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Global Exponential Stability of a Neural Network for Inverse Variational Inequalities

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Abstract We investigate the convergent properties of a projected neural network for solving inverse variational inequalities. Under standard assumptions, we establish the exponential stability of the proposed neural network. A discrete version of the proposed neural network is considered, leading to a new projection method for solving inverse variational inequalities, for which we obtain the linear convergence. We illustrate the effectiveness of the proposed neural network and its explicit discretization by considering applications in the road pricing problem arising in transportation science. The results obtained in this paper provide a positive answer to a recent open question and improve several recent results in the literature.

Keywords Dynamic programming · Variational inequality · Neural network · Exponential stability · Road pricing problem

Mathematics Subject Classification (2000) 47J20 · 49J40 · 65P40

1 Introduction

Let Ω be a nonempty closed convex subset of a Euclidean space $\mathbb{R}^n$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator. The inverse variational inequality (IVI) is defined as: Find $x^* \in \mathbb{R}^n$ such that $F(x^*) \in \Omega$ and

$$(v - F(x^*))^T x^* \geq 0 \quad \forall v \in \Omega.$$

We denote the inverse variational inequality problem associated with F and Ω by IVI(F, Ω). IVI(F, Ω) could be transformed as a regular variational inequality [8, 9] if an inverse function F^{-1} of F exists. Nevertheless, it is not a trivial task to obtain the explicit form of the inverse function of F^{-1} in reality.

IVI has broad applications in various disciplines. He et al. [3] applied IVI to a market equilibrium problem in economics. In addition, a number of normative flow control problems, appearing in transportation, telecommunication networks and policy design problems, could be interpreted by IVI(F, Ω) [1, 4]. Algorithms for solving IVI(F, Ω) have been developed. He et al. [3] proposed a proximal point based algorithm for solving the IVIs that was applied to solve a bipartite market equilibrium problem therein. The authors in [4] investigated a projection-type method for solving IVIs. Existence of solution and alternating contraction projection methods were studied in [5].

Recently, Zou et al. [12] proposed a neural network (also known as a dynamical system in the mathematical literature) for solving IVIs. Under specific conditions, the authors established asymptotic stability and exponential stability of the proposed neural network. Unfortunately, as pointed out by Xu et al. in

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[11], there are certain fatal mistakes in [12], which cannot be fixed. Xu et al. also established some stability results when F is a symmetric gradient mapping, that is, $F = \nabla g$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a convex and continuously differentiable function. Moreover, the authors posted an open question on the exponential stability of the neural network for general operator F [11, Problem 1].

It is well-known that there is a close connection between neural network (dynamical system) and its time discretization. A time discretization of a neural network gives an algorithm for solving the corresponding problem and vice versa [2, 9, 10].

The aim of this paper is twofold: We first provide a positive answer to the open question posed in [11]. Then we consider a discretization of the proposed neural network, which leads to a relaxed projection method for solving IVI(F, Ω). The relaxed projection method is general enough so the method proposed in [4] can be regarded as a special case. Under standard assumptions, we prove that the iterative sequence generated by the new projection method converges linearly to a solution of IVI(F, Ω). In addition, we obtain global upper and lower error bounds, which allow one to have an estimate of the distance between an arbitrary vector to the solution set of IVI(F, Ω). The theoretical results are confirmed by some numerical experiments on an application to the road pricing problem in transportation science.

In Section 2, we recall some basic definitions and results as well as the neural network proposed in [12]. We establish the exponential stability of the neural network in Section 3, which solves the open problem posted in [11]. Section 4 describes the explicit discretization of the neural network and the linear convergence of the corresponding algorithm. Numerical illustrations are presented in Section 5.

2 Preliminaries

One often considers IVI(F, Ω) with some additional properties imposed on the operator F such as Lipschitz continuity, monotonicity and strong monotonicity of F. Let us recall some well-known definitions.

Definition 1 *[6] The operator F is said to be*

(a) strongly monotone with modulus γ if there exists $\gamma > 0$ such that

$$(F(x) - F(y))^T(x - y) \geq \gamma\|x - y\|^2 \quad \forall x, y \in \mathbb{R}^n;$$

(b) monotone if

$$(F(x) - F(y))^T(x - y) \geq 0 \quad \forall x, y \in \mathbb{R}^n;$$

(c) Lipschitz continuous with modulus L if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \forall x, y \in \mathbb{R}^n.$$

Remark 1 *If the operator F is strongly monotone with modulus γ and Lipschitz continuous with modulus L then it follows from the Cauchy-Schwarz inequality that*

$$\begin{aligned}\gamma\|x - y\|^2 &\leq (F(x) - F(y))^T(x - y)\\ &\leq \|F(x) - F(y)\|\|x - y\|\\ &\leq L\|x - y\|^2,\end{aligned}$$

which implies $\gamma \leq L$.

Remark 2 *If the operator F is strongly monotone and Lipschitz continuous, then IVI(F, Ω) has a unique solution [5, 11].*

Next we recall the metric projection. For each $x \in \mathbb{R}^n$, there exists a unique point in Ω, (see, e.g., [8]), denoted by $P_\Omega(x)$, such that

$$\|x - P_\Omega(x)\| \leq \|x - y\| \quad \forall y \in \Omega,$$

where $\|\cdot\|$ denotes the l_2-norm of $\mathbb{R}^n$. Some well-known properties of the metric projection $P_\Omega : \mathbb{R}^n \to \Omega$ are given in the following lemma.

