UNIVERSITY OF SOUTHAMPTON
FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES
DEPARTMENT OF MATHEMATICS

Universal q-gonal Tessellations

by

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ABSTRACT

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The universal triangular map is the Farey map $\hat{\mathscr{M}}_3$ as shown by David Singerman in 1988. The orientation-preserving automorphism group of $\hat{\mathscr{M}}_3$ is the classical modular group $\Gamma = PSL(2, \mathbb{Z})$ and $\hat{\mathscr{M}}_3$ is universal in the sense that every triangular map on an orientable surface is a quotient of $\hat{\mathscr{M}}_3$ by a subgroup of Γ. In this thesis we describe tessellations $\hat{\mathscr{M}}_q$ of the upper-half complex plane $\mathbb{H}$ which are universal for q-gonal maps. The orientation-preserving automorphism group of $\hat{\mathscr{M}}_q$ is the Hecke groups H_q and we show that every q-gonal map on an orientable surface is of the form $\hat{\mathscr{M}}_q/H$ where H is a subgroup of Hecke groups H_q.

Chapter 1 is devoted to a brief outline of map theory. We define algebraic maps and topological maps, explaining the connections between them.

Chapter 2 is devoted to discussing the modular group and Hecke groups, and describing their fundamental regions.

Chapter 3 is devoted to describing the Farey map and the universal q-gonal tessellations $\hat{\mathscr{M}}_q$ and showing that $\hat{\mathscr{M}}_q$ is universal, in the sense that every q-gonal map on an orientable surface is a quotient of $\hat{\mathscr{M}}_q$ by a subgroup of H_q.

Chapter 4 is devoted to discussing the principal congruence subgroups of the Hecke groups H_q and the quotients of $\hat{\mathscr{M}}_q$ by these subgroups. An important result gives the index of these subgroups in the Hecke groups in the cases $q = 4$ and 6, a result given previously by Parsons with a different proof. We then discuss many of these maps for $q = 4$ and 6, and also study the combinatorics and geometry of these maps, including the graphical distance, diameter, stars and poles particularly nice example is a quotient of $\hat{\mathscr{M}}_4$ corresponding to Bring's curve.

Chapter 5 is devoted to considering the Petrie paths for $\hat{\mathscr{M}}_q$. These project to Petrie polygons on the quotient maps and we relate the sizes of the Petrie polygons on these maps to the period of the Hecke-Fibonacci sequence modulo n.

Contents

Academic thesis: Declaration of authorship

I, **Doha A. Kattan,** declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Universal q-gonal Tessellations

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Either none of this work has been published before submission, or parts of this work have been published as:

Signed: ....

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Acknowledgements

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Dedication

Completion of any challenging work demands determination, self-sacrifice and support, especially from those who are closest to our hearts. My humble efforts are dedicated to the most precious people in my life! My family! They have suffered much to stay with me, to provide me with comfort, and to create a calm environment in which for me to complete my studies.

I extend my apologies and love to my children, Raghad, Lamar, Meral, Layan and Abdulelah because you have borne so much to allow me to achieve my life's goal. My studies have required you to make sacrifices too. I know I have been very preoccupied with my PhD, but I always trusted that you would be proud of my achievement one day. I deeply appreciate your help and support through the many challenging days I experienced throughout this journey. I wish to thank you all, and hope that I now make you proud of your super mum!

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For me, always, family comes first!

Thanks be to God for this Jewel, which is my life!

Chapter 1

Maps

The majority of the content in this section is drawn from [Jon81] and [JS78].

1.1 Topological maps

A topological map $\mathscr{M}$ is a connected graph $\mathscr{G}$ imbedded (without crossing) in a surface $\mathscr{S}$, which we assume is connected, oriented and without a boundary, where each face of $\mathscr{M}$ (as a connected component of $\mathscr{S} \setminus \mathscr{G}$) is homeomorphic to an open disc. Simply put, it represents a decomposition of $\mathscr{S}$ into simply-connected polygonal cells, called faces (Platonic solids embedded into the Riemann sphere Σ are examples of topological maps).

If a topological map has an underlying surface $\mathscr{S}$, and an associated graph $\mathscr{G}$, with a vertex set $\mathscr{V}$, then that map is represented with a triple $(\mathscr{G}, \mathscr{V}, \mathscr{S})$. Then $\mathscr{S}$ is homeomorphic to surface $\mathscr{S}_\gamma$ which comprises of a sphere with γ handles attached, for some integers $\gamma \geq 0$; we refer to γ as the *genus* of $\mathscr{M}$.

Whenever an edge of $\mathscr{M}$ intersects a vertex, we draw an arrow at the edge facing that vertex, as shown in Figure 1.1; we call every such vertex-edge pair a *dart* of $\mathscr{M}$. Usually each edge of $\mathscr{M}$ carries two darts corresponding to the two vertices on the edge, but it is convenient to allow edges that carry just one dart as in Figure 1.1 (i.e. edges for which only one end is incident with a vertex) and we call them *free edges*.

Let Ω be the set of all the darts formed from $\mathscr{M}$ in this way. Thus, a topological map $\mathscr{M}$ is composed of vertices, edges, faces and darts.

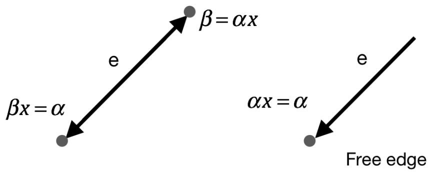
Throughout this chapter the symbol $\mathscr{M}$ will denote a topological map.

1.2 Algebraic maps

From a topological map we can describe how to obtain a permutation group. If Ω denotes the set of all darts in $\mathscr{M}$, we can then define three permutations x, y, z of Ω as follows:

- If an edge of $\mathscr{M}$ carries two darts, α and β we define a permutation x of order 2 of Ω by $\alpha x = \beta$ and hence $\beta x = \alpha$. We write our maps on the right, so αx results from applying x to α. However for a free edge α we must have $\alpha x = \alpha$. We do this so that we can always construct a topological map from an algebraic map, even if the two cycle has fixed points (see below).
  The permutation x is a product of two-cycles and one-cycles (fixed points), so $x^2 = 1$.

FIGURE 1.1



- The permutation y cyclically permutes the darts directed towards each vertex v in an anti-clockwise direction. There will therefore be one cycle of y for each vertex $v \in \mathscr{M}$; thus, y will be a product of k cycles, where k is the number of vertices of $\mathscr{M}$. The order m of y is the least common multiple m of the valencies of the vertex , so $y^m = 1$. (See Figure 1.2)

- A further permutation $z = y^{-1}x$ of Ω, that permutes the darts around every face of $\mathscr{M}$, whereby the composition is taken from left to right, so $xyz = x^2 = 1$. (See Figure 1.2)

Thus

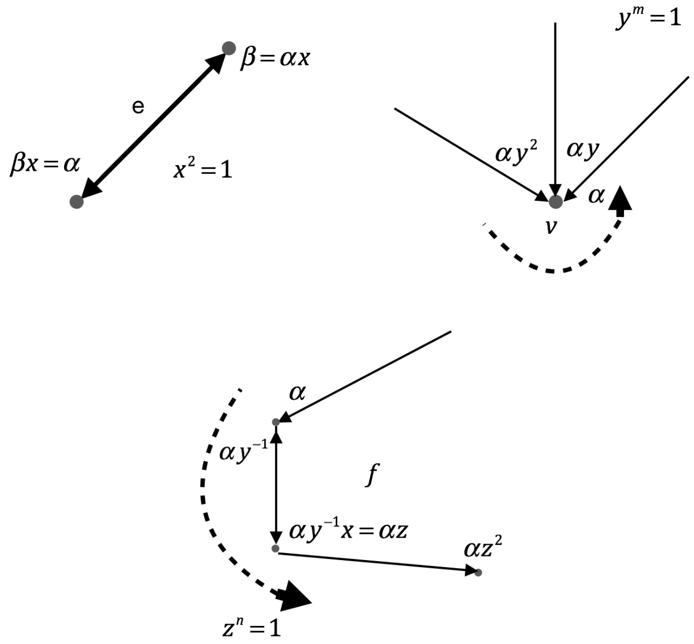
$$x^2 = y^m = z^n = xyz = 1$$

where n is the least common multiple of the face valencies, where the valency of a face is determined by the number of sides of the face. We say that $\mathscr{M}$ has type (m, n) if $m, n \in \mathbb{N} \cup \{\infty\}$ are the orders of y and z respectively. Also we say $\mathscr{M}$ has finite type if m and n are finite, and we say $\mathscr{M}$ is finite if Ω is finite; thus every finite map has finite type. (in some works, e.g Coxeter and Moser [CM57, Chapter 4] m is the least common multiple of the face valencies, and n is the least common multiple of the vertex valencies). It is convenient to allow one or both of m or n to be ∞. For example if

$m = \infty$ in $\mathscr{M}$ then we don not care about the vertex valencies as the relation $y^{\infty} = 1$ is regarded as being vacuous, hence

$$x^2 = z^n = xyz = 1.$$

FIGURE 1.2: Permutations



Note that,

$$cycles\, of\, x \leftrightarrow edges\, of\, \mathscr{M}$$

$$cycles\, of\, y \leftrightarrow vertices\, of\, \mathscr{M}$$

$$cycles\, of\, z \leftrightarrow faces\, of\, \mathscr{M}$$

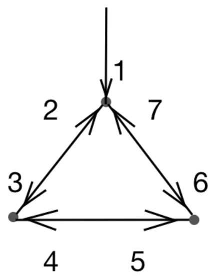
Let G be the subgroup generated by x, y and z (or by any combination of two of these, since $xyz = 1$) in the group S^{Ω} of all the permutations of Ω. Since $\mathscr{G}$ is connected, G is transitive.

Definition 1.1. An algebraic map $\mathscr{A}$ is a quadruple (G, Ω, x, y), where Ω is a set, and x, y are permutations of Ω; such that:
$x^2 = 1$ (the identity on Ω) and $G = \langle x, y \rangle$ is transitive on Ω.

We define $\mathscr{A}$ to have type (m, n) if y and $z = y^{-1}x$ have orders $m, n \in \mathbb{N} \cup \{\infty\}$, $\mathscr{A}$ has finite type if $m, n \in \mathbb{N}$, and is finite if Ω is finite.

Example 1.2. Viewing Figure 1.3 consider the set of darts $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$ with

FIGURE 1.3



the following two permutations in S^7:

$$x = (1)(2\ 3)(4\ 5)(6\ 7),$$

$$y = (7\ 1\ 2)(4\ 3)(6\ 5),$$

then

$$z = y^{-1}x = (1\ 6\ 4\ 2)(3\ 5\ 7).$$

The map has type $(6, 12)$, since

$$x^2 = y^6 = z^{12} = 1.$$

(Here 1 is a free edge).

We define an associated algebraic map $\mathscr{A} = (G, \Omega, x, y)$ of a topological map $\mathscr{M}$ to be $Alg\mathscr{M}$ where $G = \langle x, y\rangle$ is a transitive permutation group, and Ω is a set of darts as defined above.

1.3 Triangle groups

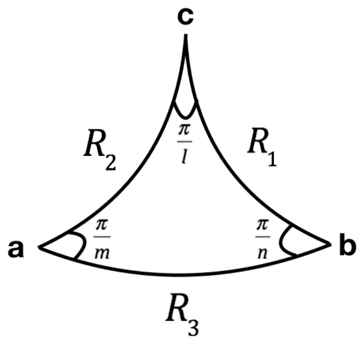
Let l, m, n be the integers $\geqslant 2$, and let T be a triangle with the angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$. If

$$\begin{cases} \frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1, \textit{then } T \textit{ is a Euclidean triangle} \\ \frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1, \textit{then } T \textit{ is a spherical triangle} \\ \frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1, \textit{then } T \textit{ is a hyperbolic triangle} \end{cases}$$

Let R_1, R_2, R_3 be the reflections on the sides of the T as in Figure 1.4.
Let $\Gamma^*(l, m, n)$ be the group generated by the reflections R_1, R_2, R_3. It has a presentation, as in the notation of Coxeter and Moser [CM57, Section 4.3],

$$\langle R_1, R_2, R_3 \mid R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^l = (R_2R_3)^m = (R_3R_1)^n = 1\rangle.$$

FIGURE 1.4: Hyperbolic Triangle



Let $\Gamma(l,m,n)$ denote the subgroup of index 2 in $\Gamma^*(l,m,n)$ consisting of transformations designed to preserve orientation. This group is generated by the rotations;

$$X = R_1R_2,\ Y = R_2R_3,\ Z = R_3R_1$$

and has the presentation

$$\langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1\rangle,$$

where X, Y and Z are rotations. $\Gamma(l,m,n)$ is called a *triangle group* and $\Gamma^*(l,m,n)$ is called an *extended triangle group.*

1.4 $\hat{\mathscr{M}}(m,n)$ and $\hat{\mathscr{A}}$

The word "tessellation" usually refers to some of the well-known regular maps on the Euclidean or the hyperbolic plane.
The universal topological map $\hat{\mathscr{M}}(m,n)$ is the tessellation of one of the three simply connected Riemann surfaces $\mathscr{U}$, that is

$$\mathscr{U} = \begin{cases} \Sigma = \mathbb{C}\cup\{\infty\} & (\textit{Riemann sphere}) \quad if\ \frac{1}{m}+\frac{1}{n} > \frac{1}{2}, \\ \mathbb{C} & (\textit{complex plane}) \quad if\ \frac{1}{m}+\frac{1}{n} = \frac{1}{2}, \\ \mathbb{H} & (\textit{hyperbolic plane}) \quad if\ \frac{1}{m}+\frac{1}{n} < \frac{1}{2}. \end{cases}$$

by regular n-gons with m meet at each vertex.

- In the first case, $\frac{1}{m}+\frac{1}{n} > \frac{1}{2}$ so if $m, n \geq 2$ then $(m,n) = (3,5), (3,4), (3,3), (5,3), (4,3), (2,n), (m,2)$. These figures correspond to the dodecahedron, cube, tetrahedron, icosahedron, octahedron, dihedron, and hosohedron (beach ball). The first five are the platonic solids.

- In the second case, $\frac{1}{m}+\frac{1}{n}=\frac{1}{2}$ we get the Euclidean tessellations $(4,4),(3,6),(6,3)$. For example $(3,6)$ is the honeycomb tessellation. (See Figure 1.5)

- In the third case, $\frac{1}{m}+\frac{1}{n}<\frac{1}{2}$ we get hyperbolic tessellations. For example, Figure 1.6 is a tessellation of type $(3,7)$.

FIGURE 1.5: Honeycomb Tessellation

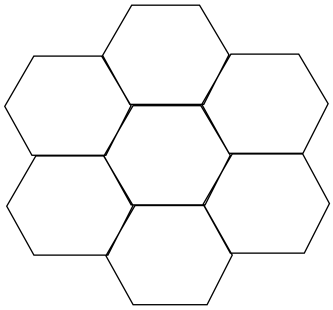
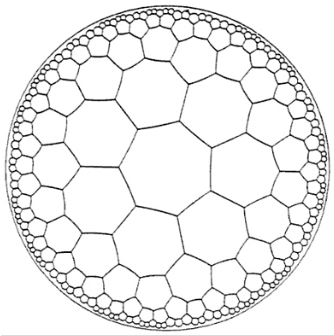


FIGURE 1.6: Tessellation of type (3,7)



Definition 1.3. [Universal algebraic maps] For each $m,n\geq 2\in\mathbb{N}\cup\{\infty\}$, let $\Gamma=\Gamma(m,n)$ be the triangle group $\Gamma(2,m,n)$. We then define the universal algebraic map of type (m,n) as $\hat{\mathscr{A}}(m,n)=(\Gamma,|\Gamma|,X,Y)$, where $|\Gamma|$ denotes the underlying set of Γ, and each $g\in\Gamma$ acts on $|\Gamma|$ by right-multiplication, $g:h\mapsto hg$ for all $h\in|\Gamma|$.

1.4.1 Map-subgroups

The triangle group $\Gamma(2,m,n)$ has a presentation of the form

$$\Gamma(2,m,n)=\langle X,Y,Z\,|\,X^2=Y^m=Z^n=XYZ=1\rangle. \tag{1.4.1}$$

If $\mathscr{M}$ is a map of type (m,n) with corresponding algebraic map $\mathscr{A}=(G,\Omega,x,y)$, then there is an epimorphism $\theta:\Gamma\to G$ given by $X\mapsto x$, $Y\mapsto y$, $Z\mapsto z$, so Γ performs a transitive action on Ω. If $G_\alpha=\{g\in G\,|\,\alpha g=\alpha\}$ for any $\alpha\in\Omega$, then $M=\theta^{-1}(G_\alpha)$, and is known as the *map-subgroup* associated to $\mathscr{M}$. We can identify Ω with the set of right M-cosets in $\Gamma(2,m,n)$ by the bijection

$$Mh \longmapsto \alpha(h\theta) \tag{1.4.2}$$

where $h \in \Gamma(2,m,n)$. (The map 1.4.2 is well defined, since if $Mg = Mh$ then $gh^{-1} \in M$ and so $(gh^{-1})\theta \in G_\alpha$, which implies that $\alpha(g\theta) = \alpha(h\theta)$).
The map-subgroups play a similar role to the fundamental groups in topology; see theorem 1.5.

1.4.2 Map automorphisms

If $\mathscr{M}_i = (\mathscr{G}_i, \mathscr{V}_i, \mathscr{S}_i)$ are topological maps $(i = 1, 2)$, we can then define a *morphism* $\theta : \mathscr{M}_1 \to \mathscr{M}_2$ as the covering of surfaces $\theta : \mathscr{S}_1 \to \mathscr{S}_2$ preserving orientation, such that:
(1) $\theta^{-1}(\mathscr{G}_2) = \mathscr{G}_1$ and $\theta^{-1}(\mathscr{V}_2) = \mathscr{V}_1$;
(2) all branch-points have a finite order.
We say the topological map $\mathscr{M}_1$ covers $\mathscr{M}_2$ if there is a morphism $\mathscr{M}_1 \to \mathscr{M}_2$.
More generally, there is a morphism $\mathscr{M}_1 \to \mathscr{M}_2$ if and only if $g^{-1}M_1g \leqslant M_2$ for some $g \in \Gamma$.
Two topological maps are isomorphic if there is a morphism between them, such that the covering of surfaces θ is a homeomorphism. Thus an automorphism of a topological map is a self-morphism induced by a homeomorphism of the underlying surface to itself. We can also define morphisms between algebraic maps: an algebraic morphism between $Alg\mathscr{M}_1 = (G_1, \Omega_1, x_1, y_1)$ and $Alg\mathscr{M}_2 = (G_2, \Omega_2, x_2, y_2)$ is a pair (δ, σ) of functions $\delta : \Omega_1 \to \Omega_2$, $\sigma : G_1 \to G_2$, where σ is a group homomorphism, $x_1\sigma = x_2$, $y_1\sigma = y_2$ and the diagram in Figure 1.7 commutes (the horizontal arrows in the diagram represent the actions of G_1 and G_2). Thus we require that $(\alpha g)\delta = (\alpha\delta)(g\sigma)$ for all $g \in G_1$,$\alpha \in \Omega_1$. Two algebraic maps are then isomorphic if there exist an algebraic morphism (δ, σ) between them, where σ is a group isomorphism and δ is a bijection.

FIGURE 1.7

$$\begin{tikzcd}
\Omega_1 \times G_1 \arrow[r] \arrow[d, "(\delta,\sigma)"] & \Omega_1 \arrow[d, "\delta"] \\
\Omega_2 \times G_2 \arrow[r] & \Omega_2
\end{tikzcd}$$

We say the algebraic map $\mathscr{A}_1$ covers the algebraic map $\mathscr{A}_2$ if there is a morphism $\mathscr{A}_1 \to \mathscr{A}_2$.

The set of topological automorphisms of a map forms an infinite group, since its edges can be continuously deform, and each vertex can be perturbed in some small neighborhood. We therefore follow Jones and Singerman [JS78, Section 3] in defining the automorphism group $Aut\mathscr{M}$ of a map $\mathscr{M}$ as the group of algebraic automorphisms of its associated algebraic map $Alg\mathscr{M}$.

1.4.3 Quotient maps and universal maps

Let $\mathscr{A} = (G, \Omega, x, y)$ be an algebraic map. If T is a group of automorphisms of $\mathscr{A}$, then T induces an equivalence relation $\sim$ on Ω, that is, $\alpha \sim \beta$ implies $\alpha t \sim \beta t$ for all $t \in T$. There is an action of G on the quotient set $\bar{\Omega} = \Omega/\sim$ given by $g : [\alpha] \mapsto [\alpha g]$, and if K is the kernel of this action, then $K \triangleleft G$ and $\bar{G} = G/K$ acts faithfully and transitively on $\bar{\Omega}$. Setting $\bar{x} = Kx$ and $\bar{y} = Ky$, we call $\bar{A} = (\bar{G}, \bar{\Omega}, \bar{x}, \bar{y})$ the quotient map of A by T.
Any subgroup $M \leq \Gamma(2, m, n)$ acts as a group of automorphisms of $\hat{\mathscr{A}}$ by left action $a : h \mapsto a^{-1}h$ for all $h \in |\Gamma|$, $a \in M$. We can therefore form the quotient algebraic map

$$\hat{\mathscr{A}}/M = (\Gamma/M^*, \Gamma/M, M^*X, M^*Y) \tag{1.4.3}$$

where the M-cosets, i.e. $\Gamma/M = \{Mg \,|\, g \in \Gamma\}$ is the underlying set of $\hat{\mathscr{A}}/M$, M^* is the core of M (the intersection of all conjugates of M in Γ i.e. $M^* = \bigcap_{g\in\Gamma} g^{-1}Mg$), and Γ/M^* acts on Γ/M by $M^*g : Mh \mapsto Mhg$ for all $g, h \in \Gamma$.

