UNIVERSITY OF SOUTHAMPTON
SCHOOL OF ENGINEERING SCIENCES
AERODYNAMICS & FLIGHT MECHANICS GROUP

A Modification of Amiet's Classical Trailing Edge Noise Theory for Strictly Two Dimensional Flows

by

Richard D. Sandberg and Neil D. Sandham

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The aim of this report is to derive theoretical expressions for the far-field pressure generated by disturbances convecting over a trailing edge. First, a general calculation of the far-field pressure is discussed. Then the classical theory of Amiet (1976*b*) is reviewed, listing the most relevant assumptions. Amiet's theory is then revised for two-dimensional flows.

1 General calculation of far-field pressure

The starting point is Goldstein's formulation (Goldstein, 1976, chapter 4) of the acoustic analogy which represents the fundamental equation governing the generation of aerodynamic sound in the presence of solid boundaries in a moving reference frame

$$\begin{aligned} p(\underline{x},t) &= \int_{-T}^{T}\int_{\nu(\tau)}\frac{\partial^2 G}{\partial y_i \partial y_j}T_{ij}^{'}(\underline{y},\tau)d\underline{y}d\tau \\ &+ \int_{-T}^{T}\int_{S(\tau)}\frac{\partial G}{\partial y_i}f_i dS(\underline{y})d\tau + \int_{-T}^{T}\int_{S(\tau)}\rho_0 V_n^{'}\frac{\partial G}{\partial \tau}d\tau \ . \end{aligned} \tag{1}$$

$f_i = -n_i(p-p_0) + n_j\tau_{ij}$ is the i^{th} component of the force per unit area exerted by the boundaries on the fluid, τ_{ij} is the viscous stress tensor and n_i denotes the i^{th} component of the inward normal vector $\underline{n}$. Here, the streamwise, the wall-normal and the spanwise directions are denoted by the subscripts $i = 1, 2, 3$, respectively. $\underline{x} = x_i$ and $\underline{y} = y_i$ are the far field and the surface co-ordinates, respectively.

Neglecting the viscous stress-tensor components, the force acting on the fluid per unit length by the airfoil is given by $f_i = -n_i p_t(\underline{y},\tau)$, where $p_t(\underline{y},\tau)$ is the unsteady pressure disturbance on the airfoil surface. It is assumed that the volume-quadrupole sources generated by the shear-stress components in the boundary layer are negligible compared with the dipole sources on the airfoil surface. Furthermore, the airfoil is considered rigid, i.e., the third term of equation (1) represents a steady pressure which does not radiate sound. Thus, equation (1) reduces to

$$p(\underline{x},t) = -\int_{-T}^{T}\int_{S}\Delta p_t(\underline{y},\tau)n_i\frac{\partial}{\partial y_i}G(\underline{x},t;\underline{y},\tau)dS(\underline{y})d\tau \ , \tag{2}$$

where $\Delta p_t(\underline{y},\tau)$ denotes the pressure difference from the top and bottom side of the airfoil or flat plate, and $G(\underline{x},t;\underline{y},\tau)$ is the radiation Green's function.

Fourier transforming the far-field pressure with respect to time gives the acoustic pressure at a single frequency due to the unsteady loading on the airfoil

$$\begin{aligned} \overline{p}(\underline{x},\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}p(\underline{x},t)e^{\mathrm{i}\omega t}dt \\ &= -\frac{1}{2\pi}\int_{-T}^{T}\int_{S}\Delta p_t(\underline{y},\tau)n_i\frac{\partial}{\partial y_i}\int_{-\infty}^{\infty}G(\underline{x},t;\underline{y},\tau)dtdS(\underline{y})d\tau \\ &= \frac{1}{2\pi}\int_{T}^{-T}\int_{S}\Delta p_t(\underline{y},\tau)I(\underline{x},\underline{y},\omega)dS(\underline{y})e^{\mathrm{i}\omega\tau}d\tau \ , \end{aligned} \tag{3}$$

with

$$I(\underline{x},\underline{y},\omega) = -n_i\frac{\partial}{\partial y_i}\overline{G}(\underline{x},\underline{y},\omega) \ . \tag{4}$$

Defining

$$\Delta p_t(\underline{y},\tau) = \int_{-\infty}^{\infty} \Delta \overline{p}_t(\underline{y},\omega_0) e^{-\mathbf{i}\omega_0 t} d\omega_0 \ , \tag{5}$$

where an overbar defines a quantity in the frequency domain, and applying the relation

$$\delta(\omega_0 - \omega) = \frac{1}{2\pi} \int_{-T}^{T} e^{-\mathbf{i}(\omega_0-\omega)\tau} d\tau \ , \tag{6}$$

we can write (3) as

$$\overline{p}(\underline{x},\omega) = \int_S \Delta \overline{p}_t(\underline{y},\omega) I(\underline{x},\underline{y},\omega) dS(\underline{y}) \ . \tag{7}$$