Lemma 1 *[8] Assume that the set Ω is a closed convex subset of $\mathbb{R}^n$. Then*

(a) *$P_\Omega(.)$ is a nonexpansive operator, i.e., for all $x, y \in \mathbb{R}^n$, it holds*

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|.$$

(b) *For any* $x \in \mathbb{R}^n$ *and* $y \in \Omega$, *it holds*

$$(x - P_\Omega(x))^T(y - P_\Omega(x)) \leq 0.$$

Lemma 2 *[3] Let* $\alpha > 0$, *then* $x^* \in \mathbb{R}^n$ *is a solution of IVI*(F, Ω) *if and only if*

$$F(x^*) = P_\Omega(F(x^*) - \alpha x^*).$$

We consider the following projected neural network

$$\frac{dx}{dt} = \lambda\{P_\Omega(F(x) - \alpha x) - F(x)\}, \tag{1}$$

where $\lambda > 0$ and $\alpha > 0$ are two scaling factors.

We first recall some stability concepts of an equilibrium point of a neural network.

Definition 2 *[2, 10]*

(a) x^* *is an equilibrium point of a neural network* (1) *if*

$$P_\Omega(F(x^*) - \alpha x^*) = F(x^*),$$

i.e., x^* *is a solution of IVI*(F, Ω).

(b) *An equilibrium point* x^* *of* (1) *is stable if, for any* $\epsilon > 0$, *there is a* $\delta > 0$ *such that, for every* $x_0 \in B(x^*, \delta)$, *the solution* $x(t)$ *of the neural network* (1) *with* $x(0) = x_0$ *is defined and* $x(t) \in B(x^*, \epsilon)$ *for all* $t > 0$, *where* $B(x^*, r)$ *is the open ball with center* x^* *and radius* r.

(c) *A stable equilibrium point* x^* *is asymptotically stable if there is a* $\delta > 0$ *such that, for every solution* $x(t)$ *with* $x(0) \in B(x^*, \delta)$, *one has*

$$\lim_{t \to +\infty} x(t) = x^*.$$

(d) *An equilibrium point* x^* *is exponentially stable if there is a* $\delta > 0$ *and constants* $\mu > 0$ *and* $\eta > 0$ *such that, for every solution* $x(t)$ *with* $x(0) \in B(x^*, \delta)$, *one has*

$$\|x(t) - x^*\| \leq \mu \|x(0) - x^*\| e^{-\eta t} \quad \forall t \geq 0, \tag{2}$$

x^* *is globally exponentially stable if (2) holds true for all solutions* $x(t)$ *of the neural network.*

The neural network (1), also known as dynamical system in the literature, was proposed by Zou et al. in [12] for solving IVI(F, Ω). Under various conditions, the authors established the asymptotic stability and globally exponential stability of the projected neural network (1). We recall below some of their results.

Theorem 1 *[12, Theorem 4] Suppose that the operator* F *is L-Lipschitz continuous,* β*- strongly monotone and*

$$\alpha^2 + 1 + L^2 - (2\alpha + 1)\beta < 0. \tag{3}$$

Then the neural network (1) *is globally exponentially stable at a solution* u^*.

Theorem 2 *[12, Theorem 5] Suppose that the operator* F *is L-Lipschitz continuous,* β*- strongly monotone and*

$$\alpha + L - \beta < 0. \tag{4}$$

Then the neural network (1) *is globally exponentially stable at a solution* u^*.

As commented clearly in Xu et al. [11, Section 4], since $\beta \leq L$ (see Remark 1), condition (4) is is never satisfied in any circumstances and condition (3) can only hold if $\beta > 1$. Hence, the results obtained in [12] are incomplete and Xu et al. posted the following open question:

Problem 1: *Assume that* F *is L-Lipschitz and* β*-strongly monotone with* $0 < \beta \leq 1$ *and there exists a unique equilibrium point* x^* *of the neural network* (1). *Is* x^* *globally exponentially stable?*

In Section 3, we will provide a positive answer for this question.

3 Global Exponential Stability

We are now in a position to establish the globally exponential stability of neural network (1) without restrictive condition (3) as imposed in [12].

Theorem 3 *Assume that F is L-Lipschitz and β-strongly monotone and $\alpha > \frac{L^2}{4\beta}$. Then the unique equilibrium point x^* of neural network* (1) *is globally exponentially stable.*

Proof: Note that under assumptions made, it follows from Remark 2 that IVI(F, Ω) has a unique solution x^*, which is also the equilibrium point of the neural network (1).
Let $x \in \mathbb{R}^n$, denoting $y := P_\Omega(F(x) - \alpha x) \in \Omega$, we have

$$(y - F(x^*))^T x^* \geq 0,$$

hence

$$(y - F(x^*))^T (\alpha x^*) \geq 0. \tag{5}$$

On the other hand, using Lemma 1(b) we have

$$(F(x) - \alpha x - y)^T (z - y) \leq 0 \quad \forall z \in \Omega.$$

Substituting $z = F(x^*) \in \Omega$ into the latter inequality yields

$$(F(x) - \alpha x - y)^T (y - F(x^*)) \geq 0.$$

Combining the last inequality with (5) we obtain

$$(F(x) - y - \alpha(x - x^*))^T (y - F(x^*)) \geq 0,$$

or equivalently

$$(F(x) - y - \alpha(x - x^*))^T (F(x) - F(x^*) + y - F(x)) \geq 0.$$

Therefore

$$\alpha(x - x^*)^T (y - F(x)) \leq -\|y - F(x)\|^2 - \alpha(x - x^*)^T (F(x) - F(x^*)) + (F(x) - y)^T (F(x) - F(x^*)).$$

