HIERARCHICALLY HYPERBOLIC GROUPS, PRODUCTS OF CAT(-1) SPACES, AND VIRTUAL TORSION-FREENESS

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Abstract. We prove that a group acting geometrically on a product of proper minimal CAT(-1) spaces without permuting isometric factors is a hierarchically hyperbolic group. As an application we construct hierarchically hyperbolic groups which are not virtually torsion-free.

1. Introduction

Hierarchically hyperbolic groups (HHGs) and spaces (HHSs) were introduced by Behrstock, Hagen and Sisto in [BHS17a]. The class of hierarchically hyperbolic groups shares a number of properties with the class of groups acting properly cocompactly on CAT(0) spaces (CAT(0) *groups*). For instance, groups in both classes have quadratic Dehn function [BHS19, Theorem 7.5] and are semihyperbolic [HHP20, Corollary F] (see also [DMS20]). In particular, they both have undistorted abelian subgroups, solvable conjugacy problem and are type FP_∞. However, there are some differences between the classes. Indeed, mapping class groups of surfaces of genus at least 3 are hierarchically hyperbolic groups [BHS17b] but not CAT(0) groups [KL96], and the $(3,3,3)$-triangle group is a CAT(0) group but is not a hierarchically hyperbolic group [PS20]. For an introduction to CAT(0) groups and spaces see [BH99].

Hierarchically hyperbolic groups are known to satisfy a number of other properties such as having finite asymptotic dimension [BHS17b, Theorem A], having a uniform bound on the conjugator length of Morse elements [AB19], and for virtually torsion-free HHGs, their uniform exponential growth is well understood [ANS19].

In this paper we investigate the hierarchical hyperbolicity of groups acting properly cocompactly on products of CAT(-1) spaces without permuting isometric factors. To do this we adopt the perspective of studying uniform lattices in products of their full isometry groups. Let X be a proper CAT(0) space. Let $H = \mathrm{Isom}(X)$, then H is a locally compact group with the topology given by uniform convergence on compacta. Let Γ be a discrete subgroup of H. We say Γ is a *uniform lattice* if X/Γ is compact.

We will assume some non-degeneracy conditions on the CAT(0) spaces to avoid many technical difficulties associated with the CAT(0) condition (see [CM09, Section 1.B] for a thorough explanation). A group H acting on a CAT(0) space X is *minimal* if there is

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no H-invariant closed convex subset $X' \subset X$. If $\mathrm{Isom}(X)$ is minimal, then we say X is minimal.

Theorem A (Theorem 3.1). *Let $H \leqslant \mathrm{Isom}(\mathbb{E}^n) \times \prod_{i=1}^m \mathrm{Isom}(X_i)$ be a closed subgroup, where each X_i is a proper irreducible non-elementary $\mathrm{CAT}(-1)$ space. Assume H acts minimally and cocompactly on $X = \mathbb{E}^n \times \prod_{i=1}^m X_i$ and let Γ be a uniform H-lattice. If $\pi_{\mathrm{O}(n)}(\Gamma)$ is trivial, then Γ is a hierarchically hyperbolic group.*

To prove a converse to Theorem A one may need to investigate the commensurators of maximal abelian subgroups of a hierarchically hyperbolic group Γ. Indeed, the CAT(0) not biautomatic groups introduced by Leary–Minasyan [LM19] and the groups constructed by the author in [Hug21] have maximal abelian subgroups which have infinite index in their commensurator and are not virtually normal. All of these groups have a non-discrete projection to $\mathrm{O}(n) \leqslant \mathrm{Isom}(\mathbb{E}^n)$.

Question 1.1. *Is a maximal abelian subgroup A of a hierarchically group Γ either finite index in its commensurator $\mathrm{Comm}_\Gamma(A)$ or virtually normal?*

As an application of Theorem A, we construct a group Γ which acts faithfully and cocompactly on a product of two locally-finite trees. We show that Γ is actually a uniform lattice in the product of the automorphism groups of the trees. We also show that Γ has no finite index torsion-free subgroups.