When dealing with turbulent flows, it is preferred to use statistical quantities (e.g. Amiet, 1975). Therefore, the cross power spectral density (PSD) is oftentimes used and is defined as

$$\begin{aligned} S_{pp}(\underline{x},\omega) &= \lim_{T\to\infty} \frac{\pi}{T} E\left\{\overline{p}(\underline{x},\omega)\overline{p}^*(\underline{x},\omega)\right\} \\ &= \lim_{T\to\infty} \frac{\pi}{T} \int_{S_y}\int_{S_z} E\left\{\Delta\overline{p}_t(\underline{y},\omega)\Delta\overline{p}_t^*(\underline{z},\omega)\right\} I(\underline{x},\underline{y},\omega) I^*(\underline{x},\underline{z},\omega) dS(\underline{y}) dS(\underline{z}) \\ &= \underline{\underline{\int_{S_y}\int_{S_z} S_{QQ}(\underline{y},\underline{z},\omega) I(\underline{x},\underline{y},\omega) I^*(\underline{x},\underline{z},\omega) dS(\underline{y}) dS(\underline{z})}} \ , \end{aligned} \tag{8}$$

where * denotes the complex conjugate, $E\{\}$ is the expected value, and it was assumed that $I(\underline{x},\underline{y},\omega)$ and $dS(\underline{y})$ are not time-dependent.
Hence, the input for the far-field pressure cross-PSD is the cross-PSD of the total pressure distribution (comprised of the incident *and* the scattered pressure field) on the surface (S_{QQ}). There are two different options for obtaining S_{QQ}:

1. The total pressure distribution on the surface $\Delta\overline{p}_t(\underline{y},\omega)$ is known. This is the case for DNS calculations where the entire time-series of the surface pressure is written to file. Then

$$\underline{S_{QQ}(\underline{y},\underline{z},\omega) = \lim_{T\to\infty} \frac{\pi}{T} E\left\{\Delta\overline{p}_t(\underline{y},\omega)\Delta\overline{p}_t^*(\underline{z},\omega)\right\}} \ . \tag{9}$$

2. The total pressure distribution on the surface $\Delta\overline{p}_t(\underline{y},\omega)$ is not known. For example, in an experiment the pressure at the trailing edge will be hard, if not impossible, to measure. Here, only the incident pressure field farther upstream can be measured. According to Amiet (1976*a*), the total pressure field at the trailing edge can be determined from the incident pressure field through a transfer function $\overline{g}(\underline{y},\omega)$

$$\Delta\overline{p}_t(\underline{y},\omega) = \overline{g}(\underline{y},\omega)\overline{p}_i(\underline{y},\omega) \ . \tag{10}$$

   Thus, we can compute $S_{QQ}(\underline{y},\underline{z},\omega)$ from

$$\underline{S_{QQ}(\underline{y},\underline{z},\omega) = \overline{g}(\underline{y},\omega)\overline{g}^*(\underline{z},\omega) S_{qq}(\underline{y},\underline{z},\omega)} \ , \tag{11}$$

   where $S_{qq}(\underline{y},\underline{z},\omega) = \lim_{T\to\infty} \frac{\pi}{T} E\left\{\overline{p}_i(\underline{y},\omega)\overline{p}_i^*(\underline{z},\omega)\right\}$.

1.1 Derivation of 3-D Green's function in moving reference frame

The integral equation (1) is derived for a moving reference frame, hence the Green's function G needs to be a solution for the wave equation with mean flow. The convected wave equation takes the form

$$\left[\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial \tau} + U\frac{\partial}{\partial y_1}\right)^2\right] G_3(\underline{x}, \underline{y}, t, \tau) = -\delta(\underline{y} - \underline{x})\delta(\tau - t) \ . \tag{12}$$

Using the relation $\beta^2 = 1 - M^2$ with $M = U/c$ results in

$$\left[\beta^2 \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \frac{2U}{c^2}\frac{\partial^2}{\partial y_1 \partial \tau}\right] G_3(\underline{x}, \underline{y}, t, \tau) = -\delta(\underline{y} - \underline{x})\delta(\tau - t) \ . \tag{13}$$

The following coordinate transformation is applied in order to obtain the wave equation for a stationary medium

$$\begin{aligned} Y_1 &= y_1 \ , \quad Y_2 = \beta y_2 \ , \quad Y_3 = \beta y_3 \ , \\ T &= \beta\tau + \frac{MY_1}{\beta c} \ , \quad \widetilde{\omega} = \frac{\omega}{\beta} \ , \quad \widetilde{k} = \frac{k}{\beta^2} \ , \quad \widetilde{c} = c\beta \ . \end{aligned} \tag{14}$$

