appropriate teaching methods was essential. He indicated in his interview that teachers generally have the content knowledge for calculus teaching and that knowing the references and a selection of materials, exercises, or activities are valuable, but if not delivered through an appropriate method, these can be less effective. Interestingly, in addition to commenting on his own teaching methods, Alex placed a great deal of responsibility on the students, suggesting that they should be prepared for the lesson by reading and maintaining attention to the lecturer's explanations. In his interview, Alex suggested that he used many teaching methods and he was able to use the deductive method, inductive method and cooperative learning. He also told the students that they would prepare a lesson and explain it to their classmates.

John and Alex appeared to regard instructional strategies as important for students in their practice, they attempted teaching calculus ideas using a systematic approach, based on a solid grounding in logic. Both teachers identified and discussed other instructional strategies used in their teaching such as lecture, collaborative learning and discussion. Both provided reasons for their use of cooperative learning in order to encourage their students to cooperate among themselves when solving problems and exploring information.

With Sam, nowhere was his demonstration of PCK more profound than in his systematic approach to teaching and he was very clear that this was his foundation and justification for the way that he approached his teaching. This systematic sequencing was demonstrated often in his lectures, for example he highlighted the difficulties with both constructing and evaluating calculus concepts in order to ensure that the students were aware of areas that were particularly challenging. Sam’s linear progression through the objectives of the lessons, rather than facilitating or checking the students' understanding of the concepts, presented some challenges. In his interview Sam indicated that he valued group work and collaboration, but in many of the lectures that I observed he did most of the speaking. Sam's linear approach to meeting course and lesson objectives somewhat deviated from the questioning strategies subcategory as outlined in the theoretical model.

Tom considered that he did not have particular instructional strategies or teaching methods that he specifically employed. He suggested that this was because he did not have enough experience in this field. This was supported by his responses in the questionnaire. In terms of his instructional methods, Tom tended to follow the same strategy throughout. While Tom may not have necessarily acknowledged that the methods he utilised were thought out in detail, from the lessons observed there was some evidence of Tom using instructional strategies, however, in his interview he did not recognise that he had done so.

6.6.3.2 Questioning Strategies in Calculus

In terms of questioning strategies, John's teaching practice showed that if the students were incorrect or did not answer, he continued to use the questioning strategies to determine where the fault in the students’ logic occurred and then redirected them, either back to the original problem or to another one that addressed the fault in the logic. This type of active learning is not generally a typical strategy used in education in the Middle East, where passive learning is often employed. He further alluded to instances where he had discussed questioning strategies. John's questioning was not only about the students’ understanding but used to involve the students in the lesson (i.e. to maintain focus) and to evoke active participation. Sometimes, John was observed to ask rhetorical questions and did not expect an answer from the students. While Alex's use of questioning strategies was not always interactive, he was able to engage the students in other ways. This was demonstrated in Lecture 3 when he first gave the students the opportunity to work in pairs and the students were verbally working together to solve the problem that Alex posed. This group work was much more collaborative than the class questioning strategies and allowed Alex to ultimately explain a misconception. The students could then describe where they were going wrong and come to a logical conclusion. While Alex was not seeking answers to specific questions, the discussions elicited critical thinking skills and offered the students the opportunity to reflect on a specific case. Alex was consistently asking the question ‘why’ of the students.

In contrast, Sam and Tom did not employ questioning strategies consistently. Despite Sam's lack of questioning strategies, he did use a significant number of examples to help him to explain the concepts that he was trying to teach. Sam was more forthright in his responses to the survey and interview components. Tom used questioning techniques to ensure that the students understood the various points of the lesson. The use of questioning strategies, however, were not consistently evident in their practice, suggesting some discrepancy between both Sam and Tom's pedagogical understanding and their practices.

6.6.3.3 Use of Pivotal Examples or Counter-examples in Calculus

In terms of the use of pivotal examples and counterexamples, John was able to pair different skill areas in his explanations and in the problems that he chose, so that the students were getting different stimuli in simultaneous instances. There were some obvious uses of pivotal examples and counter-examples. Along the way, his questioning strategies also allowed for assessment of students' knowledge and largely constituted appropriate instructional methods. The use of pivotal examples and counterexamples allowed the students to think about the general case, and to make sure that their conclusions mirrored the views of the lesson; John was able to check their understanding and develop, to some extent, their critical thinking skills. While Alex, Sam, and Tom always used pivotal examples they did not mention counterexamples in their practice and in their responses in the survey and interview. On the other hand, Alex moved to high level examples and made links between mathematical and everyday use of terms through examples, while Sam focused on using the pre-prepared examples to help focus and explain the ideas of the lesson. Sam's direct line of thinking may have influenced which examples could be seen as pivotal for the students and which ones were simply helpful within the context of the lesson. In contrast, Tom used many pivotal examples; moreover, there were many obvious situations where he demonstrated knowledge of using examples to focus on key ideas in calculus in practice. However, these examples were largely directed at students who were weaker. He suggested that the examples were essential, as they related to explaining and clarifying calculus concepts and that examples are generally better used after the definition has been presented.

6.6.3.4 Mathematical Representations in Calculus

In the use of mathematical representation, all of the teachers were linking visual, symbolic, and verbal ideas in calculus. Their use of mathematical representation was consistent. All four teachers were not only offering a verbal explanation of the concept, but also employing other strategies as well, including numeric and algebraic presentations on a whiteboard when pairing visual representation with the spoken explanation. Such choices aimed to get the students to really understand the model and to avoid misrepresentation of symbols. By doing so all of teachers were linking visual, symbolic, and verbal ideas in calculus.

6.6.4 Knowledge of Calculus Connections in Calculus

In order to consider the knowledge of calculus connections, two second-level subcategories are identified; the real-world applications of calculus, and calculus in academic subjects.

6.6.4.1 Real-world Applications of Calculus

All four teachers made some effort in this instance, but for many it was simply mentioned in passing. For example, John attempted to highlight real-world applications of calculus, but had difficulty in identifying real-world connections that the students would understand. His approach required a great deal of trust of the students that this would occur. It was unclear how relevant examples of stocks and buildings were to the students’ own experiences John's agreement with the survey statement: “I am not interested in how calculus is taught at other (similar) university institutions in other parts of the world.” (Part 5, Statement 7) suggests a somewhat closed approach to his teaching and does not really tell the students much about how the derivatives might be used in other contexts, but simply that they do. While Tom generally did so in a rather vague way, he was quite aware of the challenges that students could face trying to link the abstractness of some of the calculus concepts with everyday use.

In contrast, Alex was able to demonstrate applicability and mentioned that people benefit every day from the application of calculus, such as the applications of differentiation in health science. For students, this link to real-world applicability was beneficial because it offered an opportunity for them to see the applied value, rather than thinking of calculus in the abstract. It also put the entirety of the lesson in perspective for the students through the use of visualisation which could benefit some students to visually link the examples. Not only did Alex attempt to link the concepts to real-life examples, such as waves and vibrations, but he also attempted to explain, and to demonstrate, to students how calculus fits within everyday usage.

6.6.4.2 Calculus in Academic Subjects

Alex’s choice of examples related well to other academic subjects which is a subset of the demonstration of PCK. Sam, in certain instances, made reference to ideas that went beyond the actual learning of the materials associated with calculus, to facilitate the students' understanding of calculus because knowing the origin could provide contexts for future learning. Sam also focused on linking calculus to the real-world by discussing the applications of differentiation. In this scenario, Sam was highlighting not only the real-world components but also how enjoyable this lesson was because of its practicality. Sam indicated in his interview that he felt that representations and images were of particular importance when teaching and suggested that examples gave context to students, and this assisted in the avoidance of misconceptions. While a suggestion of real-world examples is necessarily a component of the PCK theoretical model that has been identified, it is generally posited that if a teacher has more interest in a specific area of study, they may be more inclined to show real-world applications. Sam spent some time talking about the Hubble telescope and used a pivotal example to identify relationships between mathematics and application, being able to hold the students' interest and express value in the application of the topic.

6.7 Cross-Case Analysis Summary

This section presented the elements identified in the cross-case analysis which draw on the characteristics set out in the framework proposed in Chapter 4. The characteristics to emerge from the observations, questionnaire and interview are how calculus teachers articulate and demonstrate their PCK to achieve their teaching goals, to enable students' mathematical understanding, to apply instructional strategies, to deliver their lesson, and to utilise calculus connections. The participant teachers did, thought, and talked about how they articulate and demonstrate their PCK. The interview and questionnaire responses and analysing observation were labelled under ten sub-categories (see Table 6.1), either deductive, which have already been built from the proposed model of PCK for teaching calculus with characteristics (see Table 5-7), or inductive, based on the codes and categories.

There are new characteristics extracted from the data from the observations, interview, and questionnaire. These were where the teachers articulated and demonstrated their PCK in their teaching or responses. The information collected through the instruments was classified under the ten sub-categories of the proposed model. Modifying lessons or syllabus, awareness of misconceptions of teachers coming with preconceived notions, and simplifying definitions provided in lectures compared with those outlined in the students' textbooks are considered as characteristics of knowledge of students' misconceptions and difficulties in calculus. Providing contexts for future learning is considered as relating to establishing appropriate learning goals for calculus. In the relationship between instruction and student ideas in calculus were found new six characteristics, which are using 'think, pair, share' tactic, knowing that group work influences students' collaborative learning, students taking the initiative to facilitate understanding, students explaining the lesson's concepts to their classmates, expecting students to self-direct their learning to improve insufficient previous knowledge, and placing a great deal of responsibility on the students (mentioned by some of the teachers). The categories and first level sub-categories are presented below. The second level sub-categories are presented in Table 6-1. Some elements appear in more than one characteristic column. The elements shaded in beige in Table 6-1 are those identified from the cross-case analysis that meet the second level sub-categories but do not meet the characteristics, indicating that they are proposed by this study.

Categories

6.7.1 Knowledge of Content and Students when Teaching Calculus

6.7.1.1 Learners' Cognition of Calculus

6.7.1.2 Developmental Aspects of Calculus Curriculum

- expecting all students to have the same fundamental concepts;
- directing students to deal with particular challenges;
- being aware of students' misconceptions;
- focusing on the basics to enable students to successfully understand and utilise basic elements of calculus to build their understanding;
- using learning goals to facilitate students' understanding;
- analysing each calculus topic using the definitions, theorems, proofs and examples.

6.7.2 Knowledge of Content and Teaching

- having content knowledge for calculus teaching;
- having a background in teaching and experience;
- identifying discrepancies between pedagogical understanding and practical application.

6.7.2.1 Instructional Strategies

- knowing instructional strategies are important for students in their practice;
- having instructional strategies or teaching methods;
- selecting appropriate teaching methods;
- using teacher-talk and explaining;

- following methodical approach and appropriate instructional methods;
- linking visual, symbolic, and verbal ideas in calculus;
- scaffolding approach to systematically build students' knowledge;
- using different types of sequencing of ideas related to the topic e.g. starting with a problem and/or a definition;
- using lecturing style;
- using strategy of collaborative learning and involving the students in the lesson;
- using cooperative learning/ opportunity to work in pairs;
- encouraging deductive and inductive learning;
- using a linear progression through the objectives of the lessons rather than not checking for the students' understanding of the concepts;
- teaching strategies to implement calculus topics;
- being supportive;
- using homework as a foundation to build on.

6.7.2.2 Knowledge of Calculus Connections

- if a teacher has more interest in a specific area of study, they may be more inclined to show real-world applications.

Table 6-1: Second Level Sub-Categories with their Identified Elements

(A1) Knowledge of students' misconceptions and difficulties in calculus	***(A2) Knowledge of students thinking about calculus concepts***	***(B3) Establishing appropriate learning goals for calculus***	***(B4) Identifying the key ideas in learning calculus***	***(C5) Relationship between instruction and student ideas in calculus***	***(C6) Questioning strategies in calculus***	***(C7) Use of pivotal examples or counter-examples in calculus***	***(C8) Mathematical representations in calculus***	***(D9) Real-world applications of calculus***	***(D10) Calculus in academic subject***
modify lessons or syllabus A1	making concepts understandable and easy to grasp A2	linking between theory and practice B3	identifying the key ideas in learning calculus B4	encouraging cooperation C5	discuss basic concepts of calculus C6	introducing salient points followed by more detailed explanations C7	recognising that visual representations are easier for students' understanding and effective in avoiding misunderstanding (visual) C8	presenting real-world challenges D9	applying derivatives in other contexts D9 & D10
using diagnostic tests to identify learners' cognition A1	having knowledge of learners' cognition A2	ensuring that students are challenged B3	establishing relationships between instruction and students' ideas in calculus B4	seeking collaboration C5	using discussion C6	analysing each calculus topic using the definitions, theorems, proofs and examples C7	explaining the definition and supporting it with a graph and example (visual) C8	making links between mathematical and everyday use of terms through examples D9	relating well to other academic subjects D10
being aware of students' misconceptions A1	Ensuring that the students are aware of areas that are particularly challenging A2	providing contexts for future learning B3	placing a great deal of responsibility on the students B4	providing opportunities for students' participation C5	using questioning strategies to assess students' knowledge C6 & A2	understanding of how to choose the calculus topics for instruction C7	delivering through an appropriate method e.g. pairing visual representation with the verbal explanation of the concept (visual, verbal) C8	offering opportunities for students to see the applied value D9	referring to ideas that go beyond actual learning of the materials associated with calculus D10

identify learning difficulties of students A1	identifying the difficulties with both constructing and evaluating calculus concepts A2		providing a 'route' to understandin g calculus ideas B4	directing students to deal with particular challenges C5	asking rhetorical questions C6	using examples as a foundation C7	facilitating understanding of a model to avoid misrepresentation of symbols C8	making lessons enjoyable through practical means D9	
discussing what the students know A1				provide students with material on calculus functions that they can consciously evaluate C5	questioning strategies to determine where the fault in the students' logic occurs C6 & A2		giving students different stimuli in simultaneous instances C8	using representations and images to emphasis particular importance C8 & D9	

(A1) Knowledge of students' misconceptions and difficulties in calculus	***(A2) Knowledge of students thinking about calculus concepts***	***(B3) Establishing appropriate learning goals for calculus***	***(B4) Identifying the key ideas in learning calculus***	***(C5) Relationship between instruction and student ideas in calculus***	***(C6) Questioning strategies in calculus***	***(C7) Use of pivotal examples or counter-examples in calculus***	***(C8) Mathematical representations in calculus***	***(D9) Real-world applications of calculus***	***(D10) Calculus in academic subject***
awareness of misconceptions of teachers coming with preconceived notions A1	using discussion to elicit critical thinking skills A2 & C6			using 'think, pair, share' tactic C5	using discussion to elicit critical thinking skills A2 & C6	using examples and counterexamples C7	using visualisation to provide the appropriate perspective for students C8	highlighting real-world applications of calculus D9	
simplifying definitions provided in lectures than those outlined in the students' textbooks A1	using questioning strategies to assess students' knowledge C6 & A2			getting students to read the theorem and proof before explaining it C5		knowing that examples are essential as they relate to explaining and clarifying calculus concepts C7	using representations and images to emphasis particular importance C8 & D9	identifying real-world connections D9	
highlighting ideas that students tend to struggle with A1	facilitating students' understanding of calculus A2 & C5			knowing that group work influences students' collaborative learning C5		focusing on use of pre-prepared examples C7	using examples and diagrams as a tool C8	applying derivatives in other contexts D9 & D10	

establishing relationships between instruction and students' learning difficulties in calculus A1	using questioning strategies to determine where the fault in the students' logic occurs A2 & C7			students will take the initiative to facilitate understanding C5		using examples as they are generally better used after definition C7		demonstrating the applicability and benefit from the application of calculus in the everyday D9	
being aware of the challenges that students face trying to link the abstractness of some calculus concepts with everyday use A1 & D9	checking understanding and development A2			students will reflect on a specific case if given the opportunity C5				explaining and demonstrate to students how calculus fits within everyday usage D9	

(A1) Knowledge of students' misconceptions and difficulties in calculus	*(A2) Knowledge of students thinking about calculus concepts*	*(B3) Establishing appropriate learning goals for calculus*	*(B4) Identifying the key ideas in learning calculus*	*(C5) Relationship between instruction and student ideas in calculus*	*(C6) Questioning strategies in calculus*	*(C7) Use of pivotal examples or counter-examples in calculus*	*(C8) Mathematical representations in calculus*	*(D9) Real-world applications of calculus*	*(D10) Calculus in academic subject*
assisting in the avoidance of misconceptions A1	expecting students to self-direct their learning to improve insufficient previous knowledge A2			students explain the lesson's concepts to their classmates C5				providing examples that give context D9	
flagging specific areas where weaker students could focus their attention A1				expecting students to self-direct their learning to improve insufficient previous knowledge B3 & C5				identifying relationships between mathematics and application D9	
				placing a great deal of responsibility on the students B3 & C5				being aware of the challenges that students face trying to link the abstractness of some calculus concepts with everyday use A1 & D9	

				facilitating students' understanding of calculus A2 & C5				holding students' interest and express value in the application of the topic D9	
				having a solid grounding in logic and its associated linguistic expressions C5					
				the use of methodical and systematic approach C5					
				using a linear progression through the objectives of the lessons rather than not checking for the students' understanding of the concepts C5					

Table 6-1: Second Level Sub-Categories with their Identified Elements

6.8 Chapter Summary

The teachers in this study displayed their PCK in a variety of different ways by taking different approaches in their instructional strategies with some choosing a more top-down lecturing approach to teaching while others favoured a more collaborative bottom-up approach. While all the teachers showed their PCK in relation to how they taught calculus, it was also clear that not all aspects of PCK were equally evident among them. Some were more inclined to focus on specific instructional strategies to target learners needs (e.g. the use of discussion over lecture). Others chose to highlight students' misconceptions about calculus in different ways (e.g. a discussion about what students already know, versus an all-around overview on basic mathematical concepts in order to solidify key foundational mathematics skills).

However, more importantly this cross-case analysis has provided the foundations to identify, in extensive detail, how teachers use their PCK: to develop learners' cognition of calculus; to set their teaching aims; deliver the building blocks to construct and enable their students' mathematical understanding; to develop their strategies to deliver their teaching aims and objectives and to use their PCK to apply calculus connections. The next chapter discusses these findings in terms of the existing research.

Chapter 7 Discussion

7.1 Overview

The previous chapter, the cross-case analysis, presented the similarities and differences among the teachers in this study in terms of the outcomes from the observations, interviews and survey. All of the teachers in this study demonstrated most of the aspects of PCK that fell within the theoretical model, and it was also clear that not all aspects of PCK were equally evident among them. Yet they were able to demonstrate their PCK in a multitude of different ways. What was demonstrated in the analysis was, largely, that the first research question posed at the beginning of this thesis has been fully addressed. Despite this fulfilment, there are still areas where the PCK framework may not fully document the nature of PCK and the differences among the teachers.

This discussion chapter positions the findings from the current study within the context of previous literature. For the purpose of this study, a PCK framework was employed as the means of analysis. It is necessary not only to highlight the successes and the links to the literature, but also to indicate any areas that could benefit from future research. In this chapter, the findings are interpreted and are presented within five key syntheses and the discussion relates to the research objectives and questions, theory, data, and existing research. In this section, the first-level subcategories, which form the framework for the above case studies for each of the four participants, is used as a foundation for the discussion.

7.2 Synthesis 1: Learners' Cognition of Calculus

Students' misconceptions and learning difficulties in calculus

Beginning with learners' cognition of calculus, this section seeks to demonstrate that there were many different approaches to both the identification of students' misconceptions of learning calculus as well as the details surrounding the knowledge of students' thinking about calculus.

Calculus is one of the most complex fields in mathematics for students to understand (Kashefi et al., 2012). However, all four teachers were able to identify their students' difficulties with both constructing and evaluating calculus concepts. What did vary among the teachers were the particular topics that they felt to be imperative to focus on. For example, Alex's particular concern was with any misconception that developed as a result of a lack of previous knowledge as this may inhibit students' future understanding. This was highlighted in Lecture 1 when he directed the students to deal with particularly challenging definitions. He asked his students to define the function and they said: “It is a class of ordered pairs”. Alex stated that: “This is a simple definition, and I consider that as a poor definition ... " (Lecture 1, Episode 3 (26m22s-34m15s)), which is in line with Buck's view (1970, p. 255) "that 'a function is a class of ordered pairs' is one which imposes severe limitations upon the student and provides a poor preparation for any further work with functions ...". Students’ intuitive ideas are in conflict with the formal definition of the calculus concepts (Sierpinska, 2013; Davis & Vinner, 1986; Cornu, 1991; Williams, 1991; Tall, 1993), a point that Alex showed awareness of that indicates that a suitable definition was probably more difficult than the students could handle, or that a simpler explanation would reduce the number of difficulties that were being experienced by them. In this sense Alex was demonstrating his knowledge of students’ misconceptions and learning difficulties and, as Kidron (2014) suggests, challenges and difficulties can diminish when teachers provide the formal definition of the concepts to their students. In this way, while it is acknowledged by Kidron (2014) that a definition can make it more straightforward for students to understand the material, Alex is suggesting that the definition needs to be appropriately detailed and targeted at the students’ current level of understanding to be effective. A misconception within the field of calculus might include an idea or belief that is founded on incorrect or erroneous information about some aspect or detail relating to calculus theory (Olivier, 1989; Robert & Speer, 2001; Jones & Alcock, 2014). Alex considered that in order to ensure that his students were only receiving material on calculus functions that they could consciously evaluate, he considered that students could access practical solutions to overcome their lack of previous knowledge. He encouraged his students to utilise a variety of resources.

It is reasonable to make the assumption that calculus teachers need to know something about students' thinking and misconceptions, otherwise they are unlikely to be able to devise a suitable diagnostic test which poses appropriate questions. The notion of misconception often arises because pre-existing concepts must exist for students to function in first year calculus (i.e. students must have a certain level of understanding about mathematics in order to be successful in calculus). Challenges arise, however, when teachers’ preconceptions about students’ knowledge differs from the actual competencies. According to Jones and Alcock (2014), preconceptions are pivotal in the link between pre-calculus knowledge and new knowledge, which Alex uniquely, in this study, presented. At the beginning of their observed first lectures, both he and John gave a diagnostic assessment in order to ensure a clear interpretation of where the problem areas existed. The assessments identified that most of the problems the students encountered with calculus conceptions came from their previous experiences (Bressoud et al., 2016). Both teachers used their knowledge of learners’ cognitions to address anticipated questions and students' misconceptions and identified the students’ formation of mathematical concepts in calculus.

All four teachers presented clear knowledge of what their students had learned at the secondary school level and all indicated that the students did not have much prior calculus knowledge upon entering university. Bressoud et al. (2016) argue that the transition from secondary level mathematical education, and content requirements, to that at post-compulsory level is inhomogeneous. As a result, students lack foundational concepts and knowledge for the effective transition and thus experience difficulties in grasping calculus concepts which can lead to misconceptions or unsuitable preconceptions that cause many difficulties (Gruenwald & Klymchu, 2003, p. 2). Interestingly, one of the teachers in this study, Sam, suggested that teachers also come to a class with a certain set of preconceived notions about what students are expected to know. If a student's knowledge is very basic, then the construction of the calculus concepts, necessary for the class, may not be fully understood, leading to significantly more difficulty. In addressing his students' cognition of calculus, Sam's interview response indicated that his focus is on his students' preconceived knowledge, or their lack of knowledge, prior to entrance into his class. For the teachers in this study, the outcome was not only the need to have their students learn the material of the course, but also to really understand the 'why' associated with the fundamental concepts that exist for each calculus topic taught. This concept is elegantly summarised by an extract from Kidron (2014):

> *The cognitive difficulties that accompany the learning of central notions like functions, limit, tangent, derivative, and integral at the different stages of mathematics education are well reported in the research literature on calculus learning. These concepts are key concepts that appear and reappear in different contexts in calculus. The students meet some of these central topics at school, then the same topics appear again, with a different degree of depth at university. We might attribute the high school students' cognitive difficulties to the fact that the notions were presented to them in an informal way* (p.70).

In considering misconceptions, the way that the four teachers addressed their students also differed. For example, in the case of Tom, he identified that the weak students were likely to experience considerable difficulties with the foundations of calculus and because, possibly, of a lack of previous knowledge, they would be unsuccessful. Tom indicated his level of uncertainty about how to address this gap. In contrast, Alex's strategy of using homework in multiple different ways suggested that he was confident that he would be able to use the student's homework solutions to identify misconceptions and challenges among the students. This indicates a valuable demonstration of PCK, according to the proposed model, by constructing a representation of the problem to address students' difficulties (Park & Oliver, 2008). This goes beyond the notion of general misconceptions. What Alex was doing, in this case, was using the learning difficulties of a specific group of students rather than assuming that this group of calculus students would have the same misconceptions as those who took the course previously. In addition to the above challenges, Tarmizi (2010) suggests that students can overcome their difficulties when learning calculus through building upon visualizing, which constructs a representation of the problem. Other teachers in the study did not express such levels of uncertainty; instead, they encouraged students to seek out other means of support in an attempt to draw out the weaker students. Sam, for example, wanted the students to employ certain study skills based on his own preconceived notions on what students were likely to know. Overall, while differences were observed between two teachers (Alex and John) and the other two teachers, these could be related to number of factors including the length of their teaching experience and or experience with the particular course taught, it is interesting to note that on checking their demographic information both Alex and John hold educational diplomas.

According to Weller et al. (2004), students must find resolutions to their 'cognitive issues'. The literature previously highlighted on this topic clearly indicates that learners' cognition of calculus is a shift in thinking. In order to determine how the calculus teachers addressed this issue, it was necessary to look to the strategies they employed to help students overcome the cognitive challenges. As the teachers were primarily teaching first year calculus classes, this would be the first time that the students would be working through some of the challenging concepts in calculus. The findings showed that the teachers worked more slowly through certain concepts, such as derivatives and limits, as the teachers indicated that they knew these were common areas where students' tended to struggle. All the teachers agreed that students face difficulties in understanding the subject of limits, especially Evaluating Limits of Indeterminate Forms (Tall & Vinner, 1981; Bressoud et al., 2016). The choice to focus on these areas appears to reaffirm what has been identified in the literature as a cognitive challenge. Moreover, all four teachers acknowledged that their students had difficulties with the formal epsilon delta definition of limits. This aligns with Kung's (2010) view that "limits have proved to be extremely difficult for students to learn, especially in the modern epsilon-delta definition" (p.148).

Examining learners' cognitions of calculus, especially students' misconceptions and learning difficulties in calculus, under the PCK framework, has identified that teachers use their PCK to anticipate using diagnostic tests to identify learners' cognition and by being aware of students' misconceptions. The teachers identified students' learning difficulties and discussed what the students know and provide simpler definitions in lectures than those in the students' textbooks, highlighting ideas that students tend to struggle with. Establishing relationships between instruction and students' learning difficulties in calculus and being aware of the challenges that students face in order to assist in the avoidance of misconceptions, flags specific areas where weaker students can focus their attention and use more than subcategory when linking the abstractness of some calculus concepts with everyday use.

In addition, using their PCK allows teachers to anticipate students' questions along with their misconceptions. By asking appropriately tailored questions, teachers engender mathematical thinking and activity (Hawkins et al., 2012; Petropoulou et al., 2016). If one considers the types of questions posed by the teachers, it is evident that some of the teachers consistently posed the questions that would foster cognitive stimulation. These questions were supported by other means of evidence such as visual representation and collaborative learning.

Knowledge of students' thinking about calculus concepts

Calculus concepts often ask students to consider the abstract, which can be difficult for those who are accustomed to learning facilitated by tangible outcomes and links to real world events. In order for teachers to demonstrate PCK, they must have knowledge of students' thinking about calculus. One way that this can be demonstrated is through the identification of characteristics of external, empirical and deductive concepts of calculus. According to Kashefi et al. (2012), two major barriers for student education are the manipulation of algebraic concepts and a poor understanding of such concepts. These are, essentially, the very basics required for an understanding of mathematics, and core to the ideas of calculus. If students are unable to demonstrate competence in the development of concepts, they are likely to face significant difficulties (Rasmussen, 2012). Based on these notions from the literature, it seems fairly evident that teachers must not only be aware of the way students identify the concepts of calculus, but they must also be able to navigate the structure of the course to ensure that students are able to identify the corresponding characteristics. In the findings from this study, the students were noted by both Alex and Sam to have varying levels of calculus competency (Sofronas & DeFranco, 2010). Therefore, ways that PCK was demonstrated took this into consideration. For example, Sam encouraged struggling students to seek out extra support beyond the classroom if they were having challenges with the most basic representation. In addition, Alex consistently pointed out areas that were fundamental to the students' understanding, specifically in relation to algebraic concepts (in Lecture 6). In both of these cases, the teachers are essentially ensuring that the students have the external, empirical and deductive concepts of calculus in order to successfully complete the lessons.

The outcome from exploring how calculus teachers use their knowledge of students' thinking about calculus concepts, shows that teachers might make concepts understandable and easy to grasp by having knowledge of learners' cognition (Sofronas & DeFranco, 2010). They ensure that the students are aware of areas that are particularly challenging and identify the difficulties with both constructing and evaluating calculus concepts. Using discussion to elicit critical thinking skills and using questioning strategies to assess students' knowledge facilitates students' understanding of calculus (Tataroğlu-Taşdan & Çelik, 2016). Additionally, by using questioning strategies, teachers determine where the fault in the students' logic occurs, check understanding and development, and lead students to self-direct their learning to improve insufficient previous knowledge. What can be drawn from the data provided by Alex and Sam is evidence linking teaching practices and thoughts about knowledge of students thinking about calculus concepts (Stacey, 2008). This was not true in every instance, as certainly, some examples provided clearer links than others, but overall it is possible to link teaching practices with knowledge of students thinking.

Another characteristic that was both prominent in the literature, and in the findings, related to students' formation of mathematical concepts in calculus. From a theoretical perspective, this characteristic seems largely to describe knowledge about how students are thinking during a lesson (and beyond when completing a calculus-related task, such as homework). According to Lachner and Nuckles (2016), students of calculus are generally unprepared to learn and, because students tend not to be able to easily overcome this lack of preparedness, they often indicate levels of dissatisfaction and unhappiness. If the KSA intends to be a player on a global scale, this general unhappiness could be mitigated by teaching strategies that aim to overcome the frustration associated with cognitive strategies employed by students. What the literature seems to lack, in this instance, is specific innovative strategies that teachers can utilise to encourage the shift in cognitive thinking among students (Stacey, 2008; Sofronas & DeFranco, 2010). In the case of Alex, his questioning strategy specifically targeted the level of understanding "why". This differs from the binary yes/no response to the general 'do you understand?' question, which John used constantly. While the shift is minimal it is a significant one because using Alex's, approach assists in determining which mathematical concepts are understood completely, which ones are understood partially, and which ones require more work.

Overall, the outcome from exploring how calculus teachers use their PCK of learners' cognition of calculus, as it relates to the research question of this study, requires a comprehensive answer. The issue with pedagogical knowledge is that it requires various aspects of innovation and a real understanding of the challenges in the discipline. In this case, the teachers all indicated that they knew where the potential challenges arose (Burton, 1984). Some of the teachers based this information on what had occurred in previous years, while others used questioning strategies in the classroom to obtain real time information about the comprehension of the students. What is evident from the findings is that this study contributes to the literature already in existence because it has highlighted different strategies that the teachers use in the classroom to determine learners' cognition (Sofronas & DeFranco, 2010). Moreover, by highlighting the fact that the teachers all used different strategies in the classroom and still experienced success, there are indications that the small steps they have taken towards innovation have been positively received by learners.

7.3 Synthesis 2: Developmental Aspects of the Calculus Curriculum

Establishing appropriate learning goals in calculus

Moving into the developmental aspects of calculus, learning goals were a common feature among the participants, as all four teachers' demonstrated strategies for presenting the lecture's aims and objectives particularly in Lecture 1. Sam was the most linear of the four participants as his lectures followed a very similar format with the learning goals most explicitly presented at the beginning of each lecture. Not only did Sam lay out the goals for each lecture, but he also laid out the overall objectives for the course (Tall, 2010; Pritchard, 2015). Similarly, the other teachers also demonstrated places where learning goals were highlighted. All used their knowledge of establishing appropriate learning goals for calculus for linking between theory and practice, ensuring that students would be challenged, and provided with contexts for future learning. For Alex, the learning goals seemed to be more flexible, depending upon the students' understanding. For example, he made many links between the learning goals and the topics and the way that students were progressing through the course (Pritchard, 2015; Petropoulou et al., 2016). John, who used the learning goals to facilitate his students' understanding and comprehension, mirrored this strategy. Tom pointed out that important objectives were set in the course syllabus and he used them in his teaching (Speer & Smith, 2010). Ultimately, all four teachers identified learning aims, objectives, and/or learning goals within the lesson, demonstrating application of PCK in this instance. The idea that learning objectives may be shared with students in the lesson is an important one (Hannah et al., 2011) and the literature suggests that obvious objectives make students actively want to participate to gain concepts. Stating learning objectives also makes more sense for the teacher's actions and that students retain more knowledge (Morgan, 2014; Petropoulou et al., 2016).

Literature on establishing appropriate learning goals for calculus is somewhat sparse. According to Sonnert et al. (2015), students who are taking calculus courses at university demonstrate a stymied motivation with regard to mathematical courses. This generally has a negative impact on how they perceive their own aims and course goals. In order for calculus learning, especially calculus 1, to be successful students have to demonstrate the motivation and dedication to pursue a different and sometimes challenging line of thinking. For example, Sam was quite expressive of the learning goals in each lesson. His methodical approach to stating the goals at the beginning seemed to lay out the plan for his students, thus identifying an end point (Morgan, 2014). By having the students know where they were to finish for the day, it was possible that Sam was inherently contributing to their motivation, as his step-by-step process encouraged focused attention on core areas. Alex took on learning goals in an entirely different way. For Alex, the learning goals were highlighted by encouraging learners to think in certain ways, yet this was not completed in a prescriptive way, but rather in way that allowed the students ample opportunities to learn in a way that best suited their needs. Tom encouraged his students to refer to the syllabus, where the learning goals were clearly outlined. In this way he was also encouraging motivation, but in a different way (Tall, 2004; 2008). These three teachers displayed differing, but successful methods of imparting the lessons' learning goals, to enable the students to progress in their calculus learning.