Using the strong monotonicity, the Lipschitz continuity of F and Cauchy Schwarz inequality, we obtain from the last inequality that

$$\begin{aligned} \alpha(x - x^*)^T (y - F(x)) &\leq -\|y - F(x)\|^2 - \alpha\beta\|x - x^*\|^2 + \|F(x) - y\|\|F(x) - F(x^*)\| \\ &\leq -\|y - F(x)\|^2 - \alpha\beta\|x - x^*\|^2 + L\|F(x) - y\|\|x - x^*\| \\ &\leq -\|y - F(x)\|^2 - \alpha\beta\|x - x^*\|^2 + \frac{L^2}{4}\|x - x^*\|^2 + \|F(x) - y\|^2 \\ &= -\left(\alpha\beta - \frac{L^2}{4}\right)\|x - x^*\|^2. \end{aligned} \tag{6}$$

Hence

$$(x - x^*)^T (y - F(x)) \leq -\left(\beta - \frac{L^2}{4\alpha}\right)\|x - x^*\|^2. \tag{7}$$

Consider the Lyapunov function

$$V = \frac{1}{2}\|x(t) - x^*\|^2 \quad \forall x(t) \in \Omega.$$

From (1) and (7), the time derivative of V can be expressed as

$$\begin{aligned} \frac{dV}{dt} &= (x - x^*)^T \frac{dx}{dt} \\ &= \lambda\left((x - x^*)^T (y - F(x))\right) \\ &\leq -\lambda\left(\beta - \frac{L^2}{4\alpha}\right)\|x - x^*\|^2 \\ &= -\gamma\|x - x^*\|^2, \end{aligned}$$

where

$$\gamma := \lambda \left(\beta - \frac{L^2}{4\alpha} \right) > 0.$$

Therefore

$$\|x(t) - x^*\| \leq \|x(0) - x^*\| e^{-\gamma t}.$$

This means that the equilibrium solution x^* of the neural network (1) is globally exponentially stable. □

As a result of Theorem 3, we obtain the following important global error bound. Given an arbitrary trajectory $x \in \mathbb{R}^n$, this error bound measures how close x is to the unique solution x^* of the neural network depending solely on x and the data of (1). This error bound can also be used to construct practical stopping rules for numerical method so that the final iterate will satisfy any prescribed level of accuracy.

Corollary 1 *Assume that F is L-Lipschitz and β-strongly monotone and $\alpha > \frac{L^2}{4\beta}$. Let x^* be the unique solution of neural network* (1)*. Then, for every arbitrary vector $x \in \mathbb{R}^n$, it holds that*

$$\|x - x^*\| \leq \frac{4\alpha}{4\alpha\beta - L^2} \|F(x) - P_\Omega(F(x) - \alpha x)\|. \tag{8}$$

Proof: It follows from (7) and the Cauchy Schwarz inequality that

$$\begin{aligned} \left(\beta - \frac{L^2}{4\alpha} \right) \|x - x^*\|^2 &\leq -(x - x^*)^T (y - F(x)) \\ &\leq \|x - x^*\| \|y - F(x)\| \\ &= \|x - x^*\| \|F(x) - P_\Omega(F(x) - \alpha x)\|, \end{aligned}$$

which implies (8). □

4 Explicit Discretization

The explicit discretization of neural network (1) with respect to the time variable t, with step size $h_n > 0$ and initial point $x_0 \in \mathbb{R}^n$, yields the following iterative scheme:

$$\frac{x_{n+1} - x_n}{h_n} = \lambda \left(P_\Omega(F(x_n) - \alpha x_n) - F(x_n) \right)$$

or equivalently

$$x_{n+1} = x_n - \lambda h_n F(x_n) + \lambda h_n P_\Omega(F(x_n) - \alpha x_n).$$

Setting $\lambda_n = \lambda h_n$, we obtain the following algorithm for solving IVI(F, Ω).

$$x_{n+1} = x_n - \lambda_n F(x_n) + \lambda_n P_\Omega(F(x_n) - \alpha x_n). \tag{9}$$

When $\lambda_n = \frac{1}{\alpha}$ for all n, algorithm (9) is nothing but the projection method considered in [4] when F is co-coercive on Ω, i.e., there exists $\mu > 0$ such that

$$(F(x) - F(y))^T (x - y) \geq \mu \|F(x) - F(y)\|^2 \quad \forall x, y \in \Omega.$$

We will prove that when F is strongly monotone and Lipschitz continuous, the sequence $\{x_n\}$ generated by (9) converges linearly to a solution of IVI(F, Ω). To do so, we need the following estimate.