With

$$\begin{aligned} \frac{\partial^2}{\partial y_1^2} &= \frac{\partial^2}{\partial Y_1^2} + 2\frac{M}{\widetilde{c}}\frac{\partial^2}{\partial Y_1 \partial T} + \frac{M^2}{\widetilde{c}^2}\frac{\partial^2}{\partial T^2} \ , \quad \frac{\partial^2}{\partial y_2^2} = \beta^2\frac{\partial^2}{\partial Y_2^2} \ , \quad \frac{\partial^2}{\partial y_3^2} = \beta^2\frac{\partial^2}{\partial Y_3^2} \ , \\ \frac{\partial^2}{\partial y_1 \partial \tau} &= \beta\frac{\partial^2}{\partial Y_1 \partial T} + \frac{\beta M}{\widetilde{c}}\frac{\partial^2}{\partial T^2} \ , \quad \frac{\partial^2}{\partial \tau^2} = \beta^2\frac{\partial^2}{\partial T^2} \ , \end{aligned} \tag{15}$$

the wave equation becomes

$$\left[\frac{\partial^2}{\partial Y_1^2} + \frac{\partial^2}{\partial Y_2^2} + \frac{\partial^2}{\partial Y_3^2} - \frac{1}{\widetilde{c}^2}\frac{\partial^2}{\partial T^2}\right] G_3(\underline{X}, \underline{Y}, \widetilde{t}, T) = -\delta(\underline{Y} - \underline{X})\delta\left(\frac{T}{\beta} - \frac{MY_1}{\beta\widetilde{c}} - \widetilde{t}\right) \ . \tag{16}$$

The Green's function for the above three-dimensional wave equation is known as (e.g. Crighton, 1975)

$$\overline{G}_3(\underline{X}, \underline{Y}, \widetilde{\omega}) = \frac{1}{4\pi\widetilde{R}} e^{\mathrm{i}[\widetilde{\omega}T + \widetilde{k}\widetilde{R}]} \ , \tag{17}$$

with $\widetilde{R} = \sqrt{(Y_1 - X_1)^2 + (Y_2 - X_2)^2 + (Y_3 - X_3)^2}$.
Inverting the above transformation coordinates and substituting them into the stationary medium Green function results in

$$\underline{\underline{\overline{G}_3(\underline{x}, \underline{y}, \omega) = \frac{1}{4\pi R} e^{\mathrm{i}\omega\left\{\tau + \frac{1}{\beta^2 c}[M(y_1 - x_1) + R]\right\}}}}, \tag{18}$$

with $R = \sqrt{(y_1 - x_1)^2 + \beta^2\left[(y_2 - x_2)^2 + (y_3 - x_3)^2\right]}$.

1.2 Derivation of 2-D Green's function in moving reference frame

In two dimensions, the convected wave equation takes the form

$$\left[\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial \tau} + U\frac{\partial}{\partial y_1}\right)^2\right] G_2(\underline{x},\underline{y},t,\tau) = -\delta(\underline{y}-\underline{x})\delta(\tau - t)\ . \tag{19}$$

Using the relation $\beta^2 = 1 - M^2$ with $M = U/c$ gives

$$\left[\beta^2\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} - \frac{2U}{c^2}\frac{\partial^2}{\partial y_1 \partial \tau}\right] G_2(\underline{x},\underline{y},t,\tau) = -\delta(\underline{y}-\underline{x})\delta(\tau - t)\ . \tag{20}$$

The following coordinate transformation is applied in order to obtain the wave equation for a stationary medium

$$\begin{aligned} Y_1 &= y_1\ , \quad Y_2 = \beta y_2\ , \\ T &= \beta\tau + \frac{MY_1}{\beta c}\ , \quad \widetilde{\omega} = \frac{\omega}{\beta}\ , \quad \widetilde{k} = \frac{k}{\beta^2}\ , \quad \widetilde{c} = c\beta\ . \end{aligned}$$

The wave equation becomes

$$\left[\frac{\partial^2}{\partial Y_1^2} + \frac{\partial^2}{\partial Y_2^2} - \frac{1}{\widetilde{c}^2}\frac{\partial^2}{\partial T^2}\right] G_2(\underline{X},\underline{Y},\widetilde{t},T) = -\frac{1}{\beta}\delta(\underline{Y}-\underline{X})\delta\left(\frac{T}{\beta} - \frac{MY_1}{\beta^3\widetilde{c}} - \widetilde{t}\right)\ . \tag{21}$$

The Green's function for the two-dimensional stationary medium wave equation in frequency space is known as (c.f. Crighton, 1975)

$$\overline{G}_2(\underline{X},\underline{Y},\widetilde{\omega}) = \frac{1}{4\mathbf{i}}H_0^{(2)}\left[\widetilde{\omega}T + \widetilde{k}\widetilde{R}\right]\ , \tag{22}$$

with $\widetilde{R} = \sqrt{(Y_1 - X_1)^2 + (Y_2 - X_2)^2}$ and $H_0^{(2)}$ denoting a Hankel zeroth order function of the second kind. Note, that $H_\nu^{(2)}(z) = J_\nu(z) - \mathbf{i}Y_\nu(z)$.
Inverting the above transformation coordinates and substituting them into the stationary medium Green function results in the two-dimensional Green's function in frequency space

$$\underline{\underline{\overline{G}_2(\underline{x},\underline{y},\omega) = \frac{1}{4\beta\mathbf{i}}H_0^{(2)}\left\{\omega\tau + \frac{\omega}{\beta^2 c}\left[M(y_1 - x_1) + R\right]\right\}}}\ , \tag{23}$$

with $R = \sqrt{(y_1 - x_1)^2 + \beta^2(y_2 - x_2)^2}$.