Theorem 1.4 ([JS78, Theorem 3.4]). *Every algebraic map $\mathscr{A}$ of type (m, n) is isomorphic to a quotient of the universal algebraic map $\hat{\mathscr{A}}$ of type (m, n) by a map-subgroup M corresponding to the map as defined above.*

Theorem 1.5 ([JS78, Theorem 3.6]). *If $\mathscr{A}_1, \mathscr{A}_2$ are algebraic maps, then $\mathscr{A}_1$ covers $\mathscr{A}_2$ if and only if we can find map-subgroups $M_i \leq \Gamma(m, n)$ for $\mathscr{A}_i$ $(i = 1, 2)$ with $M_1 \leq M_2$.*

According to Proposition 5.5 in [JS78], the categories $\mathbf{TM}(m, n)$ and $\mathbf{AM}(m, n)$ of topological and algebraic maps are equivalent. Thus, we can restate Theorem 1.5 as following, and Theorem 1.4 also applied to topological maps.

Proposition 1.6 ([JS78, Theorem 3.7]). *Two topological maps $\mathscr{M}_1$ and $\mathscr{M}_2$ are isomorphic if and only if their map-subgroups M_1 and M_2 are conjugate in Γ.*

Theorem 1.7. *If $\mathscr{M}_1, \mathscr{M}_2$ are topological maps, then $\mathscr{M}_1$ covers $\mathscr{M}_2$ if and only if we can find map-subgroups $M_i \leq \Gamma(m, n)$ for $\mathscr{M}_i$ $(i = 1, 2)$ with $M_1 \leq M_2$.*

The proof of the following theorem follows by using the proof of [JS78, Corollary 5.2] and (1.4.3).

Theorem 1.8. *Let $\hat{\mathscr{M}}$ and $\hat{\mathscr{A}}$ be the universal topological and algebraic maps of type (m, n) respectively, with $M \leq \Gamma(2, m, n)$. Then $Alg(\hat{\mathscr{M}}/M) \cong \hat{\mathscr{A}}/M$.*

We explain this as follows, by the definition of the algebraic map $Alg(\hat{\mathscr{M}}/M) = (G, \Omega, x^*, y^*)$. The darts of $\hat{\mathscr{M}}$ are permuted by $\Gamma = \Gamma(2, m, n)$, and the set of darts of $\hat{\mathscr{M}}/M = \Gamma/M = \Omega$ (the set of M-cosets of Γ i.e. $|\Gamma/M|$). As there is an epimorphism

$\theta : \Gamma \to G$ given by $X \mapsto x$, $Y \mapsto y$, $Z \mapsto z$. If $G_\alpha = \{g \in G \,|\, \alpha g = \alpha\}$ for any $\alpha \in \Omega$, and if $M = \theta^{-1}(G_\alpha)$, then $ker(\theta) = M^*$, inducing a permutation group Γ/M^* which we may identify with G, as well as identifying x^* with M^*X and y^* with M^*Y. Hence the algebraic map corresponds to the quotient $\hat{\mathscr{M}}/M$ is isomorphic to the quotient of the algebraic map $\hat{\mathscr{A}}/M = (\Gamma/M^*, \Gamma/M, M^*X, M^*Y)$.

Theorem 1.9. *If $\mathscr{A}$ is an algebraic map, then there is a topological map $\mathscr{M}$ such that $Alg\mathscr{M} \cong \mathscr{A}$.*

Proof. Let $\mathscr{A}$ be an algebraic map of type (m,n). Then according to Theorem 1.4 we have $\mathscr{A} \cong \hat{\mathscr{A}}/M$, where $\hat{\mathscr{A}}$ is the universal algebraic map of type (m,n) and M is some subgroup of $\Gamma(2,m,n)$.
If $\hat{\mathscr{M}}$ is the universal topological map of type (m,n), then according to Theorem 1.8 we have $Alg(\hat{\mathscr{M}}/M) \cong \mathscr{A}$. Therefore the required map is $\mathscr{M} \cong \hat{\mathscr{M}}/M$. □

If $\mathscr{M}$ is a topological map of type (m,n), with a map-subgroup $M \leq \Gamma(2,m,n)$, then the isomorphism $\mathscr{M} \cong \hat{\mathscr{M}}/M$ defines an embedding of $\mathscr{M}$ into the Riemann surface $X = \mathscr{U}/M$. In this way, every topological map can be embedded naturally into some Riemann surface.

Definition 1.10. A map $\mathscr{M}$ is *regular* if $Aut\mathscr{M}$ acts transitively on the darts of $\mathscr{M}$.

Theorem 1.11 ([JS78, Theorem 6.3])**.** *The following are equivalent:*
(i) $\mathscr{M}$ is regular;
(ii) (G,Ω) is a regular permutation group, that is, $G_\alpha = 1$, for all $\alpha \in \Omega$;
(iii) $M \trianglelefteq \Gamma$.

Corollary 1.12. *Every finite map of type (m,n) is the quotient of a finite regular map of type (m,n) by a group of automorphisms.*

This follows because every subgroup of a finite index in Γ contains a normal subgroup of a finite index in Γ.

Remark 1.13. With a topological map $\mathscr{M}$ we can readily construct an algebraic map $\mathscr{A}$. We simply define Ω as the set of darts, and x, y as permutations of Ω corresponding to the edges and vertices respectively. This enables us to go from a topological map $\mathscr{M}$ to an algebraic map $\mathscr{A}$. (Using Lemmas 2.1a, 2.1b and 2.1c in [JS78].)
The converse process is given by Theorem 1.9.

Chapter 2

Modular group and Hecke groups

2.1 Modular group

The majority of the content of this section is taken from [JS87].

Definition 2.1. The Projective Special Linear Group $PSL(2, \mathbb{R})$.
Let $\mathbb{R}$ be the field of real numbers. The special linear group $SL(2, \mathbb{R})$ is defined as the group of all 2×2 matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

The center of $SL(2, \mathbb{R})$ consists of the matrices $\pm I$, where I denotes the 2×2 unit matrix. The quotient group of $SL(2, \mathbb{R})$, with respect to its center is the projective special linear group $PSL(2, \mathbb{R})$.
In addition, we can consider the group $PSL(2, \mathbb{R})$ as the group of real Möbius transformations,

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}, z \in \mathbb{H} \cup \mathbb{R} \cup \{\infty\}.$$

This is because $-I$ acts trivially on $\mathbb{H}$. Note: $PSL(2, \mathbb{R})$ also has an action on $\mathbb{R} \cup \{\infty\}$, and that $PSL(2, \mathbb{R})$ acts on $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$.

Classifying the elements of $PSL(2, \mathbb{R})$. Before commencing geometric classification of $PSL(2, \mathbb{R})$, we need to recall that if A is a matrix then the trace of A is defined as the sum of the diagonal entries of A. That is, if

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ then } \quad \text{Trace(A)} = \alpha + \delta.$$

Additionally, to find the fixed points of any $A \in PSL(2,\mathbb{R})$, we need to solve

$$z = A(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

We have $A(\infty) = \frac{\alpha}{\gamma}$, thus A fixes ∞, if and only if, $\gamma = 0$. If $\gamma \neq 0$, then $z \in \mathbb{R}$ is a fixed point of A, if and only if

$$\gamma z^2 + (\delta - \alpha)z - \beta = 0;$$

therefore A has two fixed points in $\mathbb{C}$, $z = \frac{(\alpha-\delta) \pm \sqrt{(\delta-\alpha)^2 + 4\beta\gamma}}{2\gamma}$ unless $(\delta - \alpha)^2 + 4\beta\gamma = 0$, in which case A, then has a unique fixed point. Using $\alpha\delta - \beta\gamma = 1$, the condition becomes $(\alpha + \delta)^2 - 4 = 0$, so A has a single fixed point if and only if $(\alpha + \delta)^2 = 4$.
If $\gamma = 0$ then A fixes ∞, and we have $\alpha\delta = 1$ and $A(z) = \alpha^2 z + \alpha\beta$, so there is a second fixed point $z = \frac{\alpha\beta}{(1-\alpha^2)} \neq \infty$ if and only if $\alpha^2 \neq 1$, or equivalently $(\alpha + \delta)^2 \neq 4$.
The next step is then to classify the elements of $PSL(2,\mathbb{R})$ in terms of their trace and fixed points;
A non-identity element is *parabolic* if $|\alpha + \delta| = 2$, and so A has a single fixed point in $\mathbb{R} \cup \{\infty\}$,
an element is *hyperbolic* if $|\alpha + \delta| > 2$, then A has two fixed points in $\mathbb{R} \cup \{\infty\}$,
and an element is *elliptic* if $|\alpha + \delta| < 2$, and so A has a single fixed point in $\mathbb{H}$.

Definition 2.2. Fuchsian Groups
A Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$. An example of such a Fuchsian group is the modular group $\Gamma = PSL(2,\mathbb{Z})$.

The results derived from this section come from [JS87, sections 5.7 and 6.8]
Thus, from now on we will define Γ as the modular group consisting of all transformations

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma = 1, \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z}.$$

Γ is generated by the transformations

$$\left.\begin{array}{r} X : z \mapsto \frac{-1}{z} \\ \text{and} \qquad \\ Z : z \mapsto z + 1 \end{array}\right\}$$

Putting $Y = XZ : z \mapsto \frac{-1}{(z+1)}$ we see that X and Y generate Γ and that satisfies the relations

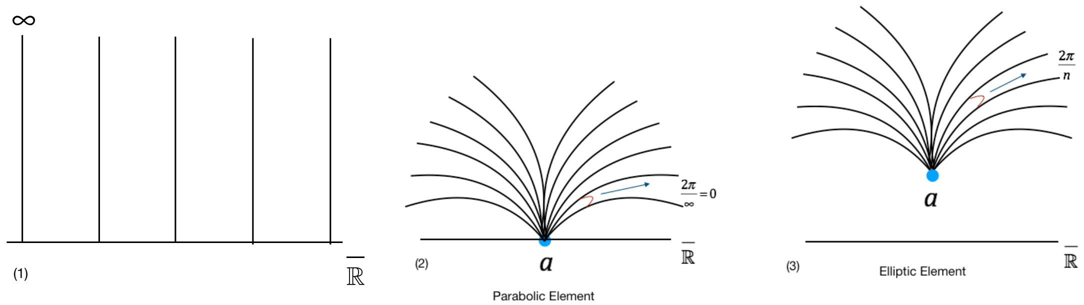
$$X^2 = Y^3 = 1.$$

In fact Γ has the presentation

$$\Gamma = \langle X, Y \mid X^2 = Y^3 = 1 \rangle, \tag{2.1.1}$$

and so there are no other relations are necessary to define Γ, see [JS87, Corollary 6.8.6]. The generator $Z = XY : z \mapsto z+1$ is represented by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ which is parabolic, and Z fixes ∞ (see Figure 2.1 (1)). The parabolic elements of Γ are the conjugates of the non-identity powers of Z, and so they take the form TZ^kT^{-1} where $k \in \mathbb{Z}\setminus\{0\}$, such that $T(\infty) = a \in \mathbb{R}$, $T \in \Gamma$ (see Figure 2.1 (2)). An elliptic element is conjugate to a hyperbolic rotation and fixes a point in $\mathbb{H}$, as in Figure 2.1 (3). If the elliptic element has an order n, (i.e. there are n darts meeting at this fixed point), then the angles between the hyperbolic geodesics in Figure 2.1 (3) are $\frac{2\pi}{n}$. In Figure 2.1 (2) the angles are $0 = \frac{2\pi}{\infty}$. Consequently, a parabolic element can be considered to be an elliptic element of infinite order.

Figure 2.1: Treating parabolic elements as being elliptic elements of order ∞



From (2.1.1) , $\Gamma = PSL(2,\mathbb{Z})$ can be considered the triangle group $\Gamma(2,\infty,3)$. The group Γ is an example of a free product. If

$$\langle X \,|\, X^2 = 1\rangle \cong C_2$$

and

$$\langle Y \,|\, Y^3 = 1\rangle \cong C_3,$$

then we have

$$\Gamma \cong C_2 * C_3.$$

Given a Fuchsian group Λ, we define $z, w \in \mathbb{H}$ as a congruent modulo Λ, written as $w \sim z$, if $w = \lambda z$, for some $\lambda \in \Lambda$.

Definition 2.3. F is a *Fundamental region* for a Fuchsian group Λ if F is a closed set in $\mathbb{H}$ such that

(i) $\bigcup\limits_{T\in\Lambda} T(F) = \mathbb{H}$, i.e. every point $z \in \mathbb{H}$ is congruent to some point in F.

(ii) $\mathring{F} \cap T(\mathring{F}) = \varnothing$ for all $T \in \Lambda \setminus \{I\}$, where $\mathring{F}$ is the interior of F, i.e. no pair of points in the interior of F are congruent and each orbit of every $z \in \mathbb{H}$, meets $\mathring{F}$ at most once.

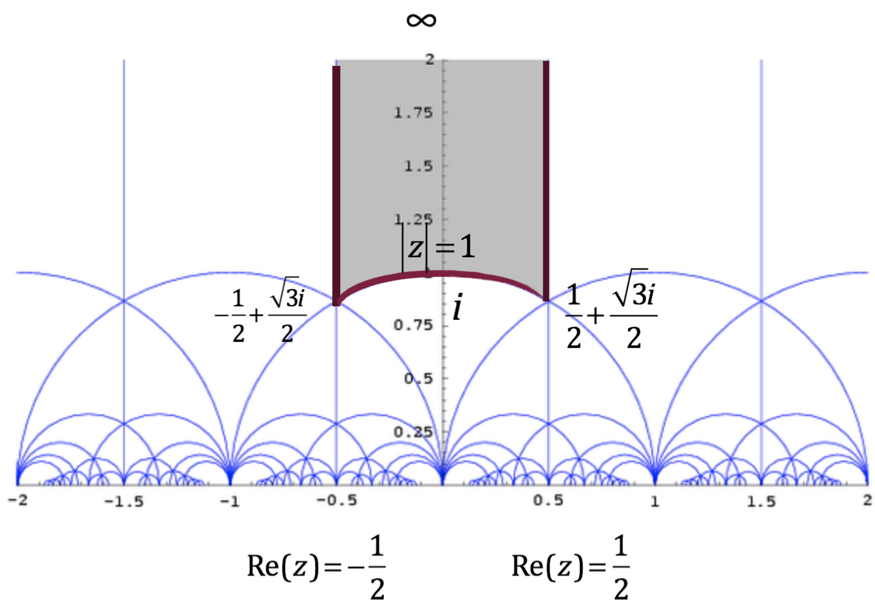
Theorem 2.4 ([JS87, Theorem 5.8.4]).

$$F = \{z \in \mathbb{H} \mid |z| \geq 1 \text{ and } |Re(z)| \leq \frac{1}{2}\}$$

is a fundamental region for the modular group.

The fundamental region here is bounded by the vertical lines $Re(z) = \frac{1}{2}$ and $Re(z) = \frac{-1}{2}$, and the circle $|z| = 1$. This region is a hyperbolic triangle; it has three vertices in $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ at $\frac{1}{2} + \frac{i\sqrt{3}}{2}$, $\frac{-1}{2} + \frac{i\sqrt{3}}{2}$, which are the fixed points of the elliptic generators and ∞, which is the unique parabolic fixed point in the region, as shown in Figure 2.2 below;

FIGURE 2.2: The fundamental region for the modular group



This region is determined in many books, e.g. [Ran77], [For51], [JS87].

2.2 The Hecke groups H_q

In [Hec36], Hecke introduced the groups H_q which are generated by two real Möbius transformations:

$$S(z) = \frac{-1}{z} \text{ and } T(z) = z + \lambda,$$

where λ is a fixed positive real number. Moreover, we can represent S and T as

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Let $R(z) = TS(z) = \frac{\lambda z - 1}{z}$.
Hecke showed H_q is properly discontinuous, i.e. it is a Fuchsian group, when
(i) $\lambda \geq 2$, or when
(ii) $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$ where q is an integer ≥ 3.
We are particularly interested in the case (ii) as we are considering $\Gamma(2,\infty,q)$ groups and using them to study q-gonal maps. Case (i) might be interesting to study for some Hecke group problems but not in the context of this thesis.
Before we start introducing the representations of the Hecke groups, we first need to show that

$$R^q = \left(\begin{smallmatrix} \lambda_q & -1 \\ 1 & 0 \end{smallmatrix}\right)^q = -I.$$

To accomplish this we diagonalize R. i.e. find a diagonal matrix conjugate to R. The elements on the diagonal are the eigenvalues of R. These are easily computed to be $\cos\frac{\pi}{q} \pm i\sin\frac{\pi}{q}$. That is $e^{\frac{\pi}{q}i}$ and $e^{\frac{-\pi}{q}i}$. Thus

$$R^q = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -I,$$

which defines the identity in $PSL(2,\mathbb{R})$.
The Hecke groups in (ii) have the following presentation in terms of the triangle group $\Gamma(2,\infty,q)$;

$$H_q = \langle S, R \,|\, S^2 = R^q = I \rangle.$$

These groups are isomorphic to the free product of two finite cyclic groups of orders 2 and q, which is the same argument given for the modular group, see [JS87, Section 6.8]; i.e.

$$H_q \cong C_2 \star C_q.$$

These facts are proven using the same method as that described for the modular group, see [JS87, Section 6.8].

We have the following table of the values of λ_q for small q;

q	3	4	5	6
λ_q	1	$\sqrt{2}$	$\frac{1+\sqrt{5}}{2}$	$\sqrt{3}$

The best known example here is when $q = 3$, and H_3 is the modular group $\Gamma = PSL(2,\mathbb{Z})$.

2.2.1 The elements of H_q

An element of the Hecke groups H_q takes the form

$$\begin{pmatrix} p_1(\lambda_q) & p_2(\lambda_q) \\ p_3(\lambda_q) & p_4(\lambda_q) \end{pmatrix}$$

where the p_i , $(i = 1, ..., 4)$ are elements of $\mathbb{Z}[\lambda_q]$ and these p_i are determined modulo the minimal polynomial of λ_q, and thus, these matrices are elements of $PSL(2, \mathbb{Z}[\lambda_q])$. This means we have the inclusion $H_q \leq PSL(2, \mathbb{Z}[\lambda_q])$. One of the principal difficulties when analysing the elements of H_q is to determine whether or not a given matrix of $PSL(2, \mathbb{Z}[\lambda_q])$ belongs to the Hecke group H_q.
The most well known Hecke groups are those for $q = 3, 4$ and 6. In these cases $\lambda_3 = 1$, $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$. Therefore the underlying fields are $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$, i.e. they are quadratic extensions of the field $\mathbb{Q}$ of the rationals.
The Hecke groups H_4 and H_6 are of particular interest, since they are the only Hecke groups, apart from the modular group Γ, whose elements are completely known.

For $q = 4$ and 6, H_q consists a set of all matrices of the following two types:
(i) the elements of type

$$\begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \; a, b, c, d \in \mathbb{Z} \; , \; ad - mbc = 1,$$

(ii) the elements of type

$$\begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix} \; a, b, c, d \in \mathbb{Z} \; , \; mad - bc = 1.$$

Where $m = 2$ or 3 for $q = 4$ and 6 respectively. The elements of type (i) are called *even*, while those of type (ii) are *odd*. For $q = 4, 6$ the even elements are a subgroup of index 2 of H_q denoted by H_q^e, i.e.

$$|H_q : H_q^e| = 2 \tag{2.2.1}$$

while the set of the odd elements is the other coset of H_q^e in H_q, denoted by H_q^o. Hence we have

$$H_q^e = \left\{ A = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \mid A \in H_q \right\}, \tag{2.2.2}$$

and

$$H_q^o = \left\{ A = \begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix} \mid A \in H_q \right\}. \tag{2.2.3}$$

Note that if we consider the multiplication of these elements, we have

odd.odd=even.even=even

even.odd=odd.even=odd.

Thus, we know all the elements of $H_3 = \Gamma$, H_4 and H_6. In other cases the elements of H_q are worked out by Rosen [Ros54], using λ-fractions given the required conditions for the substitution of an element of H_q, but for a description of the elements of H_5 see [Ros63], where the underlying field is once more the quadratic extension of $\mathbb{Q}$. These four Hecke groups are the only ones where λ_q is a root of a polynomial of a degree less than three which can be easily calculated. However, for $q \geq 7, q \in \mathbb{N}$, the algebraic number λ_q is a root of a polynomial of degree ≥ 3. Therefore it is not possible to make the determination of λ_q for $q \geq 7$ as effectively as in the first four cases. Consequently, we shall examine the minimal polynomial of λ_q instead of λ_q itself. The following table presents minimal polynomials of λ_q for $3 \leq q \leq 8$;

TABLE 2.1: The minimal polynomials of λ_q for $3 \leq q \leq 8$

q	The minimal polynomial
3	$x - 1$
4	$x^2 - 2$
5	$x^2 - x - 1$
6	$x^2 - 3$
7	$x^3 - x^2 - 2x + 1$
8	$x^4 - 4x^2 + 2$

Because of the importance of being able to determine the elements of H_q in our work, we can recall Rosen's ideas;
As shown in [JS87, p. 296-298] elements of Γ can be expressed in terms of the generators, and any relation in Γ can be written in the form of a word in X and Y. Here we do so for the elements of H_q, which could be expressed in terms of the generators S and T, as

$$V(z) = \frac{az+b}{cz+d} = T^{r_0}ST^{r_1}S...ST^{r_n}(z), \tag{2.2.4}$$

where the r_i's, $(0 < i < n)$ are integers, and only r_0 and r_n may be zero. The method Rosen used is the development of a class of continued fractions, which arise naturally from H_q. In [Hec36], Hecke gave the following result:

Theorem 2.5. *When $\lambda \geq 2$ and is real, or when $\lambda = \lambda_q = 2\, cos\frac{\pi}{q}$, $q \in \mathbb{N}$, $q \geq 3$, the set*

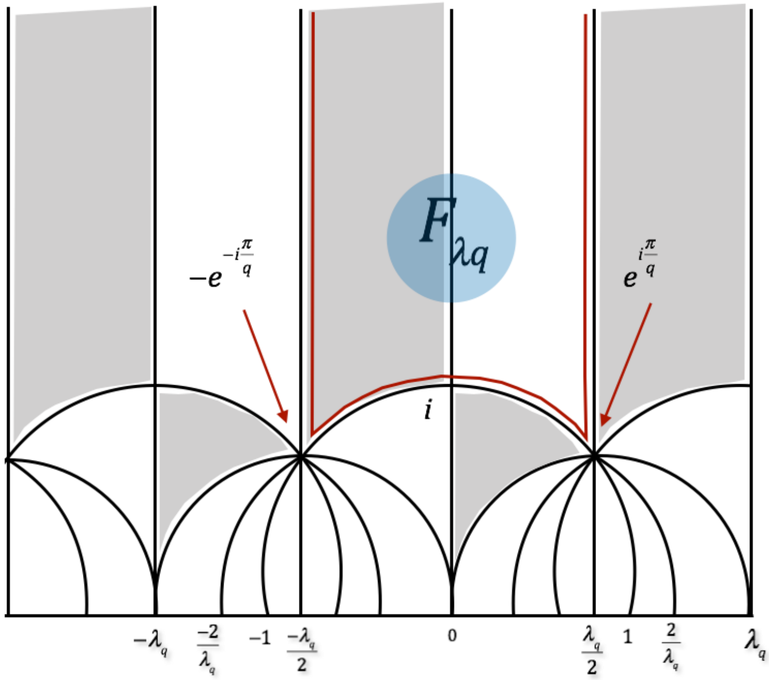
$$F_{\lambda_q} = \{z \in \mathbb{H} \mid |Re(z)| \leq \frac{\lambda}{2},\ |z| \geq 1\} \tag{2.2.5}$$

is a fundamental region for the group H_q.

