POISSONIAN OCCUPATION TIMES OF SPECTRALLY NEGATIVE LÉVY PROCESSES WITH APPLICATIONS

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Abstract. In this paper, we introduce the concept of *Poissonian occupation times* below level 0 of a spectrally negative Lévy process. In this case, occupation time is accumulated only when the process is observed to be negative at arrival epochs of an independent Poisson process. Our results extend some well known continuously observed quantities involving occupation times of spectrally negative Lévy processes. As an application, we establish a link between Poissonian occupation times and insurance risk models with Parisian implementation delays.

1. Introduction

Let $X = (X_t)_{t\geq 0}$ be a spectrally negative Lévy process, that is a càdlàg process with stationary and independent increments and no positive jumps. We define the first passage time above b,

$$\tau_b^+ = \inf\{t > 0 \colon X_t > b\},$$

with the convention $\inf\emptyset = \infty$. For ease of notation, denote $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{N}_+ = \{1, 2, 3, ...\}$. We define the following discrete observation scheme $\{\xi_n\}_{n\in\mathbb{N}}$ where $\xi_0 = 0$, $\xi_1 = \mathrm{e}_\lambda^1$ and, for $n = 2, 3, 4...$,

$$\xi_n - \xi_{n-1} = \begin{cases} \mathrm{e}_\lambda^n, & \text{if } X_{\xi_{n-1}} \geq 0, \\ \tau_0^+ \circ \theta_{\xi_{n-1}} + \mathrm{e}_\lambda^n, & \text{if } X_{\xi_{n-1}} < 0, \end{cases} \tag{1}$$

where θ is the Markov shift operator ($X_s \circ \theta_t = X_{s+t}$) and $\left\{\mathrm{e}_\lambda^n\right\}_{n\in\mathbb{N}_+}$ is a sequence of independent and identically distributed (iid) exponential random variables with rate $\lambda > 0$ (frequency rate of observations). Under the discrete observation scheme (1), the surplus process is first monitored discretely at Poisson arrival times with rate λ until the surplus is observed to be negative. Hereafter, the risk process will be observed continuously until it leaves $(-\infty, 0)$.

In this paper, we define the Poissonian occupation time below 0 over a finite-time horizon, by

$$\mathcal{O}_{t,\lambda}^X = \sum_{n\in\mathbb{N}_+} (\tau_0^+ \circ \theta_{\xi_n}) \mathbf{1}_{\left\{X_{\xi_n} < 0, \xi_n < t\right\}}, \tag{2}$$

which is the total time elapsed during the continuous observation (see Figure 1). We will first study the joint Laplace transform of $\left(\tau_b^+, \mathcal{O}_{\tau_b^+,\lambda}^X\right)$ and, as a consequence, we examine the Laplace transform as well as the distribution of Poissonian occupation time over an infinite-time horizon, that is,

$$\mathcal{O}_{\infty,\lambda}^X = \sum_{n\in\mathbb{N}_+} (\tau_0^+ \circ \theta_{\xi_n}) \mathbf{1}_{\left\{X_{\xi_n} < 0\right\}}.$$

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The analysis of quantities involving the duration of negative surplus (also called the *time in red*) under continuous monitoring $\mathcal{O}_t^X = \int_0^t \mathbf{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s$ has been well studied in the literature. The Laplace transform of $\mathcal{O}_\infty^X = \int_0^\infty \mathbf{1}_{(-\infty,0)}(X_s)\,\mathrm{d}s$ was first derived by Landriault et al. [13]. Loeffen et al. [22] derived the joint Laplace transform of $\left(\tau_b^+, \mathcal{O}_{\tau_b^+}^X\right)$, generalizing the results in [13]. For a more general treatment, Li and Palmowski [15] studied weighted occupation times. Wang et al. [24] studied the joint Laplace transform of occupation times over disjoint intervals for spectrally negative Lévy processes with a taxation structure. Recently, Landriault et al. [11] obtained an analytical expression for the distribution of the occupation time below level 0 up to an (independent) exponential horizon for a (refracted) spectrally negative Lévy risk processes.

In the aforementioned works, the study of continuously observed identities involving occupation times became challenging when surplus process has paths of unbounded variation. In [13], the ϵ-*approximation* approach, which consists of a spatial shift of the sample paths of the underlying process, was used to avoid the problem caused by the infinity activity of the process and such that classical conditioning continue to hold. A second approach consists of introducing a sequence of bounded variation processes, $(X_n)_{n\geq 1}$, which converge to the unbounded variation process X as n goes to infinity (see Loeffen et al. [22], Guérin and Renaud [7]). An important contribution of the Poisson observation approach is the unified proof for processes with bounded or unbounded variation paths, whereas these two cases need to be treated separately using the approximation approaches explained above and, when the Poisson observation rate goes to infinity, we recover results for occupation times under continuous monitoring (see e.g. Albrecher et al. [2] and Li et al. [17]).

1.1. **Motivation.** In actuarial mathematics, occupation times can naturally be used as a measure of the risk inherent to an insurance portfolio. For instance, the time spent below a solvency threshold helps quantify the risk related to an insurance surplus process. The analysis of the duration of the negative surplus is also related to Parisian ruin models in which insurers are granted a period of time to re-emerge above the threshold level before ruin is deemed to occur (see e.g. [20, 21]) and ([13, 14]). Two types of Parisian ruin are strongly related to $\mathcal{O}_t^X$: cumulative Parisian ruin and Parisian ruin with exponential delays. In the first case, ruin occurs when the surplus process stays cumulatively below a critical level longer than a pre-determined time period. Guérin and Renaud [8] studied the probability of cumulative Parisian ruin for the Cramér–Lundberg risk model with exponential claims by giving an explicit representation for the distribution of $\mathcal{O}_t^X$. A general treatment for Lévy risk processes has been recently studied by Landriault et al. [11]. On the other hand, the probability of Parisian ruin with exponential delays was first studied in [13] through the following connection between the occupation time $\mathcal{O}_\infty^X$ and this type of Parisian ruin, for $x \in \mathbb{R}$,

$$\mathbb{P}_x(\rho_\lambda < \infty) = 1 - \mathbb{E}_x\left[\mathrm{e}^{-\lambda \mathcal{O}_\infty^X}\right], \tag{3}$$

where

$$\rho_\lambda = \inf\left\{t > 0 \mid t - g_t > \mathrm{e}_\lambda^{g_t}\right\}, \tag{4}$$

and $g_t = \sup\{0 \leq s \leq t \colon X_s \geq 0\}$, while $\mathrm{e}_\lambda^{g_t}$ is an exponentially distributed random variable with rate $\lambda > 0$ and independent of X (see also Baurdoux et al. [3]). Also, it is known that Parisian ruin time in (4) corresponds to the first passage time when X is observed below 0 at

Poisson arrival times, that is

$$T_0^- = \min\{\xi_i \colon X_{\xi_i} < 0,\ i \in \mathbb{N}\}. \tag{5}$$

This paper is partly motivated by the study of Parisian ruin with Erlang implementation delays. Such type of Parisian ruin has already been studied for Lévy insurance risk models (see, e.g. Albrecher and Ivanovs [1], Landriault et al. [14] and Frostig and Keren-Pinhasik[6]). However, in contrast to Parisian ruin in (4), the relationship in Equation (3) does not hold any more when the delay follows an Erlang-distributed implementation delays. In this paper, we establish a link between Poissonian occupation time and Parisian ruin with Erlang-distributed implementation delays (see Section 4). First, we study Parisian ruin time with implementation delays given by a sum of two independent exponential random variables with different rates. Thus, we extend the study to Parisian ruin with Erlang$(2, \lambda)$ implementation delays. We will also derive several fluctuation identities as well as the Gerber-Shiu distribution. For a more general case, namely Parisian ruin with Erlang(n, λ) implementation delays, a recursive expression will be derived using the n^{th}-Poissonian occupation time.

The rest of the paper is organized as follows. In Section 2, we present the necessary background material on spectrally negative Lévy processes and scale functions. The main results, as well as the proofs, are presented in Section 3. Applications to Parisian insurance risk models are presented in section 4. In the Appendix, a few well known fluctuation identities with delays are presented.

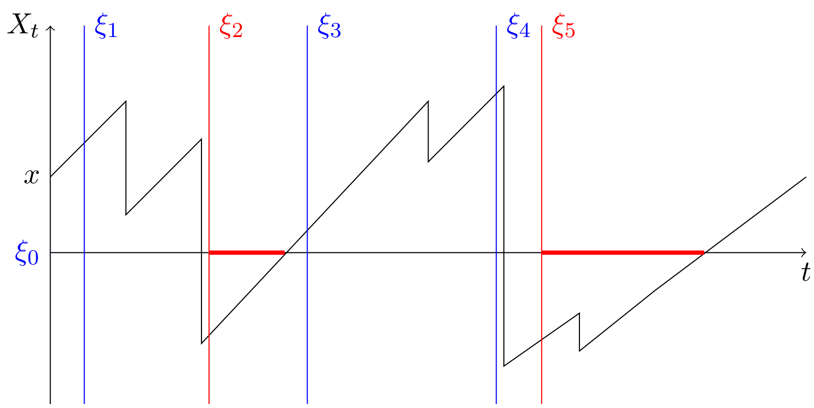

Figure 1. Illustration of Poissonian occupation time below 0.

2. Preliminaries

In this section, we present the necessary background material on spectrally negative Lévy processes.