2 Amiet's theory for three-dimensional flow

Stationary turbulent flow is considered. If the total pressure difference on the airfoil cannot be determined, a surface pressure jump relation needs to be obtained. The incident pressure difference Δp_i convects in the streamwise direction over a plate $(-2 < y_1 < 0)$ and produces a scattered pressure field Δp_s which radiates into the far-field. Note that all co-ordinates x_i and y_i are nondimensionalized with the semi-chord b in the following. The total pressure difference on the airfoil is the sum of the incident pressure difference and the scattered pressure difference

$$\Delta p_t = \Delta p_i + \Delta p_s \ . \tag{24}$$

At the trailing edge, the Kutta condition is assumed to hold, i.e. $\Delta p_t = 0$. The condition of no-flow across the airfoil is also imposed. With the above assumptions, Amiet (1976*a*) uses the Schwartzschild solution to derive an airfoil surface pressure jump (for incident pressure fluctuations on one side of the airfoil)

$$\begin{aligned} \Delta\hat{p}_t(y_1,k_3,\omega,U_c) &= H_D(y_1,k_3,\omega,U_c)\hat{p}_i(y_1,k_3,\omega,U_c) \\ H_D(y_1,k_3,\omega,U_c) &= f_i(y_1,k_3,\omega,U_c) + H_S(y_1,k_3,\omega,U_c) \\ H_S(y_1,k_3,\omega,U_c) &= \left\{(1+\mathbf{i})E^*\left[-\left(\sqrt{\mu_0^2-\left(\frac{k_3}{\beta}\right)^2}+\mu_0 M+\frac{\omega b}{U_c}\right)y_1\right]-1\right\} \ , \end{aligned} \tag{25}$$

where the reduced frequency $\mu_0 = \omega b/(c\beta^2)$, and f_i scales the incident pressure. The function E^* is a combination of Fresnel integrals

$$E^*(x) = \int_0^x \frac{1}{\sqrt{2\pi\xi}} e^{-\mathbf{i}\xi} d\xi \ . \tag{26}$$

When a spectral component of the incident pressure disturbance can be represented as

$$\hat{p}_i(y_1,k_3,\omega,U_c) = p_0 e^{\mathbf{i}[\omega(t-y_1 b/U_c)-k_3 y_3]} \ , \tag{27}$$

the total pressure jump (normalized with p_0) can be evaluated as

$$\Delta\overline{p}_t(y_1,k_3,\omega,U_c) = H_D(y_1,k_3,\omega,U_c) e^{-\mathbf{i}(\omega y_1 b/U_c + k_3 y_3)} \ . \tag{28}$$

Note that the total surface pressure difference is also a function of the convection speed when computed using the surface pressure jump function. Originally, f_i was set to unity to account for the incident pressure. However, in a later paper, Amiet (1978) introduced a convergence factor ε to gradually decrease the incident pressure towards the leading edge

$$f_i(y_1,k_3,\omega,U_c) = e^{\varepsilon\omega b y_1/U_c} \ . \tag{29}$$

The airfoil pressure jump due to all spanwise wavenumber components is therefore

$$\Delta\overline{p}_t(y_1,y_3,\omega,U_c) = \int H_D(y_1,k_3,\omega,U_c)\hat{p}_i(y_1,k_3,\omega,U_c) e^{\mathbf{i}k_3 y_3} dk_3 \ , \tag{30}$$

and the cross-PSD of the total surface pressure difference becomes

$$S_{QQ}(y_1, y_3, z_1, z_3, \omega) = \int H_D(y_1, k_3, \omega, U_c) H_D^*(z_1, k_3, \omega, U_c) S_{qq}(y_1, z_1, k_3, \omega, U_c) e^{\mathbf{i}k_3(y_3 - z_3)} dk_3 \ . \tag{31}$$

With the 3-D Green's function known and the pressure jump function given, an expression for the farfield pressure can be derived as a function of the incident pressure spectrum. $I(\underline{x}, \underline{y}, \omega)$ for a flat plate becomes

$$I(\underline{x}, \underline{y}, \omega) = -n_2 \frac{\partial \overline{G}_3(\underline{x}, \underline{y}, \omega)}{\partial y_2} = -\frac{y_2 - x_2}{R} \left[\frac{\beta^2}{R} - \mathbf{i}\frac{\omega b}{c} \right] \overline{G}_3(\underline{x}, \underline{y}, \omega) \ . \tag{32}$$

