

# Renormalisation of the energy-momentum tensor in three-dimensional scalar $SU(N)$ theories using the Wilson flow

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## Abstract

A non-perturbative determination of the energy-momentum tensor is essential for understanding the physics of strongly coupled systems. The ability of the Wilson Flow to eliminate divergent contact terms makes it a practical method for renormalising the energy-momentum tensor on the lattice. In this paper, we utilise the Wilson Flow to define a procedure to renormalise the energy-momentum tensor for a three-dimensional massless scalar field in the adjoint of  $SU(N)$  with a  $\varphi^4$  interaction on the lattice. In this theory the energy momentum tensor can mix with  $\varphi^2$  and we present numerical results for the mixing coefficient for the  $N = 2$  theory.

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## I. INTRODUCTION

The energy-momentum tensor (EMT) plays a fundamental role in quantum field theories, by virtue of being the collection of Noether currents related to space-time symmetries. It acts as the source for space-time curvature in the Einstein field equations, and its expectation value encodes the energy and momentum carried by quantum excitations. One of the motivations for this study comes from the application of holography to cosmology [1]. In this holographic approach, cosmological observables, such as the Cosmic Microwave Background (CMB) power spectra, can be described in terms of correlators of the EMT of a dual three-dimensional quantum field theory (QFT) with no

gravity. The dual theories introduced in [1] comprise three-dimensional Yang-Mills theory, coupled to massless scalars  $\varphi$  in the adjoint of  $SU(N)$  with a  $\varphi^4$  interaction. Perturbative calculations of the correlators have been performed [2–5] and the predictions of holographic cosmology were tested favourably against Planck data in [6]. The results in [6] however also implied that a non-perturbative evaluation of the EMT is required in order to fully exploit the duality in the low-multipole regime.

Here we initiate the computation of non-perturbative effects by means of lattice QFT. A fundamental limitation of the lattice framework is the fact that space-time symmetries, such as Poincaré invariance, are explicitly broken at finite lattice spacing; these symmetries are restored only in the continuum limit. Consequently, the Ward identities associated to translations are violated, and the EMT, which generates such transformations, has to be defined with care. On the lattice, the EMT has to be renormalized by tuning the coefficients of a linear combination of all operators with dimension not greater than the space-time dimension  $d$ , which are compatible with lattice symmetries. This ensures that the Ward identities are recovered in the continuum limit, up to cut-off effects. Perturbative analytic calculations using this method have been discussed extensively in [7, 8].

Various strategies have been proposed to non-perturbatively renormalize the EMT on the lattice (*cf.* [9], and references therein) such as the shifted boundary condition [10–13], and the Gradient Flow for probe operators [14–17], which is the strategy considered in this paper. The Wilson Flow originated from [18], and the idea here is to construct probes from fields at some positive flowtime, which are non-local in the elementary fields, that can eliminate the divergent contact terms present in the correlators. The divergence properties and regularization of Ward identities of flowed gauge fields are discussed extensively in [19].

In this paper we are particularly interested in renormalizing the EMT of a simpler version of the holographic dual theory, which is the class of  $3d$  massless scalar QFTs with  $\varphi$  in the adjoint of  $SU(N)$  and a  $\varphi^4$  interaction, regularized on a Euclidean space-time lattice [20]. This class of massless, super-renormalisable QFT, with the coupling  $g$  of mass dimension one, suffers from severe infrared (IR) divergences in perturbation theory. Perturbative calculations of correlation functions and renormalisation parameters, such as the critical mass or the EMT renormalisation coefficients, contain IR logarithms, which makes the results dependent on the IR regulator. The non-perturbative IR finiteness of super-renormalisable theories, where the dimensionful coupling constant acts as the IR regulator, has been conjectured and discussed in [21, 22], and has been confirmed non-perturbatively for the theory under consideration in [23]. This allows us to renormalise the theory non-perturbatively without IR ambiguity. In this paper we focus on the  $N = 2$  theory; theories with  $N > 2$  and the large  $N$  limit will be discussed in a later publication.

This paper is organised as follows. In Sec. II we first introduce the scalar  $SU(N)$  theory in the continuum and on the lattice, and we define the EMT operator and correlators. We also define the Wilson Flow, as well as the relevant correlators at finite flowtime. In Sec. III we list the parameters of the simulated ensembles for this study, and summarise the results of the critical mass determined non-perturbatively in [23]. In Sec. IV we discuss the procedure to renormalise the EMT using flowed correlators, and finally present the numerical results for the  $N = 2$  theory. We have also included a number of appendices. In appendix 1 we summarise the method to evaluate massless lattice scalar integrals in  $3d$ . In appendices 2, 3 and 4, we present the lattice perturbation theory calculations for the EMT operator mixing, correlators at vanishing flowtime, and correlators at finite flowtime respectively.

## II. GENERALITIES/DEFINITIONS

### A. Continuum and lattice $SU(N)$ scalar action

The theory under consideration here is a three-dimensional Euclidean scalar  $\mathfrak{su}(N)$  valued  $\varphi^4$  theory,

$$S[\varphi] = \int d^3x \text{Tr} \left[ (\partial_\mu \varphi(x))^2 + (m^2 - m_c^2) \varphi(x)^2 + \lambda \varphi(x)^4 \right], \quad (1)$$

with fields  $\varphi = \varphi^a(x) T^a$  where  $\varphi^a(x)$  is real, and  $T^a$  are the generators of  $SU(N)$ , which are normalised so that  $\text{Tr} [T^a T^b] = \frac{1}{2} \delta_{ab}$ . Here  $\lambda$  is the  $\varphi^4$  coupling constant with mass dimension one (which does not renormalise),  $m^2$  is the bare mass. Since the mass of the theory renormalises additively, we include the mass counterterm, or *critical mass*  $m_c^2(g)$ , *i.e.* the value of the bare mass such that the renormalised theory is massless. To make the 't Hooft scaling explicit, hereafter the following rescaled version of the action will be used,

$$S[\phi] = \frac{N}{g} \int d^3x \text{Tr} \left[ (\partial_\mu \phi(x))^2 + (m^2 - m_c^2) \phi(x)^2 + \phi(x)^4 \right], \quad (2)$$

which can be obtained by identifying  $\phi = \sqrt{g/N} \varphi$  and  $\lambda = g/N$  from Eq. (1).

The theory is discretised on a three-dimensional Euclidean lattice by replacing the action with

$$S[\phi] = \frac{a^3 N}{g} \sum_{x \in \Lambda^3} \text{Tr} \left[ \sum_{\mu} (\delta_\mu \phi(x))^2 + (m^2 - m_c^2) \phi(x)^2 + \phi(x)^4 \right]. \quad (3)$$

Here  $\delta_\mu$  is the forward finite difference operator defined by,  $\delta_\mu \phi(x) = a^{-1} [\phi(x + a\hat{\mu}) - \phi(x)]$ , where  $\hat{\mu}$  is the unit vector in direction  $\mu$ ,  $\Lambda^3$  is a lattice with cubic geometry containing  $N_L^3$  points (with periodic boundary conditions), and  $a$  the lattice spacing.

## B. Energy-momentum tensor

In the continuum theory, the energy-momentum tensor (EMT)  $T_{\mu\nu}$  is defined as the conserved current of space-time symmetries. For our scalar  $SU(N)$  theory, it is given by [24]

$$T_{\mu\nu} = \frac{N}{g} \text{Tr} \left\{ 2(\partial_\mu\phi)(\partial_\nu\phi) - \delta_{\mu\nu} \left[ \sum_\rho (\partial_\rho\phi)^2 + (m^2 - m_c^2)\phi^2 + \phi^4 \right] + \xi \left( \delta_{\mu\nu} \sum_\rho (\partial_\rho\phi)^2 - (\partial_\mu\phi)(\partial_\nu\phi) \right) \right\}. \quad (4)$$

Here the term multiplying  $\xi$  is the improvement term. In the continuum theory, due to translational invariance, the EMT satisfies Ward-Takahashi identities (WI) of the form

$$\langle \partial^\mu T_{\mu\nu}(x) P(y) \rangle = - \left\langle \frac{\delta P(y)}{\delta \phi(x)} \partial_\nu \phi(x) \right\rangle \quad (5)$$

where  $P(y)$  is any composite operator inserted at point  $y$ . If  $P$  is such that the RHS of Eq. (5) is finite for separated points  $x \neq y$ , the LHS correlation function, which contains the divergence of the EMT, is finite up to contact terms. For this theory, it can be shown that the insertion of  $T_{\mu\nu}$  does not introduce new UV divergences (as discussed in more detail in appendix 2). The improvement term is identically conserved and trivially satisfies Eq. (5). Therefore  $\xi$  will be set to 0 for the remainder of the text.

