

# Color-dressed string disk amplitudes and the descent algebra

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Inspired by the definition of color-dressed amplitudes in string theory, we define analogous *color-dressed permutations* replacing the color-ordered string amplitudes by their corresponding permutations. Decomposing the color traces into symmetrized traces and structure constants, the color-dressed permutations define *BRST-invariant permutations*, which we show are elements of the inverse Solomon descent algebra. Comparing both definitions suggests a duality between permutations in the inverse descent algebra and kinematics from the higher  $\alpha'$  sector of string disk amplitudes. We analyze the symmetries of the  $\alpha'$  disk corrections and obtain a new decomposition for them, leading to their dimensions given by sums of Stirling cycle numbers. The descent algebra also leads to the interpretation that the  $\alpha'^2 \zeta_2$  correction is orthogonal to the field-theory amplitudes as well as their respective tails of BCJ-preserving interactions. In addition, we show how the superfield expansion of BRST invariants of the pure spinor formalism corresponding to  $\alpha'^2$  corrections are encoded in the descent algebra.

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## 1. Introduction

The color-dressed string disk amplitude gives rise to an interesting harmony between color and the kinematics of the  $\alpha'^2 \zeta_2$  correction to the string disk amplitude [1]. The analysis relies on the decomposition of color traces into a basis involving symmetrized traces and structure constants following the decomposition algorithm of [2]. The color-trace decomposition problem was recently solved in closed form [3] involving the Solomon (or Eulerian) idempotents, and it became apparent that a beautiful mathematical framework governs the brute-force expressions used in the arguments of [1]. Together with some interesting observations shared by Oliver Schlotterer [4] about color-dressed permutations (to be defined in section 3), understanding the combinatorics of the harmony first seen in [1] became the motivation for this paper. As we will see, these matters are tightly knit with the mathematical framework of the Solomon descent algebra [5,6,7,8,9,10,11].

To isolate the combinatorics within the color-dressed amplitudes we will define the *color-dressed permutations*

$$P_n = \sum_{\sigma \in S_n, \sigma(1)=1} \text{Tr}(T^\sigma) \sigma, \quad T^\sigma := T^{\sigma(1)} T^{\sigma(2)} \dots T^{\sigma(n)} \quad (1.1)$$

where each permutation  $\sigma \in S_n$  is weighted by the trace of some Lie algebra generators, the *color trace*. When the closed formula for the color-trace decomposition from [3] is plugged into (1.1), the permutations appearing as coefficients with respect to a basis of color factors define what we call *BRST invariant permutations*  $\gamma_{1|P_1, \dots, P_k}$  with  $k = 1, \dots, n-1$ . We will then show that  $\gamma_{1| \dots}$  belong to the inverse Solomon algebra<sup>1</sup> and we will find a closed formula for them, namely

$$\gamma_{1|P_1, \dots, P_k} = 1 \cdot \mathcal{E}(P_1) \sqcup \mathcal{E}(P_2) \sqcup \dots \sqcup \mathcal{E}(P_k) \quad (1.2)$$

where  $\mathcal{E}(P)$  is the *Berends-Giele idempotent* and it is the result of mapping the permutations of the Solomon idempotent [12] into its inverse. We demonstrate that  $\mathcal{E}(R \sqcup S) = 0$  for  $R, S \neq \emptyset$  in section 3.1 while the justification for the terminology appears later in section 4.4.3.

We then turn back to the color-dressed string disk amplitude in section 4 where we obtain, following the results of [1], a correspondence between the above permutations and

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<sup>1</sup> After stripping off the leading label 1 and relabeling  $i \rightarrow i - 1$ , see section 3.0.1.

kinematics from the string disk amplitudes. More precisely, in addition to the natural  $C_{1|X,Y,Z} \leftrightarrow \frac{1}{6}\gamma_{1|X,Y,Z}$  the duality from [1] is refined to

$$\begin{aligned} A^{\text{SYM}}(1, 2, \dots, n) &\longleftrightarrow \gamma_{123\dots n}^{(1)}, \\ A^{F^4}(1, 2, \dots, n) &\longleftrightarrow \gamma_{123\dots n}^{(3)}, \end{aligned} \quad (1.3)$$

where  $\gamma^{(1)}$  and  $\gamma^{(3)}$  are orthogonal idempotents of the (inverse) descent algebra constructed from linear combinations of  $\gamma_{1|P_1, \dots, P_k}$  with  $k = 1$  and  $k = 3$  respectively. This interpretation is important because we can borrow a theorem from the work of Garsia and Reutenauer [6] for the right action of  $E_\mu \circ I_p$  with various combinations of partitions  $\mu$  and compositions  $p$ . Here  $E_\mu$  are the building blocks of the Reutenauer orthogonal idempotents and  $I_p$  are the idempotent basis of Solomon's descent algebra, see the review in section 2.

Using a proof from section 3.2 in which we find that the inverse  $\mathcal{I}_p$  of the idempotent basis  $I_p$  satisfies

$$\mathcal{I}_{p_1 p_2 \dots p_k}(P_1, P_2, \dots, P_k) = \mathcal{E}(P_1) \sqcup \mathcal{E}(P_2) \sqcup \dots \sqcup \mathcal{E}(P_k), \quad |P_i| = p_i \quad (1.4)$$

we are able to relate the (inverse of the) idempotent basis  $I_p$  of the theorem from [6] with the BRST-invariant permutations (1.2). This relationship allow us to obtain the Kleiss-Kuijf (KK) [13] and KK-like [14,15] symmetries of the field-theory  $A^{\text{SYM}}$  amplitudes and its  $\alpha'^2 \zeta_2$  correction  $A^{F^4}$  formulated as statements in the descent algebra. As we will see, these symmetries are decomposed according to the number of parts in the composition  $p \models n-1$ , giving rise to the *descent algebra decomposition* of their symmetries.

The decomposition of the symmetries of field-theory  $A^{\text{SYM}}$  and  $\alpha'^2 \zeta_2$  correction  $A^{F^4}$  can be justified by the dualities (1.3), but we also investigate the symmetries of the higher  $\alpha'$  corrections to the disk amplitudes (refined by their MZV content [16,17,18]) from this point of view. Using general arguments from their descent algebra decomposition and the formula (1.2), we find that the symmetries of the different corrections according to their MZV content is closely related to the discussion of the abelian  $Z$ -theory derivation of NLSM amplitudes [19]. In particular, we discover that the leading MZV contribution from abelian  $Z$  integrals governs the non-vanishing of  $A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_2^m \zeta_M}$  for the maximal partition  $k = n-1$ , where  $\zeta_M$  denotes all MZVs that do not contain factors of  $\zeta_2$  in the basis used in [20].

The descent algebra decomposition of the symmetries leads naturally to the counting of their dimensions in terms of Stirling cycle numbers due to the shuffle symmetries of  $\mathcal{E}(P)$  and the formula (1.2) for the BRST invariant permutations. Indeed one gets

$$\begin{aligned}\#\left(A^{\text{string}}(1, 2, \dots, n)\big|_{\text{FT}}\right) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = (n-2)! \\ \#\left(A^{\text{string}}(1, 2, \dots, n)\big|_{\zeta_2 \zeta_M}\right) &= \begin{bmatrix} n-1 \\ 3 \end{bmatrix} \\ \#\left(A^{\text{string}}(1, 2, \dots, n)\big|_{\zeta_2^m \zeta_M}\right) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + \begin{bmatrix} n-1 \\ 3 \end{bmatrix} + \dots + \begin{bmatrix} n-1 \\ 2m+1 \end{bmatrix}, \quad m \geq 2,\end{aligned}\tag{1.5}$$

where the first two lines correspond to the well-known dimensions of the KK [13] and KK-like [14,15] relations, while the third line is new.

Of course, the string theory version [21,22] of the Bern-Carrasco-Johansson (BCJ) tree-level relations [23], reduce all these dimensions to  $(n-3)!$  but they involve Mandelstam invariants  $s_{ij}$  and are (naively) outside the scope of the descent algebra relations. So in this paper we will not be concerned about BCJ identities. For the explicit string-theory basis reduction involving  $\alpha'$  corrections, see [24].

In this paper an important role is played by the so-called BRST invariants  $C_{1|P,Q,R}$  of the pure spinor formalism [25]. They were firstly derived at low multiplicities in [1] and were subsequently studied in different contexts and given general recursive algorithms, see [26,27,28] and references therein.

These studies induce one to suspect that the BRST invariants not only simplify the  $\alpha'^2$  corrections of the string color-dressed amplitude as in the original motivation in [1], but that they might also have a deeper combinatorial significance. For instance, under the cohomology of pure spinor superspace they satisfy various change of basis identities [27] systematized in an intriguing algorithm in the appendix A of [28]. In addition, they can be expanded in terms of SYM tree amplitudes as in the algorithm described in the appendix B of [26], which uses a new Lie bracket in the dual space of Lie polynomials [29]. Their superfield composition in terms of Berends-Giele currents admits a recursive construction [27] whose particulars suggest a combinatorial origin; especially given the relation between Berends-Giele currents and planar binary trees [30]. This paper will add to the growing pile of evidence that the BRST invariants are, in essence, combinatorial objects which found an explicit representation in the computation of string scattering amplitudes. For more on this, see section 4.4.2.

## 2. The Solomon descent algebra

We review the salient features of the Solomon descent algebra [5,6,7,8,9,10,11]. In particular, we discuss different bases and highlight the orthogonal idempotents discovered by Reutenauer, as they will be related to  $\alpha'$  corrections to string amplitudes in later sections.

### 2.0.1. Conventions

Words from the alphabet  $\mathbb{N} = \{1, 2, \dots\}$  will be denoted interchangeably either by capital Latin letters or, especially when viewed as elements of the permutation group, by lower case Greek letters. Words  $P$  of length  $n$  are acted upon by elements  $\sigma$  of the symmetric group  $S_n$  via a right-action multiplication defined by [11]

$$P \circ \sigma = p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(n)}, \quad (2.1)$$

where  $p_i$  denotes the  $i$ th letter of  $P$ . For example  $abcd \circ 3124 = cabd$ . The inverse  $\sigma^{-1}$  of a permutation  $\sigma$  of length  $n$  is such that  $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = 12 \dots n$ . For example,  $(2314)^{-1} = 3124$ . For typographical convenience, we will write a generic explicit permutation  $\sigma$  as  $W_\sigma$ , for instance 4213 becomes  $W_{4213}$ .

### 2.1. Descent classes and the Solomon descent algebra

The *descent set*  $D(\sigma)$  and the *descent number*  $d_\sigma$  of a permutation  $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$  in  $S_n$  are defined by

$$D(\sigma) = \{i \in \{1, 2, \dots, n-1\} \mid \sigma_i > \sigma_{i+1}\}, \quad d_\sigma = \#(D(\sigma)). \quad (2.2)$$

For example, the permutation  $\sigma = 546132$  has descent set  $D(\sigma) = \{1, 3, 5\}$  and descent number  $d_\sigma = 3$ . The collection of permutations with a given descent set  $S$  is called a *descent class*,

$$D_S = \sum_{D(\sigma)=S} \sigma. \quad (2.3)$$

For example, the permutations in  $S_3$  are distributed into four descent classes,

$$D_\emptyset = W_{123}, \quad D_{\{1\}} = W_{213} + W_{312}, \quad D_{\{2\}} = W_{132} + W_{231}, \quad D_{\{1,2\}} = W_{321}. \quad (2.4)$$

In general, the permutations of  $S_n$  decompose into  $2^{n-1}$  distinct descent classes; all the subsets in the powerset of  $\{1, 2, \dots, n-1\}$  since the last  $n$ -th position is never a descent.

Solomon showed the remarkable property that descent classes are closed under the right action (2.1)

$$D_S \circ D_T = \sum_{U \subseteq \{1, 2, \dots, n-1\}} c_{S,T,U} D_U \quad (2.5)$$

where the coefficients  $c_{S,T,U}$  are non-negative integers [5]. The descent classes therefore form a  $2^{n-1}$  dimensional algebra, the so-called *Solomon's descent algebra*  $\mathcal{D}_n$  [5,6,8,9,10,11].

As an example of (2.5), consider the permutations in  $S_4$ . Its 24 elements are organized into 8 descent classes as follows

$$\begin{aligned} D_{\{\emptyset\}} &= W_{1234}, & D_{\{1,2\}} &= W_{3214} + W_{4213} + W_{4312}, \\ D_{\{1\}} &= W_{2134} + W_{3124} + W_{4123}, & D_{\{3\}} &= W_{1243} + W_{1342} + W_{2341}, \\ D_{\{2\}} &= W_{1324} + W_{1423} + W_{2314} + W_{2413} + W_{3412}, & D_{\{2,3\}} &= W_{1432} + W_{2431} + W_{3421}, \\ D_{\{1,3\}} &= W_{2143} + W_{3142} + W_{3241} + W_{4132} + W_{4231}, & D_{\{1,2,3\}} &= W_{4321}. \end{aligned} \quad (2.6)$$

It is straightforward to multiply the permutations among these descent classes using the right-action of the symmetric group (2.1). For example,

$$\begin{aligned} D_{\{1\}} \circ D_{\{2\}} &= W_{1234} + W_{1243} + W_{1324} + W_{1342} + W_{1423} + W_{1432} + W_{2314} \\ &\quad + W_{2341} + W_{2413} + W_{2431} + W_{3214} + W_{3412} + W_{3421} + W_{4213} + W_{4312} \\ &= D_{\{\emptyset\}} + D_{\{1,2\}} + D_{\{2\}} + D_{\{2,3\}} + D_{\{3\}}, \end{aligned} \quad (2.7)$$

where the last line follows from the remarkable property (2.5) which ensures that the permutations in (2.7) are themselves a sum of descent classes.

## 2.2. Bases of the descent algebra

Apart from the descent classes  $D_S$  indexed by descent sets  $S$ , there are other convenient bases of the descent algebra [6].