R. Evans gave an elementary proof of this fact in [Eva73].

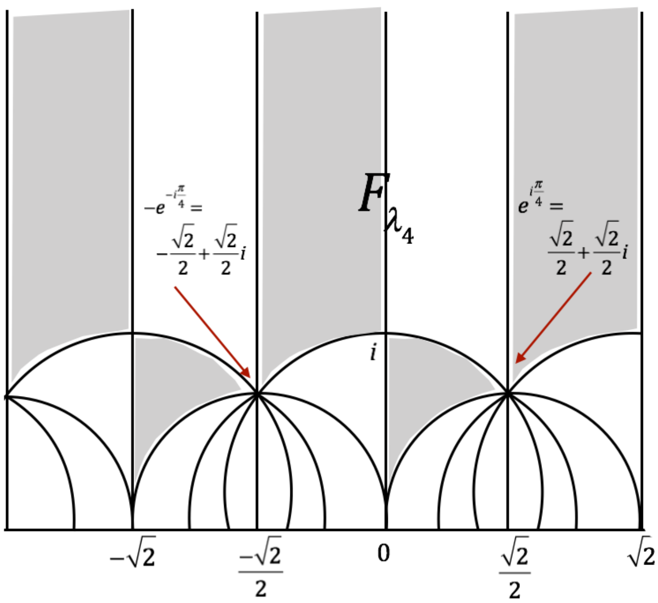
This fundamental region as shown in Figure 2.3, is bounded by the lines $Re(z) = \frac{-\lambda_q}{2}$, $Re(z) = \frac{\lambda_q}{2}$ and the unit circle which form a hyperbolic quadrilateral with four vertices in $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, two of which are the fixed points of two elliptic generators S and R, these are i and $e^{i\frac{\pi}{q}}$ respectively, the third is infinity which is the unique parabolic point

FIGURE 2.3: The shaded and white region together form a fundamental region for H_q



for this region and $-e^{-i\frac{\pi}{q}}$. The fixed points for the parabolic elements are called the *cuspset*, and these are the images of ∞, under the elements of H_q.

FIGURE 2.4: The fundamental region for H_4



Chapter 3

The Universal q-gonal Map

3.1 The Farey map

In simple terms, the Farey map $\hat{\mathscr{M}}(\infty, 3)$, abbreviated here to $\hat{\mathscr{M}}_3$, is a tessellation of the upper half-plane. We construct the Farey map as follows. The vertex set is the set of the extended rational numbers $\mathbb{Q} \cup \{\infty\}$ and two rationals $\frac{a}{b}$ and $\frac{c}{d}$, where a and b, and c and d are coprime pairs, $a, b, c, d \in \mathbb{Z}$, are joined by an edge if and only if $ad - bc = \pm 1$. The edges are given by hyperbolic geodesics. This map has the following properties.

1. There is a triangle with vertices $\frac{1}{0}, \frac{1}{1}, \frac{0}{1}$ called the principal triangle.

2. The modular group Γ acts by Möbius transformations as a group of automorphisms of the Farey map, as follows, if

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma, \text{ then } \quad A(\frac{a}{c}) = (\frac{\alpha a + \beta c}{\gamma a + \delta c}).$$

3. Every triangle has vertices of the form $\frac{a}{c}, \frac{a+b}{c+d}, \frac{b}{d}$ for some integers a, b, c and d.

Constructing the Farey map $\hat{\mathscr{M}}_3$. The edges of $\hat{\mathscr{M}}_3$ are the images of the imaginary axis under Γ. The vertices are the images of ∞ under Γ, and the faces are the images of the principal face under Γ.

Figure 3.1: The edges are the images of the imaginary axis under Γ

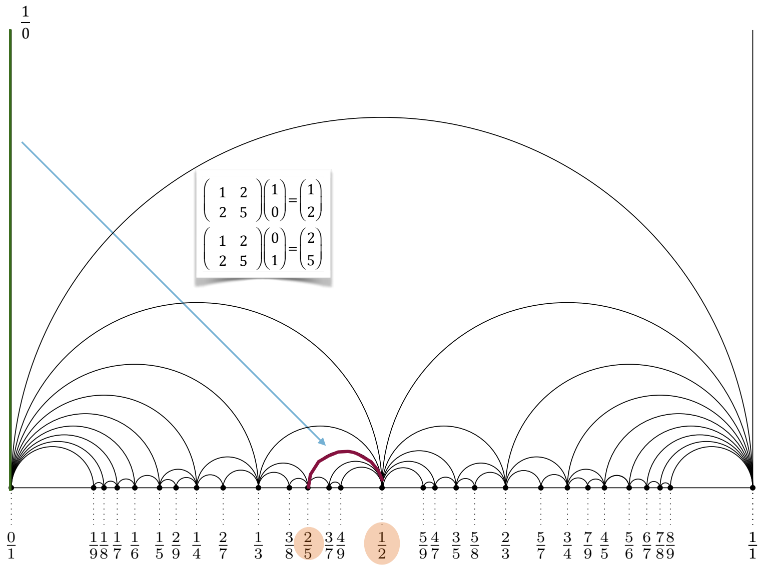


Figure 3.2: The vertices are the images of ∞ under Γ

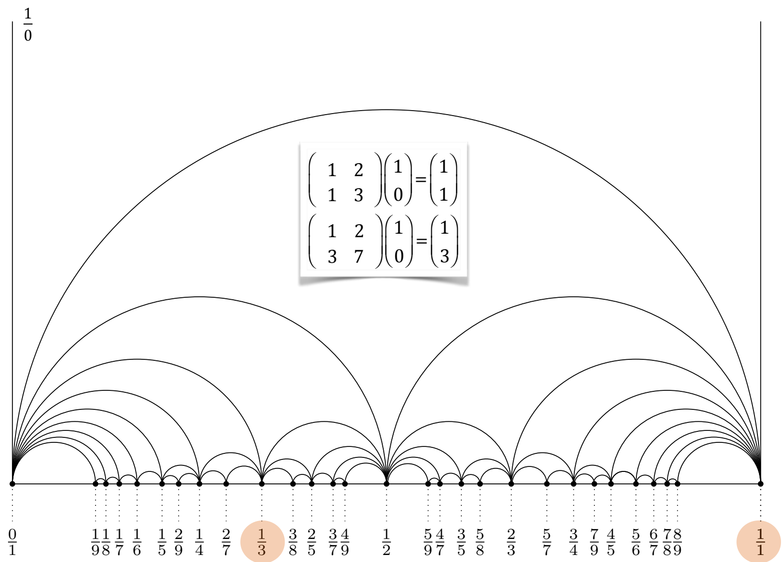


FIGURE 3.3: The principal face

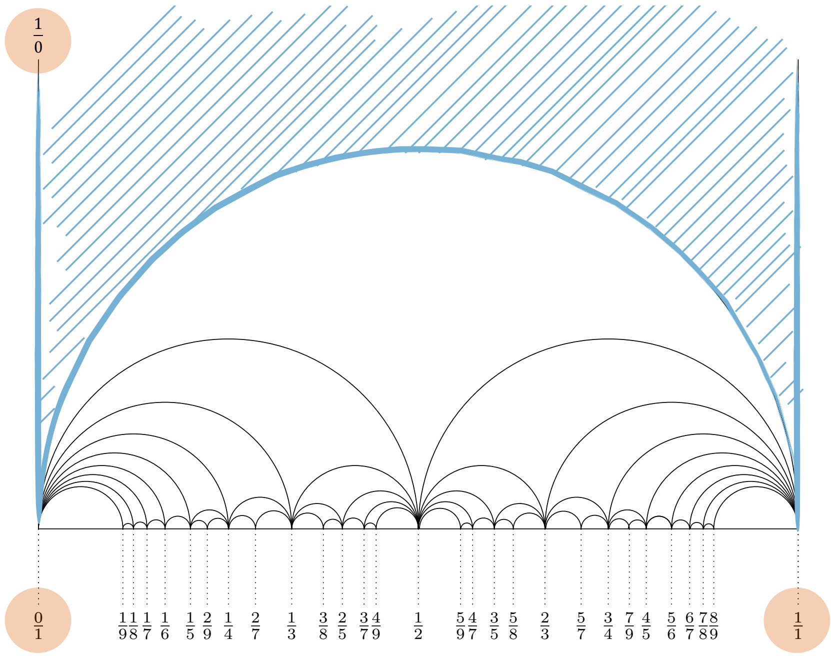
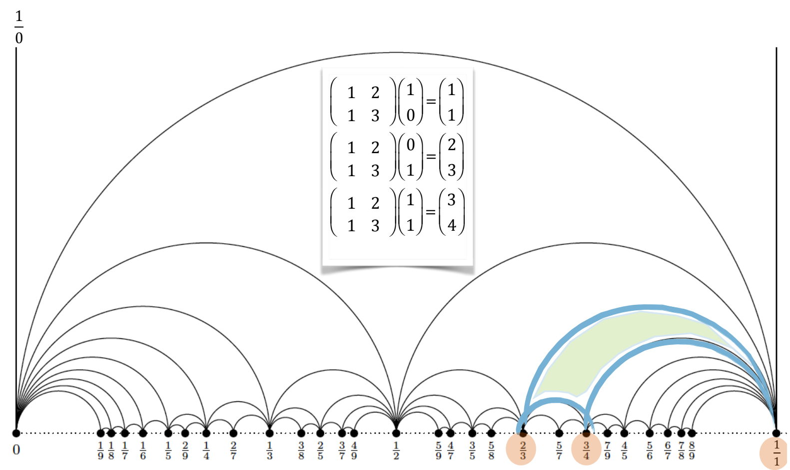


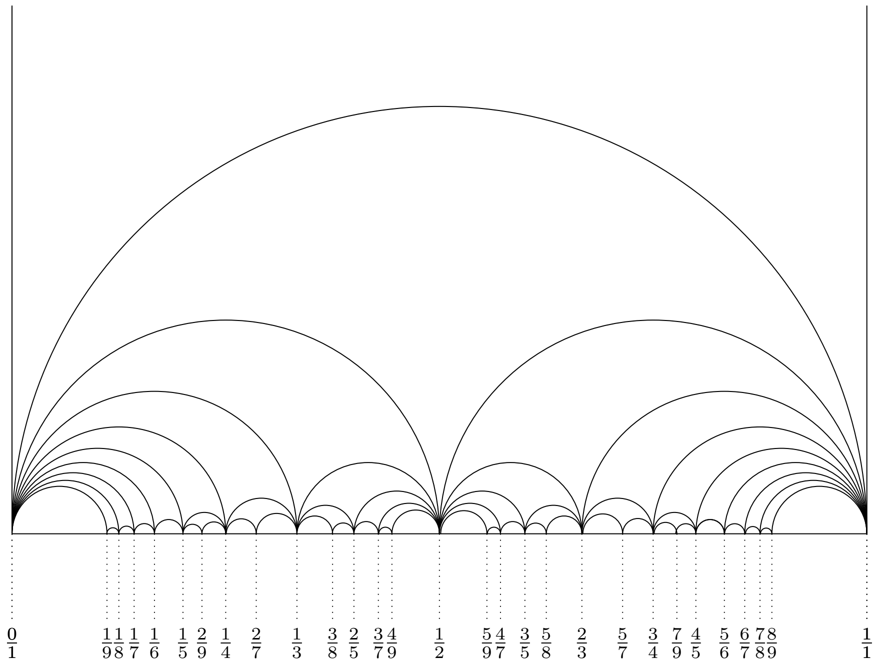
FIGURE 3.4: The faces are the images of the principal face under Γ



Thus the Farey map $\hat{\mathscr{M}}_3$ $(= \hat{\mathscr{M}}(\infty, 3))$ (Figure 3.5) is a *triangular map*, meaning that it is a map in which all the face are triangles. [Sin88] demonstrates this is the universal triangular map; in the sense that any triangular map on an orientable surface is a quotient of the Farey map $\hat{\mathscr{M}}_3$ by a subgroup of the modular group Γ. In other words the Farey map $\hat{\mathscr{M}}_3$ covers all triangular maps on orientable surfaces, and by using the extended modular group, we can extend this theory to non-orientable surfaces.

We give a slightly different proof from [Sin88] (for orientable surfaces) which more easily generalizes to the corresponding result for universal q-gonal maps as in Theorem 3.4.

FIGURE 3.5: Farey map



Theorem 3.1 ([Sin88, Theorem 1])**.** *Let T be a triangular map. Then there is a subgroup $M \leq \Gamma(2, \infty, 3)$, such that $\hat{\mathscr{M}}_3/M \simeq T$.*

Proof. Let x, y be permutations of the darts as described in Section 1.2. Let T be any triangular map, that is a topological map of type $(m, 3)$, then $x^2 = y^3 = 1$, so there is a homomorphism $\theta : \Gamma \to G =< x, y >$ as given by $X \mapsto x$, $Y \mapsto y$. Let $M = \theta^{-1}(G_\alpha)$ be a map-subgroup for T as described in section 1.4. We show that the action of Γ on the darts of $\hat{\mathscr{M}}_3$ is regular (i.e transitive and fixed-point free), as described in Theorem 1.11. It is transitive because, if $\frac{a}{c} \to \frac{b}{d}$ is a dart in $\hat{\mathscr{M}}_3$ then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ takes the principal dart $\delta = (\infty, 0) = (\frac{1}{0}, \frac{0}{1})$ to $\frac{a}{c} \to \frac{b}{d}$. To use transitivity to prove there are no fixed points, we only need to prove that $\Gamma_\delta = \{e\}$. This is because if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ fixes 0 then $b = 0$ and if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ fixes ∞ then $c = 0$, then as $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$, $a = d = \pm 1$, $b = c = 0$ so $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ the identity in Γ. Thus the stabilizer of a dart of $\hat{\mathscr{M}}_3$ is the trivial group $\{e\}$. If T is any triangular map with map-subgroup $M < \Gamma$ then as $\{e\} < M$, it follows from Theorem 1.7 that $\hat{\mathscr{M}}_3$ covers T. Now by Theorem 1.4 every algebraic map $\mathscr{A}$ of type $(m, 3)$ is isomorphic to a quotient of the universal algebraic map $\hat{\mathscr{A}}$ of type $(m, 3)$ by a map-subgroup M. As observed in Section 1.4.3, the categories of algebraic maps and topological maps are equivalent, thus the same result holds for topological maps, that is $\hat{\mathscr{M}}_3/M \simeq T$. □

This means that $\hat{\mathscr{M}}_3$ is the universal triangular map as it covers every other triangular map.

3.2 The universal q-gonal maps

In this section we generalize the concept of the Farey map to the universal q-gonal map, involving Hecke groups H_q. The universal topological q-gonal map $\hat{\mathscr{M}}(\infty, q)$, which we abbreviate to $\hat{\mathscr{M}}_q$, is a tessellation of the upper half-plane. First we construct the universal q-gonal map $\hat{\mathscr{M}}_q$. The edges (the hyperbolic geodesics) of $\hat{\mathscr{M}}_q$ are the images of the imaginary axis I (which is the edge between the two vertices $\frac{0}{1}$ and ∞) under H_q. The imaginary axis I consists of two darts one with a vertex ∞ and one with a vertex 0. As $S(z) = \frac{-1}{z}$ takes 0 to ∞ and ∞ to 0, these two darts lie in the same H_q orbit. Thus the darts are the images of the directed edge $\hat{\alpha} = (0, \infty)$ along the imaginary axis from 0 to ∞ i.e. the darts are all of the form $(g(0), g(\infty))$ where $g \in H_q$. It follows that H_q has a transitive action on the darts of $\hat{\mathscr{M}}_q$. The vertex set is the images of ∞ under H_q. The principal face in $\hat{\mathscr{M}}_q$ has vertices, that are the image of ∞ under the powers of the elements R, as defined by $R(z) = TS(z) = \frac{\lambda z - 1}{z}$ of order q. The faces are then the images of the principal face under H_q.

FIGURE 3.6: The edges are the images of the imaginary axis under H_q

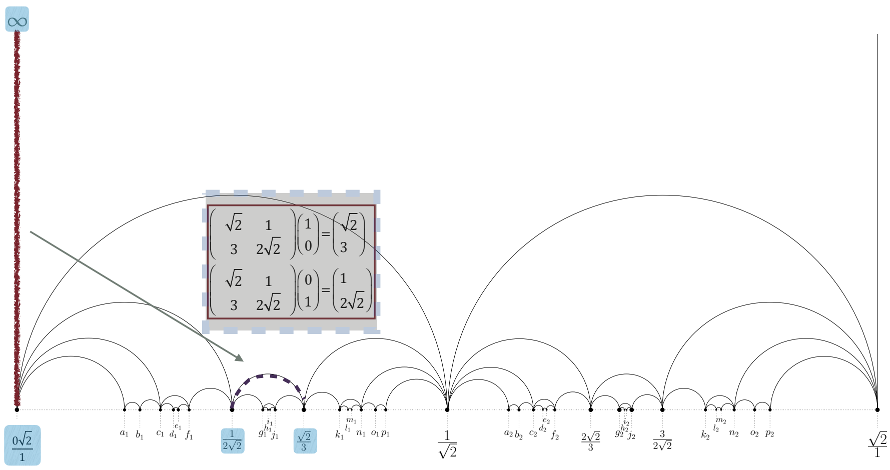
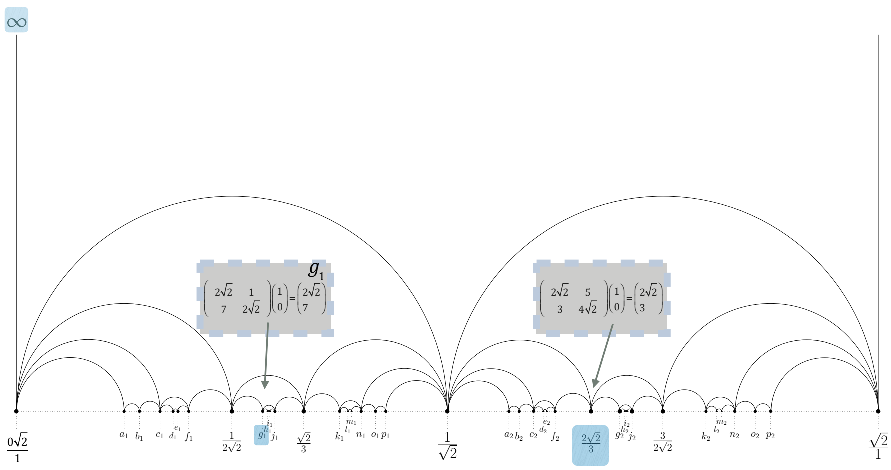


FIGURE 3.7: The vertices are the images of ∞ under H_q

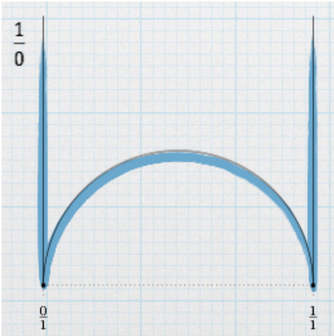


Example 3.2. (a) The vertices of the principal face in $\hat{\mathscr{M}}_3$ are

$$R(\infty) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, R^2(\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, R^3(\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

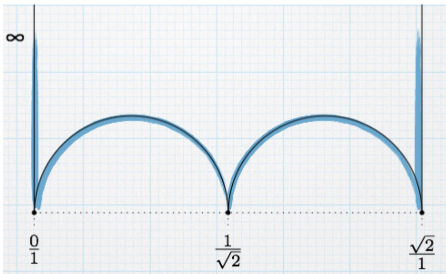
as shown in Figure 3.8.

FIGURE 3.8: Principal face when $q = 3$



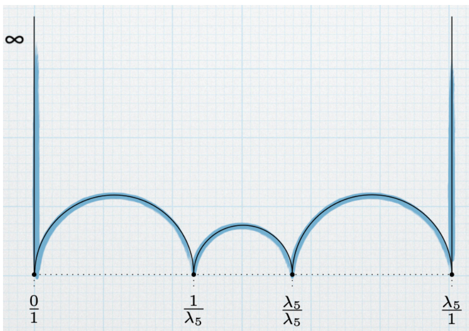
(b) The vertices of the principal face in $\hat{\mathscr{M}}_4$, $\lambda_4 = \sqrt{2}$, are $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{2}}{1}$ as shown in Figure 3.9.

FIGURE 3.9: Principal face when $q = 4$



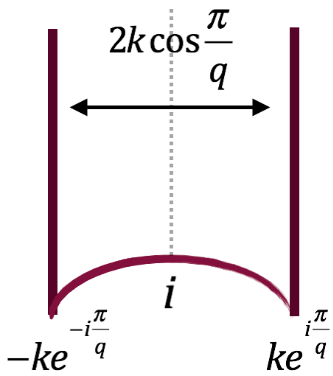
(c) The vertices of the principal face in $\hat{\mathscr{M}}_5$, $\lambda_5 = \frac{1+\sqrt{5}}{2}$, are $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{\lambda_5}$, $\frac{\lambda_5}{\lambda_5}$, $\frac{\lambda_5}{1}$ as shown in Figure 3.10.

FIGURE 3.10: Principal face when $q = 5$



To generalize Theorem 3.1 to q-gonal maps, we require the following Lemma.