On the lattice, the continuous translational symmetry is broken into the discrete subgroup of lattice translations; because of this a naïve discretisation of the EMT on the lattice,

$$T_{\mu\nu}^0 = \frac{N}{g} \text{Tr} \left\{ 2(\bar{\partial}_\mu\phi)(\bar{\partial}_\nu\phi) - \delta_{\mu\nu} \left[ \sum_\rho (\bar{\partial}_\rho\phi)^2 + (m^2 - m_c^2)\phi^2 + \phi^4 \right] \right\}, \quad (6)$$

which is obtained by replacing the partial derivatives  $\partial_\mu\phi(x)$  with the central finite difference  $\bar{\partial}_\mu\phi(x) = \frac{1}{2a} [\phi(x + a\hat{\mu}) - \phi(x - a\hat{\mu})]$  (this is chosen in order to obtain a Hermitian EMT), does not satisfy the WI Eq. (5). Now, the WI on the lattice includes an additional term [7],

$$\langle \bar{\partial}^\mu T_{\mu\nu}^0(x) P(y) \rangle = - \left\langle \frac{\bar{\delta} P(y)}{\bar{\delta} \phi(x)} \bar{\partial}_\nu \phi(x) \right\rangle + \langle X_\nu(x) P(y) \rangle. \quad (7)$$

Here  $\frac{\bar{\delta} P(y)}{\bar{\delta} \phi(x)}$  is obtained by replacing the fields and derivatives in the continuum functional derivative  $\frac{\delta P(y)}{\delta \phi(x)}$  with their lattice counterparts, and  $X_\nu$  is an operator proportional to  $a^2$ , which classically vanishes in the continuum limit. However, radiative corrections cause the expectation value  $\langle X_\nu(x) P(y) \rangle$  to produce a linearly  $a^{-1}$  divergent contribution to the WI. Therefore, the naïvely discretised EMT will not reproduce the continuum WI when the regulator is removed;  $T_{\mu\nu}$  has to be renormalised by adjusting the coefficients of a linear combination of lower-dimensional operators which satisfy the same symmetries.

In four dimensions, it has been shown in [7] that  $T_{\mu\nu}$  potentially mixes with five lower-dimensional operators, which can generate such divergences. However, in three dimensions, dimensional counting indicates that divergent mixing can only occur with  $O_3 = \delta_{\mu\nu} \frac{N}{g} \text{Tr } \phi^2$ . The *renormalised EMT* on the lattice can therefore be defined as an operator mixing,

$$T_{\mu\nu}^R = T_{\mu\nu}^0 - C_3 \delta_{\mu\nu} \frac{N}{g} \text{Tr } \phi^2. \quad (8)$$

$C_3$  has to be tuned to satisfy the continuum WI up to discretisation effects when the regulator is removed.

At leading order (LO)  $O(g)$  (*i.e.* one-loop) in lattice perturbation theory,  $C_3$  is shown to be

$$C_3^{1\text{-loop}} = \frac{g}{a} c_3^{1\text{-loop}}, \quad (9)$$

where

$$c_3^{1\text{-loop}} = \left(2 - \frac{3}{N^2}\right) \left(\frac{6W_0 - 1}{12}\right), \quad (10)$$

$$Z_0 = a \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^3 k}{(2\pi)^3} \frac{1}{\hat{k}^2} = 0.252731\dots, \quad (11)$$

for lattice momentum  $\hat{k} = \frac{2}{a} \sin(ka/2)$ , see appendix 2. In the continuum limit,  $a \rightarrow 0$ , the value of  $C_3^{1\text{-loop}}$  diverges. To account for this leading behaviour, we define

$$C_3 = \frac{g}{a} c_3, \quad (12)$$

and by determining the value of  $c_3$  non-perturbatively, we are able to renormalise the EMT on the lattice. As mentioned in the introduction, the two-loop contribution diverges logarithmically with the IR regulator.

Before discussing the strategy to obtain the value of  $c_3$  non-perturbatively, we define an EMT correlator which will be useful in our analysis. Consider the momentum-space two-point correlator,

$$C_{\mu\nu}(q) = \frac{N}{g} a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle T_{\mu\nu}^R(x) \text{Tr } \phi^2(0) \rangle. \quad (13)$$

Here  $q = \frac{2\pi}{aN_L} n$  is the momentum where  $n$  is a vector with integer components. This particular correlator is chosen since  $\text{Tr } \phi^2$  is the lowest dimension non-vanishing scalar operator in the theory. By inserting the definitions in Eqs. (8) and (12), we obtain

$$C_{\mu\nu}(q) = C_{\mu\nu}^0(q) - \frac{g}{a} c_3 \delta_{\mu\nu} C_2(q), \quad (14)$$

where

$$C_{\mu\nu}^0(q) = \frac{N}{g} a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle T_{\mu\nu}^0(x) \text{Tr } \phi^2(0) \rangle, \quad (15)$$

$$C_2(q) = \left( \frac{N}{g} \right)^2 a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle \text{Tr } \phi^2(x) \text{Tr } \phi^2(0) \rangle. \quad (16)$$

The superscript <sup>0</sup> is used to distinguish the naïvely discretised EMT from the renormalised one.

On the lattice, the correlator  $C_{\mu\nu}(q)$  has a contact term which arises when the operators coincide in position space; in momentum space, this manifests as a constant (momentum-independent) contribution  $C_{\mu\nu}(0)$  which needs to be subtracted before the proper continuum limit can be obtained,

$$\hat{C}_{\mu\nu}(q) = C_{\mu\nu}(q) - C_{\mu\nu}(0). \quad (17)$$

By dimensional counting,  $C_{\mu\nu}(0)$  has a leading  $a^{-1}$  divergent contribution. We therefore define

$$C_{\mu\nu}(0) = \frac{\kappa}{a} \delta_{\mu\nu}. \quad (18)$$

Lattice perturbation theory at next-to-leading order (NLO) gives the following results for the various expressions from above (details can be found in appendix 3):

$$\hat{C}_{\mu\nu}^{1\text{-loop}}(q) = \frac{N^2 q}{64} \left( 1 - \frac{1}{N^2} \right) \pi_{\mu\nu} + \mathcal{O}(a), \quad (19)$$

$$\hat{C}_{\mu\nu}^{2\text{-loop}}(q) = -\frac{N^2 q}{256} g_{\text{eff}} \left( 1 - \frac{1}{N^2} \right) \left( 2 - \frac{3}{N^2} \right) \pi_{\mu\nu} + \mathcal{O}(a), \quad (20)$$

$$\kappa = -\frac{N^2}{2} \left( 1 - \frac{1}{N^2} \right) \left( \frac{6W_0 - 1}{12} \right), \quad (21)$$

where  $g_{\text{eff}} = \frac{g}{|q|}$  is the *effective coupling*, and  $\pi_{\mu\nu} = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$  the *transverse projector*. It can be seen that  $\hat{C}_{\mu\nu}(q)$  has a leading  $N^2 q$  behaviour; an overall  $q$  is expected from  $\hat{C}_{\mu\nu}(q)$  being a dimension one correlator, where at LO (*i.e.* one-loop) there is no coupling constant dependence, and at NLO (*i.e.* two-loops) we encounter the first order expansion in the effective coupling  $g_{\text{eff}}$ . In both terms, the planar diagram contributes to the leading  $N^2$  factor, whereas non-planar diagrams can be seen as  $\frac{1}{N^2}$  corrections to the leading planar diagram. The fact that the finite piece of  $\hat{C}_{\mu\nu}(q)$  is proportional to the transverse projector is a consequence of the WI.

### C. Wilson flow

From above, we see that the correlator  $C_{\mu\nu}^0(q)$  contains divergent contributions in terms of  $\frac{g}{a} c_3$  from the operator mixing, as well as  $\frac{\kappa}{a}$  due to the contact term. In order to non-perturbatively

renormalise the EMT operator, we need to isolate the contact term from the operator mixing, and we will utilize the method of the Wilson Flow [18] to achieve this. For our scalar field  $\phi(x)$ , define a flowed field  $\rho(t, x)$  governed by the flow equations,

$$\partial_t \rho(t, x) = \partial^2 \rho(t, x), \quad \rho(t, x)|_{t=0} = \phi(x), \quad (22)$$

where  $\partial^2 = \sum_\mu \partial_\mu^2$  is the Laplacian, and  $t$  is the *flowtime*, a new parameter introduced into the theory. Solving by means of Fourier transformation, one finds

$$\tilde{\rho}(t, k) = e^{-k^2 t} \tilde{\phi}(k), \quad (23)$$

where  $\tilde{\rho}(t, k)$  is the Fourier transform of  $\rho(t, x)$ ; the flow effectively smears the field with radius  $\sqrt{4t}$ .