Corollary B (Corollary 4.3). *There exist hierarchically hyperbolic groups which are not virtually torsion-free.*

To the author's knowledge this is the first explicit example of an HHG which is not virtually torsion-free. The question of the existence of such an example, although not explicitly asked in the literature, is of interest to specialists since a number of theorems about HHGs require the assumption of virtual torsion-freeness (e.g. [ANS19, Theorem 1.1] [RS20, Theorem 1.2(3')]). Moreover, by [HHP20, Theorem G] an HHG has only finitely many conjugacy classes of finite subgroups, in particular, every residually finite HHG is virtually torsion-free. The author suspects that it is possible to apply the results of Hagen–Susse [HS20] to Wise's examples in [Wis07] to obtain an HHG which is not virtually torsion-free, however, the construction presented here is much more elementary and gives an explicit HHG structure.

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2. Definitions

In this section we will give the relevant background on HHSs and HHGs for our endeavours. The definitions are rather technical so we will only focus on what we need, for a full account the reader should consult [BHS19, Definition 1.1, 1.2.1]. We will follow the treatment in [PS20, Section 2]. To this end, a *hierarchically hyperbolic space (HHS)* is pair $(X, \mathfrak{S})$ where X is an ϵ-quasigeodesic space and $\mathfrak{S}$ is a set with some extra data which essentially functions as a coordinate system on X where each coordinate entry is a hyperbolic space. The relevant parts of the axiomatic formalisation are described as follows:

- For each *domain* $U \in \mathfrak{S}$, there is a hyperbolic space $\mathcal{C}U$ and *projection* $\pi_U : X \to \mathcal{C}U$ that is coarsely Lipschitz and coarsely onto [BHS19, Remark 1.3].
- $\mathfrak{S}$ has a partial order $\sqsubseteq$, called *nesting*. Nesting chains are uniformly finite, and the length of the longest such chain is called the *complexity* of $(X, \mathfrak{S})$.
- $\mathfrak{S}$ has a symmetric relation $\perp$, called *orthogonality*. The complexity bounds pairwise orthogonal sets of domains.
- The relations $\sqsubseteq$ and $\perp$ are mutually exclusive. The complement of $\sqsubseteq$, $\perp$ and $=$ is called *transversality* and denoted $\pitchfork$.
- If $U \in \mathfrak{S}$ and there is some domain orthogonal to U, then there is some $W \in \mathfrak{S}$ such that $V \sqsubset W$ whenever $V \perp U$. We call W an *orthogonal container*.
- Whenever $U \pitchfork V$ or $U \sqsubset V$ there is a bounded set $\rho_V^U \subset \mathcal{C}V$. These sets, and projections of elements $x \in X$, are *consistent* in the following sense:
  - *ρ-consistency:* Let $U, V, W \in \mathfrak{S}$ such that $U \sqsubset V$ and ρ_W^V is defined, then ρ_W^U coarsely agrees with ρ_W^V;
  - If $U \pitchfork V$ then $\min\{d_{\mathcal{C}U}(\pi_U(x), \rho_U^V), d_{\mathcal{C}V}(\pi_V(x), \rho_V^U)\}$ is bounded.

All coarseness may taken to be uniform so we can and will fix a uniform constant ϵ [BHS19, Remark 1.6].

We remind the reader that these axioms for an HHS are not a complete set but only recall the structure we will need. For the full definition the reader should consult [BHS19, Definition 1.1, 1.2.1]. The following definition of an HHG is however complete.

Let X be the Cayley graph of a group Γ and suppose $(X, \mathfrak{S})$ is an HHS, then $(\Gamma, \mathfrak{S})$ is a *hierarchically hyperbolic group structure (HHG)* if it also satisfies the following:

(1) Γ acts cofinitely on $\mathfrak{S}$ and the action preserves the three relations. For each $g \in G$ and each $U \in \mathfrak{S}$, there is an isometry $g : \mathcal{C}U \to \mathcal{C}gU$ and these isometries satisfy $g \cdot h = gh$;
(2) for all $U, V \in \mathfrak{S}$ with $U \pitchfork V$ or $V \sqsubset U$ and all $g, x \in \Gamma$ there is equivariance of the form $g\pi_U(gx) = \pi_{gU}(gx)$ and $g\rho_U^V = \rho_{gU}^{gV}$.