Lemma 3.3. *The transformation $U_k : z \mapsto kz$, is not an element of H_q, for any $q \in \mathbb{Z} \geq 3$ where $\lambda_q = 2\cos\frac{\pi}{q}$ and $k > 0$, $k \neq 1 \in \mathbb{R}$.*



Proof. Suppose that the transformation $U_k : z \mapsto kz \in H_q$ for some real $k > 0$, $k \neq 1$. Replacing U_k with its inverse if necessary, we may assume that $k > 1$. Let F be the fundamental region for H_q shown in Figure 2.3. As F has width λ_q, it follows $U_k(F)$ has width $k\lambda_q > \lambda_q$. Thus we can find $z_0 \in U_k(F)$ such that $z_0, z_0 + \lambda_q$ both lie in $U_k(F)$. As $U_k \in H_q$ then $U_k(F)$ would be a fundamental region which is a contradiction as we have found two points of the same orbit in $U_k(F)$. $\square$

If a real Möbius transformation fixes 0 and ∞ then it has the form $z \mapsto kz$, $k > 0$. (Suppose $z \mapsto \frac{az+b}{cz+d}$, $ad - bc = 1$, fixes 0 and ∞. Then $b = c = 0$ so we have $z \mapsto \frac{az}{d}$. As $ad - bc = 1$ then $d = \frac{1}{a}$ so $z \mapsto a^2 z$ and $z \mapsto kz$, $k > 0$.)

Thus, Lemma 3.3 implies that only the identity element in H_q fixes the principal dart $\delta = (\infty, 0)$.

Theorem 3.4. *Let $\mathscr{M}$ be any q-gonal map. Then there is a subgroup $M \leq H_q$ such that $\hat{\mathscr{M}}_q/M \simeq \mathscr{M}$.*

Proof. Now H_q acts as a group of automorphisms of $\hat{\mathscr{M}}_q$ and we have seen that its action on the darts of $\hat{\mathscr{M}}_q$ is regular. That is transitive and free (i.e. the stabilizer of the principal dart δ is $\{e\}$), as shown in Lemma 3.3. Also there is a bijection $\theta : H_q = \Gamma(2, \infty, q) \to G =< x, y >$ as given by $S \mapsto x$, $R \mapsto y^{-1}x$, where x, y are the permutations of the darts as described in Section 1.2, such that $x^2 = y^m = (y^{-1}x)^q = 1$ in G. We may identify the set of darts $\hat{\Omega}$ of $\hat{\mathscr{M}}_q$ with $|H_q|$ by means of the bijection $g \mapsto \hat{\alpha}g$, $g \in H_q$, so that H_q acts transitively on the set of darts by right multiplication. Therefore the algebraic map that is associated to the universal topological q-gonal map $\hat{\mathscr{M}}_q$ is just the universal algebraic q-gonal map $\hat{\mathscr{A}}_q$ defined by $\hat{\mathscr{A}}_q = (H_q, |H_q|, S, R)$, as introduced in Definition 1.3, where $|H_q|$ denotes the underlying set of $H_q (= \Gamma(2, \infty, q) = \langle S, R \,|\, S^2 = R^q = I \rangle$, as defined in Section 2.2).

Let $\mathscr{M}$ be any q-gonal map (i.e. $\mathscr{M}$ is a topological map of type (m, q)). By Theorem 1.4 every algebraic map $\mathscr{A}$ of type (m, n) is isomorphic to a quotient of the universal algebraic map $\hat{\mathscr{A}}$ of type (m, n) by a map-subgroup M. As observed in Section 1.4.3, the categories of algebraic maps and topological maps are equivalent, thus the same result holds for topological maps, that is $\hat{\mathscr{M}}_q/M \simeq \mathscr{M}$. As the map-subgroup for the universal q-gonal map is trivial and $\{e\} < M$, then by Theorem 1.7 it follows that $\hat{\mathscr{M}}_q$ covers $\mathscr{M}$ which is isomorphic to $\hat{\mathscr{M}}_q/M$. □

We have shown that $\hat{\mathscr{M}}_q$ is a universal object in the category of q-gonal maps and their morphisms. This justifies us calling $\hat{\mathscr{M}}_q$ the universal q-gonal map.

Example 3.5. When we look at $\hat{\mathscr{M}}_4$ as illustrated in Figure 3.11, we know that ∞ and 0 are vertices of $\hat{\mathscr{M}}_4$ but it is convenient to write $\infty = \frac{1}{0\sqrt{2}}$ and $0 = \frac{0\sqrt{2}}{1}$. The vertices of $\hat{\mathscr{M}}_4$ as in Table 3.1 are of two forms. The first have the form $\frac{a}{c\sqrt{2}}$, where a is odd and $(a, c) = 1$, these vertices called the *even vertices* which are the images of ∞ when we apply elements of H_4^e. The second have the form $\frac{b\sqrt{2}}{d}$, where d is odd and $(b, d) = 1$, these vertices called the *odd vertices* which are the images of ∞ when we apply elements of H_4^o. It follows that the subgroup H_4^e of even elements in H_4 maps even vertices to even vertices and odd vertices to odd vertices, while the set H_4^o of odd elements in H_4 maps even vertices to odd vertices. As $\infty = \frac{1}{0\sqrt{2}}$ is an even vertex and $0 = \frac{0\sqrt{2}}{1}$ is an odd vertex, and the edges of $\hat{\mathscr{M}}_4$ are images under H_4 of the edge joining $\frac{1}{0\sqrt{2}}$ to $\frac{0\sqrt{2}}{1}$, the edges of $\hat{\mathscr{M}}_4$ join even and odd vertices. Therefore, $\frac{a}{c\sqrt{2}}$ is joined to $\frac{b\sqrt{2}}{d}$ if and only if $ad - 2bc \equiv \pm 1 \bmod n$. Also as the principal face of $\hat{\mathscr{M}}_4$ has vertices $\frac{1}{0\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{1}{1\sqrt{2}}, \frac{1\sqrt{2}}{1}$, the faces of $\hat{\mathscr{M}}_4$ are the images of this principal face under H_4.

Example 3.6. Figures 3.11, 3.12, 3.13 and 3.15 below illustrate the universal q-gonal tessellation for $q = 4, 5, 6, 7$ and their tables of correspondence for each case. The vertices of the maps are polynomials in $\mathbb{Q}(\lambda_q)$, as apparent from the tables. It is evident from Table 3.4 that the vertices for the case $q = 7$ are polynomials of degree 2, replacing each λ_7^3 with its equivalent value $\lambda_7^2 + 2\lambda_7 - 1$, using Table 2.1.