The Wilson flow suppresses high-momentum modes exponentially, and thereby regulates the divergent contact term present in the EMT correlator  $C_{\mu\nu}^0(q)$ . We are therefore able to isolate the divergent mixing  $c_3$  from the divergent contact term. There have been extensive discussions of various implementations of the Wilson Flow for renormalising the EMT, which can be found in [12, 14–17, 19].

In our case, we are interested in determining the flowed correlator

$$C_{\mu\nu}(t, q) = \frac{N}{g} a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle T_{\mu\nu}^R(x) \text{Tr} \rho^2(t, 0) \rangle, \quad (24)$$

at finite flowtime. Here we replaced the operator  $\text{Tr} \phi^2(x=0)$  with the operator  $\text{Tr} \rho^2(t, x=0)$  at finite flowtime  $t$ , and kept the renormalised EMT operator  $T_{\mu\nu}^R(x)$  at flowtime  $t=0$ . By definition,  $C_{\mu\nu}(0, q) = C_{\mu\nu}(q)$ . Since the operator mixing  $c_3$  is local to the EMT operator  $T_{\mu\nu}(x)$ , it is not affected by replacing the probe  $\text{Tr} \phi^2(x=0)$  with the one at finite flowtime  $\text{Tr} \rho^2(t, x=0)$ . On the other hand, the divergent contact term  $C_{\mu\nu}(t, q=0)$  is suppressed. More explicitly we similarly define

$$\hat{C}_{\mu\nu}(t, q) = C_{\mu\nu}(t, q) - C_{\mu\nu}(t, 0), \quad (25)$$

$$C_{\mu\nu}(t, 0) = \delta_{\mu\nu} K(t). \quad (26)$$

As recorded in Eqs. (18) and (21), at vanishing flowtime,  $K(t=0) = \frac{\kappa}{a}$ . However, as calculated in Eq. (A.33), at small finite flowtime,

$$K(t) = \frac{\omega}{\sqrt{t}} + \mathcal{O}(\sqrt{t}), \quad (27)$$



where at LO in perturbation theory,

$$\omega = -\frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) \left(\frac{\sqrt{2}}{24\pi^{3/2}}\right). \quad (28)$$

We utilise this small  $t$  expansion to remove the contact term contribution in our correlation function in order to obtain the value of  $c_3$ . The strategy will be explained in further detail in section Sec. IV.

In analogy to Eqs. (14)–(16) we have the relations

$$C_{\mu\nu}(t, q) = C_{\mu\nu}^0(t, q) - \frac{g}{a} c_3 \delta_{\mu\nu} C_2(t, q), \quad (29)$$

where

$$C_{\mu\nu}^0(t, q) = \frac{N}{g} a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle T_{\mu\nu}^0(x) \text{Tr} \rho^2(t, 0) \rangle, \quad (30)$$

$$C_2(t, q) = \left(\frac{N}{g}\right)^2 a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle \text{Tr} \phi^2(x) \text{Tr} \rho^2(t, 0) \rangle. \quad (31)$$

Having defined the above correlation functions, we can now non-perturbatively renormalize the EMT on the lattice. The renormalisation scheme is defined by first imposing the Ward Identity

$$\bar{q}_\mu \hat{C}_{\mu\nu}(t, q) = 0 \quad (32)$$

on all lattice ensembles. Here  $\bar{q} = \frac{1}{a} \sin(aq)$  is the lattice momentum. This condition is imposed on a specific choice of momentum  $aq$  which is present in all ensembles, as constrained by the quantized momentum modes. This step gives a value of  $c_3$  for each choice of mass, volume and 't Hooft coupling. We then extrapolate the value  $c_3$  towards the massless and infinite volume limit to obtain  $\bar{c}_3$ . This defines a massless renormalisation scheme, which is independent of the volume. We will also investigate the dependence of  $c_3$  on the value of the 't Hooft coupling  $ag$ . The implementation of the scheme and the numerical fits results will be explained in Sec. IV.

### III. LATTICE SIMULATIONS

#### A. Simulation setup

The theory is simulated using the Hybrid Monte Carlo algorithm [25], which was implemented using the Grid library [26, 27]. For this paper, we will focus on the  $N = 2$  theory. The simulated volumes  $N_L^3$ , 't Hooft coupling in lattice unit  $ag$  (or equivalently the dimensionless lattice-spacing),

$ag$	$(am)^2$	$N_L^3$	trajectories	sample frequency
0.1	-0.0305, -0.031	$64^3$	1,500,000	50
0.2	-0.061, -0.062	$128^3$	500,000	50
0.3	-0.092, -0.091	$256^3$	200,000	100

Table I. For each 't Hooft coupling  $ag$ , two bare masses are simulated in three volumes

and bare masses  $(am)^2$  are listed in Table I. For each of the three 't Hooft couplings, two bare masses in the vicinity of the critical mass have been simulated (see Table II).

Correlation function computations are performed using the Hadrons library [28] and the data analysis is based on the LatAnalyze library [29]. The data and analysis code are available at [30–32]. Data analysis is performed using bootstrap resampling [33], and only every 50th or 100th trajectory is sampled in order to reduce auto-correlation. The first 5000 trajectories are discarded to ensure the ensembles are thermalized. A representative example of the value of the observable  $M^2 = \text{Tr} \left( a^3 \sum_{x \in \Lambda^3} \phi(x) \right)^2$  across one HMC simulation ( $ag = 0.1, N_L = 128, (am)^2 = -0.031$ ) is shown in Fig. 1.

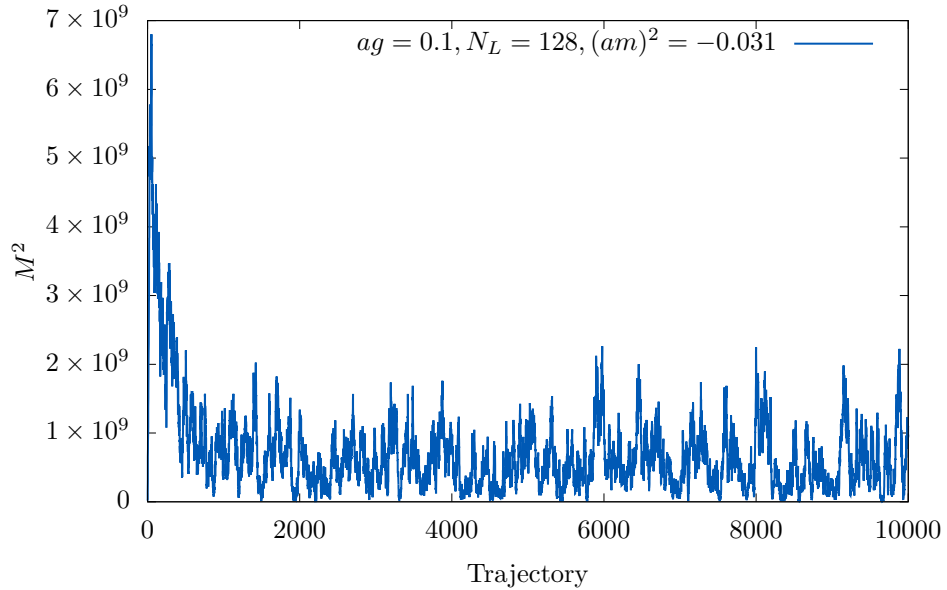


Figure 1. Example of observable  $M^2 = \text{Tr} \left( a^3 \sum_{x \in \Lambda^3} \phi(x) \right)^2$  for ensemble with  $ag = 0.1, N_L = 128, (am)^2 = -0.031$

## B. Critical mass determination

To extrapolate to the massless point, the renormalised mass of the ensembles have to be determined, which requires the critical masses for each lattice spacing as input. These have been determined in [23, 34, 35] at two-loops in lattice perturbation theory, as well as non-perturbatively by analysing the finite-size scaling of the Binder cumulant. The relevant masses are summarised in Table II.

$ag$	1-loop	2-loop	Non-perturbative
0.1	-0.03159	-0.03125	-0.0313408(38)
0.2	-0.06318	-0.06194	-0.0622974(98)
0.3	-0.09477	-0.09208	-0.092935(16)

Table II. The critical masses  $(am_c)^2$  in the infinite volume limit are calculated at NLO in lattice perturbation theory, as well as non-perturbatively in [23], which are listed for each 't Hooft coupling  $ag$ . These are used in the later global fit to obtain  $c_3$  in the massless limit.