2.1. **Lévy insurance risk processes.** A Lévy insurance risk process X is a process with stationary and independent increments and no positive jumps. To avoid trivialities, we exclude the case where X has monotone paths. As the Lévy process X has no positive jumps, its Laplace transform exists: for all $\lambda, t \geq 0$,

$$\mathbb{E}\left[\mathrm{e}^{\lambda X_t}\right] = \mathrm{e}^{t\psi(\lambda)},$$

where

$$\psi(\lambda)=\gamma\lambda+\frac{1}{2}\sigma^2\lambda^2+\int_0^\infty\left(\mathrm{e}^{-\lambda z}-1+\lambda z\mathbf{1}_{(0,1]}(z)\right)\Pi(\mathrm{d}z),$$

for $\gamma\in\mathbb{R}$ and $\sigma\geq 0$, and where Π is a σ-finite measure on $(0,\infty)$ called the Lévy measure of X such that

$$\int_0^\infty(1\wedge z^2)\Pi(\mathrm{d}z)<\infty.$$

We will use the standard Markovian notation: the law of X when starting from $X_0=x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x=0$. We recall the definition of standard first-passage time below level $b\in\mathbb{R}$,

$$\tau_b^-=\inf\{t>0\colon X_t<b\}.$$

with the convention $\inf\emptyset=\infty$.

2.2. **Scale functions.** We now present the definition of the scale functions W_q and Z_q of X. First, recall that there exists a function $\Phi\colon[0,\infty)\to[0,\infty)$ defined by $\Phi_q=\sup\{\lambda\geq 0\mid\psi(\lambda)=q\}$ (the right-inverse of ψ) such that

$$\psi(\Phi_q)=q,\quad q\geq 0.$$

Now, for $q\geq 0$, the q-scale function of the process X is defined as the continuous function on $[0,\infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda y}W_q(y)\mathrm{d}y=\frac{1}{\psi_q(\lambda)},\quad \text{for } \lambda>\Phi_q, \tag{6}$$

where $\psi_q(\lambda)=\psi(\lambda)-q$. This function is unique, positive and strictly increasing for $x\geq 0$ and is further continuous for $q\geq 0$. We extend W_q to the whole real line by setting $W_q(x)=0$ for $x<0$. We write $W=W_0$ when $q=0$. It is known that

$$\lim_{b\to\infty}\frac{W_q(x+b)}{W_q(b)}=\mathrm{e}^{\Phi_q x}. \tag{7}$$

and when $\psi'(0+)>0$, the *terminal value* of W is given by

$$\lim_{x\to\infty}W(x)=\frac{1}{\psi'(0+)}=\frac{1}{\mathbb{E}[X_1]}. \tag{8}$$

We also define another scale function $Z_q(x,\theta)$ by

$$Z_q(x,\theta)=\mathrm{e}^{\theta x}\left(1-\psi_q(\theta)\int_0^x\mathrm{e}^{-\theta y}W_q(y)\mathrm{d}y\right),\quad x\geq 0, \tag{9}$$

and $Z_q(x,\theta)=\mathrm{e}^{\theta x}$ for $x<0$. For $\theta=0$, we set

$$Z_q(x)=Z_q(x,0)=1+q\int_0^x W_q(y)\mathrm{d}y,\quad x\in\mathbb{R}. \tag{10}$$

Using (6), we can also re-write the scale function $Z_q(x,\theta)$ as follows

$$Z_q(x,\theta)=\psi_q(\theta)\int_0^\infty\mathrm{e}^{-\theta y}W_q(x+y)\mathrm{d}y,\quad x\geq 0,\ \theta\geq\Phi_q. \tag{11}$$

We also recall the *second generation* scale function, that is, for $p,p+q\geq 0$ and $x\in\mathbb{R}$, we have

$$\mathcal{W}_a^{(p,q)}(x)=W_p(x)+q\int_a^x W_{p+q}(x-y)\,W_p(y)\,\mathrm{d}y$$

$$= W_{p+q}(x) - q\int_0^a W_{p+q}(x-y)\,W_p(y)\,\mathrm{d}y, \tag{12}$$

where the second equation follows using the following identity obtained in [22]

$$(q-p)\int_0^x W_p(x-y)W_q(y)\mathrm{d}y = W_q(x) - W_p(x). \tag{13}$$

A slightly generalized version of Eq. (13) has been derived in [16]: for $s \geq 0$, $x > a$,

$$(s-(p+q))\int_a^x W_s(x-y)\mathcal{W}_a^{(p,q)}(y)\,\mathrm{d}y = \mathcal{W}_a^{(p,s-p)}(x) - \mathcal{W}_a^{(p,q)}(x). \tag{14}$$

For later use, we have from Lkabous and Renaud [19] (see Eq. (9))

$$\int_0^\infty \mathrm{e}^{-\theta z}\mathcal{W}_a^{(p,s)}(a+z)\,\mathrm{d}z = \frac{Z_p(a,\theta)}{\psi_{p+s}(\theta)}, \quad \theta > \Phi_{p+s}. \tag{15}$$

For the sake of compactness of the results, we will use the following function defined in [1]: for $x \in \mathbb{R}$ and $\alpha, \beta \geq 0$, we have

$$\tilde{Z}_q(x,\alpha,\beta) = \frac{\psi_q(\alpha)Z_q(x,\beta) - \psi_q(\beta)Z_q(x,\alpha)}{\alpha-\beta}, \tag{16}$$

and when $\alpha = \beta$, we obtain

$$\tilde{Z}_q(x,\alpha,\alpha) = \psi'(\alpha)Z_q(x,\alpha) - \psi_q(\alpha)Z_q'(x,\alpha), \tag{17}$$

where Z_q' is the derivative of Z_q taken with respect to the second argument. We write $\tilde{Z} = \tilde{Z}_0$ when $q = 0$.
Finally, we recall Kendall's identity that provides the distribution of the first upward crossing of a specific level (see [4, Corollary VII.3]): on $(0,\infty)\times(0,\infty)$, we have

$$r\mathbb{P}(\tau_z^+ \in \mathrm{d}r)\mathrm{d}z = z\mathbb{P}(X_r \in \mathrm{d}z)\mathrm{d}r. \tag{18}$$

We refer the reader to [10] and [4] for more details on spectrally negative Lévy processes. More examples and numerical computations related to scale functions can be found in e.g. [9] and [23].

3. Main results

We are now ready to state our main results. First, we provide an expression for the joint Laplace transform of $\left(\tau_b^+, \mathcal{O}^X_{\tau_b^+,\lambda}\right)$.

Theorem 1. *For $p, q \geq 0$, $\lambda > 0$ and $x \leq b$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+ - p\mathcal{O}^X_{\tau_b^+,\lambda}}\mathbf{1}_{\{\tau_b^+<\infty\}}\right] = \frac{\tilde{Z}_q(x,\Phi_{\lambda+q},\Phi_{p+q})}{\tilde{Z}_q(b,\Phi_{\lambda+q},\Phi_{p+q})}. \tag{19}$$

Proof. See the proof in Appendix A ∎

As an immediate consequence of Theorem 1, we obtain the Laplace transform of $\mathcal{O}^X_{\infty,\lambda}$.

Corollary 2. *For $\lambda, p > 0$, $\mathbb{E}[X_1] > 0$ and $x \in \mathbb{R}$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\infty,\lambda}}\right] = \mathbb{E}[X_1]\frac{\Phi_p\Phi_\lambda}{\lambda p}\tilde{Z}(x,\Phi_\lambda,\Phi_p). \tag{20}$$

Proof. Using (11), (7) and the dominated convergence theorem, we have

$$\lim_{b\to\infty}\frac{Z(b,\theta)}{W(b)}=\frac{\psi(\theta)}{\theta},\quad \textit{for } \theta>0.$$

which leads to

$$\lim_{b\to\infty}\frac{\tilde{Z}\left(b,\Phi_\lambda,\Phi_p\right)}{W(b)}=\frac{\lambda p}{\Phi_\lambda\Phi_p}.$$

Then, using (8), we obtain

$$\lim_{b\to\infty}\frac{\tilde{Z}\left(x,\Phi_\lambda,\Phi_p\right)/W(b)}{\tilde{Z}\left(b,\Phi_\lambda,\Phi_p\right)/W(b)}=\mathbb{E}\left[X_1\right]\frac{\Phi_p\Phi_\lambda}{\lambda p}\tilde{Z}\left(x,\Phi_\lambda,\Phi_p\right).$$

■

Remark 3. Our fluctuation identities are arguably compact and have a similar structure as classical fluctuation identities (under continuous monitoring). Indeed, from Theorem 1 we recover, by letting $\lambda\to\infty$, the identity in [22, Corollary 2.(ii)] which is the joint Laplace transform of $\left(\tau_b^+,\mathcal{O}^X_{\tau_b^+}\right)$ given by

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+-p\mathcal{O}^X_{\tau_b^+}}\mathbf{1}_{\left\{\tau_b^+<\infty\right\}}\right]=\frac{Z_q\left(x,\Phi_{p+q}\right)}{Z_q\left(b,\Phi_{p+q}\right)},$$

which follows immediately from the dominated convergence theorem and $\Phi_\lambda\to\infty$ when $\lambda\to\infty$, to prove $\lim_{\lambda\to\infty}Z_q(b,\Phi_{\lambda+q})/\lambda=0$. We also recover the Laplace transform of $\mathcal{O}^X_\infty$ by letting $\lambda\to\infty$ in (20). Indeed, we have

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_\infty}\right]&=\lim_{\lambda\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\infty,\lambda}}\right]=\lim_{\lambda\to\infty}\mathbb{E}\left[X_1\right]\frac{\Phi_p\Phi_\lambda}{\lambda p}\tilde{Z}\left(x,\Phi_\lambda,\Phi_p\right)\\&=\lim_{\lambda\to\infty}\mathbb{E}\left[X_1\right]\frac{Z\left(x,\Phi_\lambda\right)\Phi_\lambda}{\lambda}\left(\frac{\Phi_p}{\Phi_p-\Phi_\lambda}\right)-\lim_{\lambda\to\infty}\mathbb{E}\left[X_1\right]\Phi_\lambda\frac{\Phi_pZ\left(x,\Phi_p\right)}{p\left(\Phi_p-\Phi_\lambda\right)}\\&=\mathbb{E}\left[X_1\right]\frac{\Phi_p}{p}Z\left(x,\Phi_p\right),\end{aligned}\tag{21}$$

using

$$\lim_{\lambda\to\infty}\frac{\Phi_\lambda Z\left(x,\Phi_\lambda\right)}{\lambda}=\lim_{\lambda\to\infty}\Phi_\lambda\int_0^\infty\mathrm{e}^{-\Phi_\lambda y}W(x+y)\mathrm{d}y=\lim_{y\to0}W(x+y)=W(x),$$

where in the second equality we used the Initial Value Theorem of Laplace transform.