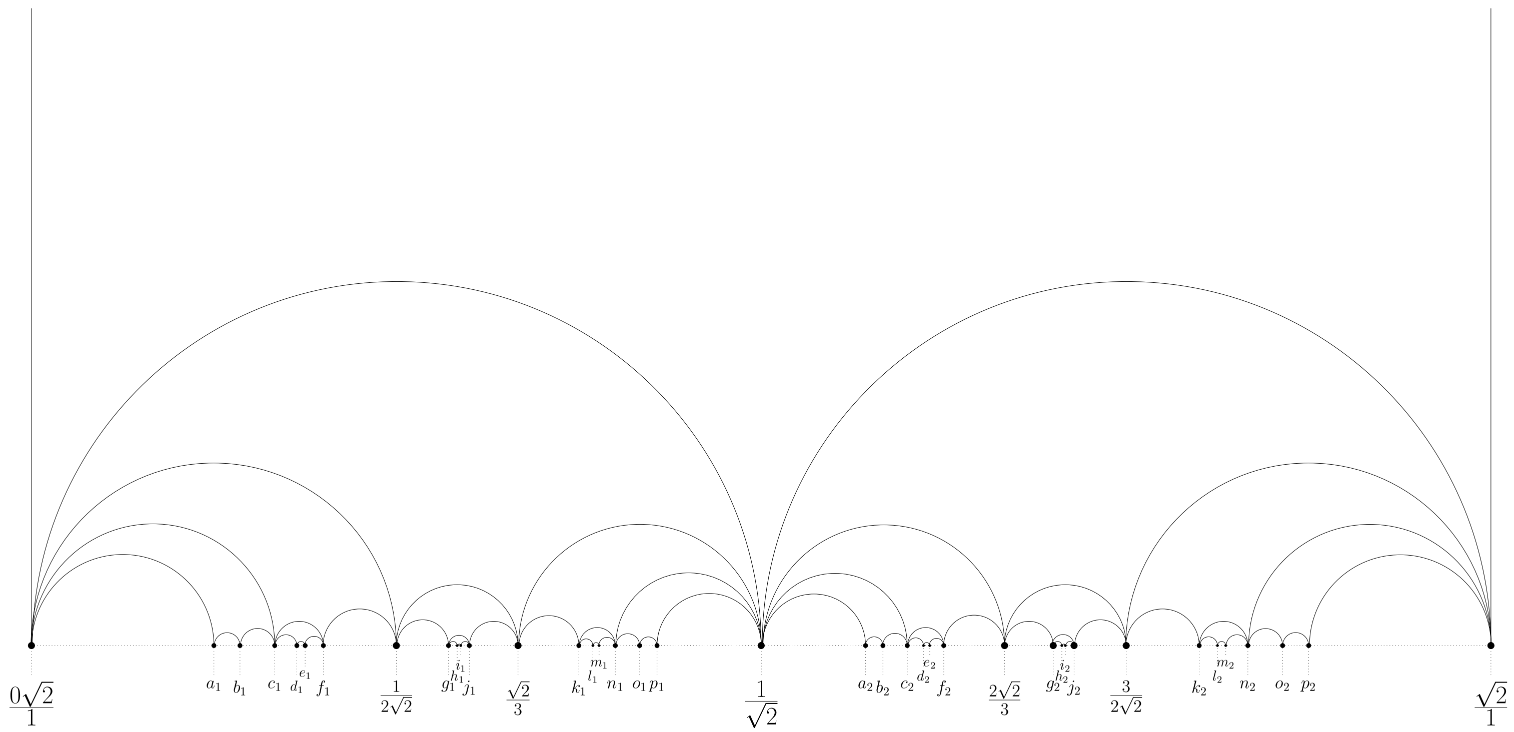


FIGURE 3.11: The universal 4-gonal tessellation $\hat{\mathscr{M}}_4$

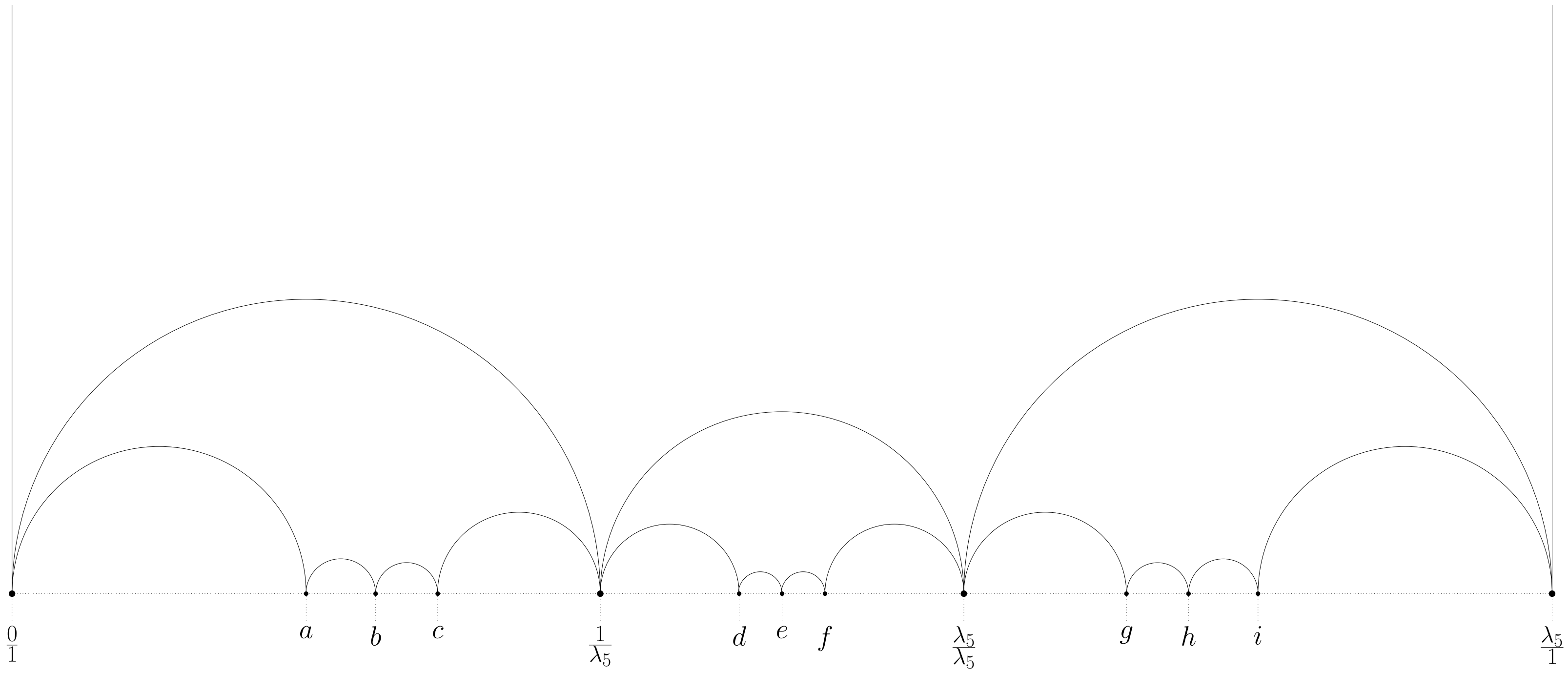


FIGURE 3.12: The universal 5-gonal tessellation $\hat{\mathscr{M}}_5$

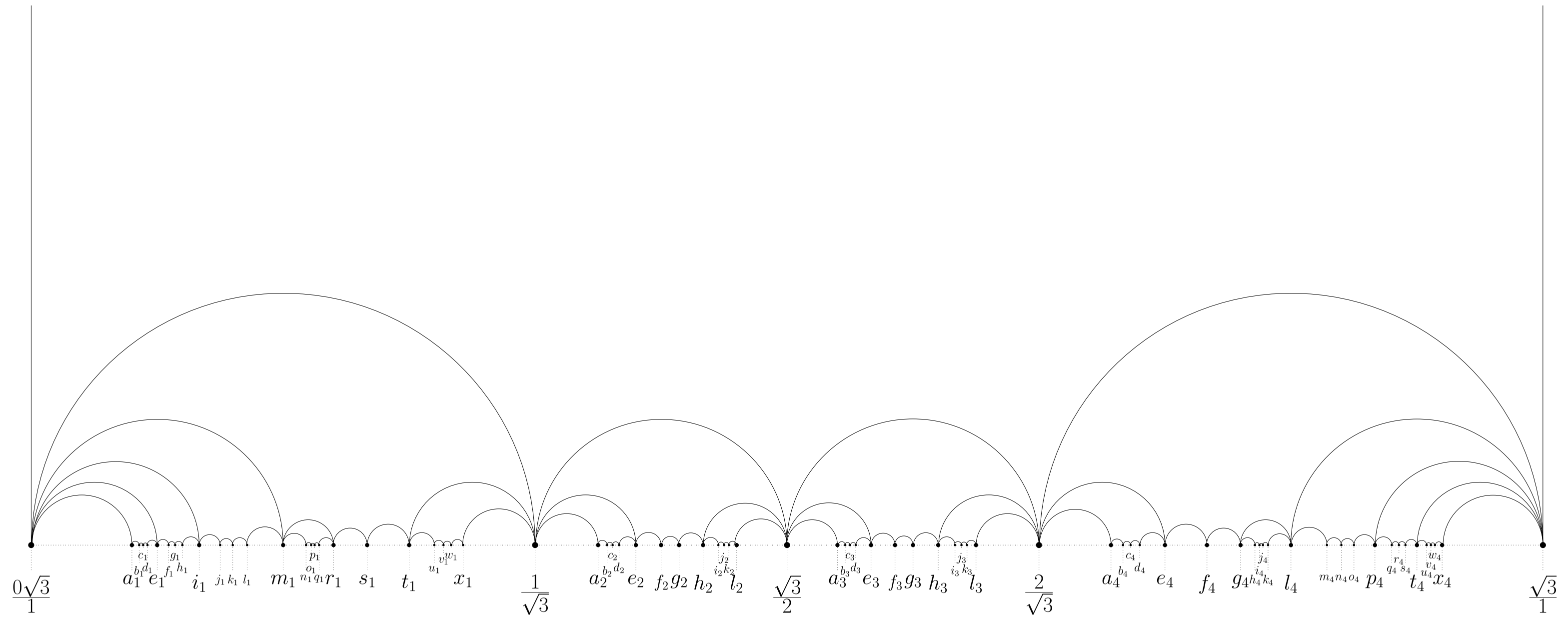


FIGURE 3.13: The universal 6-gonal tessellation $\hat{\mathscr{M}}_6$

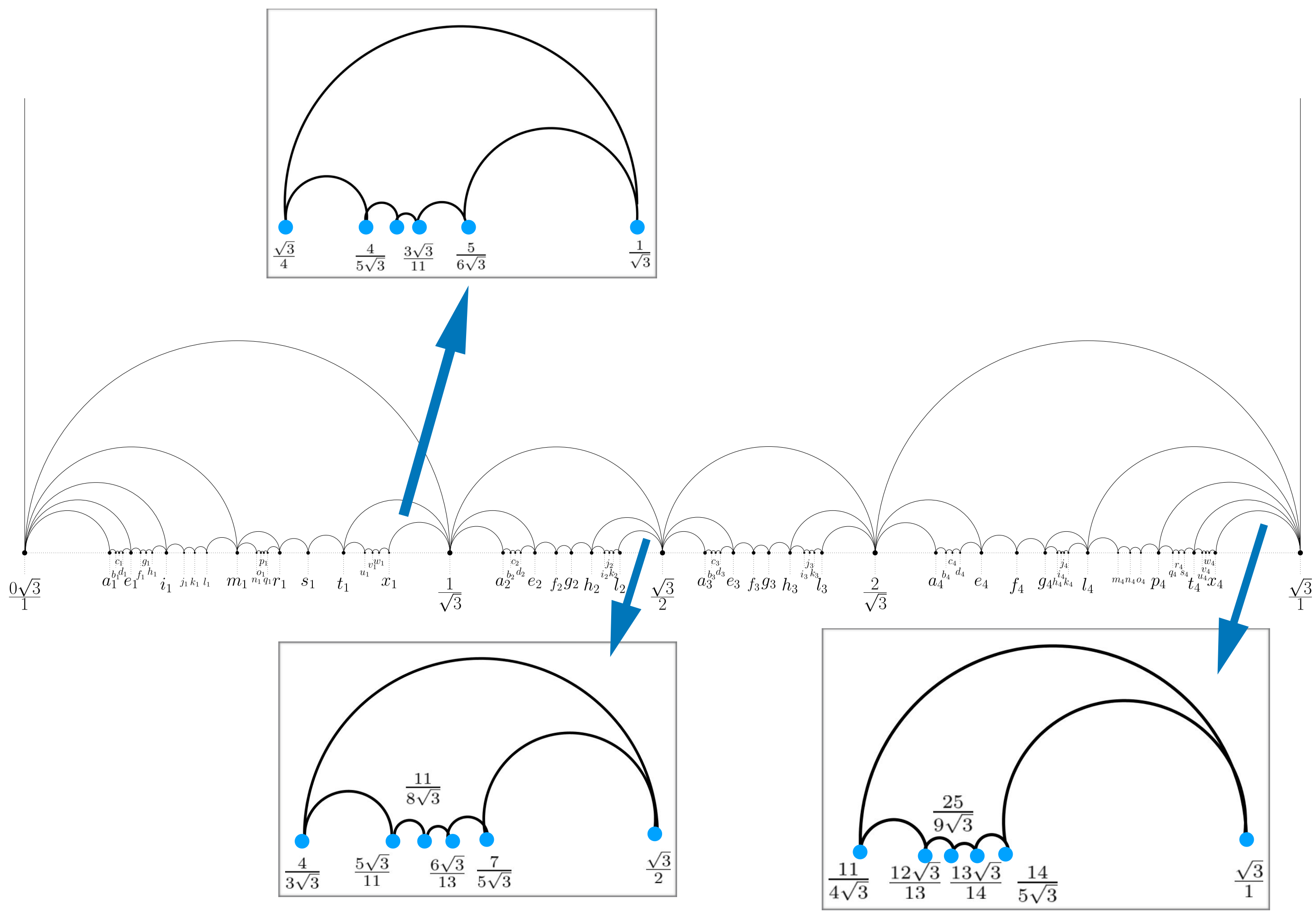


FIGURE 3.14: Magnifying sections of $\hat{\mathscr{M}}_6$

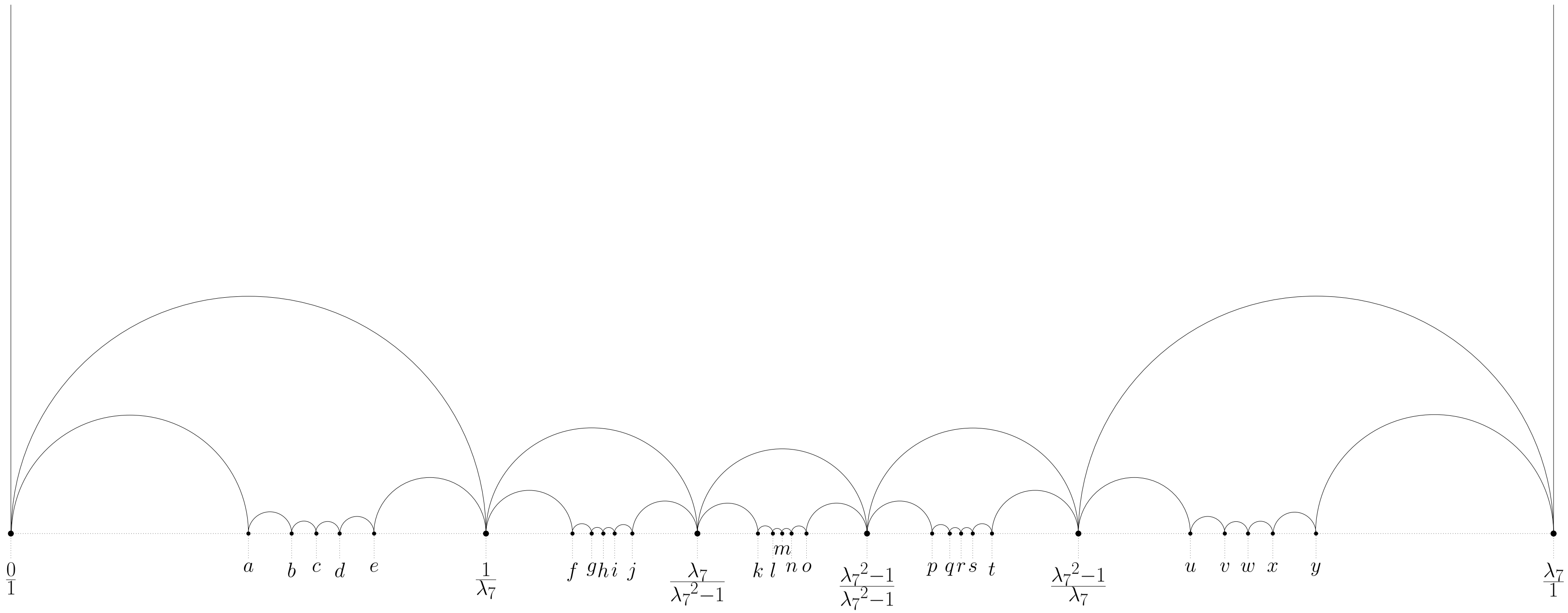


FIGURE 3.15: The universal 7-gonal tessellation $\hat{\mathscr{M}}_7$

TABLE 3.1: Table of Correspondence for $\hat{\mathscr{M}}_4$

$a_1 : \frac{1}{4\sqrt{2}}$	$i_1 : \frac{5\sqrt{2}}{17}$	$a_2 : \frac{4\sqrt{2}}{7}$	$i_2 : \frac{17}{12\sqrt{2}}$
$b_1 : \frac{\sqrt{2}}{7}$	$j_1 : \frac{3}{5\sqrt{2}}$	$b_2 : \frac{7}{6\sqrt{2}}$	$j_2 : \frac{5\sqrt{2}}{7}$
$c_1 : \frac{1}{3\sqrt{2}}$	$k_1 : \frac{3}{4\sqrt{2}}$	$c_2 : \frac{3\sqrt{2}}{5}$	$k_2 : \frac{4\sqrt{2}}{5}$
$d_1 : \frac{2\sqrt{2}}{11}$	$l_1 : \frac{5\sqrt{2}}{13}$	$d_2 : \frac{11}{9\sqrt{2}}$	$l_2 : \frac{13}{8\sqrt{2}}$
$e_1 : \frac{3}{8\sqrt{2}}$	$m_1 : \frac{7}{9\sqrt{2}}$	$e_2 : \frac{8\sqrt{2}}{13}$	$m_2 : \frac{9\sqrt{2}}{11}$
$f_1 : \frac{\sqrt{2}}{5}$	$n_1 : \frac{2\sqrt{2}}{5}$	$f_2 : \frac{5}{4\sqrt{2}}$	$n_2 : \frac{5}{3\sqrt{2}}$
$g_1 : \frac{2\sqrt{2}}{7}$	$o_1 : \frac{5}{6\sqrt{2}}$	$g_2 : \frac{7}{5\sqrt{2}}$	$o_2 : \frac{6\sqrt{2}}{7}$
$h_1 : \frac{7}{12\sqrt{2}}$	$p_1 : \frac{3\sqrt{2}}{7}$	$h_2 : \frac{12\sqrt{2}}{17}$	$p_2 : \frac{7}{4\sqrt{2}}$

TABLE 3.2: Table of Correspondence for $\hat{\mathscr{M}}_5$

$a : \frac{1}{2\lambda_5}$	$d : \frac{2\lambda_5}{2\lambda_5+1}$	$g : \frac{2\lambda_5+1}{\lambda_5+2}$
$b : \frac{\lambda_5}{2\lambda_5+1}$	$e : \frac{2\lambda_5+1}{2\lambda_5+2}$	$h : \frac{2\lambda_5+2}{2\lambda_5+1}$
$c : \frac{\lambda_5}{\lambda_5+2}$	$f : \frac{\lambda_5+2}{2\lambda_5+1}$	$i : \frac{2\lambda_5+1}{2\lambda_5}$

TABLE 3.3: Table of Correspondence for $\hat{\mathscr{M}}_6$

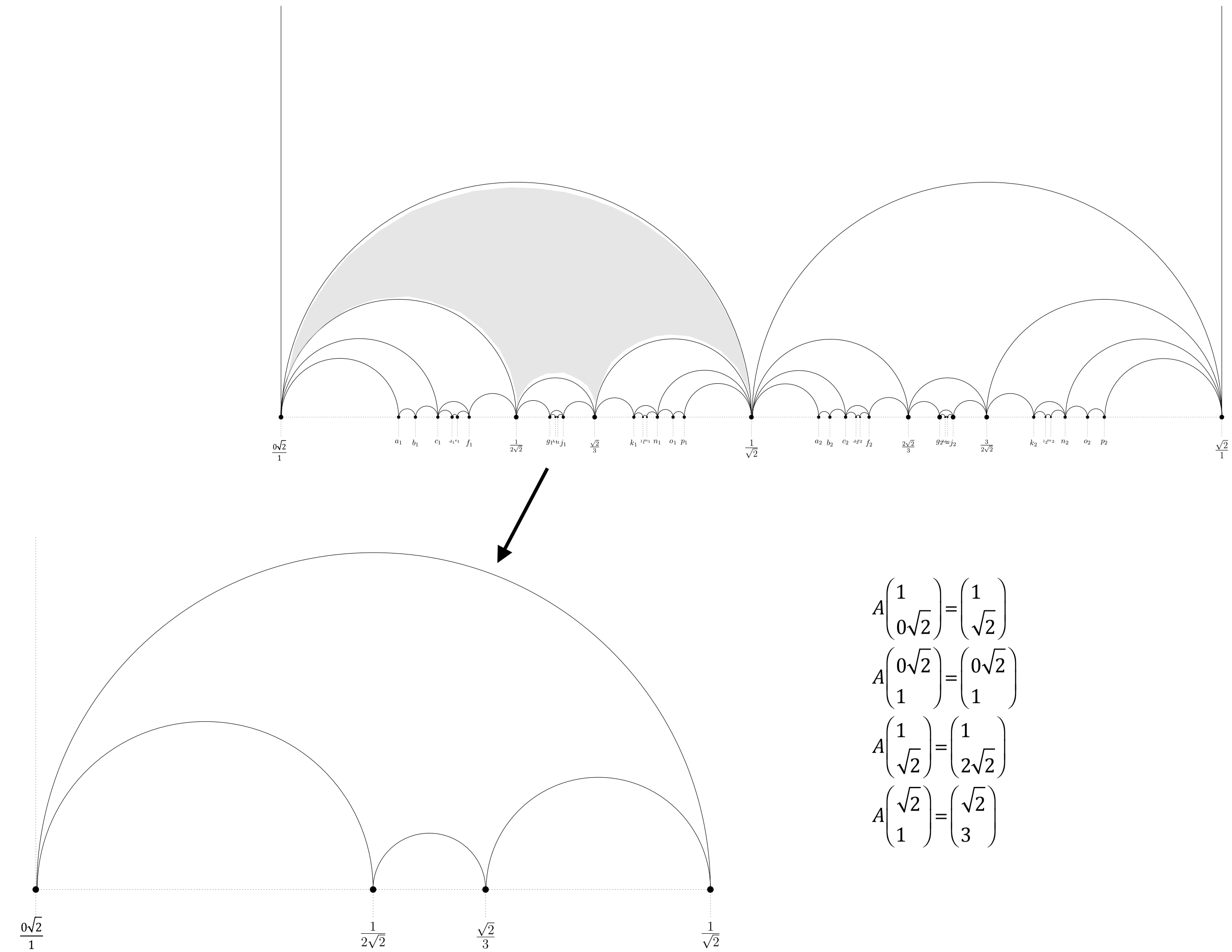
$a_1 : \frac{1}{5\sqrt{3}}$	$m_1 : \frac{1}{2\sqrt{3}}$	$a_2 : \frac{3\sqrt{3}}{8}$	$a_3 : \frac{8}{5\sqrt{3}}$	$a_4 : \frac{5\sqrt{3}}{7}$	$m_4 : \frac{6\sqrt{3}}{7}$
$b_1 : \frac{\sqrt{3}}{14}$	$n_1 : \frac{2\sqrt{3}}{11}$	$b_2 : \frac{8}{7\sqrt{3}}$	$b_3 : \frac{7\sqrt{3}}{13}$	$b_4 : \frac{13}{6\sqrt{3}}$	$n_4 : \frac{13}{5\sqrt{3}}$
$c_1 : \frac{2}{9\sqrt{3}}$	$o_1 : \frac{5}{9\sqrt{3}}$	$c_2 : \frac{5\sqrt{3}}{13}$	$c_3 : \frac{13}{8\sqrt{3}}$	$c_4 : \frac{8\sqrt{3}}{11}$	$o_4 : \frac{7\sqrt{3}}{8}$
$d_1 : \frac{\sqrt{3}}{13}$	$p_1 : \frac{3\sqrt{3}}{16}$	$d_2 : \frac{7}{6\sqrt{3}}$	$d_3 : \frac{6\sqrt{3}}{11}$	$d_4 : \frac{11}{5\sqrt{3}}$	$p_4 : \frac{8}{3\sqrt{3}}$
$e_1 : \frac{1}{4\sqrt{3}}$	$q_1 : \frac{4}{7\sqrt{3}}$	$e_2 : \frac{2\sqrt{3}}{5}$	$e_3 : \frac{5}{3\sqrt{3}}$	$e_4 : \frac{3\sqrt{3}}{4}$	$q_4 : \frac{9\sqrt{3}}{10}$
$f_1 : \frac{\sqrt{3}}{11}$	$r_1 : \frac{\sqrt{3}}{5}$	$f_2 : \frac{5}{4\sqrt{3}}$	$f_3 : \frac{4\sqrt{3}}{7}$	$f_4 : \frac{7}{3\sqrt{3}}$	$r_4 : \frac{19}{7\sqrt{3}}$
$g_1 : \frac{2}{7\sqrt{3}}$	$s_1 : \frac{2}{3\sqrt{3}}$	$g_2 : \frac{3\sqrt{3}}{7}$	$g_3 : \frac{7}{4\sqrt{3}}$	$g_4 : \frac{4\sqrt{3}}{5}$	$s_4 : \frac{10\sqrt{3}}{11}$
$h_1 : \frac{\sqrt{3}}{10}$	$t_1 : \frac{\sqrt{3}}{4}$	$h_2 : \frac{4}{3\sqrt{3}}$	$h_3 : \frac{3\sqrt{3}}{5}$	$h_4 : \frac{17}{7\sqrt{3}}$	$t_4 : \frac{11}{4\sqrt{3}}$
$i_1 : \frac{1}{3\sqrt{3}}$	$u_1 : \frac{4}{5\sqrt{3}}$	$i_2 : \frac{5\sqrt{3}}{11}$	$i_3 : \frac{11}{6\sqrt{3}}$	$i_4 : \frac{13\sqrt{3}}{16}$	$u_4 : \frac{12\sqrt{3}}{13}$
$j_1 : \frac{\sqrt{3}}{8}$	$v_1 : \frac{3\sqrt{3}}{11}$	$j_2 : \frac{11}{8\sqrt{3}}$	$j_3 : \frac{8\sqrt{3}}{13}$	$j_4 : \frac{22}{9\sqrt{3}}$	$v_4 : \frac{25}{9\sqrt{3}}$
$k_1 : \frac{2}{5\sqrt{3}}$	$w_1 : \frac{5}{6\sqrt{3}}$	$k_2 : \frac{6\sqrt{3}}{13}$	$k_3 : \frac{13}{7\sqrt{3}}$	$k_4 : \frac{9\sqrt{3}}{11}$	$w_4 : \frac{13\sqrt{3}}{14}$
$l_1 : \frac{\sqrt{3}}{7}$	$x_1 : \frac{2\sqrt{3}}{7}$	$l_2 : \frac{7}{5\sqrt{3}}$	$l_3 : \frac{5\sqrt{3}}{8}$	$l_4 : \frac{5}{2\sqrt{3}}$	$x_4 : \frac{14}{5\sqrt{3}}$

TABLE 3.4: Table of Correspondence for $\hat{\mathscr{M}}_7$

$a : \frac{1}{2\lambda_7}$	$f : \frac{2\lambda_7}{2{\lambda_7}^2-1}$	$k : \frac{2{\lambda_7}^2-1}{2{\lambda_7}^2+\lambda_7-2}$	$p : \frac{2{\lambda_7}^2+2\lambda_7-2}{{\lambda_7}^2+2\lambda_7-1}$	$u : \frac{{\lambda_7}^2+2\lambda_7-1}{{\lambda_7}^2+1}$
$b : \frac{\lambda_7}{2{\lambda_7}^2-1}$	$g : \frac{2{\lambda_7}^2-1}{2{\lambda_7}^2+2\lambda_7-2}$	$l : \frac{2{\lambda_7}^2+2\lambda_7-2}{2{\lambda_7}^2+2\lambda_7-1}$	$q : \frac{2{\lambda_7}^2+2\lambda_7-1}{2{\lambda_7}^2+\lambda_7}$	$v : \frac{2{\lambda_7}^2+\lambda_7}{{\lambda_7}^2+2\lambda_7-1}$
$c : \frac{{\lambda_7}^2-1}{2{\lambda_7}^2+\lambda_7-2}$	$h : \frac{2{\lambda_7}^2+\lambda_7-2}{2{\lambda_7}^2+2\lambda_7-1}$	$m : \frac{2{\lambda_7}^2+2\lambda_7-1}{2{\lambda_7}^2+2\lambda_7}$	$r : \frac{2{\lambda_7}^2+2\lambda_7}{2{\lambda_7}^2+2\lambda_7-1}$	$w : \frac{2{\lambda_7}^2+2\lambda_7-1}{2{\lambda_7}^2+\lambda_7-2}$
$d : \frac{{\lambda_7}^2-1}{{\lambda_7}^2+2\lambda_7-1}$	$i : \frac{{\lambda_7}^2+2\lambda_7-1}{2{\lambda_7}^2+\lambda_7}$	$n : \frac{2{\lambda_7}^2+\lambda_7}{2{\lambda_7}^2+2\lambda_7-1}$	$s : \frac{2{\lambda_7}^2+2\lambda_7-1}{2{\lambda_7}^2+2\lambda_7-2}$	$x : \frac{2{\lambda_7}^2+2\lambda_7-2}{2{\lambda_7}^2-1}$
$e : \frac{\lambda_7}{{\lambda_7}^2+1}$	$j : \frac{{\lambda_7}^2+1}{{\lambda_7}^2+2\lambda_7-1}$	$o : \frac{{\lambda_7}^2+2\lambda_7-1}{2{\lambda_7}^2+\lambda_7-2}$	$t : \frac{2{\lambda_7}^2+\lambda_7-1}{2{\lambda_7}^2-1}$	$y : \frac{2{\lambda_7}^2-1}{2\lambda_7}$

To draw a diagram for the universal q-gonal tessellation $\hat{\mathscr{M}}_q$, we must first find the principal face, F_0. Let $R = \begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}$, then R is an elliptic element of order q, as shown in Section 2.2. The vertices of F_0 are $R^k(\infty)$, $(k = 0, ..., q-1)$, as apparent from the Example 3.2. Then, all the other faces are images of the principal face F_0 under the elements A of H_q, of the form AF_0. Whenever the entries of the matrix involve the polynomial of λ_q for $q \geq 5$, we reduce this polynomial using the minimal polynomial for each λ_q. Overall, symmetry occurs around the center of each principal face for every $\hat{\mathscr{M}}_q$.

Example 3.7. Applying $A = \begin{pmatrix} 1 & 0\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \in H_4^e$ to the principal face F_0 of $\hat{\mathscr{M}}_4$, we get the shaded face, as shown in Figure 3.16.



$$A\begin{pmatrix}1\\0\sqrt{2}\end{pmatrix}=\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}$$
$$A\begin{pmatrix}0\sqrt{2}\\1\end{pmatrix}=\begin{pmatrix}0\sqrt{2}\\1\end{pmatrix}$$
$$A\begin{pmatrix}1\\\sqrt{2}\end{pmatrix}=\begin{pmatrix}1\\2\sqrt{2}\end{pmatrix}$$
$$A\begin{pmatrix}\sqrt{2}\\1\end{pmatrix}=\begin{pmatrix}\sqrt{2}\\3\end{pmatrix}$$

FIGURE 3.16: A magnifying section of $\hat{\mathscr{M}}_4$

Chapter 4

Principal congruence maps $\mathscr{M}_q$

4.1 The congruence subgroups of the modular group

We mentioned in Section 2.1 that the triangle group $\Gamma(2, \infty, 3)$ is considered to be $\Gamma = PSL(2, \mathbb{Z})$, which is defined by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} / \{\pm I\}.$$

Now for any positive integer n we define the *principal congruence subgroup* of level n of the modular group Γ as follows

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm I \mod n \right\}$$

that is, the group of elements of the matrices of Γ which are congruent to $\pm I$ mod n, while remembering to identify each element of Γ with its negative. Any subgroup G of Γ containing a principal congruence subgroup $\Gamma(n)$ is referred to as a *congruence subgroup*, and the least n such that $G \geq \Gamma(n)$ is called the *level* of G.
Some examples of congruence subgroups of special interest are

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \mod n \right\}$$

and

$$\Gamma_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv \pm 1 \mod n \text{ and } c \equiv 0 \mod n \right\};$$

each can be defined for any positive integer n. Observe that the congruence $d \equiv \pm 1 \bmod n$ in the definition of $\Gamma_1(n)$ can be deduced from the congruences $a \equiv \pm 1 \bmod n,\ c \equiv 0 \bmod n$ and the condition $ad - bc = 1$ of the determinant.

Clearly the inclusions

$$\Gamma(n) \leq \Gamma_1(n) \leq \Gamma_0(n) \leq \Gamma$$

hold for any positive integer n. Furthermore, we have

1. $\Gamma(n)$ is a normal subgroup of Γ, i.e. $\Gamma(n) \triangleleft \Gamma$ and so it is a normal subgroup of $\Gamma_0(n)$ and of $\Gamma_1(n)$.

2. $\Gamma_1(n)$ is a normal subgroup of $\Gamma_0(n)$, i.e. $\Gamma_1(n) \triangleleft \Gamma_0(n)$.

For the indices of the above inclusions we have the following formulae, [Ran77],[DS05];

$$|\Gamma : \Gamma_0(n)| = n \prod_{p|n} (1 + \frac{1}{p}) \tag{4.1.1}$$

$$|\Gamma : \Gamma_1(n)| = \frac{n^2}{2} \prod_{p|n} (1 - \frac{1}{p^2}) \tag{4.1.2}$$

$$|\Gamma : \Gamma(n)| = \frac{n^3}{2} \prod_{p|n} (1 - \frac{1}{p^2}) \tag{4.1.3}$$

$$|\Gamma_1(n) : \Gamma(n)| = n \tag{4.1.4}$$

$$|\Gamma_0(n) : \Gamma_1(n)| = \frac{\phi(n)}{2} \tag{4.1.5}$$

for any $n > 2$ and with the product running over the distinct prime divisors of n. For the proof of the above indices see [IS05, Section 2.1]. The case $n = 2$ is treated in Section 4.3.

4.2 The congruence subgroups of the Hecke groups

Let I be an ideal of $\mathbb{Z}[\lambda_q]$. We define

$$PSL(2, \mathbb{Z}[\lambda_q], I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_q]) \mid a-1, b, c, d-1 \ \in I \right\}.$$

Similarly, we define $PSL_1(2, \mathbb{Z}[\lambda_q], I)$ and $PSL_0(2, \mathbb{Z}[\lambda_q], I)$, as follows,

$$PSL_1(2, \mathbb{Z}[\lambda_q], I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_q]) \mid a-1, c, d-1 \ \in I \right\},$$

and

$$PSL_0(2, \mathbb{Z}[\lambda_q], I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_q]) \mid c \in I \right\}.$$

Now for any ideal I of $\mathbb{Z}[\lambda_q]$ we define the *principal congruence subgroup* of the Hecke group H_q as,

$$H_q(I) = PSL(2, \mathbb{Z}[\lambda_q], I) \cap H_q,$$

that is, the subgroup of H_q consisting of elements in $PSL(2, \mathbb{Z}[\lambda_q])$. Moreover, analogous definitions of $(H_q)_1(I)$ and $(H_q)_0(I)$ of the form;

$$(H_q)_1(I) = PSL_1(2, \mathbb{Z}[\lambda_q], I) \cap H_q,$$

and

$$(H_q)_0(I) = PSL_0(2, \mathbb{Z}[\lambda_q], I) \cap H_q.$$

Clearly,

$$H_q(I) \leqslant (H_q)_1(I) \leqslant (H_q)_0(I) \leqslant H_q.$$

For the rest of this chapter we take the special case when $I = (n)$ and $2 < n \in \mathbb{Z}^+$, i.e. (n) is a principal ideal of $\mathbb{Z}[\sqrt{m}]$ where $m = 2, 3$ for $q = 4, 6$. In particular

$$H_q(n) \leqslant (H_q)_1(n) \leqslant (H_q)_0(n) \leqslant H_q. \tag{4.2.1}$$

The even principal congruence subgroup is now

$$H_q^e(n) = \left\{ \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \in H_q^e \mid a \equiv d \equiv \pm 1 \bmod n,\ b \equiv c \equiv 0 \bmod n \right\}. \tag{4.2.2}$$

We define the even subgroups of $(H_q)_1(n)$ and $(H_q)_0(n)$ as follows;

$$(H_q)_1^e(n) = \left\{ \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \in H_q^e \mid a \equiv d \equiv \pm 1 \bmod n, c \equiv 0 \bmod n \right\}, \tag{4.2.3}$$

$$(H_q)_0^e(n) = \left\{ \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \in H_q^e \mid c \equiv 0 \bmod n \right\}. \tag{4.2.4}$$

Also,

$$(H_q)_0^o(n) = \left\{ \begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix} \in H_q^o \mid c \equiv 0 \bmod n \right\} \tag{4.2.5}$$

is the set of odd elements and the other coset of $(H_q)_0^e(n)$ in $(H_q)_0(n)$.
As $a\sqrt{m}$ and $d\sqrt{m}$ are not congruent to ± 1 mod n we can not have odd elements in

$H_q(n)$ and $(H_q)_1(n)$. Hence

$$H_q(n) = H_q^e(n), \text{ i.e. } |H_q(n) : H_q^e(n)| = 1, \tag{4.2.6}$$

and

$$(H_q)_1(n) = (H_q)_1^e(n), \text{ i.e. } |(H_q)_1(n) : (H_q)_1^e(n)| = 1. \tag{4.2.7}$$

Thus

$$H_q^e(n) \leqslant (H_q)_1^e(n) \leqslant (H_q)_0^e(n) \leqslant H_q^e. \tag{4.2.8}$$

An *even element* is an element of the even subgroups (4.2.2), (4.2.3) and (4.2.4) otherwise the element is *odd*.
Note that if we consider the multiplication of these elements, we have

$$odd.odd{=}even.even{=}even$$

$$even.odd{=}odd.even{=}odd.$$

Our aim is to compute the index $|H_q : H_q(n)|$ for $q = 4, 6$ and so in the following pages we compute the intermediate indices $|(H_q)_1(n) : H_q(n)|$, $|(H_q)_0(n) : (H_q)_1(n)|$ and $|H_q : (H_q)_0(n)|$.