Identifying the key ideas in learning calculus

There is much literature on calculus instruction (e.g. Biza et al., 2016; Bressoud et al., 2013; Bressoud et al., 2016; Schoenfeld, 1995; Robert & Speer, 2001). Rasmussen et al. (2014, p.508) classify the existing literature as focusing on: (1) identifying and studying student difficulties and cognitive obstacles; (2) investigations of the processes by which students learn particular concepts; (3) classroom studies, including the effects of curricular and pedagogical innovations on student learning, and more recently, (4) research on teachers' beliefs, and practices. What was noted in the literature review in Chapter 2 were many methods that were employed in the calculus classroom and the extent to which these would be applicable in the KSA context. The KSA is not only attempting to demonstrate proficiency but is attempting to excel in this field; the KSA MOE has identified innovation in teaching as a fundamental component to success.

Yet while there is disagreement about the pedagogical approach in calculus, and no best way for teaching, researchers on calculus (e.g. Biza et al., 2016; Bressoud et al., 2016; Bressoud et al., 2016) agree that there are a set of key ideas for learning calculus. These include definitions, relating a definition to an example, axioms, theorems, proofs, examples, and diagrams (Alcock, 2014). The fundamental piece of placing these key ideas into the PCK theoretical framework is the requirement of teachers to provide, and make available, definitions, theorems and proofs to students as well as to provide relationships between mathematical and everyday use of terms. The teachers in the current study used different methods to demonstrate these key ideas and illustrated how the teacher could use their knowledge of calculus teaching to sequence the building blocks of mathematical theories of the concepts of calculus.

Alex, for example, when describing theorems in Lecture 3, used a 'think, pair, share' tactic that allowed students to have a bit of time to discuss their thoughts about the key ideas before having to produce output in front of the entire class (Escudero & Sanchez, 2007). Contrastively, but still pedagogically relevant, Sam, John, and Alex also used an example as a foundation to present the theorem and went on to explain the definition, theorem and proof, and the relevance to the aims of the lesson (Bardelle & Ferrari, 2011; Klymchuk, 2012; Wagner, 2017). In yet another example, Tom, in Lecture 3, began with a definition and specifically highlighted ideas that students tended to struggle with in relation to this concept. In this example, Tom was demonstrating aspects of signposting (Biggs, 2003), which assists in creating the building blocks for students' learning. This outcome not only can assist students in identifying key ideas, but also flags specific areas where weaker students can focus their attention.

What can be gleaned from these examples of building blocks of mathematical theories is that they can be used in several ways, such as the teacher suggesting that definitions are essential as they relate to formal calculus theory. The lectures observed showed that the teachers used the similar structure of (1) providing a definition, (2) providing an example/graphic, and (3) explaining the value of the concept throughout (Alcock, 2014). To deliver the building blocks to construct and enable students' mathematical understanding, calculus teachers use their PCK in a number of ways. Calculus teachers use their knowledge for identifying the key ideas in learning calculus, establishing relationships between instruction and students' ideas in calculus, placing a great deal of responsibility on the students, and providing a 'route' to understanding calculus ideas.

7.4 Synthesis 3: Instructional Strategies of Teaching Calculus

Relationship between instruction and students' ideas in calculus

There is inevitably a relationship between what the calculus teacher teaches and the calculus ideas that students have. What becomes more interesting in the framework of PCK is how the relationship between instruction and students' ideas is connected to logic. It is expected that calculus teachers be somewhat systematic in their approach to teaching, as the content of calculus follows a series of building blocks, such as definitions, theorems, and proofs (Alcock, 2014). Furthermore, recent literature has highlighted that students are influenced by teacher content knowledge, preparation, use of routines and content coverage (Rowan et al., 2002; Weber, 2015), indicating worthwhile connections between instruction and students' ideas in this area.

In shifting the focus to the instructional strategies in calculus, one of the areas of focus is on the relationship between instruction and students' ideas in calculus. In terms of the systematic characteristic that underlies this approach, John and Sam were both able to demonstrate this, as their lessons were pre-planned and methodical. This allowed them to present what they felt were sequencing problems for students in order to facilitate scaffolded learning (Vygotsky, 1987) to assess whether the students were able to employ the thinking skills that would allow them to move to the next level of calculus understanding. In doing this, the students seemed largely to understand what they were supposed to learn and when they were supposed to ask questions. Alex was also methodical in his approach to teaching calculus. Unlike the other teachers, he provided the students with a 'readiness' assessment in the first week. From this assessment, he designed the course so that students could more easily understand the structure of certain calculus connections (Speer et al., 2010). Additionally, John employed sequencing that made it easier for his students to provide feedback on areas of concern, thus strengthening the relationship between himself and his students (Kashefi et al., 2012; Bergsten, 2012). John also applied a not otherwise seen strategy when asking students to read the proof before explaining it. This strategy is akin to what Weber and Mejia-Ramos (2011) pointed out, that the aim of mathematicians in reading a proof is not only to understand but also to get the techniques and ideas of the proof. Three of the case teachers particularly commented on the value of this logical progression. In contrast, Tom did not identify his own personal teaching methods as a way to build a relationship with his students. He indicated that he did not seek out additional teaching strategies and was not entirely aware of teaching methods that would assist his pedagogical instruction further than what it was (see Sullivan, 2011). Despite this difference in perspective, Tom employed teaching methods in the classroom that were to some extent similar to those of the other teachers. He used examples consistently, described the lesson aims, and attempted to scaffold the learning of the students. Therefore, while Tom may not have been aware of his teaching methods, the ones that he employed could generally be deemed as suitable. Tom mentioned a major point in distinguishing mathematics from many other subjects that "mathematics does not encourage ... memorisation". Tom's strategy, when reminding students about memorisation, is consistent with how the other teachers represented calculus in the classroom. An alternative to stressing memorisation might entail the teachers expecting students to employ critical thinking skills, which can be achieved through higher order thinking. This alternative is not something that is often promoted in the KSA because, in general, there is an emphasis on passive lecture-led styles, yet this lack of stress on memorisation presents something unique in how calculus is taught in the classrooms in the KSA. Teachers' PCK is realised through the conscious shift in perspective when teaching calculus concepts.

One of the challenges with this subcategory, in terms of the theoretical model of PCK, is the somewhat abstract concepts that contribute to the list of characteristics. For example, encouraging cooperation, seeking collaboration, providing opportunities for students' participation, directing students to deal with particular challenges, providing students with material on calculus functions so that they can consciously evaluate, using 'think, pair, share' tactics. By getting students to read the theorem and proof before explaining it and knowing that group work influences students' collaborative learning, students will take the initiative to facilitate understanding. They can reflect on a specific case if given the opportunity and explain the lesson's concepts to their classmates.

Expecting students to self-direct their learning to improve insufficient previous knowledge places a great deal of responsibility on the students. However, facilitating their understanding of calculus gives them a solid grounding in logic and its associated linguistic expressions. Using a methodical and systematic approach and using a linear progression through the objectives of the lessons, rather than not checking for students' understanding of the concepts, is a characteristic of this subcategory. A demonstration of 'appropriate' instructional methods and 'leading students to more easily see the structure of certain calculus concepts' requires subjective bias (Holton, 2001; Nardi et al., 2005). Sofronas and DeFranco (2010) highlight that teachers' knowledge affects instruction and those comparisons can be made to enhance the field of pedagogical logic within the field of calculus.

When considering the background and nature of the instructional experiences the teachers have, it is evident that John can appropriately express the structure of his lessons. This is not surprising as his educational qualifications, including his degree in mathematics education and his extensive experience working in international contexts, would suggest that he is likely to have a sophisticated pedagogical framework for understanding how to offer different strategies for teaching.

The main notion behind making use of discussion in the classroom is different from lecture and from question/answer sessions, as discussions give students the opportunity to share both what they know and what they are unsure about. Discussions do not have to be entire group sessions, nor do they need to be led by the teacher; they can be small group or pair-work activities designed to determine comprehension in some way of a particular concept. The participant teachers did not always share the same perspective on the value of discussions. John has four years of teaching experience and a diverse background in education; he obtained his undergraduate degree from his home country and studied abroad extensively. While his PhD and Master's degree did not offer any specific training in pedagogy related to mathematics, his undergraduate degree was in mathematics education. John has not participated in research or academic conferences related to the scholarship of teaching and learning, related to mathematics. For both John and Alex, discussion was seen as a means to solidify information that already existed, indicating that it was likely students would be able to learn from each other and that they required the teacher to be the guidance for all knowledge. Interestingly, and based upon the observed lessons, John is not discussing the macro view of the lecture (i.e. talking about the lecture from start to finish), instead, what he is referring to are mini lessons within the larger lesson framework. When reviewing these lessons, the definition process for John begins a cycle – of which there might be multiple cycles within the same lecture. John seems to follow the cycle: 1) definition, 2) representation, 3) lecturing, and 4) discussion throughout the course of the class. Alex sees the inclusion of tasks and assignments given to students as a different and more innovative way to teach. There is no definitive reasoning as to why this approach seems to be the strategy for Alex. Alex received his graduate degrees from overseas, though also had educational experiences in KSA. While his undergraduate degree was in mathematics education, signalling some pedagogical understanding of the connection between teaching and calculus, his post-graduate degrees centred on applied mathematics. During his four-year teaching career in calculus, he has never attended a conference specifically related to the teaching and learning of calculus. His comments are another way to consider explanation. From a pedagogical perspective, he is not only ensuring that the explanations he is providing are clear, but he is also supporting students in being open and honest about the lessons they are participating in. This is another form of assessment, this time by students, which allows Alex to reflect upon his own teaching practice in order to ensure that the learning objectives are being achieved.

Tom has eight years of teaching experience after having achieved his undergraduate degree and both post-graduate degrees from a university from his home country. His PhD in mathematics is not supported by any pedagogical background courses, nor has he studied abroad or engaged in any sort of academic conferences related to the teaching and learning of calculus. Sam obtained much of his schooling overseas, living there more than ten years while obtaining both undergraduate and post-graduate degrees (Master and PhD). In his six years of teaching, he has not participated in any conference related to mathematics education nor has he studied any pedagogy associated with mathematics education. Looking at his background information, it can be gleaned that he has never had any formal pedagogical training and generally shows no interest in attending teaching and learning events related to educational development. For Tom, there seems to be a comfort that lecturing will provide students with the knowledge that they need to understand the concepts of calculus in a way that they may not get from their peers. As previously noted, Alex highlighted the need for detailed explanations in order to facilitate student understanding. While this was demonstrated by Alex and something he commented on during the interviews, Sam also applied a similar strategy in practice. But Sam did not necessarily use explanations as prominently as Alex. However, explanations still featured prominently in the lessons of Sam, especially surrounding challenging areas, such as theorems. Sam tended to use a lecture approach in most of his practice.

Questioning strategies in calculus

Another area of similarity among the teachers was the use of questioning strategies. All four teachers employed questioning strategies in the classroom, though the purpose of the questions and the amount of questions asked, differed. For John, questioning and student assessment went hand in hand. If the students were able to answer the question correctly, John moved on to the next topic. If students were unsuccessful, John either asked the question in a different way or moved on to an easier question that would allow the students to build up their knowledge (Stolk, 2013). Alex, on the other hand, used questions much less frequently than John and modified future questions based upon the responses from the students. Instead, when Alex asked students to respond to questions, these came from his knowledge of critical thinking skills (Mills, 2013). Sam and Tom also indicated in the survey that they frequently used questions, but compared to the other teachers, this was not entirely true in practice. Furthermore, for Sam and Tom, questioning strategies were used sparsely, and they largely related to the homework, which is another approach altogether. In all, questioning strategies are complex and can be deployed in a variety of different ways (Boaler & Brodie, 2004). In this study, all the teachers demonstrated the characteristics of PCK as they align with the theoretical model. In examining questioning strategies in calculus, under the PCK framework, teachers using PCK would anticipate discussing basic concepts of calculus, using discussion, using questioning strategies to assess students' knowledge, asking rhetorical questions, questioning strategies to determine where the fault in the students' logic occurs, using discussion to elicit critical thinking skills.

The use of these types of questioning strategies are clear indications that the Saudi system of education is shifting its focus towards student engagement. In the last decade, rote learning has been a fundamental component in the Saudi education system (Alshahrani & Ally, 2016), and as a result the KSA has received criticism on this rigidity (Smith & Abouammoh, 2013). Most tasks in calculus require significant cognitive engagement, and scholars such as Abu Asaad (2010) have identified the problems that 'rote learning' can engender in the university context. Seeing all four teachers employing questioning strategies suggests that even in the few years since the publication of these articles, the KSA is attempting to shift its focus from an entirely rote learning-based model of instruction. As this relates to the PCK theoretical framework, it is evident that with the characteristic that encourages teachers to 'actively encourage students to think about the case' – the participants of this study are meeting this requirement. This in turn allows for the development of critical thinking skills, which have been identified as fundamental for the more abstract case of calculus.

Use of pivotal examples or counterexamples in calculus

In building upon the expansion from rote learning, the literature highlights that students who are taught through rote learning are generally unlikely to be able to link formal theory to the solution to their problem (Sofronas & DeFranco, 2010). This is problematic because it means that students are not be able to take what they have been taught and then apply it to their own examples (Weber, 2004). Therefore, teachers must not only employ potentially useful examples or counter examples

(Gruenwald & Klymchuk, 2003; Klymchuk, 2005) but must do so in a way that encourages critical thinking (Klymchuk, 2010).

Some teachers in this study employed examples as a means to both assess students’ challenges and mitigate certain instances of students' misconceptions (Gruenwald & Klymchuk, 2003). For example, in his interview, John highlighted that he used examples in the classroom setting and that he found this to be a valuable tool in addressing the different learning styles of the students (Klymchuk, 2005). In practice, John demonstrated this approach by allowing his students to submit their answers in class and then working with the problematic areas by providing multiple more examples in order to ensure that the students could demonstrate the knowledge required to move on to the next level. Alex was also able to demonstrate this effectiveness through the use of examples. Like John, he tailored his examples specifically to the class. Alex was not just classifying all university students as having certain misconceptions about calculus, but instead he was focusing on this particular cohort and the challenges that existed in this group. This is particularly valuable in the academic setting, as no two cohorts are likely to present with exactly the same types of misconceptions.

While the use of examples was a common theme among the teachers, not all of the teachers used the examples in the same way. In the case of Sam, examples were used methodically and in a systematic way (Gruenwald & Klymchuk, 2003). In terms of his lecture, these examples were pre-prepared. Alex, Sam and Tom used examples as homework as a means to assess students’ misconceptions, and then configured their lectures so these challenges were addressed (Klymchuk, 2010). In addition to this, they used pre-selected examples that they felt would meet the requirements for the students' learning. For any issue that fell beyond Alex's own teaching became the responsibility of the student to overcome. Alex would suggest the use of YouTube or other materials that would allow the student to take the initiative to catch up on the required material (Jones & Cuthrell, 2011). The fact that the students did not know some aspects of mathematics, prior to entering the course, was not unusual, though teachers like John took a different approach. With John, when students demonstrated deficiencies, he simply told them, somewhat sarcastically, that they should have learned the material in secondary school, though unlike Sam, he did not offer any support that students could have used to address their weaknesses. This outcome demonstrates that while all the teachers used examples in some form, they do not all use them in the same sort of way or for the same sorts of functions.

In terms of the PCK theoretical model, this subcategory captures a focus on the key ideas in calculus when introducing salient points followed by more detailed explanations and analysing each calculus topic using the definitions, theorems, proofs and examples, understanding of how to choose the calculus topics for instruction, using examples as a foundation, using examples and counterexamples, knowing that examples are essential as they relate to explaining and clarifying calculus concepts, focusing on use of pre-prepared examples, and using examples as they are generally better used after definition. Having previously established that the teachers in this study are competent in identifying these key ideas, the characteristics that have been highlighted are clearly demonstrated by them, in this study. This is consistent with the literature, which suggests the benefits of examples and counter examples as links to effective critical thinking (Bardelle & Ferrari, 2011; Peled & Zaslavsky, 1997)

Mathematical representation in calculus

The final aspect of instructional strategies that requires discussion is mathematical representations in calculus. The literature suggests that visual representations are deemed to be particularly helpful in the mathematics classroom (Vincent et al., 2015). Calculators and other mathematical tools (e.g. computer programs) are deemed to be useful in assisting students to see the modelling of functions. Furthermore, students have the ability to obtain mathematical representations and visual representations outside the classroom, after the end of lectures (Kumsa et al., 2017; Holton, 2001). This is because there are many resources available that can facilitate learning on a more regular basis. Most of the teachers were able to link the visual, symbolic, and verbal ideas in calculus in ways that facilitated students' understanding (Biggs, 2003). Alex demonstrated this most prominently in Lecture 1 with trigonometric functions, while John tended to focus on visual representations paired with verbal explanation. Certainly, all four teachers were able to communicate ideas that assisted students in problem solving, another fundamental characteristic of the theoretical framework.

Yet mathematical representations, as they relate to the PCK theoretical framework, go beyond the link between visual and verbal ideas (Ostebee & Zorn, 2002; Przenioslo, 2004). It also encompasses the flexibility that goes along with the creation of such ideas. In the context of this study, flexibility was demonstrated by all of the teachers in how they encouraged students to think about calculus (Speer et al., 2010). Alex used different coloured pens on the whiteboard to not only portray the representation, but to identify the different steps (Lecture 3). Moreover, some of the teachers used diagrams to ensure that learning was facilitated more regularly (Tall & Vinner, 1981). For Alex, visual representations were independently applied, and he used these to ensure that the students were taking the representations that had been provided in class and using them in their own context. All the teachers also used linking visual, symbolic, and verbal as a way to ensure that students were receiving calculus ideas in different ways (Ostebee & Zorn, 2002; Przenioslo, 2004). It is recognised that visual representations enable students' understanding and are effective in avoiding (visual) misunderstandings. They can also explain the definition and support it with a (visual) graph and example, delivered through an appropriate method (e.g. pairing visual representation with the verbal explanation of the concept (visual, verbal)). This can facilitate understanding of a model to avoid misrepresentation of symbols, giving students different stimuli in simultaneous instances. Using visualisation to provide the appropriate perspective for students, using representations and images to emphasis particular importance, and using examples and diagrams as a tool, fit within the larger model of PCK, suggesting that there is strong evidence that the characteristics assigned to this subcategory are applicable. In this study, teachers' pedagogical strategies that related to mathematical representations generally coincided with the findings that were previously presented in the literature.

Classroom activities vary by teacher and by class, as certain tasks are better suited to meet specific learning objectives than others. Classroom activities serve multiple purposes; first, they are valuable for the teacher in gaining knowledge about students' thinking about calculus concepts, and more specifically, identifying students' progression in understanding typical calculus concepts. Second, classroom activities often seek to focus on key ideas in calculus by using pivotal examples and counter examples as a form of instructional strategy (Klymchuk, 2014), specifically from an active learning perspective. Third, classroom activities and tasks that require students to flexibly use a wide range of representations generally offer benefits to both students and teachers. In this instance, teachers are able to identify student progress and comprehension relatively quickly through the use of classroom activities, allowing subsequent lessons (and even parts of the same lesson) to be appropriately adapted.

Tom was quite focused on leading the class through the lecture, which is evident in his teacher-centred approach to instruction, where the teacher is responsible for directing the learning to a particular concept. This macro view of the classroom continues with Tom's inclusion of definitions in the classroom. In his self-described response, he indicates a focus on providing students with the larger view without necessarily highlighting each individual component. Both Tom and Sam tended to prioritise teacher-talk-time, leaving virtually no room for in-class discussions or assessment of knowledge. It is possible that this type of teacher-centred instruction may not necessarily be the norm in all of Sam's lectures or that he is even aware of his own teacher-centred focus, though the outcome does coincide with what is known about a more teacher focused pedagogical approach. This is a very different approach to the other teachers (e.g. Tom) because in this case, Sam is indicating that it is not only about retention, but it is about understanding. He makes the link between the use of representations, the learning outcomes, and the understanding of the students

In the case of Alex and John, the idea of discussion among groups at the end of the lesson seems like a plausible approach to assess comprehension, though in practice, this strategy did not appear in some of the observed lessons. Both John and Alex are using students' misconceptions to frame the concept in a way a student would be able to understand, but John is scaffolding students learning, culminating in the use of the theorem followed by an explanation, lecture, or discussion. This was again demonstrated in the classroom setting; it was especially evident in the way that the lecture was built to make the students self-sufficient by the end of the class. In the observation of some of John's lessons, after setting out the expectations at the beginning and having students work through the different problems, time was left at the end of the class for students to undertake problems with very little guidance from the teacher. Time was left at the end for any questions that had arisen from this self-directed activity.

Both John and Alex offered different strategies for teaching the same material, suggesting that they had a similar learning objective in mind in their attempt to guide students to achieve it. They had the aim of successfully completing the lesson. With a lecturing style approach, students are often very comfortable to become passive learners in the classroom, as this is typically how they have been taught in secondary school, particularly in Arabic countries (Alrashidi & Phan 2015). The shift to a more active and collaborative approach may offer students greater opportunities to put their knowledge into practice, which can offer significant benefits in the long run. Interestingly, John and Alex suggested that they felt they did not teach mathematics in a purely 'mathematical way' when responding to the question in their interviews. One area of future research might be to examine what exactly these teachers understand to be a 'mathematical way' and to determine how that response fits within the larger components of the curriculum.

Alex and John highlighted the value of group work as beneficial to better comprehension for the students as long as it is structured – in this case by a worksheet. Bringing the focus to a single problem, or objective, directs the students to a particular task, which would align with the curriculum objectives. From a pedagogical perspective, this seems very practical, as students see the benefit of the task and are forced to explain their thinking to others, which should solidify the concept within long term memory. In addition, Alex gave his students the opportunity to work with other students of their choosing. By allowing students to select their own groups, it is likely that the discussion will be more free flowing, as there may be less social or cultural implications that would hinder the discussion process.

7.5 Synthesis 4: Knowledge of Calculus Connections

Real world applications of calculus

The final element that requires consideration is the knowledge of calculus connections, which was analysed by assessing the real-world application of calculus. The PCK model includes characteristics as a way of interpreting experience or human activity as well as the application of calculus in everyday use.

In this study, all four teachers made some effort in this instance, but for many it was simply mentioned in passing. For John, he mentioned it in the interview and then linked this to real world problems in Lecture 6; for Tom, the application to the real world was only discussed in the interview. It was Alex who made the most connections, demonstrating some real-world applications, which Khakbaz (2016, p.192) suggests makes for “a coherent and meaningful content”.

In terms of this topic, there were weak links between the participants and the theoretical model of PCK. Yet despite the fact that the links were weak, there were places where the theoretical aspects of calculus were linked with elements that were physical in nature, which gave the students a more well-rounded interpretation of the topic overall. What can be gleaned from this category is that the interpretation of experiences and relations to human activity can broadly be understood by the interactions that occur in the classroom (Harcharras & Mitrea, 2007). The teachers were trying to present real-world challenges, making links between mathematical and everyday use of terms through examples and offering opportunities for the students to see the applied value. By making lessons enjoyable through practical means and using representations and images to emphasise particular importance, highlights real-world applications of calculus, identifies real-world connections, and apply derivatives in other contexts. Demonstrating the applicability and benefit from the application of calculus in the everyday explains and demonstrates to students how calculus fits within everyday usage. Providing examples that give context identifies relationships between mathematics and application and being aware of the challenges that students face trying to link the abstractness of some calculus concepts with everyday use can hold students’ interest and express value in the application of the topic.

Working together to solve a problem and collaborating with others who demonstrate various strengths and weaknesses is of paramount importance when it relates to everyday life (Harcharras & Mitrea, 2007). These skills are likely new to the Saudi university classroom because in a rote learning setting, the ability for interaction would be minimal (Neill & Shuard, 1982; Harcharras & Mitrea, 2007). According to the literature, many university students view calculus as just another class where memorisation of equations is required in order to pass (Bresoud et al., 2013) and they recognise that calculus does not fit easily into the real world. It is possible that more can be done to ensure that students see the connections to subjects such as engineering, physical, business, and economics, but aspects of PCK are being addressed.

Calculus in academic subjects

It is difficult to compare academics across countries (Barnes, 2007) because of the differences between students, learning, and teaching in different contexts, but while these challenges exist across countries, one of the components of PCK is whether calculus can be relatable to other subjects. The answer, in this instance, is only minimally. Mainly, the closest link to other academic subjects was identified to be physics, which is arguably another science that asks students to think cognitively in a slightly different way than they would in other classes (Khakbaz, 2016). In this study, Sam made a passing reference to James Gregory in Lecture 3, which highlighted one instance in the entire observed section that went beyond references to physics (Harcharras & Mitrea, 2007). Additionally, there were some basic references made to physics, among the four teachers, but generally, the link to other academic subjects was minimal.

In terms of the PCK model, the characteristic for this subcategory is the demonstration of calculus in various academic subjects (Neill & Shuard, 1982; Harcharras & Mitrea, 2007). If the key word in this sentence is 'various,' then the applicability of this subcategory to the teachers in this study is not fully addressed. This is because there was simply not the evidence that this exists. This outcome does not indicate that the teachers lacked PCK, but rather that more needs to be done to possibly redefine this subcategory in the PCK model. If calculus requires students to think in a way that is different from many of their other classes, then it seems unlikely that useful comparisons could be made in this way, such as applying derivatives in other contexts, relating well to other academic subjects, and referring to ideas that go beyond actual learning of the materials associated with calculus.

7.6 Synthesis 5: Additional Issues

Discrepancies between intended or declared practice and actual practice

What was made evident by the findings of this study is that teachers do not always undertake in practice the things that they indicate are important to them in 'theory'. There were several instances where the survey responses and the observed outcomes from the lectures did not coincide. This outcome could simply have been due to instances where the teachers completed certain tasks in lectures that were not observed, but it could also demonstrate that the teachers had the pedagogical knowledge but were unable to accurately demonstrate this knowledge in practice. More research is required in this area for clarification.

From the literature, it is evident that calculus teachers can be confident about students' misconceptions, but that this could lead to preconceived notions about students' capabilities (Eichler & Erens 2014). This was largely consistent among the teachers in this study, especially with those who had taught the lesson many times previously. For example, Sam indicated that he frequently used questioning strategies in his lectures, but this was not generally observed. Tom acknowledged that he used questioning strategies, but this was also not observed in his classes. These examples indicate that the calculus teachers have a set of beliefs about their calculus teaching, and these beliefs were provided in detail in both the interviews and the survey. In another example Tom suggested that he did not have particular ISs or teaching methods that he specifically employed, this supports the issue that there is a divide between what constitutes knowledge and what constitutes a belief (Phillip et al., 2007). The PCK theoretical framework offers some support for this connection through the description of the various characteristics. Additionally, the methodological approach of using triangulation has provided a more well-rounded picture of how these findings fit within the larger realm of PCK. The difficulty occurs when attempting to link the beliefs expressed to what is being undertaken in the classroom. More research is required in this area to clarify the practice of these beliefs.

Calculus teachers' technological pedagogy

Recent literature on the subject of calculus has suggested new methods that can be utilised to facilitate learning among students. One study by Kashefi et al. (2012) suggests that a combination of both face-to-face learning and e-learning would be valuable for students. Kashefi et al.'s study suggests that IT and web-based assistance could provide the innovation necessary to encourage students to use their critical thinking skills along with new ways of approaching problem solving in mathematics. The teachers in the current study used technology to varying degrees. The classrooms were generally equipped with standard technology which included a chalkboard, OHP, overhead camera, and projection screen, indicating a somewhat limited availability of technological innovations in the classroom. However, the teachers encouraged the students to use external sources to enhance their learning; for example, both Alex and Sam suggested that students utilise YouTube videos to support their learning. Alex also used Maplesoft as a means to enhance the visual experience for students. Finally, all the teachers required the use of calculators in the classroom, as these were seen as a necessary tool to facilitate the calculation/graphing process, providing further justification that the teachers felt technology was a beneficial tool for their students.

The PCK theoretical framework designed for this study, does not entirely specify areas of technological advancement that would be helpful to the teachers' pedagogical methods. Although the characteristics of each of the subcategories allow for considerable flexibility, there were areas where technology could be placed or discussed within a subcategory. Adding a category that identifies specific technological related undertakings could possibly be a future addition to the PCK theoretical framework.

Knowledge of mathematical procedures

The participants in this study demonstrated knowledge of mathematics procedures in many instances. The teaching practice of teachers usually takes into account the various learning approaches of students, especially when it refers to the visual component. According to Weber (2004), students learn about calculus concepts in three distinctly different ways. These include the natural learning approach, the formal learning approach and the rote/procedural learning approach (pp. 129-130). Students take what has been learned in class and apply it to their own examples (Weber, 2004, p. 130), but they may have challenges linking formal theory to their problem. This is a fairly consistent outcome in calculus and one that is considered problematic.

In the case of the participants of this study, there were still many places where knowledge of mathematics procedures approach was implemented. However, what stood out among these four teachers were the other instances in the classroom where they attempted to take the rote learning examples and employ collaborative work, such as group work, to facilitate problem-solving processes. One example in particular was the case of John, who employed a significant amount of teacher talk within his classroom, and from his teaching approach, he was very systematic in the delivery of the lessons. Yet John and Alex also used collaborative work to encourage students to use their critical thinking skills and employed questioning strategies so that the students could demonstrate success at their own level.

While the teachers in this study did not employ large demonstrations of teaching, specific to the natural and formal approach, they did attempt to move beyond rote learning, thus knowledge of mathematics procedures could be an area for future research.

7.7 Successes and Limitations of the PCK Model

PCK in the theoretical model proposed in Chapter 4 has been divided into four first-level subcategories. This is in line with Shulman's (1987, p.8) definition of PCK as "a special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding", with the suggestion of Marks (1990) that mathematics offers a unique take on how PCK could be implemented in the classroom, and with the views of Hill et al. (2008) that PCK in calculus is particularly unique because teachers must demonstrate mathematics content knowledge, specialised calculus knowledge, and pedagogical knowledge. Looking back upon this final point, after the completion of this study, it is clear that calculus instruction is specifically unique because it is a specialised focus of mathematics. It is acknowledged that, for students, calculus requires different cognitive actions, but from a teaching perspective it is clear how this relates to PCK. There were indications from this study that teachers can demonstrate specifically how to get students to think in a certain way through building mathematical theories, using mathematical representations, and calculus connections. The misconceptions the teachers in this study identified among their students could be measured, as could the challenging areas of learning; yet the actual strategy to teach the cognitive development/shift was little apparent. This is not to say that the teachers were not doing something to encourage this development, but rather evident from the model it requested a higher level of knowledge. Additionally, while this is noted, there were underpinnings that the teachers were successful in their undertakings. The students, as observed in the lectures, generally seemed to be passing the course, thus indicating that they were following the building blocks as outlined by the teachers. It is suggested that more research is required on the student experience in order to determine whether this component could be an addition when exploring PCK of calculus teachers.

Overall, when judging how the model functioned in establishing evidence of PCK in the Saudi classroom, the four first-level subcategories offered useful evidence that could be linked to each of the teachers in some way. More research is required on the PCK of calculus teachers in order to further justify the workings of the characteristics.

7.8 Reflections on the Methodology

The methodology chosen for this research draws on a range of perspectives identified by researchers. It challenges the view articulated by Crasnow (2011, p.28) that studying cases can be held in 'low regard' in research, though her viewpoint relates specifically to using just one or only a few cases to draw firm conclusions. This research does not set out to draw conclusions, but rather to illuminate and understand an existing situation. This recognition of the potential limitation of studying cases, as a foundation to a piece of research, is challenged by Yin (2004) who recognises that studying cases can be a successful research method when 'a how or why question is being asked about a contemporary set of events over which the investigator has little or no control' (p.9). This is certainly the case in this research as the researcher was not able to control the context in which the research took place. During the early stages of the development of this research, this provided a challenge that the researcher needed to overcome. It was clear that the context and situation needed to be interpreted and understanding this through multiple case studies was the most appropriate method. The approach is also supported by Rowley (2002) who recognises that one key strength of studying cases is the strong contextualisation with 'real life'. This study seeks to understand a real-life context of the experiences and understanding of calculus teachers. By choosing an instrumental case approach and applying multiple methods, the research was able to incorporate the most suitable aspects of several approaches and build a successful and rigorous methodology. During the coding process the researcher paid specific attention to grain size and inferring. When starting with the second level sub-categories, then going to first level, this was clear when analysing the observations. When focusing on small grain size there were more colours and more overlap, when using big grain size there would be less overlap, but this would not provide the detail about the phenomenon. Another issue was the inferring. For example, when analysing the interviews, it was important to understand how the interviewees 'felt' about a particular situation or context through what they inferred about it. Through this process, it became clear that the findings of the research would not be accurate if only explicit thoughts and opinions were considered. There was value to be found in what the participants hinted at or did not explore fully. Such issues were important to identify before the research took place. It was decided what would be included and what would not and these clear 'rules' guided the research methods.

7.9 Chapter Summary

What the discussion chapter has indicated is that the research questions have been fully addressed. It has been shown that the teachers have used their PCK to develop unique and innovative strategies in order to target some of the misconceptions' students have. This means that some of the teachers are employing strategies that move away from the traditional and passive style of instruction. In the context of KSA, this is unique and innovative and therefore a significant contribution of the four cases. The teachers made their aims for the lessons clear by regularly stating these at the beginning of the lesson and by consistently referring to the aims and objectives at various key points throughout the lecture. The teachers focused their attention on scaffolded learning and cooperative learning in order to ensure that students have the understanding necessary to continue and they encouraged students to seek outside/additional support when needed. Each teacher demonstrated his own strategy to deliver the lesson, but all the teachers focused on providing examples, formulae, and definitions in a way that they felt best assisted the students' learning. While most of the lectures were predominantly teacher-focused and had a high percentage of teacher-talk time, there were indications that these timings were less than in previous research studies. Finally, there were also places where the teachers used their PCK to apply calculus connections, referring to real-world scenarios and to other academic subjects, such as physics.