Making the far field assumption $|\underline{x}| >> |\underline{y}|$ leads to

$$R \approx \sigma - \frac{x_1 y_1 + \beta^2 x_3 y_3}{\sigma} \ , \tag{33}$$

with $\sigma = \sqrt{x_1^2 + \beta^2(x_2^2 + x_3^2)}$. For large R, $I(\underline{x}, \underline{y}, \omega)$ then simplifies to

$$I(\underline{x}, \underline{y}, \omega) = \frac{\mathbf{i}\omega x_2 b}{4\pi c \sigma^2} e^{\mathbf{i}\frac{\omega b}{c\beta^2}\left[\sigma - \frac{x_1 y_1 + \beta^2 x_3 y_3}{\sigma} + M(y_1 - x_1)\right]} \ . \tag{34}$$

Inserting the above expression into the far-field pressure equation (3) and assuming that the far-field sound is produced by a point force of strength per unit area $F(y_1, y_3, \omega)e^{\mathbf{i}\omega t}$ in a stream of Mach number M, Amiet (1975) gives

$$\begin{aligned} \overline{p}(\underline{x}, \omega) &= \int\int \frac{1}{2\pi} F(y_1, y_3, \omega) e^{\mathbf{i}\omega t} \frac{\mathbf{i}\omega x_2 b}{4\pi c\sigma^2} e^{\mathbf{i}\frac{\omega b}{c\beta^2}\left[\sigma - \frac{x_1 y_1 + \beta^2 x_3 y_3}{\sigma} + M(y_1 - x_1)\right]} \int e^{\mathbf{i}\omega\tau} d\tau dy_1 dy_3 \\ &= \frac{\mathbf{i}\omega x_2 b}{4\pi c\sigma^2} \int\int F(y_1, y_3, \omega) e^{\mathbf{i}\omega b\left[\frac{t}{b} + \frac{M(y_1 - x_1) + \sigma}{c\beta^2} - \frac{x_1 y_1 + \beta^2 x_3 y_3}{\sigma c \beta^2}\right]} dy_1 dy_3 \end{aligned} \tag{35}$$

In order to get the PSD, multiply with the complex conjugate

$$\begin{aligned} S_{pp}(\underline{x}, \omega) &= \left(\frac{\mathbf{i}\omega x_2 b}{4\pi c\sigma^2}\right)^2 \int\int\int\int S_{QQ}(y_1, y_3, z_1, z_3, \omega) \\ &\quad e^{\mathbf{i}\omega b\left[\frac{t}{b} + \frac{M(y_1 - x_1) + \sigma}{c\beta^2} - \frac{x_1 y_1 + \beta^2 x_3 y_3}{\sigma c\beta^2}\right]} e^{-\mathbf{i}\omega b\left[\frac{t}{b} + \frac{M(z_1 - x_1) + \sigma}{c\beta^2} - \frac{x_1 z_1 + \beta^2 x_3 z_3}{\sigma c\beta^2}\right]} dy_1 dy_3 dz_1 dz_3 \\ &= -\left(\frac{\omega x_2 b}{4\pi c\sigma^2}\right)^2 \int\int\int\int S_{QQ} e^{\mathbf{i}\frac{\omega b}{c}\left[\frac{(y_1 - z_1)}{\beta^2}\left(M - \frac{x_1}{\sigma}\right) + \frac{x_3}{\sigma}(z_3 - y_3)\right]} dy_1 dy_3 dz_1 dz_3 \ . \end{aligned} \tag{36}$$

If the total surface pressure cross-PSD (S_{QQ}) is known, the above relation can be used directly.

In case only the incident pressure field is available, inserting S_{QQ} from equation (31) into the far-field pressure PSD gives

$$\begin{aligned} S_{pp}(\underline{x}, \omega) &= -\left(\frac{\omega x_2 b}{4\pi c\sigma^2}\right)^2 \int\int\int H_D(y_1, k_3, \omega, U_c) H_D^*(z_1, k_3, \omega, U_c) \int \hat{\hat{S}}_{qq}(k_1, k_3, \omega, U_c) dk_1 \\ &\quad e^{\mathbf{i}\frac{\omega b}{c}\left[\frac{(y_1 - z_1)}{\beta^2}\left(M - \frac{x_1}{\sigma}\right)\right]} dy_1 dz_1 \int\int e^{\mathbf{i}(z_3 - y_3)\left(\frac{\omega b}{c}\frac{x_3}{\sigma} - k_3\right)} dy_3 dz_3 dk_3 \ . \end{aligned} \tag{37}$$