In the next theorem, we give an explicit expression for the density of $\mathcal{O}^X_{\infty,\lambda}$.

Theorem 4. *For $\lambda>0$, $r\geq0$ and $x\in\mathbb{R}$, we have*

$$\begin{aligned}\mathbb{P}_x\left(\mathcal{O}^X_{\infty,\lambda}\in\mathrm{d}r\right)&=\mathbb{E}\left[X_1\right]\frac{\Phi_\lambda}{\lambda}Z\left(x,\Phi_\lambda\right)\delta_0\left(\mathrm{d}r\right)+\mathbb{E}\left[X_1\right]\frac{\Phi_\lambda^2}{\lambda}\Gamma_\lambda\left(r\right)Z\left(x,\Phi_\lambda\right)\mathrm{d}r\\&\quad-\mathbb{E}\left[X_1\right]\Phi_\lambda\left(\Gamma_\lambda\left(r\right)W\left(x\right)+\int_0^r\Gamma_\lambda\left(r-s\right)\Lambda'\left(x,s\right)\mathrm{d}s\right)\mathrm{d}r,\end{aligned}\tag{22}$$

where

$$\Gamma_\lambda(r)=\int_0^\infty\mathrm{e}^{\Phi_\lambda z}\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right),$$

and

$$\Lambda'(x,r) = \int_0^\infty W'(x+z)\,\frac{z}{r}\mathbb{P}(X_r \in \mathrm{d}z)\,.$$

Proof. See the proof in Appendix A ■

Remark 5. We can rewrite formula (22) as

$$\mathbb{P}_x\left(\mathcal{O}^X_{\infty,\lambda} \in \mathrm{d}r\right) = \mathbb{P}_x\left(T_0^- = \infty\right)\delta_0(\mathrm{d}r) + \mathbb{P}_x\left(\mathcal{O}^X_{\infty,\lambda} \in \mathrm{d}r, T_0^- < \infty\right),$$

where,

$$\mathbb{P}_x\left(T_0^- = \infty\right) = \mathbb{E}[X_1]\,\frac{\Phi_\lambda}{\lambda} Z(x,\Phi_\lambda),$$

which can be found in [13] (see Corollary 1), and

$$\begin{aligned}\mathbb{P}_x\left(\mathcal{O}^X_{\infty,\lambda} \in \mathrm{d}r, T_0^- < \infty\right) &= \mathbb{E}[X_1]\,\frac{\Phi_\lambda\Gamma_\lambda(r)}{\lambda}\left(\Phi_\lambda Z(x,\Phi_\lambda) - \lambda W(x)\right)\mathrm{d}r \\ &\quad -\mathbb{E}[X_1]\,\Phi_\lambda\left(\int_0^r \Gamma_\lambda(r-s)\,\Lambda'(x,s)\,\mathrm{d}s\right)\mathrm{d}r.\end{aligned}$$

4. Applications : Parisian ruin

In this section, we want to provide a link between Poissonian occupation times and Parisian ruin models with Erlang distributed implementation delays.

4.1. **Parisian ruin with implementation delays given by a sum of two independent exponential random variables.**

Under the Poissonian occupation time, occupation time is accumulated once the process is observed to be negative at Poisson arrival times. In other words, occupation time is accumulated when process stays below 0 for a period of time equal to a copy of an exponentially distributed r.v. e_λ with rate $\lambda > 0$. Now, if we consider implementation clock given by an exponentially distributed r.v. e_p with rate $p > 0$ (independent of X and e_λ), then $\mathbb{P}_x\left(\mathcal{O}^X_{\infty,\lambda} > \mathrm{e}_p\right)$ corresponds to the probability of Parisian ruin with implementation delays modelled by the sum of e_p and e_λ. Hence, we have established the following connection

$$\mathbb{P}_x\left(\rho^{(p,\lambda)} < \infty\right) = \mathbb{P}_x\left(\mathcal{O}^X_{\infty,\lambda} > \mathrm{e}_p\right) = 1 - \mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\infty\lambda}}\right], \tag{23}$$

where $\rho^{(p,\lambda)}$ is the Parisian ruin time given by

$$\rho^{(p,\lambda)} = \inf\left\{t > 0 \mid t - g_t > \mathrm{e}_p^{g_t} + \mathrm{e}_\lambda^{g_t}\right\}, \tag{24}$$

and for the finite-time case,

$$\mathbb{P}_x\left(\rho^{(p,\lambda)} \le t\right) = 1 - \mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{t,\lambda}}\right].$$

One can prove this relationship using similar steps as in the proof of Proposition 2.4 in [8].

Using the relationship in Equation (23) together with Corollary 2, we obtain the following expression for $\mathbb{P}_x\left(\rho^{(p,\lambda)} < \infty\right)$.

Corollary 6. *Let $p, \lambda > 0$ where $p \neq \lambda$, $x \in \mathbb{R}$ and $\mathbb{E}[X_1] > 0$,*

$$\mathbb{P}_x\left(\rho^{(p,\lambda)} < \infty\right) = 1 - \mathbb{E}[X_1]\,\frac{\Phi_\lambda\Phi_p}{\lambda p}\tilde{Z}(x,\Phi_\lambda,\Phi_p)\,. \tag{25}$$

Remark 7. Using the discussion in Remark 3, one immediately recovers the expression for the probabilities of Parisian ruin with exponentially distributed implementation delays,

$$\lim_{p\to\infty} \mathbb{P}_x\left(\rho^{(p,\lambda)}=\infty\right)=\mathbb{E}\left[X_1\right]\frac{\Phi_\lambda}{\lambda}Z\left(x,\Phi_\lambda\right)=\mathbb{P}_x\left(\rho_\lambda=\infty\right),$$

$$\lim_{\lambda\to\infty} \mathbb{P}_x\left(\rho^{(p,\lambda)}=\infty\right)=\mathbb{E}\left[X_1\right]\frac{\Phi_p}{p}Z\left(x,\Phi_p\right)=\mathbb{P}_x\left(\rho_p=\infty\right),$$

and finally, (25) reduces to the probability of classical ruin in (59) since

$$\begin{aligned}\lim_{p\to\infty}\lim_{\lambda\to\infty}\mathbb{E}\left[X_1\right]\frac{\Phi_p\Phi_\lambda}{\lambda p}\tilde{Z}\left(x,\Phi_\lambda,\Phi_p\right) &= \lim_{\lambda\to\infty}\lim_{p\to\infty}\mathbb{E}\left[X_1\right]\frac{\Phi_p\Phi_\lambda}{\lambda p}\tilde{Z}\left(x,\Phi_\lambda,\Phi_p\right)\\ &= \mathbb{E}\left[X_1\right]W\left(x\right)=\mathbb{P}_x\left(\tau_0^-=\infty\right).\end{aligned}$$

This gives an interpretation of the symmetry in p and λ in (25). Indeed, if we reverse the roles of e_p and e_λ, that is, supposing arrival times wiht rate $p>0$ and e_λ as the implementation clock, we also have

$$\mathbb{P}_x\left(\rho^{(p,\lambda)}<\infty\right)=1-\mathbb{E}_x\left[\mathrm{e}^{-\lambda\mathcal{O}^X_{\infty,p}}\right].$$

In the next theorem, we give further fluctuation identities involving the Parisian ruin time $\rho^{(p,\lambda)}$.