If $q = 4$ and $m = 2$, then we have the following lemma;

Lemma 4.1. *If n is even then there are no odd elements in $(H_4)_0(n)$.*

Proof. If n is even and as $c \equiv 0 \mod n$ i.e. $n|c$ by (4.2.5) then c is even. Therefore $2ad - bc$ is even, which is a contradiction. Hence if n is even then $(H_4)_0(n)$ contains no odd elements. □

Now let $q = 6$ and $m = 3$, then

Lemma 4.2. *If $3|n$ then there are no odd elements in $(H_6)_0(n)$.*

Proof. If $3|n$ and as $c \equiv 0 \mod n$ i.e. $n|c$ by (4.2.5) then $3|c$. Therefore $3|3ad - bc$, which is a contradiction. Hence there are no odd elements in $(H_6)_0(n)$ when n is a multiple of 3. □

Remark 4.3. If $m = 2$ then $(n, m) = m$ if and only if n is even. If $m = 3$ then $(n, m) = m$ if and only if $3|n$. Then

Proposition 4.4. *If $(n,m)=m$, then $(H_{2m})_0(n)=(H_{2m})_0^e(n)$ (i.e. there are no odd elements in $(H_{2m})_0(n)$) . If $(n,m)=1$, then $(H_{2m})_0^e(n)$ is a subgroup of index two in $(H_{2m})_0(n)$, for $m=2,3$.*

Proof. To prove this we just need to show that $(H_{2m})_0(n)$ contains an odd element if $(n,m)=1$. Suppose $m=2$, then $A=\begin{pmatrix}1\sqrt{2} & 1\\ n & \frac{n+1}{2}\sqrt{2}\end{pmatrix}$ is an odd element in $(H_4)_0(n)$. Similarly, if $m=3$ and $n\equiv -1 \bmod 3$, then $B=\begin{pmatrix}1\sqrt{3} & 1\\ n & \frac{n+1}{3}\sqrt{3}\end{pmatrix}$ and if $n\equiv 1 \bmod 3$ then we get $C=\begin{pmatrix}\frac{1-n}{3}\sqrt{3} & 1\\ -n & 1\sqrt{3}\end{pmatrix}$, where B and C are odd elements in $(H_6)_0(n)$. □

Example 4.5. If $m=2$ and $n=3$, then the following is the set of cosets representatives for the subgroup $(H_4)_0(3)$

$$\{\pm\begin{pmatrix}1 & 0\sqrt{2}\\ 0\sqrt{2} & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 1\sqrt{2}\\ 0\sqrt{2} & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 2\sqrt{2}\\ 0\sqrt{2} & 1\end{pmatrix},$$
$$\pm\begin{pmatrix}1\sqrt{2} & 1\\ 3 & 2\sqrt{2}\end{pmatrix}, \pm\begin{pmatrix}2\sqrt{2} & 1\\ 3 & 1\sqrt{2}\end{pmatrix}, \pm\begin{pmatrix}5\sqrt{2} & 3\\ 3 & 1\sqrt{2}\end{pmatrix}\}.$$

In the following table we determine the indices for the vertical inclusions in Figure 4.1

TABLE 4.1

$\lvert H_q(n):H_q^e(n)\rvert=1$	by (4.2.6)
$\lvert (H_q)_1(n):(H_q)_1^e(n)\rvert=1$	by (4.2.7)
$$\lvert (H_q)_0(n):(H_q)_0^e(n)\rvert=\begin{cases}2 & if\ (n,m)=1\\ 1 & if\ (n,m)=m.\end{cases} \quad (4.2.9)$$	by Proposition 4.4
$\lvert H_q:H_q^e\rvert=2$	by (2.2.1)

$$\begin{array}{ccccccc} H_q(n) & \leqslant & (H_q)_1(n) & \leqslant & (H_q)_0(n) & \leqslant & H_q \\ \vee & & \vee & & \vee & & \vee \\ H_q^e(n) & \leqslant & (H_q)_1^e(n) & \leqslant & (H_q)_0^e(n) & \leqslant & H_q^e \end{array}$$

FIGURE 4.1: The subgroups lattices of H_q

Our aim is to determine the index $|H_q : H_q(n)|$ by determining the index of each inclusion in the lower chain of inclusions in Figure 4.1. As a consequence we can determine the index of each inclusion in the upper chain of inclusions in Figure 4.1 using Table 4.1 i.e. the indices $|(H_q)_0(n) : (H_q)_1(n)|$, $|H_q : (H_q)_0(n)|$ and $|(H_q)_1(n) : H_q(n)|$.

As

$$H_q^e(n) \leqslant (H_q)_1^e(n) \leqslant (H_q)_0^e(n) \leqslant H_q^e \leqslant H_q, \tag{4.2.10}$$

we will show

1. $H_q^e(n) \trianglelefteq H_q$, and so $H_q^e(n)$ is normal in $(H_q)_1^e(n)$, $(H_q)_0^e(n)$ and H_q^e.
2. $(H_q)_1^e(n) \trianglelefteq (H_q)_0^e(n)$.

To prove (1) suppose $A = \begin{pmatrix} a & \sqrt{m}b \\ \sqrt{m}c & d \end{pmatrix} \in H_q^e(n)$, then we conjugate A by the generators of H_q

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{T} = \begin{pmatrix} 1 & \sqrt{m} \\ 0 & 1 \end{pmatrix}$$

respectively, we get

$$SAS^{-1} = \begin{pmatrix} d & -\sqrt{m}c \\ -\sqrt{m}b & a \end{pmatrix} \equiv \pm I \mod n,$$

so $SAS^{-1} \in H_q^e(n)$, and

$$TAT^{-1} = \begin{pmatrix} a+mc & \sqrt{m}(b+d-a-mc) \\ \sqrt{m}c & d-mc \end{pmatrix} \equiv \pm I \mod n$$

so $TAT^{-1} \in H_q^e(n)$. To prove (2), let $\mathbb{U}(n)$ denotes the group of units modulo n and consider the ring homomorphism

$$\theta : (H_q)_0^e(n) \to \mathbb{U}(n)/\{\pm 1\}$$

defined by $\theta \begin{pmatrix} a & \sqrt{m}b \\ \sqrt{m}nc & d \end{pmatrix} \equiv \pm d \bmod n$.

Because

$$\theta(\begin{pmatrix} a & \sqrt{m}b \\ \sqrt{m}nc & d \end{pmatrix} \begin{pmatrix} a' & \sqrt{m}b' \\ \sqrt{m}nc' & d' \end{pmatrix}) = \theta \begin{pmatrix} aa'+mnc'b & \sqrt{m}(ab'+bd') \\ n\sqrt{m}(ca'+dc') & mncb'+dd' \end{pmatrix}$$

$$\equiv \pm(mncb'+dd') \bmod n \equiv \pm dd' \bmod n$$

$$\equiv \theta \begin{pmatrix} a & \sqrt{m}b \\ \sqrt{m}nc & d \end{pmatrix} \theta \begin{pmatrix} a' & \sqrt{m}b' \\ \sqrt{m}nc' & d' \end{pmatrix},$$

θ is a homomorphism. To prove θ is epimorphism, let $d \in \mathbb{U}(n)/\{\pm 1\}$ and we need to find $A \in (H_q)_0^e(n)$ such that $\theta(A) \equiv \pm d \bmod n$. We first consider the case $m = 2$; if n is even, then d is odd as $(n, d) = 1$. If n is odd, by replacing d with $d+n$ if needed, we may assume as well that d is odd. Then d is coprime to $2n$ and by Bézout's identity, there exist integers a, b such that $ad + 2bn = 1$. Then the matrix $\begin{pmatrix} a & b\sqrt{2} \\ -n\sqrt{2} & d \end{pmatrix} \in (H_4)_0^e(n)$ has determinant 1 and we have $\theta(A) \equiv \pm d \bmod n$ as desired. For $m = 3$; if $3|n$, then $3 \nmid d$ as $(n, d) = 1$. If $3 \nmid n$, by replacing d with $d+n$ when $d \equiv 2 \bmod 3$ and replacing d with $(d+1)+n$ when $d \equiv 1 \bmod 3$, if needed, we may assume as well that $3 \nmid d$. Then d is coprime to $3n$ and by Bézout's identity, there exist integers a, b such that $ad+3bn = 1$. Then the matrix $\begin{pmatrix} a & b\sqrt{3} \\ -n\sqrt{3} & d \end{pmatrix} \in (H_6)_0^e(n)$ has determinant 1 and we have $\theta(A) \equiv \pm d \bmod n$. Hence θ is an epimorphism.

If $d \equiv \pm 1 \bmod n$, then because $ad \equiv 1 \bmod n$, then $a \equiv \pm 1 \bmod n$, showing that the kernel of θ is $(H_q)_1^e(n)$ and hence $(H_q)_1^e(n) \trianglelefteq (H_q)_0^e(n)$ with index $\frac{\phi(n)}{2}$.

For the indices of the quotient groups in the above chain (4.2.10), working out the index of $H_q^e(n)$ in $(H_q)_1^e(n)$, we define a homomorphism

$$\psi : (H_q)_1^e(n) \to \mathbb{Z}_n$$

by

$$\psi(\pm \begin{pmatrix} an+1 & \sqrt{m}b \\ cn\sqrt{m} & dn+1 \end{pmatrix}) \equiv b \bmod n.$$

ψ is homomorphism because

$$\psi(\begin{pmatrix} an+1 & \sqrt{m}b \\ \sqrt{m}cn & dn+1 \end{pmatrix} \begin{pmatrix} a'n+1 & \sqrt{m}b' \\ \sqrt{m}c'n & d'n+1 \end{pmatrix})$$

$$\begin{aligned} &= \psi \begin{pmatrix} n(aa'n + a + a' + mbc') + 1 & \sqrt{m}(b'an + b' + bd'n + b) \\ \sqrt{m}n(cna' + c + c'dn + c') & n(mcb' + dd'n + d + d') + 1 \end{pmatrix} \\ &\equiv (b'an + b' + bd'n + b) \mod n \equiv (b' + b) \mod n \\ &\equiv \psi \begin{pmatrix} an+1 & \sqrt{m}b \\ \sqrt{m}cn & dn+1 \end{pmatrix} \psi \begin{pmatrix} a'n+1 & \sqrt{m}b' \\ \sqrt{m}c'n & d'n+1 \end{pmatrix}. \end{aligned}$$

Also given any $b \in \mathbb{Z}_n$, then $\begin{pmatrix} 1 & \sqrt{m}b \\ 0 & 1 \end{pmatrix} \in (H_q)_1^e(n)$ and is mapped to b under ψ, hence ψ is clearly an epimorphism. The kernel consists of those elements with $b \equiv 0 \bmod n$. As $b \equiv 0 \bmod n$, $ad \equiv \pm 1 \bmod n$. Then the kernel is $H_q^e(n)$ which shows that

$$|(H_q)_1^e(n) : H_q^e(n)| = n. \tag{4.2.11}$$

Using (4.2.6) and (4.2.7), as there are no odd elements in $H_q(n)$ and $(H_q)_1(n)$, we get

$$|(H_q)_1(n) : H_q(n)| = n. \tag{4.2.12}$$

From (2) as the kernel is $(H_q)^e_1(n)$, so the index of $(H_q)^e_1(n)$ in $(H_q)^e_0(n)$ is

$$|(H_q)^e_0(n) : (H_q)^e_1(n)| = \frac{\phi(n)}{2}. \tag{4.2.13}$$

Now as

$$|(H_q)_0(n) : (H_q)_1(n)| = |(H_q)_0(n) : (H_q)^e_0(n)||(H_q)^e_0(n) : (H_q)^e_1(n)|,$$

using (4.2.7), (4.2.9) and (4.2.13) , we get

$$|(H_q)_0(n) : (H_q)_1(n)| = \begin{cases} \phi(n) & if\ (n, m) = 1 \\ \frac{\phi(n)}{2} & if\ (n, m) = m. \end{cases} \tag{4.2.14}$$

The index of $(H_q)_0(n)$ in H_q was worked out by Keskin [Kes98, Lemma 3]. In order to show the proof of this lemma we need to mention the following proposition.

Proposition 4.6. *The groups H^e_4 , H^e_6 are conjugate to the congruence subgroups $\Gamma_0(2)$ and $\Gamma_0(3)$ respectively inside $PSL(2,\mathbb{R})$.*

Proof. For $q = 4, 6$ and $m = 2, 3$. Let

$$A = \begin{pmatrix} m^{\frac{-1}{4}} & 0 \\ 0 & m^{\frac{1}{4}} \end{pmatrix} \in PSL(2,\mathbb{R}) \text{ and } B = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \in H^e_q,$$

then

$$ABA^{-1} = \begin{pmatrix} a & b \\ cm & d \end{pmatrix} \in \Gamma_0(m).$$

Then the mapping $\chi : H^e_q \to \Gamma_0(m)$ defined by

$$\chi(\begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}) = \begin{pmatrix} a & b \\ cm & d \end{pmatrix} \in \Gamma_0(m)$$

shows that H^e_q is conjugate to $\Gamma_0(m)$. □

Theorem 4.7.

$$|H_q : (H_q)_0(n)| = \begin{cases} n\prod_{p|n}(1 + \frac{1}{p}) & if\ (n, m) = 1 \\ 2n\prod_{p|n,p\neq m}(1 + \frac{1}{p}) & if\ (n, m) = m. \end{cases} \tag{4.2.15}$$

Proof. First assume that $(m, n) = m$, where $p \neq m$. It is clear that $(H_q)^e_0(n) \subset H^e_q \subset H_q$. Let χ be the mapping defined in Proposition 4.6. Then χ is a mapping from H^e_q to $\Gamma_0(m)$.

Also since the map χ is conjugation by an element of $PSL(2,\mathbb{R})$ then χ is an isomorphism and $\chi((H_q)_0^e(n)) = \Gamma_0(mn)$. Since $\Gamma_0(mn) \subset \Gamma_0(m) \subset \Gamma$, we get

$$|H_q : (H_q)_0(n)| = |H_q : H_q^e||H_q^e : (H_q)_0(n)| = 2|H_q^e : (H_q)_0^e(n)|,$$

where

$$|H_q^e : (H_q)_0^e(n)| = |\Gamma_0(m) : \Gamma_0(mn)| = \frac{|\Gamma : \Gamma_0(mn)|}{|\Gamma : \Gamma_0(m)|} = n \prod_{p|n, p\neq m} (1 + \frac{1}{p}).$$

Now if $m = 2$ and $p \neq 2$, we have

$$|H_4 : (H_4)_0(n)| = 2|H_4^e : (H_4)_0^e(n)| = 2|\Gamma_0(2) : \Gamma_0(2n)| = 2\frac{|\Gamma : \Gamma_0(2n)|}{|\Gamma : \Gamma_0(2)|} = 2n \prod_{p|n, p\neq 2} (1+\frac{1}{p}).$$

Similarly when $m = 3$ and $p \neq 3$, we have

$$|H_6 : (H_6)_0(n)| = 2|H_6^e : (H_6)_0^e(n)| = 2|\Gamma_0(3) : \Gamma_0(3n)| = 2\frac{|\Gamma : \Gamma_0(3n)|}{|\Gamma : \Gamma_0(3)|} = 2n \prod_{p|n, p\neq 3} (1+\frac{1}{p}).$$

For the case where $(m,n) = 1$, since $(H_q)_0^e(n) \subset H_q^e$ and $(H_q)_0^e(n) \subset (H_q)_0(n)$. It is clear that

$$|H_q : H_q^e| = |(H_q)_0(n) : (H_q)_0^e(n)| = 2.$$

Thus we have

$$|H_q : (H_q)_0(n)| = \frac{|H_q : (H_q)_0^e(n)|}{|(H_q)_0(n) : (H_q)_0^e(n)|}$$

$$= \frac{|H_q : H_q^e||H_q^e : (H_q)_0^e(n)|}{|(H_q)_0(n) : (H_q)_0^e(n)|} = |H_q^e : (H_q)_0^e(n)|.$$

Using Proposition 4.6 and the map χ as defined above, we have

$$|H_q^e : (H_q)_0^e(n)| = |\Gamma_0(m) : \Gamma_0(mn)| = \frac{|\Gamma : \Gamma_0(mn)|}{|\Gamma : \Gamma_0(m)|} = n \prod_{p|n} (1 + \frac{1}{p}).$$

□

If $(n,m) = 1$, using the first factors given by Theorem 4.7 and (4.2.14) with $\phi(n) = n\prod_{p|n}(1 - \frac{1}{p})$, we have

$$|H_q : (H_q)_1(n)| = |H_q : (H_q)_0(n)||(H_q)_0(n) : (H_q)_1(n)| = n^2 \prod_{p|n} (1 - \frac{1}{p^2}).$$

As the case $(n,m) = m$ can be calculated in a similar way, we get

$$|H_q : (H_q)_1(n)| = \begin{cases} n^2 \prod_{p|n}(1 - \frac{1}{p^2}) & if\ (n,m) = 1 \\ n^2(1 - \frac{1}{m}) \prod_{p|n, p\neq m}(1 - \frac{1}{p^2}) & if\ (n,m) = m. \end{cases} \tag{4.2.16}$$

Now as a conclusion, using Theorem 4.7, (4.2.14) and (4.2.12) as defined above we can derive the following index formulae.
For $(n, m) = 1$, we have

$$
\begin{aligned}
|H_q : H_q(n)| &= |H_q : (H_q)_0(n)||(H_q)_0(n) : (H_q)_1(n)||(H_q)_1(n) : H_q(n)| \\
&= n\prod_{p|n}(1+\frac{1}{p}).n\prod_{p|n}(1-\frac{1}{p}).n \\
&= n^3\prod_{p|n}(1-\frac{1}{p^2}).
\end{aligned}
$$

For $(n, m) = m$, we have

$$
\begin{aligned}
|H_q : H_q(n)| &= 2n\prod_{p|n, p\neq m}(1+\frac{1}{p}).\frac{n}{2}\prod_{p|n}(1-\frac{1}{p}).n \\
&= n^3(1-\frac{1}{m})\prod_{p|n, p\neq m}(1-\frac{1}{p^2}).
\end{aligned}
$$

Let us define $\mu_q(n)$ as follows:

$$
\mu_q(n) = \begin{cases} n^3\prod_{p|n}(1-\frac{1}{p^2}) & if\ (n, m) = 1 \\ n^3(1-\frac{1}{m})\prod_{p|n, p\neq m}(1-\frac{1}{p^2}) & if\ (n, m) = m. \end{cases} \tag{4.2.17}
$$

Now we have the following theorem which confirms Parson's result [Par76, Theorem 2.3].

Theorem 4.8. $\mu_q(n) = |H_q : (H_q)(n)|$.

As we know this is valid for $q = 4$ and 6, and only for $n > 2$. In the next section we discuss the corresponding results for H_3,H_4 and H_6 where $n = 2$.

4.3 The case $n = 2$

Every element of $H_3 = \Gamma$ is congruent mod(2) to precisely one of:

$$
e = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, s_1 = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, s_2 = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
$$

$$
s_3 = \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, r_1 = \pm\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, r_2 = \pm\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.
$$

The group $\Gamma/\Gamma(2)$ is generated by s_1, r_2 which obey the relations

$$
s_1^2 = r_2^3 = (s_1 r_2)^2 = e,
$$

so $\Gamma/\Gamma(2)$ is isomorphic to the dihedral group D_3 of order 6. Also for $n = 2$ we have $\Gamma_0(2) = \Gamma_1(2)$, a subgroup of index 3 in Γ.

Every element of H_4 is congruent mod(2) to precisely one of:

$$e = \pm \begin{pmatrix} 1 & 0\sqrt{2} \\ 0\sqrt{2} & 1 \end{pmatrix}, s_1 = \pm \begin{pmatrix} 1 & 1\sqrt{2} \\ 0\sqrt{2} & 1 \end{pmatrix}, s_2 = \pm \begin{pmatrix} 1 & 0\sqrt{2} \\ 1\sqrt{2} & 1 \end{pmatrix},$$

$$s_3 = \pm \begin{pmatrix} -1 & -1\sqrt{2} \\ 1\sqrt{2} & 1 \end{pmatrix}, s_4 = \pm \begin{pmatrix} 1\sqrt{2} & 1 \\ 1 & 1\sqrt{2} \end{pmatrix}, r_1 = \pm \begin{pmatrix} 1\sqrt{2} & 1 \\ -1 & 0\sqrt{2} \end{pmatrix},$$

$$s_5 = \pm \begin{pmatrix} 0\sqrt{2} & -1 \\ 1 & 0\sqrt{2} \end{pmatrix}, r_2 = \pm \begin{pmatrix} 0\sqrt{2} & -1 \\ 1 & 1\sqrt{2} \end{pmatrix}.$$

The group $H_4/H_4(2)$ is generated by s_1, r_1 which obey the relations

$$s_1^2 = r_1^4 = (s_1 r_1)^2 = e,$$

so $H_4/H_4(2)$ is isomorphic to the dihedral group D_4 of order 8.