Now, define

$$\mathcal{L}(x_1,k_3,\omega,U_c)=\int_{-2}^{0}H_D(y_1,k_3,\omega,U_c)e^{-\mathbf{i}\mu_0 y_1\left(M-\frac{x_1}{\sigma}\right)}dy_1 \ . \tag{38}$$

A closed form analytical solution exists for (38) and is given in Amiet (1976*b*). The far-field pressure PSD simplifies to

$$S_{pp}(\underline{x},\omega)=-\left(\frac{\omega x_2 b}{4\pi c\sigma^2}\right)^2\int|\mathcal{L}|^2\int\hat{\hat{S}}_{qq}(k_1,k_3,\omega,U_c)dk_1\int\int e^{\mathbf{i}(z_3-y_3)(\frac{\omega b}{c}\frac{x_3}{\sigma}-k_3)}dy_3dz_3dk_3 \ . \tag{39}$$

Integrating over the spanwise coordinate ($-d\leq x_3\leq d$), which is also nondimensionalized with b, and assuming infinite span

$$\int_{-d}^{d}\int_{-d}^{d}e^{\mathbf{i}(z_3-y_3)(\frac{\omega b}{c}\frac{x_3}{\sigma}-k_3)}dy_3dz_3=4\frac{\sin^2\left[d\left(\frac{\omega b x_3}{c\sigma}-k_3\right)\right]}{\left(\frac{\omega b x_3}{c\sigma}-k_3\right)^2} \ ,$$

$$\lim_{d\to\infty}\left.\frac{\sin^2\left[d\left(\frac{\omega b x_3}{c\sigma}-k_3\right)\right]}{\left(\frac{\omega b x_3}{c\sigma}-k_3\right)^2\pi d}\right|_{x_3=0}\to\delta\left.\left(\frac{\omega b x_3}{c\sigma}-k_3\right)\right|_{x_3=0}=\delta(-k_3) \ , \tag{40}$$

we get

$$\underline{\underline{S_{pp}(\underline{x},\omega)=\left(\frac{\omega x_2 b}{2\pi c\sigma^2}\right)^2\pi d|\mathcal{L}(x_1,0,\omega,U_c)|^2\int\hat{\hat{S}}_{qq}(k_1,0,\omega,U_c)dk_1}} \ . \tag{41}$$

Brooks version (Brooks & Hodgson, 1981) of Corcos's spectrum is

$$\hat{\hat{S}}_{qq}(k_1,k_3,\omega,U_c)=S_0(\omega)S_1(k_1)S_3(k_3) \ . \tag{42}$$

The frozen spectrum $\hat{S}_{qq}$ can be related to the non-frozen spectrum $\hat{\hat{S}}_{qq}$ as

$$\hat{S}_{qq}\left(\frac{\omega b}{U_c},k_3\right)=\int_{-\infty}^{\infty}\hat{\hat{S}}_{qq}(k_1,k_3,\omega,U_c)d\omega \ , \tag{43}$$

with $k_1=\frac{\omega b}{U_c}$ for frozen turbulence. Therefore,

$$\int_{-\infty}^{\infty}\hat{\hat{S}}_{qq}(k_1,0,\omega,U_c)dk_1=\frac{b}{U_c}\int_{-\infty}^{\infty}\hat{\hat{S}}_{qq}(k_1,0,\omega,U_c)d\omega=\frac{b}{U_c}\hat{S}_{qq}(k_1,0) \ . \tag{44}$$

Employing the approximation (Amiet, 1975)

$$\hat{S}_{qq}(k_1,0)=\frac{1}{\pi}\frac{l_y(\omega)}{b}U_cS_{qq}(\omega,0) \ , \tag{45}$$

with $l_y(\omega)$ denoting the frequency-dependent spanwise correlation length, we finally arrive at

$$\int_{-\infty}^{\infty}\hat{\hat{S}}_{qq}(k_1,0,\omega,U_c)dk_1=\frac{1}{\pi}l_y(\omega)S_{qq}(\omega,0) \ . \tag{46}$$

The far-field spectrum for an observer in the $x_3=0$ plane therefore is

$$\underline{\underline{S_{pp}(\underline{x},\omega)=\left(\frac{\omega x_2 b}{2\pi c\sigma^2}\right)^2 d|\mathcal{L}(x_1,0,\omega,U_c)|^2 l_y(\omega)S_{qq}(\omega,0)}} \ , \tag{47}$$

corresponding to the classical result of Amiet (1976*b*).