Lemma 3 *Assume that F is L-Lipschitz and β-strongly monotone and $\alpha > \frac{L^2}{4\beta}$. Let $y := P_\Omega(F(x) - \alpha x)$ then*

$$\|F(x) - y\|^2 \leq -\frac{4\alpha^2\beta}{4\alpha\beta - L^2} (x - x^*)^T (y - F(x)). \tag{10}$$

Proof: It follows from (6) that

$$\begin{aligned}\alpha(x-x^*)^T(y-F(x)) &\le -\|y-F(x)\|^2-\alpha\beta\|x-x^*\|^2+L\|F(x)-y\|\|x-x^*\|\\ &\le -\|y-F(x)\|^2-\alpha\beta\|x-x^*\|^2+\frac{L^2}{4\alpha\beta}\|y-F(x)\|^2+\alpha\beta\|x-x^*\|^2\\ &= -\left(1-\frac{L^2}{4\alpha\beta}\right)\|y-F(x)\|^2,\end{aligned}$$

which implies (10). □

As a consequence of Lemma 3, we have the following global lower error bound.

Corollary 2 *Assume that F is L-Lipschitz and β-strongly monotone and $\alpha > \frac{L^2}{4\beta}$. Let x^* be the unique solution of neural network* (1). *Then, for every arbitrary vector $x \in \mathbb{R}^n$, it holds that*

$$\frac{4\alpha\beta-L^2}{4\alpha^2\beta}\|F(x)-P_\Omega(F(x)-\alpha x)\| \le \|x-x^*\|.$$

Proof: Follows directly from (10) and the Cauchy Schwarz inequality. □

The convergence analysis of (9) is stated in Theorem 4 below. For simplicity of the analysis, we fix $\lambda_n = \lambda$ for all n.

Theorem 4 *Assume that F is L-Lipschitz and β-strongly monotone. Assume also that*

$$\alpha > \frac{L^2}{4\beta} \quad \text{and} \quad \lambda < \frac{4\alpha\beta-L^2}{2\alpha^2\beta}. \tag{11}$$

Then the sequence $\{x_n\}$ generated by

$$x_{n+1} = x_n - \lambda F(x_n) + \lambda P_\Omega(F(x_n)-\alpha x_n) \tag{12}$$

converges linearly to the unique solution x^* of IVI(F,Ω).

Proof: Setting $y_n = P_\Omega(F(x_n)-\alpha x_n)$, we have from (10) and (7) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(x_n-x^*)+\lambda(y_n-F(x_n))\|^2\\ &= \|x_n-x^*\|^2+2\lambda(x_n-x^*)^T(y_n-F(x_n))+\lambda^2\|y_n-F(x_n)\|^2\\ &\le \|x_n-x^*\|^2+\lambda\left(2-\frac{4\alpha^2\beta\lambda}{4\alpha\beta-L^2}\right)(x_n-x^*)^T(y_n-F(x_n))\\ &\le \left[1-2\lambda\left(1-\frac{2\alpha^2\beta\lambda}{4\alpha\beta-L^2}\right)\left(\beta-\frac{L^2}{4\alpha}\right)\right]\|x_n-x^*\|^2.\end{aligned}$$

Hence

$$\|x_{n+1}-x^*\| \le q\|x_n-x^*\|,$$

where

$$q = \sqrt{1-2\lambda\left(1-\frac{2\alpha^2\beta\lambda}{4\alpha\beta-L^2}\right)\left(\beta-\frac{L^2}{4\alpha}\right)} \in (0,1),$$

which means that $\{x_n\}$ converges linearly to x^*. □

Remark 3 *Let us choose $\lambda = \frac{1}{\alpha}$, then* (12) *becomes the following projection method studied in [4].*

$$x_{n+1} = x_n - \frac{1}{\alpha}F(x_n) + \frac{1}{\alpha}P_\Omega(F(x_n)-\alpha x_n).$$

In this case, condition (11) *reduces to $\alpha > \frac{L^2}{2\beta}$ and the linear convergence is guaranteed as*

$$\|x_{n+1}-x^*\| \le \sqrt{1-\frac{2\alpha\beta-L^2}{2\alpha^2}}\|x_n-x^*\|.$$

Observe that the rate $q = \sqrt{1-\frac{2\alpha\beta-L^2}{2\alpha^2}}$ attains the minimum value when $f(\alpha) := \frac{2\alpha^2}{2\alpha\beta-L^2}$ attains smallest value on $(0,+\infty)$. It is not difficult to check that this is occurred when $\alpha = \alpha^* = \frac{L^2}{\beta}$ and the best value of q is $q = q^* = \sqrt{1-\frac{\beta^2}{2L^2}}$.

5 Numerical Illustrations

This section presents a numerical example, adopted from [4], to illustrate the effectiveness of the proposed projected neural network algorithm (9). Consider a continuous-time road pricing problem, where a traffic management authority seeks to manage the vehicular flows f_i by imposing link tolls x_i on a set of links $i \in \mathcal{L}$. The imposed tolls are expected to leverage the link flows to predetermined intervals denoted by $\Omega = \{f_i | a_i \leq f_i(x) \leq b_i, i \in \mathcal{L}\}$. These constraints are designated such that the resulting traffic pattern could be maintained at a desired level, for instance, close to a system optimum traffic pattern with emission constraints.