### 2.2.1. Composition basis $B_p$

The composition  $p$  of  $n$ , denoted  $p \models n$ , is a  $k$ -tuple of positive integers with sum  $n$ ,

$$p = (p_1, p_2, \dots, p_k), \quad p_1 + p_2 + \dots + p_k = n. \quad (2.8)$$

There is a bijection between compositions  $p \models n$  and subsets  $S$  of  $\{1, 2, \dots, n-1\}$

$$p = (p_1, p_2, \dots, p_k) \mapsto \{p_1, p_1 + p_2, \dots, p_1 + p_2 + \dots + p_{k-1}\} := S(p), \quad (2.9)$$

$$S = \{i_1, i_2, \dots, i_k\} \mapsto (i_1, i_2 - i_1, \dots, i_k - i_{k-1}, n - i_k) := C_n(S). \quad (2.10)$$

Thus the total number of compositions of  $n$  is  $2^{n-1}$ , the cardinality of the powerset of  $\{1, 2, \dots, n-1\}$ . Note that the map  $C_n(S) = p$  depends on the order  $n$  of the permutation group  $S_n$  for  $C_3(\{1, 2\}) = (1, 1, 1)$  but  $C_4(\{1, 2\}) = (1, 1, 2)$ . In particular,  $C_n(\emptyset) = (n)$ .

The basis  $B_p$  is indexed by compositions  $p$  rather than subsets and is defined by [6]

$$B_p = D_{\subseteq S(p)}, \quad (2.11)$$

with  $S(p)$  given by (2.9). For example, the  $D_S$  basis elements (2.6) become

$$\begin{aligned} B_{1111} &= D_{\emptyset} + D_{\{1\}} + D_{\{2\}} + D_{\{3\}} + D_{\{1,2\}} & B_{13} &= D_{\emptyset} + D_{\{1\}}, \\ &+ D_{\{1,3\}} + D_{\{2,3\}} + D_{\{1,2,3\}}, & B_{22} &= D_{\emptyset} + D_{\{2\}}, \\ B_{112} &= D_{\emptyset} + D_{\{1\}} + D_{\{2\}} + D_{\{1,2\}}, & B_{31} &= D_{\emptyset} + D_{\{3\}}, \\ B_{121} &= D_{\emptyset} + D_{\{1\}} + D_{\{3\}} + D_{\{1,3\}}, & B_4 &= D_{\emptyset}. \\ B_{211} &= D_{\emptyset} + D_{\{2\}} + D_{\{3\}} + D_{\{2,3\}}, \end{aligned} \quad (2.12)$$

The inverse of (2.11) is given by Lemma 8.18 in [11]

$$D_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} D_{\subseteq T}. \quad (2.13)$$

For example (in  $S_4$ ),  $D_{\{1,2\}} = B_{112} - B_{13} - B_{22} + B_4$ ,  $D_{\{1\}} = B_{13} - B_4$ ,  $D_{\{2\}} = B_{22} - B_4$ , and  $D_{\emptyset} = B_4$ , from which we verify that  $D_{\emptyset} + D_{\{1\}} + D_{\{2\}} + D_{\{1,2\}} = B_{112}$ .

The permutations within a basis element  $B_p$  can be found via [31,6]

$$B_{p_1 p_2 \dots p_k} = \theta(X_1 \sqcup X_2 \sqcup \dots \sqcup X_k), \quad 12 \dots n = X_1 \dots X_k, \quad |X_i| = p_i. \quad (2.14)$$

where the inverse map  $\theta$  is given by

$$\theta(\sigma) \mapsto \sigma^{-1}. \quad (2.15)$$

For example, if  $p = (1, 1, 2)$  then  $X_1 = 1$ ,  $X_2 = 2$  and  $X_3 = 34$  and we get

$$\begin{aligned} B_{112} &= \theta(1 \sqcup 2 \sqcup 34) = W_{1234} + W_{1324} + W_{1423} + W_{2134} + W_{2314} + W_{2413} \\ &+ W_{3124} + W_{3214} + W_{3412} + W_{4123} + W_{4213} + W_{4312}. \end{aligned} \quad (2.16)$$

### 2.2.2. Multiplication table for $B_p \circ B_q$

There is a closed formula for the multiplication of  $B_p \circ B_q$  [9,6,32]. Let  $M$  be a matrix with non-negative integer entries  $m_{ij}$  whose *row sum*  $r(M)$  and *column sum*  $c(M)$  are vectors defined by

$$r(M)_i := \sum_j m_{ij}, \quad c(M)_j := \sum_i m_{ij}. \quad (2.17)$$

Then

$$B_p \circ B_q = \sum_{\substack{c(M)=p \\ r(M)=q}} B_{\text{co}(M)} \quad (2.18)$$

where  $\text{co}(M)$  denotes the composition obtained by reading the matrix  $M$  row by row from top to bottom while excluding the zero entries  $m_{ij} = 0$ . This product is associative and  $B_n$  is a multiplicative identity for compositions of  $n$  [32].

For example, let us recover the result (2.7) for  $D_{\{1\}} \circ D_{\{2\}}$  using the above multiplication table (2.18) in  $S_4$ . Given that  $D_{\{1\}} = B_{13} - B_4$  and  $D_{\{2\}} = B_{22} - B_4$ , the only non-trivial product we need is  $B_{13} \circ B_{22}$  since  $B_4$  is the identity for compositions of  $n = 4$ . The set of integer matrices  $M$  with  $c(M) = (1, 3)$  and  $r(M) = (2, 2)$  is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}. \quad (2.19)$$

Thus  $B_{13} \circ B_{22} = B_{112} + B_{211}$  and  $D_{\{1\}} \circ D_{\{2\}} = (B_{13} - B_4) \circ (B_{22} - B_4)$  implies

$$D_{\{1\}} \circ D_{\{2\}} = B_{112} + B_{211} - B_{13} - B_{22} + B_4 = D_\emptyset + D_{\{1,2\}} + D_{\{2\}} + D_{\{2,3\}} + D_{\{3\}} \quad (2.20)$$

where we used the conversions (2.12).

### 2.2.3. The Eulerian idempotent

The Eulerian (or Solomon) idempotent is defined by [12,7,31,33] (see also [34])

$$E_n = \sum_{\sigma \in S_n} \kappa_\sigma \sigma, \quad \kappa_\sigma = \frac{(-1)^{d_\sigma}}{|\sigma| \binom{|\sigma|-1}{d_\sigma}} \quad (2.21)$$

where  $d_\sigma$  denotes the descent number (2.2) of the permutation  $\sigma$ . For example,

$$E_2 = \frac{1}{2}(W_{12} - W_{21}), \quad E_3 = \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321}. \quad (2.22)$$

Apart from being an idempotent satisfying  $E_n \circ E_n = E_n$ , the definition (2.21) is also a Lie polynomial [7]. Therefore its coefficients  $\kappa_\sigma$  must satisfy the shuffle symmetry [35]

$$\kappa_{R \sqcup S} = 0, \quad R, S \neq \emptyset. \quad (2.23)$$

As usual, the definition (2.21) in terms of the fixed alphabet  $\mathbb{N}$  in  $S_n$  can be turned into a function of an arbitrary word  $P$  by the right action (2.1) of the symmetric group [11,36],

$$E(P) = E^P := P \circ E_n, \quad |P| = n. \quad (2.24)$$

For example,  $E(i, j, k) = ijk \circ E_3 = \frac{1}{3}W_{ijk} - \frac{1}{6}W_{ikj} - \frac{1}{6}W_{jik} - \frac{1}{6}W_{jki} - \frac{1}{6}W_{kij} + \frac{1}{3}W_{kji}$ .

#### 2.2.4. The idempotent basis $I_p$

The idempotent basis  $I_p$  of the descent algebra  $\mathcal{D}_n$  satisfying  $I_p \circ I_p = I_p$  was introduced in [6] and it is indexed by the compositions of  $n$

$$I_{p_1 p_2 \dots p_k}(P) = \sum_{\substack{X_1, \dots, X_k \\ |X_i| = p_i}} \langle P, X_1 \sqcup X_2 \sqcup \dots \sqcup X_k \rangle E^{X_1} E^{X_2} \dots E^{X_k}, \quad (2.25)$$

where the sum is constrained by the length of  $X_i$  being equal to the corresponding  $p_i$  in the composition  $p$  and  $E^{X_i}$  denote the Eulerian idempotent function (2.24). For example, with canonical  $P = 12 \dots n$  we have

$$\begin{aligned} I_{11} &= W_{12} + W_{21}, \quad I_2 = \frac{1}{2}(W_{12} - W_{21}), \\ I_{111} &= W_{123} + W_{132} + W_{213} + W_{231} + W_{312} + W_{321}, \\ I_{21} &= \frac{1}{2}W_{123} + \frac{1}{2}W_{132} - \frac{1}{2}W_{213} + \frac{1}{2}W_{231} - \frac{1}{2}W_{312} - \frac{1}{2}W_{321}, \\ I_{12} &= \frac{1}{2}W_{123} - \frac{1}{2}W_{132} + \frac{1}{2}W_{213} - \frac{1}{2}W_{231} + \frac{1}{2}W_{312} - \frac{1}{2}W_{321}, \\ I_3 &= \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321}. \end{aligned} \quad (2.26)$$

#### 2.2.5. $I_p$ to $B_p$

The idempotent basis elements  $I_p$  for  $p = p_1 p_2 \dots p_k$  can be expanded in terms of compositions  $B_q$  using an algorithm discussed in [6]. First one defines *moments*  $e_m$  as a polynomial in non-commuting variables  $t_i$  for  $i = 1, 2, \dots$  from the generating series

$$\sum x^n e_n = \log(1 + \sum t_i x^i) \quad (2.27)$$

where  $x$  is a commuting parameter. For example, from (2.27) it follows that

$$\begin{aligned} e_1 &= t_1, \quad e_2 = t_2 - \frac{1}{2}t_1^2, \quad e_3 = t_3 - \frac{1}{2}(t_1 t_2 + t_2 t_1) + \frac{1}{3}t_1^3 \\ e_4 &= -\frac{1}{4}t_1^4 + \frac{1}{3}t_1^2 t_2 + \frac{1}{3}t_1 t_2 t_1 - \frac{1}{2}t_1 t_3 + \frac{1}{3}t_2 t_1^2 - \frac{1}{2}t_2^2 - \frac{1}{2}t_3 t_1 + t_4 \end{aligned} \quad (2.28)$$

Then to convert the  $I_p$  basis elements to the composition basis  $B_q$  one uses [6]

$$I_p = \delta(e_{p_1} e_{p_2} \dots e_{p_k}), \quad \text{with } \delta(t_{i_1} t_{i_2} \dots t_{i_k}) := B_{i_1 i_2 \dots i_k}. \quad (2.29)$$

For example,

$$I_4 = -\frac{1}{4}B_{1111} + \frac{1}{3}B_{112} + \frac{1}{3}B_{121} - \frac{1}{2}B_{13} + \frac{1}{3}B_{211} - \frac{1}{2}B_{22} - \frac{1}{2}B_{31} + B_4. \quad (2.30)$$

### 2.3. Reutenauer orthogonal idempotents

A partition  $\lambda$  of  $n$ , denoted  $\lambda \vdash n$ , is a  $k$ -tuple of positive integers with sum  $n$  satisfying  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ . If  $p \models n$  is a composition of  $n$ , the *shape*  $\lambda(p)$  of  $p$  is the partition of  $n$  obtained by rearranging the parts of  $p$  in decreasing order. Also,  $k(p)$  is the *number of parts* of the composition  $p$ . For example,  $p = (2, 3, 1, 2)$  implies  $\lambda(p) = 3221$  and  $k(p) = 4$ .

Given a partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  into  $k$  parts, theorem 3.1 of [6] shows that

$$E_\lambda := \frac{1}{k!} \sum_{\lambda(p)=\lambda} I_p, \quad \sum_{\lambda \vdash n} E_\lambda = W_{12\dots n}. \quad (2.31)$$

Note that when the partition  $\lambda$  of  $n$  has only one part,  $E_\lambda = I_n$  coincides with the Eulerian idempotent  $E_n$  (2.21), so this notation is not ambiguous. For example,  $E_1 = I_1$  and

$$\begin{aligned} E_2 &= I_2, & E_3 &= I_3, & E_{111} &= \frac{1}{3!} I_{111}, \\ E_{11} &= \frac{1}{2} I_{11}, & E_{21} &= \frac{1}{2} (I_{12} + I_{21}), & E_{211} &= \frac{1}{3!} (I_{112} + I_{121} + I_{211}). \end{aligned} \quad (2.32)$$

one can readily verify  $E_3 + E_{21} + E_{111} = W_{123}$  using the expansions listed in the appendix A.

The Reutenauer idempotents  $E^{(m)}$  are defined in the alphabet  $\{1, 2, \dots\}$  as the sum over all permutations of  $E_\lambda$  from (2.31) such that  $\lambda$  is a partition of  $n$  with  $m$  parts, i.e.,

$$E^{(m)} = \sum_{\substack{\lambda \vdash n \\ k(\lambda)=m}} E_\lambda \quad (2.33)$$

For example,

$$\begin{aligned} n=2 \quad E^{(1)} &= E_2, & E^{(2)} &= E_{11} \\ n=3 \quad E^{(1)} &= E_3, & E^{(2)} &= E_{21}, & E^{(3)} &= E_{111} \\ n=4 \quad E^{(1)} &= E_4, & E^{(2)} &= E_{31} + E_{22}, & E^{(3)} &= E_{211}, & E^{(4)} &= E_{1111} \end{aligned} \quad (2.34)$$

It was shown in [6,7] that (2.33) are orthogonal idempotents which sum to the identity permutation

$$\sum_{i=1}^n E^{(i)} = W_{123\dots n}, \quad E^{(i)} E^{(j)} = \begin{cases} E^{(i)} & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases} \quad (2.35)$$

An alternative definition of the Reutenauer idempotents in terms of a generating function can be found in [11].