## IV. RENORMALISATION OF THE EMT

The renormalisation condition Eq. (32) implies that  $\hat{C}_{\mu\nu}(t, q)$  is purely transverse, *i.e.* ,

$$\hat{C}_{\mu\nu}(t, q) = F(t, q)\bar{\pi}_{\mu\nu} \quad (33)$$

where  $\bar{\pi}_{\mu\nu} = \delta_{\mu\nu} - \frac{\bar{q}_\mu \bar{q}_\nu}{\bar{q}^2}$  is the transverse projector with lattice momentum  $\bar{q}$ . In other words,  $\hat{C}_{\mu\nu}$  vanishes in the direction with purely longitudinal momentum. For example, picking the momentum to be purely in the direction  $q_l = (q_0, q_1, q_2) = (0, 0, q_2)$ ,

$$\hat{C}_{22}(t, q_l) = 0. \quad (34)$$

Substituting the definition of  $\hat{C}_{\mu\nu}(t, q_l)$  from Eqs. (25) and (29), we obtain

$$\hat{C}_{22}(t, q_l) = C_{22}(t, q_l) - C_{22}(t, 0) = C_{22}^0(t, q_l) - \frac{g}{a}c_3 C_2(t, q_l) - K(t) = 0 \quad (35)$$

$$\rightarrow c_3 = \frac{a}{g} \frac{C_{22}^0(t, q_l)}{C_2(t, q_l)} - f_g(g\sqrt{t}, q_l), \quad (36)$$

where

$$f_g(g\sqrt{t}, q_l) = \frac{a}{g} \frac{K(t)}{C_2(t, q_l)}. \quad (37)$$

Using the one-loop perturbative expressions for  $K(t)$  and  $C_2(t, q)$  from Eqs. (A.27) and (A.33), this gives

$$f_g(g\sqrt{t}, q_l) = \frac{\omega'(q_l)}{g\sqrt{t}} + \mathcal{O}\left(q_l\sqrt{t}\right), \quad (38)$$

where  $\omega'(q) = \frac{\sqrt{2}(aq)}{3\pi^{3/2}}$ . (Details can be found in appendix 4). The strategy to obtain the value of  $c_3$  is to first flow the correlators to a range of small finite flowtimes, picking a fixed momentum  $aq_l$ . Then, utilising Eq. (36), we fit the ratio on the left hand side of

$$\frac{a}{g} \frac{C_{22}^0(t, q_l)}{C_2(t, q_l)} = c_3 + f_g(g\sqrt{t}, q_l), \quad (39)$$

as a function of the *physical flowtime*  $g\sqrt{t}$ . We have tested a range of fit functions for  $f_g$ , and have found that the fit ansatz

$$\frac{a}{g} \frac{C_{22}^0(t, q_l)}{C_2(t, q_l)} = c_3 + \frac{\Omega}{g\sqrt{t}} \quad (40)$$

provides a very good fit to the data. Here we keep the first term linear in the *inverse physical flowtime*  $\frac{1}{g\sqrt{t}}$  from Eq. (38), and leave  $\Omega$  and  $c_3$  as fit parameters. From the fit we can extrapolate  $c_3$  from the  $y$ -intercept.

### A. Numerical results

Picking the fit ranges for the physical flowtime  $g\sqrt{t}$  requires special attention. They must first be sufficiently small to justify the small flowtime expansion of Eq. (38). This also ensures the smearing radius is sufficiently smaller than the length of the lattice ( $gL = gaN_L$ ) such that there will be small finite volume contributions from the boundaries. The physical flowtime must also be larger than the lattice spacing ( $ag$ ) such that actual smearing occurs across lattice points. We therefore impose the range to be between  $ag < g\sqrt{t} < 1$ . The momentum  $q_l$  is chosen by the lowest discrete momentum from the smallest volume for each lattice spacing.

The fits are shown in Fig. 2. The fit results including the choice of momentum and the values of  $c_3$  for each ensemble are summarised in Table III.

In order to assess the mass, volume and lattice-spacing dependence of the value of  $c_3$ , we perform a global fit

$$c_3(\overline{m_R^2}, gL, ag) = \overline{c}_3 + p_0 \overline{m_R^2} + \frac{p_1}{gL} + p_2(ag), \quad (41)$$

where  $\overline{m_R^2} = (m^2 - m_c^2)/g^2$  is the dimensionless renormalised mass (The values of  $m_c^2$  are summarised in Table II),  $gL$  is the dimensionless length of the lattice, and  $ag$  the dimensionless lattice spacing.

$ag$	$N_L$	$(am)^2$	$a q_t $	dof	$\chi^2/dof$	$p$ -value	$c_3$
0.1	64	-0.0305	0.098	4	1.60	0.17	0.0531(35)
0.1	64	-0.031	0.098	3	0.15	0.93	0.0467(39)
0.1	128	-0.0305	0.098	5	0.06	1.00	0.0334(90)
0.1	128	-0.031	0.098	5	1.00	0.42	0.0445(85)
0.1	256	-0.0305	0.098	3	0.18	0.91	0.015(25)
0.1	256	-0.031	0.098	3	1.26	0.28	0.033(23)
0.2	64	-0.061	0.098	5	1.14	0.34	0.0466(23)
0.2	64	-0.062	0.098	5	1.64	0.15	0.0519(24)
0.2	128	-0.061	0.098	5	0.70	0.62	0.0464(53)
0.2	128	-0.062	0.098	5	0.67	0.65	0.0402(41)
0.2	256	-0.061	0.098	2	0.27	0.77	0.050(14)
0.2	256	-0.062	0.098	2	0.04	0.96	0.059(12)
0.3	64	-0.091	0.098	5	0.56	0.73	0.0478(19)
0.3	64	-0.092	0.098	5	0.85	0.51	0.0488(15)
0.3	128	-0.091	0.098	6	0.52	0.79	0.0484(29)
0.3	128	-0.092	0.098	5	0.85	0.52	0.0430(39)
0.3	256	-0.091	0.098	3	0.14	0.94	0.0643(97)
0.3	256	-0.092	0.098	3	0.52	0.67	0.0645(89)

Table III. For each simulation, we perform the fit for the value of  $c_3$  using Eq. (40). The momentum chosen is the first discrete momentum from the smallest volume in each lattice spacing, and the flowtime fit range is bounded by  $ag < g\sqrt{t} < 1$ .

From the fit data, it was observed that when two or more parameters are included in the global fits, the fits are not improved since the parameters are already compatible with 0; we therefore only retain the models with no extra parameter dependence (i.e. a constant fit), and models with one parameter. The fit values for  $\bar{c}_3$  using different models, along with their corresponding definitions, are summarised in Table IV. The constant fit for model 1 is shown in Fig. 3, and the fits in the corresponding free parameter for models 2 - 4, i.e.  $\overline{m_R^2}$ ,  $gL$ , and  $ag$ , are shown in Fig. 4.

In Fig. 5, the fit values of  $\bar{c}_3$  for the different models from Table IV, along with the one-loop value  $c_3^{1\text{-loop}}$  from Eq. (10) are plotted. The  $p$ -values for each of the fit models are acceptable ( $p > 0.05$ ), but the extra parameters are all compatible with zero; there is only very weak dependence on the mass, volume and lattice spacing for our simulated ensembles. Again, it is worth pointing out that the finiteness of this value in the infinite volume limit is a non-perturbative feature of the theory. In perturbation theory, all terms of  $\mathcal{O}(g^2)$  are IR divergent and depend on the IR regulator; but