Also for $n = 2$ we have $(H_4)_0(2) = (H_4)_1(2)$, is a subgroup of index 4.

Every element of H_6 is congruent mod(2) to precisely one of:

$$e = \pm \begin{pmatrix} 1 & 0\sqrt{3} \\ 0\sqrt{3} & 1 \end{pmatrix}, s_1 = \pm \begin{pmatrix} 1 & 1\sqrt{3} \\ 0\sqrt{3} & 1 \end{pmatrix}, s_2 = \pm \begin{pmatrix} 1 & 0\sqrt{3} \\ 1\sqrt{3} & 1 \end{pmatrix},$$

$$s_3 = \pm \begin{pmatrix} 2 & 1\sqrt{3} \\ 1\sqrt{3} & 2 \end{pmatrix}, r_1 = \pm \begin{pmatrix} 5 & 1\sqrt{3} \\ 3\sqrt{3} & 2 \end{pmatrix}, r_2 = \pm \begin{pmatrix} 2 & 1\sqrt{3} \\ 3\sqrt{3} & 5 \end{pmatrix},$$

$$s_4 = \pm \begin{pmatrix} 1\sqrt{3} & 10 \\ 2 & 7\sqrt{3} \end{pmatrix}, s_5 = \pm \begin{pmatrix} 1\sqrt{3} & 7 \\ 2 & 5\sqrt{3} \end{pmatrix}, s_6 = \pm \begin{pmatrix} 0\sqrt{3} & -1 \\ 1 & 0\sqrt{3} \end{pmatrix},$$

$$s_7 = \pm \begin{pmatrix} 5\sqrt{3} & 2 \\ 7 & 1\sqrt{3} \end{pmatrix}, r_3 = \pm \begin{pmatrix} 1\sqrt{3} & 1 \\ -1 & 0\sqrt{3} \end{pmatrix}, r_4 = \pm \begin{pmatrix} 0\sqrt{3} & 1 \\ -1 & 1\sqrt{3} \end{pmatrix}.$$

The group $H_6/H_6(2)$ is generated by s_1, r_1 which obey the relations

$$s_1^2 = r_1^6 = (s_1 r_1)^2 = e,$$

so $H_6/H_6(2)$ is isomorphic to the dihedral group D_6 of order 12. Also for $n = 2$ we have $(H_6)_0(2) = 2(H_6)_1(2)$, is a subgroup of index 3.

The following table summarizes the corresponding indices;

	Γ	H_4	H_6
$\lvert (H_q)_1(2) : H_q(2) \rvert$	2	2	2
$\lvert (H_q)_0(2) : (H_q)_1(2) \rvert$	1	1	2
$\lvert H_q : (H_q)_0(2) \rvert$	3	4	3
$\lvert H_q : H_q(2) \rvert$	6	8	12
$H_q/H_q(2) \cong$	D_3	D_4	D_6

As a conclusion, when $n = 2$, $q = 4, 6$ and $m = 2, 3$, the index $|H_q : H_q(2)| = 4m$.

4.4 The principal congruence maps $\mathscr{M}_3(n)$

The majority of the content in this section is drawn from [IS05].

Definition 4.9. Let $\mathscr{M}_3(n)$ be the triangular map $\hat{\mathscr{M}}_3/\Gamma(n)$.
We call $\mathscr{M}_3(n)$ the principal congruence map, PC-map or the Farey map modulo n.

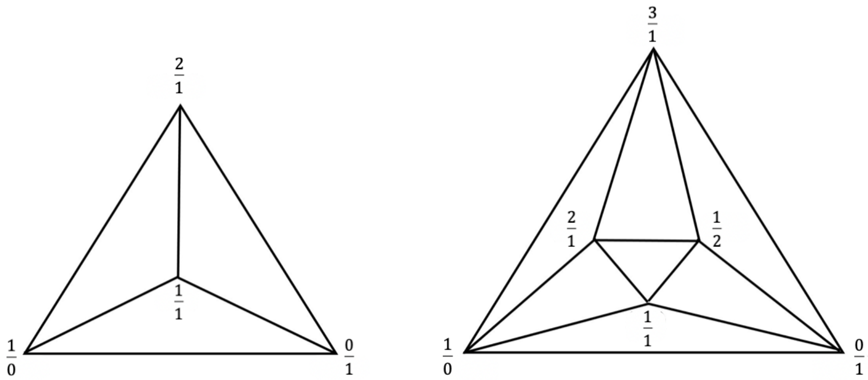
The vertices of this map are the *Farey fractions* mod n. These are of the form $\frac{a}{c}$ where $a, c \in \mathbb{Z}$ and $(a, c, n) = 1$, excluding $\frac{0}{0}$. We can think of these vertices as fractions $\frac{a}{c}$ where now $a, c \in \mathbb{Z}_n$ (not both zeros), and we identify $\frac{a}{c}$ with $\frac{-a}{-c}$. Essentially these are ordered pairs (a, c) where $(a, c, n) = 1$, $(a, c) \neq (0, 0)$ and where (a, c) is identified with $(-a, -c)$. The edges and triangular faces of $\mathscr{M}_3(n)$ are the projections of the edges and triangles of the universal triangulation $\hat{\mathscr{M}}_3$; thus two vertices are joined by an edge if and only if $ad - bc \equiv \pm 1 \mod n$, and $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$ are the vertices forming triangular faces with $\frac{a}{b}$ and $\frac{c}{d}$. These are the only faces in the map.

Example 4.10. The vertices of $\mathscr{M}_3(4)$ are $\frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{1}{2}, \frac{0}{1}$.
The vertices of $\mathscr{M}_3(5)$ are $\frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \frac{2}{0}, \frac{0}{1}$.
As we can notice that $\frac{2}{2}$ is a vertex in $\mathscr{M}_3(5)$ as $(2, 2, 5) = 1$ but not a vertex in $\mathscr{M}_3(4)$ as $(2, 2, 4) = 2$.

Example 4.11. If $n = 2$ there are three vertices $\frac{1}{0}, \frac{0}{1}, \frac{1}{1}$. Any two vertices are joined by an edge and so $\mathscr{M}_3(2)$ is a triangle embedded in the sphere. For $n = 3, 4$ we obtain a tetrahedron and octahedron respectively, as in Figure 4.2. Also for $n = 5$ we obtain an icosahedron; thus all the regular triangular maps on the sphere are PC-maps. Moreover, for $n \geq 6$ the maps $\mathscr{M}_3(n)$ give some interesting geometric objects, such as the map of type $\{6, 3\}_{2,2}$ of genus 1 on a torus for $n = 6$. Also Klein's Riemann surface of genus 3 and the Fricke-Klein surface of genus 5 when $n = 7$ and 8 respectively.

The vital statistics for PC-maps $\mathscr{M}_3(n)$ were discussed in [IS05]. As $\mathscr{M}_3(n)$ is a regular map, using $|\Gamma : \Gamma(n)|$ as defined in (4.1.3), we have: the number of edges is equal to $|\Gamma : \Gamma(n)|/2$, the number of faces is equal to $|\Gamma : \Gamma(n)|/3$, and the number of vertices

FIGURE 4.2: Left: $\mathscr{M}_3(3)$; tetrahedron. Right: $\mathscr{M}_3(4)$; octahedron



is equal to $|\Gamma : \Gamma(n)|/n$. Every vertex has valency n. The genus $g_3(n)$ of $\mathscr{M}_3(n)$ can be found using the following well-known formula [Ran77]

$$g_3(n) = 1 + \frac{n^2}{24}(n-6)\prod_{p|n}(1-\frac{1}{p^2}) \quad (n > 2).$$

The case $n = 2$ was discussed in Section 4.3. The following table gives some information (number of vertices and edges, valency of faces, genus, etc.) for $n = 2, 3, 4, 5, 6, 7$.

n	$\|V\|$	$\|E\|$	$\|F\|$	$g_3(n)$	$\|\Gamma : \Gamma(n)\|$	$Aut(\mathscr{M}_3(n))$	Description of $\mathscr{M}_3(n)$	Riemann surface
2	3	3	2	0	6	S_3	Triangle	Riemann sphere
3	4	6	4	0	12	A_4	Tetrahedron	Riemann sphere
4	6	12	8	0	24	S_4	Octahedron	Riemann sphere
5	12	30	20	0	60	A_5	Icosahedron	Riemann sphere
6	12	36	24	1	72	$PSL_2(\mathbb{Z}_6)$	$\{6,3\}_{2,2}$	Hexagonal torus
7	24	84	56	3	168	$PSL_2(\mathbb{Z}_7)$	Klein's surface	Klein's quartic

TABLE 4.2: Vital statistics for some $\mathscr{M}_3(n)$ maps

Here $Aut(\mathscr{M}_3(n))$ is the automorphism group of the maps $\mathscr{M}_3(n)$. Also S_n and A_n are the symmetric and alternating groups on n letters, respectively.

4.5 The principal congruence maps $\mathscr{M}_q(I)$

In this section we generalize the concept of the Farey map modulo n to the Hecke groups H_q. We deal with ideals in $\mathbb{Z}[\lambda_q]$ of the form $I = (n)$, where n is an integer, and with the cases $q = 4$ and 6.

Definition 4.12. If $\hat{\mathscr{M}}_q$ is the universal q-gonal map, and I is an ideal of $\mathbb{Z}[\lambda_q]$ then a PC-map is a map of the form $\hat{\mathscr{M}}_q/H_q(I)$.
We call $\mathscr{M}_q(I)$ a principal congruence map or PC-map.

The maps $\hat{\mathscr{M}}_q/H_q(n)$ for $q=4$ and 6 lie on the Riemann surface $\mathbb{H}^*/H_q(n)$, where $\mathbb{H}^*=\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$ and $\mathbb{H}^*/H_q(n)$ denotes the quotient surface $\mathbb{H}/H_q(n)$ compactified by adding the points in $\mathbb{Q}(\sqrt{m})\cup\{\infty\}$ to the plane $\mathbb{H}$.

4.5.1 Vital Statistics for PC-maps $\mathscr{M}_4(n)$

As described in Example 3.5 the vertices of $\hat{\mathscr{M}}_4$ are classified as even and odd vertices, and the same applies in $\mathscr{M}_4(n)$. The even vertices of $\mathscr{M}_4(n)$ form the orbit of $\infty=\frac{1}{0\sqrt{2}}$ under even elements of H_4. For $a,b,c,d\in\mathbb{Z}_n$, even vertices have the form $\frac{a}{c\sqrt{2}}$ where $(a,c,n)=1$, and a is odd whenever n is even. The odd vertices form the orbit of $0=\frac{0\sqrt{2}}{1}$ under even elements of H_4. Odd vertices have the form $\frac{b\sqrt{2}}{d}$ where $(b,d,n)=1$ and d is odd whenever n is even. Also we identify $\frac{a}{c\sqrt{2}}$ with $\frac{-a}{-c\sqrt{2}}$. As $\infty=\frac{1}{0\sqrt{2}}$ is an even vertex which is joined to $0=\frac{0\sqrt{2}}{1}$ that is an odd vertex, so edges of $\mathscr{M}_4(n)$ must join even and odd vertices $\frac{a}{c\sqrt{2}}$ and $\frac{b\sqrt{2}}{d}$ if and only if $ad-2bc\equiv\pm1 \mod n$. Because the edges never join two vertices of the same type we have a bipartite graph. The faces of $\mathscr{M}_4(n)$ are just the images of the principal face of $\hat{\mathscr{M}}_4$ under H_4.

Elements of H_4 has been discussed in Section 2.2. H_4 acts transitively on the darts of $\hat{\mathscr{M}}_4$. A dart of $\hat{\mathscr{M}}_4$ is an ordered pair $(\frac{a}{c\sqrt{2}},\frac{b\sqrt{2}}{d})$ and elements of H_4 map the principal dart $\delta=(\frac{1}{0\sqrt{2}},\frac{0\sqrt{2}}{1})$ to $(\frac{a}{c\sqrt{2}},\frac{b\sqrt{2}}{d})$ and so $H_4/H_4(n)$ acts transitively on the darts of $\hat{\mathscr{M}}_4/H_4(n)$. By Definition 1.10 $\hat{\mathscr{M}}_4$ is a regular map and for a similar reason so is $\mathscr{M}_4(n)$. As $\frac{1}{0\sqrt{2}}$ is joined to $\frac{k\sqrt{2}}{1}$ in $\mathscr{M}_4(n)$ for $k\in\mathbb{Z}_n$, it follows that $\frac{1}{0\sqrt{2}}$ has valency n and hence by regularity every vertex of $\mathscr{M}_4(n)$ has valency n. Using Theorem 4.8 helps us in working out the numbers of darts, edges, faces and vertices of $\mathscr{M}_4(n)$, therefore we have:

The number of darts =	$\mu_4(n)$
The number of edges =	$\mu_4(n)/2$
The number of faces =	$\mu_4(n)/4$
The number of vertices =	$\mu_4(n)/n$

Here $|H_4:(H_4)(n)|=\mu_4(n)$ as defined immediately before the Theorem 4.8. If $g_4(n)$ is the genus of the map $\mathscr{M}_4(n)$ then the Euler characteristic is given by

$$2-2g_4(n)=\mu_4(n)(\frac{1}{n}-\frac{1}{2}+\frac{1}{4})=\mu_4(n)(\frac{4-n}{4n}),$$

$$g_4(n)=\mu_4(n)(\frac{n-4}{8n})+1,$$

from which we deduce the following formulae for the genus of $\mathscr{M}_4(n)$

$$g_4(n) = \begin{cases} 0 & if\ n = 2 \\ 1 + \frac{n^2}{8}(n-4)\prod_{p|n}(1-\frac{1}{p^2}) & if\ (n,2) = 1 \\ 1 + \frac{n^2}{16}(n-4)\prod_{p|n,p\neq 2}(1-\frac{1}{p^2}) & if\ (n,2) = 2. \end{cases} \tag{4.5.1}$$

$g_4(n)$ is also the genus of the surface $\mathbb{H}^*/H_4(n)$ which carries the map $\mathscr{M}_4(n)$.

Let us see what these vertices are for low values of n, say $n = 2, 3, ..., 8$. As shown in Figures 4.4, 4.6, 4.7 and 4.3, these maps give interesting geometric shapes.

TABLE 4.3: $\mathscr{M}_4(n)$ vertices for low values of n

n	$\mathscr{M}_4(n)$ vertices
2	$\frac{1}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{1\sqrt{2}}{1}$
3	$\frac{1}{0\sqrt{2}}, \frac{0}{1\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{2\sqrt{2}}{1}, \frac{1\sqrt{2}}{0}, \frac{1\sqrt{2}}{1}$
4	$\frac{1}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{3}, \frac{2\sqrt{2}}{1}$
5	$\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}, \frac{0}{1\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{2}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{4}{1\sqrt{2}}, \frac{0}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{2}{2\sqrt{2}}, \frac{4}{2\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{0\sqrt{2}}{2}, \frac{1\sqrt{2}}{0}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{2}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{4}, \frac{2\sqrt{2}}{0}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{2}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{4}$
6	$\frac{1}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{5}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}, \frac{1}{3\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{5}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{5}, \frac{3\sqrt{2}}{1}$
7	$\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}, \frac{3}{0\sqrt{2}}, \frac{0}{1\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{2}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{4}{1\sqrt{2}}, \frac{5}{1\sqrt{2}}, \frac{6}{1\sqrt{2}}, \frac{0}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{2}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{4}{2\sqrt{2}}, \frac{5}{2\sqrt{2}},$ $\frac{6}{2\sqrt{2}}, \frac{0}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}, \frac{3}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{5}{3\sqrt{2}}, \frac{6}{3\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{0\sqrt{2}}{2}, \frac{0\sqrt{2}}{3}, \frac{1\sqrt{2}}{0}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{2}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{4}, \frac{1\sqrt{2}}{5}, \frac{1\sqrt{2}}{6}, \frac{2\sqrt{2}}{0}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{2}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{4}, \frac{2\sqrt{2}}{5},$ $\frac{2\sqrt{2}}{6}, \frac{3\sqrt{2}}{0}, \frac{3\sqrt{2}}{1}, \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{3}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{5}, \frac{3\sqrt{2}}{6}$
8	$\frac{1}{0\sqrt{2}}, \frac{3}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{5}{1\sqrt{2}}, \frac{7}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}, \frac{7}{2\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{3}{3\sqrt{2}}, \frac{5}{3\sqrt{2}}, \frac{7}{3\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{3}{4\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{0\sqrt{2}}{3}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{5}, \frac{1\sqrt{2}}{7}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{5}, \frac{2\sqrt{2}}{7}, \frac{3\sqrt{2}}{1}, \frac{3\sqrt{2}}{3}, \frac{3\sqrt{2}}{5}, \frac{3\sqrt{2}}{7}, \frac{4\sqrt{2}}{1}, \frac{4\sqrt{2}}{3}$

The corresponding maps $\mathscr{M}_4(n)$ for these values of n are as follows:

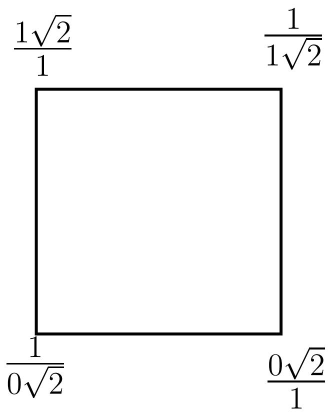


FIGURE 4.3: $\mathscr{M}_4(2)$; Square embedded in a sphere

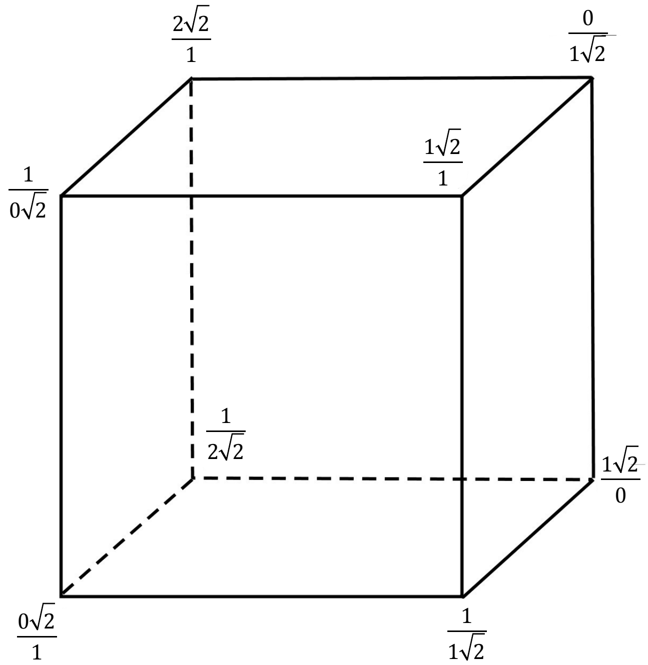


FIGURE 4.4: $\mathscr{M}_4(3)$; Cube embedded in a sphere

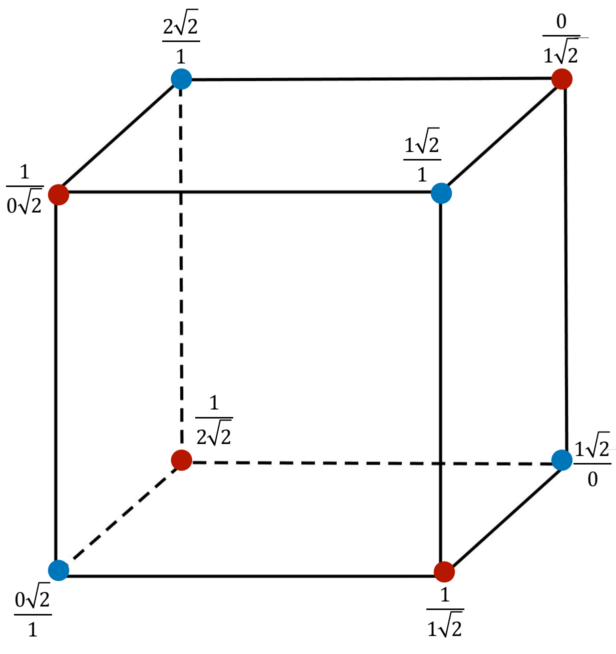


FIGURE 4.5: The cube with its bipartite structure

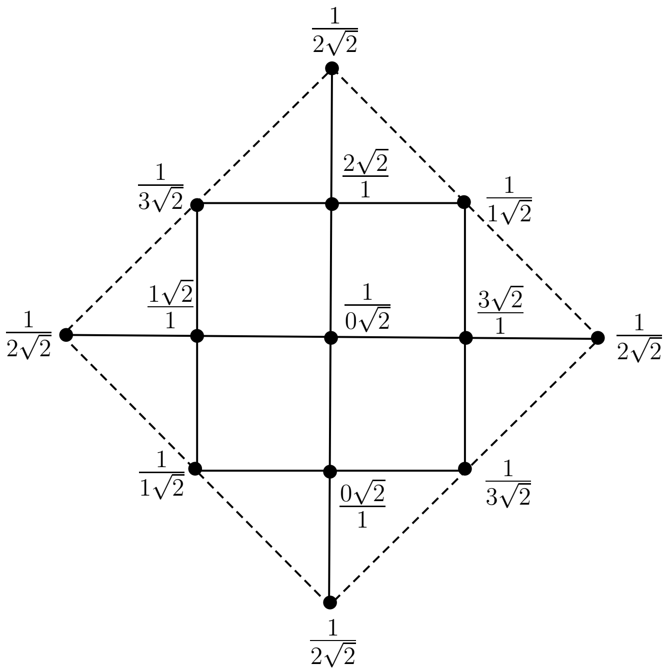


FIGURE 4.6: $\mathscr{M}_4(4)$; $\{4,4\}_{2,2}$ embedded in a square torus of genus 1
[The square torus is obtained by identifying the opposite sides of the outer square].