### 3. The combinatorics of color-dressed permutations

In this section we will investigate the combinatorics of the *color-dressed permutations*  $P_n$

$$P_n = \sum_{\sigma \in S_n, \sigma(1)=1} \text{Tr}(T^\sigma) W_\sigma, \quad T^\sigma := T^{\sigma(1)} T^{\sigma(2)} \cdots T^{\sigma(n)}, \quad (3.1)$$

which, as we will see, represents the replacement  $A^{\text{string}}(\sigma) \rightarrow \sigma$  in the color-dressed amplitude (4.1). This investigation will be done by decomposing [2] the traces of color factors into symmetrized traces and structure constants  $f^{abc}$  of the gauge group, where

$$d^{12\dots k} := \frac{1}{k!} \sum_{\sigma \in S_k} \text{Tr}(T^\sigma), \quad d^{12} := \frac{1}{2} \delta^{12}, \quad [T^a, T^b] = i f^{abc} T^c. \quad (3.2)$$

More explicitly, we use the basis of color factors from the general trace decomposition found in [3] (for technical reasons the alphabet  $\{0, 1, 2, \dots\}$  is used momentarily)

$$\text{Tr}(T^0 T^1 \cdots T^{n-1}) = \sum_{S_{n-1} \ni \sigma = \sigma_1 \cdots \sigma_k} i^{n-1-k} \kappa_{\sigma_1} \cdots \kappa_{\sigma_k} d^{0a_1 \cdots a_k} F_{a_1}^{\sigma_1} \cdots F_{a_k}^{\sigma_k}. \quad (3.3)$$

where  $\sigma = \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_k$  denotes the decreasing Lyndon factorization of the word  $\sigma$  to be defined below and the coefficients  $\kappa_\sigma$  were defined in (2.21). The basis of color factors from (3.3) is given by  $i^{n-k} d^{0a_1 \cdots a_k} F_{a_1}^{\sigma_1} \cdots F_{a_k}^{\sigma_k}$  where the factors  $F_a^\sigma$  for a word  $\sigma$  and a letter  $a$  are defined recursively by

$$F_a^{Pj} = F_b^P f^{bj a}, \quad F_a^i = \delta_a^i. \quad (3.4)$$

The *decreasing Lyndon factorization* (dLf) of a word  $\sigma$  is defined as [37,6]

$$\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_k \quad (3.5)$$

representing the unique deconcatenation of  $\sigma$  into subwords  $\sigma_1, \dots, \sigma_k$  such that  $\sigma_1 > \dots > \sigma_k$  in the lexicographical order of the alphabet  $\mathbb{N} = \{1, 2, \dots\}$ . In addition, each  $\sigma_j$  for  $1 \leq j \leq k$  is a Lyndon word, which for a permutation with no repeated letters means that first letter in  $\sigma_j$  is the minimum among its letters. Representing the concatenation by a dot to distinguish the subwords  $\sigma_i$  in the dLF factorization of  $\sigma$ , we have

$$1432 = 1432, \quad 2134 = 2.134, \quad 54132 = 5.4.132, \quad 42671835 = 4.267.1835. \quad (3.6)$$

**Definition (BRST-invariant permutation).** Written in the basis of color factors  $i^{n-k} d^{1a_1 \dots a_k} F_{a_1}^{\sigma_1} \dots F_{a_k}^{\sigma_k}$ , the color-dressed permutation (3.1) is given by

$$P_n = \sum_{\sigma \in S_{n-1}} i^{n-1-k} d^{1a_1 \dots a_k} F_{a_1}^{\sigma_1} \dots F_{a_k}^{\sigma_k} \gamma_{1|\sigma_1, \dots, \sigma_k}, \quad (3.7)$$

where the coefficients  $\gamma_{1|\sigma_1, \dots, \sigma_k}$  are denoted BRST-invariant permutations.

The reason for this terminology will become clear in section 4.1 when  $\gamma_{1|\sigma_1, \sigma_2, \sigma_3}$  will be related to the BRST-invariant superfields  $C_{1|\sigma_1, \sigma_2, \sigma_3}$  of the pure spinor formalism. We will see later in (3.30) that  $\gamma_{1|\sigma_1, \dots, \sigma_k}$  is totally symmetric under exchanges of  $\sigma_i \leftrightarrow \sigma_j$  and that it satisfies shuffle symmetries in each  $\sigma_i$ .

For example, plugging  $\text{Tr}(T^1 T^2 T^3) = d^{1a_1 a_2} F_{a_1}^2 F_{a_2}^3 + \frac{i}{2} d^{1a} F_a^{23}$  into (3.1) yields

$$\begin{aligned} P_3 &= \text{Tr}(T^1 T^2 T^3) W_{123} + \text{Tr}(T^1 T^3 T^2) W_{132} \\ &= d^{1ab} F_a^2 F_b^3 \gamma_{1|2,3} + i d^{1a} F_a^{23} \gamma_{1|23}, \end{aligned} \quad (3.8)$$

with

$$\gamma_{1|2,3} = W_{123} + W_{132}, \quad \gamma_{1|23} = \frac{1}{2} W_{123} - \frac{1}{2} W_{132}. \quad (3.9)$$

Repeating the same exercise for  $n = 4$  using (3.3)

$$\begin{aligned} \text{Tr}(T^1 T^2 T^3 T^4) &= d^{1abc} F_a^2 F_b^3 F_c^4 \\ &+ \frac{i}{2} d^{1ab} F_a^2 F_b^{34} + \frac{i}{2} d^{1ab} F_a^{23} F_b^4 + \frac{i}{2} d^{1ab} F_a^{24} F_b^3 \\ &- \frac{1}{3} d^{1a} F_a^{234} + \frac{1}{6} d^{1a} F_a^{243} \end{aligned} \quad (3.10)$$

we obtain

$$\begin{aligned} P_4 &= d^{1abc} F_a^2 F_b^3 F_c^4 \gamma_{1|2,3,4} \\ &+ i d^{1ab} F_a^{23} F_b^4 \gamma_{1|23,4} + i d^{1ab} F_a^{24} F_b^3 \gamma_{1|24,3} + i d^{1ab} F_a^2 F_b^{34} \gamma_{1|2,34} \\ &+ i^2 d^{1a} F_a^{234} \gamma_{1|234} + i^2 d^{1a} F_a^{243} \gamma_{1|243} \end{aligned} \quad (3.11)$$

where the BRST-invariant permutations are given by

$$\begin{aligned} \gamma_{1|2,3,4} &= W_{1234} + W_{1243} + W_{1324} + W_{1342} + W_{1423} + W_{1432} \\ \gamma_{1|23,4} &= \frac{1}{2} W_{1234} + \frac{1}{2} W_{1243} - \frac{1}{2} W_{1324} - \frac{1}{2} W_{1342} + \frac{1}{2} W_{1423} - \frac{1}{2} W_{1432} \\ \gamma_{1|2,34} &= \frac{1}{2} W_{1234} - \frac{1}{2} W_{1243} + \frac{1}{2} W_{1324} + \frac{1}{2} W_{1342} - \frac{1}{2} W_{1423} - \frac{1}{2} W_{1432} \\ \gamma_{1|24,3} &= \frac{1}{2} W_{1234} + \frac{1}{2} W_{1243} + \frac{1}{2} W_{1324} - \frac{1}{2} W_{1342} - \frac{1}{2} W_{1423} - \frac{1}{2} W_{1432} \\ \gamma_{1|234} &= \frac{1}{3} W_{1234} - \frac{1}{6} W_{1243} - \frac{1}{6} W_{1324} - \frac{1}{6} W_{1342} - \frac{1}{6} W_{1423} + \frac{1}{3} W_{1432} \\ \gamma_{1|243} &= -\frac{1}{6} W_{1234} + \frac{1}{3} W_{1243} - \frac{1}{6} W_{1324} + \frac{1}{3} W_{1342} - \frac{1}{6} W_{1423} - \frac{1}{6} W_{1432} \end{aligned} \quad (3.12)$$

For  $n = 5$  we obtain

$$\begin{aligned}
P_5 = & d^{1abcd} F_a^2 F_b^3 F_c^4 F_d^5 \gamma_{1|2,3,4,5} \\
& + id^{1abc} F_a^{23} F_b^4 F_c^5 \gamma_{1|23,4,5} + id^{1abc} F_a^{24} F_b^3 F_c^5 \gamma_{1|24,3,5} + id^{1abc} F_a^{25} F_b^3 F_c^4 \gamma_{1|25,3,4} \\
& + id^{1abc} F_a^2 F_b^{34} F_c^5 \gamma_{1|2,34,5} + id^{1abc} F_a^2 F_b^{35} F_c^4 \gamma_{1|2,35,4} + id^{1abc} F_a^2 F_b^3 F_c^{45} \gamma_{1|2,3,45} \\
& + i^2 d^{1ab} F_a^{234} F_b^5 \gamma_{1|234,5} + i^2 d^{1ab} F_a^{243} F_b^5 \gamma_{1|243,5} + i^2 d^{1ab} F_a^{235} F_b^4 \gamma_{1|235,4} \\
& + i^2 d^{1ab} F_a^{245} F_b^3 \gamma_{1|245,3} + i^2 d^{1ab} F_a^{253} F_b^4 \gamma_{1|253,4} + i^2 d^{1ab} F_a^{254} F_b^3 \gamma_{1|254,3} \\
& + i^2 d^{1ab} F_a^{345} F_b^2 \gamma_{1|345,2} + i^2 d^{1ab} F_a^{354} F_b^2 \gamma_{1|354,2} \\
& + i^2 d^{1ab} F_a^{23} F_b^{45} \gamma_{1|23,45} + i^2 d^{1ab} F_a^{24} F_b^{35} \gamma_{1|24,35} + i^2 d^{1ab} F_a^{25} F_b^{34} \gamma_{1|25,34} \\
& + i^3 d^{1a} F_a^{2345} \gamma_{1|2345} + i^3 d^{1a} F_a^{2354} \gamma_{1|2354} + i^3 d^{1a} F_a^{2435} \gamma_{1|2435} \\
& + i^3 d^{1a} F_a^{2453} \gamma_{1|2453} + i^3 d^{1a} F_a^{2534} \gamma_{1|2534} + i^3 d^{1a} F_a^{2543} \gamma_{1|2543}
\end{aligned} \tag{3.13}$$

where the various  $\gamma_{1|A_1, A_2, \dots, A_k}$  are listed in the appendix A.

Note that the label 1 plays a special role due to the choice of the color basis where it always appear inside the symmetrized trace;  $d^{1\dots}$  [1]. The total number of  $\gamma_{1|A_1, A_2, \dots, A_k}$  with  $k$  parts are given by the Stirling cycle number  $\begin{bmatrix} n-1 \\ k \end{bmatrix}$  while the total number of terms in the expansion of  $P_n$  is given by  $\sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} = (n-1)!$ .

### 3.0.1. Relating the BRST-invariant permutations with the descent algebra

We wish to understand the systematics of the permutations in each  $\gamma_{1|A_1, A_2, \dots, A_k}$  and find an algorithm to generate them directly. In the next sections we will see that these permutations are related to the descent algebra reviewed in section 2. To see the relation consider  $\gamma_{1|23,4}$  from (3.12), relabel  $i \rightarrow i-1$  and strip off the leading “0” to obtain

$$\gamma_{\times|12,3} = \frac{1}{2} \underbrace{W_{123}}_{\in D_\emptyset} + \frac{1}{2} \underbrace{W_{132}}_{\in D_{\{2\}}} - \frac{1}{2} \underbrace{W_{213}}_{\in D_{\{1\}}} - \frac{1}{2} \underbrace{W_{231}}_{\in D_{\{2\}}} + \frac{1}{2} \underbrace{W_{312}}_{\in D_{\{1\}}} - \frac{1}{2} \underbrace{W_{321}}_{\in D_{\{1,2\}}}, \tag{3.14}$$

where  $\times$  indicates the entry “0”. Note that the resulting permutations are *not* in the descent algebra  $\mathcal{D}_3$  since permutations in the same descent class (as indicated below each permutation) have different coefficients – alternatively one can use proposition 2.1 from [38]. However, the inverse permutations in  $\theta(\gamma_{\times|12,3})$  do belong to the same descent classes:

$$\begin{aligned}
\theta(\gamma_{\times|12,3}) &= \frac{1}{2} \underbrace{W_{123}}_{\in D_\emptyset} + \frac{1}{2} \underbrace{W_{132}}_{\in D_{\{2\}}} - \frac{1}{2} \underbrace{W_{213}}_{\in D_{\{1\}}} - \frac{1}{2} \underbrace{W_{312}}_{\in D_{\{1\}}} + \frac{1}{2} \underbrace{W_{231}}_{\in D_{\{2\}}} - \frac{1}{2} \underbrace{W_{321}}_{\in D_{\{1,2\}}} \\
&= \frac{1}{2} D_\emptyset - \frac{1}{2} D_{\{1\}} + \frac{1}{2} D_{\{2\}} - \frac{1}{2} D_{\{1,2\}}.
\end{aligned} \tag{3.15}$$

as can be verified using the explicit permutations in  $D_S$  from (2.4).

So we see that after relabeling  $i \rightarrow i - 1$ , stripping off the leading letter from the permutation words and considering the inverse permutations, the result can be described in terms of the Solomon descent algebra. This means that the BRST-invariant permutations belong to the *inverse* descent algebra  $\mathcal{D}'_n := \theta(\mathcal{D}_n)$ . This motivates us to consider the inverse permutations of the Eulerian idempotent (2.21).

### 3.1. The Berends-Giele idempotent

Let us consider the inverse  $\theta(E_n)$  of the Eulerian idempotent (2.21) and, because the inverse of an idempotent is also idempotent, call it the *Berends-Giele idempotent*:

$$\mathcal{E}_n = \sum_{\sigma \in S_n} \kappa_{\sigma^{-1}} \sigma, \quad \mathcal{E}(P) = \mathcal{E}_P := P \circ \mathcal{E}_n, \quad |P| = n. \quad (3.16)$$

The reason for the *Berends-Giele* terminology is the correspondence in (4.36) with the standard Berends-Giele current of Yang-Mills theory [39]. The first few examples of (3.16) are

$$\begin{aligned} \mathcal{E}(1) &= W_1, & \mathcal{E}(12) &= \frac{1}{2}(W_{12} - W_{21}), \\ \mathcal{E}(123) &= \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321}, \end{aligned} \quad (3.17)$$

while the expansion of  $\mathcal{E}_{1234}$  can be found in the appendix A.0.1.