Same as [4], we assume that traffic flows follow user's equilibrium (UE) and the link performance function is strongly monotone. The continuous-time road pricing problem faced by the authority can be interpreted as: Find $x_i(t)$, such that $\lim_{t\to\infty} f(x(t)) \in \Omega$. Following the discussion in [4], we can derive the Lagrangian function as: $L(f, \gamma, \mu) = \sum_{i\in\mathcal{L}}[\gamma_i(b_i - f_i(x)) + \mu_i(f_i(x) - a_i)]$ with multiplier vectors γ_i and μ_i. The equilibrium traffic flows f_i^* satisfy the KKT conditions:

$$\begin{cases} \lambda_i^* \geq 0; & b_i - f_i^* \geq 0; \quad \lambda_i^*(b_i - f_i^*) = 0 \qquad \forall i \in \mathcal{L}, \\ \mu_i^* \geq 0; & f_i^* - a_i \geq 0; \quad \mu_i^*(f_i^* - a_i) = 0 \qquad \forall i \in \mathcal{L}. \end{cases} \tag{13}$$

As $\lambda_i > 0$ and $\mu_i > 0$ are mutually exclusive, the control variable (i.e., toll) can be defined as $x_i = \lambda_i - \mu_i$ for each link $i \in \mathcal{L}$. By the KKT condition (13) and the definition x_i, if $f_i^* = b_i$ then $x_i^* \geq 0$; if $f_i^* = a_i$ then $x_i^* \leq 0$; if $a_i < f_i^* < b_i$ then $x_i^* = 0$. Therefore, the equilibrium traffic flows and link tolls can be regarded as an IVI problem that is to find an optimal toll pattern x^*, such that the responsive link flows $f(x^*) \in \Omega$ and

$$(f(x) - f(x^*))^T x^* \leq 0, \qquad \forall f(x) \in \Omega.$$

Note that link flows $f_i(x)$ are implicit functions of link tolls $x = \{x_i\}$, which can be observed from the field or obtained by solving a traffic assignment problem. In order to manage the flows to converge to the preferred pattern Ω, the continuous-time dynamic tolls $x_i(t)$ can be designed by following the projected neural network (1).

Let us consider a network shown in Figure 1, consisting of eight nodes and sixteen links connecting sixteen origin-destination (OD) pairs. Links 1, 2, 3, and 4 represent four bridges over a river, connecting four origins, O_1 to O_4, on one side to four destinations, D_1 to D_4, on the other side of the river. The authority plans to manage the volume/capacity ratios of three main bridges (links 1, 2 and 3) by imposing tolls on them. The goal is to maintain the traffic volumes on the three bridges lying in the bounds $100 \leq x_1 \leq 150$, $50 \leq x_2 \leq 100$, and $250 \leq x_3 \leq 300$.

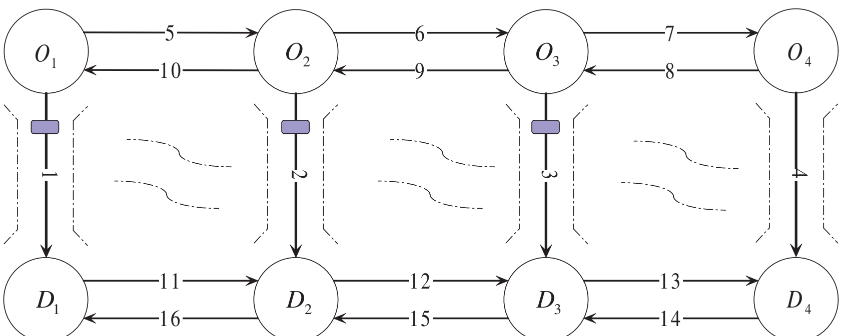

Fig. 1: A four-bridge road pricing problem

As shown by the projected neural network (1), the dynamics of tolls depend on the value of the resulting link flows $f_i(x)$, which are determined by solving a fixed-demand user equilibrium traffic assignment in this study. In the traffic assignment, the OD demands are given in Table 1; the free flow travel times and the link capacities are provided in Table 2; and the link travel time t_i on link i follows the Bureau of Public Roads (BPR) function as:

$$t_i(f_i) = t_i^0 \left[1 + 0.15 \left(\frac{f_i}{c_i}\right)^4\right],$$

Table 1: Origin-destination demand table

Demand	O_1	O_2	O_3	O_4
D_1	60	30	20	15
D_2	50	160	45	30
D_3	20	30	20	10
D_4	20	15	15	40

Table 2: Link free flow travel time and capacity

Link i	1	2	3	4	5	6	7	8	9	10	11
t_i^0	60	40	60	40	20	20	20	20	20	20	20
c_i	150	100	300	200	300	300	300	300	300	300	300

where f_i, t_i^0, and c_i denote link flow, free flow travel time, and capacity on link i, respectively. Excluding a small neighborhood of zero, the BPR function satisfies the strongly monotone property. Thus, we can apply the proposed projected neural network algorithm (9) to solve the dynamic road pricing problem.

The numerical study was focused on investigating the effects of scaling factors λ and α on the stability of the dynamical system. We implemented the explicit discretization of the neural network presented in Section 4 where scaling factor λ is associated with the time step size h. In the first testing scenario, we set $\alpha = 50$ and $\lambda_n = \frac{1}{\alpha} = 0.02, \forall n$. Under this setting, the discretization algorithm (9) is the projection method proposed in [4] as discussed in Section 4. Denote $r_n = \|\lambda_n(F(x_n) - P_\Omega(F(x_n) - \alpha x_n))\|$ as the residents of the projected neural network (1). Clearly, the system is at equilibrium if $r_n = 0$. Therefore, r_n can be used to illustrate convergence rate or as a proxy of asymptotic stability (8). Figure 2 illustrates the evolution of the traffic flows on the three bridges with toll. In the first 20 time steps, traffic flows on bridges 1 and 3 converge quickly to their lower bounds of desired flow control intervals. After 100 time steps, the traffic flow on bridge 2 also converges to the upper bound of the desired level. The evolution of the residue, shown on the right side of the figure, demonstrates the linear convergence in Theorem 4, given the values of α and λ.