In summary, this study has highlighted many of the main ideas identified in the literature and the discussion of the findings have considered the success of PCK and provided reflections on the methodology and grain size and inferring.

Chapter 8 Conclusion

8.1 Introduction

This research project set out to examine the PCK of four university level calculus teachers through the use of survey, interviews, and observations. The triangulation of the data allowed for a comprehensive picture to be established surrounding their PCK of learners' cognition, teaching and the way that teaching is specifically implemented in the classroom. The data from this study were analysed using a specially designed framework. While this qualitative research took on ambitious goals, the outcome is a study steeped in fine detail, in order to appropriately address the research questions. Furthermore, the findings from this research provide a platform for future research in the field of PCK not only of calculus teachers, but also of teachers of other areas of mathematics at the university level. The contribution this research makes paves the way for the future development of calculus teachers and students and provides a model that can be developed and used widely within the field. Although this research was situated within the KSA university system, which is therefore the focus, it also makes a global contribution to the knowledge and understanding of calculus teaching in universities.

This final chapter begins with the key findings, which indicate how, and in what way, the study findings have addressed the research questions. This is followed by the study limitations, which are presented for clarity and cohesiveness, and is followed by the implications for future research. It is acknowledged that this study is only a first step in the connection between PCK and calculus teachers, and thus there are many future opportunities for development. A section on the reflections of the researcher with an overview of the role of the researcher outlines the growth and development that has occurred along this challenging research journey.

8.2 Summary of Findings Related to the Research Questions

RQ1: What could be a model of PCK for teaching calculus?

To address this research question, the researcher drew upon a number of frameworks of teacher knowledge (Lesseig, 2016; Khakbaz, 2016; COACTIV, 2004; TEDS-M, 2008) leading to a new two-two-pronged framework for PCK for teaching calculus being devised and adopted for this study. Lesseig (2016) organises her framework into two categories: *knowledge of content and students* and *knowledge of content and teaching*. This present study's framework, however, differentiates between the categories of knowledge of content and students when teaching calculus on the one hand, and knowledge of content and teaching calculus on the other. These categories are underpinned by a number of first level and second level sub-categories (See Figure 4-4). Figure 8-1 shows this framework as a model of PCK for teaching calculus. The key feature of this model is that there is no prescribed point of entry, as each element within the categories and sub-categories has equal significance. Furthermore, as the point of entry may be at any point, it can be used, not just at university 1 level alone.

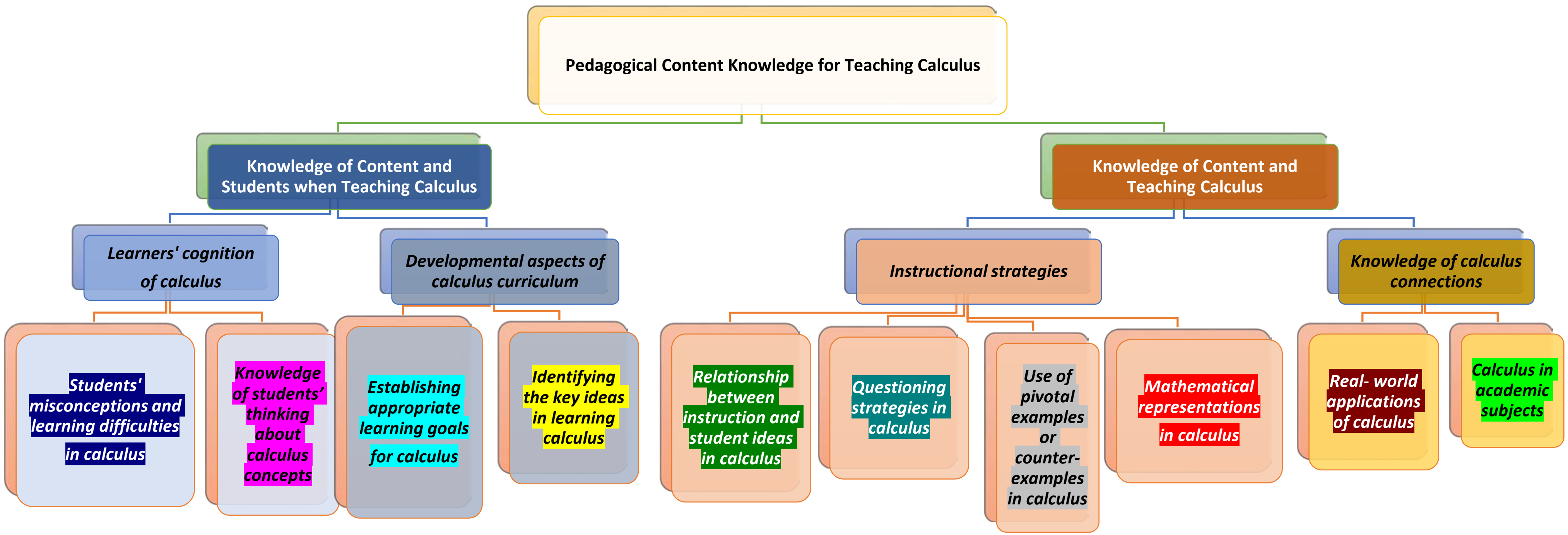

Figure 8-1: The Proposed Model of PCK for Teaching Calculus.

RQ2: Using this model of PCK, how do calculus teachers articulate and demonstrate their PCK?

Learners’ cognition of calculus

This research question has been addressed throughout the findings and analysis chapters, and the answer is, broadly, that teachers demonstrate and use their PCK in many different ways. It is evident that the sample group of teachers are aware of the difficulties their students face, largely through preconceived notions from previous classes taught. In order to ensure that their students' misconceptions are addressed, a variety of strategies are employed. Examining learners’ cognitions of calculus, especially students' misconceptions and learning difficulties in calculus under the PCK framework, has identified that teachers using PCK anticipate using diagnostic tests to identify learners' cognition and are aware of students' misconceptions. It is clear teachers identify students' learning difficulties and discuss what the students know and simplify definitions provided in lectures, further than those outlined in the students’ textbooks, highlighting ideas that students tend to struggle with. They establish relationships between instruction and students' learning difficulties in calculus and are aware of the challenges that students face in order to assist in the avoidance of misconceptions, flag specific areas where weaker students can focus their attention and use more than a subcategory when linking the abstraction of some calculus concepts with everyday use. Definitions on challenging topics are universally provided by the teachers in separate instances and while they do not focus on the definitions in the same way, they take what they know about their students and attempt to encourage them to address the misconceptions in a way appropriate for the context. What is fairly evident, from these teachers, is the use of specific strategies that can be helpful to their learners, thus encouraging them to advance through the course material. From the way the teachers in this study demonstrate and use their knowledge of students’ thinking about calculus concepts, indicates that by having knowledge of learners’ cognition of concepts gives teachers the ability to make these concepts understandable and easier to grasp. Teachers need to ensure that their calculus students are aware of areas that are particularly challenging and identify the difficulties with both constructing and evaluating calculus concepts.

Developmental aspects of the calculus curriculum

For many of the case teachers, the topics that were being discussed were pre-assigned and the students needed to learn the material in the course and have a strong foundation in the course in order to proceed onto the more advanced classes. While the outcomes may have been pre-assigned, however, the teachers in this study were free to design lessons that allocated certain amounts of time to each concept. The participants in this study were clear when providing their students information regarding the aims of the lesson. These aims were either highlighted at the beginning of the lesson or flagged at fundamental points throughout the lesson to ensure that students were focusing their attention on key concepts, and all the teachers demonstrated places where learning goals were highlighted. All used their knowledge of establishing appropriate learning goals for calculus for linking between theory and practice, ensuring that their students would be challenged and provided with contexts for future learning.

The teachers addressed the aims in different ways, which ranged from a very rigid approach to the course material, being methodical and structured, to more fluid ways of delivering these aims. If the aim was not achieved by the end of the lesson it would be carried over, whereas a more rigid approach ensured that the aim was covered. In addition, building blocks are fundamental to this project and the teachers used various techniques to ensure that the students understood the material and consequently were able to move on to the next concept. The fundamental reason for placing these key ideas into the PCK theoretical framework is the requirement of teachers to provide, and make available, definitions, theorems and proofs to students as well as to provide relationships between mathematical and everyday use of terms. The teachers in the current study used different methods to demonstrate these key ideas and illustrates how a teacher can use their knowledge of calculus teaching to sequence the building blocks of mathematical theories (BBMT) of the concepts of calculus.

Instructional strategies of teaching calculus

This category was somewhat challenging, as some of the characteristic descriptions associated with the strategies were vague (i.e. the use of ‘appropriate instructional methods’). The calculus teachers were mainly systematic in their delivery of the calculus lessons to the students. While all demonstrated ways that they used their PCK in lessons, it was clearly evident that a number of strategies used focused around using ‘think, pair, share’ tactics. These strategies encourage cooperation, collaboration and students' participation. By directing and providing students with material on calculus functions they can consciously evaluate enables them to deal with particular challenges. By encouraging students to read the theorem and proof before explaining it and knowing that group work influences students' collaborative learning and they will take the initiative, in a number of ways, to facilitate understanding. It was noted that students would reflect on a specific case, if given the opportunity, and explain the lesson's concepts to their classmates. Facilitating their understanding of calculus, gives them a solid grounding in logic and its associated linguistic expressions. Expecting students to self-direct their learning, rather than checking for their understanding to improve insufficient previous knowledge, places a great deal of responsibility on them. However, when teachers use a methodical and systematic approach and linear progression through the objectives of the lessons meets the characteristic of this subcategory.

It was also evident that the calculus teachers in this study tried to get their students involved in the lesson, to varying degrees. In the past, calculus teaching mainly took the form of a lecture-type teaching strategy, where knowledge was imparted to the students and where teachers wrote on blackboards while students sat quietly and copied down the examples. In the observed classes, while this was still a common method for instruction, there were many more instances where students were encouraged to actively get involved in the class. This was often identified through group work activities or other collaborative undertakings (e.g. group homework). In terms of teaching strategy, the inclusion of students within the learning process facilitates their critical thinking skills and can minimize misconceptions through the use of practical learning strategies.

Most of the case teachers demonstrated the characteristics of PCK described in the theoretical model. In examining questioning strategies in calculus, under the PCK framework, the teachers in this study show that utilising PCK anticipates discussing basic concepts of calculus. By using questioning strategies to assess students' knowledge and asking rhetorical questions together with questioning strategies to determine where the fault in the students' logic occurs allows for the use of discussion to elicit critical thinking skills. Sometimes teachers used a combination of both. As calculus is unusual in its application (i.e. a student must understand the basic formulas and concepts before proceeding to the next topic), ensuring that students actually firmly grasp ideas before moving on, is fundamental.

Additionally, pivotal examples or counter examples were used to ensure that students achieved the level of understanding required to continue. In terms of the PCK theoretical model, the focus for this subcategory is to ensure a focus on the key ideas in calculus when introducing salient points followed by more detailed explanations and analysing each calculus topic using definitions, theorems, proofs and examples. Teachers are expected to understand how to choose the calculus topics for instruction, use examples as a foundation, use examples and counterexamples, know that examples are essential as they relate to explaining and clarifying calculus concepts, focus on use of pre-prepared examples, and use examples as they are generally better used after definition. Having previously established that the teachers in this study are competent in identifying these key ideas, the characteristics that have been highlighted are clearly demonstrated by them. This is consistent with the literature, which suggests the benefits of examples and counter examples as links to effective critical thinking.

The participants in this study utilized mathematical representations, recognising that visual representations enable students' understanding and are effective in avoiding misunderstandings.

Explaining the definition and supporting it with a graph and example (visual), delivered through an appropriate method e.g. pairing visual representation with the verbal explanation of the concept (visual, verbal), can facilitate understanding of a model to avoid misrepresentation of symbols, giving students different stimuli in simultaneous instances. Using visualisation to provide the appropriate perspective for students, using representations and images to emphasise particular importance, and using examples and diagrams as a tool, provide strong evidence that these characteristics, assigned to this subcategory, fit within the larger model of PCK. In this study, these were met by the teachers and their pedagogical strategies, related to mathematical representations and generally coincide with the findings that have previously presented in the literature.

Knowledge of calculus connections

While the teachers in this study were able to apply their PCK to calculus connections, this was an area of the research where there were not many instances of application during the observations. It was not quite clear that the teachers were able to link calculus to other academic subjects, except physics. This outcome does not necessarily indicate that the teachers lacked PCK, but rather that more needs to be done to possibly redefine this subcategory in the PCK model. If calculus requires students to think in a way that is different from many of their other classes, then it seems unlikely that useful comparisons could be made in this way, such as applying derivatives in other contexts, relating well to other academic subjects, and referring to ideas that go beyond actual learning of the materials associated with calculus. Some teachers were able to tie the concepts to everyday ‘real-world’ situations (e.g. waves or vibrations), but the amount of time spent applying these practical ideas to the lessons was minimal. This is not necessarily an indication of a fault of the teachers or a lack of PCK usage. Certainly, the number of times something is mentioned is less important than when it is mentioned, and the participants of this study attempted to highlight instances of connections at the most opportune moments. Therefore, the teachers were trying to present real-world challenges, making links between mathematical and everyday use of terms through examples and offering opportunities for the students to see the applied value. By making lessons enjoyable through practical means and using representations and images to emphasis particular importance, highlights real-world applications of calculus, identifies real-world connections, and apply derivatives in other contexts. Demonstrating the applicability and benefit from the application of calculus in the everyday, explains and demonstrates to students how calculus fits within everyday usage. Providing examples that give context identifies relationships between mathematics and application and being aware of the challenges that students face trying to link the abstractness of some calculus concepts with everyday use can hold students’ interest and express value in the application of the topic.

Finally, based on the results from the observations, interview, and survey this research question was fully addressed.

8.3 Justification of Theoretical and Methodological Choices

The justification for the research theories in this study is based upon the notion that students who use rote/procedural learning may have difficulty linking theory to practice (Sofronas & DeFranco, 2010). One of the reasons why this was deemed to be the case was due to the teacher-centred approach that often exists within the calculus/mathematics classroom. The underlying premise is that teachers are not provided with the pedagogical training to offer effective and innovating strategies in the classroom, and while they are generally likely to have content knowledge, they may not always have PCK. Because of this, there is a need to establish a theoretical framework that can demonstrate whether PCK exists within the calculus teachers' practice, at university level. Given that calculus is a subject area that has not previously been examined in relation to PCK, it becomes a worthwhile undertaking, especially in the case of the KSA. For a nation that is predominantly centred around lecture-style approaches and memorisation, calculus offers a break from the norm because memorisation is not so effective for this problem-based subject area. As a result, not only does the examination of PCK allow for a glimpse into what other subject areas might look like as teachers venture into a more active style of teaching and learning, but PCK also gives current calculus teachers an understanding of why calculus instruction is so important to the larger understanding of pedagogy.

In order for this to occur, the methods of interviews, survey, and observations were employed to answer the research questions. This methodology not only provided a solid framework for the triangulation of data, but it also linked what teachers said they did in the classroom to what they actually did. This provided significant benefit when writing up the findings, as the multi-methods approach made the outcomes more thorough and rigorous.

8.4 Contributions of the Study

8.4.1 Research Contribution to Teacher Education

University teacher education in KSA has leaned towards the integrative aspect of the PCK spectrum because there is very little teacher education prior to being able to teach at the university level. Teachers at the university level, in many subjects not just in mathematics, tend to lecture rather than to integrate students into the learning process. As a result, there has been a recent push in university teacher education in Saudi that encourages teachers to be more innovative in the classroom. The outcomes from this study suggest that teachers are gradually making changes to their teaching practice that allows for such innovation to occur. In terms of how this contributes to research in a larger perspective, is that clearly more work needs to be done in this area. According to Shulman (1987), the interconnectedness of content and pedagogy is lacking, and it is necessary to go beyond the subject matter when considering how teachers can facilitate learning. Now, over three decades on from Shulman's writings, calculus teaching has not made great strides in this area. Therefore, one of the main areas where this research can contribute is not only to identify places where improvement can occur, but also to identify the elements of PCK that are now, encouragingly, being utilised in the classroom.

8.4.2 Research Contribution to Educational Practice and Theory

The Saudi MOE has been tasked with the implementation of Vision 2030 and the Saudi government has spent a significant amount of financial resources developing higher education within the KSA (Al-Aqeel, 2018). The focus on educational practice has been targeted at the creation of a centralised system of control and support, ensuring that education is state-funded (i.e. it is free for Saudi citizens). Furthermore, the government has spent considerable efforts on the Horizon project, which is a higher education system that includes all major stakeholders including the government, individual universities, industry, and community representatives. This contributes to educational practice because it puts KSA as a contender on the world stage, a fundamental goal of the Ministry of Education and of the Saudi government.

In order for such large goals to be obtained, it is imperative that the impact of change related to pedagogy be documented. This study is essentially one step in this process. If the KSA wants to be competitive on a global scale and to be able to liaise with other universities, industry and community representatives, actual data must be obtained to show that the changes being implemented are successfully leading towards the goals outlined. What has been demonstrated by this study is that changes are being made in the classroom within the field of calculus. Teachers are, to varying degrees, demonstrating PCK in each one of the subcategories.

Furthermore, because this research project is based upon the PCK framework and a specific set of characteristics, the outcomes from this project are easily identifiable and could contribute to a much wider participant pool. As a result, not only did these specific teachers (i.e. the participants involved in the study) benefit from reflection on their own PCK, but the study offers insight into the future development of PCK in the KSA, as the context of this study. There is also the global contribution to the knowledge and understanding of calculus teaching in universities. Not only does this study demonstrate that the framework can be applied in calculus in the context of this study, but it can also be replicated elsewhere. This could in turn contribute to the overall Vision 2030 project.

8.4.3 Methodological Contribution

The use of qualitative data analysis is the foundation for this research. Qualitative research is particularly useful when considering a particular concept and attempting to obtain in-depth information from the participants. However, qualitative data can also be associated with researcher bias. Every effort was taken to ensure that the results were presented as objectively as possible, as multiple data sources can often contribute to an increase in validity and reliability. In this study it was deemed advantageous to use triangulation as it is highly beneficial in ensuring validity; if one of the methods is weak, the remaining methodological approaches will maintain strength in the results (Cunningham, 1997). Different calculus classes were observed in this study, and to ensure that the structured observations were valid, method triangulation was used. The two other methods were semi-structured interviews with the four participants and questionnaires. In addition, the researcher used video recording to allow for re-examination of any uncertain or unclear interpretation by the researcher. Video recordings also allowed the researcher also allowed the researcher to go back to the participants for clarifications.

Up until this research study, much of the previous literature on this topic has focused on the cognitive approach to learning. Previous research literature largely followed the pattern of (1) identifying the difficulties of students, (2) investigating how students learn a particular concept, (3) evolving the classroom to address this concept, and (4) research on the teacher, instructor, teaching assistant, or graduate student (Rasmussen, Marrongelle & Borba, 2014). The frameworks which were developed for these patterns were inefficient at addressing the links between pedagogy and content knowledge (Hitt & Gonzalez-Martin, 2016). Identifying the links between pedagogy and content knowledge through the application of multiple methods, this study has made a methodological contribution. Mathematical knowledge development is a complex phenomenon and teachers must address a wide range of topics through opportunities that create an immersive learning environment for students. Therefore, strategies that highlight these opportunities and their modes of development are fundamental.

The PCK model that has been developed makes a significant contribution to research because it links both pedagogy and content, but then breaks these two concepts into manageable units that can be individually addressed. This is important, as it allows for replication across universities and topics in mathematics. Thus, the justification that PCK can be identified through the use of the theoretical framework is particularly helpful.

8.5 Limitations of the Study

Every study has limitations in one way or another, and this study is no different. Choices had to be made along the way that would allow this project to proceed, but sacrifices resulted. These sacrifices were not problematic, but should nonetheless be identified as, in the next section, implications for future research can thus materialise.

The first limitation relates to sample size. The very nature of research that studies cases does not allow for significant numbers of participants. Instead, it asks a very select group to provide extensive and detailed information in order to contribute to a very specific research question. The sample size limitation restricts the generalisability of the study's findings. Yet, despite this limitation, the study of cases offered intricate details into the link between theory and practice. Furthermore, because this study sample were all teaching calculus in one particular country, the expectations of these teachers may be significantly different than other parts of the world, making the participant profiles an additional limitation.

Another limitation of this study was the creation of the instruments. This was the first attempt by the researcher to create instruments that would be implemented in a research study, and so every effort was taken to ensure that the instruments could stand up to a test of rigour. Every care was taken to make sure that the observations were carefully documented, and the researcher tested more than three observation schedules, the survey was piloted, and that the interviews were annotated appropriately, the design of the questions and the delivery of these questions, during both the survey and interview, were conducted by a novice researcher.

Additionally, in terms of the data collection process, timing and financial resource limitations were significant in the comprehensiveness of this research project. With more time, the participant pool could have been expanded beyond four. This would have provided more data for this study and further contributed to answering the research questions. Yet, time links to financial resources. Not only was time limited by the deadlines imposed on this project, but at some point, the researcher had to recognise that sufficient time had been spent on this research project.

8.6 Recommendations for Future Research

The comprehensive literature review undertaken for this study revealed a paucity of research on the PCK of calculus teachers, and while this was challenging for the current study, it identified a gap in knowledge regarding calculus teaching at university level (Biza et al., 2016; Bressoud et al., 2016; Petropoulou et al., 2016; Rasmussen et al., 2014). As a result of this research project, there appear to be three possible options for the development of future research. These include further examination of the model, an expansion of the sample group, and comparisons made among university level teachers.

In the first instance, the model for PCK can be further examined. When using this framework, there were certain areas where the characteristics outlined provided a definition that was not sufficiently accurate for the purpose of this study. Teachers tended to fall into a category no matter what they did or how many times they did it. For example, a teacher who employed questioning strategies in a lecture could be either a teacher who used questioning extensively, or a teacher who used questioning only sparingly. Both are deemed to be demonstrating PCK, but in very different ways. Yet the theoretical framework did not account for this. Further, some of the phrases, such as 'appropriate' require subjective interpretation by the researcher, which limits the reliability of the results. These misgivings could be remedied through further research on this topic and more extensive use of the theoretical framework.

The second area of development relates to the expansion of the sample group. The initial study was small, using only four participants in one single country. Future research could examine the relationship between PCK and calculus teachers in other countries around the globe, as it is likely that the subjects being taught in first year calculus are largely similar worldwide. Furthermore, the current study could be adapted for application with a larger participant pool. For expansion to occur, time must be considered; therefore, if the interviews with participants were omitted and the survey instrument refined for clarity and to include some markers from the interview questions, the study could be expanded. In this way, through the use of survey and observations, the PCK framework could still be applied but the amount of time for transcription and analysis should be appropriate for the larger participant pool. This type of research expansion would only further justify the theoretical framework that has been developed and the link between its applicability among different contexts and/or participants.

Lastly, one of the interesting points that emerged about university teachers in this study is that they are rarely trained in pedagogy. It is much more common for teachers to be content level experts and then be instructed to teach students what they know. Yet this is not true for all teachers, as some have both content knowledge and a pedagogical background. One area that lacks current data is the comparison of PCK among teachers with and without formal pedagogical experience. By connecting teachers who have training with those that do not, or by comparing teachers with many years of experience to novice teachers, the PCK framework may become more developed. Furthermore, it may show the movement or development among teachers in mathematics who are seeking to further advance their professional careers.

8.7 Researcher's Reflections

> *Any man who keeps working is not a failure. He may not be a great writer, but if he applies the old-fashioned virtues of hard, constant labour, he'll eventually make some kind of career for himself as writer. – Ray Bradbury*

This research process has been long and somewhat arduous, filled with challenges and struggles that were expected initially, but not fully appreciated until much later in the study. However, the outcome of this thesis has allowed for growth; this is true both intellectually and emotionally. From an intellectual perspective, I had anticipated what I thought this research programme was going to include. I knew what the chapters 'looked like' and felt that I could, at least somewhat easily, apply this to my own context. I had research questions, and what I thought was a clear and straightforward path to follow. Yet, what I have learnt from this process is that the reality can be much different from the perceptions. Collection of the data took much longer than I had anticipated; organisation and analysis took much longer than that. As time slipped away, there was a point where I felt that the end would never come. As this occurred, a transition began. Pieces of the research process began to come together. First, I finalised the methodology and began to work with the research instruments. I identified a plausible theoretical framework that could be applied in the context of mathematics, which was an exciting first step. I obtained results that I felt I could use in the writing up process, and then found a presentation style that eventually worked for my purposes.

I gained confidence in my ability and achieved much more understanding about my topic. This does not mean that I came to be an expert researcher. Sure, I am further along than I was at the beginning of this process, but there are still things that I can identify where I lack the knowledge to pursue this type of methodology. However, as I peruse through my notebooks full of scribbles and my failed attempts at chapter designs, the output reads much like a diary. I can see days of frustration, especially in the middle of this project, but I also can see progress. I also can see my notebooks after each meeting with my supervisor, which was a great guide.

From an emotional standpoint, this project has placed me both in the depths of despair and at points of jubilation, and this rollercoaster of performance has meant that I have had to deal with many challenging issues. Data collection, data analysis, and writing a thesis is somewhat of a lonely process. These are because while people can help you with general advice, no one really understands your topic in the way you do, meaning that the explanation of the topic is only ever done in a simplified manner and this, to a point, takes away from the value of the research. I hope that as the reader continued through this thesis, he or she was be able to see the confidence and passion that undertaken this thesis has given me.

8.8 Chapter Summary

As has been shown in this research project, the research questions have been fully addressed. Yet what is more important is that the findings and analysis of the data within this thesis have led to even more questions in the field of PCK research. These findings, therefore, make it helpful for leaders, decision makers, policy makers, teachers, and other researchers to focus their attention on particular aspects of PCK that are most relevant to their situations. Calculus, it has been noted in the findings, is a challenging topic for students and has been one where the method of instruction has remained largely unchanged in the university setting. Yet, as has been shown in this project, steps are being taken to modify aspects of the learning experience to include students more actively in the classroom. Many useful outcomes have been suggested by the participants in this study as ways to negotiate the calculus classroom. Change is not an easy thing to come by, and it requires considerable work. By addressing the aims and objectives of this project and through the detailed findings provided, this research project has demonstrated a valuable contribution to knowledge.

Appendix A **The context**

A.1 Introduction

It is necessary that the context is identified to give a well-rounded description into the nature of the study. This appendix describes the nature and location of the university with the Saudi context before discussing the location, classes examined, and the participants.

A.2 The University (University X) in Saudi Arabia

Saudi Arabia has 28 public and 10 private universities. All but two cater to both men and women with segregated areas for each gender. All of the private universities and 16 of the public universities have been created in the last decade (Smith & Abouammoh, 2013). The university in this study is one of these anciently created universities

The University X (UX) is one of the most well-known education establishments based in Saudi Arabia. It offers a high level of education to the students and is consistently rated among the top schools in the region by external university assessment agencies. While each top university in Saudi Arabia has multiple colleges and/or institutes, this university has over 100 departments, five institutes and multiple centres with a combined count of over hundred academic departments. The university offers both graduate and undergraduate programs for both male and female students. Over 100,000 students are enrolled in different colleges and institutions of the university, and nearly 5000 faculty members are responsible for teaching courses there (MOE, 2017). UX has extensive calculus programs which are considered a vital part of the college of mathematics (Ministry of Education, 2017).

There are many reasons for choosing this particular college from university (UX) to base this research study on. They are:

- “The university greatly contributes to the enrichment of human knowledge” (MOE, 2017). This college helps students to expand their knowledge. Recently, in 2013 this university has started to offer the preparatory pre-calculus courses (MOE, 2017). The Ministry of Higher Education, (Date N.K) states that the aim of this preparatory year is to expand the students’ knowledge in mathematics (and more specifically calculus preparation) and prepare them for specialism in their field. Thus, it is worthwhile to investigate a high-level university so that a variety of results can be obtained. The university has a good standard in education however; it has just recently started the preparatory pre-calculus courses which identify that there is scope for investigation. The University X competes against other universities in Saudi Arabia because other universities also offer this type of preparation programming, though not to the same level of development.

- The selected college has seen tremendous growth (MOE, 2017). This confirms that these universities have expanded to a great extent and therefore there is the potential to obtain useful information along at an important time in the development of the mathematics programming, which is beneficial to this research.

Classes at this college are widespread; with over 12 academic departments, the majors/minors available to students are extensive. Yet, regardless of which program or specialization area that undergraduate students choose to pursue, they must take certain core requirements that demonstrate breadth (i.e. a science student must take a certain number of humanities and arts courses in order to obtain a degree). In addition to fulfilling the breadth requirement, many undergraduate students must also take first-year calculus because it is a pre-requisite for many programs, including but not limited to, Mathematics, Engineering, Chemistry, Physics, Biology, Computer Science, and Environmental Sciences.

Students are welcome to enrol in any calculus course, though are encouraged to select the calculus course in alignment with their major (i.e. Engineering students would be encouraged to take Fundamentals of Integral Calculus for Engineers Students). Only one section of each course is offered per term. Classes at University X generally run from 8 in the morning to 8 at night; students select their schedule based on course requirements and availability. Students are required to take a pre-test prior to enrolment. Because of the rapid changes to the university model in Saudi Arabia, there are currently some concerns that the high school model does not always teach the necessary components that prepare students for the mathematics they will be taught at university. Students who do not meet the criteria for enrolment must take a foundational year pre-calculus program (there are also other foundational year courses on offer at University X) in order to ensure that they are more likely to be successful and have the necessary skills required to pursue their degree.

Class sizes vary considerably from one programme to another and also by year. While these calculus classes are capped at 100, some of the first-year nature science courses are capped at 150, while other first year courses might have a capacity of only 50. As students move up into second, third, or fourth year, class sizes are generally reduced with most having a cap of 40 students (Al-Dakhil, 2011).

The student population at University X is largely homogenous with most of the students being Saudi nationals. Student demographics provided by the institution generally suggest that there are slightly more female students (56%) to male students (44%), that most students are between the ages of 18 and 25, and that over 90% of the student population was receiving some sort of funding package to attend the university (MOE, 2017). The funding (generally in the form of scholarships) could have been provided by the government, the institution, or a combination of the two, though students do not pay fees to attend public universities in Saudi Arabia.

The faculty population at University X is more diverse than the student population. As indicated previously, the university environment has rapidly expanded in Saudi Arabia over the last decade. As a result, it has been difficult for universities to find and employ professors of high educational quality and with the appropriate qualifications necessary to teach at the university level (Smith & Abouammoh, 2013). The mathematics faculty are a diverse group demographically, with many having degrees in mathematics as well as in other subjects. Some have studied abroad while others have received their schooling solely within the borders of home countries. Despite their wide range of backgrounds, none of the teachers has any formal teacher training (in the form of degrees, diplomas, or certificates), though many have taken Professional Development (PD) courses.

A.3 Overview of class selection

A.3.1 Overview of the Course

Students studying at University X have the opportunity to select from a range of first year calculus courses. These courses are often targeted at offering skills development relating to a specific major (e.g. First-year calculus for nature science students). Despite having different names for each of the courses, the materials that the students use and the course description are largely similar across all first-year courses. One of the reasons why the university has chosen to operate under this framework is that it puts all the potential students hoping to enter the same major into the same first-year calculus class, creating direct competition among students (i.e. all the mathematics students are together and study under the same teacher). In this way, upper year professors can gain insight into student performance based on first-year results.

The class chosen for review was titled Fundamentals of Integral Calculus and specific track was identified; this was for BSc in Mathematics. In the course description for this first-year class, the only pre-requisite was high school mathematics. Enrolment was capped by the university at 70 students per class (Al-Dakhil, 2011).

In this course, the traditional method of lecturing was employed, with no online components taught (though the textbook offered some online problem sets). Lectures were held twice weekly for two hours each session and the course lasted for 14 weeks, making the contact hours in the class equal to 56 hours. Lecturers were also required to maintain weekly office hours with 2 hours being officially scheduled and communicated to students per week. In addition to the class time and office hours, students were required to complete a one-hour tutorial weekly with a Teaching Assistant. The total contact time students would have with the Teaching Assistant was 12 hours. Students were also required to complete work outside of the class. The syllabus indicated that the students were expected to spend 12 hours per week on additional private study/learning hours. The required textbook for this course was Calculus Early Transcendental, International Edition, and seventh Edition, edited by James Stewart (UX). This is a common textbook across university programs in Saudi Arabia and is used worldwide.

The ‘purpose’ for this course was outlined in the syllabus as follows:

- Student should mature in their understanding of calculus through the study of limits, derivatives, and integrals and their applications.
- Student acquires knowledge by learning derivatives and integrals of the logarithmic, exponential, inverse trigonometric, hyperbolic functions.
- Student studies the techniques of integration, finding the area between two curves, volumes of revolution, and volumes of a solid with known cross sections and find the length of a curve.
- Student knows the limit of sequences, sum of infinite series and finding Maclaurian, Taylor expansion of functions in one variable.
- Student acquires cognitive skills through thinking and problem solving.
- Student becomes responsible for their own learning through solutions of assignments and time management.