3 Deriving Amiet's theory for two-dimensional flow

In two dimensions, equation (1) reduces to

$$p(\underline{x}, t) = -\int_{-T}^{T} \int_{C(\tau)} \Delta p_t(\underline{y}, \tau) \frac{\partial}{\partial y_2} G_2(\underline{x}, t; \underline{y}, \tau) dy_1 d\tau \ . \tag{48}$$

when making the same assumptions as for the three dimensional case. The acoustic pressure in the far-field becomes

$$\overline{p}(\underline{x}, \omega) = \int_C \Delta\overline{p}_t(\underline{y}, \omega) I_2(\underline{x}, \underline{y}, \omega) dy_1 \ , \tag{49}$$

with

$$I_2(\underline{x}, \underline{y}, \omega) = -\frac{\partial}{\partial y_2} \overline{G}_2(\underline{x}, \underline{y}, \omega) \ . \tag{50}$$

The PSD is

$$\underline{\underline{S_{pp}(\underline{x}, \omega) = \int_{C_y} \int_{C_z} S_{QQ}(\underline{y}, \underline{z}, \omega) I_2(\underline{x}, \underline{y}, \omega) I_2^*(\underline{x}, \underline{z}, \omega) dy_1 dz_1}} \ . \tag{51}$$

As for the 3-D case, if the total pressure distribution on the surface is known, S_{pp} can be evaluated directly. Otherwise, the transfer function for the surface pressure jump needs to be used.

The airfoil surface pressure jump (due to Amiet, 1976a, 1978) is given as

$$\begin{aligned}
\Delta\overline{p}_t(y_1, \omega, U_c) &= H_D(y_1, \omega, U_c)\overline{p}_i(y_1, \omega, U_c) \\
H_D(y_1, \omega, U_c) &= e^{\varepsilon\omega y_1 b/U_c} + H_S(y_1, \omega, U_c) \\
H_S(y_1, \omega, U_c) &= \left\{(1+\mathbf{i})E^*\left[-\left(\mu_0(1+M) + \frac{\omega b}{U_c}\right) y_1\right] - 1\right\} \ ,
\end{aligned} \tag{52}$$

where the reduced frequency is again defined as $\mu_0 = \omega b/(c\beta^2)$. Considering a spectral component of the incident pressure disturbance

$$\overline{p}_i(y_1, \omega, U_c) = p_0 e^{\mathbf{i}\omega(t - y_1/U_c)} \ , \tag{53}$$

the total pressure (normalized with p_0) becomes

$$\Delta\overline{p}_t(y_1, \omega, U_c) = H_D(y_1, \omega, U_c) e^{-\mathbf{i}(\omega y_1 b/U_c)} \tag{54}$$

The PSD of the total surface pressure difference becomes

$$S_{QQ}(y_1, z_1, \omega) = H_D(y_1, \omega, U_c) H_D^*(z_1, \omega, U_c) S_{qq}(y_1, z_1, \omega, U_c) \ . \tag{55}$$

With the 2-D Green's function known and the pressure jump function given, an expression for the farfield pressure can now be derived as a function of the incident pressure spectrum. $I_2(\underline{x}, \underline{y}, \omega)$ for a flat plate becomes

$$\begin{aligned}
I_2(\underline{x}, \underline{y}, \omega) &= -\frac{\partial}{\partial y_2}\left[\frac{1}{4\beta\mathbf{i}} H_0^{(2)}\left\{\mu_0\left[M(y_1 - x_1) + R\right]\right\}\right] \\
&= -\frac{\mathbf{i}(y_2 - x_2)\omega b}{4\beta c R} H_1^{(2)}\left\{\mu_0\left[M(y_1 - x_1) + R\right]\right\} \ .
\end{aligned} \tag{56}$$

Expecting that the form of the 2-D Green's function (Hankel function) does not allow for an analytical solution similar to that of $\mathcal{L}$ in the 3-D case, the far field assumption $|\underline{x}| >> |\underline{y}|$ is not made in the 2-D case. Therefore, the final solution will be valid in the entire integration domain (outside of the boundary layer, because the Green's function was derived for a convection velocity U) and not restricted to the far-field. Inserting $I_2(\underline{x}, \underline{y}, \omega)$ into the far-field pressure equation (49) and assuming that the far-field sound is produced by a point force of strength $F(y_1, \omega)e^{\mathbf{i}\omega t}$ per unit chord in a stream of Mach number M (Amiet, 1975) we obtain

$$\begin{aligned} \overline{p}(\underline{x}, \omega) &= \int \frac{1}{2\pi} F(y_1, \omega) e^{\mathbf{i}\omega t} \frac{\mathbf{i}(y_2 - x_2)\omega b}{4\beta c R} H_1^{(2)} \left\{ \mu_0 \left[M(y_1 - x_1) + R \right] \right\} \int e^{\mathbf{i}\omega\tau} d\tau dy_1 \\ &= \frac{\mathbf{i}(y_2 - x_2)\omega b}{4\beta c} e^{\mathbf{i}\omega t} \int \frac{1}{R} F(y_1, \omega) H_1^{(2)} \left\{ \mu_0 \left[M(y_1 - x_1) + R \right] \right\} dy_1 \ . \end{aligned} \tag{57}$$