Theorem 8. *For $p,\lambda>0$, $a,b,q,\theta\geq 0$ and $x\leq b$,*

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)}+\theta X_{\rho^{(p,\lambda)}}}\mathbf{1}_{\left\{\rho^{(p,\lambda)}<\tau_b^+\right\}}\right]\\ &\qquad=\frac{p}{\psi_{q+\lambda}\left(\theta\right)\psi_{q+p}\left(\theta\right)}\left(\mathcal{E}^{(q,\lambda)}\left(x,\theta\right)-\frac{\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}\mathcal{E}^{(q,\lambda)}\left(b,\theta\right)\right),\end{aligned}$$

where

$$\mathcal{E}^{(q,\lambda)}\left(x,\theta\right)=\lambda Z_q\left(x,\theta\right)-\psi_q\left(\theta\right)Z_q\left(x,\Phi_{\lambda+q}\right).$$

For $x\in\mathbb{R}$,

$$\begin{aligned}&\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)}+\theta X_{\rho^{(p,\lambda)}}}\mathbf{1}_{\left\{\rho^{(p,\lambda)}<\infty\right\}}\right]\\ &=\frac{p}{\psi_{q+\lambda}\left(\theta\right)\psi_{q+p}\left(\theta\right)}\left(\mathcal{E}^{(q,\lambda)}(x,\theta)-\frac{\psi_q\left(\theta\right)\left(\Phi_{q+\lambda}-\theta\right)\left(\Phi_{p+q}-\Phi_q\right)}{p\left(\theta-\Phi_q\right)}\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)\right).\end{aligned}\tag{26}$$

For $-a\leq x\leq b$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\left\{\tau_b^+<\rho^{(p,\lambda)}\wedge\tau_{-a}^-\right\}}\right]=\frac{\tilde{W}_q^{(p,\lambda)}\left(x,a\right)}{\tilde{W}_q^{(p,\lambda)}\left(b,a\right)},\tag{27}$$

where

$$\tilde{W}_q^{(p,\lambda)}\left(x,a\right)=\lambda\mathcal{W}_x^{(q,p)}\left(x+a\right)W_{q+\lambda}(a)-p\mathcal{W}_x^{(q,\lambda)}\left(x+a\right)W_{p+q}(a).$$

When $a\to\infty$, we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\left\{\tau_b^+<\rho^{(p,\lambda)}\right\}}\right]=\frac{\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}.\tag{28}$$

Proof. See the proof in Appendix A ■

Remark 9. Following the same steps as in computation of the limit in (21), we get

$$\lim_{p\to\infty}\frac{\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}=\frac{Z_q\left(x,\Phi_{\lambda+q}\right)}{Z_q\left(b,\Phi_{\lambda+q}\right)}.$$

Thus,

$$\begin{aligned}\lim_{p\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)}+\theta X_{\rho^{(p,\lambda)}}}\mathbf{1}_{\left\{\rho^{(p,\lambda)}<\tau_b^+\right\}}\right] &= \frac{1}{\lambda-\psi_q(\theta)}\left(\mathcal{E}^{(q,\lambda)}\left(x,\theta\right)-\frac{Z_q\left(x,\Phi_{\lambda+q}\right)}{Z_q\left(b,\Phi_{\lambda+q}\right)}\mathcal{E}^{(q,\lambda)}\left(b,\theta\right)\right)\\ &= \frac{\lambda}{\lambda-\psi_q(\theta)}\left(Z_q\left(x,\theta\right)-Z_q\left(x,\Phi_{\lambda+q}\right)\frac{Z_q\left(b,\theta\right)}{Z_q\left(b,\Phi_{\lambda+q}\right)}\right)\\ &= \mathbb{E}_x\left[\mathrm{e}^{-qT_0^-+\theta X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right],\end{aligned}$$

which corresponds to identity (62).

Next is an expression of the Gerber–Shiu distribution at the Parisian ruin time $\rho^{(p,\lambda)}$.

Theorem 10. *For $p,\lambda>0$, $q\geq 0$, $x\leq b$ and $y\leq 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)}}\mathbf{1}_{\left\{X_{\rho^{(p,\lambda)}}\in\mathrm{d}y,\rho^{(p,\lambda)}<\tau_b^+\right\}}\right]=p\left(\mathcal{E}_y^{(q,\lambda)}\left(x\right)-\frac{\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}\mathcal{E}_y^{(q,\lambda)}\left(b\right)\right)\mathrm{d}y,\quad(29)$$

where

$$\begin{aligned}\mathcal{E}_y^{(q,\lambda)}\left(x\right) &= \lambda\frac{\mathcal{W}_x^{(q,\lambda)}\left(x-y\right)-\mathcal{W}_x^{(q,p)}\left(x-y\right)}{\lambda-p}\\ &\quad -Z_q\left(x,\Phi_{\lambda+q}\right)\left(\frac{\lambda W_{q+\lambda}(-y)-pW_{p+q}\left(-y\right)}{\lambda-p}\right).\end{aligned}$$

Remark 11. The Gerber–Shiu distribution in (29) has a similar structure as the one in [3] (although it is hard to show the convergence when $p\to\infty$), that is,

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbf{1}_{\left\{X_{T_0^-}\in\mathrm{d}y,T_0^-<\tau_b^+\right\}}\right]=\lambda\left(\frac{Z_q\left(x,\Phi_{\lambda+q}\right)}{Z_q\left(b,\Phi_{\lambda+q}\right)}\mathcal{W}_b^{(q,\lambda)}\left(b-y\right)-\mathcal{W}_x^{(q,\lambda)}\left(x-y\right)\right)\mathrm{d}y.\quad(30)$$

Using the relationship between the Parisian time $\rho^{(p,\lambda)}$ and Poissonian occupation time, we obtain the following expression for the Laplace transform of $\mathcal{O}^X_{\mathrm{e}_q,\lambda}$ where e_q is an exponential random variable with rate $q>0$ that is independent of X.

Corollary 12. *For $p\geq 0$, $q,\lambda>0$ and $x\in\mathbb{R}$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\mathrm{e}_q,\lambda}}\right]=1-\frac{p}{(\lambda+q)\left(p+q\right)}\left(\mathcal{E}^{(q,\lambda)}(x,0)-\frac{q\Phi_{q+\lambda}\left(\Phi_{p+q}-\Phi_q\right)}{p\Phi_q}\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)\right).\quad(31)$$

Proof. First, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\mathrm{e}_q,\lambda}}\right]=\mathbb{P}_x\left(\mathcal{O}^X_{\mathrm{e}_q,\lambda}<\mathrm{e}_p\right)=\mathbb{P}_x\left(\rho^{(p,\lambda)}>\mathrm{e}_q\right)=1-\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)}}\right].\quad(32)$$

Then, using (26) for $\theta = 0$, we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\mathrm{e}_q,\lambda}}\right] = 1 - \frac{p}{(\lambda+q)(p+q)}\left(\mathcal{E}^{(q,\lambda)}(x,0) - \frac{q\Phi_{q+\lambda}\left(\Phi_{p+q}-\Phi_q\right)}{p\Phi_q}\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)\right). \quad (33)$$

■

Remark 13. When $\lambda \to \infty$, we recover the Laplace transform of $\mathcal{O}^X_{\mathrm{e}_q}$,

$$\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\mathrm{e}_q}}\right] = \lim_{\lambda\to\infty}\mathbb{E}_x\left[\mathrm{e}^{-p\mathcal{O}^X_{\mathrm{e}_q,\lambda}}\right] = 1 - \frac{p}{(p+q)}\left(Z_q(x) - \frac{q\left(\Phi_{p+q}-\Phi_q\right)}{p\Phi_q}Z_q\left(x,\Phi_{p+q}\right)\right), \quad (34)$$

which can be found in [11].

4.1.1. *Parisian ruin with Erlang$(2,\lambda)$ implementation delays.* As an application of the previous results, we study Parisian ruin with Erlang$(2,\lambda)$ implementation delays denoted by $\rho^{(2)}_\lambda$. Hence, letting $p \to \lambda$ in Corollary 6, we obtain the following expression for the probability of Parisian ruin with Erlang$(2,\lambda)$ implementation delays which has a similar structure as the one in (61).

Corollary 14. *For $\lambda > 0$, $x \in \mathbb{R}$ and $\mathbb{E}[X_1] > 0$, we have*

$$\mathbb{P}_x\left(\rho^{(2)}_\lambda < \infty\right) = 1 - \mathbb{E}[X_1]\frac{\Phi^2_\lambda}{\lambda^2}\tilde{Z}\left(x,\Phi_\lambda,\Phi_\lambda\right). \quad (35)$$

Similarly, from Theorem 8, we obtain the following results.

Corollary 15. *For $q,\lambda > 0$, $x \le b$ and $y \le 0$,*

$$\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(2)}_\lambda}\mathbf{1}_{\left\{X_{\rho^{(2)}_\lambda}\in\mathrm{d}y,\,\rho^{(2)}_\lambda<\tau^+_b\right\}}\right] = \lambda\left(\mathcal{E}^{(\lambda)}_y(x) - \frac{\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{\lambda+q}\right)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{\lambda+q}\right)}\mathcal{E}^{(\lambda)}_y(b)\right)\mathrm{d}y, \quad (36)$$

where

$$\mathcal{E}^\lambda_y(x) = \lambda\frac{\partial\mathcal{W}^{(q,\lambda)}_x(x-y)}{\partial\lambda} - Z_q\left(x,\Phi_{\lambda+q}\right)\left(W_{q+\lambda}(-y) + \lambda\frac{\partial W_{q+\lambda}(-y)}{\partial\lambda}\right).$$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(2)}_\lambda+\theta X_{\rho^{(2)}_\lambda}}\mathbf{1}_{\left\{\rho^{(2)}_\lambda<\tau^+_b\right\}}\right] = \frac{\lambda}{\left(\psi_{\lambda+q}(\theta)\right)^2}\left(\mathcal{E}^{(q,\lambda)}(x,\theta) - \frac{\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{\lambda+q}\right)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{\lambda+q}\right)}\mathcal{E}^{(q,\lambda)}(b,\theta)\right), \quad (37)$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau^+_b}\mathbf{1}_{\left\{\tau^+_b<\rho^{(2)}_\lambda\right\}}\right] = \frac{\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{\lambda+q}\right)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{\lambda+q}\right)}.$$

For $x \in \mathbb{R}$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(2)}_\lambda+\theta X_{\rho^{(2)}_\lambda}}\mathbf{1}_{\left\{\rho^{(2)}_\lambda<\infty\right\}}\right] = \frac{\lambda}{\left(\psi_{\lambda+q}(\theta)\right)^2}\left(\mathcal{E}^{(q,\lambda)}(x) - \frac{\psi_q(\theta)\left(\Phi_{q+\lambda}-\theta\right)\left(\Phi_{\lambda+q}-\Phi_q\right)\tilde{Z}_q\left(x,\Phi_{\lambda+q},\Phi_{\lambda+q}\right)}{\lambda\left(\theta-\Phi_q\right)}\right). \quad (38)$$

Remark 16. Setting $\theta = x = 0$ in (38), we recover Equation (22) in [1], that is

$$\mathbb{E}\left[\mathrm{e}^{-q\rho_\lambda^{(2)}}\mathbf{1}_{\left\{\rho_\lambda^{(2)}<\infty\right\}}\right] = \frac{\lambda}{(\lambda+q)} - \frac{\lambda}{(\lambda+q)^2}\frac{\Phi_{\lambda+q}\left(\Phi_{\lambda+q}-\Phi_q\right)}{\Phi_q\Phi'_{\lambda+q}},$$

where Φ'_λ is the derivative of Φ_λ with respect to the sub-index λ.