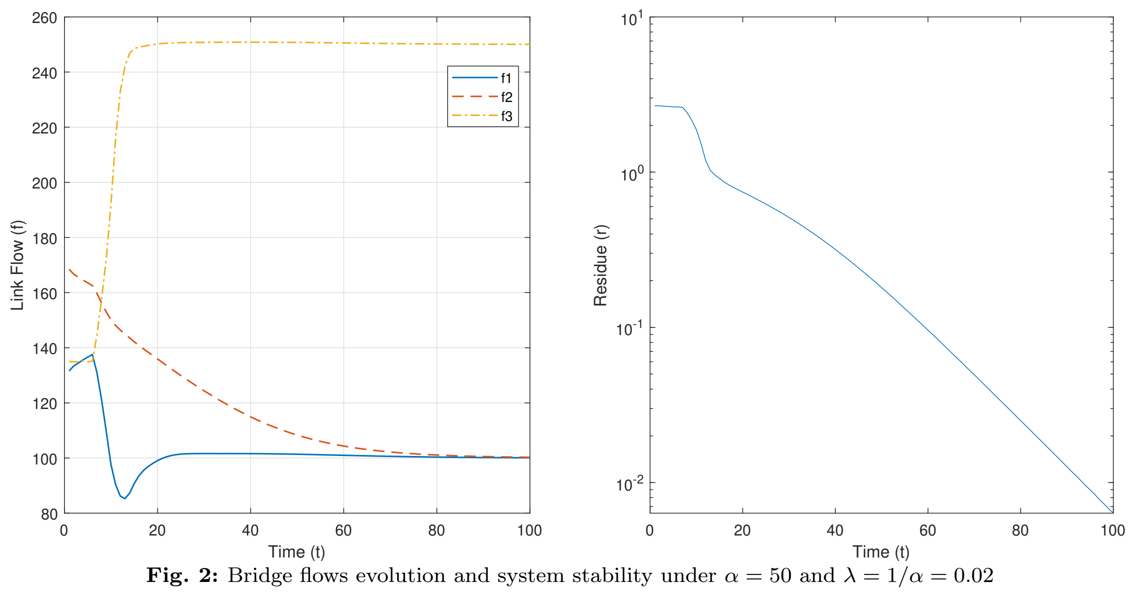

Fig. 2: Bridge flows evolution and system stability under $\alpha = 50$ and $\lambda = 1/\alpha = 0.02$

Note that the value of scaling factor λ affects the convergence performance of the explicit discretization of neural networks. If $\lambda = 0.01$ while $\alpha = 50$, the traffic flows evolve slower toward the equilibrium state. Comparing the traffic flows shown in Figure 3 to those in Figure 2, we can observe that the traffic flow on bridge 2 is quite far away from the desired level after 100 time steps, resulting in a larger value of residue.

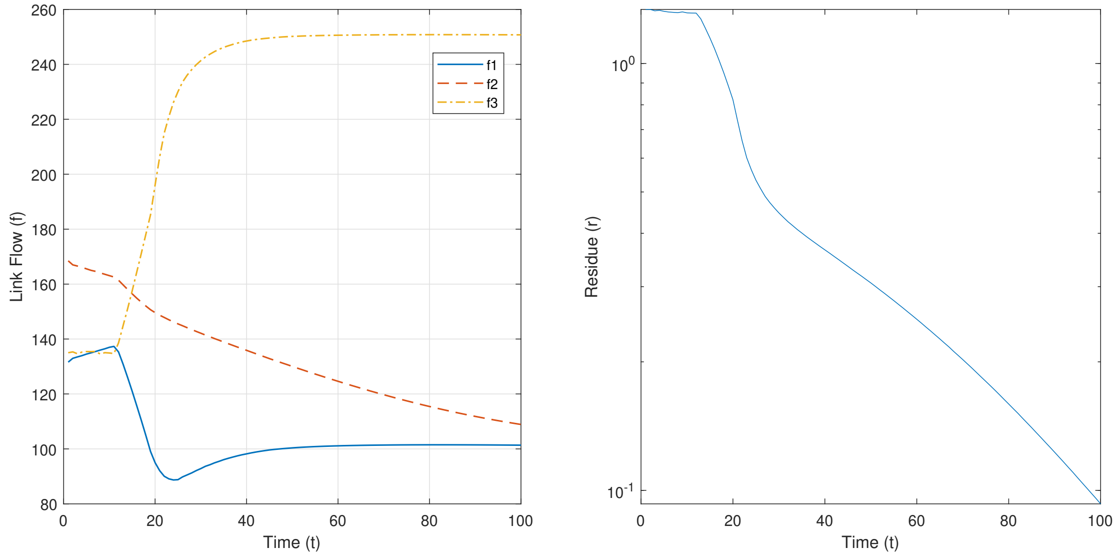

Fig. 3: Bridge flows evolution and system stability under $\alpha = 50$ and $\lambda = 0.01$

However, if we increase $\lambda = 0.03$, the larger value of scaling factor λ will drive the system to converge quickly toward equilibrium. As illustrated by Figure 4, the traffic flows on the three bridges converge close to the desired levels in 60 time steps. The residue is reduced to 0.01 at about 60 time steps.