In order to get the PSD, multiply with the complex conjugate

$$\begin{aligned} S_{pp}(\underline{x}, \omega) &= -\left(\frac{(y_2 - x_2)\omega b}{4\beta c} \right)^2 \int\int \frac{1}{R_{y_1} R_{z_1}} S_{QQ}(y_1, z_1, \omega) H_1^{(2)} \left\{ \mu_0 \left[M(y_1 - x_1) + R_{y_1} \right] \right\} \\ & \quad H_1^{(1)} \left\{ \mu_0 \left[M(z_1 - x_1) + R_{z_1} \right] \right\} dy_1 dz_1 \ , \end{aligned} \tag{58}$$

where $R_\phi = \sqrt{(\phi - x_1)^2 + \beta^2 (y_2 - x_2)^2}$, $H_\nu^{(1)}(z) = J_\nu(z) + \mathbf{i}Y_\nu(z)$, and $H_\nu^{(2)}(z) = J_\nu(z) - \mathbf{i}Y_\nu(z)$. Provided the total surface pressure cross-PSD (S_{QQ}) is known, the above relation can be used directly.

In case only the incident pressure field is available, inserting S_{QQ} from equation (55) into the far-field pressure PSD gives

$$\begin{aligned} S_{pp}(\underline{x}, \omega) &= -\left(\frac{(y_2 - x_2)\omega b}{4\beta c} \right)^2 \int\int \frac{1}{R_{y_1} R_{z_1}} H_D(y_1, \omega, U_c) H_1^{(2)} \left\{ \mu_0 \left[M(y_1 - x_1) + R_{y_1} \right] \right\} \\ & \quad H_D^*(z_1, \omega, U_c) H_1^{(1)} \left\{ \mu_0 \left[M(z_1 - x_1) + R_{z_1} \right] \right\} \int \hat{S}_{qq}(k_1, \omega, U_c) dk_1 dy_1 dz_1 \ . \end{aligned} \tag{59}$$

Defining

$$\mathcal{M} = \int_{-2}^{0} \frac{1}{R} H_D(y_1, \omega, U_c) H_1^{(2)} \left\{ \mu_0 \left[M(y_1 - x_1) + R \right] \right\} dy_1 \ , \tag{60}$$

we can write

$$\underline{S_{pp}(\underline{x}, \omega) = -\left(\frac{\omega(y_2 - x_2) b}{4\beta c} \right)^2 |\mathcal{M}|^2 \int \hat{S}_{qq}(k_1, \omega, U_c) dk_1} \ . \tag{61}$$

Finally, assuming a frozen disturbance field where $S_1(k_1) \to \delta(k_1 - \omega b/U_c)$,

$$\int_{-\infty}^{\infty} \hat{S}_{qq}(k_1, \omega, U_c) dk_1 = \int_{-\infty}^{\infty} S_0(\omega) \delta(k_1 - \omega b/U_c) dk_1 = \int_{-\infty}^{\infty} S_0(\omega) \delta(\omega - k_1 U_c/b) d\omega = S_0(\omega) \ , \tag{62}$$

we arrive at the final result for the two dimensional far-field pressure PSD

$$\underline{\underline{S_{pp}(\underline{x}, \omega) = -\left(\frac{\omega(y_2 - x_2) b}{4\beta c} \right)^2 |\mathcal{M}|^2 S_0(\omega)}} \ . \tag{63}$$

Recall that, unlike for the 3-D case, this solution is valid in the entire area outside the boundary layer as no far-field approximation was made.

3.1 Single frequency formulation

Conducting a single frequency calculation (such as a Tollmien-Schlichting wave over a trailing edge), the evaluation of the PSD is not necessary. Instead, the computation of the farfield pressure $\overline{p}(\underline{x}, \omega)$ is sufficient.

By setting $F(y_1, \omega)e^{\mathbf{i}\omega t} = \Delta\overline{p}_t(y_1, \omega, U_c) = H_D(y_1, \omega, U_c)\overline{p}_i(y_1, \omega, U_c)$, equation (57) becomes,

$$\overline{p}(\underline{x}, \omega) = -\frac{\mathbf{i}\omega(y_2 - x_2)b}{4\beta c}\int_{-2}^{0}\frac{1}{R}H_D(y_1, \omega, U_c)\overline{p}_i(y_1, \omega, U_c)H_1^{(2)}\left\{\mu_0\left[M(y_1 - x_1) + R\right]\right\}dy_1\ . \tag{64}$$

Placing the surface at $y_2 = 0$ and using the streamwise wavenumber $K_x = \omega b/U_c$ the final expression is

$$\overline{p}(\underline{x}, \omega) = \frac{\mathbf{i}\omega b x_2}{4\beta c}\int_{-2}^{0}\frac{1}{R}H_D(y_1, K_x)\overline{p}_i(y_1, K_x)H_1^{(2)}\left\{\mu_0\left[M(y_1 - x_1) + R\right]\right\}dy_1\ . \tag{65}$$

Note again that this solution is also applicable in the near-field.

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