Note that this is not the original definition of a HHG as given in [BHS19]. Instead, we have adopted the simpler axioms from [PS20], the axioms we have given imply the original axioms, however, by [DHS20, Section 2.1] they are in fact equivalent.

3. Proof of Theorem A

In this section we will prove Theorem A from the introduction.

Theorem 3.1 (Theorem A)**.** *Let $H \leqslant \mathrm{Isom}(\mathbb{E}^n) \times \prod_{i=1}^m \mathrm{Isom}(X_i)$ be a closed subgroup, where each X_i is a proper irreducible non-elementary* $\mathrm{CAT}(-1)$ *space. Assume H acts minimally and cocompactly on $X = \mathbb{E}^n \times \prod_{i=1}^m X_i$ and let Γ be a uniform H-lattice. If $\pi_{\mathrm{O}(n)}(\Gamma)$ is trivial, then Γ is a hierarchically hyperbolic group.*

Proof. Let q be a Γ-equivariant quasi-isometry $\mathrm{Cay}(\Gamma, \mathrm{A}) \to \mathrm{X}$ given by the Švarc-Milnor Lemma [BH99, p. I.8.19]. For $i \in \{1-n, \ldots, 0\}$ let $X_i = \mathbb{R}$ and $H_i = \mathrm{Isom}(\mathbb{E})$. Let $i \in \{1-n, \ldots, m\}$. Now, products of HHSs are HHSs so $(X, \mathfrak{S})$ is an HHS [BHS19, Proposition 8.27]. Moreover, by the description given in the proof of [BHS19, Proposition 8.27] every domain of $\mathfrak{S}$ is either bounded (in fact a point) or some X_i.

Note that $\mathfrak{S}$ is finite and the action on $\mathfrak{S}$ is trivial. Every domain of the structure is either bounded (in fact a point) or one of the X_i. In the first case the Γ action is trivial and in the second case Γ acts via π_{H_i}. This immediately yields the first axiom.

For the second axiom consider the following diagram where the vertical arrows are given by applying the obvious group action:

$$\begin{array}{ccc} \Gamma \times \mathrm{Cay}(\Gamma, \mathrm{A}) & \xrightarrow{(\pi_{H_i}, \pi_{X_i} \circ q)} & \pi_{H_i}(\Gamma) \times X_i \\ \downarrow & & \downarrow \\ \mathrm{Cay}(\Gamma, \mathrm{A}) & \xrightarrow[\pi_{X_i} \circ q]{} & X_i. \end{array}$$

We will verify the diagram commutes. Let $x \in \mathrm{Cay}(\Gamma, \mathrm{A})$ and $g \in \Gamma$. First, we evaluate the composite map going down then across, we have

$$(g, x) \mapsto gx \mapsto \pi_{X_i}(q(gx)).$$

Going the other way we have

$$(g, x) \mapsto (\pi_{H_i}(g), \pi_{X_i}(q(x))) \mapsto (\pi_{H_i}(g), \pi_{X_i}(q(x)) = \pi_{X_i}(gq(x)) = \pi_{X_i}(q(gx))$$

where the last equality is given by the Γ-equivariance of q. In particular, $g\pi_{X_i}(x) = \pi_{gX_i}(gx) = \pi_{X_i}(gx)$. The other condition for equivariance is established immediately since any two domains that are not points are orthogonal to each other. □

Corollary 3.2. *Let Γ be a group acting properly cocompactly by isometries on a finite product of proper irreducible minimal* $\mathrm{CAT}(-1)$*-spaces without permuting isometric factors, then Γ is a hierarchically hyperbolic group.*

Proof. The group Γ splits as a short exact sequence

$$\{1\} \rightarrowtail F \rightarrowtail \Gamma \twoheadrightarrow \Lambda \twoheadrightarrow \{1\},$$

where Λ satisfies the conditions of the previous theorem and F is the kernel of the action onto the product space. Since F acts trivially on the product space, it acts trivially on the HHG structure for Λ. The epimorphism $\varphi : \Gamma \twoheadrightarrow \Lambda$ induces an equivariant quasi-isometry ψ on the associated Cayley graphs. Thus, we may precompose every map in the previous theorem with φ or ψ to endow Γ with the structure of a HHG. □

4. Non-virtually torsion-free HHGs

In this section we will construct a hierarchically hyperbolic group which is not virtually torsion-free.