4.1.2. *Examples.* In this subsection, we compute the probability of Parisian ruin with Erlang$(2,\lambda)$ delays using Eq. (35) for the cases of Brownian risk process and Cramér-Lundberg process with exponential claims.

1. Brownian risk process. Let X be a Brownian risk process, i.e.

$$X_t = x + \mu t + \sigma B_t,$$

where $\mu > 0$, $\mu > 0$ and $B = \{B_t, t \geq 0\}$ is a standard Brownian motion. Then, we have $\psi(\theta) = \mu\theta + \frac{\sigma^2}{2}\theta^2$ and $\Phi_\lambda = \left(\sqrt{\mu^2 + 2\sigma^2\lambda} - \mu\right)\sigma^{-2}$. In this case, for $x \geq 0$ and $q > 0$, the scale functions of X are given by

$$W(x) = \frac{1}{\mu}\left(1 - \mathrm{e}^{-2\mu x/\sigma^2}\right),$$

and

$$Z(x,\theta) = \frac{\psi(\theta)}{\mu}\left(\frac{1}{\theta} - \frac{\mathrm{e}^{-2\mu\sigma^{-2}x}}{\theta + 2\mu\sigma^{-2}}\right).$$

The derivative of Z with respect to θ is given by

$$\begin{aligned}
Z'(x,\theta) &= \frac{\psi'(\theta)}{\mu}\left(\frac{1}{\theta} - \frac{\mathrm{e}^{-2\mu\sigma^{-2}x}}{\theta + 2\mu\sigma^{-2}}\right) \\
&\quad + \frac{\psi(\theta)}{\mu}\left(\frac{\mathrm{e}^{-2\mu\sigma^{-2}x}}{(\theta + 2\mu\sigma^{-2})^2} - \frac{1}{\theta^2}\right) \\
&= \psi'(\theta)\frac{Z(x,\theta)}{\psi(\theta)} + \frac{\psi(\theta)}{\mu}\left(\frac{\mathrm{e}^{-2\mu\sigma^{-2}x}}{(\theta + 2\mu\sigma^{-2})^2} - \frac{1}{\theta^2}\right).
\end{aligned}$$

Using the expression in (17), we obtain

$$\tilde{Z}(x,\Phi_\lambda,\Phi_\lambda) = \frac{\lambda}{\mu}\left(\frac{1}{\Phi_\lambda^2} - \frac{\mathrm{e}^{-2\mu\sigma^{-2}x}}{(\Phi_\lambda + 2\mu\sigma^{-2})}\right).$$

Putting all the terms together, we get the following expression for the probability of Parisian ruin with Erlang$(2,\lambda)$ delay,

$$\mathbb{P}_x\left(\rho_\lambda^{(2)} < \infty\right) = 1 - \frac{\Phi_\lambda^2}{\lambda}\left(\frac{1}{\Phi_\lambda} - \frac{\mathrm{e}^{-2\mu\sigma^{-2}x}}{(\Phi_\lambda + 2\mu\sigma^{-2})}\right).$$

2. Cramér-Lundberg process with exponential claims. Let X be a Cramér-Lundberg risk processes with exponentially distributed claims, i.e.

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity $\eta > 0$, and where $\{C_1, C_2, \dots\}$ are independent and exponentially distributed random variables with parameter α. The Poisson process and the random variables are mutually independent. Then, we have

$$\psi(\theta) = c\theta - \eta + \frac{\alpha\eta}{\theta + \alpha} \quad \text{and} \quad \Phi_\lambda = \frac{1}{2c}\left(\lambda + \eta - c\alpha + \sqrt{(\lambda + \eta - c\alpha)^2 + 4c\alpha\lambda}\right).$$

The scale functions of X are given by

$$W(x) = \frac{1}{c - \eta/\alpha}\left(1 - \frac{\eta}{c\alpha}\mathrm{e}^{(\frac{\eta}{c} - \alpha)x}\right),$$

and

$$Z(x, \theta) = \frac{\psi(\theta)}{c - \eta/\alpha}\left(\frac{1}{\theta} - \frac{\eta}{c\alpha}\frac{\mathrm{e}^{(\eta/c - \alpha)x}}{\theta + \alpha - \eta/c}\right).$$

The derivative of Z with respect to θ is given by

$$Z'(x, \theta) = \frac{\psi'(\theta)}{\psi(\theta)}Z(x, \theta) + \frac{\psi(\theta)}{c - \eta/\alpha}\left(\frac{\eta}{c\alpha}\frac{\mathrm{e}^{(\eta/c - \alpha)x}}{(\theta + \alpha - \eta/c)^2} - \frac{1}{\theta^2}\right),$$

and consequently,

$$\tilde{Z}(x, \Phi_\lambda, \Phi_\lambda) = \frac{\lambda}{c - \eta/\alpha}\left(\frac{1}{\Phi_\lambda^2} - \frac{\eta}{c\alpha}\frac{\mathrm{e}^{(\eta/c - \alpha)x}}{(\Phi_\lambda + \alpha - \eta/c)^2}\right).$$

Putting all the pieces together, we obtain

$$\mathbb{P}_x\left(\rho_\lambda^{(2)} < \infty\right) = 1 - \frac{\Phi_\lambda^2}{\lambda}\left(\frac{1}{\Phi_\lambda^2} - \frac{\eta}{c\alpha}\frac{\mathrm{e}^{(\eta/c - \alpha)x}}{(\Phi_\lambda + \alpha - \eta/c)^2}\right).$$

4.2. **Parisian ruin with Erlang(m, λ) implementation delays.** Now, we suppose that occupation time is accumulated after two consecutive observations of the surplus process below 0, we denote the Poissonian occupation time by

$$\mathcal{O}^X_{\infty,\lambda,2} = \sum_{n \geq 2} \tau_0^+ \circ \theta_{\xi_n^{(2)}} \mathbf{1}_{\{\sup_{t \in \left[\xi_{n-1}^{(2)}, \xi_n^{(2)}\right]}(X_t) < 0\}},$$

where discrete observation scheme $\left\{\xi_n^{(2)}\right\}_{n \in \mathbb{N}}$ is defined such as $\xi_0^{(2)} = 0$, $\xi_1^{(2)} = \mathrm{e}_\lambda^1$, $\xi_2^{(2)} - \xi_1^{(2)} = \mathrm{e}_\lambda^2$ and for $n \geq 3$

$$\xi_n^{(2)} - \xi_{n-1}^{(2)} = \begin{cases} \mathrm{e}_\lambda^n, & \text{if } \sup_{t \in \left[\xi_{n-2}^{(2)}, \xi_{n-1}^{(2)}\right]}(X_t) \geq 0, \\ \tau_0^+ \circ \theta_{\xi_{n-1}^{(2)}} + \mathrm{e}_\lambda^n, & \text{if } \sup_{t \in \left[\xi_{n-2}^{(2)}, \xi_{n-1}^{(2)}\right]}(X_t) < 0, \end{cases} \tag{39}$$

The Laplace transform of $\mathcal{O}^X_{\infty,\lambda,2}$ can be computed using the following a standard probabilistic decomposition

$$\mathbb{E}_x\left[\mathrm{e}^{-q\mathcal{O}^X_{\infty,\lambda,2}}\right] = \mathbb{P}_x\left(\rho_\lambda^{(2)} = \infty\right) + \mathbb{E}\left[\mathrm{e}^{-q\mathcal{O}^X_{\infty,\lambda,2}}\right]\mathbb{E}_x\left[\mathrm{e}^{\Phi_q X_{\rho_\lambda^{(2)}}}\mathbf{1}_{\left\{\rho_\lambda^{(2)} < \infty\right\}}\right]. \tag{40}$$

Letting $q \to \lambda$ in (40) combined with the results in the previous subsection (4.1), one can obtain an expression for probability of Parisian ruin with Erlang$(3, \lambda)$ implementation delays.