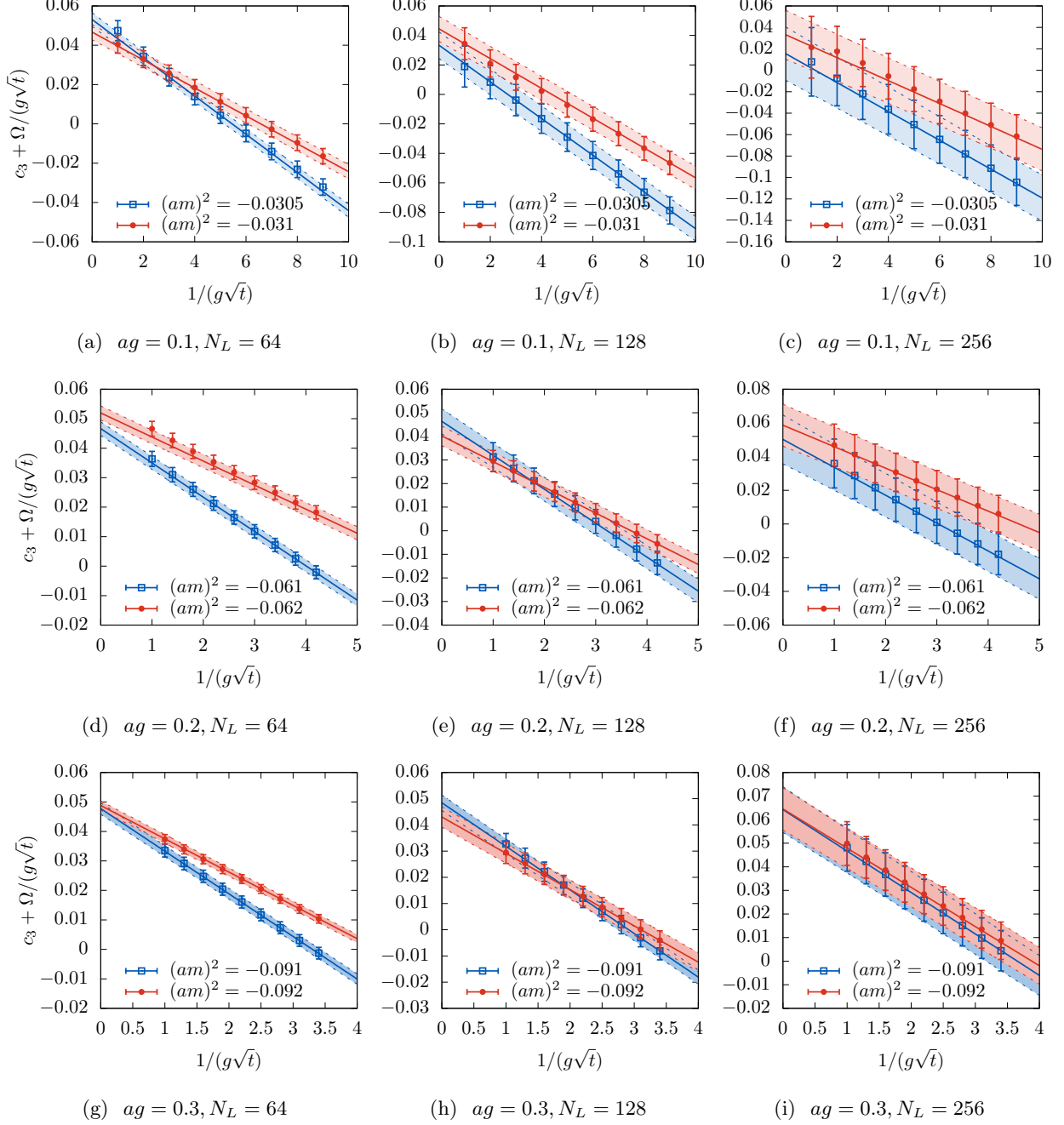


Figure 2. Plots showing  $c_3$  against the inverse physical flowtime  $\frac{1}{g\sqrt{t}}$  using Eq. (40) for three 't Hooft couplings and 3 volumes. The red and blue data points are for the lighter and heavier mass simulations respectively, and the corresponding error bands in the fit are from statistical uncertainty. The value of  $c_3$  is the  $y$ -intercept on the fit.

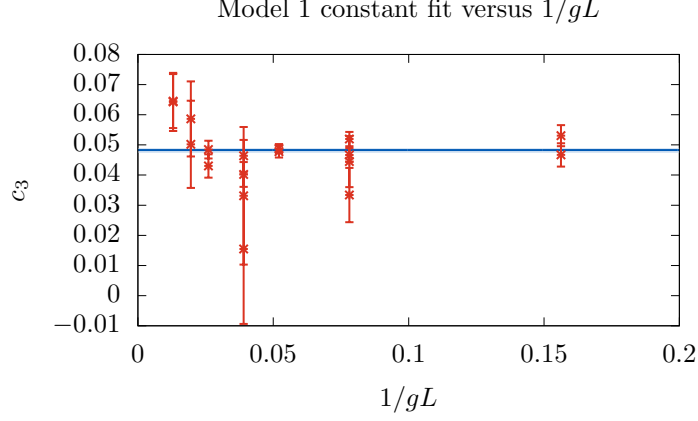


Figure 3.  $c_3$  global fit using Model 1 (constant fit). The value of  $c_3$  is plotted against  $\frac{1}{gL}$

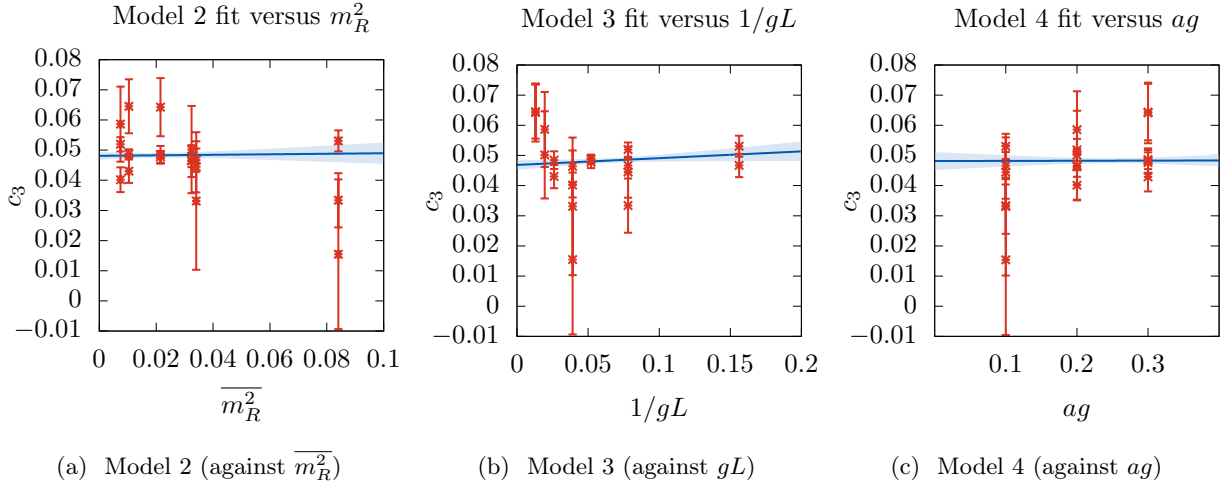


Figure 4.  $c_3$  global fits model 2, 3, 4, as defined in Table IV. Each plot is plotted against the respective free fitting parameter for each model, *i.e.*  $\overline{m_R^2}$ ,  $gL$ , and  $ag$ ; the value for  $\overline{c_3}$  is the  $y$ -intercept of the fit line.

model	fit function	$\overline{c_3}$	$p_0$	$p_1$	$p_2$	dof	$\chi^2/\text{dof}$	$p$ -value
1	$c_3 = \overline{c_3}$	0.04828(81)				17	1.40	0.12
2	$c_3 = \overline{c_3} + p_0 \overline{m_R^2}$	0.0481(12)	0.009(44)			16	1.49	0.09
3	$c_3 = \overline{c_3} + \frac{p_1}{gL}$	0.0469(17)		0.022(23)		16	1.43	0.12
4	$c_3 = \overline{c_3} + p_2(ag)$	0.0481(30)			0.001(12)	16	1.49	0.09

Table IV. For each global fit model, we include one or no parameters from Eq. (41) to fit for the value of  $\overline{c_3}$ . For reference, the 1-loop perturbative value, Eq. (10), gives  $c_3^{1\text{-loop}} \approx 0.05379$

as shown in [23] the theory is in fact non-perturbatively IR finite, where the dimensionful coupling effectively acts as the IR regulator in the infinite volume limit. Comparing the non-perturbative result for  $\bar{c}_3$  with the one-loop perturbative value, the non-perturbative value is approximately 10% smaller than the one-loop result. This is qualitatively expected, as the higher order terms in perturbation theory (with the IR regulator replaced by the coupling) changes sign at every order, and the two-loop result is a correction of the opposite sign to the one-loop value.

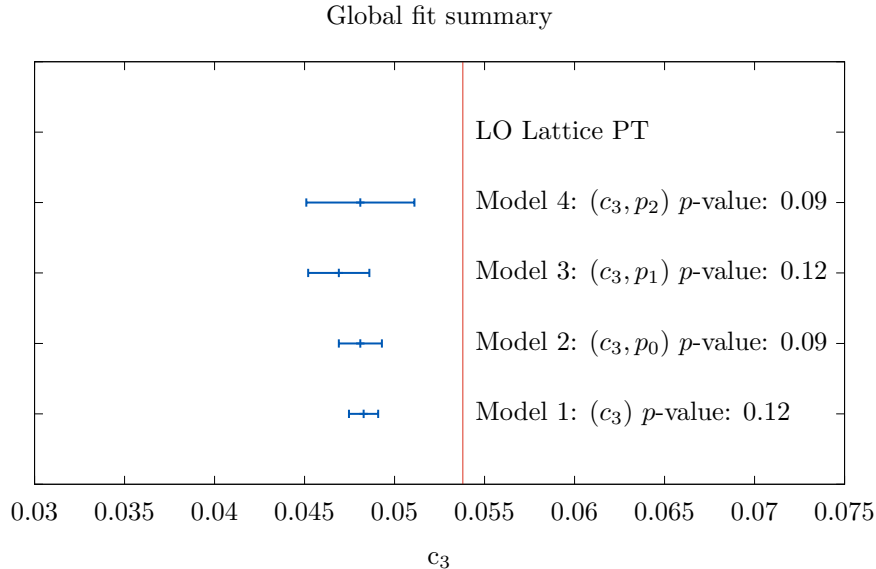


Figure 5. The value of  $\bar{c}_3$  from each of the model in Table IV is represented with a blue point, and the red line shows the 1-loop perturbation theory value from Eq. (10).