**Proposition (Shuffle Symmetry).** *The Berends-Giele idempotent (3.16) satisfies*

$$\mathcal{E}(R \sqcup S) = 0. \quad (3.18)$$

*Proof.* Since the sum in (3.16) is over all permutations we rename  $P \circ \sigma = \tau$  and sum over  $\tau$ . Notice that  $\sigma^{-1} = \tau^{-1} \circ P$ , so  $\mathcal{E}(P) = \sum_{\sigma} \kappa_{\sigma^{-1}} P \circ \sigma = \sum_{\tau} \kappa_{(\tau^{-1} \circ P)} \tau$  and therefore

$$\mathcal{E}(R \sqcup S) = \sum_{\tau} \kappa_{(\tau^{-1} \circ (R \sqcup S))} \tau = \sum_{\tau} \kappa_{(\tau^{-1}(R) \sqcup \tau^{-1}(S))} \tau = 0 \quad (3.19)$$

where the last equality follows from (2.23) and the crucial observation in (1.5) of [6] that<sup>2</sup>  $\sigma^{-1} \circ (R \sqcup S) = \sigma^{-1}(R) \sqcup \sigma^{-1}(S)$ , where  $\sigma^{-1}(R)$  denotes the word obtained by replacing each letter in  $R$  by its image under  $\sigma^{-1}$ .  $\square$

In a tangencial point for this paper, we note that since the proof above only depends on the shuffle symmetries of the coefficients  $\kappa_P$  and we know from [35] that any Lie polynomial can be expanded as  $\sum_{\sigma} M_{\sigma} \sigma$  with  $M_{R \sqcup S} = 0$  for nonempty  $R, S$ , we conclude:

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<sup>2</sup> The order of multiplications is crucial since  $(R \sqcup S) \circ \sigma$  is not itself a proper shuffle.

**Corollary.** *If  $\Gamma$  is a Lie polynomial then the word function  $\mathcal{F}(P) := P \circ \theta(\Gamma)$  satisfies the shuffle symmetry  $\mathcal{F}(R \sqcup S) = 0$  for  $R, S \neq \emptyset$ .*

For example consider the Lie polynomial  $\Gamma = [[1, 2], 3] = 123 - 213 - 312 + 321$ . We get  $\theta(\Gamma) = 123 - 213 - 231 + 321$  which means  $F(a, b, c) = abc \circ \theta(\Gamma) = abc - bac - bca + cba = \rho^\ell(a, b, c)$ , and it is well known that  $\rho^\ell(R \sqcup S) = 0$  [35,40].

### 3.2. Inverse idempotent basis and shuffle symmetries

Following the motivation in section 3.0.1 in which we learned that the BRST-invariant permutations are related to the inverse of the descent algebra, it will be convenient to define the inverse of the idempotent basis  $I_p$  as

$$\mathcal{I}_{p_1 p_2 \dots p_k}(P_1, P_2, \dots, P_k) := \theta(I_{p_1 p_2 \dots p_k})(P), \quad |P_i| = p_i, \quad P_1 P_2 \dots P_k = P \quad (3.20)$$

where the map  $\theta$  is defined in (2.15). For example

$$\mathcal{I}_{21}(12, 3) = \frac{1}{2} (W_{123} + W_{132} - W_{213} - W_{231} + W_{312} - W_{321}). \quad (3.21)$$

See (A.5) for the explicit permutations in  $\mathcal{I}_{22}(12, 34)$ .

The reason for separating the arguments into words of length  $|P_i| = p_i$  corresponding to the parts  $p_i$  of the composition will become clear after we prove the following:

**Proposition.** *The inverse of the idempotent basis (3.20) satisfies*

$$\mathcal{I}_{p_1 p_2 \dots p_k}(P_1, P_2, \dots, P_k) = \mathcal{E}(P_1) \sqcup \mathcal{E}(P_2) \sqcup \dots \sqcup \mathcal{E}(P_k), \quad |P_i| = p_i \quad (3.22)$$

where  $P = P_1 \dots P_k$  is the factorization of  $P$  with  $P_i$  of length  $p_i$ .

*Proof.* The proof will be based on the following observations collected from [11], which should be consulted for more details as the equation numbers below refer to it. First, the adjoint of an arbitrary function  $F(P) = P \circ F$  of a word  $P$  is given by  $\theta(F)(P) = P \circ \theta(F)$ , see (3.3.5). Second, the adjoint of  $F_{p_1} \star F_{p_2} \dots \star F_{p_k}$  is  $\theta(F_{p_1}) \star' \theta(F_{p_2}) \dots \star' \theta(F_{p_k})$  where  $\star$  and  $\star'$  are the convolution operators defined in (1.5.7) and (1.5.8) and  $\theta(F_j)$  is the adjoint of  $F_j$  when viewed as a function by the right-action (2.1), see proof of Lemma 3.13. Third, for permutations  $F_{p_i}$  of length  $p_i$  one can show (by adapting the proof of Lemma 3.13)

$$(F_{p_1} \star' \dots \star' F_{p_k})(P) = F_{p_1}(P_1) \sqcup \dots \sqcup F_{p_k}(P_k) \quad (3.23)$$

where the functions are defined via a right action as  $F_{p_i}(P_i) := P_i \circ F_{p_i}$ . The proof of (3.22) then follows from the observation by (1.5.4) and (1.5.7) that the idempotent basis  $I_p$  (2.25) can be rewritten as a convolution  $I_{p_1 \dots p_k}(P) = (E_{p_1} \star \dots \star E_{p_k})(P)$  where  $E_p$  is the Eulerian idempotent (2.21). Therefore its adjoint  $\theta(I_{p_1 \dots p_k})(P)$  is given by

$$\begin{aligned} \theta(I_{p_1 \dots p_k})(P) &= \theta(E_{p_1})(P_1) \star' \dots \star' \theta(E_{p_k})(P_k), \quad P = P_1 \dots P_k, \quad |P_i| = p_i \\ &= \mathcal{E}(P_1) \sqcup \dots \sqcup \mathcal{E}(P_k) \end{aligned} \quad (3.24)$$

where we used (3.23) and  $\mathcal{E}(P_i) = \theta(E_{p_i})(P_i)$ .  $\square$

If  $Q = q_1 \dots q_n$  then its reversal is the word  $\tilde{Q} := q_n \dots q_1$ . If a function satisfies the shuffle symmetry or, in other words, belongs to the dual space of Lie polynomials [29], then  $\tilde{F}(P) = (-1)^{|P|-1} F(P)$  and we conclude from (3.22):

**Corollary.** *The reversal of words in  $\mathcal{I}_{p_1 \dots p_k}(P_1, \dots, P_k)$  is given by*

$$\tilde{\mathcal{I}}_{p_1 \dots p_k}(P_1, \dots, P_k) = (-1)^{\#\text{even}(p)} \mathcal{I}_{p_1 \dots p_k}(P_1, \dots, P_k) \quad (3.25)$$

where  $\#\text{even}(p)$  denotes the number of even parts in the composition  $p$ .

We now see the reason for splitting the word  $P$  into  $k$  slots in the function  $\mathcal{I}_{p_1 \dots p_k}(P_1, \dots, P_k)$  as it satisfies the *shuffle symmetry* and it is symmetric under any  $i \leftrightarrow j$ :

$$\begin{aligned} \mathcal{I}_{\dots p_i \dots}(\dots, R \sqcup S, \dots) &= 0, \quad R, S \neq \emptyset, \quad |R| + |S| = p_i, \\ \mathcal{I}_{\dots p_i \dots p_j \dots}(\dots, P_i, \dots, P_j, \dots) &= \mathcal{I}_{\dots p_j \dots p_i \dots}(\dots, P_j, \dots, P_i, \dots). \end{aligned} \quad (3.26)$$

These symmetries will provide a major consistency check when we propose a duality between the descent algebra and kinematics of the string scattering amplitudes, see (4.7).

### 3.3. A closed formula for the BRST-invariant permutations

A closed formula for the BRST-invariant permutations is obtained by using a modification of the formula (3.3) for general permutations as [41]

$$\text{Tr}(T^0 T^\sigma) = \sum_{\tau=\tau_1 \dots \tau_k} i^{n-k} \kappa_{\rho_1} \dots \kappa_{\rho_k} d^{0a_1 \dots a_k} F_{a_1}^{\tau_1} \dots F_{a_k}^{\tau_k}, \quad |\sigma| = n. \quad (3.27)$$

Here  $\rho_i$  for  $i = 1, \dots, k$  is defined by  $\sigma^{-1} \circ \tau := \rho_1 \dots \rho_k$  with the constraint  $|\rho_i| = |\tau_i|$  given by the decreasing Lyndon factorization (3.5) of  $\tau = \tau_1 \dots \tau_k$ . For example, to find the coefficient of the term  $d^{0a_1 a_2 a_3} F_{a_1}^4 F_{a_2}^2 F_{a_3}^{13}$  in the expansion of  $\text{Tr}(T^0 T^{3214})$  corresponding

to  $\sigma = 3214$  in (3.27) we first determine the dLf of  $\tau = 4213$  as 4.2.13 to obtain  $|\tau_1| = 1$ ,  $|\tau_2| = 1$  and  $|\tau_3| = 2$ . From  $\sigma^{-1} \circ \tau = 4231 := \rho_1 \rho_2 \rho_3$  we get  $\rho_1 = 4$ ,  $\rho_2 = 2$  and  $\rho_3 = 31$ . Therefore the formula (3.27) gives  $-\frac{i}{2} d^{0a_1 a_2 a_3} F_{a_1}^4 F_{a_2}^2 F_{a_3}^{13}$  since  $\kappa_4 = \kappa_2 = 1$  and  $\kappa_{31} = -\frac{1}{2}$ .

Plugging in (3.27) into the color-dressed permutation  $P_{n+1}$  yields

$$P_{n+1} = \sum_{\sigma \in S_n} \text{Tr}(T^0 T^\sigma) W(0, \sigma) = \sum_{\sigma \in S_n} i^{n-k} d^{0a_1 \dots a_k} F_{a_1}^{\sigma_1} \dots F_{a_k}^{\sigma_k} \gamma_{0|\sigma_1, \dots, \sigma_k} \quad (3.28)$$

where  $\sigma = \sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_k$  is the dLf (3.5) of  $\sigma$  and [41]

$$\gamma_{0|\sigma_1, \dots, \sigma_k} = \sum_{\tau = \tau_1 \dots \tau_k} \kappa_{\rho_1} \dots \kappa_{\rho_k} W(0, \tau), \quad (3.29)$$

where  $\sigma^{-1} \circ \tau := \rho_1 \dots \rho_k$  such that  $|\rho_i| = |\tau_i|$  from the dLf of  $\tau$ . Conjecturally, a more convenient representation for the BRST-invariant permutations is given by

$$\gamma_{1|P_1, \dots, P_k} = 1 \cdot \mathcal{I}_{p_1 \dots p_k}(P_1, \dots, P_k) = 1 \cdot \mathcal{E}(P_1) \sqcup \mathcal{E}(P_2) \sqcup \dots \sqcup \mathcal{E}(P_k) \quad (3.30)$$

The shuffle symmetry (3.18) of  $\mathcal{E}(P_i)$  can be used to fix the first letter of  $P_i$  and the commutativity of the shuffle product implies total symmetry in word exchanges, so the number of components of  $\gamma_{1|P_1, \dots, P_k}$  with  $n-1$  letters distributed in the  $k$  words is given by the Stirling<sup>3</sup> cycle numbers [42]

$$\#(\gamma_{1|P_1, P_2, \dots, P_k}) = \begin{bmatrix} n-1 \\ k \end{bmatrix}, \quad \sum_{i=1}^k |P_i| = n-1. \quad (3.31)$$

### 3.4. BRST-invariant permutations and orthogonal idempotents

Since the BRST-invariant permutations have been related to the idempotent basis of the (inverse) descent algebra in (3.30) we may construct orthogonal idempotents as in section 2.3. To this effect we define the inverse of the Reutenauer idempotents (2.33)

$$\gamma_{12 \dots n}^{(i)} := 1 \cdot \theta(E^{(i)}) \quad (3.32)$$

where the labels in  $\theta(E^{(i)})$  must be shifted as  $i \rightarrow i+1$  prior to the left concatenation with the letter 1. Equivalently, from (2.31), (2.33), and (3.30) we obtain

$$\gamma_{12 \dots n}^{(k)} = \sum_{12 \dots n = P_1 \dots P_k} \frac{1}{k!} \gamma_{1|P_1, \dots, P_k}. \quad (3.33)$$

---

<sup>3</sup> We also note the appearance of Stirling cycle numbers in [7].

From the discussion of section 2.3 it follows that (3.32) are orthogonal idempotents in the inverse descent algebra  $\mathcal{D}'_n$  satisfying ( $\delta^{ij}$  is the Kronecker delta)

$$\sum_{k=1}^{n-1} \gamma_{12\dots n}^{(k)} = W_{12\dots n}, \quad \gamma_{12\dots n}^{(i)} \gamma_{12\dots n}^{(j)} = \delta^{ij} \gamma_{12\dots n}^{(i)} \quad (3.34)$$

For example, from the BRST-invariant permutations in (3.9) we get

$$\gamma_{123}^{(1)} \equiv \gamma_{1|23} = \frac{1}{2}W_{123} - \frac{1}{2}W_{132}, \quad \gamma_{123}^{(2)} \equiv \frac{1}{2}\gamma_{1|2,3} = \frac{1}{2}W_{123} + \frac{1}{2}W_{132}, \quad (3.35)$$

which satisfy

$$\gamma_{123}^{(1)} + \gamma_{123}^{(2)} = W_{123}, \quad \gamma_{123}^{(i)} \gamma_{123}^{(j)} = \delta^{ij} \gamma_{123}^{(i)}. \quad (3.36)$$

Similarly, at multiplicity four the definition (3.33) leads to

$$\gamma_{1234}^{(1)} = \gamma_{1|234}, \quad \gamma_{1234}^{(2)} = \frac{1}{2}(\gamma_{1|23,4} + \gamma_{1|2,34}), \quad \gamma_{1234}^{(3)} = \frac{1}{3!}\gamma_{1|2,3,4}, \quad (3.37)$$

yielding

$$\begin{aligned} \gamma_{1234}^{(1)} &= \frac{1}{3}W_{1234} - \frac{1}{6}W_{1243} - \frac{1}{6}W_{1324} - \frac{1}{6}W_{1342} - \frac{1}{6}W_{1423} + \frac{1}{3}W_{1432}, \\ \gamma_{1234}^{(2)} &= \frac{1}{2}W_{1234} - \frac{1}{2}W_{1432}, \\ \gamma_{1234}^{(3)} &= \frac{1}{6}W_{1234} + \frac{1}{6}W_{1243} + \frac{1}{6}W_{1324} + \frac{1}{6}W_{1342} + \frac{1}{6}W_{1423} + \frac{1}{6}W_{1432}. \end{aligned} \quad (3.38)$$

It is straightforward but tedious to check that the above are orthogonal idempotents

$$\gamma_{1234}^{(1)} + \gamma_{1234}^{(2)} + \gamma_{1234}^{(3)} = W_{1234}, \quad \gamma_{1234}^{(i)} \gamma_{1234}^{(j)} = \delta^{ij} \gamma_{1234}^{(i)}. \quad (3.39)$$

At multiplicity five the orthogonal idempotents are given by

$$\begin{aligned} \gamma_{12345}^{(1)} &= \gamma_{1|2345}, & \gamma_{12345}^{(3)} &= \frac{1}{3!}(\gamma_{1|23,4,5} + \gamma_{1|2,34,5} + \gamma_{1|2,3,45}), \\ \gamma_{12345}^{(2)} &= \frac{1}{2}(\gamma_{1|234,5} + \gamma_{1|23,45} + \gamma_{1|2,345}), & \gamma_{12345}^{(4)} &= \frac{1}{4!}\gamma_{1|2,3,4,5}, \end{aligned} \quad (3.40)$$

whose expansions can be found in the appendix A and [43]

$$\sum_{k=1}^4 \gamma_{12345}^{(k)} = W_{12345}, \quad \gamma_{12345}^{(i)} \gamma_{12345}^{(j)} = \delta^{ij} \gamma_{12345}^{(i)}. \quad (3.41)$$

In the next section we will argue that the idempotents  $\gamma^{(1)}$  and  $\gamma^{(3)}$  defined in (3.33) correspond to SYM amplitudes  $A^{\text{SYM}}$  and the  $\alpha'^2$  correction  $A^{F^4}$  of the string disk amplitude and we will exploit the consequences of their generalizations to higher  $\alpha'$  corrections.