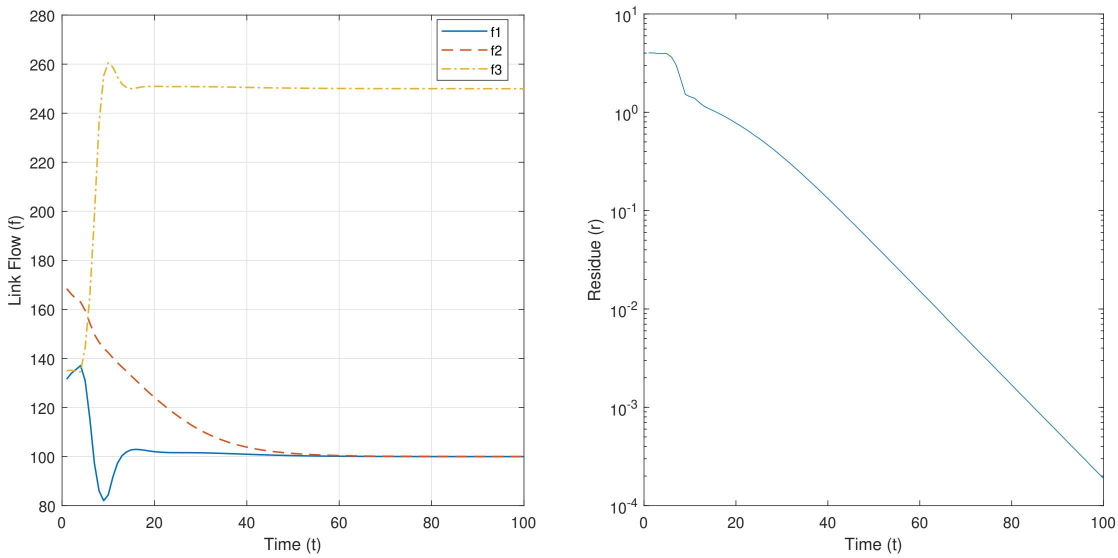

Fig. 4: Bridge flows evolution and system stability under $\alpha = 50$ and $\lambda = 0.03$

We further investigate the effect of the other scaling factor α. Consider the case that $\alpha = 10$ and $\lambda = 0.01$. As illustrated by Figure 5, the traffic flows converge to the equilibrium with a pattern similar to that shown in Figure 3. It seems that the system convergence is less sensitive to α than to λ.

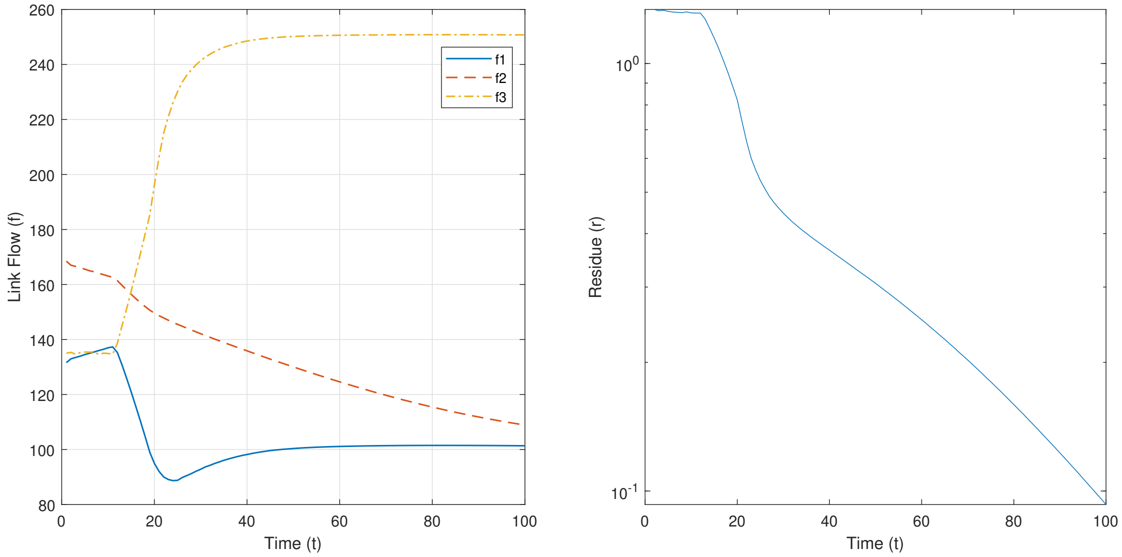

Fig. 5: Bridge flows evolution and system stability under $\alpha = 10$ and $\lambda = 0.01$

If we increase $\lambda = 0.03$ while keeping $\alpha = 10$, the system converges to equilibrium as shown in Figure 6. The convergence performance is similar to the case in Figure 4.