## V. CONCLUSION AND OUTLOOK

We have presented a procedure to non-perturbatively renormalise the EMT on the lattice for a three-dimensional scalar QFT with a  $\varphi^4$  interaction and field  $\varphi$  in the adjoint of  $SU(N)$ . We have also presented numerical results of the EMT operator mixing for the theory with  $N = 2$ . The method utilises the Wilson Flow to define a probe at positive flowtime, which can eliminate the divergent contact term present in EMT correlator. This allows us to determine the mixing coefficient with the lower-dimensional operator  $\delta_{\mu\nu} \frac{N}{g} \text{Tr} \phi^2$ . This ensures that the Ward Identity can be restored in the continuum limit, up to cut-off effects.

The context of our investigation is to predict the CMB power spectrum for holographic cosmological models, and to test them against observational data. The next step of the investigation is to determine the renormalised EMT two-point function,  $C_{\mu\nu\rho\sigma}(q) = \langle T_{\mu\nu}(q) T_{\rho\sigma}(-q) \rangle$ , for this class



of scalar theories. This two-point function can be used to compute the primordial CMB power spectra in the holographic cosmology framework. On the lattice, this correlator contains a large contact term of order  $\mathcal{O}(a^{-3})$ . This large contact term presents significant statistical noise to the signal of the renormalised two-point function. We are currently exploring using the Wilson Flow to eliminate the presence of such a contact term, which will allow us to make a fully non-perturbative prediction for the CMB power spectra with the  $SU(N)$  scalar theory as the dual theory.

We are also working towards simulating and performing the renormalisation of the EMT for three-dimensional QFTs with adjoint  $SU(N)$  scalars coupled to gauge fields. This is the class of theories preferred by the fit of the perturbative predictions to Planck data [6]. Much work has been performed in studying the EMT on the lattice for gauge theories [36] and gauge theories with fermions [8]. The implementation of the Wilson Flow for renormalising the EMT has also been studied for gauge theories [16, 19]. We are exploring related methods to perform renormalisation of the EMT for theories with scalar fields coupled to gauge fields. This will take us closer to fully testing the viability of holographic cosmological models as a description of the very early Universe.

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## Appendix: Lattice perturbation theory calculations

In this appendix we present the details of the lattice perturbation theory (LPT) calculations in Sec. II. We will first evaluate two lattice scalar integrals in appendix 1, which are necessary to calculate the EMT  $c_3$  coefficient mixing in appendix 2, the correlators  $C_2(q)$ ,  $C_{\mu\nu}(q)$  at vanishing flowtime in appendix 3, and the correlators  $C_2(t, q)$ ,  $C_{\mu\nu}(t, q)$  at finite flowtime in appendix 4.

### 1. Massless lattice integrals: $V(q)$ and $I_{\mu\nu}(q)$

To evaluate the relevant massless lattice integrals, we generalise the method used in [37] to three dimensions. Using a set of recursion relations, any massless, one-loop lattice scalar integrals in three dimensions of the form

$$\mathcal{B}_\varepsilon(p; n) = \lim_{\delta \rightarrow 0} \int_{-\pi/a}^{\pi/a} \frac{d^3 k}{(2\pi)^3} \frac{\hat{k}_0^{2n_0} \hat{k}_1^{2n_1} \hat{k}_2^{2n_2}}{(\hat{k}^2 + \varepsilon^2)^{p+\delta}} \quad \text{with} \quad \varepsilon \ll 1, \quad p \in \mathbb{Z}, \quad n \in \mathbb{Z}_+^3, \quad (\text{A.1})$$

can be reduced to a linear combination of two constants,

$$Z_0 = \mathcal{B}_0(1; \{0, 0, 0\}) \approx 0.252731009858663 \quad \text{and} \quad Z_1 = \frac{\mathcal{B}_0(1; \{1, 1, 0\})}{3} \approx 0.181058342883210. \quad (\text{A.2})$$

Here,  $\hat{k} = \frac{2}{a} \sin(ka/2)$  is the lattice momentum. These two constants have been determined to high precision using the Lüscher–Weisz coordinate-space method [38].

The two momentum-dependent scalar lattice integrals required for the following LPT calculations are

$$V(q) = \int_{-\pi/a}^{\pi/a} \frac{d^3 k}{(2\pi)^3} \frac{1}{(\hat{k}^2 + m^2)(\widehat{(q-k)}^2 + m^2)}, \quad (\text{A.3})$$

$$I_{\mu\nu}(q) = \int_{-\pi/a}^{\pi/a} \frac{d^3 k}{(2\pi)^3} \frac{\bar{k}_\mu \overline{(q-k)}_\nu}{(\hat{k}^2 + m^2)(\widehat{(q-k)}^2 + m^2)}, \quad (\text{A.4})$$

where  $\bar{k} = \frac{1}{a} \sin(ka)$ . By expanding the expressions in powers of the external momenta [39, 40] and using the recursion relations, in the massless limit, these evaluate to

$$\lim_{m \rightarrow 0} V(q) = \frac{1}{8q} + a \left( \frac{14Z_0 + 9Z_1 - 4}{96} \right) + a^3 q^2 \left( \frac{34Z_0 - 9Z_1 + 4}{27648} \right) + \mathcal{O}(a^5 q^4) \quad (\text{A.5})$$

$$\begin{aligned}
\lim_{m \rightarrow 0} I_{\mu\nu}(q) &= \frac{\delta_{\mu\nu}}{a} \left( \frac{1-6Z_0}{12} \right) + \frac{q}{64} (2\delta_{\mu\nu} - \pi_{\mu\nu}) + a \left[ \frac{Z_0}{16} \delta_{\mu\nu} q_\mu^2 + q^2 (2\pi_{\mu\nu} - 3\delta_{\mu\nu}) \left( \frac{10Z_0 - 9Z_1 + 4}{1152} \right) \right] \\
&\quad + \frac{a^3}{552960} \left[ 4(\delta_{\mu\nu}(q_1^4 + q_2^4 + q_3^4) + 6q^2 q_\mu^2 \delta_{\mu\nu} + 4q_\mu q_\nu (q_\mu^2 + q_\nu^2)) (52 - 86Z_0 - 117Z_1) \right. \\
&\quad \left. + 360\delta_{\mu\nu} q_\mu^4 (6Z_0 + 9Z_1 - 4) + q^4 (4\pi_{\mu\nu} - 5\delta_{\mu\nu}) (236 - 298Z_0 - 531Z_1) \right] + \mathcal{O}(a^5 q^6). \quad (\text{A.6})
\end{aligned}$$

## 2. EMT operator mixing: $c_3$

Here we calculate the perturbative renormalisation of  $T_{\mu\nu}$  on the lattice. The naïve discretisation of the EMT is

$$T_{\mu\nu}^0 = \frac{N}{g} \text{Tr} \left\{ 2(\bar{\delta}_\mu \phi)(\bar{\delta}_\nu \phi) - \delta_{\mu\nu} \left[ \sum_\rho (\bar{\delta}_\rho \phi)^2 + (m^2 - m_c^2) \phi^2 + \phi^4 \right] + \xi \left( \delta_{\mu\nu} \sum_\rho (\bar{\delta}_\rho \phi)^2 - (\bar{\delta}_\mu \phi)(\bar{\delta}_\nu \phi) \right) \right\} \quad (\text{A.7})$$

Here the term multiplying  $\xi$  is the improvement term. Since the improvement term is identically conserved and satisfies the Ward Identities,  $\xi$  has been taken to be 0 in the main text. The calculations here retraces the steps taken for the  $4d$  case in [7].

By considering operators which have a lower dimension than  $T_{\mu\nu}$ , the only operator capable of producing divergent mixing is  $\delta_{\mu\nu} \text{Tr} \phi^2$ . We therefore defined the renormalised EMT in Eq. (8) as

$$T_{\mu\nu}^R = T_{\mu\nu}^0 - C_3 \delta_{\mu\nu} \frac{N}{g} \text{Tr} \phi^2, \quad (\text{A.8})$$

with  $C_3$  being the divergent mixing coefficient. To calculate the mixing coefficient perturbatively, consider the insertion of  $T_{\mu\nu}$  in the two-point correlator, *i.e.*  $\langle \phi^a(x_1) \phi^b(x_2) T_{\mu\nu}(x_3) \rangle$ . The one-loop diagrams are shown in Fig. 6. Both in the continuum and on the lattice, diagram (a) in Fig. 6 is finite, and contributes to the WI. However, for diagram (b), as a result of the breaking of translational invariance, the LPT result diverges, even though in the continuum the PT result is finite (this could be calculated by replacing the lattice momenta  $\hat{q}$  with the continuum momenta  $q$ , and the integration limit by  $\int_{-\infty}^{\infty}$ ).