## 4. Duality between kinematics and idempotents of the descent algebra

We find a duality between the permutations in the inverse descent algebra and kinematics from different  $\alpha'$  sectors of the string disk amplitudes. Two orthogonal idempotents are directly related to the field-theory and  $\alpha'^2$  corrections and we rederive their dimensions from a theorem in the descent algebra literature. This duality suggests a new decomposition of the symmetries of the higher  $\alpha'$  components of the disk amplitude, which we explicitly check to high orders. This decomposition yields their dimensions as a sum of Stirling cycle numbers, generalizing the field-theory and  $\alpha'^2$  formulas. In addition, we show that the superfield expansion of the BRST invariants  $C_{1|X,Y,Z}$  from the pure spinor formalism is encoded in the permutations of the descent algebra, hinting of a new direction to explain their rich combinatorial properties.

### 4.1. Color-dressed string amplitude and permutations: a duality

The color-dressed string disk amplitude

$$M_n(\alpha') = \sum_{\sigma \in S_{n-1}} \text{Tr}(T^1 T^{\sigma(2)} \cdots T^{\sigma(n)}) A^{\text{string}}(1, \sigma(2), \dots, \sigma(n)), \quad (4.1)$$

is a sum over the different disk orderings of the open string amplitude weighted by traces of color factors. The explicit form of the disk amplitudes is a linear combination of field-theory amplitudes  $A^{\text{SYM}}$  of ten-dimensional super-Yang-Mills [44] given by [45,46]

$$A^{\text{string}}(P) = \sum_{Q, R \in S_{n-3}} Z(P|1, R, n, n-1) S[R|Q]_1 A^{\text{SYM}}(1, Q, n-1, n) \quad (4.2)$$

where  $S(P|Q)_1$  is the field-theory KLT kernel [47,48,49] conveniently computed recursively by  $S[A, j|B, j, C]_i = (k_{iB} \cdot k_j) S[A|B, C]_i$ , with base case  $S[\emptyset|\emptyset]_i := 1$  [19,29]. In addition,  $Z(P|Q)$  are the non-abelian  $Z$ -theory amplitudes of [50,46]

$$Z(P|Q) := \alpha'^{n-3} \int_{D(P)} \frac{dz_1 dz_2 \cdots dz_n}{\text{vol}(SL(2, \mathbb{R}))} \frac{\prod_{i < j}^n |z_{ij}|^{\alpha' s_{ij}}}{z_{q_1 q_2} z_{q_2 q_3} \cdots z_{q_n q_1}}, \quad (4.3)$$

where  $D(P) \equiv \{(z_1, z_2, \dots, z_n) \in \mathbb{R}^n \mid -\infty < z_{p_1} < z_{p_2} < \dots < z_{p_n} < \infty\}$  is the domain of the iterated integrals. The first terms in the  $\alpha'$  expansion of (4.3) yield

$$A^{\text{string}}(1, 2, \dots, n) = A^{\text{SYM}}(1, 2, \dots, n) + \zeta_2 \alpha'^2 A^{F^4}(1, 2, \dots, n) + \mathcal{O}(\alpha'^3), \quad (4.4)$$

where the notation  $A^{F^4}$  is a reminder of the interaction of four field-strengths in the effective action [51]. Using arguments of locality and BRST invariance in pure spinor superspace, it was argued in [1] (see also the appendix B of [52]) that the  $\alpha'^2$  correction  $A^{F^4}$  could be written in terms of BRST-closed combinations of superfields  $C_{1|X,Y,Z}$  as:

$$A^{F^4}(1, 2, \dots, n) = \sum_{12\dots n=XYZ} C_{1|X,Y,Z}. \quad (4.5)$$

The BRST invariants satisfy shuffle symmetries  $C_{1|R \sqcup S, Y, Z} = 0$  for  $R, S \neq \emptyset$  and are totally symmetric under exchanges of any pairs  $X \leftrightarrow Y$  etc. So there are  $\left[ \begin{smallmatrix} n-1 \\ 3 \end{smallmatrix} \right]$  independent BRST invariants at  $n$  points. It is not a coincidence that this coincides with the number of components (3.31) in the BRST-invariant permutation  $\gamma_{1|X,Y,Z}$ . Example decompositions of (4.5) at four and five points are given by  $A^{F^4}(1, 2, 3, 4) = C_{1|2,3,4}$  and  $A^{F^4}(1, 2, 3, 4, 5) = C_{1|23,4,5} + C_{1|2,34,5} + C_{1|2,3,45}$ .

Plugging in the color-trace decomposition (3.10) into the color-dressed amplitude (4.1) for  $n = 4, 5$  and using the decomposition (4.5) in terms of BRST invariants leads to

$$\begin{aligned} M_4(\alpha') &= -\frac{1}{2} (f^{12a} f^{a34} A^{\text{SYM}}(1, 2, 3, 4) + f^{13a} f^{b24} A^{\text{SYM}}(1, 3, 2, 4)) \\ &\quad + 6\zeta_2 \alpha'^2 d^{1234} C_{1|2,3,4} + \mathcal{O}(\alpha'^3), \\ M_5(\alpha') &= -\frac{i}{2} A^{\text{SYM}}(1, 2, 3, 4, 5) f^{12a} f^{a3b} f^{b45} + \text{sym}(234) \\ &\quad + 6i\zeta_2 \alpha'^2 \left( C_{1|23,4,5} f^{23a} d^{a145} + C_{1|24,3,5} f^{24a} d^{a135} + C_{1|25,3,4} f^{25a} d^{a134} \right. \\ &\quad \left. + C_{1|34,2,5} f^{34a} d^{a125} + C_{1|35,2,4} f^{35a} d^{a124} + C_{1|45,2,3} f^{45a} d^{a123} \right) \end{aligned} \quad (4.6)$$

with similar expansions at higher points [1].

Comparing the color-dressed permutations  $P_4$  and  $P_5$  in (3.11) and (3.13) with the above color-dressed amplitudes suggests the following dualities ( $\gamma^{(i)}$  is defined in (3.33))<sup>4</sup>:

$$C_{1|X,Y,Z} \longleftrightarrow \frac{1}{6} \gamma_{1|X,Y,Z}, \quad (4.7)$$

$$A^{\text{SYM}}(1, 2, \dots, n) \longleftrightarrow \gamma_{123\dots n}^{(1)}, \quad (4.8)$$

$$A^{F^4}(1, 2, \dots, n) \longleftrightarrow \gamma_{123\dots n}^{(3)}. \quad (4.9)$$

where the deconcatenations (4.5) and (3.33) have been used to obtain the duality between the  $\alpha'^2$  correction  $A^{F^4}(1, 2, \dots, n)$  and the orthogonal idempotent  $\gamma_{12\dots n}^{(3)}$ .

---

<sup>4</sup> Rewriting the  $F^{\dots\dots}$  factors of  $P_n$  in the DDM basis [53] and using the shuffle symmetries of  $\gamma_{1|P}$  leads to the same permutations on both sides of the  $\gamma_{1|\sigma n} \leftrightarrow A^{\text{SYM}}(1, \sigma, n)$  correspondence.

## 4.2. Descent algebra decomposition of $\alpha'$ -correction symmetries

The duality between the descent algebra and kinematics suggested above is exploited to show that the symmetries of the  $\alpha'$  corrections to disk amplitudes satisfy surprising new KK-like symmetries dictated by a theorem of the descent algebra.

### 4.2.1. Field theory and $\alpha'^2$ corrections

For a partition  $\mu$  and a composition  $p$ , theorem 4.2 of [6] states that

$$E_\mu \circ I_p = 0, \quad \text{if } \lambda(p) \neq \mu. \quad (4.10)$$

Under the dualities (4.8) and (4.9) this relation implies the symmetries

$$\begin{aligned} A^{\text{SYM}}(\gamma_{1|P_1, \dots, P_k}) &= 0, \quad k \neq 1 \\ A^{F^4}(\gamma_{1|P_1, \dots, P_k}) &= 0, \quad k \neq 3 \end{aligned} \quad (4.11)$$

To see this we use that  $P \cdot (\sigma \circ \tau) = ((P \cdot \sigma) \circ (P \cdot \tau))$  if  $P$  has no common letters with  $\sigma$  and  $\tau$  and  $\theta(\sigma \circ \tau) = \theta(\tau) \circ \theta(\sigma)$  to show that the theorem (4.10) implies  $(1 \cdot I_p) \circ (1 \cdot E_\mu^\theta) = 0$  for  $\lambda(p) \neq \mu$  leading to (4.11) since  $A^{F^4}$  corresponds to a sum of  $E_\mu^\theta$  with partitions with three parts  $k(\mu) = 3$  and  $A^{\text{SYM}}$  to  $E_\mu^\theta$  with  $k(\mu) = 1$ .

The number of independent  $n$ -point  $A^{\text{SYM}}(1, \sigma)$  and  $A^{F^4}(1, \sigma)$  follows from (4.11) by elementary properties of the Stirling cycle numbers,

$$\begin{aligned} \#(A^{\text{SYM}}(1, 2, \dots, n)) &= (n-1)! - \sum_{\substack{k=1 \\ k \neq 1}}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} = (n-2)! \\ \#(A^{F^4}(1, 2, \dots, n)) &= (n-1)! - \sum_{\substack{k=1 \\ k \neq 3}}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ 3 \end{bmatrix}, \end{aligned} \quad (4.12)$$

since the number of components of  $\gamma_{1|P_1, \dots, P_k}$  is  $\begin{bmatrix} n-1 \\ k \end{bmatrix}$  by (3.31) and  $\sum_{k=1}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} = (n-1)!$ , so they correspond to the non-vanishing  $k$  in (4.11). These are the well-known dimensions of  $A^{\text{SYM}}$  and  $A^{F^4}$  under the KK [13] and KK-like relations [14,15]. Therefore the symmetries (4.11) constitute a *descent algebra decomposition* of the KK and KK-like identities.

The dualities (4.8) and (4.9) together with the properties of the Reutenauer idempotents (2.35) suggest “idempotent” identities for the  $A^{\text{SYM}}$  and  $A^{F^4}$  amplitudes viewed as symmetry relations of their respective functions:

$$\begin{aligned} A^{\text{SYM}}(\gamma_{12\dots n}^{(1)}) &= A^{\text{SYM}}(1, 2, \dots, n), \\ A^{F^4}(\gamma_{12\dots n}^{(3)}) &= A^{F^4}(1, 2, \dots, n), \end{aligned} \quad (4.13)$$

which can be explicitly checked to hold true for various values of  $n$ .

#### 4.2.2. Higher $\alpha'$ corrections to string disk amplitudes

In this section we investigate the symmetries of higher  $\alpha'$  corrections of the string disk amplitude and show that the descent algebra also decomposes them and provides new insight into their structure. The explicit data in Table 1 was collected using the  $\alpha'$  corrections to disk amplitudes obtained in [45,54,55,46], see also [56,57,14,58] and references therein.

We begin by labelling the higher  $\alpha'$  corrections of string disk amplitudes by their conjectural MZV basis content written in the form  $\zeta_2^n \zeta_M$  for  $n = 0, 1, 2, \dots$ . This is the same organization found in the motivic decomposition of the disk amplitudes [18]

$$\begin{aligned} F = & (1 + \zeta_2 P_2 + \zeta_2^2 P_4 + \zeta_2^3 P_6 + \zeta_2^4 P_8 + \dots) \\ & \times (1 + \zeta_3 M_3 + \zeta_5 M_5 + \frac{1}{2} \zeta_3^2 M_3^2 + \zeta_7 M_7 + \zeta_3 \zeta_5 M_5 M_3 + \frac{1}{5} \zeta_{3,5} [M_5, M_3] + \dots). \end{aligned} \quad (4.14)$$

The descent algebra decomposition organizes the search for symmetries of string disk amplitudes (4.2) by the BRST invariant permutations (3.7). More precisely, denoting by  $|\zeta_2^n \zeta_M|$  the restriction to a particular element of the set of MZVs, one checks whether

$$A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_2^n \zeta_M}, \quad p = (p_1, \dots, p_k) \models n-1, \quad |P_i| = p_i \quad (4.15)$$

vanishes or not. The number of checks at  $n$  points would appear to grow exponentially, but luckily a vanishing outcome of (4.15) seems to depend only on the number of parts  $k(p)$  of the composition, independently of  $n$ . These have been observed experimentally with data up to  $n = 8$  and will be assumed in general. Thus each time  $n$  increases by one it suffices to test (4.15) for a *single* case of maximum  $k = n-1$ , if  $k$  is odd (see (4.21)) or, in other words, if  $n$  is even. That is  $\gamma_{1|2,3,\dots,n} = 1.(2 \sqcup 3 \sqcup \dots \sqcup n)$  by (3.30). Therefore (4.15) becomes a sum over all cyclic orderings of the  $n$ -point string disk amplitude (4.2),

$$A^{\text{string}}(\gamma_{1|2,3,\dots,n}) = \sum_{Q,R \in S_{n-1}} Z_{\times}(1, R, n, n-1) S[R|Q]_1 A^{\text{SYM}}(1, Q, n-1, n) \quad (4.16)$$

where  $Z_{\times}(Q) := \sum_{\sigma \in S_{n-1}} Z(1, \sigma|Q)$  are the *abelian Z-theory amplitudes* of [19]. As alluded to above, the proof (4.21) implies that when  $n$  is odd (4.15) always vanishes. This agrees with the statement that NLSM amplitudes vanish for  $n$  odd. Therefore the  $\alpha'$  expansion of the maximal case  $k = n-1$  can be obtained from the methods of [19,59]<sup>5</sup>.