Based on these criteria for student learning, the Course Description for Fundamentals of Calculus was as follows:

Week	**Topic to be Covered**
1	Functions and Models.
2	Limits and Derivatives
3	Differentiation Rules

4	Implicit differentiation, derivatives of logarithmic, linear approximations and differentials, hyperbolic function.
5	Applications of Differentiation
6	Integrals, Areas and distances, Integration by substitution, Definite integral, The fundamental Theorem of calculus, Definite integral by Substitution.
7	Midterm Exam #1 Applications of integration.
8	Techniques of Integration, Integration by parts, Trigonometric Integrals.
9	Trigonometric Substitutions, Integrating Rational fractions.
10	Improper Integrals, Sequences, Monotone Sequences, Infinite Series
11	Further Applications of integration and Differential Equations
12	Alternating Series; Conditional convergence.
13	Maclaurin and Taylor series and Power Series.
14	Maclaurin and Taylor polynomials and Applications of Taylor polynomials.
15	Review for Final Exam

Appendix A1: 1 Course Description for the First-Year Calculus Course

The syllabus for the course also targeted specific course learning outcomes that were provided to students including topics of (a) knowledge, (b) cognitive skills, (c) interpersonal skills and responsibility and (d) communication, information technology and numeracy. This information can be found (Mathematics Department).

Appendix B **Sources of the Model**

Categories	First-Level Subcategories	*Second-Level Subcategories*	Characteristics (from all of these studies)
Knowledge of content and students when teaching calculus (from Lesseig 2016)	- *Learners' cognition of calculus.* (from coactive data)	- *Students' misconceptions and learning difficulties in calculus.* (from coactive and Khakbaz 2016)	- identifying students' difficulties with both constructing and evaluating calculus concepts; - identifying students' difficulties, including their inability to state definitions, not knowing how a proof should begin, inadequate concept images, and an inability or disinclination to generate and use examples; - Using knowledge of learners' cognitions to address anticipated questions and students' misconceptions.
		Knowledge of students' thinking about calculus concepts. (from COACTIVE data and Lesseig 2016)	- identifying the characteristics of external, empirical and deductive concepts of calculus; - identifying students' formation of mathematical concepts in calculus. - identifying students' progression in understanding typical calculus concepts.
	- *Developmental aspects of the calculus curriculum.* (from Lesseig 2016)	*Establishing appropriate learning goals in calculus.*(from TEDS-M)	- demonstrating ways to define and explain the lesson's goal and objectives to the students.
		- *Identifying the key ideas in learning calculus.* (from TEDS-M)	- providing and making available definitions, theorems and proofs to students; - providing forms of argumentation appropriate for students' levels; - identifying relationships between mathematical and everyday use of terms; - demonstrating "routes" to explain calculus ideas, examples, or proofs.
Knowledge of content and teaching calculus. (from Lesseig 2016)	- *Instructional strategies.* (from Lesseig 2016)	*Relationship between instruction and students' ideas in calculus.* (from Lesseig 2016)	- teaching calculus ideas using a systematic approach based on a solid grounding in logic and its associated linguistic expressions; - presenting and sequencing problems that can lead students to more easily see the structure of certain calculus concepts; - demonstrating methods of answering questions, responding to students' ideas, using examples; - Demonstrating use of appropriate instructional methods.
		Questioning strategies in calculus. (from Lesseig 2016)	- obtaining justification beyond just procedures; - encouraging thinking about the general case; - actively involving students in the lesson through questioning; - develop critical thinking skills, by asking why; - Checking understanding on completion of work.
		Use of pivotal examples or counter-examples in calculus. (came from Lesseig 2016)	- *Using examples or counter-examples to focus on key ideas in calculus*
		- *Mathematical Representation in calculus.* (from Lesseig 2016 and coactive 2004)	- demonstrating the role of representations as recognised in manipulating mathematical objects, communicating ideas, and assisting in problem solving; - drawing strong connections between the representations students use and their understanding; - using tasks that require a flexible use of different representations; - linking visual, symbolic, and verbal ideas in calculus
	- *Knowledge of calculus connections.* (from Lesseig 2016, Khakbaz 2016)	- *Real- world applications of calculus.* (from Lesseig 2016; Khakbaz 2016)	- emphasising mathematics as a way of interpreting experience or as a human activity; - indicating that people benefit from the applications of calculus every day; - Linking between calculus concepts and application of calculus in everyday use.
		- *Calculus in academic subjects.* (from Lesseig 2016; Khakbaz 2016)	Demonstrating calculus in various academic subjects.

Appendix C Example of data identified from previous research in the literature

	PCK of teaching calculus conceptual framework categories	Examples from existing research		Examples from my data PCK of teaching calculus
		Study	Example	Example
Knowledge of Content and Students for calculus *Learners' cognatic of calculus*	***students' misconceptions and learning difficulties in calculus***	(Muzangwa and Chifamba 2012)	What is the derivative of $y = x^x$ Misconception of x^n all students wrongly answered item 7 on the derivative of x^x. All students chose Option1, a misconception of applying the rule for x^n where n is a real number. Instructors should emphasise the difference between those two terms. -----------------	
		----------	What is the correct expanded form of cos(x+y) ? Misconception of distributive law. -----------	
		(Muzangwa and Chifamba 2012)	Data analysis revealed that not all students' misconceptions in Calculus 1 are related to Calculus. Many of the difficulties experienced by students were algebraic or non-calculus related. Misconceptions are formed as learners construct their own meaning from ideas communicated to them. Students accumulate pieces of information from past experiences, thus building up concept images in their own minds.	

		(Rouhani and Lewkowicz, 2014)	Tucker (1999) articulated a need to bridge the gap between the knowledge and experiences of college students and mathematics faculty. Many students who experienced differentiated instruction at the K-12 level encounter what Kirst (2004) has termed the "high school-college disconnect," to which he attributes the significant percentage of college freshman who fail to complete a bachelor's degree. Further, large class sizes and course pacing, both	
		(Sofronas and DeFranco, 2010)	difficult for many first-year college students, contribute to the disconnect between high school and college contexts for learning mathematics (Chronicle, 2003; Hillel, 2001; Saunders & Bauer, 1998; Wood, 2001; Zevenbergen, 2001). ----------------- For example, in the subdomain of research on the learning and teaching of derivative, early research focused on students' difficulties and under-developed conceptual understandings of derivative (Orton 1983; Ferrini-Mundy and Graham 1991). An even earlier paper by Morgan and Warnock (1978) reported on an investigation of student difficulties as a result of calculating derivatives on a calculator. Similarly, early research on the learning and teaching of limit detailed a range of student difficulties and misconceptions (e.g., Davis and Vinner 1986; Furinghetti and Paola 1988; Tall and Vinner 1981). ------------	
		Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016).	Functions Vandebrouck (2011) addressed the evolution of function thinking from secondary school to university; he strengthened the difficulties for students to transit from a pointwise and global perspective on functions to a more local perspective as required by the formal Calculus world of the university. According to this study, this transition requires a complex conceptualisation of the function concept in terms of process and object duality. This conceptualization required an early start for the development of the variational thinking in students (Warren 2005; Dooley 2009; Warren et al. 2013).	

Appendix D **Ethical Approval for Pilot Study**

Your Ethics Submission (Ethics ID:24058) has been reviewed and approved

ERGO [ergo@soton.ac.uk]

To: Alzubaidi I.A.A.

Inbox

Tuesday, December 06, 201

Submission Number: 24058

Submission Name: Pedagogical Content Knowledge of Calculus Teachers at University Level.

This is email is to let you know your submission was approved by the Ethics Committee.

You can begin your research unless you are still awaiting specific Health and Safety approval (e.g. for a Genetic or Biological Materials Risk Assessment)

Comments

1.We strongly advise you to also obtain local ethics approval, which some countries require. Please check with your local university or national ethics board to see if local ethics approval is required.

Click here to view your submission

Coordinator: Ibrahim Alzubaidi

ERGO : Ethics and Research Governance Online

http://www.ergo.soton.ac.uk

DO NOT REPLY TO THIS EMAIL

Appendix E **Pilot Study**

E.1 The Pilot Survey Instrument

The survey was one of the main components used in this research and is, perhaps, the instrument that required the most care. The survey went through several stages before the final draft was distributed teachers. The information below outlines how this process occurred.

The first step in the process was to ensure that the PhD Supervisor had an input into the design. Once the researcher had read significant information of the design of surveys and questionnaires, initial discussions were had with the Supervisor in order to start the brainstorming process. A first draft of the survey was then designed (in English) and given to the PhD Supervisor to review. Discussions were had on formatting, language and the motivation behind some of the statements. The survey was then revised, translated into Arabic and given to the two pilot participants (Arabic speaking teachers) to complete. It is important to note that these two participants did not participate in the final study.

Each of the Arabic speakers was a colleague of the researcher. They were initially briefed on the research proposal and asked to comment on any statements that they did not understand. The researcher met with each pilot participant individually, and a record was kept of their queries. Once the two participants had completed the survey, the group met as a whole for a small-group type discussion. This was in order to alleviate any conflicting views and it was important for the researcher to get the underlying reasoning from the participants in order to best improve this survey.

As the small group session was completed, the researcher then modified the survey based on the responses from their points and small group session. The researcher then gave the revised survey to the same 2 pilot participants and asked for feedback.

E.1.1 Changes made to the survey instrument as a result of the pilot study

The survey instrument was one of the most influential components of the study as it not only set the benchmark for the calculus teachers’ pedagogical knowledge, but it provided a benchmark from which the rest of the findings could relate to. As it was such an influential anchor, getting it to be just right was a challenging process. Because it was so challenging, a significant amount of effort went into getting it right before the pilot study commenced.

The researcher provided the survey to two participants. These were the same two participants who also completed the pilot interviewing and who were observed during the lessons. These participants were chosen primarily out of convenience. Both were experienced teachers with a strong familiarity with calculus. The survey was provided to each participant under similar conditions to what would occur in the main study. There were four parts and participants needed to respond with an answer ranging between strongly agree (5) and strongly disagree (1).

Changes made to this part of the pilot survey were minimal but important. The participants stressed that the first statement: “I have sufficient knowledge about calculus” might be insulting to the teachers who were actually teaching the course. It was particularly challenging to translate ‘sufficient knowledge’ into Arabic. As a result, the wording was changed to ..."had enough knowledge". Some of the feedback provided by the pilot participants included:

> *This survey is very long and I am not enjoying the completion of it. I understand that it will go to first year calculus teachers but I teach a range of mathematics students, not just first year ones. With the upper year ones, I know that there is a certain level of mathematics knowledge because they have taken other courses with our professors, so I am much more confident answering the survey questions when I think of them. With my first year students, it is much more difficult to assess.* (Pilot Participant 2)

> *This survey is interesting but very long. I am tired!* (Pilot Participant 1)

Based up on this feedback, the researcher further strengthened the position that first year calculus teachers are in a unique position for PCK because of the uncertainty associated with first year students.

In part 2, pilot participants suggested that the statement: “I can anticipate my students’ prior calculus knowledge before the lecture” was not a good statement because the anticipated knowledge would differ between the time a teacher met the students for the first time and the time period later in the course. As such, the pilot participants suggested that this statement would not provide accurate feelings. They also indicated that the two statements at the end of section 2 relating to homework could be amalgamated into one. As a result, these statements were reworded to: “I assign enough homework for students to work through the points while facilitating an understanding of calculus”. Other than these changes and the addition of numbering for each of the statements, the Likert scale part of this survey remained the same. A final version of the survey instrument can be found as Appendix H.1, H.2.

E.2 The Pilot Interview Schedule

The interview was the secondary form of data collection though in the longitudinal timeframe of this research it came first. There were two interviews held - one with each of the participants in the pilot study. Both interviews were held in the Arabic language. In the first section, the primary focus was to ensure the questions flowed logically, as this section was largely comprised of demographic questions. This meant that the researcher needed to be prepared to restate these questions in order of logical flow, and to take into consideration that when translated into English, the flow might not necessarily 'flow' in the same way. The second, third and fourth parts of the interview targeted questions on aims of teaching calculus, instructional strategies, analysing calculus. Again, in these sections the researcher was looking to determine if there was a good flow, but in addition, it was necessary to ensure that the questions provided were particularly relevant to the field of mathematics. This was also true of section five, which asked about questions on learning. As the participants had a background in mathematics, they were able to comment on aspects of this and to make suggestions that would help the researcher to ensure the appropriateness of the questions.

The interview was semi-structured, and questions were designed in order to gain information about the teachers while providing detailed information about aspects of PCK. It was also necessary for the researcher to listen to the questions and concerns of the participants about the design and delivery of the questions. According to McNamara (2009), interviewing is a skill that requires practice. As such, these interviews gave the researcher the opportunity to practice delivery of the questions and to hone in on some of the non-verbal communication aspects of interviewing, both on the part of the researcher and of the participants.

Question preparation occurred several weeks in advance of the interview. Questions were initially written and given to the PhD Supervisor for discussion. These questions were then discussed at a supervision meeting (the final interview questions can be found in the Appendix J.1, J.2). Initially, the researcher had five main sections, each which had multiple questions. The PhD Supervisor had previously indicated that this was perhaps a lot of work for participants.

After the discussion with the PhD Supervisor, the questions were revised in order to ask for more detailed responses, though one of the challenges for this research was that the interviews were conducted fully in Arabic and the PhD Supervisor did not speak Arabic and perhaps did not fully understand the cultural aspects of the interviewing process. Thus, it was particularly important that the two interview participants provided as much feedback as possible in order to ensure that the questions were appropriate, necessary, and linked to the research questions.

The two pilot interviews were then conducted with the two participant teachers at University X. In these pilot interviews, all three sections of the interview were asked, and the participants were asked not only to respond to the questions but to indicate any associated problems, concerns, or confusion. As these interviews were audio recorded, the researcher had records of what issues were identified, these could then be amended before the final study. Further, the researcher was able to answer all of the questions posed by the participants in relation to the study. As these pilot interviews were successful, the final questions are drawn up for the final interviews in the main study with the calculus teacher participants.

E.2.1 Changes made to the interview schedule as a result of the pilot study

Two interviews were conducted, one with each participant. The researcher began by going through the questions that were outlined in the pilot instrument in order to get an idea about the amount of time that would take. In both instances, the timing was about 45 minutes to one hour. Once the interview was completed, the researcher asked each participant to comment on any issues associated with these questions. None of the questions were particularly problematic. They had been translated appropriately into Arabic and the participants indicated that the questions were clear and relatively easy to answer.

While the questions were answerable, there was some concern by both participants that the research questions were not fully addressed as a result of the questions that were asked. This did not seem to be particularly problematic, as many of the research questions were more appropriately addressed within other areas (e.g. the survey). It was possible that the researcher had not translated the questions appropriately into Arabic, or that some of the meaning was getting lost in translation. This specifically had to do with the 'Questions on Teaching' section of the semi-structured interviews. One participant responded:

> *Why are you asking me these questions? I thought that your research had to do with how much I know about calculus and about teaching. What does it matter about how I feel about mathematics? Of course, I like mathematics, but that does not mean that I like every course I teach. It also does not mean that I do a bad job or teach differently in the courses that I don't like as much. It is my job to teach and we cannot always like our job all of the time.* (**Pilot Participant 1)**

Based on this response, the researcher removed the second question from the 'Questions on Teaching' section of the semi-structured interview. This allowed for more discussion to occur within the remaining questions. A question was also added which asked participants to list all courses that they were teaching in that year (both in the term examined and the previous academic term). This was in response to the participants indicating that they were responsible for several mathematics classes - as would likely be true of the main participants in the study.

In response to the section on 'Questions on learning' or Part 3 of the semi-structured interviews, the pilot participants both indicated that the learning preconceptions really depended on the mathematical level of the students (i.e. first year students versus upper year students). Both participants indicated that with upper year students the class sizes are smaller and so there is more time available for the professor to ensure that the students are on track. There are also opportunities for the professors to know the students. One participant indicated:

For our third-year students, there are only twenty students. By this point, I know their names and for many of them I can identify the areas where they are weak. If there is a problem for multiple students, I can address it in the class and make sure that everyone is staying on track. With the first-year students, the classes are much larger, and I cannot possibly know all the students. I can guess about what they know or what they do not know, but if many students struggle with an issue, I cannot spend time on it. All I can suggest that they do is study that area really hard. I have to keep up with the other classes and stay on track, so there is much less time to work with students. While I am interested in having the students learn, I am faced with many more obstacles in these situations. (**Pilot Participant 2)**

In this instance, this issue could be mitigated by asking the participants about what other classes they are teaching. It is acknowledged that class sizes differ and that within the context of this project, the first-year model comes with a syllabus that is typically adopted by all professors teaching the same course - with some flexibility related to the different disciplines (e.g. calculus for chemistry, mathematics, and engineering students). This notion will be considered in relation to the final study and choose the participants who teach calculus for mathematics students only.

Some of the more general feedback from the two participants (obtained after the interview was completed) included statements such as:

You are asking a lot of questions about a typical calculus 1 class, but I am unsure if such a thing really exists…my classes are all very different. It is not only the size of the class but the personalities of the students. When I think of a typical class, context is really important. (Pilot Participant 1)

I enjoy talking about my classroom teaching with you in this way. It gives me time to reflect on my own teaching and my own students…maybe this takes up a lot of my time though…I could talk about some of these points a lot more and I feel somewhat rushed by you. (Pilot Participant 2)

In response to these qualitative comments, the researcher confirmed that, at least for the interviews, harm was avoided (in terms of ethics) and that there was some benefit to asking the participants to undertake this study. As this is an essential component of this research, this response was particularly invigorating. This was paired with a more negative comment by Participant 2 that the researcher was somehow rushing him to finish. This seems somewhat typical of novice interviewers, as both Creswell (2013) and McNamara (2009) suggest that interviewing is a skill that requires considerable practice and is a learned behaviour. Prior to the final interviews, the researcher continues to practice asking questions (with colleagues) and focused on limiting non-verbal communication during the interview proceedings.

After receiving this feedback from the pilot participants, the researcher reviewed the translation of the questions. A few modifications were made to some of the words to ensure that the core of the question was being maintained. The researcher then brought the English and Arabic translations (versions 1 and 2) to a colleague who spoke excellent English and Arabic. A conversation ensued about the best ways to word the translation and the colleague made some further suggestions. These suggestions were then implemented in the final interview schedule.

The researcher also took the time to practice asking questions outside of the formal research questions. The purpose of the semi-structured model was that its design allowed the researcher to pursue participants' responses when the need arose. The researcher needed to be able to do this fluidly in Arabic and this was a considerable challenge, as many of the researcher's colleagues did not speak Arabic and therefore practicing this skill became difficult. To overcome this challenge, the researcher obtained a list of general interview topics from the internet and practiced interviewing friends and family members (in Arabic) with unstructured interviews. Constructive feedback was then requested from these friends and family members. Through this practice, the researcher was able to further develop interviewing skills to a more comprehensive level.

Ultimately, the researcher needed to not only improve the interview instrument but needed to develop personal skills within this model that allowed a certain standard to be maintained. Language also arose as a considerable challenge. In summary, the interview schedule was changed by removing question from the 'Questions on Teaching' section and by adding a question asking for all courses taught. The 'Questions on Learning' section remained unchanged in the English version, though the Arabic translation was modified. The researcher also took on professional development opportunities to improve the interviewing strategies and to ensure that non-verbal communication was considered during the interview process.

E.3 The Pilot Observation Schedule

The observation schedule was perhaps the instrument that required the most work as the initial schedules created had multiple different components within each subsection, and the researcher had concerns that this may create too many categories during the coding process. The researcher began with three different observation schedules, as it was unclear which one would be most appropriate. The researcher aimed to use different observation schedules in the process of designing a final appropriate observation schedule. It was hoped that by collecting the data and then analysing it according to the three different schedules, the outcomes would best represent the research questions posed.

There were also concerns in relation to the coding process. The initial concerns were that coding and sub-coding for this instrument would require too many codes to be produced. As a result, the findings would be difficult to analyse in any sort of useful way.

Once an initial draft of each of the observation schedules was created, the researcher approached the PhD Supervisor for an initial discussion relating to these concerns. On one hand, the researcher wanted to use aspects from previous research to justify the creation of these instruments. On the other hand, the researcher was unsure how this would work within the overarching framework of this multiple methods study. As such, the researcher required both comments on the observation schedules and actual implementation. In first step, the researcher used the PhD supervisor's idea when he suggested using YouTube to develop the observation schedule.

The researcher began by approaching other PhD candidates, also within the field of education at the same institution, who were also using observation schedules. The researcher is part of a study group with some of the other students and so this discussion offered initial insight into the concerns of the researcher over the complexity and how this might apply in practice. This discussion occurred prior to the researcher leaving for the Pilot Study in December. As a result of this discussion, the researcher decided that since the observations were being recorded, the multiple categories and sub-categories were not necessarily problematic, as these codes did not necessarily have to translate into actual codes within the quantitative data framework being used. That said, the researcher did not want to employ too many codes as to overcomplicate the research or stray away from the research questions posed. The researcher chose to maintain several categories as they were and to see how difficult the coding process was following the pilot.

The researcher approached the two participants by email and explained about the pilot study and the observation schedule. Both participants indicated that there would be an exam break in January and so if the researcher wanted to conduct these observations that it would need to be done early in the visit. As such, the researcher conducted the observations for the pilot study prior to the interview. In the final study, the researcher had planned to conduct the survey and interviews prior to the observations.

The researcher videotaped six lectures during the pilot study phase, each lasting approximately two hours. For Pilot Participant 1, the researcher observed them on December 19th, 20th and 27th. The first two observations occurred at different times in the afternoon while the final observation occurred in the morning. For Pilot Participant 2, the researcher also conducted three observations of lectures (approximately 2 hours each). These were held on December 21st, 26th and 28th, and again the first 2 observations took part in the afternoon and the final observation took place in the morning. The researcher did not have control over the timings, as these were pre-scheduled classes being held within the academic term.

For the observations, the researcher knew that the audio feeds would need to focus on the podium, but that also the researcher would need to pick up audio feeds from the points near the blackboards. During classes, the professor often wears a microphone so that he can communicate with the students either at the lectern or when doing work at the blackboard. As such, the researcher needed to position the camera so that it took into account both the blackboards and the lectern. In practice, this proved to be much more difficult than in theory. When the researcher set up the camera, there were some issues. For example, in order for the researcher to get the entire blackboard section into the camera picture, the text that was being written was much too small and the researcher could not make out the formulas being examined. This formulaic information would be essential in the research process, so the researcher needed to choose a clear indication of the most appropriate place to locate the recording equipment and researcher within the classroom, so the researcher needed to implement a two-camera system (see Figure Appendix E 2).

In the set up for the observation, the rooms that each participant was in were different. While they still operated under the general premise that the instructor stood at the front and worked on the board, one classroom was considerably larger than the other was. This was not something that the researcher had taken into consideration. However, with the implementation of another camera, the researcher could position one camera of the left of the room that would encompass most of the blackboard, and a camera on the right of the room that would capture the lectern and the remaining bit of blackboard not picked up by the other camera. In this way, the researcher could not only read the formulas and information written on the boards but could pick up better audio feeds at both ends of the room to ensure for better transcription. The researcher chose a location that allowed him to see the full classroom, and the researcher used the observation schedule to write notes.

The researcher applied this double camera strategy in both of the observations undertaken. First, the two participants were told about the camera strategy and that two cameras would be positioned in their classroom. They were told that the purpose of the observation was largely to test whether the instrument (designed by the researcher) was appropriate. Each class lasted for approximately 2 hours and the researcher video recorded each of these sessions. Once completed, the researcher was able to review the footage with footnotes of direct observation and determine the appropriateness of the observation schedule.

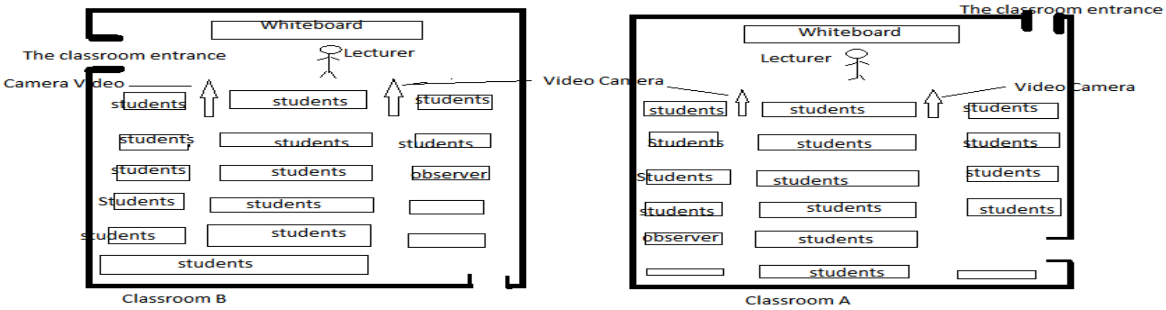

Figure Appendix E 8-2: The location under calculus teacher observation of the pilot of the study

E.3.1 Changes made to the observation schedule as a result of the pilot study

Changes made to the pilot study observation schedule required two elements. Initially, the researcher needed to determine which observation schedule was preferable and also whether or not the data could be appropriately coded.

First, in the classroom, the first observation, the researcher used version 2 and the second observation the researcher used version 3 and the third observation used version 1. Second, the researcher needed to consider how the video recordings would be used. There were two cameras, and so employing a strategy that effectively documented the lecture was an initial hurdle. The researcher used timeline (see table below) and began by quickly reviewing the footage to determine where the lecturer stood during the majority of the teaching time with his footnotes of direct observation. This footage then became the primary focus for the observation. As both videos were time-stamped, the researcher could pull up footage from the secondary camera any time that the lecturer moved out of range or if alternative accommodations needed to be made (e.g. if a student asked a question that could not be heard on the primary camera). The researcher chose to use a timeline approach (see table 1) for all six of the observations. This choice was made fairly easily once the footage was obtained because the footage was so detailed and could be clearly aligned with several of the observation schedules being utilized.

Once the researcher had made this initial decision, the first two observations were coded using all three instruments. Based upon the choice of a timeline model, the third observation schedule was discounted as being inappropriate because it focused on General Pedagogical Knowledge and it looks like to require a ‘tick box’ framework that did not align well with the research questions or presentation of the data, except the fifth point in the second section (explanation of the subject) which is (provide alternative explanations of difficult point). This left two models to be assessed. These two schedules were then applied to the second observation sessions for each lecturer. By this time, the researcher was becoming more familiar with both of these research instruments and with the organization of the material within the schedule. Based on this familiarity, the second observation schedule was discounted because the information that needed to be recorded in each of the boxes was overly qualitative. It also focused on

GPK. In primary attempts to code this data, there were too many discrepancies and it did not produce outcomes that were feasible in answering the research questions or aligning with the other research instruments. therefore, the researcher picked question two from this observation schedule which is (Explain how the calculus teacher makes lecture objectives clear to students). As a result, the second observation schedule was discounted, leaving the researcher with the first observation schedule as the preferred option with some changes. The researcher produced a new observation schedule (Appendix J, J.1), informed a little by version 2 and 3 and much by version 1.

The researcher then organised the final observations from the third video recording for each lecturer and used the selected observation schedule to record the results. These observations could then be used to pilot the implementation of the schedule. Moreover, from the pilot study the researcher decides to use a camera system and knows how choose the suitable position in classroom.

At this stage, the researcher was simply trying to learn how the main data analysis might be approached given the theoretical framework and research questions. Some of the observations were not linked to the actual lesson (e.g. the planning stages) - this material was either obtained from the professor prior to class or was listed on the course syllabus (e.g. learning outcomes). Therefore, while this information was important for organizational purposes, it was not considered a part of the final observation schedule. For the pedagogical practices section, the researcher was able to apply a code to some of the 10 sub-categories and input time stops at each instance where one of these occurred. Again, in this instance, some of these required pre-class information (such as the seating arrangements). These were removed from the final observation schedule (though noted by the researcher).

Title: function	Day/data teacher …. Room: ….. case:
0-10 Aims and objectives of the course, significance of the calculus, diagnostic test. 10-20 Warm up activity on definition of function. 20-30 Q+A on function; uncovering common mistake codomain= range with some examples 30-40 Graph sketching. 40-50 Representation of one to one function definition algebraically and visually with some examples+ using a reverse example (question strategies). 50-60 Some commonly-used function+ verbally, Graph sketching, and algebraically representation 60-70 Graph sketching and verbal representation of Piecewise-defined function 70-80 The geometric significance of even and odd function. 80-90 Counter-examples in odd and even function. 90-100 Identifying the key ideas in Increasing and decreasing functions 100-110 Verbal and algebraic representation 110-120 Assessing students' understanding - some exercises (classroom activity) + homework.	

Table-Appendix E 8-1: Timeline (A typical lecture) (Functions and models).

Appendix F Ethical Approval for Main Study

Your Ethics Submission (Ethics ID:25305) has been reviewed and approved

Ad

ERGO [ergo@soton.ac.uk]

To: Alzubaidi I.A.A.

Inbox

Thursday, February 16, 2017 4:34

Submission Number: 25305

Submission Name: Pedagogical Content Knowledge of Calculus Teachers at University Level (Main Study)

This is email is to let you know your submission was approved by the Ethics Committee.

You can begin your research unless you are still awaiting specific Health and Safety approval (e.g. for a Genetic or Biological Materials Risk Assessment)

Comments

1.There is a minor issue with regard to some uncertainty about how consent will be obtained on the ethics form (point13). However it is clear within the rest of the submission how this will be achieved and it is therefore unnecessary to resubmit for approval but please alter the ethics form to correspond to the rest of your documentation prior to commencing your data collection. Good luck with your research.

Click here to view your submission

Coordinator: Ibrahim Alzubaidi

ERGO : Ethics and Research Governance Online

http://www.ergo.soton.ac.uk

DO NOT REPLY TO THIS EMAIL

Appendix G **Letter for Data Collection**

UNIVERSITY OF Southampton

PRIVATE and CONFIDENTIAL
KSA Cultural Bureau in London
630 Chiswick High Road
London W4 5RY
United Kingdom

23 February 2017

Dear KSA Cultural Bureau in London

IBRAHIM ABDAH ALI ALZUBAIDI
Southampton University Student ID: 26456028
Southampton University email address: iaaa1g15@soton.ac.uk
Saudi Bureau reference number UMU527
Date of birth 03-06-1981
UK Phone number 07448794697 KSA phone number 00966503700022
UK address 49 Ripstone Gardens Southampton.
Sauid address ALQUNFUDA – Makkah Al-Mukarmh 210912 P.O 483

I am writing to confirm that Ibrahim Alzubaidi, a registered full-time PhD student at the Education School of the University of Southampton, UK, has gained University of Southampton Ethics approval for his main data collection for his PhD study. As his project entails collecting data in KSA, his plan is to travel to Saudi Arabia to collect data from 15 April 2017 to 01 August 2017. He has already obtained an authorization letter to conduct his research project at the Mathematics Department, University College Alqunfuda, Umm Al-Qura University.

As his PhD supervisor, I hope these arrangements meet with your approval. I take this opportunity to thank you for all the support you provide for him for his PhD research.

Yours sincerely

David Keith Jones
Associate Professor
Direct tel: +44 (0)23 8059 2449
email: d.k.jones@southampton.ac.uk

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Appendix H **Approval from Mathematics Department in University X**

المملكة العربية السعودية
وزارة التعليم العالي
جامعة أم القرى

To Whom it May Concern

This letter confirms that **Mr.Ibrahim Abdah Alzubaidi** is authorized to conduct the research project entitled " **Pedagogical Content Knowledge of Calculus Teachers at University Level**". The study will conducted at our department during the second semester, and the summer semester of current academic year 2016/2017.

Please do feel free to contact me by email, should you need any further information.

Respectfully submitted,

Dr. Alhossain A. Alnashri,
Head of the Mathematics Department,
Alqunfudah University college,
Umm Alqura University, KSA.
E-mail: (hossa-1427@hotmail.com)

Appendix I Data Collection Instrument - Survey (in English)

Final Survey Instrument

Part one: Demographic Information

1. What university did you attend?

2. Which country was it located in?

3. What age range do you fall into?

☐ 35 and under, ☐ 36-50, ☐ above 50.

4. What qualifications do your currently possess?

5. Please indicate degrees, diplomas, and certificates?؟

6. What was your major at university?

7. Did you also complete a minor? If yes, what field was it in?

8. How long have taught calculus at university level?

9. What did you do prior to working at this university?

10. Have you ever studied abroad? If yes, please provide location/dates .

11. Have you ever attended an academic conference about teaching and learning of calculus? If yes, please provide location(s)/dates.

Part 2: Knowledge of aims for teaching calculus

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I always know how to organise the main aims of each calculus lesson that I teach					
I never consider individual differences among students when planning my calculus lessons					
I always select lesson objectives for each calculus lesson by considering suitable methods for teaching					
I am aware of using a wide range of knowledge in planning my calculus lessons					
At the start of each calculus lesson I never define the aims of the lesson to students					
I often change my plan for a calculus lesson while I am teaching the lesson					

Part 3: Knowledge of instructional strategies for calculus

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I have experienced and investigated different ways of teaching calculus					
I do not know how to choose the teaching strategies to achieve the aims of the calculus topics that I teach					
I always use a very mathematical way of teaching calculus					
I only use examples and diagrams after having introduced the formal calculus theory					

I avoid using a wide range of teaching approaches in a classroom setting					
I often use examples and diagrams as a tool for introducing formal calculus theory					
I always use a variety of ways and strategies to develop students' understanding of calculus					

Part 4: Knowledge of student understanding within calculus

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I always seek feedback from students on their understanding of calculus and adapt my teaching accordingly					
I never adjust my progress through the calculus syllabus to take account of common student misunderstandings and misconceptions					
I always select teaching approaches that build on student thinking and learning in calculus					
I anticipate my students' prior calculus knowledge before the lesson					
I rarely show my students different approaches to look at the same calculus questions.					
I always ask questions to evaluate my students' understanding of the calculus topic that I am teaching					
I only assign homework and/or assignments in order to better facilitate student understanding of calculus					
I always avoid assigning too much calculus homework for my students so they can work through the points without being overwhelmed with work					

Part 5: Knowledge of curriculum for calculus

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I analyse each calculus topic by building blocks of mathematical theories using axioms, definitions, theorem, and proof.					
I never explain the proof of formal theory in calculus					
The university requirements play no role in my design of the calculus course					
I do not know where to direct the students if they need assistance with a particular mathematical concept					
I am aware of how the calculus material I teach fits within the bigger university context.					
I know how calculus is taught at other (similar) university institutions in Saudi Arabia.					
I am not interested in how calculus is taught at other (similar) university institutions in other parts of the world.					