More generally, we denote the m^{th}-Poissonian occupation time by $\mathcal{O}^X_{t,\lambda,m}$. In this case, occupation time will be accumulated when the process X has been observed to be negative at the last m Poisson arrival times. Then, for $m \in \mathbb{N}$, we define

$$\mathcal{O}^X_{\infty,\lambda,m} = \sum_{n\geq m} \tau_0^+ \circ \theta_{\xi_n^{(m)}} \mathbf{1}_{\left\{\sup_{t\in\left[\xi^{(m)}_{n-m+1},\xi^{(m)}_n\right]}(X_t)<0\right\}},$$

where $\xi^{(m)}_n - \xi^{(m)}_{n-1} = \mathrm{e}^n_\lambda$ for $n \leq m$, and

$$\xi^{(m)}_n - \xi^{(m)}_{n-1} = \begin{cases} \mathrm{e}^n_\lambda, & \text{if } \sup_{t\in\left[\xi^{(m)}_{n-m},\xi^{(m)}_{n-1}\right]}(X_t) \geq 0, \\ \tau_0^+ \circ \theta_{\xi^{(m)}_{n-1}} + \mathrm{e}^n_\lambda, & \text{if } \sup_{t\in\left[\xi^{(m)}_{n-m},\xi^{(m)}_{n-1}\right]}(X_t) < 0, \end{cases} \tag{41}$$

for $n \geq m+1$. Also, we denote $\rho^{(m)}_\lambda$ as the Parisian ruin time with Erlang(m,λ) implementation delay, that is,

$$\rho^{(m)}_\lambda = \inf\left\{t > 0 \mid t - g_t > T^{m,g_t}_\lambda\right\},$$

where T^{m,g_t}_λ follows an Erlang(n,λ) distribution. In particular, we have $\rho^{(0)}_\lambda = \tau_0^-$ and $\rho^{(1)}_\lambda = T_0^-$. Similarly, we have the following connection

$$\mathbb{P}_x\left(\rho^{(m)}_\lambda < \infty\right) = 1 - \mathbb{E}_x\left[\mathrm{e}^{-\lambda\mathcal{O}^X_{\infty,\lambda,m}}\right]. \tag{42}$$

Thus, we obtain the following recursive formula for the probability of Parisian ruin with Erlang(m,λ) implementation delays.

Proposition 17. *For $m \geq 1$, $\lambda > 0$ and $x \in \mathbb{R}$,*

$$\mathbb{P}_x\left(\rho^{(m)}_\lambda = \infty\right) = \mathbb{P}_x\left(\rho^{(m-1)}_\lambda = \infty\right) + \mathbb{P}\left(\rho^{(m-1)}_\lambda = \infty\right)\frac{\mathbb{E}_x\left[\mathrm{e}^{\Phi_\lambda X_{\rho^{(m-1)}_\lambda}}\mathbf{1}_{\left\{\rho^{(m-1)}_\lambda<\infty\right\}}\right]}{1-\mathbb{E}\left[\mathrm{e}^{\Phi_\lambda X_{\rho^{(m-1)}_\lambda}}\mathbf{1}_{\left\{\rho^{(m-1)}_\lambda<\infty\right\}}\right]}. \tag{43}$$

Remark 18. When m tends to infinity, it is possible to approximate the probability of Parisian ruin with fixed delay, that is $\mathbb{P}\left(\kappa_r < \infty\right)$ where

$$\kappa_r = \inf\left\{t > 0\colon t - g_t > r\right\}, \tag{44}$$

and $g_t = \sup\left\{0 \leq s \leq t\colon X_s \geq 0\right\}$, by the probability of Parisian ruin with Erlang$(m, \lambda = m/r)$ distributed implementation delays (see Bladt et al. [5] and Landriault et al. [13]).

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Appendix A. Proofs

A.1. **Proof of Theorem 1.** First, we set

$$g(x) = \mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+ - p\mathcal{O}^X_{\tau_b^+,\lambda}}\mathbf{1}_{\left\{\tau_b^+<\infty\right\}}\right].$$

For $x<0$, by the strong Markov property and the spectral negativity of X,

$$g\left(x\right)=\mathbb{E}_{x}\left[\mathrm{e}^{-(p+q)\tau_{0}^{+}}\right]g\left(0\right). \tag{45}$$

For $0\leq x\leq b$, again by the strong Markov property,

$$g\left(x\right)=\mathbb{E}_{x}\left[\mathrm{e}^{-q\tau_{b}^{+}}\mathbf{1}_{\left\{\tau_{b}^{+}<T_{0}^{-}\right\}}\right]+\mathbb{E}_{x}\left[\mathrm{e}^{-qT_{0}^{-}}g\left(X_{T_{0}^{-}}\right)\mathbf{1}_{\left\{T_{0}^{-}<\tau_{b}^{+}\right\}}\right]. \tag{46}$$

Hence, plugging (45) into (46) and using (58), we obtain

$$g\left(x\right)=\mathbb{E}_{x}\left[\mathrm{e}^{-q\tau_{b}^{+}}\mathbf{1}_{\left\{\tau_{b}^{+}<T_{0}^{-}\right\}}\right]+\mathbb{E}_{x}\left[\mathrm{e}^{-qT_{0}^{-}+\Phi_{p+q}X_{T_{0}^{-}}}\mathbf{1}_{\left\{T_{0}^{-}<\tau_{b}^{+}\right\}}\right]g\left(0\right) \tag{47}$$

Letting $x=0$ in (47) and using (62) and (64), we obtain

$$g\left(0\right)=\frac{\mathbb{E}\left[\mathrm{e}^{-q\tau_{b}^{+}}\mathbf{1}_{\left\{\tau_{b}^{+}<T_{0}^{-}\right\}}\right]}{1-\mathbb{E}\left[\mathrm{e}^{-qT_{0}^{-}+\Phi_{p+q}X_{T_{0}^{-}}}\mathbf{1}_{\left\{T_{0}^{-}<\tau_{b}^{+}\right\}}\right]}=\frac{\lambda-p}{\left(\Phi_{\lambda+q}-\Phi_{p+q}\right)}\frac{1}{\tilde{Z}_{q}\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}$$

Thus, substituting $g(0)$ into (47) combined with (62) and (64), yields

$$\begin{aligned}g\left(x\right)&=\frac{Z_{q}\left(x,\Phi_{\lambda+q}\right)}{Z_{q}\left(b,\Phi_{\lambda+q}\right)}+\left(\frac{\lambda-p}{\left(\Phi_{\lambda+q}-\Phi_{p+q}\right)}\frac{1}{\tilde{Z}_{q}\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}\right)\\&\quad\times\frac{\lambda}{\lambda-p}\left(\frac{Z_{q}\left(x,\Phi_{p+q}\right)Z_{q}\left(b,\Phi_{\lambda+q}\right)-Z_{q}\left(x,\Phi_{\lambda+q}\right)Z_{q}\left(b,\Phi_{p+q}\right)}{Z_{q}\left(b,\Phi_{\lambda+q}\right)}\right)\\&=\frac{\tilde{Z}_{q}\left(x,\Phi_{\lambda+q},\Phi_{p+q}\right)}{\tilde{Z}_{q}\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}.\end{aligned}$$

A.2. **Proof of Theorem 4.** From (20), we have

$$\begin{aligned}\mathbb{E}_{x}\left[\mathrm{e}^{-p\mathcal{O}_{\infty,\lambda}^{X}}\right]&=\mathbb{E}\left[X_{1}\right]\frac{\Phi_{p}\Phi_{\lambda}}{\lambda p}\left(\frac{pZ\left(x,\Phi_{\lambda}\right)-\lambda Z\left(x,\Phi_{p}\right)}{\Phi_{p}-\Phi_{\lambda}}\right)\\&=\mathbb{E}\left[X_{1}\right]\frac{Z\left(x,\Phi_{\lambda}\right)\Phi_{\lambda}}{\lambda}\left(1+\frac{\Phi_{\lambda}}{\Phi_{p}-\Phi_{\lambda}}\right)-\mathbb{E}\left[X_{1}\right]\Phi_{\lambda}\frac{\Phi_{p}Z\left(x,\Phi_{p}\right)}{p\left(\Phi_{p}-\Phi_{\lambda}\right)}.\end{aligned}$$

Using Kendall's identity, Tonelli's Theorem and (58), we obtain

$$\int_{0}^{\infty}\mathrm{e}^{-pr}\Gamma_{\lambda}(r)\mathrm{d}r=\frac{1}{\Phi_{p}-\Phi_{\lambda}},\quad p>\lambda. \tag{48}$$

We also have the following identity from Landriault et al. [11] (see Eq. (16)),

$$\frac{\Phi_{p}Z\left(x,\Phi_{p}\right)}{p}=\int_{0}^{\infty}\mathrm{e}^{-py}\left(\Lambda'\left(x,y\right)+W\left(x\right)\delta_{0}\left(\mathrm{d}y\right)\right)\mathrm{d}y, \tag{49}$$

By Laplace inversion, we deduce

$$\begin{aligned}\mathbb{P}_{x}\left(\mathcal{O}_{\infty,\lambda}^{X}\in\mathrm{d}r\right)&=\mathbb{E}\left[X_{1}\right]Z\left(x,\Phi_{\lambda}\right)\frac{\Phi_{\lambda}}{\lambda}\delta_{0}\left(\mathrm{d}r\right)+\mathbb{E}\left[X_{1}\right]Z\left(x,\Phi_{\lambda}\right)\frac{\Phi_{\lambda}^{2}}{\lambda}\Gamma_{\lambda}\left(r\right)\mathrm{d}r\\&\quad-\mathbb{E}\left[X_{1}\right]\Phi_{\lambda}\left[\Gamma_{\lambda}\left(r\right)W\left(x\right)+\int_{0}^{r}\Gamma_{\lambda}\left(r-s\right)\Lambda'\left(x,s\right)\mathrm{d}s\right]\mathrm{d}r.\end{aligned}$$