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<sup>5</sup> I thank Oliver Schlotterer for discussions on this point.

$k$	$A^{\text{string}}(\gamma_{1 P_1, \dots, P_k}) = 0$	$A^{\text{string}}(\gamma_{1 P_1, \dots, P_k}) \neq 0$
7	$\zeta_7, \zeta_5, \zeta_3, \zeta_3^2, \zeta_2, \zeta_2\zeta_5, \zeta_2\zeta_3, \zeta_2^2, \zeta_2^2\zeta_3$	$\zeta_2^3$
6	$\forall \zeta_M$	$\times$
5	$\zeta_7, \zeta_5, \zeta_3, \zeta_3^2, \zeta_2, \zeta_2\zeta_5, \zeta_2\zeta_3$	$\zeta_2^2, \zeta_2^2\zeta_3, \zeta_2^3$
4	$\forall \zeta_M$	$\times$
3	$\zeta_7, \zeta_5, \zeta_3, \zeta_3^2$	$\zeta_2, \zeta_2\zeta_5, \zeta_2\zeta_3, \zeta_2^2, \zeta_2^2\zeta_3, \zeta_2^3$
2	$\forall \zeta_M$	$\times$
1	$\zeta_2, \zeta_2\zeta_3, \zeta_2\zeta_5$	$\zeta_7, \zeta_5, \zeta_3, \zeta_3^2, \zeta_2^2, \zeta_2^2\zeta_3, \zeta_2^3$

**Table 1.** Overview of the descent algebra symmetries of higher  $\alpha'$  corrections to string disk amplitudes of up to  $n = 8$  points displayed by their MZV content of weight  $w \leq 7$ . The entries depend only on the number of parts  $k$  of the composition of  $n-1$ . However, a partition with  $k$  parts cannot be probed by disk amplitudes with fewer than  $k+1$  points.

#### 4.2.3. Even zeta value symmetry classes

The descent algebra decomposition of the symmetries of the string disk amplitudes (4.15) will be classified by their  $\zeta_2^n$  content; this is because the second line of (4.14) respects both the KK and BCJ *field-theory* amplitude relations [19,16]<sup>6</sup>. Since they lead to the lower bound of  $(n-3)!$  degrees of freedom they are not expected to modify the dimensions of a given  $\zeta_2^n$  symmetry class, leading to all components  $\zeta_2^m \zeta_M$  sharing the same symmetries.

Indeed, Table 1 shows that the components  $\zeta_7, \zeta_5, \zeta_3, \zeta_3^2$  have the same KK symmetries of  $A^{\text{SYM}}$ , while the components  $\zeta_2, \zeta_2\zeta_3, \zeta_2\zeta_5$  have the same KK-like symmetries of  $A^{F^4}$ . This confirms the argument above and we can state:

$$\begin{aligned} A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_M} &= 0, \quad k \neq 1, \\ A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_2\zeta_M} &= 0, \quad k \neq 3. \end{aligned} \quad (4.17)$$

Therefore their dimensions are counted by the same  $\left[ \begin{smallmatrix} n-1 \\ 1 \end{smallmatrix} \right]$  and  $\left[ \begin{smallmatrix} n-1 \\ 3 \end{smallmatrix} \right]$  as before.

The symmetries of the  $\zeta_2^n$  classes for  $n \geq 2$  have never been studied before. The experimental data collected in Table 1 indicates the *non-vanishing* cases as

$$\begin{aligned} A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_2^2\zeta_M} &\neq 0, \quad k = 1, 3, 5, \\ A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_2^3\zeta_M} &\neq 0, \quad k = 1, 3, 5, 7. \end{aligned} \quad (4.18)$$

<sup>6</sup> The string monodromy relations give rise to deformations of the field-theory BCJ relations by even powers of  $\alpha'$  accompanied by factors of  $\zeta_2^m$ . These are given by the first line of (4.14) so the second line preserves the field-theory KK and BCJ amplitude relations.

The analysis of the NLSM amplitudes from [19,59] and the descent algebra decomposition patterns mentioned above give rise to a conjecture for all  $\zeta_2^m \zeta_M$  components with  $m \geq 2$ :

$$A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_2^m \zeta_M} \neq 0, \quad k = 1, 3, 5, \dots, 2m+1, \quad m \geq 2 \quad (4.19)$$

Unlike (4.12), the dimensions of (4.19) are given by a *sum* of Stirling cycle numbers

$$\#(A^{\text{string}}(1, 2, \dots, n)|_{\zeta_2^m \zeta_M}) = \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + \begin{bmatrix} n-1 \\ 3 \end{bmatrix} + \dots + \begin{bmatrix} n-1 \\ 2m+1 \end{bmatrix} \quad (4.20)$$

By construction, (4.19) and the results displayed in Table 1 of the NLSM analysis of [19] match when  $k = n-1$  for  $n \geq 4$ . For example, the  $\zeta_2^3$  components do not vanish for  $k = 1, 3, 5, 7$  corresponding to non-vanishing entries for  $n = 4, 6, 8$  in the table 1 of [19]. The vanishing of  $\zeta_2^3$  for  $k = 9$  and higher from (4.19) corresponds to the vanishing of  $\zeta_2^3$  for  $n = 10$  (and higher) in [19]. The conjectural status of (4.19) hinges on the observation that once the symmetry of a particular  $\zeta_2^m \zeta_M$  component is established from the analysis done at  $k = n-1$ , it remains valid for the same  $k$  when  $n$  is increased, see the paragraph after (4.15).

Parity of the amplitude  $A^{\text{string}}(1, \dots, n) = (-1)^n A^{\text{string}}(n, \dots, 1)$  explains the vanishing of (4.15) for even  $k$  as observed in Table 1. A quick counting argument suggests why this is so as  $\sum_k \begin{bmatrix} n-1 \\ 2k \end{bmatrix} = \frac{1}{2}(n-1)!$  is the upper bound in the dimension of string disk amplitudes from properties of the string worldsheet alone [22]. More precisely:

**Proposition.** *If  $k$  is even then the  $n$ -point disk amplitude satisfies*

$$A^{\text{string}}(\gamma_{1|P_1, \dots, P_k}) = 0, \quad (4.21)$$

where  $\gamma_{1|P_1, \dots, P_k}$  is the BRST-invariant permutation (3.7).

*Proof.* The parity of  $A^{\text{string}}$  at  $n$  points can be written as  $A^{\text{string}}(1, \sigma) = (-1)^n A^{\text{string}}(1, \tilde{\sigma})$  by cyclicity. This means, by (3.30), that  $A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})$  will vanish whenever the parity of  $A^{\text{string}}$  at  $n$  points is opposite to the parity of  $\mathcal{I}_p$  for  $p \models n-1$ . To see why this is true consider the example of  $A^{\text{string}}(\gamma_{1|23,4})$  with the expression for the BRST-invariant permutation in (3.12). The terms can be rearranged as

$$A^{\text{string}}(\gamma_{1|23,4}) = \frac{1}{2}(A_{1234}^{\text{string}} - A_{1432}^{\text{string}}) + \frac{1}{2}(A_{1243}^{\text{string}} - A_{1342}^{\text{string}}) + \frac{1}{2}(A_{1423}^{\text{string}} - A_{1324}^{\text{string}}) = 0 \quad (4.22)$$

which vanishes by parity  $A_{1234}^{\text{string}} = A_{1432}^{\text{string}}$ . Notice that this happens because the parity of  $\mathcal{I}_{21}(23, 4)$  from (3.25) is the opposite of the string disk amplitude;  $\tilde{\mathcal{I}}_{21}(23, 4) = -\mathcal{I}_{21}(23, 4)$ . The proposition can now be proven by considering the two cases when  $n$  is even or odd.

For  $n$  even the parity of the  $n$ -point disk amplitude is  $+$  so  $A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})$  vanishes if the parity of  $\mathcal{I}_p$  is  $-$  for a composition  $p$  of  $n-1$ . By (3.25) this means that there must be an odd number of even parts in the composition  $p$  (which sum to even). But since  $n-1$  is odd, there must be an odd number of odd parts in  $p$  (which sum to odd). Therefore the number of parts  $k(p)$  is even ( $= \text{odd} + \text{odd}$ ). Similarly, when  $n$  is odd the number of parts  $k(p)$  in the composition of  $p$  is also even (from even + even). This finishes the proof.  $\square$

#### 4.2.4. Idempotent properties of higher $\alpha'$ corrections

Experimentally, the idempotent properties (4.13) generalize to their  $\zeta_M$  and  $\zeta_2\zeta_M$  symmetry classes at higher  $\alpha'$ :

$$A^{\text{string}}(\gamma_{12\dots n}^{(1)})|_{\zeta_M} = A^{\text{string}}(1, 2, \dots, n)|_{\zeta_M}, \quad (4.23)$$

$$A^{\text{string}}(\gamma_{12\dots n}^{(3)})|_{\zeta_2\zeta_M} = A^{\text{string}}(1, 2, \dots, n)|_{\zeta_2\zeta_M}, \quad (4.24)$$

For example, one can check using the string five-point disk amplitudes at order  $\alpha'^7\zeta_2\zeta_5$  that  $A^{\text{string}}(\gamma_{12345}^{(3)})|_{\zeta_2\zeta_5} = A^{\text{string}}(1, 2, 3, 4, 5)|_{\zeta_2\zeta_5}$ , or

$$\begin{aligned} A_{12345}^{\text{string}}|_{\zeta_2\zeta_5} = & \frac{1}{12} \left( 3A_{12345}^{\text{string}} + A_{12354}^{\text{string}} + A_{12435}^{\text{string}} + A_{12453}^{\text{string}} + A_{12534}^{\text{string}} - A_{12543}^{\text{string}} \right. \\ & + A_{13245}^{\text{string}} - A_{13254}^{\text{string}} + A_{13425}^{\text{string}} + A_{13452}^{\text{string}} - A_{13524}^{\text{string}} - A_{13542}^{\text{string}} \\ & + A_{14235}^{\text{string}} + A_{14253}^{\text{string}} - A_{14325}^{\text{string}} - A_{14352}^{\text{string}} + A_{14523}^{\text{string}} - A_{14532}^{\text{string}} \\ & \left. + A_{15234}^{\text{string}} - A_{15243}^{\text{string}} - A_{15324}^{\text{string}} - A_{15342}^{\text{string}} - A_{15423}^{\text{string}} - 3A_{15432}^{\text{string}} \right) \Big|_{\zeta_2\zeta_5} \end{aligned} \quad (4.25)$$

The highly non-trivial nature of such an identity provides strong support for the descent algebra decomposition of symmetries discussed above.

#### 4.3. Descent algebra symmetries and the color-dressed amplitude

The descent algebra decomposition of the different  $\zeta_2^m\zeta_M$  symmetry classes of the string disk amplitude controls the appearance of the  $\zeta_2^m\zeta_M$  corrections in the color-dressed string disk amplitude  $M_n(\alpha')$  (4.1). If  $A^{\text{string}}(\gamma_{1|P_1, \dots, P_k})|_{\zeta_2^m\zeta_M}$  does *not* vanish for a given  $k$  then the color-dressed amplitude contains  $\zeta_2^m\zeta_M d^{1a_1\dots a_k} F_{a_1}^{P_1} \dots F_{a_k}^{P_k}$  contributions. This follows from linearity using  $A^{\text{string}}(P_n) = M_n(\alpha')$  in the color-dressed permutation  $P_n$  in (3.7). For instance, from  $A^{\text{string}}(\gamma_{1|2345})|_{\zeta_2^2} \neq 0$  we get  $\zeta_2^2 d^{1a} F_a^{2345} = (\zeta_2^2/2) f^{23a} f^{a4b} f^{b51}$  corrections in the five-point color-dressed amplitude  $M_5(\alpha')$ .

#### 4.4. More consequences of the descent algebra duality

We show that theorem 4.2 of [6] leads to a derivation of the claim from [1] that the representation (4.5) is invertible. In addition, inspired by the duality (4.7) we find an algorithm to extract the superfield expansion of  $C_{1|P,Q,R}$  from the permutations of  $\gamma_{1|P,Q,R}$ .