Part 6: Knowledge of assessment for mathematics

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I am confident that student performance in calculus can be reliably assessed in the classroom.					
I unsure whether summative testing gives a reliable picture of student capability in calculus.					
I avoid assessing student learning of calculus in multiple ways.					
I always provide constructive formative feedback to calculus students.					

I.1 Data Collection Instrument - Survey (in Arabic)

Final Survey Instrument **الاستبيان**

Part one: Demographic Information الجزء الأول: المعلومات الديموغرافية:

1. What university did you attend? 1. ماهي الجامعة التي درست بها البكالوريوس والماجستير والدكتوراه؟

2. Which country was it located in? 2. بأي دولة تقع هذه الجامعات؟

3. What age range do you fall into? 5. في أي فئة عمر من الاتي يكون عمرك؟

☐ 35 and under, ☐ 36-50, ☐ above 50.

4. What qualifications do your currently possess? 4. ماهي المؤهلات التي تمتلكها حاليا؟

5. Please indicate degrees, diplomas, and certificates? 5. فضلا يرجو التنويه لكل الشهادات والدرجات التي لديك؟

6. What was your major at university? 6. ماذا كان تخصصك في الجامعة؟

7. Did you also complete a minor? If yes, what field was it in?
7. هل أكملت التخصص الدقيق؟ إذا اجابتك نعم ، بأي مجال من مجالات الرياضيات كان؟

8. How long have taught calculus at university level? 8. ما هي المدة التي درست بها حساب التفاضل والتكامل على مستوى الجامعة؟

9. What did you do prior to working at this university? 9. ما هو عملك السابق قبل أن تعمل بالجامعة؟

10. Have you ever studied abroad? If yes, please provide location/dates . 10. هل سبق لك أن درست في الخارج؟ إذا كانت الإجابة بنعم، يرجى تقديم الموقع / التواريخ

11. Have you ever attended an academic conference about teaching and learning of calculus? If yes, please provide location(s)/dates. 11. هل سبق لك حضور مؤتمر أكاديمي حول تعليم وتعلم حساب التفاضل والتكامل؟ إذا كانت الإجابة بنعم، يرجى تقديم الموقع (المواقع) / التواريخ

Part 2: Knowledge of aims for teaching calculus القسم الثاني: معرفة أهداف تدريس حساب التفاضل والتكامل

Statement	**Strongly Disagree** لا أوافق بشدة	**Disagree** لا أوافق	**Neutral** معتدل أو محايد	**Agree** أوافق	**Strongly Agree** أوافق بشدة
I always know how to organise the main aims of each calculus lesson that I teach دائما أعرف كيف أنظم الأهداف الرئيسية لكل درس من حساب التفاضل والتكامل التي أدرسها.					
I never consider individual differences among students when planning my calculus lessons لا أراعي الفروق الفردية إطلاقا بين الطلاب عند التخطيط لدرس حساب التفاضل والتكامل					
I always select lesson objectives for each calculus lesson by considering suitable methods for teaching دائما أحدد أهداف الدرس لحساب التفاضل والتكامل واختيار الأساليب المناسبة لتحقيقها					
I am aware of using a wide range of knowledge in planning my calculus lessons وأنا مدرك لاستخدام مجالات واسعة من المعارف في تخطيط دروسي في حساب التفاضل والتكامل					
At the start of each calculus lesson I never define the aims of the lesson to students في بداية كل درس حساب التفاضل والتكامل أبدا لا أحدد أهداف الدرس للطلاب					
I often change my plan for a calculus lesson while I am teaching the lesson غالبا ما أغير خطتي لدرس حساب التفاضل والتكامل خلال تدريسي الدرس					

Part 3: Knowledge of instructional strategies for calculus الجزء الثالث: معرفة الاستراتيجيات التعليمية لحساب التفاضل والتكامل

Statement	Strongly Disagree لا أوافق بشدة	Disagree لا أوافق	Neutral محايد	Agree أوافق	Strongly Agree أوافق بشدة
I have experienced and investigated different ways of teaching calculus لقد جربت وتحققت من طرق مختلفة لتدريس حساب التفاضل والتكامل					
I do not know how to choose the teaching strategies to achieve the aims of the calculus topics that I teach أنا لا أعرف كيف اختار استراتيجيات التدريس لتحقيق أهداف موضوعات حساب التفاضل والتكامل التي أدرس					
I always use a very mathematical way of teaching calculus أنا دائما استخدام طرق رياضية بحتة لتدريس حساب التفاضل والتكامل					
I only use examples and diagrams after having introduced the formal calculus theory استخدام الأمثلة والرسوم البيانية بعد أن أقدم نظرية حساب التفاضل والتكامل الرسمية					
I avoid using a wide range of teaching approaches in a classroom setting أنا تجنب استخدام مجموعة واسعة من أساليب التدريس داخل الصف					
I often use examples and diagrams as a tool for introducing formal calculus theory غالبا ما أستخدم الأمثلة والرسوم البيانية كأداة لتقديم نظرية رسمية في حساب التفاضل والتكامل					
I always use a variety of ways and strategies to develop students' understanding of calculus دائما استخدام مجموعة متنوعة من الطرق والاستراتيجيات لتطوير فهم الطلاب لحساب التفاضل والتكامل					

Appendix I

Part 4: Knowledge of student understanding within calculus الجزء الرابع: معرفة فهم الطالب في حساب التفاضل والتكامل

Statement	Strongly Disagree لا أوافق بشدة	Disagree لا أوافق	Neutral محايد	Agree أوافق	Strongly Agree أوافق بشدة
I always seek feedback from students on their understanding of calculus and adapt my teaching accordingly دائما أطلب تغذية راجعة من الطلاب عن فهمهم لحساب التفاضل والتكامل وأكيف تدريسي وفقا لذلك					
I never adjust my progress through the calculus syllabus to take account of common student misunderstandings and misconceptions أبدا لا أعدل تقدمي المحقق خلال منهج حساب التفاضل والتكامل لمراعاة سوء فهم الطلاب العامة والمفاهيم الخاطئة					
I always select teaching approaches that build on student thinking and learning in calculus دائما أحدد أساليب التدريس التي تقوم على بناء تفكير الطالب وتعلمه في حساب التفاضل والتكامل					
I anticipate my students' prior calculus knowledge before the lesson أتوقع معرفة طلابي السابقة عن حساب التفاضل والتكامل قبل الدرس					
I rarely show my students different approaches to look at the same calculus questions. نادرا ما أقدم لطلابي طرق مختلفة للإجابة عن نفس أسئلة التفاضل والتكامل					
I always ask questions to evaluate my students' understanding of the calculus topic that I am teaching دائما أسأل أسئلة لتقييم فهم طلابي لموضوع حساب التفاضل والتكامل الذي أدرسه					
I only assign homework and/or assignments in order to better facilitate student understanding of calculus أنا فقط أخصص الواجبات المنزلية و / أو المهام من أجل تسهيل فهم أفضل لطلاب حساب التفاضل والتكامل					
I always avoid assigning too much calculus homework for my students so they can work through the points without being overwhelmed with work دائما أتجنب تعيين الكثير من الواجبات الحسابية المنزلية لطلابي حتى يتمكنوا من العمل من خلال الأفكار دون أن تطغى على العمل					

Part 5: Knowledge of curriculum for calculus الجزء الخامس: معرفة منهج حساب التفاضل والتكامل

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I analyse each calculus topic by building blocks of mathematical theories using axioms, definitions, theorem, proof. أحلل كل موضوع حساب تفاضل وتكامل من خلال لبنات البناء الرياضي باستخدام البديهيات، والتعاريف، النظريات، البرهان					
I never explain the proof of formal theory in calculus لا أشرح أبدا إثبات النظرية الرسمية في حساب التفاضل والتكامل					
The university requirements play no role in my design of the calculus course متطلبات الجامعة لا تلعب أي دور في تصميمي لدروس حساب التفاضل والتكامل					
I do not know where to direct the students if they need assistance with a particular mathematical concept لا أعرف أين أوجه الطلاب إذا كانوا بحاجة إلى مساعدة مع مفهوم رياضي معين					
I am aware of how the calculus material I teach fits within the bigger university context. أنا مطلع كيف أن مادة التفاضل والتكامل التي أدرسها تتناسب مع السياق الجامعي الأكبر					
I know how calculus is taught at other (similar) university institutions in Saudi Arabia أعرف كيف يتم تدريس حساب التفاضل والتكامل في مؤسسات جامعية أخرى (مماثلة) في المملكة العربية السعودية					
I am not interested in how calculus is taught at other (similar) university institutions in other parts of the world. أنا غير مهتم في كيفية تدريس التفاضل والتكامل في مؤسسات أخرى (مماثلة) في الجامعات في أنحاء أخرى من العالم.					

Part 6: Knowledge of assessment for mathematics الجزء السادس: المعرفة بتقييم الرياضيات

Statement	Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
I am confident that student performance in calculus can be reliably assessed in the classroom أنا واثق من أن أداء الطلاب في حساب التفاضل والتكامل يمكن تقييمها بشكل موثوق في الفصول الدراسية					
I unsure whether summative testing gives a reliable picture of student capability in calculus أنا غير متأكد ما إذا كان الاختبار التجميعي يعطي صورة موثوقة لمستوى الطالب في حساب التفاضل والتكامل					
I avoid assessing student learning of calculus in multiple ways أتجنب تقييم تعلم الطلاب لحساب التفاضل والتكامل بطرق متعددة					
I always provide constructive formative feedback to calculus students دائما أقدم تغذية راجعة تكوينية بناءة للطلاب حساب التفاضل والتكامل					

أشكر لك تعاونك وشكرا لك لمجهودك وعلى الوقت اللي قضيته في إكمال هذا الاستبيان،،،

I.2 Participant Information Sheet for Survey

UNIVERSITY OF Southampton

Participant Information Sheet For the survey of the main study

Study Title: **Pedagogical Content Knowledge of Calculus Teachers at University Level.**

Researcher: Ibrahim Abdah A Alzubaidi **Ethics number: 25305**

Please read this information carefully before deciding to take part in this research. If you are happy to participate you will be asked to sign a consent form.

What is the research about?
I am a PhD student. This study is being carried out as part of a PhD in Education at the University of Southampton. This study will seek obtaining a deep understanding of calculus teachers' pedagogical content knowledge at Al-Qunfudhah University College at Umm Al-Qura University in Saudi Arabia. My questions are'' What is the calculus teacher's form of Pedagogical Content Knowledge (PCK) at university level? And What influences calculus teachers in implementing their PCK in a practical setting''. This project is sponsored by University of Southampton. My PhD is funded by Saudi Ministry of Education.

Why have I been chosen?
The participants in this project are calculus teacher at Al-Qunfudhah University College. Your voluntary participation is highly valuable to colleting the data needed for this project.

What will happen to me if I take part?
If you agree to participate, your co-operation will be highly appreciated in taking part in the study. You will be asked to complete a paper questionnaire. In the cover page of the survey, they will read information about it, and then complete it in less than 50 minutes.

Are there any benefits in my taking part?
There may not be direct benefit to you but your participation would help me to examine the main issues of this study and this might help improve teaching Calculus in Saudi Arabia. We will have a better understanding of using pedagogical content knowledge at university level. You are likely to find this study interesting and make you reflect and think in improvement Teaching Calculus in Saudi Arabia.

Are there any risks involved?
There will be no risks involved by taking part in this exercise.

Will my participation be confidential?
Yes, of course. Any information obtained in connection with this study will be kept strictly confidential and will be stored safely and later destroyed. Data coded and kept on a password protected computer and will be used just for this study. Information will be kept safe in line with UK laws (the Data Protection Act) and University of Southampton policy.

What happens if I change my mind?
Participation in this study is completely voluntary. For that reason, you have the right to withdraw up until four weeks after the end of data collection, and this will not have any effect on any of your rights.

What happens if something goes wrong?
For any concerns or complaints about this study, You can Contact Head of Research Governance (02380 595058, rgoinfo@soton.ac.uk).

Where can I get more information?
If you would like further information about the project, please contact me on either my mobile or email address:
iaaa1g15@soton.ac.uk or iaar6000@hotmail.com.
Phone: 00447448794697 or 00966503552719.
or can contact my supervisor: David Keith Jones
Email: D.K.Jones@soton.ac.uk

[Date February 2nd 2017] [Version number 1]

I.3 Consent Form to Participate in the Research: Survey

UNIVERSITY OF Southampton

CONSENT FORM For the survey of the main study

Study title: Pedagogical Content Knowledge of Calculus Teachers at University Level

Researcher name: Ibrahim Abdah A Alzubaidi

Ethics reference: 25305

Please initial the box(es) if you agree with the statement(s):

I have read and understood the information sheet (Version number 1, Date February 2nd 2017) and have had the opportunity to ask questions about the study. ☐

I agree to take part in this research project and agree for my data to be recorded and used for the purpose of this study ☐

I understand that my responses will be anonymised in reports of the research ☐

I understand my participation is voluntary and I may withdraw at any time until four weeks after the survey without my legal rights being affected ☐

Data Protection

I understand that information collected about me during my participation in this study will be stored on a password protected computer and that this information will only be used for the purpose of this study.

Name of participant (print name)..

Signature of participant..

Date..

[Date February 2nd 2017] [Version number 1]

I.4 Participant Information Sheet for Survey (in Arabic)

ورقة معلومات المشاركة في الدراسة

عنوان الدراسة: معرفة طرق تدريس محتوى حساب التفاضل والتكامل لدى معلمين التفاضل والتكامل على مستوى الجامعة.

فضلا اقرأ هذه المعلومات قبل أن تقرر المشاركة في هذا البحث، إذا قررت المشاركة سيطلب منك التوقيع على نموذج الموافقة المرفق.

ما هي فكرة البحث ولماذا؟

أنا طالب دكتوراه. يتم تنفيذ هذه الدراسة كجزء من متطلبات الحصول على شهادة الدكتوراه في التربية من جامعة ساوثمبتون. تتمحور فكرة هذه الدراسة حول الحصول على فهم عميق لمعرفة معرفة معلمي حساب التفاضل والتكامل بطرق تدريس حساب التفاضل والتكامل وكذلك المحتوى لدى أعضاء هيئة التدريس بالكلية الجامعية بالقنفذة فرع جامعة أم القرى بالمملكة العربية السعودية. أسئلتي هي " ما هو نموذج معرفة معلم حساب التفاضل والتكامل بطرق تدريسه ومحتواه (PCK) على المستوى الجامعي؟ وكيف مدرسين حساب التفاضل والتكامل يستخدمون هذه المعرفة في الممارسة العملية؟ ". ويُرعى هذا المشروع من قبل جامعة ساوثمبتون. ويتم تمويل درجة الدكتوراه من قبل وزارة التربية والتعليم السعودية.

لماذا تم اختياري؟

لان المشاركون في هذا المشروع هم معلمون حساب التفاضل والتكامل في الكلية الجامعية بالقنفذة وتطوعك بالمشاركة هو قيم للغاية لجمع البيانات اللازمة لهذا المشروع.

ماذا سيحدث لي إذا كنت جزءا في هذا البحث؟

إذا وافقت على المشاركة ، فسيكون تعاونك موضع تقدير كبير في المشاركة في الدراسة. سيُطلب منك إكمال استبيان ورقي. في صفحة الغلاف الخاصة سوف تجد كل المعلومات، سيقومون بقراءة المعلومات المتعلقة به ، ثم إكماله في أقل من 10 دقيقة

هل هناك اي فوائد لي عندما اكون جزءا في هذا البحث؟

قد لا تكون هناك فائدة مباشرة لك ولكن مشاركتكم تساعدني على دراسة القضايا الرئيسية لهذه الدراسة، وهذا قد يساعد في تحسين تدريس حساب التفاضل والتكامل في المملكة العربية السعودية. سيكون لدينا فهم أفضل لاستخدام المعرفة الخاصة بطرق التدريس للمحتوى على المستوى الجامعي. أنت من المحتمل أن تجد هذه الدراسة مثيرة للاهتمام.

هل هناك اي مخاطر تنطوي عليها مشاركتي؟

ليس هناك اي مخاطر إطلاقا.

هل ستكون مشاركتي سرية؟

نعم بالطبع. جميع البيانات سوف تكون محمية في كمبيوتر مع كلمة سر والمستندات سوف تكون في مكان آمان لا يصل له إلا الباحث فقط.

ماذا يحدث لي إذا قمت بتغيير رايي؟

لك الحق في الانسحاب من مشروع الدراسة في أي وقت لمدة أربعة أسابيع بعد جمع البيانات وببساطة قم بتبليغ الباحث او أرسل ايميل واطلب الانسحاب.

من اين يمكنني الحصول على مزيد من المعلومات ؟

إذا كنت ترغب في مزيد من المعلومات حول هذا المشروع، يرجى الاتصال بي على أي من هاتفي المحمول أو البريد الإلكتروني:

iaaa1g15@soton.ac.uk أو iaar6000@hotmail.com.

هاتف: 00447448794697 أو 00966503552719.

I.5 Consent Form to Participate in the Research: Survey (in Arabic)

نموذج إقرار بالمشاركة في
تعبئة الاستبيان

عنوان الدراسة: معرفة طرق تدريس محتوى حساب التفاضل والتكامل لدى معلمين التفاضل والتكامل على مستوى الجامعة.
اسم الباحث: ابراهيم عبده الزبيدي

بعد قراءة ورقة معلومات المشاركة في البحث أرجوا التكرم بتعبئة هذا النموذج والتوقيع علية في حالة الموافقة عل البنودً المرفقة في شاكرين لكم تعاونكم

لقد قرأت وفهمت ورقة معلومات المشاركة في البحث وحصلت عل الفرصة الكافية لطرح أسئلتي واستفساراتي عن الدراسة ☐

أوافق على المشاركة في مشروع البحث هذا عن طريق المشاركة في تعبئة الاستبان ☐

أتفهم أن تكون بياناتي في هذا البحث مجهولة المصدر ☐

أتفهم أن مشاركتي في هذا البحث هي تطوعية ولي الحق في الانسحاب لمدة أربعة أسابيع من تاريخ جمع البيانات ولن يترتب على ذلك أي حقوق أو متطلبات قانونية ☐

اسم المشارك:

توقيع المشارك:

التاريخ:

Appendix J Observation Schedule

Outline of lecture:

Title:	Data Room: case
0-10	
10-20	
20-30	
30-40	
40-50	
50-60	
60-70	
70-80	
80-90	
90-100	
100-110	
110-120	

Timeline	Case: data: room : Observation of teaching	Interview + questionnaire	Element of teaching calculus PCK
	Lecture:		
Episode1			
Episode 2			
Episode 3			
Episode 4			
Episode 5			

J.1 Participant Information Sheet for Observation

UNIVERSITY OF Southampton

Participant Information Sheet for the observation of the main study

Study Title: **Pedagogical Content Knowledge of Calculus Teachers at University Level.**

Researcher: Ibrahim Abdah A Alzubaidi **Ethics number**: **25305**

Please read this information carefully before deciding to take part in this research. If you are happy to participate you will be asked to sign a consent form.

What is the research about?

I am a PhD student. This study is being carried out as part of a PhD in Education at the University of Southampton. This study will seek obtaining a deep understanding of calculus teachers' pedagogical content knowledge at Al-Qunfudhah University College at Umm Al-Qura University in Saudi Arabia. My questions are" What is the calculus teacher's form of Pedagogical Content Knowledge (PCK) at university level? And What influences calculus teachers in implementing their PCK in a practical setting". This project is sponsored by University of Southampton. My PhD is funded by Saudi Ministry of Education.

Why have I been chosen?

As a calculus teacher at university your participation in the study will help reveal what pedagogical content knowledge do you have and how do you apply this during teaching calculus at university level.

What will happen to me if I take part?

We will arrange to meet at a time that is convenient to you within the university premises. Once you have given consent to take part, we will have a detailed discussion. In this part of the procedure involves observing calculus teachers in classroom in order to obtain as much information as possible on what an observation process involves as this will be the researcher's first attempt in observing calculus teacher's. Each observation will last for 120 minutes for eight lectures. That will be video recorded. The researcher will use an observation sheet to write down notes about the teaching in progress

Are there any benefits in my taking part?

There may not be direct benefit to you but your participation would help me to examine the main issues of this study and this might help improve teaching Calculus in Saudi Arabia. We will have a better understanding of using pedagogical content knowledge at university level. You are likely to find this study interesting and make you reflect and think in improvement Teaching Calculus in Saudi Arabia.

Are there any risks involved?

There will be no risks involved by taking part in this exercise.

Will my participation be confidential?

Yes, of course. Any information obtained in connection with this study will be kept strictly confidential and will be stored safely and later destroyed. Data coded and kept on a password protected computer and will be used just for this study. Information will be kept safe in line with UK laws (the Data Protection Act) and University of Southampton policy.

What happens if I change my mind?

Participation in this study is completely voluntary. For that reason, you have the right to withdraw up until four weeks after the end of data collection, and this will not have any effect on any of your rights.

What happens if something goes wrong?

For any concerns or complaints about this study, You can Contact Head of Research Governance (02380 595058, rgoinfo@soton.ac.uk).

Where can I get more information?

If you would like further information about the project, please contact me on either my mobile or email address:
iaaa1g15@soton.ac.uk or iaar6000@hotmail.com.
Phone: 00447448794697 or 00966503552719.
or can contact my supervisor: David Keith Jones
Email: D.K.Jones@soton.ac.uk

[Date February 2nd 2017] [Version number 1]

J.2 Consent Form to Participate in the Research – Observation

UNIVERSITY OF Southampton

CONSENT FORM For the observations of the main study

Study title: **Pedagogical Content Knowledge of Calculus Teachers at University Level**

Researcher name: **Ibrahim Abdah A Alzubaidi**

Ethics reference: **25305**

Please initial the box(es) if you agree with the statement(s):

I have read and understood the information sheet (Version number 1, Date February 2nd 2017) and have had the opportunity to ask questions about the study.	☐
I agree to take part in this research project and agree for my data to be video recorded and used for the purpose of this study	☐
I understand that my responses will be anonymised in reports of the research	☐
I understand my participation is voluntary and I may withdraw at any time until four weeks after the observations without my legal rights being affected	☐

Data Protection

I understand that information collected about me during my participation in this study will be stored on a password protected computer and that this information will only be used for the purpose of this study.

Name of participant (print name)..

Signature of participant...

Date...

Version number 1, Date February 2nd 2017

J.3 Participant Information Sheet for Observation (in Arabic)

ورقة معلومات المشاركة في الدراسة

عنوان الدراسة: معرفة طرق تدريس محتوى حساب التفاضل والتكامل لدى معلمين التفاضل والتكامل على مستوى الجامعة.

فضلا اقرأ هذه المعلومات قبل أن تقرر المشاركة في هذا البحث، إذا قررت المشاركة سيطلب منك التوقيع على نموذج الموافقة المرفق.

ما هي فكرة البحث ولماذا؟

أنا طالب دكتوراه. يتم تنفيذ هذه الدراسة كجزء من متطلبات الحصول على شهادة الدكتوراه في التربية من جامعة ساوثمبتون. تتمحور فكرة هذه الدراسة حول الحصول على فهم عميق لمعرفة معرفة معلمي حساب التفاضل والتكامل بطرق تدريس حساب التفاضل والتكامل وكذلك المحتوى لدى أعضاء هيئة التدريس بالكلية الجامعية بالقنفذة فرع جامعة أم القرى بالمملكة العربية السعودية. أسئلتي هي " ما هو نموذج معرفة معلم حساب التفاضل والتكامل بطرق تدريسه ومحتواه (PCK) على المستوى الجامعي؟ وكيف مدرسين حساب التفاضل والتكامل يستخدمون هذه المعرفة في الممارسة العملية؟ ". ويُرعى هذا المشروع من قبل جامعة ساوثمبتون. ويتم تمويل درجة الدكتوراه من قبل وزارة التربية والتعليم السعودية.

لماذا تم اختياري؟

لان المشاركون في هذا المشروع هم معلمون حساب التفاضل والتكامل في الكلية الجامعية بالقنفذة وتطوعك بالمشاركة هو قيم للغاية لجمع البيانات اللازمة لهذا المشروع.

ماذا سيحدث لي إذا كنت جزءا في هذا البحث؟

سنقوم بالترتيب للاجتماع في وقت مناسب لك داخل مبنى الجامعة. بمجرد الموافقة على المشاركة ، سنناقش مناقشة مفصلة. يتضمن هذا الجزء من الإجراء مراقبة معلمي حساب التفاضل والتكامل في الفصل الدراسي من أجل الحصول على أكبر قدر ممكن من المعلومات حول ما تنطوي عليه عملية الملاحظة لأن هذه ستكون أول محاولة للباحث في مراقبة مدرس التفاضل والتكامل. ستستمر كل ملاحظة لمدة 120 دقيقة لمدة ثماني محاضرات. سيكون هذا الفيديو المسجل. سوف يستخدم الباحث ورقة ملاحظة لكتابة ملاحظات حول التدريس قيد التنفيذ

هل هناك اي فوائد لي عندما اكون جزءا في هذا البحث؟

قد لا تكون هناك فائدة مباشرة لك ولكن مشاركتكم تساعدني على دراسة القضايا الرئيسية لهذه الدراسة، وهذا قد يساعد في تحسين تدريس حساب التفاضل والتكامل في المملكة العربية السعودية. سيكون لدينا فهم أفضل لاستخدام المعرفة الخاصة بطرق التدريس للمحتوى على المستوى الجامعي. أنت من المحتمل أن تجد هذه الدراسة مثيرة للاهتمام.

هل هناك اي مخاطر تنطوي عليها مشاركتي؟

ليس هناك اى مخاطر إطلاقا.

هل ستكون مشاركتي سرية؟

نعم بالطبع. جميع البيانات سوف تكون محمية في كمبيوتر مع كلمة سر والمستندات سوف تكون في مكان أمان لا يصل له إلا الباحث فقط.

ماذا يحدث لي إذا قمت بتغيير رايي؟

لك الحق في الانسحاب من مشروع الدراسة في أي وقت لمدة أربعة أسابيع بعد جمع البيانات وببساطة قم بتبليغ الباحث او ارسل ايميل واطلب الانسحاب.

من اين يمكنني الحصول على مزيد من المعلومات ؟

إذا كنت ترغب في مزيد من المعلومات حول هذا المشروع، يرجى الاتصال بي على أي من هاتفي المحمول أو البريد الإلكتروني:

iaaa1g15@soton.ac.uk أو iaar6000@hotmail.com.

هاتف: 00447448794697 أو 00966503552719.

J.4 Consent Form to Participate in the Research - Observation (in Arabic)

نموذج إقرار بالمشاركة في الملاحظة

عنوان الدراسة: معرفة طرق تدريس محتوى حساب التفاضل والتكامل لدى معلمين التفاضل والتكامل على مستوى الجامعة.

اسم الباحث: ابراهيم عبده الزبيدي

بعد قراءة ورقة معلومات المشاركة في البحث أرجوا التكرم بتعبئة هذا النموذج والتوقيع علية في حالة الموافقة عل البنود المرفقة في شاكرين لكم تعاونكم

لقد قرأت وفهمت ورقة معلومات المشاركة في البحث وحصلت عل الفرصة الكافية لطرح أسئلتي واستفساراتي عن الدراسة ☐

أوافق على المشاركة في مشروع البحث هذا عن طريق ملاحظتي داخل الفصل وتسجيل فيديو للبيانات التي تخدم مشروع البحث ☐

أتفهم أن تكون بياناتي في هذا البحث مجهولة المصدر ☐

أتفهم أن مشاركتي في هذا البحث هي تطوعية ولي الحق في الانسحاب لمدة أربعة أسابيع من تاريخ جمع البيانات ولن يترتب على ذلك أي حقوق أو متطلبات قانونية ☐

اسم المشارك:

توقيع المشارك:

التاريخ:

Appendix K Data Collection Instrument - Interview Schedule (in English)

K.1 Final Interview Schedule

The Semi-Structured Interview Schedule for Calculus Teachers.

Thank you again for agreeing to participate in my research project. This part of the project includes an interview that should last approximately 40 minutes. Let us begin the interview now.

Questions on knowledge of teaching calculus.

1. What are your most important objectives when teaching calculus? Why? How do you intend to reach them?
2. How do you teach your typical calculus class?
3. How do you create a plan for a calculus lesson prior to teaching it? Do you consider learning objectives or learning outcomes for each lesson?
4. Are you aware of using a range of knowledge in planning your calculus lessons? If so, what are some examples?
5. How do you select your calculus topic aims? To what extent do you explain your calculus topic aims to your students at the beginning of the lesson?
6. How do you select the textbook and materials used in the calculus class?
7. How do you know that your calculus lesson objectives have been achieved?
8. What do you consider the most important ideas in this instructional topic for students at this level?
9. Have you used a range of teaching approaches in your calculus classroom setting? If yes, could you provide examples?
10. To what extent are various mathematical representations (real-life examples, physical models and manipulatives, digital manipulatives and visuals, pictures and diagrams, graphs, algebraic symbols) important when you are teaching Limits? Derivatives? Integrals? Which, if any, are most important? Which, if any, are least important?
11. What are your primary instructional strategies when teaching calculus?
12. How do you know that your calculus teaching is effective?
13. Do you employ the use of technology when teaching calculus? If yes, what and how?

Questions on knowledge of student understanding within calculus.

1. What difficulties in learning calculus do you remember having as a student? Do you take these into consideration when teaching your students? If yes, how?
2. What preconceptions do learners have about learning calculus? How, if at all, do you address these preconceptions?

3. What is it about calculus that makes learning it easy or difficult?

4. In order for students to be successful in your calculus course, what prior knowledge do they require? Do they generally come prepared with this knowledge?

5. What is the primary difficulty learners experience in calculus in your classes? How do you address these difficulties?

6. How do you know whether or not the learners have understood the calculus material you have taught in a lesson?

7. How are students assessed on their calculus knowledge? (i.e. midterms, Final exams, Assignments, etc.)

8. Do learners provide feedback on your course? (e.g. end of term evaluations) Are there any suggestions that students consistently make?

Questions on the sequencing of building blocks of mathematical theories in calculus?

1. Based on your own opinions and experience, how do you see the topic of calculus within the field of mathematics?

2. How do you decide which calculus topics to teach and the order in which these topics are sequenced in your lessons to students?

3. To what extent is there a usual sequence of teaching calculus topics? If there is, what are your comments on this usual sequence of teaching?

4. Do you explain the nuances of every calculus proof? Why, or why not?

5. What calculus topics do you find easiest to teach? Why?

6. What calculus topics do you find most difficult to teach? Why?

7. Identify three characteristics that you think a good calculus teacher should have. Why are these important?

Thank you very much for taking part in this interview. The interview is now finished.

K.2 Data Collection Instrument - Interview Schedule (in Arabic)

A.1 Final Interview Schedule

The Semi-Structured Interview Schedule for Calculus Teachers.

Thank you again for agreeing to participate in my research project. This part of the project includes an interview that should last approximately 40 minutes. Let us begin the interview now.

Questions on knowledge of teaching calculus.

1. What are your most important objectives when teaching calculus? Why? How do you intend to reach them?

   ما هي أهم اهدافك عند تدريس حساب التفاضل والتكامل؟ ولماذ؟ كيف يمكن تعد لتحقيقها؟

2. How do you teach your typical calculus class?

   كيف تدرس طلابك حساب التفاضل والتكامل عمليا؟

3. How do you create a plan for a calculus lesson prior to teaching it? Do you consider learning objectives or learning outcomes for each lesson?

   كيف تخطط لدرس التفاضل والتكامل قبل تدريسه؟ هل تضع في اعتبارك اهداف التعلم أو مخرجات التعلم لكل درس؟

4. Are you aware of using a range of knowledge in planning your calculus lessons? If so, what are some examples?

   هل تستخدم مجموعة أو تشكيلة من المعارف في تخطيط دروس حساب التفاضل والتكامل الخاص بك؟ إذا كان الأمر كذلك، ما هي بعض الأمثلة؟

5. How do you select your calculus topic aims? To what extent do you explain your calculus topic aims to your students at the beginning of the lesson?

   كيف تحدد أهداف درس التفاضل والتكامل؟ إلى أي مدى تشرح أو توضح اهداف محاضرتك إلى طلابك في بداية الدرس؟

6. How do you select the textbook and materials used in the calculus class?

   كيف تحدد أو تختار الكتاب المقرر والمادة العلمية المستخدمة في تدريس حساب التفاضل والتكامل؟

7. How do you know that your calculus lesson objectives have been achieved?

   كيف تعرف أن أهداف درسك في حساب التفاضل قد تحققت؟

8. What do you consider the most important ideas in this instructional topic for students at this level?

   ما اهم الأفكار التي تضعها في الاعتبار في ترتيب موضوع حساب التفاضل والتكامل للطلاب في هذا المستوى؟

9. Have you used a range of teaching approaches in your calculus classroom setting? If yes, could you provide examples?

   هل سبق وان أستخدمت مجموعة من طرائق التدريس في تدريس التفاضل والتكامل؟ اذا نعم فضلا اعطني بعض الأمثلة؟

10. To what extent are various mathematical representations (real-life examples, physical models and manipulatives, digital manipulatives and visuals, pictures and diagrams, graphs, algebraic symbols) important when you are teaching Limits? Derivatives? Integrals? Which, if any, are most important? Which, if any, are least important?