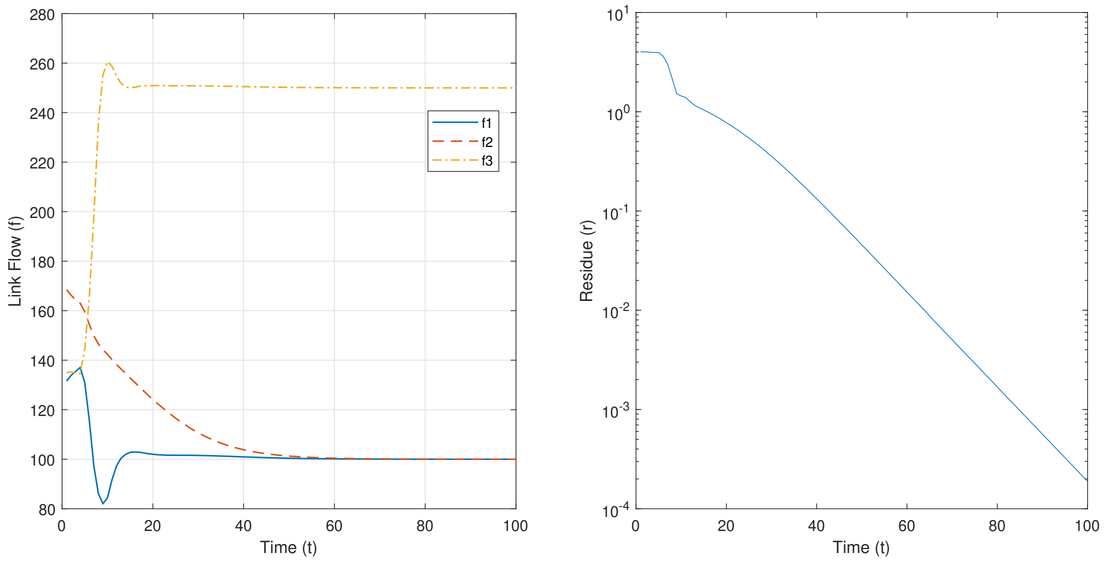

Fig. 6: Bridge flows evolution and system stability under $\alpha = 10$ and $\lambda = 0.03$

However, if we keep increasing $\lambda = 1/\alpha = 0.1$, the system cannot converge as shown in Figure 7. In this case, λ violates the condition in Theorem 4, causing an unstable system evolution.

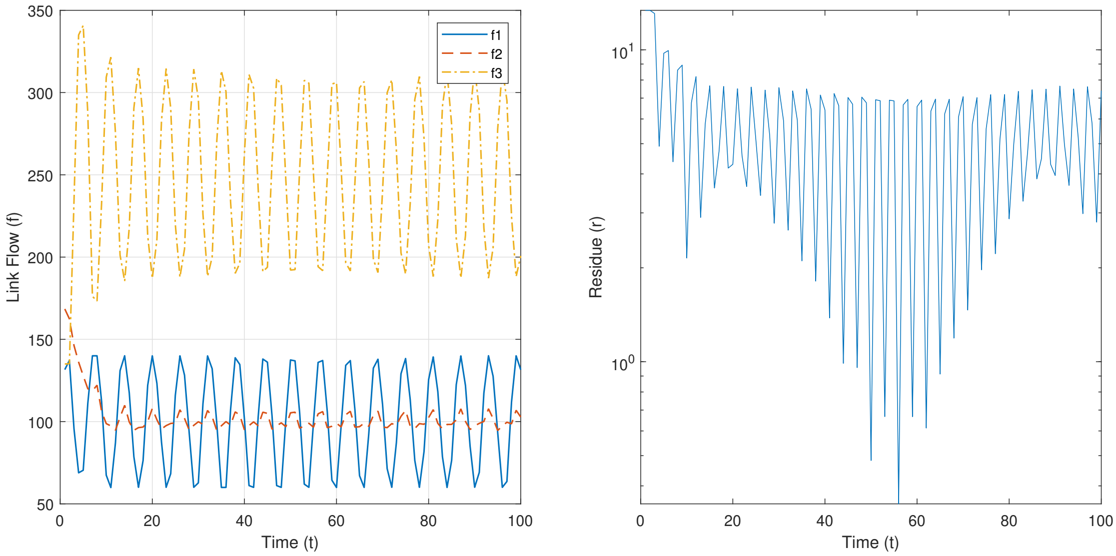

Fig. 7: Bridge flows evolution and system instability under $\alpha = 10$ and $\lambda = 0.1$

The illustrations so far emphasize the importance of scaling factors α and λ on system stability. They are expected to satisfy conditions 11, which depend on the strong monotonicity and Lipschitz modulus of F, i.e., β and L. However, it is common in practice that mapping F does not have an explicit form, for example, representing a traffic assignment problem herein. The implicit form prevents an explicit derivation of appropriate α and λ. Identifying the global values β and L on Ω remains a costly computational task that requires further analysis.

6 Conclusion

We have provided a positive answer to a recent question on neural networks for solving IVI problem. In addition, we established new exponential stability and linear convergence results as well as global error bounds. Numerical experiments in real life application problems have been performed confirming the theoretical results obtained. Observed that all the results presented in this paper also hold in infinite dimensional Hilbert spaces without any additional assumption. It is interesting to investigate whether the strong monotonicity condition could be relaxed to strong pseudo monotonicity as in the regular variational inequality problems [2, 7] or not.

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