##### 4.4.1. BRST invariants from $A^{F^4}$

An immediate consequence of the duality (4.7) is that the representation of  $A^{F^4}$  in terms of  $C_{1|P_1,P_2,P_3}$  given in (4.5) is invertible. To see this one uses Theorem 4.2 of [6],

$$E_\mu \circ I_p = I_p, \quad \text{if } \lambda(p) = \mu \quad (4.26)$$

where  $\lambda(p)$  is the shape of the composition  $p$  and  $E_\mu$  is defined in (2.31). This implies  $(1 \cdot \mathcal{I}_p) \circ (1 \cdot \theta(E_\mu)) = (1 \cdot \mathcal{I}_p)$  for  $\lambda(p) = \mu$  or, using the function interpretation of the right action  $\sigma \circ F := F(\sigma)$  with a partition with three parts  $k(\mu) = 3$

$$A^{F^4}(\gamma_{1|P_1,P_2,P_3}) = 6C_{1|P_1,P_2,P_3}, \quad |P_i| = p_i, \quad P_1 P_2 P_3 = 23 \dots n, \quad (4.27)$$

where we used the identifications (3.30), (3.32) and duality (4.9) on the left-hand side and the duality (4.7) on the right-hand side. For example, from  $\gamma_{1|23,4,5}$  of (A.1) we get

$$\begin{aligned} & 6C_{1|23,4,5} = \\ & \frac{1}{2} \left( A_{12345}^{F^4} + A_{12354}^{F^4} + A_{12435}^{F^4} + A_{12453}^{F^4} + A_{12534}^{F^4} + A_{12543}^{F^4} - A_{13245}^{F^4} - A_{13254}^{F^4} \right. \\ & - A_{13425}^{F^4} - A_{13452}^{F^4} - A_{13524}^{F^4} - A_{13542}^{F^4} + A_{14235}^{F^4} + A_{14253}^{F^4} - A_{14325}^{F^4} - A_{14352}^{F^4} \\ & \left. + A_{14523}^{F^4} - A_{14532}^{F^4} + A_{15234}^{F^4} + A_{15243}^{F^4} - A_{15324}^{F^4} - A_{15342}^{F^4} + A_{15423}^{F^4} - A_{15432}^{F^4} \right). \end{aligned} \quad (4.28)$$

Plugging in  $A^{F^4}(1, 2, 3, 4, 5) = C_{1|23,4,5} + C_{1|2,34,5} + C_{1|2,3,45}$  and using shuffle symmetries of  $C_{1|...}$  these terms collapse to a single term,  $6C_{1|23,4,5}$ , in agreement with (4.27). The theorem (4.26) therefore justifies the indirect arguments of [1]. In addition, in view of the relations (3.33) and (4.27), the idempotent identity (4.13) yields the decomposition of the  $\alpha'^2$  correction of the disk amplitude (4.5) found in [1]

$$A^{F^4}(1, 2, \dots, n) = \frac{1}{6} \sum_{12\dots n = P_1 P_2 P_3} A^{F^4}(\gamma_{1|P_1,P_2,P_3}) = \sum_{12\dots n = P_1 P_2 P_3} C_{1|P_1,P_2,P_3}. \quad (4.29)$$

#### 4.4.2. The superfield expansion of $C_{1|P,Q,R}$ from the permutations of $\gamma_{1|P,Q,R}$

The algorithm proposed in [27] generates the superfield expansion of the scalar BRST invariant recursively as

$$C_{i|P,Q,R} = M_i M_{P,Q,R} + M_i \cdot [C_{p_1|p_2 \dots p_{|P|}, Q, R} - C_{p_{|P|}|p_1 \dots p_{|P|-1}, Q, R} + (P \leftrightarrow Q, R)] \quad (4.30)$$

starting from  $C_{i|j,k,l} = M_i M_{j,k,l}$  with the dot representing concatenation,  $M_i \cdot M_A := M_{iA}$ . For example, the first few expansions are given by

$$C_{1|2,3,4} = M_1 M_{2,3,4}, \quad (4.31)$$

$$C_{1|23,4,5} = M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5},$$

$$\begin{aligned} C_{1|234,5,6} = & M_1 M_{234,5,6} + M_{12} M_{34,5,6} + M_{123} M_{4,5,6} - M_{124} M_{3,5,6} \\ & - M_{14} M_{23,5,6} - M_{142} M_{3,5,6} + M_{143} M_{2,5,6}, \end{aligned}$$

$$\begin{aligned} C_{1|23,45,6} = & M_1 M_{23,45,6} + M_{12} M_{45,3,6} - M_{13} M_{45,2,6} + M_{14} M_{23,5,6} - M_{15} M_{23,4,6} \\ & + M_{124} M_{3,5,6} - M_{134} M_{2,5,6} + M_{142} M_{3,5,6} - M_{152} M_{3,4,6} \\ & - M_{125} M_{3,4,6} + M_{135} M_{2,4,6} - M_{143} M_{2,5,6} + M_{153} M_{2,4,6}. \end{aligned}$$

It is not difficult to suspect that such a systematic generation of terms indicate a hidden combinatorial structure. We now propose how these terms can be extracted from the permutations of the BRST-invariant permutations  $\gamma_{1|P,Q,R}$  of the inverse descent algebra (further justifying the terminology of  $\gamma_{1|...}$ ). The steps are as follows:

1. Sum over the cyclic permutations of all permutations in  $\gamma_{1|P,Q,R}$ :

$$W_\sigma \rightarrow W_\sigma + \text{cyclic}(\sigma) \quad (4.32)$$

2. Decompose  $W_\sigma$  into all possible four-word deconcatenations:

$$W_\sigma = \sum_{XYZW=\sigma} W_X.W_Y.W_Z.W_W \quad (4.33)$$

3. Move label 1 to the front by repeatedly commuting  $W_C.W_{A1B} = W_{A1B}.W_C$  if necessary and write the result in terms of Berends-Giele superfields:

$$W_{A1B}.W_C.W_D.W_E := \frac{1}{4!} M_{A1B} M_{C,D,E} \quad (4.34)$$

The resulting expressions have been explicitly checked<sup>7</sup> for all topologies of BRST invariants up to eight points. In addition, using the descent duality (4.7) one may also derive the change of basis identities for  $C_{i \neq 1|...} = \sum C_{1|...}$  from [27,28] by choosing a different label to be singled-out in the color-dressed permutation (3.1) [41].

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<sup>7</sup> The shuffle symmetry  $AiB = (-1)^{|A|} i \tilde{A} \sqcup B$  [38] is needed to rewrite words in a Lyndon basis.

#### 4.4.3. The descent algebra dual to the standard Berends-Giele current

In view of the Berends-Giele formula to compute SYM tree amplitudes [39] and the shuffle identities obeyed by the standard Berends-Giele currents [60,61], the symmetry relations (4.11) suggest an interesting duality.

To see this we consider the descent algebra symmetry  $A^{\text{SYM}}(\gamma_{1|234,5}) = 0$  for  $k = 2$  in (4.11) from a new perspective. We have the Berends-Giele formula  $A^{\text{SYM}}(1, P) = s_P J_1^m J_P^m$  on the one hand and on the other hand we have the duality  $A^{\text{SYM}}(1, P) \leftrightarrow \gamma_{1|P}$  as in (4.8). However, the BRST-invariant permutation with a single part is given by (3.30) as  $\gamma_{1|P} = 1 \cdot \mathcal{E}_P$ . In summary,

$$s_P J_1^m J_P^m = A^{\text{SYM}}(1, P) \longleftrightarrow \gamma_{1|P} = 1 \cdot \mathcal{E}_P \quad (4.35)$$

suggesting the duality and the origin for the terminology of  $\mathcal{E}_P$ <sup>8</sup>:

$$J_P^m \longleftrightarrow \mathcal{E}_P. \quad (4.36)$$

And we now obtain  $A^{\text{SYM}}(\gamma_{1|234,5}) = s_{2345} J_1^m J_{\mathcal{E}_{234} \sqcup \mathcal{E}_5}^m = 0$  using  $\gamma_{1|234,5} = 1 \cdot \mathcal{E}_{234} \sqcup \mathcal{E}_5$  and the shuffle symmetry  $J_{R \sqcup S}^m = 0$  of the Berends-Giele currents.

## 5. Conclusion

In this paper we investigated the combinatorial properties of the permutations appearing in the *color-dressed permutations* (3.1) using the basis of color factors from the color trace decomposition of [3] and we found closed formulas that generate them. This led us to the study of descent algebras and we pointed out the relation between these permutations and the (inverse) descent algebra. In particular, we showed how the color-dressed permutations give rise to orthogonal idempotents in the descent algebra which sum to the identity permutation. These idempotents have been extensively studied in the mathematics literature, most notably by Reutenauer and Garsia [7,31,6].

After discovering closed formulas for various permutations appearing in the color-dressed permutations we turned our attention to the formulation of the color-dressed string

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<sup>8</sup> When using the pure spinor representation of the SYM tree amplitude  $A^{\text{SYM}}(1, P) = \langle M_1 E_P \rangle$ , where  $E_P$  is the ghost-number two superfield defined in [62], the correspondence (4.36) can be stated even more suggestively as  $E_P \leftrightarrow \mathcal{E}_P$ .

disk amplitudes (4.1) including  $\alpha'$  corrections. We then pointed out, inspired by [1], the correspondence between the permutations from the (inverse) descent algebra and kinematics from the string disk amplitudes. Knowing that the permutations can be related to the descent algebra, we exploited the consequences of some theorems from the mathematics literature on descent algebra to the  $\alpha'$  corrections of the string disk amplitudes. This led to the discovery of a new descent algebra decomposition of the KK-like symmetries of the  $\alpha'$  corrections organized by their MZV content and counts their dimensions in terms of Stirling cycle numbers, leading to (4.19). Abelian  $Z$ -theory arguments together with some observations yield lend credence to their validity in general. Besides, these claims have been explicitly checked using various data points in string theory up to  $n = 8$  and  $\alpha'^7$ .

Furthermore, the duality with the descent algebra also suggests non-obvious KK-like identities among  $\alpha'$  corrections (4.23) and (4.24) arising from the “idempotent” property of certain orthogonal permutations studied by Garsia and Reutenauer [7,6].

In addition, we found an algorithm to extract the superfield content of the BRST invariants in the pure spinor formalism from the BRST-invariant permutations in the inverse descent algebra, hinting about their combinatorial origin.

### 5.0.1. Outlook and future directions

The duality between kinematics and idempotents of the inverse descent algebra in (4.7), (4.8) and (4.9) and the orthogonality property of the Reutenauer idempotents (2.35) led to the symmetries of various  $\alpha'$  corrections of the string disk amplitude. But this consequence was based purely on the functional interpretation of the right action of permutations as  $\sigma \circ F := F(\sigma)$ . A more intriguing possibility is to interpret the  $\alpha'^{1/2}$  correction  $A^{F^4}$  and the tree amplitude  $A^{\text{SYM}}$  as “orthogonal” and “idempotents” directly in kinematic space:

$$\begin{aligned} A^{F^4}(1, 2, \dots, n) \circ A^{\text{SYM}}(1, 2, \dots, n) &= 0, \\ A^{\text{SYM}}(1, 2, \dots, n) \circ A^{F^4}(1, 2, \dots, n) &= 0, \\ A^{\text{SYM}}(1, 2, \dots, n) \circ A^{\text{SYM}}(1, 2, \dots, n) &= A^{\text{SYM}}(1, 2, \dots, n), \\ A^{F^4}(1, 2, \dots, n) \circ A^{F^4}(1, 2, \dots, n) &= A^{F^4}(1, 2, \dots, n). \end{aligned} \tag{5.1}$$

However, it is not clear how to define the right-action action of the kinematic variables of polarizations and momenta among themselves.

In trying to find the right-action on kinematics one may consider the Lie-polynomial form of the string disk amplitudes from [29]. In this setting the field-theory and  $\zeta_2$  corrections at four and five points become ( $12 := [1, 2]$ ,  $123 := [[1, 2], 3]$  etc)

$$\begin{aligned}
A^{\text{SYM}}(1, 2, 3, 4) &= \frac{[12, 3]}{s_{12}} + \frac{[1, 23]}{s_{23}} \\
A^{F^4}(1, 2, 3, 4) &= s_{23}[12, 3] + s_{12}[1, 23] \\
A^{\text{SYM}}(1, 2, 3, 4, 5) &= \frac{[123, 4]}{s_{12}s_{123}} + \frac{[321, 4]}{s_{23}s_{123}} + \frac{[12, 34]}{s_{12}s_{34}} + \frac{[1, 432]}{s_{34}s_{234}} + \frac{[1, 234]}{s_{23}s_{234}}, \\
A^{F^4}(1, 2, 3, 4, 5) &= -\frac{1}{s_{12}}(s_{45}[12, 34] + s_{34}[123, 4]) - \frac{1}{s_{23}}(s_{45}[1, 234] + s_{15}[321, 4]) \\
&\quad - \frac{1}{s_{34}}(s_{12}[1, 432] + s_{15}[12, 34]) - \frac{1}{s_{45}}(s_{12}[321, 4] + s_{23}[123, 4]) \\
&\quad - \frac{1}{s_{51}}(s_{23}[1, 432] + s_{34}[1, 234]) \\
&\quad + [1, 234] + [12, 34] + [13, 24] + [321, 4]
\end{aligned} \tag{5.2}$$

where we omitted the last leg  $n$  which multiplies from the right. These Lie-polynomial amplitudes give rise to the standard  $A^{\text{SYM}}$  and  $A^{F^4}$  amplitudes upon dressing the numerators with BCJ-satisfying polarizations [61,63], for example  $[12, 3]4 \rightarrow A_{[12, 3]}^m A_4^m$ . Indexing the Mandelstams and Lie monomials  $\Gamma$  from  $A^{\text{SYM}}$  and  $A^{F^4}$  with the subscripts (0) and (2) we define the following kinematical right-action multiplications

$$\begin{aligned}
s_{ij}^{(m)} \circ s_{kl}^{(n)} &= \delta_{ik}\delta_{jl}s_{ij}, & \frac{1}{s_{ij}^{(m)}} \circ \frac{1}{s_{kl}^{(n)}} &= \delta_{ik}\delta_{jl}\frac{1}{s_{ij}}, & \frac{1}{s_{ij}^{(m)}} \circ s_{ij}^{(n)} &= s_{ij}^{(n)} \circ \frac{1}{s_{ij}^{(m)}} = 1, \\
\Gamma^{(i)} \circ \Gamma^{(i)} &= \Gamma^{(i)}, & \Gamma^{(i)} \circ \Sigma^{(j)} + \Sigma^{(i)} \circ \Gamma^{(j)} &= 0.
\end{aligned} \tag{5.3}$$

Then preliminary analysis shows that up to the purely local terms in  $A^{F^4}$  we have  $A^{\text{SYM}}(1, 2, \dots, n) \circ A^{F^4}(1, 2, \dots, n) = 0$ , tested up to  $n = 6$ . It would be interesting to see whether these observations can be made to include the local terms, and how to transfer these rules to the kinematics in terms of polarizations and momenta.