Let Λ be a Burger-Mozes simple group [BM97; BM00a; BM00b] acting on $\mathcal{T}_1 \times \mathcal{T}_2$ splitting as an amalgamated free product $F_n *_{F_m} F_n$ with embeddings $i, j : F_m \to F_n$. This defines a groups Λ which embeds discretely into the product of $T_1 = \mathrm{Aut}(\mathcal{T}_1)$ and $T_2 = \mathrm{Aut}(\mathcal{T}_2)$ with compact quotient. For instance one may take Rattaggi's example of a lattice in the product of an 8-regular and 12-regular tree which splits as $F_7 *_{F_{73}} F_7$ [Rat07b] (see also [Rat07a] or one of Radu's examples [Rad20]).

Define $A = \mathbb{Z}_p \rtimes F_n$ for p prime such that the F_n-action is non-trivial. Consider the embeddings $\widetilde{i}, \widetilde{j} : F_m \rightarrowtail F_n \rightarrowtail A$ given by the composition of i or j with the obvious inclusion. Now, we build a group Γ as an amalgamated free product $A *_{F_m} A$, note that Γ surjects onto the original Burger-Mozes group Λ with kernel the normal closure of the torsion elements. Let $\mathcal{T}_3$ denote the Bass-Serre tree of Γ and let T_3 denote the corresponding automorphism group.

Proposition 4.1. *Γ is a uniform $(T_1 \times T_3)$-lattice which does not permute the factors.*

This can be easily deduced by endowing Γ with a graph of lattices structure in the sense of [Hug21, Defintion 3.1] and then applying [Hug21, Theorem A]. Instead we will provide a direct proof.

Proof. The group Γ acts on Bass-Serre tree $\mathcal{T}_3$ and also on $\mathcal{T}_1$ via the homomorphism $\psi : \Gamma \to T_1$ defined by taking the surjection $\Gamma \twoheadrightarrow \Lambda$. The diagonal action on the product space $\mathcal{T}_1 \times \mathcal{T}_3$ is properly discontinuous cocompact and by isometries. The kernel of the action is trivial, since the only elements which could act trivially are the torsion elements. However, these all clearly act non-trivially on $\mathcal{T}_3$. Thus, the action is faithful. We conclude that Γ is a uniform $(T_1 \times T_3)$-lattice. □

It remains to show Γ is not virtually torsion-free.

Proposition 4.2. *Γ is not virtually torsion-free.*

The author thanks Yves de Cornulier for the following argument.

Proof. Note that F_n normally generates A. The finite residual $\Gamma^{(\infty)}$ of Γ contains the Burger-Mozes simple group Λ. Thus, both copies of F_n are contained in $\Gamma^{(\infty)}$. As F_n normally generates A, it follows $\Gamma^{(\infty)} = \Gamma$. Since, A is not torsion-free, we conclude Γ is not virtually torsion-free. □

Corollary 4.3 (Corollary B)**.** *Γ is a hierarchically hyperbolic group which is not virtually torsion-free.*

Proof. By Proposition 4.1 and Theorem 3.1 we see Γ is a hierarchically hyperbolic group. By Proposition 4.2 we see Γ is not virtually torsion-free. □

Remark 4.4. In [Hug21, Corollary 9.9] the author gave a way to use A. Thomas's construction in [Tho06] to promote lattices in products of trees to lattices in products of "sufficiently symmetric" right-angled buildings. Applying [Hug21, Corollary 9.9] to one of the non-virtually torsion-free lattices Γ we obtain a non-virtually torsion-free lattice Λ acting on a product of "sufficiently symmetric" right-angled hyperbolic buildings each not quasi-isometric to a tree. Moreover, by Theorem 3.1 Λ is hierarchically hyperbolic.

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