Using LPT, diagram (b) in Fig. 6 evaluates to

$$B_{\mu\nu}(q) = -\delta_{ab} \left( 2N - \frac{3}{N} \right) \left[ -2I_{\mu\nu}(q) + \delta_{\mu\nu} \left( \sum_\alpha I_{\alpha\alpha}(q) + (m^2 - \xi(\bar{q}_\mu \bar{q}_\nu - \delta_{\mu\nu} \bar{q}^2)) V(q) \right) \right]. \quad (\text{A.9})$$

Using [39], the divergent term of  $B_{\mu\nu}(q)$  can be isolated with  $B_{\mu\nu}(0)$ , while the remaining terms

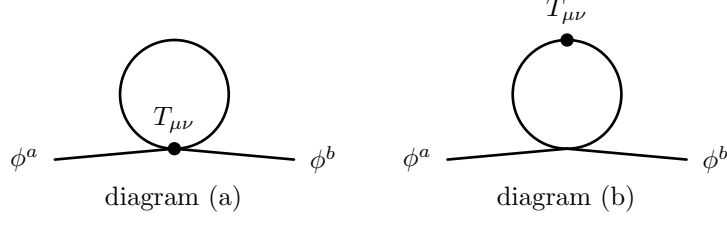


Figure 6. The insertion of  $T_{\mu\nu}$  in a two-point correlator, *i.e.*  $\langle \phi^a(x_1) \phi^b(x_2) T_{\mu\nu}(x_3) \rangle$ , up to one-loop. The black dot represents the insertion of  $T_{\mu\nu}$

are finite or vanish in the continuum limit. This evaluates to

$$\begin{aligned} B_{\mu\nu}(0) &= \frac{\delta_{\mu\nu}}{a} \left( 2N - \frac{3}{N} \right) \left( \frac{6Z_0 - 1}{12} \right) \\ &= \frac{C_3^{1\text{-loop}} N}{g} \delta_{\mu\nu}. \end{aligned} \quad (\text{A.10})$$

Using the definition from Eq. (9),

$$C_3^{1\text{-loop}} = \frac{g}{a} c_3^{1\text{-loop}}, \quad (\text{A.11})$$

to absorb the leading  $\frac{1}{a}$  behaviour, we obtain

$$c_3^{1\text{-loop}} = \left( 2 - \frac{3}{N^2} \right) \left( \frac{6Z_0 - 1}{12} \right). \quad (\text{A.12})$$

This gives the result in Eq. (10).

### 3. Correlators at vanishing flowtime: $C_2(q)$ and $C_{\mu\nu}(q)$

The first two-point correlation function to calculate is defined in Eq. (16):

$$C_2(q) = \left( \frac{N}{g} \right)^2 a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle \text{Tr} \phi^2(x) \text{Tr} \phi^2(0) \rangle. \quad (\text{A.13})$$

The one- and two-loop diagrams are shown in Fig. 7(a) and (b) respectively.

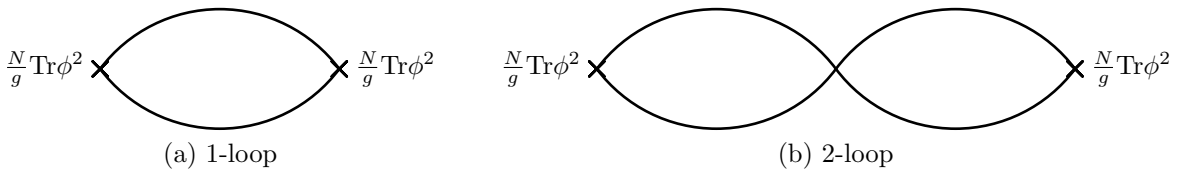


Figure 7. Perturbative expansion of  $C_2(q)$  at 1- and 2-loops.

Note that the two-loop diagram is simply the square of the one-loop diagram up to an overall colour factor. These diagrams evaluate to

$$C_2^{1\text{-loop}}(q) = \text{tr}(T_a T_b) \text{tr}(T_c T_d) (\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) V(q) = \frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) V(q), \quad (\text{A.14})$$

$$\begin{aligned} C_2^{2\text{-loop}}(q) &= -2 \left( \frac{g}{N} \right) \text{tr}(T_a T_b) \text{tr}(T_c T_d T_e T_f) \text{tr}(T_g T_h) (\delta_{ac} (16 \delta_{bd} \delta_{eg} + 8 \delta_{be} \delta_{dg}) \delta_{fh}) V(q)^2 \\ &= -N^2 g \left( 2 - \frac{5}{N^2} + \frac{3}{N^4} \right) V(q)^2. \end{aligned} \quad (\text{A.15})$$

In the massless limit, using Eq. (A.5), these yield

$$C_2^{1\text{-loop}}(q) = \frac{N^2}{16g} \left( 1 - \frac{1}{N^2} \right) \left[ (g/q) + (ag) \frac{14Z_0 + 9Z_1 - 4}{12} + (ag)^3 (q/g)^2 \frac{34Z_0 - 9Z_1 + 4}{3456} + \mathcal{O}((ag)^5) \right], \quad (\text{A.16})$$

$$\begin{aligned} C_2^{2\text{-loop}}(q) &= -\frac{N^2}{64g} \left( 2 - \frac{5}{N^2} + \frac{3}{N^4} \right) \left[ (g/q)^2 + (ag)(g/q) \left( \frac{14Z_0 + 9Z_1 - 4}{6} \right) \right. \\ &\quad \left. + (ag)^2 \frac{(14Z_0 + 9Z_1 - 4)^2}{144} + (ag)^3 (q/g) \left( \frac{34Z_0 - 9Z_1 + 4}{1728} \right) + \mathcal{O}((ag)^4) \right]. \end{aligned} \quad (\text{A.17})$$

Now we evaluate the correlation function in Eq. (13):

$$C_{\mu\nu}(q) = \frac{N}{g} a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle T_{\mu\nu}^R(x) \text{Tr} \phi^2(0) \rangle = C_{\mu\nu}^0(q) - \frac{gc_3}{a} \delta_{\mu\nu} C_2(q), \quad (\text{A.18})$$

where

$$C_{\mu\nu}^0(q) = \frac{N}{g} a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle T_{\mu\nu}^0(x) \text{Tr} \phi^2(0) \rangle, \quad (\text{A.19})$$

$$C_2(q) = \left( \frac{N}{g} \right)^2 a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle \text{Tr} \phi^2(x) \text{Tr} \phi^2(0) \rangle \quad (\text{A.20})$$

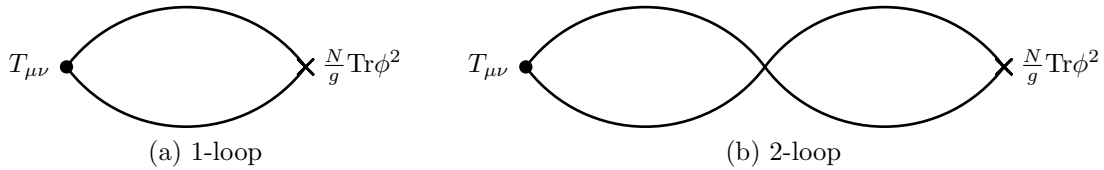


Figure 8. Perturbative expansion of  $C_{\mu\nu}^0(q)$  at 1- and 2-loops.