It would be interesting to frame the Eisenstein-Kronecker series and the expansion functions  $g^{(n)}(z, \tau)$  as defined by Brown and Levin [64] in terms of the Solomon descent algebra. The scalar generalized elliptic integrands (or *homology invariants*)  $E_{1|P, Q, R}$  of [65] satisfy the same symmetry relations obeyed by the scalar BRST invariants and their composition in terms of the functions  $g^{(n)}(z, \tau)$  can be mapped to the superfields of the BRST invariants  $C_{1|P, Q, R}$ . Since the expansion of  $C_{1| \dots}$  is encoded in the descent algebra

as in section 4.4.2, there should be a descent algebra characterization of the homology invariants as well.

In addition, we suspect that the descent algebra decomposition of KK-like symmetries are universal to color-dressed amplitudes, for they encode basic relations from the color traces. As such, there should be KK-like identities for the single-trace part at genus one. It will be interesting to see whether the possible symmetries are graded by a combination of standard MZVs and eMZVs [66,67,68], and how double traces and higher modify the analysis.

More speculative work may define a “BRST” operator  $\Delta$  acting on permutations in a similar way as the pure spinor BRST operator acts on Berends-Giele supercurrents. The permutations would then need to be distinguished by different concatenation decompositions. For example  $\Delta(123) = 1.23 + 12.3$  where the dot represents a “concatenation” operation. With additional structure such as the deconcatenation algorithm of section 4.4.2 this BRST operator may be used in a more abstract setting to study the “cohomology” of permutations that ultimately may give further insight into the combinatorics of the BRST invariants and the generalized elliptic integrands.

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## Appendix A. Explicit permutations at low multiplicities

The multiplicity-five BRST-invariant permutations (defined in (3.7)) are given by

$$\begin{aligned} \gamma_{1|2,3,4,5} &= W_{1(2\#3\#4\#5)} \tag{A.1} \\ \gamma_{1|23,4,5} &= \frac{1}{2}W_{12345} + \frac{1}{2}W_{12354} + \frac{1}{2}W_{12435} + \frac{1}{2}W_{12453} + \frac{1}{2}W_{12534} + \frac{1}{2}W_{12543} \\ &\quad - \frac{1}{2}W_{13245} - \frac{1}{2}W_{13254} - \frac{1}{2}W_{13425} - \frac{1}{2}W_{13452} - \frac{1}{2}W_{13524} - \frac{1}{2}W_{13542} \\ &\quad + \frac{1}{2}W_{14235} + \frac{1}{2}W_{14253} - \frac{1}{2}W_{14325} - \frac{1}{2}W_{14352} + \frac{1}{2}W_{14523} - \frac{1}{2}W_{14532} \\ &\quad + \frac{1}{2}W_{15234} + \frac{1}{2}W_{15243} - \frac{1}{2}W_{15324} - \frac{1}{2}W_{15342} + \frac{1}{2}W_{15423} - \frac{1}{2}W_{15432} \end{aligned}$$

$$\begin{aligned}
\gamma_{1|234,5} &= \frac{1}{3}W_{12345} + \frac{1}{3}W_{12354} - \frac{1}{6}W_{12435} - \frac{1}{6}W_{12453} + \frac{1}{3}W_{12534} - \frac{1}{6}W_{12543} \\
&\quad - \frac{1}{6}W_{13245} - \frac{1}{6}W_{13254} - \frac{1}{6}W_{13425} - \frac{1}{6}W_{13452} - \frac{1}{6}W_{13524} - \frac{1}{6}W_{13542} \\
&\quad - \frac{1}{6}W_{14235} - \frac{1}{6}W_{14253} + \frac{1}{3}W_{14325} + \frac{1}{3}W_{14352} - \frac{1}{6}W_{14523} + \frac{1}{3}W_{14532} \\
&\quad + \frac{1}{3}W_{15234} - \frac{1}{6}W_{15243} - \frac{1}{6}W_{15324} - \frac{1}{6}W_{15342} - \frac{1}{6}W_{15423} + \frac{1}{3}W_{15432} \\
\gamma_{1|23,45} &= \frac{1}{4}W_{12345} - \frac{1}{4}W_{12354} + \frac{1}{4}W_{12435} + \frac{1}{4}W_{12453} - \frac{1}{4}W_{12534} - \frac{1}{4}W_{12543} \\
&\quad - \frac{1}{4}W_{13245} + \frac{1}{4}W_{13254} - \frac{1}{4}W_{13425} - \frac{1}{4}W_{13452} + \frac{1}{4}W_{13524} + \frac{1}{4}W_{13542} \\
&\quad + \frac{1}{4}W_{14235} + \frac{1}{4}W_{14253} - \frac{1}{4}W_{14325} - \frac{1}{4}W_{14352} + \frac{1}{4}W_{14523} - \frac{1}{4}W_{14532} \\
&\quad - \frac{1}{4}W_{15234} - \frac{1}{4}W_{15243} + \frac{1}{4}W_{15324} + \frac{1}{4}W_{15342} - \frac{1}{4}W_{15423} + \frac{1}{4}W_{15432} \\
\gamma_{1|2345} &= \frac{1}{4}W_{12345} - \frac{1}{12}W_{12354} - \frac{1}{12}W_{12435} - \frac{1}{12}W_{12453} - \frac{1}{12}W_{12534} + \frac{1}{12}W_{12543} \\
&\quad - \frac{1}{12}W_{13245} + \frac{1}{12}W_{13254} - \frac{1}{12}W_{13425} - \frac{1}{12}W_{13452} + \frac{1}{12}W_{13524} + \frac{1}{12}W_{13542} \\
&\quad - \frac{1}{12}W_{14235} - \frac{1}{12}W_{14253} + \frac{1}{12}W_{14325} + \frac{1}{12}W_{14352} - \frac{1}{12}W_{14523} + \frac{1}{12}W_{14532} \\
&\quad - \frac{1}{12}W_{15234} + \frac{1}{12}W_{15243} + \frac{1}{12}W_{15324} + \frac{1}{12}W_{15342} + \frac{1}{12}W_{15423} - \frac{1}{4}W_{15432}
\end{aligned}$$

According to the deconcatenation (3.33) these BRST-invariant permutations give rise to the following orthogonal idempotents:

$$\begin{aligned}
\gamma_{12345}^{(1)} &= \frac{1}{4}W_{12345} - \frac{1}{12}W_{12354} - \frac{1}{12}W_{12435} - \frac{1}{12}W_{12453} - \frac{1}{12}W_{12534} + \frac{1}{12}W_{12543} \\
&\quad - \frac{1}{12}W_{13245} + \frac{1}{12}W_{13254} - \frac{1}{12}W_{13425} - \frac{1}{12}W_{13452} + \frac{1}{12}W_{13524} + \frac{1}{12}W_{13542} \\
&\quad - \frac{1}{12}W_{14235} - \frac{1}{12}W_{14253} + \frac{1}{12}W_{14325} + \frac{1}{12}W_{14352} - \frac{1}{12}W_{14523} + \frac{1}{12}W_{14532} \\
&\quad - \frac{1}{12}W_{15234} + \frac{1}{12}W_{15243} + \frac{1}{12}W_{15324} + \frac{1}{12}W_{15342} + \frac{1}{12}W_{15423} - \frac{1}{4}W_{15432} \\
\gamma_{12345}^{(2)} &= \frac{11}{24}W_{12345} - \frac{1}{24}W_{12354} - \frac{1}{24}W_{12435} - \frac{1}{24}W_{12453} - \frac{1}{24}W_{12534} - \frac{1}{24}W_{12543} \\
&\quad - \frac{1}{24}W_{13245} - \frac{1}{24}W_{13254} - \frac{1}{24}W_{13425} - \frac{1}{24}W_{13452} - \frac{1}{24}W_{13524} - \frac{1}{24}W_{13542} \\
&\quad - \frac{1}{24}W_{14235} - \frac{1}{24}W_{14253} - \frac{1}{24}W_{14325} - \frac{1}{24}W_{14352} - \frac{1}{24}W_{14523} - \frac{1}{24}W_{14532} \\
&\quad - \frac{1}{24}W_{15234} - \frac{1}{24}W_{15243} - \frac{1}{24}W_{15324} - \frac{1}{24}W_{15342} - \frac{1}{24}W_{15423} + \frac{11}{24}W_{15432} \\
\gamma_{12345}^{(3)} &= \frac{1}{4}W_{12345} + \frac{1}{12}W_{12354} + \frac{1}{12}W_{12435} + \frac{1}{12}W_{12453} + \frac{1}{12}W_{12534} - \frac{1}{12}W_{12543}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{12}W_{13245} - \frac{1}{12}W_{13254} + \frac{1}{12}W_{13425} + \frac{1}{12}W_{13452} - \frac{1}{12}W_{13524} - \frac{1}{12}W_{13542} \\
& + \frac{1}{12}W_{14235} + \frac{1}{12}W_{14253} - \frac{1}{12}W_{14325} - \frac{1}{12}W_{14352} + \frac{1}{12}W_{14523} - \frac{1}{12}W_{14532} \\
& + \frac{1}{12}W_{15234} - \frac{1}{12}W_{15243} - \frac{1}{12}W_{15324} - \frac{1}{12}W_{15342} - \frac{1}{12}W_{15423} - \frac{1}{4}W_{15432} \\
\gamma_{12345}^{(4)} &= \frac{1}{24}W_{1(2\sqcup 3\sqcup 4\sqcup 5)}
\end{aligned}$$

Here we list the first few expansions of  $E_\lambda$  defined in (2.31):

$$\begin{aligned}
E_2 &= \frac{1}{2}W_{12} - \frac{1}{2}W_{21}, & E_{11} &= \frac{1}{2}W_{12} + \frac{1}{2}W_{21} \\
E_3 &= \frac{1}{3}W_{123} - \frac{1}{6}W_{132} - \frac{1}{6}W_{213} - \frac{1}{6}W_{231} - \frac{1}{6}W_{312} + \frac{1}{3}W_{321} \\
E_{21} &= \frac{1}{2}W_{123} - \frac{1}{2}W_{321}, & E_{111} &= \frac{1}{6}W_{123} + \text{perm}(1, 2, 3) \\
E_{211} &= \frac{1}{4}W_{1234} + \frac{1}{12}W_{1243} + \frac{1}{12}W_{1324} + \frac{1}{12}W_{1342} + \frac{1}{12}W_{1423} - \frac{1}{12}W_{1432} \\
& + \frac{1}{12}W_{2134} - \frac{1}{12}W_{2143} + \frac{1}{12}W_{2314} + \frac{1}{12}W_{2341} + \frac{1}{12}W_{2413} - \frac{1}{12}W_{2431} \\
& + \frac{1}{12}W_{3124} - \frac{1}{12}W_{3142} - \frac{1}{12}W_{3214} - \frac{1}{12}W_{3241} + \frac{1}{12}W_{3412} - \frac{1}{12}W_{3421} \\
& + \frac{1}{12}W_{4123} - \frac{1}{12}W_{4132} - \frac{1}{12}W_{4213} - \frac{1}{12}W_{4231} - \frac{1}{12}W_{4312} - \frac{1}{4}W_{4321}
\end{aligned} \tag{A.2}$$

#### A.0.1. The Berends-Giele idempotents

The Berends-Giele idempotents  $\mathcal{E}(P)$  are defined in section 3.1 as the inverse  $\theta(E(P))$  of the Eulerian idempotent (2.21). Their expansions up to multiplicity three were given in (3.17) and now we write down the multiplicity four:

$$\begin{aligned}
\mathcal{E}(1234) &= \frac{1}{4}W_{1234} - \frac{1}{12}W_{1243} - \frac{1}{12}W_{1324} - \frac{1}{12}W_{1342} - \frac{1}{12}W_{1423} + \frac{1}{12}W_{1432} \\
& - \frac{1}{12}W_{2134} + \frac{1}{12}W_{2143} - \frac{1}{12}W_{2314} - \frac{1}{12}W_{2341} + \frac{1}{12}W_{2413} + \frac{1}{12}W_{2431} \\
& - \frac{1}{12}W_{3124} - \frac{1}{12}W_{3142} + \frac{1}{12}W_{3214} + \frac{1}{12}W_{3241} - \frac{1}{12}W_{3412} + \frac{1}{12}W_{3421} \\
& - \frac{1}{12}W_{4123} + \frac{1}{12}W_{4132} + \frac{1}{12}W_{4213} + \frac{1}{12}W_{4231} + \frac{1}{12}W_{4312} - \frac{1}{4}W_{4321}
\end{aligned} \tag{A.3}$$

As a curiosity, noting that  $E_4 = I_4$  one can derive these permutations using the conversion (2.29) together with (2.14) for the permutations in  $\theta(B_p)$  (note  $\theta^2 = 1$ ). So (2.30) yields the permutations in  $\mathcal{E}_{1234} = \theta(I_4)$  as

$$\begin{aligned}
\mathcal{E}(1234) &= -\frac{1}{4}1\sqcup 2\sqcup 3\sqcup 4 + \frac{1}{3}1\sqcup 2\sqcup 34 + \frac{1}{3}1\sqcup 23\sqcup 4 - \frac{1}{2}1\sqcup 234 \\
& + \frac{1}{3}12\sqcup 3\sqcup 4 - \frac{1}{2}12\sqcup 34 - \frac{1}{2}123\sqcup 4 + 1234.
\end{aligned} \tag{A.4}$$

As a multiplicity-four example of the inverse idempotent basis of (3.20) we have

$$\begin{aligned} \mathcal{I}_{22}(12,34) = & \frac{1}{4}W_{1234} - \frac{1}{4}W_{1243} + \frac{1}{4}W_{1324} + \frac{1}{4}W_{1342} - \frac{1}{4}W_{1423} - \frac{1}{4}W_{1432} \quad (\text{A.5}) \\ & - \frac{1}{4}W_{2134} + \frac{1}{4}W_{2143} - \frac{1}{4}W_{2314} - \frac{1}{4}W_{2341} + \frac{1}{4}W_{2413} + \frac{1}{4}W_{2431} \\ & + \frac{1}{4}W_{3124} + \frac{1}{4}W_{3142} - \frac{1}{4}W_{3214} - \frac{1}{4}W_{3241} + \frac{1}{4}W_{3412} - \frac{1}{4}W_{3421} \\ & - \frac{1}{4}W_{4123} - \frac{1}{4}W_{4132} + \frac{1}{4}W_{4213} + \frac{1}{4}W_{4231} - \frac{1}{4}W_{4312} + \frac{1}{4}W_{4321}. \end{aligned}$$

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