    إلى أي مدى تمثل مختلف التمارين الرياضية (أمثلة واقعية، نماذج مادية محسوسات وينيويات، محسوسات رقمية وبصرية، وصور، رسوم ورسومات بيانية،، رموز جبرية) مهمة عندما كنت تدرس النهايات؟ المشتقات؟ التكاملات؟ ما هي، إن وجدت، الأكثر أهمية؟ ما هي، إن وجدت، الأقل أهمية؟

11. What are your primary instructional strategies when teaching calculus?

    ما هي استراتيجياتك التعليمية الأساسية عند تدريس التفاضل والتكامل

12. How do you know that your calculus teaching is effective?

    كيف تعرف أن تدريسك للتفاضل والتكامل الخاص فعال؟

13. Do you employ the use of technology when teaching calculus? If yes, what and how?

    هل تستخدم التكنولوجيا عند تدريس التفاضل والتكامل؟ إذا كانت الإجابة بنعم، ماذا وكيف؟

Questions on knowledge of student understanding within calculus.

أسئلة حول معرفة فهم الطالب في حساب التفاضل والتكامل

1. What difficulties in learning calculus do you remember having as a student? Do you take these into consideration when teaching your students? If yes, how?

   ما هي الصعوبات في تعلم حساب التفاضل والتكامل هل تتذكرها كطالب؟ هل تأخذ هذه الصعوبات في الاعتبار عند تعليم طلابك؟ إذا كان الجواب نعم، كيف؟

2. What preconceptions do learners have about learning calculus? How, if at all, do you address these preconceptions?

   ما هي المفاهيم المسبقة التي يمتلكها المتعلمون حول تعلم التفاضل والتكامل؟ كيف، إذا كان على الإطلاق، هل توجه هذه الأفكار المسبقة؟

3. What is it about calculus that makes learning it easy or difficult?

   ماذا عن حساب التفاضل والتكامل الذي يجعل تعلمه سهل أو صعب؟

4. In order for students to be successful in your calculus course, what prior knowledge do they require? Do they generally come prepared with this knowledge?

   من أجل أن يكون الطلاب ناجحين في دورة حساب التفاضل والتكامل، ما هي المعارف السابقة التي تتطلبها؟ هل يأتون عموما مع هذه المعرفة؟

5. What is the primary difficulty learners experience in calculus in your classes? How do you address these difficulties?

   ما هي الصعوبات الاساسية التي يواجهها المتعلمون في حساب التفاضل والتكامل في قاعاتكم الدراسية؟ كيف تعالج هذه الصعوبات؟

6. How do you know whether or not the learners have understood the calculus material you have taught in a lesson?

   كيف يمكنك معرفة ما إذا كان المتعلمون قد فهموا مادة حساب التفاضل والتكامل التي تعلمتها في درس؟

7. How are students assessed on their calculus knowledge? (i.e. midterms, Final exams, Assignments, etc.)

   كيف تقييم معرفة طلابك في حساب التفاضل والتكامل؟ (أي منتصف المدة، والاختبارات النهائية، والتعيينات، وما إلى ذلك)

8. Do learners provide feedback on your course? (e.g. end of term evaluations) Are there any suggestions that students consistently make?

   هل يقوم المتعلمون بتقديم التغذية الراجعة حول المادة؟ (مثل تقييمات نهاية الترم) هل هناك أي اقتراحات يقدمها الطلاب باستمرار؟

Questions on the sequencing of building blocks of mathematical theories in calculus?

أسئلة حول تسلسل لبنات البناء من النظريات الرياضية في حساب التفاضل والتكامل؟

1. Based on your own opinions and experience, how do you see the topic of calculus within the field of mathematics?

   استنادا إلى آرائك الخاصة والخبرة، كيف ترى موضوع حساب التفاضل والتكامل في مجال الرياضيات؟

2. How do you decide which calculus topics to teach and the order in which these topics are sequenced in your lessons to students?

   كيف تقرر تدريس أي مواضيع حساب التفاضل والتكامل والترتيب الذي يتم فيه تسلسل هذه المواضيع في الدروس الخاصة بك للطلاب؟

3. To what extent is there a usual sequence of teaching calculus topics? If there is, what are your comments on this usual sequence of teaching?

   إلى أي مدى هناك تسلسل معتاد في تدريس مواضيع حساب التفاضل والتكامل؟ إذا كان هناك، ما هي تعليقاتكم على هذا التسلسل المعتاد من التدريس؟

4. Do you explain the nuances of every calculus proof? Why, or why not?

   هل تفسر وتشرح الاجزاء الدقيقة في كل برهان في التفاضل والتكامل ؟ لماذا و لماذا لا؟

5. What calculus topics do you find easiest to teach? Why?

   ما هي مواضيع حساب التفاضل والتكامل التي تجدها أسهل للتدريس؟ لماذا ؟!

6. What calculus topics do you find most difficult to teach? Why?

   ما هي مواضيع حساب التفاضل والتكامل التي تجد صعوبة في تدريسها؟ لماذا ؟!

7. Identify three characteristics that you think a good calculus teacher should have. Why are these important?

   حدد ثلاث خصائص تعتقد أنه ينبغي أن يكون لدى معلم حساب التفاضل والتكامل الجيد. لماذا هذه مهمة؟

Thank you very much for taking part in this interview. The interview is now finished.

K.3 Participant Information Sheet for Interview Schedule

UNIVERSITY OF Southampton

Participant Information Sheet for the interview of the main study

Study Title: **Pedagogical Content Knowledge of Calculus Teachers at University Level.**

Researcher: Ibrahim Abdah A Alzubaidi **Ethics number: 25305**

Please read this information carefully before deciding to take part in this research. If you are happy to participate you will be asked to sign a consent form.

What is the research about?
I am a PhD student. This study is being carried out as part of a PhD in Education at the University of Southampton. This study will seek obtaining a deep understanding of calculus teachers' pedagogical content knowledge at Al-Qunfudhah University College at Umm Al-Qura University in Saudi Arabia. My questions are" What is the calculus teacher's form of Pedagogical Content Knowledge (PCK) at university level? And What influences calculus teachers in implementing their PCK in a practical setting". This project is sponsored by University of Southampton. My PhD is funded by Saudi Ministry of Education.

Why have I been chosen?
You have been chosen purposively to take part in this study as a key respondent. As a calculus teacher, your participation in the study will help me to have a clear picture about calculus teachers' pedagogical content Knowledge.

What will happen to me if I take part?
We will arrange to meet at a time that is convenient to you within the college premises. Once you have given consent to take part. You will interview for 30 minutes approximately. That will be audio recorded. You will be asked some questions about pedagogical content knowledge. There are no right or wrong answers.

Are there any benefits in my taking part?
There may not be direct benefit to you but your participation would help me to examine the main issues of this study and this might help improve teaching Calculus in Saudi Arabia. We will have a better understanding of using pedagogical content knowledge at university level. You are likely to find this study interesting and make you reflect and think in improvement Teaching Calculus in Saudi Arabia.

Are there any risks involved?
There will be no risks involved by taking part in this exercise.

Will my participation be confidential?
Yes, of course. Any information obtained in connection with this study will be kept strictly confidential and will be stored safely and later destroyed. Data coded and kept on a password protected computer and will be used just for this study. Information will be kept safe in line with UK laws (the Data Protection Act) and University of Southampton policy.

What happens if I change my mind?
Participation in this study is completely voluntary. For that reason, you have the right to withdraw up until four weeks after the end of data collection, and this will not have any effect on any of your rights.

What happens if something goes wrong?
For any concerns or complaints about this study, You can Contact Head of Research Governance (02380 595058, rgoinfo@soton.ac.uk).

Where can I get more information?
If you would like further information about the project, please contact me on either my mobile or email address:
iaaa1g15@soton.ac.uk or iaar6000@hotmail.com.
Phone: 00447448794697 or 00966503552719.
or can contact my supervisor: David Keith Jones
Email: D.K.Jones@soton.ac.uk

[Date February 2nd 2017] [Version number 1]

K.4 Consent Form to Participate in the Research: Interview Schedule

UNIVERSITY OF Southampton

CONSENT FORM For the interviews of the main study

Study title: **Pedagogical Content Knowledge of Calculus Teachers at University Level**

Researcher name: **Ibrahim Abdah A Alzubaidi**

Ethics reference: **25305**

Please initial the box(es) if you agree with the statement(s):

Statement	Initial
I have read and understood the information sheet (Version number 1, Date February 2nd 2017) and have had the opportunity to ask questions about the study.	
I agree to take part in this research project and agree for my data to be audio recorded and used for the purpose of this study	
I understand that my responses will be anonymised in reports of the research	
I understand my participation is voluntary and I may withdraw at any time until four weeks after the interviews without my legal rights being affected	

Data Protection

I understand that information collected about me during my participation in this study will be stored on a password protected computer and that this information will only be used for the purpose of this study.

Name of participant (print name)..

Signature of participant..

Date...

Version number 1, Date February 2nd 2017

K.5 Participant Information Sheet for Interview Schedule (in Arabic)

ورقة معلومات المشاركة في الدراسة

عنوان الدراسة: معرفة طرق تدريس محتوى حساب التفاضل والتكامل لدى معلمين التفاضل والتكامل على مستوى الجامعة.

فضلا اقرأ هذه المعلومات قبل أن تقرر المشاركة في هذا البحث، إذا قررت المشاركة سيطلب منك التوقيع على نموذج الموافقة المرفق.

ما هي فكرة البحث ولماذا؟

أنا طالب دكتوراه. يتم تنفيذ هذه الدراسة كجزء من متطلبات الحصول على شهادة الدكتوراه في التربية من جامعة ساوثمبتون. تتمحور فكرة هذه الدراسة حول الحصول على فهم عميق لمعرفة معرفة معلمي حساب التفاضل والتكامل بطرق تدريس حساب التفاضل والتكامل وكذلك المحتوى لدى أعضاء هيئة التدريس بالكلية الجامعية بالقنفذة فرع جامعة أم القرى بالمملكة العربية السعودية. أسئلتي هي " ما هو نموذج معرفة معلم حساب التفاضل والتكامل بطرق تدريسه ومحتواه (PCK) على المستوى الجامعي؟ وكيف مدرسين حساب التفاضل والتكامل يستخدمون هذه المعرفة في الممارسة العملية؟ ". ويُرعى هذا المشروع من قبل جامعة ساوثمبتون. ويتم تمويل درجة الدكتوراه من قبل وزارة التربية والتعليم السعودية.

لماذا تم اختياري؟

لان المشاركون في هذا المشروع هم معلمون حساب التفاضل والتكامل في الكلية الجامعية بالقنفذة وتطوعك بالمشاركة هو قيم للغاية لجمع البيانات اللازمة لهذا المشروع.

ماذا سيحدث لي إذا كنت جزءا في هذا البحث؟

وسوف أقوم بترتيب لقاء في الوقت المناسب لك داخل الجامعة. وسوف أقدم لك خطاب الموافقة على المشاركة. ستكون المقابلة اقل من 30 دقيقة تقريبا. وسيكون ذلك مسجل تسجيل صوتي. سوف أسأل بعض الأسئلة حول المعرفة بطرق التدريس الخاصة بالتفاضل والتكامل. لا توجد إجابات صحيحة أو خاطئة.

هل هناك اي فوائد لي عندما اكون جزءا في هذا البحث؟

قد لا تكون هناك فائدة مباشرة لك ولكن مشاركتكم تساعدني على دراسة القضايا الرئيسية لهذه الدراسة، وهذا قد يساعد في تحسين تدريس حساب التفاضل والتكامل في المملكة العربية السعودية. سيكون لدينا فهم أفضل لاستخدام المعرفة الخاصة بطرق التدريس للمحتوى على المستوى الجامعي. أنت من المحتمل أن تجد هذه الدراسة مثيرة للاهتمام.

هل هناك أي مخاطر تنطوي عليها مشاركتي؟

ليس هناك اي مخاطر إطلاقا.

هل ستكون مشاركتي سرية؟

نعم بالطبع. جميع البيانات سوف تكون محمية في كمبيوتر مع كلمة سر والمستندات سوف تكون في مكان أمان لا يصل له إلا الباحث فقط.

ماذا يحدث لي إذا قمت بتغيير رأيي؟

لك الحق في الانسحاب من مشروع الدراسة في أي وقت لمدة أربعة أسابيع بعد جمع البيانات وببساطة قم بتبليغ الباحث او أرسل ايميل واطلب الانسحاب.

من أين يمكنني الحصول على مزيد من المعلومات ؟

إذا كنت ترغب في مزيد من المعلومات حول هذا المشروع، يرجى الاتصال بي على أي من هاتفي المحمول أو البريد الإلكتروني:

iaaa1g15@soton.ac.uk أو iaar6000@hotmail.com.

هاتف: 00447448794697 أو 00966503552719.

K.6 Consent Form to Participate in the Research: Interview Schedule (in Arabic)

نموذج إقرار بالمشاركة في المقابلة

عنوان الدراسة: معرفة طرق تدريس محتوى حساب التفاضل والتكامل لدى معلمين التفاضل والتكامل على مستوى الجامعة.

اسم الباحث: ابراهيم عبده الزبيدي

بعد قراءة ورقة معلومات المشاركة في البحث أرجوا التكرم بتعبئة هذا النموذج والتوقيع علية في حالة الموافقة عل البنود المرفقة في شاكرين لكم تعاونكم

لقد قرأت وفهمت ورقة معلومات المشاركة في البحث وحصلت عل الفرصة الكافية لطرح أسئلتي واستفساراتي عن الدراسة ☐

أوافق على المشاركة في مشروع البحث هذا عن طريق المقابلة وتسجيل صوتي للبيانات التي تخدم مشروع البحث ☐

أتفهم أن تكون بياناتي في هذا البحث مجهولة المصدر ☐

أتفهم أن مشاركتي في هذا البحث هي تطوعية ولي الحق في الانسحاب لمدة أربعة أسابيع من تاريخ جمع البيانات ولن يترتب على ذلك أي حقوق أو متطلبات قانونية ☐

اسم المشارك:

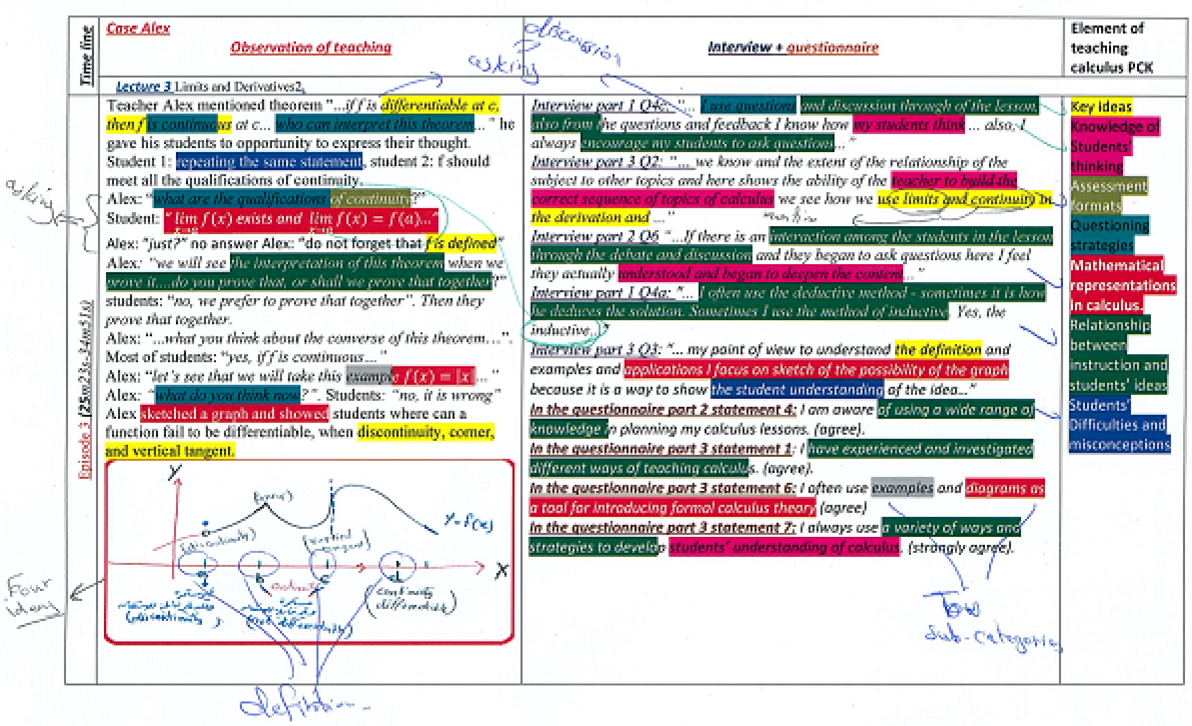
توقيع المشارك:

التاريخ:

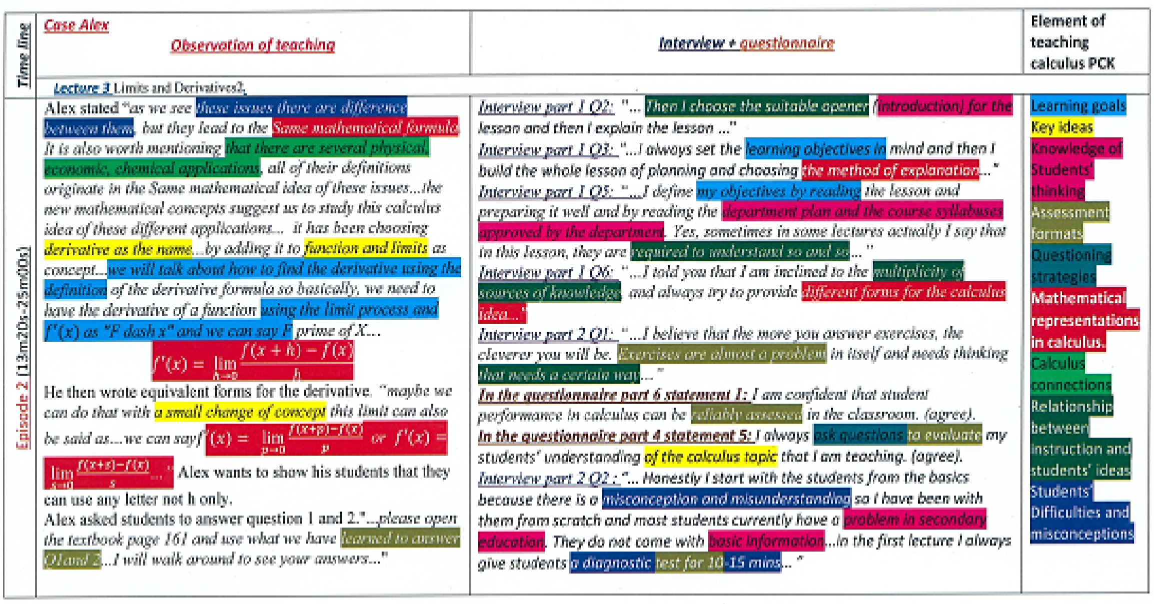
K.7 Example of Following up Interview Question after Watching Video with Case John

Episode 2 (12m	
Students gave dozens of answers.John explained the difference between them "... *when we see evaluate the limit in the question and do not give Limit Computation we do direct substitution, while if we see in the question prove we should use the is the precise definition of limit...".* Students were not interacting with their teacher when he started to prove that by using the precise definition of limit. Students had difficulties with the formal epsilon delta definition of the limit...John mentioned "...we *will take some exercises in the next lecture"*. John explained three examples by using direct substitution $\lim_{x\to 2}(\frac{3+x^2}{x+1})$; $\lim_{\theta\to\frac{\pi}{2}}(\sin\theta + 2\theta)$ and $\lim_{x\to 1}(\frac{2x^2-3x+1}{1-x^2})$?	*students' understanding of calculus topic that I am teaching. (agree).* ***In the questionnaire part 3statement 4:*** I only use examples and diagrams after having introduced the formal calculus theory agree) *Interview part 1 Q4: "...after explaining the main idea, I give an example or two examples..."* This is following up interview question after watching the video: why do you use an example and you know this question does not relate to the previous idea? *Interview following up Q1: "...you can see in the short video which you sent, I mentioned in the beginning "but it won't always work" this as counter example and would like to use it as introduction for next idea..."* ***In the questionnaire part 4 statement 2:*** *I never adjust my progress through calculus syllabus to take account of common student misunderstandings and misconceptions. (Disagree).*

Appendix L Example of empirical data collected and analysed for this study



Time line	Case Alex Observation of teaching	Interview + questionnaire	Element of teaching calculus PCK
	Lecture 3 Limits and Derivatives2,		
Episode 3 (25m23s-34m51s)	Teacher Alex mentioned theorem "...if f is differentiable at c, then f is continuous at c... who can interpret this theorem..." he gave his students to opportunity to express their thought. Student 1: repeating the same statement, student 2: f should meet all the qualifications of continuity. Alex: "what are the qualifications of continuity?" Student: " $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$..." Alex: "just?" no answer Alex: "do not forget that f is defined" Alex: "we will see the interpretation of this theorem when we prove it....do you prove that, or shall we prove that together?" students: "no, we prefer to prove that together". Then they prove that together. Alex: "...what you think about the converse of this theorem...". Most of students: "yes, if f is continuous..." Alex: "let's see that we will take this example $f(x) = \|x\|$..." Alex: "what do you think now?". Students: "no, it is wrong" Alex sketched a graph and showed students where can a function fail to be differentiable, when discontinuity, corner, and vertical tangent.	Interview part 1 Q4c: "... I use questions and discussion through of the lesson, also from the questions and feedback I know how my students think ... also, I always encourage my students to ask questions..." Interview part 3 Q2: "... we know and the extent of the relationship of the subject to other topics and here shows the ability of the teacher to build the correct sequence of topics of calculus we see how we use limits and continuity in the derivation and ..." Interview part 2 Q6 "...If there is an interaction among the students in the lesson through the debate and discussion and they began to ask questions here I feel they actually understood and began to deepen the content..." Interview part 1 Q4a: "... I often use the deductive method - sometimes it is how he deduces the solution. Sometimes I use the method of inductive. Yes, the inductive..." Interview part 3 Q3: "... my point of view to understand the definition and examples and applications I focus on sketch of the possibility of the graph because it is a way to show the student understanding of the idea..." In the questionnaire part 2 statement 4: I am aware of using a wide range of knowledge in planning my calculus lessons. (agree). In the questionnaire part 3 statement 1: I have experienced and investigated different ways of teaching calculus. (agree). In the questionnaire part 3 statement 6: I often use examples and diagrams as a tool for introducing formal calculus theory (agree) In the questionnaire part 3 statement 7: I always use a variety of ways and strategies to develop students' understanding of calculus. (strongly agree).	Key ideas Knowledge of Students' thinking Assessment formats Questioning strategies Mathematical representations in calculus. Relationship between instruction and students' ideas Students' Difficulties and misconceptions



Time line	Case Alex Observation of teaching	Interview + questionnaire	Element of teaching calculus PCK
	Lecture 3 Limits and Derivatives2,		
Episode 2 (13m20s-25m00s)	Alex stated "as we see these issues there are difference between them, but they lead to the Same mathematical formula. It is also worth mentioning that there are several physical, economic, chemical applications, all of their definitions originate in the Same mathematical idea of these issues...the new mathematical concepts suggest us to study this calculus idea of these different applications... it has been choosing derivative as the name...by adding it to function and limits as concept...we will talk about how to find the derivative using the definition of the derivative formula so basically, we need to have the derivative of a function using the limit process and f'(x) as "F dash x" and we can say F prime of X... $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ He then wrote equivalent forms for the derivative. "maybe we can do that with a small change of concept this limit can also be said as...we can say $f'(x) = \lim_{p \to 0} \frac{f(x+p)-f(x)}{p}$ or $f'(x) = \lim_{s \to 0} \frac{f(x+s)-f(x)}{s}$..." Alex wants to show his students that they can use any letter not h only. Alex asked students to answer question 1 and 2."...please open the textbook page 161 and use what we have learned to answer Q1and 2...I will walk around to see your answers..."	Interview part 1 Q2: "... Then I choose the suitable opener (introduction) for the lesson and then I explain the lesson ..." Interview part 1 Q3: "...I always set the learning objectives in mind and then I build the whole lesson of planning and choosing the method of explanation..." Interview part 1 Q5: "...I define my objectives by reading the lesson and preparing it well and by reading the department plan and the course syllabuses approved by the department. Yes, sometimes in some lectures actually I say that in this lesson, they are required to understand so and so..." Interview part 1 Q6: "...I told you that I am inclined to the multiplicity of sources of knowledge, and always try to provide different forms for the calculus idea..." Interview part 2 Q1: "...I believe that the more you answer exercises, the cleverer you will be. Exercises are almost a problem in itself and needs thinking that needs a certain way..." In the questionnaire part 6 statement 1: I am confident that student performance in calculus can be reliably assessed in the classroom. (agree). In the questionnaire part 4 statement 5: I always ask questions to evaluate my students' understanding of the calculus topic that I am teaching. (agree). Interview part 2 Q2 : "... Honestly I start with the students from the basics because there is a misconception and misunderstanding so I have been with them from scratch and most students currently have a problem in secondary education. They do not come with basic information...in the first lecture I always give students a diagnostic test for 10-15 mins..."	Learning goals Key ideas Knowledge of Students' thinking Assessment formats Questioning strategies Mathematical representations in calculus. Calculus connections Relationship between instruction and students' ideas Students' Difficulties and misconceptions

Appendix L

Episode 4 (35m [6s–46m49s])	Then, he touched upon the rules of finding derivatives. " *I want you to know some common alternative notation for the derivative such as that symbols D, d/dx, Df, f'."* Alex started this part, Alex tries providing and making available definitions, theorems and proofs to students; "*...we will take the function then will define it then will see its theorem and prove it and take an example... in the first and second lecture we had some commonly used functions... constants function did you remember the definition of that?... what is domain of it? ...the theorem 1 (constants function rule) ... I will prove this theorem... to help you to understand that let's take this example...what is identity function?... theorem 2 (identity function rule) ... the proof of this theorem is ...take this example for understanding that ..."* *Power function is ...theorem 3 (Power Rule) ...in your notebook try to prove that... focus on the board to see this example..."* Alex identified students' difficulties inadequate concept images. "*...power rule" I want you to analyse this power by taking a very basic binomial of (a+b) and we will raise it to different powers of n and analyze what is going on in this pattern so that when I get to the power rule ... I will give you method ... take (a+b) and raise it to the zero power... anything to the zero power what does it equal?"* Student 1: *1* Alex: "*... (a+b) to the first power is just itself, and (a+b)2 I know many of you have difficult... So (a+b) times (a+b) is a^2+2ab+b^2 ... then if we would like to get this trinomial and wrap it in parenthesis and multiply by (a+b) again is (a+b)^3 we will get a^3+3a^2b+3ab^2+b^3 ... I will give you rule ... the first term starts with the highest power n and then counts down, whereas the second term exponents are counting up ... So when we see (a+b)^3, the first term being a and its exponent is counting down in value ..."*	*Interview part 3 Q3: "... I focus on concepts. You must have a starting point from which focus on definitions and explain them in detail and give your students what did other books say about the concept maybe in the symbols... open their mind as you can... Because if the student does not understand the definition, it is difficult to understand what follows and then sequentially according to the course book, theory, result, proof or graph. But the usual sequence begins from the definition and then an example or theory and proof according to the importance. My point of view to understand the definition and examples and applications I focus on sketching if the possibility of the graph ..."* *I focus on concepts... must have a starting point from which focus on definitions and explain them in detail and give my students what did other books say about the concept maybe in the symbols... open their mind as I can... Because if the student does not understand the definition, it is difficult to understand what follows and then sequentially according to the textbook, theorem, result, proof or diagram. But the usual sequence begins from the definition and then an example or theorem and proof according to the importance. My point of view to understand the definition and examples with applications then I focus on sketching if the possibility of the graph* ***In the questionnaire part 4 statement 3:*** *I always select teaching approaches that build on student thinking and learning in calculus. (agree).* ***In the questionnaire part 4 statement 4:*** *I anticipate my students' prior calculus knowledge before the lesson. (agree).* ***In the questionnaire part 4 statement 6:*** *I always ask questions to evaluate my students' understanding of the calculus topic that I am teaching. (agree).* *Interview part 2 Q2: "... Honestly I start with the students from the basics because there is a misconception and misunderstanding so I have been with them from scratch and most students currently have a problem in secondary education. They do not come with previous information ... believe me they cannot analyse a basic binomial ..."* *Interview part 3 Q4: "... Here, it depends on the quality of the students. I rarely explain every proof. I teach physics students and chemistry students or any other discipline. They do not need some proofs. But students of the mathematics are the students I currently have, and you will observe me in their class, I will give them the complete proofs. They are sometimes higher than their level but I give them, but they are not required to understand all ... I give them a note that they are going to study it at higher levels and therefore when they study it in the future, ... They will be aware of how the mechanism how to prove. So that the students learn but not required to understand or master it..."* ***In the questionnaire part 5 statement 1:*** *I analyse each calculus topic by building blocks of mathematical theories using axioms, definitions, theorem, proof.(agree)* ***In the questionnaire part 5 statement 2:*** *I never explain the proof of formal theory in calculus. (disagree)*	Learning goals Key ideas Knowledge of Students' thinking Assessment formats Questioning strategies Mathematical representations in calculus. Relationship between instruction and students' ideas Students' Difficulties and misconceptions

Time line	*John* ***Observation of teaching***	***Interview + questionnaire***	Element of teaching calculus PCK
	Lecture 3 Limits and Continuity.		
Episode 2 (12m1s–22m50s)	*"...today we will use the simplest methods that we can use to evaluate limits, which is direct substitution, but it won't always work. We will see that later... I am going to be explaining computing limits algebraically and we can see how we do direct substitution, let's take first example..."* Example 1: Evaluate the limit: $\lim_{x \to 3}(x^2 + x - 5)$.  *"... What is the difference between this example and when we say prove this* $\lim_{x \to 3}(x^2 + x - 5) = 7$*?"* Students gave dozens of answers John explained the difference between them "*... when we see evaluate the limit in the question and do not give Limit Computation we do direct substitution, while if we see in the question prove we should use the is the precise definition of limit...".* Students were not interacting with their teacher when he started to prove that by using the precise definition of limit. Students had difficulties with the formal epsilon delta definition of the limit...John mentioned "*...we will take some exercises in the next lecture".* John explained three examples by using direct substitution $\lim_{x \to 2}(\frac{3+x^2}{x+1})$; $\lim_{\theta \to \frac{\pi}{2}}(\sin\theta + 2\theta)$ and $\lim_{x \to 1}(\frac{2x^2-3x+1}{1-x^2})$?	*Interview part 3 Q3: "... during the lecture, I always have an introduction and an explanation of the subject and then the application. The lesson depends on the definitions, theorems, proof and examples ..."* *Interview part 1 Q 9: "...we encourage the students to try to get the idea and try to answer some questions..."* *Interview part 2 Q1a: "...sometimes I feel that the student needs a lot of examples to absorb every idea...but I give several examples, and then I explain in more than one way to understand the lesson... student must learn how to draw this relationship between calculus ideas and mathematical representation of it..."* ***In the questionnaire part 3 statement 7:*** *I always use a variety of ways and strategies to develop students' understanding of calculus. (Agree).* ***In the questionnaire part 3 statement 7:*** *I always ask questions to evaluate my students' understanding of calculus topic that I am teaching. (agree).* ***In the questionnaire part 3statement 4:*** I only use examples and diagrams after having introduced the formal calculus theory agree) *Interview part 1 Q4: "...after explaining the main idea, I give an example or two examples..."* This is following up interview question after watching the video: why do you use an example and you know this question does not relate to the previous idea? *Interview following up Q1: "...you can see in the short video which you sent. I mentioned in the beginning "but it won't always work" this as counter example and would like to use it as introduction for next idea..."* ***In the questionnaire part 4 statement 2:*** *I never adjust my progress through calculus syllabus to take account of common student misunderstandings and misconceptions. (Disagree).*	Pivotal and counter examples Questioning strategies Key ideas Students' Difficulties and misconceptions Mathematical representations in calculus Relationship between instruction and students ideas Knowledge of Students' thinking

Appendix M List of Publications

Alzubaidi, I., & Jones, K. (2018). A case study of a university teacher of Calculus 1. In V. Durand-Guerrier, R. Hochmuth, S. Goodchild, & N. M. Hogstad (Eds.), *Proceedings of INDRUM 2018: the Second Conference of the International Network for Didactic Research in University Mathematics* (pp. 452-453). (Proceedings of the International Network for Didactic Research in University Mathematics; Vol. 2). Kristiansand, Norway: University of Agder and INDRUM.

Alzubaidi, I. and Jones, K. (2018). Poster about ‘What do I need to know about teaching to teach calculus?. Southampton University, the UK*: Doctoral Research Showcase*. (Presented).

Alzubaidi, I. and Jones, K. (2018). Poster about ‘How can I teach calculus. Southampton University Education School, the UK: *PGR Student Conference*. (Presented).

A case study of a university teacher of calculus 1

Ibrahim Alzubaidi, Keith Jones

List of References

Abdullah, F. (2006) Measuring service quality in higher education: three instruments compared, *International Journal of Research and Method in Education*, Vol. 29 No. 1, pp. 71-89.

Abu Asaad, S.A. (2010) أساليب تدريس الرياضيات *(Strategies in Teaching Mathematics)*. Amman: Dar Alshuroug.

Adams P. (2011) (Dis) continuity and the Coalition: primary pedagogy as craft and primary pedagogy as performance. *Educational Review* 63 (4), pp. 467-483.

Adams, T. L. (2003) Reading mathematics: More than words can say. *The Reading Teacher*, 56(8), pp. 786-795.

Akerson, V.L., Pongsanon, K., Rogers, M.A.P., Carter, I. and Galindo, E. (2017) Exploring the use of lesson study to develop elementary preservice teachers' pedagogical content knowledge for teaching nature of science. *International Journal of Science and Mathematics Education*, 15(2), pp. 293-312.

Alamolhodaei, A. (1996) *A Study of Learning Calculus in Higher Education and Students' Learning Styles*, Ph.D. Thesis, University of Glasgow.

Al-Amri, M. (2011) Higher education in Saudi Arabia. *Journal of Higher Education Theory and Practice*, 11(4), pp. 88-91.

Al-Aqeel, A. (2016) *Education policy in the Kingdom of Saudi Arabia*. Riyadh, Saudi Arabia: Al-Roshd Library.