The relevant one- and two-loop diagrams for the correlator  $C_{\mu\nu}^0(q)$  are shown in Fig. 8(a) and

(b) respectively, and they evaluate to

$$C_{\mu\nu}^{0\text{ 1-loop}}(q) = \frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) (\delta_{\mu\nu} \sum_{\alpha} I_{\alpha\alpha}(q) - 2I_{\mu\nu}(q)) + (\xi(\bar{q}_{\mu}\bar{q}_{\nu} - \delta_{\mu\nu}\bar{q}^2) - \delta_{\mu\nu}m^2) C_2^{1\text{-loop}}(q) \quad (\text{A.21})$$

$$C_{\mu\nu}^{0\text{ 2-loop}}(q) = -N^2 g \left(1 - \frac{5}{2N^2} - \frac{3}{2N^4}\right) (\delta_{\mu\nu} \sum_{\alpha} I_{\alpha\alpha}(q) - 2I_{\mu\nu}(q)) V(q) + (\xi(\bar{q}_{\mu}\bar{q}_{\nu} - \delta_{\mu\nu}\bar{q}^2) - \delta_{\mu\nu}m^2) C_2^{2\text{-loop}}(q). \quad (\text{A.22})$$

At one-loop, Eq. (A.21),  $C_{\mu\nu}^0$  contains only the tree-level EMT, so  $C_{\mu\nu}^{1\text{-loop}} = C_{\mu\nu}^{0\text{ 1-loop}}$ . There is no contribution coming from the operator mixing  $c_3$ , which comes with another order  $O(g)$ . However, the term  $\delta_{\mu\nu} \sum_{\alpha} I_{\alpha\alpha}(q) - 2I_{\mu\nu}(q)$  presents a divergent contact term at  $C_{\mu\nu}^{0\text{ 1-loop}}(0)$ ,

$$\begin{aligned} C_{\mu\nu}^{0\text{ 1-loop}}(0) &= -\frac{N^2}{2a} \left(1 - \frac{1}{N^2}\right) \left(\frac{6Z_0 - 1}{12}\right) \delta_{\mu\nu} \\ &= \frac{\kappa}{a} \delta_{\mu\nu}. \end{aligned} \quad (\text{A.23})$$

The integral producing this contact term is similar to that in  $c_3^{1\text{-loop}}$  in Eq. (A.10), with the only difference being the colour factor. This contact term has to be subtracted before the continuum limit of the correlator is taken.

For the two-loop expression, it can be shown that after subtracting the correlator  $\frac{gc_3}{a} \delta_{\mu\nu} C_2^{1\text{-loop}}(q)$  to renormalise the EMT from Eq. (A.18), the correlator is UV finite; no extra divergences other than the one coming from the operator expansion appear.

#### 4. Correlators at finite flowtime: $C_2(t, q)$ , $C_{\mu\nu}(t, q)$

At finite flowtime, the lattice integrals are regulated by the flowtime  $t$ . In perturbation theory, the kernel for each propagator has an extra exponential factor,  $e^{-tq^2}$ , where  $q$  is the momentum of the propagator. We first evaluate the correlator

$$C_2(t, q) = \left(\frac{N}{g}\right)^2 a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle \text{Tr } \phi^2(x) \text{Tr } \rho^2(t, 0) \rangle \quad (\text{A.24})$$

at finite flowtime. This correlator is obtained by replacing  $\text{Tr } \phi^2(x)$  with  $\text{Tr } \rho^2(t, 0)$  in  $C_2(q)$ . Since the regulated correlators are finite, we look at the continuum limit ( $a \rightarrow 0$ ) of the correlator in perturbation theory. At one-loop, this evaluates to

$$C_2^{1\text{-loop}}(t, q) = \frac{N^2}{16} \left(1 - \frac{1}{N^2}\right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{-tq^2} e^{-t(q-k)^2}}{(k^2 + m^2)((q-k)^2 + m^2)}. \quad (\text{A.25})$$

In the massless limit,

$$C_2^{1\text{-loop}}(t, q) = \frac{N^2}{16g} \left(1 - \frac{1}{N^2}\right) \left[1 - \frac{2}{\pi} \int_0^\sigma ds \frac{e^{-2s^2} \text{Erfi}(s)}{s}\right] \left(\frac{g}{q}\right), \quad (\text{A.26})$$

where  $\sigma = \sqrt{tq^2/2}$ , and  $\text{Erfi}(z) = -i \text{Erf}(iz)$  is the *imaginary error function*, which has the series expansion  $\text{Erfi}(z) = \pi^{-1/2} (2z + \frac{2}{3}z^3 + \dots)$  about  $z = 0$ . Expanding in  $\sigma$ , this evaluates to

$$C_2^{1\text{-loop}}(t, q) \approx \frac{N^2}{16g} \left(1 - \frac{1}{N^2}\right) \left[1 + \left(\frac{32q^2t}{\pi^3}\right)^{1/2} \left(\frac{5q^2t}{18} - 1\right)\right] \left(\frac{g}{q}\right) + \mathcal{O}(\sigma^5). \quad (\text{A.27})$$

Similarly, we look at the continuum limit of

$$C_{\mu\nu}^0(t, q) = \frac{N}{g} a^3 \sum_{x \in \Lambda} e^{-iq \cdot x} \langle T_{\mu\nu}^0(x) \text{Tr} \rho^2(t, 0) \rangle \quad (\text{A.28})$$

at finite flowtime. In the continuum limit, the EMT does not require renormalisation, we can therefore drop the <sup>0</sup> superscript. At one-loop,

$$C_{\mu\nu}^{1\text{-loop}}(t, q) = \frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) \int \frac{d^3k}{(2\pi)^3} \frac{\delta_{\mu\nu} k \cdot (q - k) - 2k_\mu(q - k)_\nu + \xi(q_\mu q_\nu - \delta_{\mu\nu} q^2)}{(k^2 + m^2)((q - k)^2 + m^2)} e^{-tq^2} e^{-t(q-k)^2} \quad (\text{A.29})$$

In the massless limit, this evaluates to

$$\begin{aligned} C_{\mu\nu}^{1\text{-loop}}(t, q) = & -\frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) \frac{q}{64\pi^{3/2}} \left[ \sqrt{\pi} \text{Erfi}(\sigma) (\pi_{\mu\nu}(3 + 2\sigma^2) - 2\delta_{\mu\nu}) \sigma^{-4} e^{-2\sigma^2} \right. \\ & + 8\sqrt{\pi}(1 - 4\xi) \pi_{\mu\nu} \int_0^\sigma \frac{e^{-2s^2} \text{Erfi}(s)}{s} ds \\ & \left. - 2(1 - 4\xi) \pi^{3/2} \pi_{\mu\nu} + e^{-\sigma^2} (4\delta_{\mu\nu} - 6\pi_{\mu\nu}) \sigma^{-3} \right] \quad (\text{A.30}) \end{aligned}$$

where  $\pi_{\mu\nu} = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$  is the transverse projector. To obtain the ‘flowed contact term’  $K(t)$  from Eq. (25), we utilise the fact that the contact term is the longitudinal part of the correlator  $C_{\mu\nu}(t, q)$ . We separate the above expression for  $C_{\mu\nu}(t, q)$  into a transverse part,  $C_{\mu\nu}(t, q)^{\text{transverse}}$  (which is proportional to the  $\pi_{\mu\nu}$ ), and the remaining longitudinal part  $C_{\mu\nu}(t, q)^{\text{longitudinal}}$ . The transverse part

$$\begin{aligned} C_{\mu\nu}(t, q)^{\text{transverse}} = & -\pi_{\mu\nu} \frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) \frac{q}{64\pi^{3/2}} \left[ \sqrt{\pi} \text{Erfi}(\sigma) (3 + 2\sigma^2) \sigma^{-4} e^{-2\sigma^2} \right. \\ & + 8\sqrt{\pi}(1 - 4\xi) \int_0^\sigma \frac{e^{-2s^2} \text{Erfi}(s)}{s} ds \\ & \left. - 2(1 - 4\xi) \pi^{3/2} + 6e^{-\sigma^2} \sigma^{-3} \right] \quad (\text{A.31}) \end{aligned}$$

is finite, as ensured by the WI. The remaining longitudinal part gives

$$C_{\mu\nu}(t, q)^{\text{longitudinal}} = -\delta_{\mu\nu} \frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) \frac{q}{64\pi^{3/2}} \left[ \sqrt{\pi} \text{Erfi}(\sigma) (-2)\sigma^{-4} e^{-2\sigma^2} + 4e^{-\sigma^2} \sigma^{-3} \right]. \quad (\text{A.32})$$

When expanded about  $\sigma = 0$ , the leading order term contributing to the contact term  $C_{\mu\nu}(t, q)^{\text{longitudinal}}$  is

$$\begin{aligned} C_{\mu\nu}(t, q)^{\text{longitudinal}} &\approx -\delta_{\mu\nu} \frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) \frac{q}{64\pi^{3/2}} \frac{8}{3\sigma} + \mathcal{O}(\sigma) \\ &= -\delta_{\mu\nu} \frac{N^2}{2} \left(1 - \frac{1}{N^2}\right) \frac{\sqrt{2}}{24\pi^{3/2}\sqrt{t}} + \mathcal{O}(\sigma) \\ &= \delta_{\mu\nu} K(t), \end{aligned} \quad (\text{A.33})$$

which gives us the result in Eq. (27).

Using Eqs. (A.27) and (A.32), the perturbative expression for the ratio in Eq. (37) can be calculated,

$$\begin{aligned} f_g(g\sqrt{t}, q_l) &= \frac{a}{g} \frac{K(t)}{C_2(t, q_l)} \\ &= -\frac{\sqrt{2}}{3\pi^{3/2}} \frac{aq}{g\sqrt{t}} + \mathcal{O}(\sigma), \end{aligned} \quad (\text{A.34})$$

giving the result in Eq. (38).

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