A.3. **Proof of Theorem 8.** Let

$$v(x)=\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)}+\theta X_{\rho^{(p,\lambda)}}}\mathbf{1}_{\left\{\rho^{(p,\lambda)}<\tau_b^+\right\}}\right].$$

For $x<0$, by the strong Markov property and the spectral negativity of X, we have

$$v(x)=\mathbb{E}_x\left[\mathrm{e}^{-q\mathrm{e}_p+\theta X_{\mathrm{e}_p}}\mathbf{1}_{\left\{\tau_0^+>\mathrm{e}_p\right\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-(p+q)\tau_0^+}\right]v(0). \tag{50}$$

For $0\le x<b$, using again the strong Markov property, we obtain

$$v(x)=\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}v\left(X_{T_0^-}\right)\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]. \tag{51}$$

Plugging (50) in (51), we obtain, for all $x\in\mathbb{R}$

$$\begin{aligned}v(x) &= \mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbb{E}_{X_{T_0^-}}\left[\mathrm{e}^{-q\mathrm{e}_p+\theta X_{\mathrm{e}_p}}\mathbf{1}_{\left\{\tau_0^+>\mathrm{e}_p\right\}}\right]\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]\\ &\quad+\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbb{E}_{X_{T_0^-}}\left[\mathrm{e}^{-(p+q)\tau_0^+}\right]\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]v(0).\end{aligned} \tag{52}$$

For $x<0$, $\theta>p+q$ and using the potential measure in Eq. (60), we have

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\mathrm{e}_p+\theta X_{\mathrm{e}_p}}\mathbf{1}_{\left\{\tau_0^+>\mathrm{e}_p\right\}}\right] &= \int_{-\infty}^0 \mathrm{e}^{\theta z}\mathbb{P}_x\left(X_{\mathrm{e}_p}\in\mathrm{d}z,\mathrm{e}_q\wedge\tau_0^+>\mathrm{e}_p\right)\\ &= p\int_{-\infty}^0\mathrm{e}^{\theta z}\left(\mathrm{e}^{\Phi_{p+q}x}W_{p+q}(-z)-W_{p+q}(x-z)\right)\mathrm{d}z\\ &= \frac{p\mathrm{e}^{\Phi_{p+q}x}}{\psi_{p+q}(\theta)}-\frac{\mathrm{e}^{\theta x}p}{\psi_{p+q}(\theta)}.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}\mathbb{E}_{X_{T_0^-}}\left[\mathrm{e}^{-p\mathrm{e}_q+\theta X_{\mathrm{e}_p}}\mathbf{1}_{\left\{\tau_0^+>\mathrm{e}_q\right\}}\right]\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right] &= \frac{p}{\psi_{p+q}(\theta)}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-+\Phi_{p+q}X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]\\ &\quad-\frac{p}{\psi_{p+q}(\theta)}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-+\theta X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right],\end{aligned}$$

and plugging the above expectation in (52) and using (58), the expression of $v(x)$ is now equal to

$$\begin{aligned}v(x) &= \frac{p}{\psi_{p+q}(\theta)}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-+\Phi_{p+q}X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]-\frac{p}{\psi_{p+q}(\theta)}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-+\theta X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]\\ &\quad+\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-+\Phi_{p+q}X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]v(0).\end{aligned} \tag{53}$$

For $x=0$ and using (58) and (62), we have

$$\begin{aligned}v(0) &= \frac{\frac{p}{\psi_{p+q}(\theta)}\left(\mathbb{E}\left[\mathrm{e}^{-qT_0^-+\Phi_{p+q}X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]-\mathbb{E}\left[\mathrm{e}^{-qT_0^-+\theta X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]\right)}{1-\mathbb{E}\left[\mathrm{e}^{-qT_0^-+\Phi_{p+q}X_{T_0^-}}\mathbf{1}_{\left\{T_0^-<\tau_b^+\right\}}\right]}\\ &= \frac{-p}{\psi_{p+q}(\theta)}-\frac{p}{\psi_{p+q}(\theta)}\left(\frac{(\lambda-p)}{\psi_{q+\lambda}(\theta)\left(\Phi_{\lambda+q}-\Phi_{p+q}\right)}\frac{\mathcal{E}^{(q,\lambda)}(b,\theta)}{\tilde{Z}_q\left(b,\Phi_{\lambda+q},\Phi_{p+q}\right)}\right).\end{aligned}$$

Now, plugging the last expression in (53) combined with (62), we have, after few manipulations,

$$\begin{aligned} v(x) &= \frac{p}{\psi_{p+q}(\theta)\,\psi_{\lambda+q}(\theta)}\left(\lambda Z_q(x,\theta)-\psi_q(\theta)\,Z_q(x,\Phi_{\lambda+q})\right) \\ &\quad -\frac{p}{\psi_{p+q}(\theta)\,\psi_{\lambda+q}(\theta)}\frac{\tilde{Z}_q(x,\Phi_{\lambda+q},\Phi_{p+q})}{\tilde{Z}_q(b,\Phi_{\lambda+q},\Phi_{p+q})}\left(\lambda Z_q(b,\theta)-\psi_q(\theta)\,Z_q(b,\Phi_{\lambda+q})\right) \\ &= \frac{p}{\psi_{q+\lambda}(\theta)\,\psi_{q+p}(\theta)}\left(\mathcal{E}^{(q,\lambda)}(x,\theta)-\frac{\tilde{Z}_q(x,\Phi_{\lambda+q},\Phi_{p+q})}{\tilde{Z}_q(b,\Phi_{\lambda+q},\Phi_{p+q})}\mathcal{E}^{(q,\lambda)}(b,\theta)\right) \end{aligned}$$

which proves the first identity.

To prove the second identity, we need compute the following limit

$$\lim_{b\to\infty}\frac{\mathcal{E}^{(q,\lambda)}(b,\theta)}{\tilde{Z}_q(b,\Phi_{\lambda+q},\Phi_{p+q})}.$$

Using again (11), it follows by the dominated convergence theorem,

$$\lim_{b\to\infty}\frac{Z_q(b,\Phi_{\lambda+q})}{Z_q(b,\theta)}=\frac{\lambda(\theta-\Phi_q)}{\psi_q(\theta)\,(\Phi_{\lambda+q}-\Phi_q)},$$

and then,

$$\lim_{b\to\infty}\frac{\tilde{Z}_q(b,\Phi_{\lambda+q},\Phi_{p+q})}{Z_q(b,\theta)}=\frac{\lambda p\,(\theta-\Phi_q)}{\psi_q(\theta)\,(\Phi_{\lambda+q}-\Phi_q)(\Phi_{p+q}-\Phi_q)}.$$

We finally obtain

$$\lim_{b\to\infty}\frac{\mathcal{E}^{(q,\lambda)}(b,\theta)}{\tilde{Z}_q(b,\Phi_{\lambda+q},\Phi_{p+q})}=\lim_{b\to\infty}\frac{\mathcal{E}^{(q,\lambda)}(b,\theta)/Z_q(b,\theta)}{\tilde{Z}_q(b,\Phi_{\lambda+q},\Phi_{p+q})\,/Z_q(b,\theta)}=\frac{\psi_q(\theta)\,(\Phi_{q+\lambda}-\theta)\,(\Phi_{p+q}-\Phi_q)}{p\,(\theta-\Phi_q)}.$$

Now, we prove identity in Equation (27). We set

$$w(x)=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\left\{\tau_b^+<\rho^{(p,\lambda)}\wedge\tau_{-a}^-\right\}}\right].$$

For $-a<x<0$, using (57), the strong Markov property and the spectral negativity of X, we have

$$w(x)=\mathbb{E}_x\left[\mathrm{e}^{-(p+q)\tau_0^+}\mathbf{1}_{\left\{\tau_0^+<\tau_{-a}^-\right\}}\right]w(0)=\frac{W_{p+q}(x+a)}{W_{p+q}(a)}w(0). \tag{54}$$

For $0\leq x\leq b$, using again the strong Markov property, we obtain

$$w(x)=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\left\{\tau_b^+<T_0^-\wedge\tau_{-a}^-\right\}}\right]+\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}w\left(X_{T_0^-}\right)\mathbf{1}_{\left\{T_0^-<\tau_b^+\wedge\tau_{-a}^-\right\}}\right]. \tag{55}$$

Plugging (54) in (55), we deduce, for $x\in\mathbb{R}$

$$w(x)=\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\left\{\tau_b^+<T_0^-\wedge\tau_{-a}^-\right\}}\right]+\frac{w(0)}{W_{p+q}(a)}\mathbb{E}_x\left[\mathrm{e}^{-qT_0^-}W_{p+q}\left(X_{T_0^-}+a\right)\mathbf{1}_{\left\{T_0^-<\tau_b^+\wedge\tau_{-a}^-\right\}}\right].$$

For $x=0$, using (63) and (65), we have

$$\begin{aligned} w(0) &= \frac{\mathbb{E}\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\left\{\tau_b^+<T_0^-\wedge\tau_{-a}^-\right\}}\right]}{1-\mathbb{E}\left[\mathrm{e}^{-qT_0^-}W_{p+q}\left(X_{T_0^-}+a\right)\mathbf{1}_{\left\{T_0^-<\tau_b^+\wedge\tau_{-a}^-\right\}}\right]/W_{p+q}(a)} \\ &= \frac{(\lambda-p)\,W_{p+q}(a)\,W_{q+\lambda}(a)}{\lambda\tilde{W}_q^{(p,\lambda)}(b)}. \end{aligned}$$