Alcock, L. (2014) *How to think about analysis*. Oxford: Oxford University Press.

Al-Dakhil, A.A. (2011) *Higher education: What are the pros and what are the cons?* (In Arabic). Riyadh: Al-Obeikan.

Al-Dossary, S.A. (2008) *A study of the factors affecting student retention at King Saud University, Saudi Arabia: Structural equation modelling and qualitative methods*. PhD thesis, University of Stirling.

Alebaikan, R. and Troudi, S. (2010) Blended learning in Saudi universities: Challenges and perspectives. *Research in Learning Technology*, 18(1), pp. 49-59.

Alexander R. (2004) Still no pedagogy? Principle, pragmatism and compliance in primary education. *Cambridge Journal of Education,* 34(1), pp. 7-33.

Al-Husain, D., & Hammo, B. (2015). Investigating the Readiness of college students for ICT and Mobile Learning: A case study from King Saud University. *International Arab Journal of e-Technology*, *4*(1), pp. 48–55.

Al-Khateeb, M.A. (2011). *Designing and Teaching Contemporary Mathematics Curriculum* (In Arabic). Amman, Jordan: Daralhamed.

Alshahrani, K. and Ally, M. (eds.) (2016) *Transforming education in the Gulf Region: Emerging learning technologies and innovative pedagogy for the 21st Century*. London: Routledge, Taylor & Francis Group.

Alrashidi, O. and Phan, H., (2015) Education Context and English Teaching and Learning in the Kingdom of Saudi Arabia: An Overview. *English Language Teaching*, *8*(5), pp.33-44.

Artigue M. (2001) What can we learn from educational research at the university level? In: Holton D (ed) *The teaching and learning of mathematics at university level*. An ICMI study. Kluwer, Dordrecht, pp 207–220.

Aulia, E.V., Poedjiastoeti, S. and Agustini, R., (2018) The Effectiveness of Guided Inquiry-based Learning Material on Students' Science Literacy Skills. In *Journal of Physics: Conference Series*. IOP Publishing.

Ary, D., Jacobs, L. C., and Razavieh, A. (1979) *Introduction to research in education*. London: Holt, Rinehart and Winston.

Asiri, M.J.S. (2012) Factors influencing the use of learning management system in Saudi Arabian Higher Education: A theoretical framework. *Higher Education Studies*, *2*(2), pp.125-137.

Aydin, S., Demirdogen, B., Akin, F.N., Uzuntiryaki-Kondakci, E. and Tarkin, A. (2015) The nature and development of interaction among components of pedagogical content knowledge in practicum. *Teaching and Teacher Education*, 46, 37-50.

Babbie, E. (2001) *The practice of social research.* (9th ed.). Belmont: Wadsworth.

Back, R.J. and Wright, J. (2012) *Refinement calculus: A systematic introduction*. New York: Springer Science & Business Media.

Baker, B., Cooley, L., & Trigueros, M. (2000) A Calculus graphing schema. *Journal for Research in Mathematics Education,* 31, pp 557–578.

Baker, M. and Chick, H. (2006) Pedagogical content knowledge for teaching primary mathematics: A case study of two teachers. In: P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 60-67). Sydney: MERGA.

Ball, D. L. and Bass, H. (2000) Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In: J. Boaler. (ed.) *Multiple perspectives on the teaching and learning of mathematics.* Westport, CT: Ablex*,* 83–104.

Ball, D. L. and Bass, H. (2000) Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In: D.C. Phillips (ed.), *Constructivism in education: Opinions and second opinions on controversial issues. Yearbook of the National Society for the Study of Education.* Chicago: University of Chicago Press, pp.193-224.

Ball, D. L. and Bass, H. (2003) Making mathematics reasonable in school. In: J. Kilpatrick, W.G. Martin and D. Schifter, (eds.), *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics, pp. 27–44.

Ball, D. L. (1990) The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, *90*(4), pp.449-466.

Ball, D. L., Hill, H. C, and Bass, H. (2005) Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, *30*(3), pp. 14-46.

Ball, D. L., Lubienski, S. T., Mewborn, D. S. (2001) Research on teaching mathematics: the unsolved problem of teachers' mathematical knowledge. in: Richardson, V. (ed.) *Handbook of research on teaching*, pp. 433 - 456, New York, Macmillan.

Ball, D.L. and Sleep, L. (2007) *What is mathematical knowledge for teaching, and what are features of tasks that can be used to develop MKT?* Presentation made at the Center for Proficiency in Teaching Mathematics (CPTM) pre-session of the Annual Meeting of the Association of Mathematics Teacher Educators (AMTE), 25 January.

Ball, D.L., Thames, M.H. and Phelps, G. (2008) Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), pp.389-407.

Barbour, R.S. (1998) Mixing qualitative methods: Quality assurance or qualitative quagmire? *Qualitative Health Research*, *8*(3), pp.352-361.

Barclay, A.C. (2012) *Measuring teacher immediacy and communication competence on student achievement in calculus: A sequential explanatory mixed method design.* PhD thesis, Colorado State University.

Bardelle, C. and Ferrari, P.L. (2011) Definitions and examples in elementary calculus: The case of monotonicity of functions. *ZDM*, *43*(2), pp.233-246.

Barnes, H.E. (2007) *The relationship between the subject matter knowledge of secondary pre-service mathematics teachers and classroom practice*. PhD thesis, University of Pretoria.

Barnett, J. and Hodson, D. (2001) Pedagogical context knowledge: Toward a fuller understanding of what good science teachers know. *Science Education*, *85*(4), pp.426-453.

Barrett, J.W. and Süli, E. (2012) Reflections on Dubinskii's nonlinear compact embedding theorem. *Publications de l'Institut Mathématique*, *91*(105), pp.95-110.

Bassey, M. (1999) *Case Study Research in Educational Settings*. Buckingham: Open University Press.

Baumert, J. & Kunter, M. (2013) The COACTIV Model of Teachers' Professional Competence. In: Kunter, M., Baumert, J., Blum, W. et al. (Eds), *Cognitive Activation in the Mathematics Classroom and Professional Competence of Teachers- Results from the COACTIV Project*. New York: Springer, pp.25-48.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., et al. (2010) Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, 47, pp.133-180.

Baumert, J., Kunter, M., (2004) The COACTIV Model of Teachers' Professional Competence. In: M. Kunter et al. (eds.), *Cognitive Activation in the Mathematics Classroom 25 and Professional Competence of Teachers*, Mathematics Teacher Education 8, New York, Springer.

Bednarz, N., & Proulx, J. (2009) Knowing and using mathematics in teaching: conceptual and epistemological clarifications. *For the Learning of Mathematics*, 29(3), pp. 11-17.

Bell, J. (2005) *Doing your Research Project: A guide for first-time researchers in education, health and social science*, Open University Press, New York.

BERA (2011) *Ethical guidelines for educational research.* London: BERA. Available from: https://www.bera.ac.uk/wp-content/uploads/2014/02/BERA-Ethical-Guidelines-2011.pdf [Accessed 5 March 2017].

Berg, B. L. (2007) *Qualitative research methods for the social sciences*, 7th ed. Boston: Allyn & Bacon.

Bergsten, C. (2012) A calculus lecture: What made it good?. In: *Sixth Nordic Conference on Mathematics Education (NORMA 2011), 11-14 May 2011, University of Iceland* (pp. 167-176). University of Iceland Press.

Bernstein, B. (1990) *The structuring of pedagogical discourse. Class, codes and control Volume 4.* London, Routledge.

Biggs, J.B. (2011) *Teaching for quality learning at university: What the student does*. McGraw-Hill Education (UK).

Biggs, J. (2003) *Aligning teaching for constructing learning*. York, UK: Higher Education Academy.

Biza, I., Giraldo, V., Hochmuth, R., Khakbaz, A., & Rasmussen, C. (2016) *Research on Teaching and Learning Mathematics at the Tertiary Level*. Cham: Springer.

Blackledge, D. and Hunt, B. (2018) *Sociological Interpretations of education*. London: Routledge.

Blair, J., Czaja, R.F. & Blair, E.A. (2013) *Designing surveys: A guide to decisions and procedures*. Sage Publications.

Blömeke, S. and Delaney, S. (2014) Assessment of teacher knowledge across countries: A review of the state of research. In: *International perspectives on teacher knowledge, beliefs and opportunities to learn* (pp. 541-585). Dordrecht: Springer.

Boaler, J. & Brodie, K. (2004) The importance, nature and impact of teacher questions. In: D.
Boaler, J. (2002) The development of disciplinary relationships: Knowledge, practice and identity in mathematics classrooms. *For the learning of mathematics*, *22*(1), pp.42-47.

Bode, M., Drane, D., Kolikant, Y.B.D. and Schuller, M. (2009) A clicker approach to teaching calculus. *Notices of the AMS*, *56*(2), pp.253-256.

Bodner, G. M. and Orgill, M. (2007) *Theoretical frameworks for research in chemistry/science education.* Upper Saddle River, NJ: Pearson Prentice Hall.

Borg, S. (2003) Teacher cognition in grammar teaching: A literature review. *Language Awareness*, *12*(2), pp.96-108.

Borg, S. (2003) Teacher cognition in language teaching: A review of research on what language teachers think, know, believe, and do. *Language Teaching* 36, pp.81–109.

Borg, S. and Alshumaimeri, Y. (2012) University teacher educators’ research engagement: Perspectives from Saudi Arabia. *Teaching and Teacher Education*, *28*(3), pp.347-356.

Borg, W.R. (1981) *Applying educational research: a practical guide for teachers.* New York, Longman Inc.

Boyd, C. (2000) Combining qualitative and quantitative approaches In: P.L. Munhalland and C.O. Boyd (eds.), *Nursing research: A qualitative perspective*. Boston: Jones & Bartlett, pp.454-475.

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V. and Törner, G., (2016) Teaching and learning of calculus. In: *Teaching and Learning of Calculus* (pp. 1-37). Cham: Springer.

Bressoud, D.M., Carlson, M.P., Mesa, V. and Rasmussen, C. (2013) The calculus student: Insights from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology*, *44*(5), pp.685-698.

Bromme, R. (1995) What exactly is pedagogical content knowledge? Critical remarks regarding a fruitful research program. In S. Hopmann, & K. Riquarts (Eds.) *IPN Schriftenreihe: Vol. 147.Didaktik and/or curriculum*, pp.205-216.

Bryman, A. (2004) *Social research methods*. Oxford: Oxford University Press.

Bryman, A. (2008) Barriers of integrating quantitative and qualitative research. *Journal of Mixed Methods Research, 1*(1), pp. 8-22.

Bryman, A. (2006) Integrating quantitative and qualitative research: how is it done? *Qualitative Research.* London: Sage.

Buck, R. (1970). “Functions” in mathematics education. In E. G. Begle & H. G. Richey (Eds.), *69th yearbook of the National Society for the Study of Education* (pp. 236-259). Chicago: University of Chicago.

Burns, R. B. (2000) *Introduction to research methods* (4th ed.). London: Sage.

Burton, L., (1984) Mathematical thinking: The struggle for meaning. *Journal for Research in Mathematics Education*, pp.35-49.

Cai, J., Mok, I.A., Reddy, V. and Stacey, K. (2016) *International comparative studies in mathematics.* New York: Springer International.

Campos, D.F. and Pinto, M.M.F. (2016) Mathematics teachers’ conceptions and constraints for changing teaching practices in Brazilian higher education: An analysis through activity theory. *International Journal of Mathematical Education in Science and Technology*, *47*(8), pp.1179-1205.

Carlson, M. (1997) Obstacles for college algebra students in understanding functions: what do high-performing students really know? *The AMATYC Review.* 19, pp 48-59.

Carpenter, T.P., Fennema, E., Peterson, P. and Carley, D.A. (1988) Teachers’ pedagogical content knowledge of students’ problem solving in elementary mathematics. *Research in Mathematics Education, 24*(2), pp.94-116.

Carvalho, S. & White, H., (1997) *Combining the quantitative and qualitative approaches to poverty measurement and analysis: the practice and the potential*. The World Bank.

Castillo, J.J. (2009) *Convenience sampling*. Available from: http://explorable.com/convenience-sampling [Accessed 10 February 2016].

Cazorla, I.M. (2006) *Teaching statistics in Brazil*. Brazil: ICOTS-7. Available from: http://iase-web.org/documents/papers/icots7/9A2_CAZO.pdf [Accessed 27 November 2016].

Central Department of Statistics and Information (2015) *Saudi Arabia*. Available from: http://www.cdsi.gov.sa/english [Accessed 27 November 2016].

Chapman, O. (2017) Understanding and enhancing teachers' knowledge for teaching mathematics. *Journal of Mathematics Teacher Education (JMTE)*, 20, pp.303 - 307.

Chevallard, Y. (2005) Steps towards a new epistemology in mathematics education. In: M. Bosch (Ed.), *European research in mathematics education IV. Proceedings of CERME4* (pp. 21-30): Fundemi -IQS, Universitat Ramon Llull.

Clement, L.L. (2001) What do students really know about functions*. Connecting Research to Teaching,* 94, pp 745-748.

Cochran, K. F., King, R. A., & DeRuiter, J. A. (1991) Pedagogical content knowledge: A tentative model for teacher preparation. *Paper presented at the Annual Meeting of the American Educational Research Association*, Chicago, IL.

Cochran, K.F., DeRuiter, J.A. and King, R.A., (1993). Pedagogical content knowing: An integrative model for teacher preparation. *Journal of Teacher Education*, *44*(4), pp.263-272.

Codó, E. (2008) Interviews and questionnaires In: L. Wei and M. Moyer (eds.), *The Blackwell guide to research methods in bilingualism and multilingualism.* Malden, MA: Oxford, pp.158-176.

Cohen, D., & Manion, L. (1994) *Research methods in education* (4th ed.). London: Routledge.

Cohen, L., Manion, L., & Morrison, K. (2011) *Research methods in education.* (7th ed.). London: Routledge.

Cohen, L., Manion, L. & Morrison, K. (2011) *Research methods in education* (7th ed) Chapter 9, pp.143-164.

Cohen, L., Manion, L. and Morrison, K. (2013) *Research methods in education,* 7th ed. New York: Routledge.

Cooney, T. and Wiegel, H. (2003) Examining the mathematics in mathematics teacher education. In: A., Bishop, M.A. Clements, C. Keitel, J. Kilpatrick and F. Leung (eds.), *Second International Handbook of Mathematics Education.* Dordrecht: Kluwer Academic Publishers, pp.795-828.

Cornu, B. (1991) Limits. In D.O. Tall (Ed.), *Advanced mathematical thinking*. Dordrecht: Kluwer, pp.153-166.

Coupland, M., Dunn, P. K., Galligan, L., Oates, G., & Trenholm, S. (2016, in press). Tertiary Mathematics Education. In K. Makar, S. Dole, J. Visnovska, M. Goos, A. Bennison & K. Fry (Eds.), Research in mathematics education in Australasia: 2012–2015 (RiMEA). Rotterdam: Sense Publishers.

Crasnow, S. (2011) Evidence for use: Causal pluralism and the role of case studies in political science research. *Philosophy of the Social Sciences, 41*(1), pp.26–49.

Creswell, J. W. (2014) *Research design: qualitative, quantitative and mixed methods approaches.* (4th ed.). Lincoln: Sage Publications, Inc.

Creswell, J. (2003) *Research design: Qualitative, quantitative, and mixed methods approaches* (2nd edition). Thousand Oaks, CA: Sage.

Creswell, J. et al. (2003) Advanced mixed methods research designs. In: A. Tashakkori and C. Teddlie (eds), *Handbook of mixed methods in social and behavioural research.* Thousand Oaks, CA: Sage.

Creswell, J. W. (2013) *Research design: Qualitative, quantitative and mixed methods approaches*, 4th ed. Thousand Oaks, CA: SAGE.

Creswell, J. W. (2014). *Research design: qualitative, quantitative and mixed methods approaches.* (4th ed.). Lincoln: Sage Publications, Inc.

Creswell, J.W., Hanson, W.E., Plano, V.L.C. and Morales, A. (2007) Qualitative research designs selection and implementation. *The Counseling Psychologist*, *35*(2), pp.236-264.

Croll, P. (1986) *Systematic classroom observation*. Lewes: Falmer Press.

Crotty, M. (1998) *The foundations of social research: Meaning and perspective in the research process*. Thousand Oaks, CA: SAGE.

Crow, G., Wiles, R., Heath, S. and Charles, V. (2006) Research ethics and data quality: The implications of informed consent. *International Journal of Social Research Methodology*, *9*(2), pp.83-95.

Cunningham, J. (1997) Case study principles for different types of cases. *Quality and Quantity, 31*(4), pp.401-423.

Darandari, E.Z., Al-Qahtani, S.A., Allen, I.D., Al-Yafi, W.A., Al-Sudairi, A.A. and Catapang, J. (2009) The quality assurance system for post-secondary education in Saudi Arabia: A comprehensive, developmental and unified approach. *Quality in Higher Education*, *15*(1), pp.39-50.

Darder, A. and Baltodano, M. (2003) *The critical pedagogy reader*. Psychology Press.

Davis, R.B. & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, 5, pp.281-303.

De Lange, J. (2003) Mathematics for literacy. In: The National Council on Education and the Disciplines. *Quantitative literacy: Why numeracy matters for schools and colleges*. NCED, p.80.

De Vaus, D. (1993) *Surveys in social research,* 3rd ed. London: UCL Press.

DeGeorge, B. and Santoro, A. M. (2004) Manipulatives: A hands-on approach to math. *Principal*, *84*(2), pp.28-28.

Denscombe, M. (2003) *The good research guide for small-scale research projects*, 2nd ed. Buckingham: Open University Press

Denzin, N., and Lincoln, Y. (eds.) (1994) *Handbook of qualitative research*. Thousand Oaks, CA: SAGE.

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teacher and Teacher Education*, 34, pp.12–25.

Domegan, C. and Fleming, D. (2007) *Marketing research in Ireland: Theory and practice*, 2nd ed. Dublin: Gill & Macmillan.

Doorman, M. and Van Maanen, J. (2008) A historical perspective on teaching and learning calculus. *Australian Senior Mathematics Journal, 22*(2), pp.4-14.

Dossey, J. A. (1992) The nature of mathematics: Its role and its influence. In: D. A. Grouws (Ed.) (1992), *Handbook of Research on Mathematics Teaching and Learning,* pp. 39 - 49, Reston, National Council of Teachers of Mathematics.

Drever, E. (2003) *Using semi-structured interviews in small-scale research: A teacher's guide.* Glasgow: SCRE Centre, University of Glasgow.

Driscoll, M.P. (1994) Psychology of Learning for Instruction. Needham, MA: Allyn and Bacon.

Dubinsky, E. & Wilson, R. T. (2013) High schools students' understanding of the function concept. *The Journal of Mathematical Behavior,* 32, pp 83-101.

Duff, P. (2008) *Case study research in applied linguistics.* New York: Lawrence Erlbaum Associates.

Duling, E.B. (1992) The development of pedagogical-content knowledge: Two case studies of exemplary general music teachers. PhD thesis: The Ohio State University.

Eichler, A., & Erens, R. (2014) Teachers' beliefs towards teaching calculus. *ZDM Mathematics Education*, 46, pp.647–659.

Eisenhardt, K. (1989) Building theories from case study research. *Academy of Management Review, 14*(4), pp.532–550.

Eisenhardt, K.M. and Graebner, M.E. (2007) Theory building from cases: Opportunities and challenges. *Academy of Management Journal*, *50*(1), pp.25-32.

Escudero, I., & Sánchez, V (2007) How do domains of knowledge integrate into mathematics teachers' practice? *Journal of Mathematical Behavior*, 26, pp.312-327.

ESRC (2009) *Taking stock: A summary of ESRC work to evaluate the impact of research on policy and practice*. Available from: http://www.esrc.ac.uk/_images/Taking%20Stock_tcm8-4545.pdf [Accessed 2 November 2016].

Etikan, I., Musa, S., & Alkassim, R. (2016) Comparison convenience sampling and purposive sampling. *American Journal of Theoretical and Applied Statistics*, 5(1), pp.1-4.

Even, R. (1990) Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*(6), pp.521-544.

Even, R. (1993) Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *94*-116.

Fan, L. (2014) *Investigating the pedagogy of mathematics: How do teachers develop their knowledge?* Singapore: World Scientific.

Felicio, D.M. and Pieniadz, J. (1999) Ethics in higher education: Red flags and grey areas. *Feminism & Psychology*, *9*(1), pp.53-73.

Fennema, E., & Franke, M. L. (1992) Teachers' knowledge and its impact. In: D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 147-164). New York, NY, England: Macmillan Publishing Co, Inc.

Fernandez, C. (2014) Knowledge Base for Teaching and Pedagogical Content Knowledge (PCK): Some useful models and implications for teachers training. *Problems of Education in the 21st Century*, p.*60*.

Ferrini-Mundy, J. and Graham, K.G. (1991) An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*, *98*(7), pp.627-635.

Festinger L. (1957) *A Theory of cognitive dissonance. Stanford, CA: Stanford University Press.*

Finelli, C.J., Klinger, A. and Budny, D.D. (2001) Strategies for improving the classroom environment. *Journal of Engineering Education*, *90*(4), pp.491-497.

Flyvbjerg, B. (2006) Five misunderstandings about case-study research. *Qualitative Inquiry*, *12*(2), pp.219-245.

Fowler Jr, F.J. and Cosenza, C. (2009) Design and evaluation of survey questions In: L. Bickman and D. Rog, (eds.), *The SAGE handbook of applied social research methods*. London: SAGE, pp.375-412.

Fraser, S.P. (2016) Pedagogical content knowledge (PCK): Exploring its usefulness for science lecturers in higher education. *Research in Science Education, 46*(1), pp.141-161.

French, D. (2005) Subject Knowledge and Pedagogical Knowledge, in Where will the next generation of UK mathematicians come from? Available from: http://www.maths.manchester.ac.uk/~avb/pdf/DougFrenchSubjectKnowledge.pdf - [Accessed 8 April 2019].

Gall, M.D., Borg, W.R. and Gall, J.P. (2003) Case study research. In: M. Gall, D. Borg, R. Walter and J.P. Gall, (eds.), *Educational research: An introduction,* 8th ed. White Plains, NY: Longman, pp.123-163.

Geddis, A.N. (1993) Transforming subject-matter knowledge: The role of pedagogical content knowledge in learning to reflect on teaching. *International Journal of Science Education*, *15*(6), pp.673-683.

Gess-Newsome, J. (1999). Pedagogical content knowledge: an introduction and orientation. In J. Gess-Newsome, & N. G. Lederman (Eds.), *Examining pedagogical content knowledge.* Dordrecht, Netherlands: Kluwer, pp.3-17.

Goldhaber, D. (2002) The mystery of good teaching. *Education Next*, *2*(1), pp.50-55.

Gredler, *E.M. (1997) Learning and instruction. Theory into practice. New Jersey: Merrill.*

Greene, J., Caracelli, V. and Graham, W. (1989) Toward a conceptual framework for mixed-method evaluation designs. *Educational Evaluation and Policy Analysis, 11*(3), pp.255–74.

Greene, J.C. and Caracelli, V.J. (1997) Defining and describing the paradigm issue in mixed method evaluation. *New Directions for Evaluation*, *1997*(74), pp.5-17.

Gregory, I. (2003) *Ethics in research*. London: A&C Black.

Grossman, P.L. (1990) *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.

Grossman, P.L. and Richert, A.E. (1988) Unacknowledged knowledge growth: A re-examination of the effects of teacher education. *Teaching and Teacher Education*, *4*(1), pp.53-62.

Grossman, P.L. and Yerian, S.Y. (1992) *Pedagogical content knowledge: The research agenda*. San Francisco: American Educational Research Association.

Grossman, P.L., Wilson, S. M. and Shulman, L. S. (1989) Teachers of substance: Subject matter knowledge for teaching In: M. Reynolds, (ed.) *The knowledge base for beginning teachers.* New York: Pergamon, pp.23-36.

Gruenwald, N., & Klymchuk, S. (2003) Using counter-examples in teaching calculus: Students' attitudes. *The New Zealand Mathematics Magazine, 40*(2), pp.33-41.

Gueudet, G. (2008) Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67(3), pp.237–254.

Gueudet, G. (2015) University teachers' resources and documentation work. In K. Krainer & N. Vondrová (Eds.), *Proceedings of CERME9*. Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME, pp. 2138–2144.

Gyöngyösi, E., Solovej, J.P. & Winsløw, C. (2011) Using CAS based work to ease the transition from calculus to real analysis. In: *Proceedings of CERME*, Vol. 7, pp.2002-2011.

Habre, S. and Abboud, M. (2006) Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior*, *25*(1), pp.57-72.

Hadiyanti, L.N., Widodo, A., Rochintaniawati, D. and Riandi, R. (2014) *Pedagogical content knowledge of experienced and prospective biology teachers*. UPI-UPSI Conference, At Bandung, June 2014. Volume: 2014. Conference Paper (PDF Available) Available from: https://www.researchgate.net/publication/282901356_PEDAGOGICAL_CONTENT_KNOWLEDGE_OF_EXPERIENCED_AND_PROSPECTIVE_BIOLOGY_TEACHERS [Accessed Jan 29 2018].

Halim, L. and Meerah, S.M.M. (2002) Science trainee teachers' pedagogical content knowledge and its influence on physics teaching. *Research in Science & Technological Education*, *20*(2), pp.215-225.

Hamdan, A. (2005) Women and education in Saudi Arabia: Challenges and achievements. *International Education Journal*, *6*(1), pp.42-64.

Hammersley, M. (1987) Some notes on the terms *validity* and *reliability*. *British Educational Research Journal, 13*(1), pp.73-81.

Hancock, D. and Algozzine, B. (2006) *Doing case study research: A practical guide.* New York: Teachers College Press.

Hancock, D.R. and Algozzine, B. (2017) *Doing case study research: A practical guide for beginning researchers*. Teachers College Press

Handal, B. (2003) Teachers' mathematical beliefs: A review. *The Mathematics Educator*, 13(2).

Handelsman, J., Ebert-May, D., Beichner, R., Bruns, P., Chang, A., DeHaan, R., Gentile, J., Lauffer, S., Stewart, J., Tilghman, S.M. and Wood, W.B. (2004). Scientific teaching. *Science, 304*(5670), pp.521-522.

Hannah, J., Stewart, S. and Thomas, M. (2011) Analysing lecturer practice: the role of orientations and goals. *International Journal of Mathematical Education in Science and Technology*, *42*(7), pp.975-984.

Harcharras, A., & Mitrea, D. (2007). *Calculus connections: mathematics for middle school teachers*. Upper Saddle River, NJ: Pearson.

Harrison, H. et al. (2017) Case study research: Foundations and methodological orientations. *FQS: Forum: Qualitative social research sozialforum*, *18*(1), p.19.

Hartley, J. (2004) Case study research. In: C. Cassell and G. Symon, (eds.), *Essential guide to qualitative methods in organizational research.* London: Sage, 323-333.

Hawkins, J.M., Frank, B.W., Thompson, J.R., Wittmann, M.C. and Weymss, T.M., (2012 February) Probing student understanding with alternative questioning strategies. In: *AIP Conference Proceedings, 1413*(1), pp.207-210.

Hayward, C. N., Kogan, M., & Laursen, S. L. (2015) Facilitating instructor adoption of inquiry-based learning in college mathematics. *International Journal of Research in Undergraduate Mathematics Education*, pp 1–24.

Hazzi, O. A., & Maldaon, I. (2015) A pilot study: vital methodological issues. *Verslas: Teorija ir praktika / Business: Theory and Practice,* 16(1), pp.53-62 http://dx.doi.org/10.3846/btp.2015.437.

Henze, I. and Van Driel, J.H. (2015) Toward a more comprehensive way to capture PCK in its complexity. *Re-examining pedagogical content knowledge in science education,* pp.120-134.

Hiebert, J. and Grouws, D. A. (2007) The effect of classroom mathematics teaching on students' learning. In: F.K. Lester (ed.), *Second handbook of research on mathematics teaching and learning.* Charlotte, NC: Information Age Publishing, pp.371-404.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008) Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), pp.372-400.

Hill, H. C., Rowan, B. and Ball, D. L. (2005) Effects on teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42, pp.371-406.

Hill, H. C., Schilling, S. G., Loewenberg and Ball, D. (2004) Developing Measures of Teachers' Mathematics Knowledge for Teaching. The *Elementary School Journal*, *105*(1), pp.11-30.

Hitt, F. and González-Martín, A.S. (2016) Generalization, covariation, functions, and calculus. In: *The Second Handbook of Research on the Psychology of Mathematics Education* New York Rotterdam, Sense Publishers, pp.3-38.

Hodgen, J. (2011). Knowing and identity: a situated theory of mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 27-42). London and New York: Springer.

Holton, D. (ed.) (2001) *The teaching and learning of mathematics at university level: An ICMI study,* vol. 7. New York: Springer Science & Business Media.

Huillet, D. (2009). Mathematics for teaching: an anthropological approach and its use in teacher training. *For the Learning of Mathematics*, 29(3), pp.4-10.

Hu, Jing-Jing (2014) A critical review of Pedagogical Content Knowledge' components: nature, principle and trend. *International Journal of Education and Research*, *2*(4), 411 - 424. Available from: https://www.ijern.com/journal/April-2014/36.pdf - [Accessed 10 March 2019].

Ijeh, S.B. (2012) How competent mathematics teachers develop pedagogical content knowledge in statistics teaching. PhD thesis:.University of Pretoria.

Jaworski, B., Potari, D., Rasmussen, C., Oates, G. and Kwon, O.N. (2016) Mathematics learning and teaching at university level. In *40th Conference of the International Group for the Psychology of Mathematics Education,* Vol. 1, pp.375-381.

Jaworski, B., Treffert-Thomas, S., & Bartsch, T. (2009) Characterising the teaching of university mathematics: A case of linear algebra.. In: M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (3, pp. 249–256). Thessaloniki, Greece: PME.

Johnson, E., Ellis, J. and Rasmussen, C. (2016) It's about time: the relationships between coverage and instructional practices in college calculus. *International Journal of Mathematical Education in Science and Technology*, *47*(4), pp.491-504.

Jones, I. and Alcock, L. (2014) Peer assessment without assessment criteria. *Studies in Higher Education*, *39*(10), pp.1774-1787.

Jones, T. and Cuthrell, K. (2011) YouTube: Educational potentials and pitfalls. *Computers in the Schools*, *28*(1), pp.75-85.

Jong, O.D. (2003) Exploring science teachers' PCK. In: D. Psillos, P. Karotoglou and V. Tselfeset (eds.), *Science education research in the knowledge-based society*. Dordrecht: Kluwer Academic, pp.373–381.

Jong, O.D., Van Driel, J.H. and Verloop, N. (2005) Preservice teachers' pedagogical content knowledge of using particle models in teaching chemistry. *Journal for Research in Science Teaching*, 42(8), pp.947-964.

Kabael, T. (2010) Cognitive development of applying the chain rule through three worlds of mathematics. *Australian Senior Mathematics Journal*, *24*(2), pp.14-28.

Kahan, J.A., Cooper, D.A. and Bethea, K.A. (2003) The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, *6*(3), pp.223-252.

Kaput, J.J., Schoenfeld, A.H. and Dubinsky, E. (1998) *Research in collegiate mathematics education II.*, vol. 4. Providence, RI: American Mathematical Society.

Kashefi, H., Ismail, Z. and Yusof, Y.M. (2012) Overcoming students obstacles in multivariable calculus through blended learning: A mathematical thinking approach. *Procedia-Social and Behavioral Sciences*, 56, pp.579-586.

Kashefi, H., Zaleha I. and Yudariah M. Y. (2012a) Using blended learning to support students thinking powers in multivariable calculus. *Jurnal Teknologi*, 57 (Sains Sosial), pp.1-14.

Kathirveloo, P. and Puteh, M. (2014) Effective teaching: Pedagogical Content Knowledge. In: *Proceeding of International Joint Seminar Garut, Garut, Indonesia, 21 September 2014.* Available from: https://www.researchgate.net/publication/303940850 - [Accessed 8 Jan 2019].

Kennedy, M.C., Anderson, C.W., Conti, S. and O'Hagan, A. (2006) Case studies in Gaussian process modelling of computer codes. *Reliability Engineering & System Safety*, *91*(10), pp.1301-1309.

Kerlinger, F. N. (1986) *Foundations of behavioral research* (3rd ed.). San Diego, CA: Harcourt Brace.

Khakbaz, A. (2016) Mathematics university teachers' perception of pedagogical content knowledge (PCK). *International Journal of Mathematical Education in Science and Technology*, *47*(2), pp.185-196.

Kidron, I. (2014) Calculus teaching and learning. In: Encyclopedia of mathematics education. *Springer Netherlands,* pp.69-75.

Kirk, J. and Miller, M. (1986) *Reliability, validity and qualitative research.* Beverly Hills, CA: Sage.

Klymchuk, S. (2005) Counter-examples in teaching/learning of calculus: students' performance. *The New Zealand Mathematics Magazine, 42(1), pp.31-38*.

Klymchuk, S. (2010). *Counterexamples in Calculus.* Washington, DC: MAA.

Klymchuk, S. (2012) Using Counter-Examples in teaching & learning of calculus: Students' attitudes and performance. *Mathematics teaching-research journal online*, *5*(4).

Klymchuk, S. (2014). Experience with using counterexamples in an introductory calculus class. *International Journal of Mathematical Education in Science and Technology*, 45(8), pp.1260-1265.

Klymchuk, S., 2016, July. Using Counterexamples, Puzzles and Provocations in the Teaching and Learning of Calculus. In *13th International Congress on Mathematical Education (ICME-13)*. Springer.

Kohlbacher, F. (2006) The use of qualitative content analysis in case study research. *FQS: Forum: Qualitative Social Research Sozialforschung, 7*(1), p.21.