Now, plugging the last expression in (55), we deduce

$$\begin{aligned} w(x) &= \frac{\mathcal{W}_x^{(q,\lambda)}(x+a)}{\mathcal{W}_b^{(q,\lambda)}(b+a)} + \frac{W_{q+\lambda}(a)}{\tilde{W}_q^{(p,\lambda)}(b)} \left(\mathcal{W}_x^{(q,p)}(x+a) - \mathcal{W}_x^{(q,\lambda)}(x+a)\right) \\ &\quad - \frac{W_{q+\lambda}(a)}{\tilde{W}_q^{(p,\lambda)}(b)} \left(\mathcal{W}_b^{(q,p)}(b+a) - \mathcal{W}_b^{(q,\lambda)}(b+a)\right) \frac{\mathcal{W}_x^{(q,\lambda)}(x+a)}{\mathcal{W}_b^{(q,\lambda)}(b+a)} = \frac{\tilde{W}_q^{(p,\lambda)}(x)}{\tilde{W}_q^{(p,\lambda)}(b)}, \end{aligned}$$

which concludes the proof for the third identity.
Now, taking $a \to \infty$ in (27), we have

$$\lim_{a\to\infty} \frac{\tilde{W}_q^{(p,\lambda)}(x,a)}{\tilde{W}_q^{(p,\lambda)}(b,a)} = \lim_{a\to\infty} \frac{\tilde{W}_q^{(p,\lambda)}(x,a) / \left(W_{p+q}(a) W_{q+\lambda}(a)\right)}{\tilde{W}_q^{(p,\lambda)}(b,a) / \left(W_{p+q}(a) W_{q+\lambda}(a)\right)},$$

and

$$\begin{aligned} \lim_{a\to\infty} \frac{\tilde{W}_q^{(p,\lambda)}(x,a)}{W_{p+q}(a) W_{q+\lambda}(a)} &= \lim_{a\to\infty} \frac{\lambda \mathcal{W}_x^{(q,p)}(x+a) W_{q+\lambda}(a) - p \mathcal{W}_x^{(q,\lambda)}(x+a) W_{p+q}(a)}{W_{p+q}(a) W_{q+\lambda}(a)} \\ &= \lambda Z_q(x, \Phi_{p+q}) - p Z_q(x, \Phi_{q+\lambda}), \end{aligned}$$

where in the last equality we used (7) and the fact that,

$$\lim_{a\to\infty} \frac{\mathcal{W}_x^{(q,p)}(x+a)}{W_{p+q}(a)} = \lim_{a\to\infty} \frac{W_{p+q}(x+a) - p\int_0^x W_{p+q}(x+a-y) W_q(y)\,\mathrm{d}y,}{W_{p+q}(a)} = Z_q(x, \Phi_{p+q}),$$

and similarly,

$$\lim_{a\to\infty} \frac{\mathcal{W}_x^{(q,\lambda)}(x+a)}{W_{q+\lambda}(a)} = Z_q(x, \Phi_{q+\lambda}),$$

which both follow using (7). This ends the proof.

A.4. **Proof of Theorem 10.** We have

$$\mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)} + \theta X_{\rho^{(p,\lambda)}}} \mathbf{1}_{\{\rho^{(p,\lambda)} < \tau_b^+\}}\right] = \int_{-\infty}^0 \mathrm{e}^{\theta y} \mathbb{E}_x\left[\mathrm{e}^{-q\rho^{(p,\lambda)}} \mathbf{1}_{\{X_{\rho^{(p,\lambda)}} \in \mathrm{d}y, \rho^{(p,\lambda)} < \tau_b^+\}}\right]. \tag{56}$$

More precisely, we need to compute the Laplace inverse of $\frac{\mathcal{E}^{(q,\lambda)}(x,\theta)}{\psi_{q+\lambda}(\theta)\psi_{q+p}(\theta)}$ with respect to θ. First, using (6), we have

$$\begin{aligned} \frac{\psi_q(\theta)}{\psi_{q+\lambda}(\theta)\psi_{q+p}(\theta)} &= \frac{1}{\psi_{q+p}(\theta)} + \frac{\lambda}{\psi_{q+\lambda}(\theta)\psi_{q+p}(\theta)} \\ &= \int_{-\infty}^0 \mathrm{e}^{\theta y} \left(W_{p+q}(-y) + \lambda \int_0^{-y} W_{q+\lambda}(-y-z) W_{p+q}(z)\,\mathrm{d}z\right) \mathrm{d}y \\ &= \int_{-\infty}^0 \mathrm{e}^{\theta y} \left(\frac{\lambda W_{q+\lambda}(-y) - p W_{p+q}(-y)}{\lambda - p}\right) \mathrm{d}y, \end{aligned}$$

where the last equality follows using (13). Using (15) and (14), we deduce

$$\begin{aligned} \frac{Z_q(x,\theta)}{\psi_{q+\lambda}(\theta)\psi_{q+p}(\theta)} &= \int_{-\infty}^0 \mathrm{e}^{\theta y} \left(\int_0^{-y} W_{q+\lambda}(-y-z) \mathcal{W}_x^{(q,p)}(x+z)\,\mathrm{d}z\right) \mathrm{d}y \\ &= \int_{-\infty}^0 \mathrm{e}^{\theta y} \left(\frac{\mathcal{W}_x^{(q,\lambda)}(x-y) - \mathcal{W}_x^{(q,p)}(x-y)}{\lambda - p}\right) \mathrm{d}y. \end{aligned}$$

The desired expression follows by Laplace inversion.

Appendix B. Fluctuation identities with and without delays

In this subsection , we present some of the existing fluctuation identities.
For $q > 0$ and $x \leq b$, we have

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\tau_0^-\}}\right] = \frac{W_q(x)}{W_q(b)}, \tag{57}$$

and

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<\infty\}}\right] = \mathrm{e}^{-\Phi_q(b-x)}. \tag{58}$$

Moreover, the *classical* probability of ruin is given by

$$\mathbb{P}_x\left(\tau_0^- < \infty\right) = 1 - (\mathbb{E}\left[X_1\right])_+ W(x), \tag{59}$$

where $(x)_+ = \max(x, 0)$. Also, for $x \leq a$, we have the following expression for the p-potential measure of X killed on exiting $(-\infty, a]$:

$$\int_0^\infty \mathrm{e}^{-pt}\mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_a^+\right)\mathrm{d}t = \left(\mathrm{e}^{\Phi_p(x-a)}W_p\left(a-y\right) - W_p\left(x-y\right)\right)\mathrm{d}y, \tag{60}$$

see, for instance, Chap. 8 of Kyprianou [10].

Now, Here is a collection of known fluctuation identities with delays. For $x \in \mathbb{R}$ and $\mathbb{E}[X_1] > 0$, we have form [13]

$$\mathbb{P}_x\left(T_0^- < \infty\right) = 1 - \mathbb{E}_x\left[\mathrm{e}^{-\lambda\mathcal{O}_\infty^X}\right] = 1 - \mathbb{E}[X_1]\frac{\Phi_\lambda}{\lambda}Z\left(x, \Phi_\lambda\right). \tag{61}$$

We also have the following identities has been derived by Albrecher et al. [2] (see Eq. (14) of Theorem 3.1) and by Landriault et al.[12] (see Theorem 2 and Corollary 2).

Lemma 19. *For $\lambda > 0$, $a, b, q, \theta \geq 0$ and $x \leq b$, we have*

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_0^- + \theta X_{T_0^-}}\mathbf{1}_{\{T_0^-<\tau_b^+\}}\right] = \frac{\lambda}{\lambda - \psi_q(\theta)}\left(Z_q\left(x,\theta\right) - Z_q\left(x, \Phi_{\lambda+q}\right)\frac{Z_q\left(b,\theta\right)}{Z_q\left(b, \Phi_{\lambda+q}\right)}\right). \tag{62}$$

For $a > 0$ and $-a \leq x \leq b$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^-\wedge\tau_{-a}^-\}}\right] = \frac{\mathcal{W}_x^{(q,\lambda)}\left(x+a\right)}{\mathcal{W}_b^{(q,\lambda)}\left(b+a\right)}, \tag{63}$$

and for $x \leq b$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^-\}}\right] = \frac{Z_q\left(x, \Phi_{\lambda+q}\right)}{Z_q\left(b, \Phi_{\lambda+q}\right)}. \tag{64}$$

We also have the following useful identity one can extract from [18].

Lemma 20. *For $a, b, p, q \geq 0$, $\lambda, r > 0$, $z \in reals$ and $-a \leq x \leq b$, we have*

$$\begin{aligned}\mathbb{E}_x&\left[\mathrm{e}^{-qT_0^-}W_p\left(X_{T_0^-}+z\right)\mathbf{1}_{\{T_0^-<\tau_b^+\wedge\tau_{-a}^-\}}\right] \\ &= \frac{\lambda}{p-(q+\lambda)}\frac{\mathcal{W}_x^{(q,\lambda)}\left(x+a\right)}{\mathcal{W}_b^{(q,\lambda)}\left(b+a\right)}\left(\mathcal{W}_b^{(q,p-q)}\left(b+z\right) - \mathcal{W}_b^{(q,\lambda)}\left(b+z\right)\right) \\ &\qquad - \frac{\lambda}{p-(q+\lambda)}\left(\mathcal{W}_x^{(q,p-q)}\left(x+z\right) - \mathcal{W}_x^{(q,\lambda)}\left(x+z\right)\right). \end{aligned} \tag{65}$$

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