König, J., Blömeke, S., Paine, L., Schmidt, W. H. and Hsieh, F. J. (2011) General pedagogical knowledge of future middle school teachers: On the complex ecology of teacher education in the United States, Germany, and Taiwan. *Journal of Teacher Education*, *62*(2), pp.188-201.

Krauss, S., Baumert, J. and Blum, W. (2008) Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM*, *40*(5), pp.873-892.

Krieger, Z. (2007) Saudi Arabia puts its billions behind western-style higher education. *The Chronicle of Higher Education*, *54*(3), A1.

Kumar, R. (2011) *Research Methodology: A step –by -step guide for beginners.* London: SAGE Publications Ltd.

Kumsa, A., Pettersson, K. and Andrews, P. (2017) Obstacles to students' understanding of the limit concept. In: *10th Congress of European Research in Mathematics Education.* CERME10*.*

Kung, D. T. (2010) Teaching assistants learning how students think. In F. Hitt, D. Holton & P.W. Thompson (Eds.), *Research in collegiate mathematics education VII,* pp.143-169.

Kvale, S. (2008) *Doing interviews*. London: SAGE.

Lachner, A. and Nückles, M. (2016) Tell me why! content knowledge predicts process-orientation of math researchers' and math teachers' explanations. *Instructional Science*, *44*(3), pp.221-242.

Lado, R. (1964). *Language teaching: A scientific approach*. London: McGraw Hill.

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017) Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526-550). Reston VA: National Council of Teachers of Mathematics

Lawson, A. E., Abraham, M. R., & Renner, J. W. (1989) A theory of instruction: Using the learning cycle to teach science concepts and thinking skills. *Monograph of the National Association for Research in Science Teaching, No.l,* Cincinnati, OH, NARST.

Leinhardt, G. and Smith, D.A. (1985) Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), pp.247-271.

Lesseig, K. (2016) Investigating mathematical knowledge for teaching proof in professional development. *International Journal of Research in Education and Science (IJRES)*, *2*(2), 253-270. Available from: https://files.eric.ed.gov/fulltext/EJ1105101.pdf - [Accessed 11 Dec 2018].

Lincoln, Y., Lynham, S. and Guba, E. (2011) Paradigmatic controversies, contradictions, and emerging confluences, revisited in qualitative research. In: N. Denzin and Y. Lincoln (Eds.), *The Sage handbook of* qualitative *research* (4th ed., pp.97-128). Thousand Oaks, CA: Sage.

Lincoln, Y.S. and Guba, E.G. (2000) Paradigmatic controversies, contradictions, and emerging influences. In: Lortie, D. (1975) *Schoolteacher: A Sociological Study*. London: University of Chicago Press, pp.97-128.

Lincoln, Y. S., & Guba, E. G. (1985) *Naturalistic inquiry*. Beverly Hills, CA: Sage.

Lodico, M.G., Spaulding, D.T. and Voegtle, K.H. (2010) *Methods in educational research: From theory to practice.* US: Jossey-Bass.

Loughran, J., Mulhall, P. and Berry, A. (2008) Exploring pedagogical content knowledge in science teacher education*. International Journal of Science Education, 30*(10), pp.1301-1320.

Maat, S. M., Zakaria, E., & Yusof, Y. M. (2011) Exploring mathematics teachers' pedagogical content knowledge of calculus in a technical institute. *International Journal of Learning*, 18(1), pp.25-34.

Maciejewski, W. and Star, J.R. (2016) Developing flexible procedural knowledge in undergraduate calculus. *Research in Mathematics Education*, *18*(3), pp.299-316.

Macmillan, E. D. (2007) *Macmillan English Dictionary*. Macmillan Education, 2nd edition.

Magnusson, S., Krajcik, J. and Borko, H. (1999) Nature, sources, and development of pedagogical content knowledge for science teaching*. In examining pedagogical content knowledge*. *Springer Netherlands* pp.95-132.

Mali, A., Biza, I., & Jaworski, B. (2014) Characteristics of university mathematics teaching: Use of generic examples in tutoring. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME38 and PME-NA36* (Vol. 4, pp. 161–168). Vancouver, Canada: PME.

Manouchehri, A. (1997) School mathematics reform: Implementations for mathematics teacher preparation. *Journal of Teacher Education, 48*(3), pp.197–209.

Mansour, N., Alshamrani, S.M., Aldahmash, A.H. and Alqudah, B.M. (2013) Saudi Arabian science teachers and supervisors' views of professional development needs. *Eurasian Journal of Educational Research*, 51, pp.29-44.

Marks, R. (1990) Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education*, *41*(3), pp.3-11.

Martinez-Luaces, V. & Noh, S. (2015) Teaching and Learning of Calculus. In: The Proceedings of the 12th International Congress on Mathematical Education. *Springer International Publishing,* pp.447-451.

Mason, J. (2008) PCK and beyond. In P. Sullivan, & T. Wood (Eds.), *International handbook of mathematics teacher education: Vol. 1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 301e322). Rotterdam: Sense Publishers.

Mathison, S. (1988) Why triangulate? *Educational Researcher*, *17*(2), pp.13-17.

McCall, G., (1984) Systematic field observation. *Annual review of sociology*, 10(1), pp.263-282.

McCallum, W. (2000) The goals of the calculus course. In S. Ganter (Ed.), *Calculus renewal: Issues for undergraduate mathematics education in the next decade* (pp. 11-22). New York, NY: Plenum Press.

McNamara, C. (2009) *General guidelines for conducting interviews*. Available from: http://managementhelp.org/businessresearch/interviews.htm [Accessed 18 January 2017].

Mead, G. (1934) *Mind, self and society.* Chicago, IL: University of Chicago.

Melibari, G. (2016) ELT teaching quality and practice in Saudi Arabia: A case study of the perspectives of ESP and EGP students, teachers and managers at the ELC in Umm al-Qura University. *University of Hertfordshire, Doctoral Thesis.*

Merriam, S. (1998) *Qualitative research and case study applications in education* (2nd ed.). San Francisco, CA: Jossey-Bass Publishers.

Merriam, S. B. (2009) *Qualitative research: A guide to design and implementation.* San Francisco, CA: Jossey-Bass.

Merriam, S.B. and Bierema, L.L. (2013) *Adult learning: Linking theory and practice*. John Wiley & Sons.

Mesa, V. & Burn, H. (2015) The calculus curriculum. In: D. Bressoud, V. Mesa and C. Rasmussen (2015) Insights and recommendations from the MAA National Study of College Calculus. *MAA Notes. Washington, DC: Mathematical Association of America*. in the USA.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014) *Qualitative data analysis: A methods sourcebook* (3rd ed.). Los Angeles, CA: Sage.

Miller, L.M. (2006) Pedagogical content knowledge. In: G.M. Bodner, and M. Orgill (eds.), *Theoretical frameworks for researchers in chemistry/science education.* Upper Saddle River, New Jersey: Pearson Education, pp.86-106

Mills, M. (2013) A multidimensional analysis of instructor questions in advanced mathematics lectures. Proceedings from RUME 16: *Research in Undergraduate Mathematics Education.* Denver: Colorado.

Ministry of Education (MoE) (n.d.) *University Higher Education*. Available from: www.moe.gov.sa /en/studyinside/Government-Universities/Pages/default.aspx [Accessed 10 December 2017].

Mishra, P. and Koehler, M. J. (2006) Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, 108(6), pp.1017.

Moran-Ellis, J., Alexander, V.D., Cronin, A., Dickinson, M., Fielding, J., Sleney, J. and Thomas, H., (2006) Triangulation and integration: processes, claims and implications. *Qualitative research*, *6*(1), pp.45-59.

Morgan, D. (1998) Practical strategies for combining qualitative and quantitative methods: Applications for health research. *Qualitative Health Research,* 8, pp.362–76.

Morgan, C. (2014) Understanding practices in mathematics education: structure and text. *Educational Studies in Mathematics, 87*, pp.129-143.

Morse, J. (1991) Approaches to qualitative-quantitative methodological triangulation. *Nursing Research, 40*(2), pp.20–3.

Mouly, G. (1978) *Educational research: The art and science of investigation*. Boston, Massachusetts: Allyn and Bacon, Inc.

Muijs, D. and Reynolds, D. (2000) School effectiveness and teachers' effectiveness in mathematics: Some preliminary findings from evaluation of the mathematics enhancement programme (primary). *School Effectiveness and Improvement*, *11*(3), pp.273-303.

Mullis, I. V. S., Martin, M. O. (eds.) (2017) TIMMSS 2019 Assessment Frameworks, Boston College and International Association for the Evaluation of Educational Achievement (IEA). Available from: http://timss2019.org/wp-content/uploads/frameworks/T19-Assessment-Frameworks.pdf - [Accessed 19 April 2019].

Muzangwa, J. and Chifamba, P., (2012) Analysis of Errors and Misconceptions in the Learning of Calculus by Undergraduate Students. *Acta Didactica Napocensia*, *5*(2), pp.1-10.

Myers, M. D. (2009) *Qualitative research in business and management*. London: SAGE.

Nardi, E., Ryve, A., Stadler, E. & Viirman, O. (2014) Commognitive analyses of the learning and teaching of mathematics at university level: The case of discursive shifts in the study of Calculus. *Research in Mathematics Education*, *16*(2), pp.182-198.

Nardi, N., Jaworski, B., & Hegedus, S. (2005) A Spectrum of Pedagogical Awareness for Undergraduate Mathematics: From tricks to techniques. *Journal for Research in Mathematics Education,* 36, pp.284-316.

Nardi, E. (2008) *Amongst mathematicians: Teaching and learning mathematics at university level*. New York, NY: Springer.

Nardi, E. (2011) 'Driving noticing' yet 'risking precision': University mathematicians' pedagogical perspectives on verbalisation in mathematics. In: M. Pytlak, T. Rowland, and E. Swoboda (Eds.), *Proceedings of CERME7 (7th Conference of European Research in Mathematics Education)* (pp. 2053–2062). Rzeszów, Poland: ERME.

Neill, H., & Shuard, H. (1982) *Teaching calculus*. Glasgow: Blackie.

Neuman, W. (1997) *Social research methods, qualitative and quantitative approaches* (3rd ed.). Boston: Allyn and Bacon.

Neuman, W.L. (2007) *Basics of social research: Qualitative and quantitative approaches* (2nd ed.). Canada: Pearson Education.

Ng, S.F. (1995) Malaysian pre-service primary mathematics teachers and their lecturers: Practice and beliefs about mathematics, teaching and learning. *University of Birmingham, Doctoral dissertation.* Available at online at http://ethos.bl.uk/OrderDetails.do?did=1&uin=uk.bl.ethos.365276 [Accessed 17 March, 2017].

Niess, M. L. (2005) Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education*, *21*(5), pp.509-523.

Nisbett, R. E. and Ross, L. (1980) *Human inference: Strategies and shortcomings of social judgment*. Englewood Cliffs, NJ: Prentice Hall.

Nordin, H., Davis, N. and Ariffin, T.F.T. (2013) A case study of secondary pre-service teachers' Technological Pedagogical and Content Knowledge mastery Level. *Procedia-Social and Behavioral Sciences*, *103*, pp.1-9.

O'Donoghue, T. (2006) *Planning your qualitative research project: An introduction to interpretivist research in education.* London: Routledge.

O'Leary, Z. (2017) *The essential guide to doing your research project*. Sage.

Oliver, P. *and* Jupp, V. (2006*)* Purposive sampling. In*:* V. Jupp (ed.), T*he SAGE Dictionary of Social Research Methods.* London*: SAGE*, pp. 245-246*.*

Olivier, A. (1989) *Handling pupils' misconceptions*. Presidential address delivered at the Thirteenth National Convention on Mathematics, Physical Science, and Biology Education, Pretoria, South Africa, 3-7 July 1989.

Ostebee, A., & Zorn, P. (2002). *Calculus from graphical, numerical, and symbolic points of view*. Belmont, CA: Brooks/Cole.

Palaiologos, G.T. (2011) *From pedagogy to andragogy and heutagogy: Thinking distance education and self-directed learning*. Creative Commons.

Palys, T. (2008) Purposive sampling. In: L. Given (ed.), *The SAGE encyclopedia of qualitative research methods*, (2nd ed.). London: SAGE, pp.697-698.

Park, S. and Oliver, J.S., (2008) Revisiting the conceptualization of pedagogical content knowledge (PCK): PCK as a conceptual tool to understand teachers as professionals. *Research in science Education*, *38*(3), pp.261-284.

Patton, M. (2002) *Qualitative research and evaluation methods*. London: Sage. *Practitioner-Researchers*. Mass.: Blackwell.

Peled, I. and Zaslavsky, O. (1997) Counter-examples that (only) prove and counter-examples that (also) explain. *FOCUS on Learning Problems in mathematics*, *19*(3), pp.49-61.

Pellissier, R. (2008) *Business research made easy*. Cape Town: Juta and Company Ltd.

Penso. S. (2002) Pedagogical content knowledge: How do student teachers identify and describe the causes of their pupils' learning difficulties? *Asia-Pacific Journal of Teacher Education*, *30*(1), pp.25-37.

Petropoulou, G., Jaworski, B., Potari, D. and Zachariades, T. (2016) How do research mathematicians teach Calculus? In: K. Krainer, and N. Vondrová, (eds.), *CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education*, Prague, Czech Republic, February 2015, pp.2221-2227.

Petrou, M. and Goulding, M. (2011) Conceptualising teachers' mathematical knowledge in teaching In: T. Rowland and K. Ruthven (eds.), *Mathematical knowledge in teaching*. Netherlands: Springer, pp.9-25.

Philipp, R.A., Ambrose, R., Lamb, L.L., Sowder, J.T., Schappelle, B.P., Sowder, L., Thanheiser, E. and Chauvot, J. (2007) Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, *38*(5), pp.438-476.

Plotz, M. (2007) *Criteria for effective mathematics teacher education with regard to mathematical content knowledge for teaching*, PhD dissertation, North-West University.

Polkinghorne, D.E. (2005) Language and meaning: Data collection in qualitative research. *Journal of Counseling Psychology*, *52*(2), pp.137.

Potari, D., Zachariades, T., Christou, K., Kyriazis, G. and Pitta-Padazi, D. (2007) Teachers' mathematical and pedagogical awareness in calculus teaching. In: S. Alatorre, J. Cortina, M. Saiz, and A. Mendez, (2007) *Twenty Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico, 9-12 November 2006. Mérida, Mexico: Universidad Pedagógica Nacional, pp.846-849.

Power, M.R. (1998) *Non-verbal communication*. New York: McGraw Hill.

Pritchard, D. (2015). Lectures and transition: from bottles to bonfires? In Croft, A., Grove, M., Kyle, J. & Lawson, D. (Eds.) *Transitions in Undergraduate Mathematics Education*, 5, pp. 57-69. University of Birmingham, Birmingham.

Prokop, M. (2003) Saudi Arabia: The politics of education. *International Affairs*, *79*(1), pp.77-89.

Przenioslo, M. (2004) Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics, 55*(1), pp.103–132.

Punch, K. (1998). *Introduction to social research: Quantitative and qualitative approaches*. London: Sage.

Rasmussen, C. (2012) *A report on a national study of college calculus: Who is switching out of STEM and why?* Plenary address at the 15th Conference on Research in Undergraduate Mathematics Education, Portland, OR, pp.23-25.

Rasmussen, C., Marrongelle, K. and Borba, M.C. (2014) Research on calculus: What do we know and where do we need to go? *ZDM*, 46(4), pp.507-515.

Raven, J. (2011) Emiratizing the education sector in the UAE: Contextualization and challenges. *Education, Business and Society: Contemporary Middle Eastern Issues*, *4*(2), pp.134-141.

Richards, K. (2003) *Qualitative inquiry in TESOL*. New York: Palgrave Macmillan.

Ritchie, J., Lewis, J., Nicholls, C.M. and Ormston, R. (eds.) (2013) *Qualitative research practice: A guide for social science students and researchers*. London: SAGE.

Robert, A. and Speer, N. (2001) Research on the teaching and learning of calculus/elementary analysis. In: D. Holton (ed.), *The teaching and learning of mathematics at university level.* Netherlands: Springer, pp.283-299.

Robson, C. (2002) *Real World Research: A Resource for Social Scientists and Practitioner-Researchers*. Mass.: Blackwell.

Rollnick, M. (2016) Learning about semi-conductors for teaching—the role played by content knowledge in pedagogical content knowledge (PCK) development. *Research in Science Education*, pp.1-36.

Rollnick, M., Bennett, J., Rhemtula, M., Dharsey, N. and Ndlovu, T. (2008) The place of subject matter knowledge and pedagogical content knowledge: A case study of South African teachers teaching the amount of substance and chemical equilibrium. *International Journal of Science Education, 30*(10), pp.1365–1387.

Rosnow, R.L. and Rosenthal, R. (2011) Ethical principles in data analysis: An overview. In: S. Sterba, and A.T. Panter (eds.), *Handbook of Ethics in Quantitative Methodology*. New York: Routledge, pp.37-58.

Rowan, B., Correnti, R. and Miller, R. J. (2002) What large-scale survey research tells us about teacher effects on student achievement: Insights from the Prospects Study of elementary schools. *Teachers College Record*, *104*(8), pp.1525-1567.

Rowley, J. (2002) Using case studies in research. *Management Research News, 25*(1), pp.16–27.

Rugh, W. A. (2002) Education in Saudi Arabia: Choices and constraints. *Middle East Policy*, *9*(2), 40.

Sabah, S., Fayez, M., Alshamrani, S., & Mansour, N. (2014) Continuing professional development (CPD) provision for science and mathematics teachers in Saudi Arabia: Perceptions and experiences of CPD providers. *Journal of Baltic Science Education*, *13*(3), pp.91–104.

Sapsford, R. and Jupp, V. (2006) *Data Collection and Analysis*. London: Sage Publications.

Savery J.R., Duffy, T.M. (1995) *Problem based learning: An instructional model and its constructivist framework. Educ Tech. 35, pp.31-38.*

Schmidt, W. H., Tatto, M., Bankov, K., Blomeke, S., Cedillo, T., Cogan, L., Han, S., Houang, R., Hsieh, F., Paine, L., Santillan, M. and Schwille, J. (2007) *The preparation gap: Teacher education for middle school mathematics in six countries*. Michigan: MSU Center for Research in Mathematics and Science Education. Available from: http://www.educ.msu.edu/content/downloads/sites/usteds/MT21Report.pdf [Accessed 3 October 2016].

Schoenfeld, A. H. (2006) Problem solving from cradle to grave. *Annales de Didactique et de Sciences Cognitives*,11, pp.41-73.

Schoenfeld, A.H. (1995) A brief biography of calculus reform. *UME Trends: News and Reports on Undergraduate Mathematics Education*, *6*(6), pp.3-5.

Sealey, V. (2014) A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, 33, pp.230–245.

Seldon, J., Mason, A., & Seldon, A. (1989) Can average calculus students solve non-routine problems? *Journal of Mathematical Behavior*, *8*(1), pp.45-50.

Senk, S.L., Peck, R.B., Bankov, K., & Tatto, M.T. (2008) *Conceptualizing and Measuring Mathematical Knowledge for Teaching: Issues from TEDS-M, an IEA Cross-National Study*. Available from: https://www.semanticscholar.org/paper/Conceptualizing-and-Measuring-Mathematical-for-from-Senk-Peck/71411a8df600ec183708053590c67c4810cd1c11 - [Accessed 23 March 2019].

Shank, G., & Brown, L. (2007). *Exploring educational research literacy*. New York: Routledge.

Shulman, L. S., Grossman, P. (1988). *Knowledge growth in teaching: A final report to the Spencer foundation,* Stanford, Calif., Stanford University.

Shulman, L. (1986) Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), pp.4-15.

Shulman, L. (1987) Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, *57*(1), pp.1–23.

Shulman, L. S., & Sykes, G. (1986) *A national board for teaching? In search of a bold standard.* Paper presented at the Task Force on Teaching as a Profession, Carnegie Forum on Education and the Economy, Stanford, CA.

Sierpinska, A. (1992) On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.) *The Concept of Function Aspects of Epistemology and Pedagogy*, *MAA Notes Volume 25* (pp 25-58). Washington, DC: Mathematical Association of America.

Sierpinska, A., (2013) *Understanding in mathematics*. Routledge.

Silverman, D. (2015) *Interpreting qualitative data*. London: SAGE.

Simons, H. (2009) *Case study research in practice*. Los Angeles, CA: Sage.

Simpson, M., & Tuson, J. (1995). *Using observations in small-scale research: A beginner's guide*. Edinburgh: Scottish Council for Research in Education. ERIC Document 394991.

Siyepu, S.W. (2009) The zone of proximal development in the learning of differential calculus. *Mathematical Knowledge for Teaching*, pp.136-148.

Skjelstad, E.E. (2009) *The influence of teacher practice on calculus students' motivation.* PhD dissertation, Texas Tech University.

Slavin R.E. (1994) Educational psychology: Theory and practice. Boston, USA: Allyn and Bacon, 4.

Smith, L. and Abouammoh, A. (2013) *Higher education in Saudi Arabia: Reforms, challenges and priorities*. Netherlands: Springer.

Smith, M.S. (2014) Tools as a catalyst for practitioners' thinking. In: Mathematics Teacher Educator, 3 (1), 3-7. Available from: .https://www.jstor.org/stable/10.5951/mathteaceduc.3.1.0003?seq=1#page_scan_tab_contents - [Accessed 11 Feb 2019].

Sofronas, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L., Hamelin, C. (2011) What does it mean for a student to understand the first-year calculus?. *Journal of Mathematical Behaviour*, 30, pp.131-148.

Sofronas, K.S. and DeFranco, T.C., (2010) An examination of the knowledge base for teaching among mathematics faculty teaching calculus in higher education. *Research in Collegiate Mathematics Education*, *7*, pp.171-206.

Sonnert, G., Sadler, P.M., Sadler, S.M. and Bressoud, D.M. (2015) The impact of instructor pedagogy on college calculus students' attitude toward mathematics. *International Journal of Mathematical Education in Science and Technology*, *46*(3), pp.370-387.

Sowder, J. T., Phillip, R. A., Armstrong, B. E. and Schappelle, B. P. (1998) *Middle-grade teachers' mathematical knowledge and its relationship to instruction: A research monograph.* Albany, NY: State University of New York.

Speer, N.M., Smith, J.P. and Horvath, A. (2010) Collegiate mathematics teaching: An unexamined practice. *The Journal of Mathematical Behavior*, *29*(2), pp.99-114.

Stacey, K. (2008). What is mathematical thinking and why is it important? In: *Progress report of the APEC project: Collaborative studies on innovations for teaching and learning mathematics in different cultures (II)-Lesson study focusing on mathematical thinking*, Tokyo: CRICED, University of Tsukuba. Available: http://marsigitpsiko.blogspot.com /2008/12/what-is-mathematical-thinking-andwhay.html, [Accessed 15 April 2019].

Stake, R. E. (2000). *Case studies.* In: N. K. Denzin, & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed., pp. 435-454). Thousand Oaks, CA: Sage Publications.

Stake, R. E. (2006) *Multiple case study analysis.* New York, NY: Guilford.

Stake, R.E. (1995) *The art of case study research*. London: SAGE.

Star, J. R. (2002). Re-conceptualizing procedural knowledge: The emergence of "intelligent" performances among equation solvers. In D. Mewborn, P. Sztajn, D. White, H. Wiegel, R. Bryant & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 999-1007). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Steffe, L.P., & Gale, J.E. (1995) *Constructivism in education. Hillside, NJ: Laurence Erlbaum.*

Stewart, A. (2014) Case study. In: Jane Mills & Melanie Birks (Eds.), *Qualitative methodology: A practical guide* (pp.145-159). Thousand Oaks, CA: Sage.

Stewart, James, (2012) Calculus: Early Transcendentals, 7th ed., Cengage Learning, Mason, Ohio

Stolk, K. (2013). *Types of questions that comprise a teacher's questioning discourse in a conceptually-oriented classroom.* (Master's thesis). Retrieved from Theses and Dissertations (Paper 3716).

Stylianides, A.J. and Stylianides, G.J. (2013) Seeking research-grounded solutions to problems of practice: Classroom-based interventions in mathematics education. *ZDM*, *45*(3), pp.333-341.

Sulaiman, N.A. (2011) *Exploring Kuwaiti mathematics student-teachers' beliefs toward using logo and mathematics education.* Nottingham Trent University, Doctoral Thesis.

Sullivan, P. (2011) *Teaching mathematics: Using research-informed strategies*. New York: Sage.

Tall, D. & Ramos, J. (2004) Reflecting on Post-Calculus-Reform. Plenary for *Topic Group 12: Calculus, International Congress of Mathematics Education*, Copenhagen, Denmark.

Tall, D. & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics,* Vol. 12, No. 2 pp.151-169.

Tall, D. (1993) Students' difficulties in calculus. In: C. Gaulin, B.R. Hodgson, D.H. Wheeler, and J. Egsgard (eds.), *Proceedings of the Seventh International Congress on Mathematical Education*. Quebec, 1993. Quebec: Les Presses de l'Université Laval, pp.13-28.

Tall, D. (2004) The three worlds of mathematics. *For the Learning of Mathematics*, 23(3), pp.29–33.

Tall, D. (2004) Thinking through three worlds of mathematics. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceeding of the 28th PME International Conference*, 4, 281-288. vol. 4, pp. 281–288.

Tall, D. O. (2008) The transition to formal thinking in mathematics. *Mathematics Education Research Journal, 20*(2), pp.5–24.

Tall, D. (2010) *A sensible approach to the calculus*. Plenary address at the National and International Meeting on the Teaching of Calculus, Puebla, Mexico, 23-25th September.

Tamir, P. (1988) Subject matter and related pedagogical knowledge in teacher education. *Teaching and Teacher Education*, *4*(2), pp.99-110.

Tarmizi, R.A., (2010) Visualizing student's difficulties in learning calculus. *Procedia-Social and Behavioral Sciences*, *8*, pp.377-383.

Tashakkori, A. and Teddlie, C. (1998) *Mixed methodology: Combining qualitative and quantitative approaches.* Thousand Oaks, CA: Sage.

Tataroğlu-Taşdan, B. & Çelik, A. (2016) A conceptual framework for examining mathematics teachers' pedagogical content in the context of supporting mathematical thinking. Online Submission. *European Journal of Education Studies,* 2(5), 90-120. Available from: https://eric.ed.gov/?id=ED583704.

Tatto, M. T., Schwille, J., Senk, S., Ingvarson, L., Peck, R., & Rowley, G. (2008*) Teacher Education and Development Study in Mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics. Conceptual framework*. East Lansing: Teacher Education and Development International Study Center, College of Education, Michigan State University.

Thomas, A. Saroyan, A. and Dauphinee W.D. (2011*) Evidence-based practice: a review of theoretical assumptions and effectiveness of teaching and assessment interventions in health professions. Adv Health Sci Educ Theory Pract., 16, 253-276. 10.1007/s10459-010-9251-6.*

Thompson, A. G. (1992) Teachers' beliefs and conceptions: A synthesis of the research. In: D.A. Grouws (ed.), *Handbook of research on mathematics teaching and learning.* Reston, VA: National Council of Teachers of Mathematics, pp.127–146.

Thompson, P.W. (1994) Students, functions, and the undergraduate curriculum. In E. Dubinsky, A.H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Issues in Mathematics Education Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

Thompson, P. W., Cheepurupalli, R., Hardin, B., Lienert, C. and Selden, A. (2010) *Cultivating symbol sense in your calculus class*. IM&E Workshop, San Diego, CA, 27-29 March.

Thurmond, V.A. (2001) The point of triangulation. *Journal of Nursing Scholarship*, *33*(3), pp.253-258.

Todorov, T.D., (2001) Back to classics: teaching limits through infinitesimals. *International Journal of Mathematical Education in Science and Technology*, *32*(1), pp 1-20.

Toerien, R. (2011) *A case study of the pedagogical content knowledge of in-service science teachers teaching organic chemistry in two South African secondary schools.* Cape Town: University of Cape Town Press.

Tongco, M.D.C. (2007) Purposive sampling as a tool for informant selection. *A Journal for Plant, People and Applied Research*, 5, pp.147-158.

Törner, G., Potari, D. and Zachariades, T. (2014) Calculus in European classrooms: curriculum and teaching in different educational and cultural contexts. *ZDM*, *46*(4), pp.549-560.

Triandis, H. C., & Berry, J. W. (1980) *Handbook of cross-cultural psychology (Vol. 2).* Boston: Allyn & Bacon. (Eds.).

Trochim, W.M., (2006) Qualitative measures. *Research measures knowledge base*, 361, pp.29-31.

U.S. Department of Education (1998) *Education 1998.* Washington, D.C.: U.S. Government Printing Office.

Van Driel, J., Verloop, N. and De Vos, W. (1998) Developing science teachers' pedagogical content knowledge. *Journal for Research in Science Teaching*, *35*(6), pp.673-695.

Van Teijlingen, E. and Hundley, V. (2002) The importance of pilot studies. *Nursing Standard*, *16*(40), pp.33-36.

Van Wynsberghe, R. and Khan, S. (2007) Redefining case study. *International Journal of Qualitative Methods*, 6(2), Article 6. Available from: http://www.ualberta.ca/~iiqm/backissues/6_2/vanwynsberghe.htm [Accessed 17 October 2016].

Verloop, N., Van Driel, J. and Meijer, P. (2001) Teacher knowledge and the knowledge base of teaching. *International journal of educational research*, *35*(5), pp.441-461.

Viirman, O. (2014). The functions of function discourse—University mathematics teaching from a commognitive standpoint. *International Journal of Mathematical Education in Science and Technology*, 45(4), pp.512–527.

Viirman, O. (2014) 'The function concept and university mathematics teaching' (Doctoral dissertation). Department of mathematics and computer science, Karlstad University.

Vincent, B., LaRue, R., Sealey, V. and Engelke, N. (2015) Calculus students' early concept images of tangent lines. *International Journal of Mathematical Education in Science and Technology*, *46*(5), pp.641-657.

Vistro-Yu, C.P. (2005) On pedagogical knowledge in mathematics: How secondary school mathematics teachers face the challenge of teaching a new class. *International Journal for Mathematics Teaching and Learning,* October 25th. Available from: http://www.cimt.org.uk/journal/vistroyu.pdf [Accessed 4 November 2016].

Volante, L. (2016) *The intersection of international achievement testing and educational policy: Global perspectives on large-scale reform*. New York: Routledge.

Von Glaserfeld, E. (1995) A constructivist approach to teaching. Constructivism in education. In: L.P, Steffe and J.E. Gale (eds). Hillside, NJ: Lawrence Erlbraum Associates, pp.3-15.

Vygotsky, L. (1978) Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

Vygotsky, L. (1980) *Mind in Society*, Cambridge (Massachusetts), Harvard University Press.

Vygotsky, L. (1987) Zone of proximal development. *Mind in society: The development of higher psychological processes*, *5291*, p.157.

Wagner, E.R., Orme, S.M., Turner, H.J. and Yopp, D. (2017) Encouraging example generation: A teaching experiment in first-semester calculus. *PRIMUS*, *27*(2), 212-234.

Wagner, J. F., Speer, N. M., Rossa, B. (2007) Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course, in: *Journal of Mathematical Behaviour,* 26, pp.247 – 266.

Watson, J., Callingham, R., and Donne, J. (2008) Establishing PCK for teaching statistics. In: C. Batanero, G. Burrill, C. Reading, and A. Rossman, (eds.), *Joint ICME/IASE Study: Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education,* Monterrey, Mexico, July 2008. Available from: http://www.ugr.es/~icmi/iase_study/ [Accessed 4 November 2016].

Weber, K. (2004) Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*(2), pp.115-133.

Weber, K., (2015) Mathematics professors and mathematics majors' expectations of lectures in advanced mathematics," AMS Blog on Teaching and Learning Mathematics. Retrieved from blogs.ams.org/matheducation/2015/02/10/mathematics-professors-and-mathematics-majors-expectations-of-lectures-in-advanced-mathematics/.

Weber, K. and Mejia-Ramos, J.P. (2011) Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics*, *76*(3), pp.329-344.

Weller, K., Brown, A., Dubinsky, E., McDonald, M. & Stenger, C. (2004) Intimations of infinity. *Notices of the AMS*, *51*(7), pp.741-750.

Williams, S. (1991). Models of limit held by college students. *Journal for Research in Mathematics Education*, 22, PP. 219-236.

Willis, J.W., Jost, M. and Nilakanta, R. (2007) *Foundations of qualitative research: Interpretive and critical approaches*. London: SAGE.

Wilson, S., Shulman, L. and Richert, A. (1987) “150 different ways of knowing”: Representations of knowledge in teaching. In: J. Calderhead (ed.) *Exploring teachers’ thinking.* Eastbourne, UK: Cassell, pp.104-123.

Winter, G. (2002) A comparative discussion of the notion of validity. *Qualitative and Quantitative Research,* 4(3).

Wragg, T. (1999) *An introduction to classroom observation* (Classic Edition*)*. New York: Routledge.

Wu, H. (2005) *Must content dictate pedagogy in mathematics education*? Expanded version of talk given at California State University, Northridge.

Yamani, M. (2000) *Changed identities: The challenge of the new generation in Saudi Arabia*. London: Royal Institute of International Affairs.

Yin, R. (2014). *Case study research: Design and methods.* Los Angeles, CA: Sage.

Yin, R. K. (1998) The abridged version of case study research: Design and method. In: L. Bickman and D. J. Rog (eds.) *Handbook of applied social research methods.* Thousand Oaks, CA: SAGE, pp.229-259.

Yin, R. (2004). *The case study anthology*. (1st ed.). Thousand Oaks, U.S.: SAGE Publications.

Yin, R. K. (2003) *Case study research: Design and methods*, (3rd ed.). Thousand Oaks, CA: SAGE.

Yushau, B. (2006) The effects of blended e-learning on mathematics and computer attitudes in pre-calculus algebra. *The Mathematics Enthusiast*, *3*(2), pp.176-18.

Zandieh, M. (2000) A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in Collegiate Mathematics Education*. Providence, RI: American Mathematical Society, Vol. IV, pp 103–127.