Perhaps the most famous example of a regular map of type $\{n,4\}$ corresponds to a Riemann surface of genus 4 with 120 automorphisms (having S_5 as automorphism group) when $n=5$. We can construct this by defining an epimorphism $\omega : \Gamma(2,5,4) \to S_5$. If

$$\Gamma(2,5,4) = \langle X, Y, Z \mid X^2 = Y^5 = Z^4 = XYZ = 1 \rangle,$$

then ω is given by $X \mapsto x = (1\ 5)$, $Y \mapsto y = (1\ 5\ 4\ 3\ 2)$, and $Z \mapsto z = (2\ 3\ 4\ 5)$. If K is the kernel then $\mathbb{H}/K$ is a Riemann surface with 120 automorphisms. Then by the Riemann-Hurwitz formula

$$2 - 2g_4(5) = \mu_4(5)(\frac{1}{5} - \frac{1}{2} + \frac{1}{4}) = 120(\frac{-1}{20}),$$

therefore the genus of the quotient surface is $g_4(5) = 4$. The surface underlying this map is the unique Riemann surface of genus 4 with 120 automorphisms, namely Bring's curve.

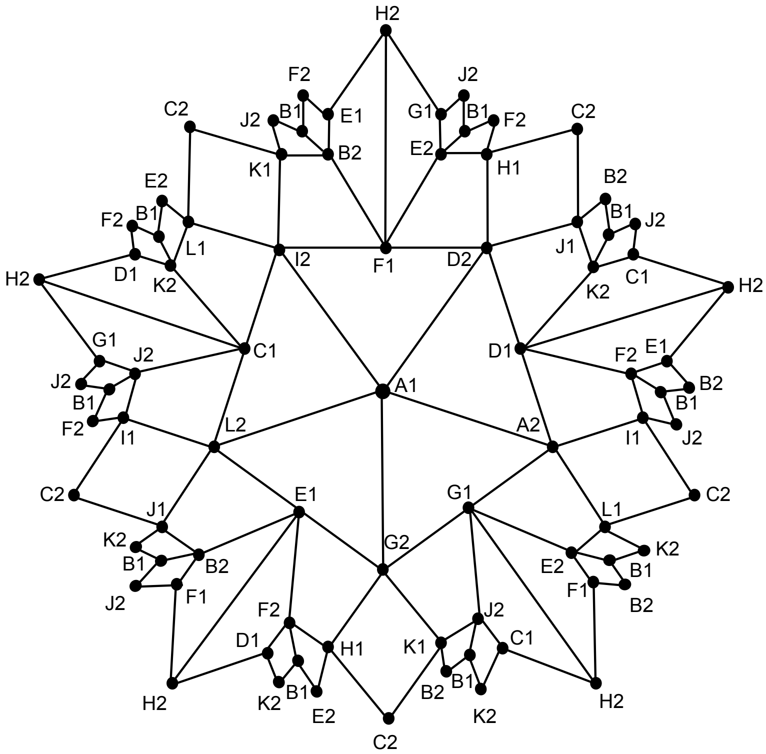


FIGURE 4.7: $\mathscr{M}_4(5)$; $\{5,4\}$ embedded in Bring's curve of genus 4

TABLE 4.4: Table of Correspondence for $\mathscr{M}_4(5)$

$A_1 : \frac{1}{0\sqrt{2}}$	$B_1 : \frac{2}{0\sqrt{2}}$	$C_1 : \frac{0}{1\sqrt{2}}$	$D_1 : \frac{1}{1\sqrt{2}}$
$E_1 : \frac{2}{1\sqrt{2}}$	$F_1 : \frac{3}{1\sqrt{2}}$	$G_1 : \frac{4}{1\sqrt{2}}$	$H_1 : \frac{0}{2\sqrt{2}}$
$I_1 : \frac{1}{2\sqrt{2}}$	$J_1 : \frac{3}{2\sqrt{2}}$	$K_1 : \frac{2}{2\sqrt{2}}$	$L_1 : \frac{4}{2\sqrt{2}}$
$A_2 : \frac{0\sqrt{2}}{1}$	$B_2 : \frac{0\sqrt{2}}{2}$	$C_2 : \frac{1\sqrt{2}}{0}$	$D_2 : \frac{1\sqrt{2}}{1}$
$E_2 : \frac{1\sqrt{2}}{2}$	$F_2 : \frac{1\sqrt{2}}{3}$	$G_2 : \frac{1\sqrt{2}}{4}$	$H_2 : \frac{2\sqrt{2}}{0}$
$I_2 : \frac{2\sqrt{2}}{1}$	$J_2 : \frac{2\sqrt{2}}{2}$	$K_2 : \frac{2\sqrt{2}}{3}$	$L_2 : \frac{2\sqrt{2}}{4}$

Bring's curve is famous because it has the largest possible automorphism group of any Riemann surface of genus 4, namely S_5. Every compact Riemann surface corresponds to a complex curve, which in this case is the complete intersection of the three hypersurfaces

$$\sum_{i=1}^{5} x_i = 0 \qquad \sum_{i=1}^{5} x_i^2 = 0 \qquad \sum_{i=1}^{5} x_i^3 = 0$$

in four-dimensional projective space $\mathbb{P}^4$ [Web05] and [RR92]. The regular map $\mathscr{M}_4(5)$ of genus 4 has 120 automorphisms as shown in Table 4.5. In [CD01] a list of regular maps of genus $2 \leq g \leq 101$ is given. From this we find that there is a unique regular map of type $\{5,4\}$ with 120 automorphisms.

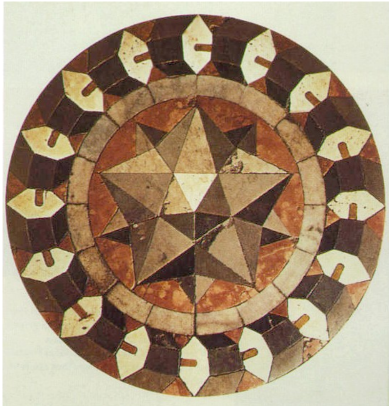


FIGURE 4.8: An early picture of Bring's curve which is associated to the small stellated dodecahedron [Web05]. It appeared as a mosaic by Paolo Uccello on the floor of San Marco cathedral, Venice ca. 1430.

This picture actually denotes a map of type $\{5,5\}$, but because of the inclusion $\Gamma(2,5,5) < \Gamma(2,4,5)$ it can be shown that the underlying Riemann surfaces are the same. In [SS03] it is shown that this map of type $\{4,5\}$ is the *medial* map of the map of type $\{5,5\}$. Start with a map $\mathscr{M}$ of type $\{5,5\}$. Two edges are adjacent if they intersect in a vertex. We define a *medial* to be a geodesic joining the midpoints of two adjacent edges. The medial map of $\mathscr{M}$, denoted by $\mathrm{Med}(\mathscr{M})$, has a vertex set the mid-points of the edges of $\mathscr{M}$, the edge set of $\mathrm{Med}(\mathscr{M})$ are the medials of $\mathscr{M}$ and there are two kinds of face centers of $\mathrm{Med}(\mathscr{M})$; firstly those at the vertices of $\mathscr{M}$ and secondly, those at the face-centers of $\mathscr{M}$. It is easy to see that the vertices of $\mathrm{Med}(\mathscr{M})$ have valency 4, and the face sizes are equal to 5, so the $\mathrm{Med}(\mathscr{M})$ has type $\{4,5\}$. Our map in Figure 4.7 of type $\{5,4\}$ is just the dual map of the medial map of type $\{5,5\}$ and so corresponds to Bring's curve.

Example 4.13. The following table provides the vital statistics for some examples of maps $\mathscr{M}_4(n)$

n	$\mu_4(n)$	$\lvert E\rvert$	$\lvert V\rvert$	$\lvert F\rvert$	$g_4(n)$
2	8	4	4	2	0
3	24	12	8	6	0
4	32	16	8	8	1
5	120	60	24	30	4
6	96	48	16	24	5
7	336	168	48	84	19
8	256	128	32	64	17

TABLE 4.5: Vital statistics for some maps $\mathscr{M}_4(n)$

The following is the technique of drawing any $\mathscr{M}_4(n)$:
To draw $\mathscr{M}_4(n)$ we do the following.

- Find out the number of edges, vertices and faces, using the formulae in (4.5.1), as demonstrated in Table 4.5.

- Knowing the valency n of each vertex, accordingly we can work out the n neighbor vertices of each vertex, applying the relations $ad - 2bc \equiv \pm 1 \bmod n$, and $2ad - bc \equiv \pm 1 \bmod n$.

- Finding the principal face as has been shown in Section 3.2, Example 3.2.

- Applying the generators $T = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}$ or words in them to the principal face to form all others equilateral faces.

- Depending on the neighborhoods and the valency of each vertex, we can arrange the quadrilateral faces to get geometric objects as in Figures 4.3, 4.4, 4.6,4.7.

Let us discuss the drawing of $\mathscr{M}_4(5)$ in details.

Example 4.14.

- Listing the 24 vertices as shown in Example 4.13, working out the number of vertices from (4.5.1). The even and odd vertices are;
  $A_1 = \frac{1}{0\sqrt{2}}$, $B_1 = \frac{2}{0\sqrt{2}}$, $C_1 = \frac{0}{1\sqrt{2}}$, $D_1 = \frac{1}{1\sqrt{2}}$, $E_1 = \frac{2}{1\sqrt{2}}$, $F_1 = \frac{3}{1\sqrt{2}}$,
  $G_1 = \frac{4}{1\sqrt{2}}$, $H_1 = \frac{0}{2\sqrt{2}}$, $I_1 = \frac{1}{2\sqrt{2}}$, $J_1 = \frac{3}{2\sqrt{2}}$, $K_1 = \frac{2}{2\sqrt{2}}$, $L_1 = \frac{4}{2\sqrt{2}}$,
  $A_2 = \frac{0\sqrt{2}}{1}$, $B_2 = \frac{0\sqrt{2}}{2}$, $C_2 = \frac{1\sqrt{2}}{0}$, $D_2 = \frac{1\sqrt{2}}{1}$, $E_2 = \frac{1\sqrt{2}}{2}$, $F_2 = \frac{1\sqrt{2}}{3}$, $G_2 = \frac{1\sqrt{2}}{4}$,
  $H_2 = \frac{2\sqrt{2}}{0}$, $I_2 = \frac{2\sqrt{2}}{1}$, $J_2 = \frac{2\sqrt{2}}{2}$, $K_2 = \frac{2\sqrt{2}}{3}$, $L_2 = \frac{2\sqrt{2}}{4}$.
  Also there are 30 faces.

- The valency of each vertex is 5. Applying the relations $ad - 2bc \equiv \pm 1 \bmod 5$, and $2ad - bc \equiv \pm 1 \bmod 5$, among the vertices to find the neighborhood vertices of each vertex. We have;

- The principal face, call it face 1, has the vertices D_2, D_1, A_2, A_1.

$A_1 \leftrightarrow A_2, D_2, I_2, L_2, G_2$	$A_2 \leftrightarrow A_1, D_1, I_1, L_1, G_1$
$B_1 \leftrightarrow B_2, E_2, F_2, J_2, K_2$	$B_2 \leftrightarrow B_1, E_1, F_1, J_1, K_1$
$C_1 \leftrightarrow H_2, I_2, J_2, K_2, L_2$	$C_2 \leftrightarrow H_1, I_1, J_1, K_1, L_1$
$D_1 \leftrightarrow A_2, D_2, F_2, H_2, K_2$	$D_2 \leftrightarrow A_1, D_1, F_1, H_1, J_1$
$E_1 \leftrightarrow B_2, F_2, G_2, H_2, L_2$	$E_2 \leftrightarrow B_1, F_1, G_1, H_1, L_1$
$F_1 \leftrightarrow B_2, D_2, E_2, H_2, I_2$	$F_2 \leftrightarrow B_1, D_1, E_1, H_1, I_1$
$G_1 \leftrightarrow A_2, E_2, G_2, H_2, J_2$	$G_2 \leftrightarrow A_1, E_1, G_1, H_1, K_1$
$H_1 \leftrightarrow C_2, D_2, E_2, F_2, G_2$	$H_2 \leftrightarrow C_1, D_1, E_1, F_1, G_1$
$I_1 \leftrightarrow A_2, C_2, F_2, J_2, L_2$	$I_2 \leftrightarrow A_1, C_1, F_1, K_1, L_1$
$J_1 \leftrightarrow B_2, C_2, D_2, K_2, L_2$	$J_2 \leftrightarrow B_1, C_1, G_1, I_1, K_1$
$K_1 \leftrightarrow B_2, C_2, I_2, G_2, J_2$	$K_2 \leftrightarrow B_1, C_1, D_1, J_1, L_1$
$L_1 \leftrightarrow A_2, C_2, E_2, I_2, K_2$	$L_2 \leftrightarrow A_1, C_1, E_1, I_1, J_1$

TABLE 4.6: Vertices Neighborhoods

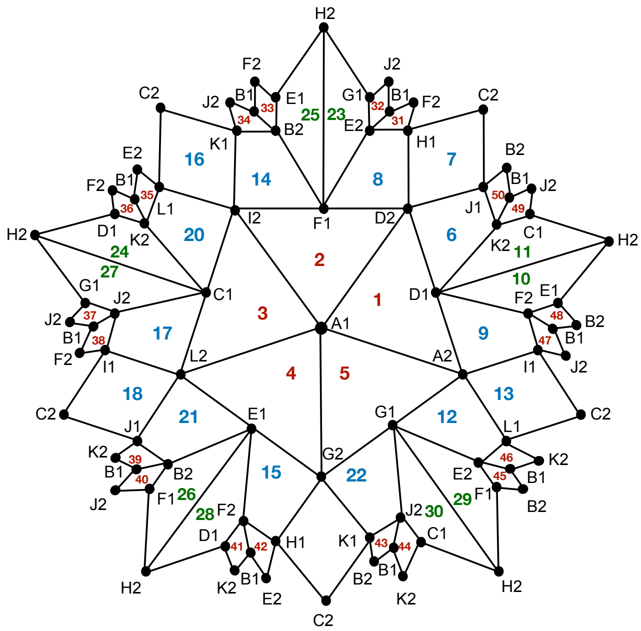


FIGURE 4.9: Faces of $\mathscr{M}_4(5)$

- Applying the generators T and R to get some faces as follow;

T over Face 1$\to$ Face 2	T^2 over Face 1$\to$ Face 3	T^3 over Face 1$\to$ Face 4
T^4 over Face 1$\to$ Face 5	R over Face 34$\to$ Face 43	R over Face 3$\to$ Face 7
R over Face 4$\to$ Face 8	R^2 over Face 2$\to$ Face 9	R^2 over Face 19$\to$ Face 35
R^2 over Face 4$\to$ Face 11	R^3 over Face 17$\to$ Face 46	R^3 over Face 4$\to$ Face 13
T over Face 6$\to$ Face 14	T over Face 46$\to$ Face 39	T over Face 31$\to$ Face 35
T over Face 7$\to$ Face 16	T^2 over Face 7$\to$ Face 18	T^3 over Face 7$\to$ Face 19
T over Face 8$\to$ Face 20	T^2 over Face 8$\to$ Face 21	T^3 over Face 8$\to$ Face 22
T over Face 10$\to$ Face 23	T^2 over Face 10$\to$ Face 24	T^3 over Face 10$\to$ Face 25
T over Face 11$\to$ Face 26	R^3 over Face 22$\to$ Face 47	R^3 over Face 27$\to$ Face 31
T over Face 39$\to$ Face 34	T^2 over Face 24$\to$ Face 30	T over Face 34$\to$ Face 38

- Arranging the faces to build up the final shape of the map, depending on the common neighbor vertices of each vertex and the edge in common.

4.5.2 Vital Statistics for PC-maps $\mathscr{M}_6(n)$

Discussion of the vital statistics for $\mathscr{M}_6(n)$ is analogous to that for $\mathscr{M}_4(n)$, with slight differences. The vertices of $\hat{\mathscr{M}}_6$ are partitioned into even and odd vertices, as in $\mathscr{M}_6(n)$. The even vertices of $\mathscr{M}_6(n)$ are the orbit of $\infty = \frac{1}{0\sqrt{3}}$ under even elements of H_6. For $a, b, c, d \in \mathbb{Z}_n$, even vertices have the form $\frac{a}{c\sqrt{3}}$ where $(a, c, n) = 1$, and $3 \nmid a$ whenever $3|n$. The odd vertices are the orbit of $0 = \frac{0\sqrt{3}}{1}$ under even elements of H_6. Odd vertices have the form $\frac{b\sqrt{3}}{d}$ where $(b, d, n) = 1$ and $3 \nmid d$ whenever $3|n$. Edges and faces are exactly the same as described in $\mathscr{M}_4(n)$. As in case $q = 4$, the edges never join two vertices of the same type this implies that the graph is bipartite. Finding the numbers of darts, edges, faces and vertices of $\mathscr{M}_6(n)$, using Theorem 4.8 we have;

The number of darts =	$\mu_6(n)$
The number of edges =	$\mu_6(n)/2$
The number of faces =	$\mu_6(n)/6$
The number of vertices =	$\mu_6(n)/n$

Here $|H_6 : (H_6)(n)| = \mu_6(n)$ as defined immediately before the Theorem 4.8. If $g_6(n)$ is the genus of the map $\mathscr{M}_6(n)$ then the Euler characteristic is given by

$$2 - 2g_6(n) = \mu_6(n)(\frac{1}{n} - \frac{1}{2} + \frac{1}{6}) = \mu_6(n)(\frac{3-n}{3n}),$$

$$g_6(n) = 1 + \mu_6(n)(\frac{n-3}{6n}),$$

from which we deduce the following formulae for the genus of $\mathscr{M}_6(n)$

$$g_6(n) = \begin{cases} 0 & if\ n = 2 \\ 1 + \frac{n^2}{6}(n-3)\prod_{p|n}(1 - \frac{1}{p^2}) & if\ (n,3) = 1 \\ 1 + \frac{n^2}{9}(n-3)\prod_{p|n, p\neq 3}(1 - \frac{1}{p^2}) & if\ (n,3) = 3. \end{cases} \tag{4.5.2}$$

The vertices for low values of n, say $n = 2, 3, ..., 6$ are;

TABLE 4.7: $\mathscr{M}_6(n)$ vertices for low values of n

n	$\mathscr{M}_6(n)$ vertices
2	$\frac{1}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}$
3	$\frac{1}{0\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{1}, \frac{2\sqrt{3}}{1}$
4	$\frac{1}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{1}{2\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{2\sqrt{3}}{1}$
5	$\frac{1}{0\sqrt{3}}, \frac{2}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{0}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{4}{2\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{0\sqrt{3}}{2}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{1\sqrt{3}}{4}, \frac{2\sqrt{3}}{0}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{2}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{4}$
6	$\frac{1}{0\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{5}{1\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{5}{2\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}$ $\frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{4}, \frac{1\sqrt{3}}{5}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{1}, \frac{3\sqrt{3}}{2}$
7	$\frac{1}{0\sqrt{3}}, \frac{2}{0\sqrt{3}}, \frac{3}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{5}{1\sqrt{3}}, \frac{6}{1\sqrt{3}}, \frac{0}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{4}{2\sqrt{3}}, \frac{5}{2\sqrt{3}},$ $\frac{6}{2\sqrt{3}}, \frac{0}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}, \frac{3}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}, \frac{6}{3\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{0\sqrt{3}}{2}, \frac{0\sqrt{3}}{3}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{1\sqrt{3}}{4}, \frac{1\sqrt{3}}{5}, \frac{1\sqrt{3}}{6}, \frac{2\sqrt{3}}{0}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{2}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{4}, \frac{2\sqrt{3}}{5},$ $\frac{2\sqrt{3}}{6}, \frac{3\sqrt{3}}{0}, \frac{3\sqrt{3}}{1}, \frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{3}, \frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{5}, \frac{3\sqrt{3}}{6}$
8	$\frac{1}{0\sqrt{3}}, \frac{3}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{5}{1\sqrt{3}}, \frac{6}{1\sqrt{3}}, \frac{7}{1\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{5}{2\sqrt{3}}, \frac{7}{2\sqrt{3}}, \frac{0}{3\sqrt{3}}, \frac{1}{3\sqrt{3}},$ $\frac{2}{3\sqrt{3}}, \frac{3}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}, \frac{6}{3\sqrt{3}}, \frac{7}{3\sqrt{3}}, \frac{1}{4\sqrt{3}}, \frac{3}{4\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{0\sqrt{3}}{3}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{1\sqrt{3}}{4}, \frac{1\sqrt{3}}{5}, \frac{1\sqrt{3}}{6}, \frac{1\sqrt{3}}{7}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{5}, \frac{2\sqrt{3}}{7}, \frac{3\sqrt{3}}{0}, \frac{3\sqrt{3}}{1},$ $\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{3}, \frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{5}, \frac{3\sqrt{3}}{6}, \frac{3\sqrt{3}}{7}, \frac{4\sqrt{3}}{1}, \frac{4\sqrt{3}}{3}$

The corresponding $\mathscr{M}_6(n)$ maps for these values of n;

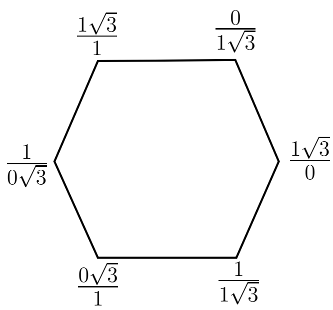


FIGURE 4.10: $\mathscr{M}_6(2)$; Hexagon embedded in a sphere

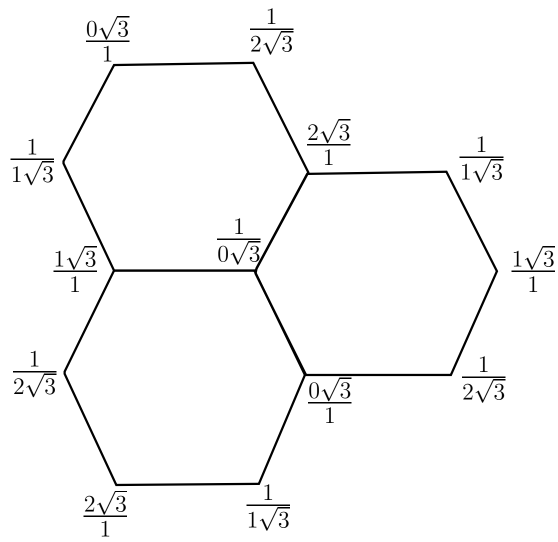


FIGURE 4.11: $\mathscr{M}_6(3)$; $\{3,6\}$ embedded in a hexagonal torus

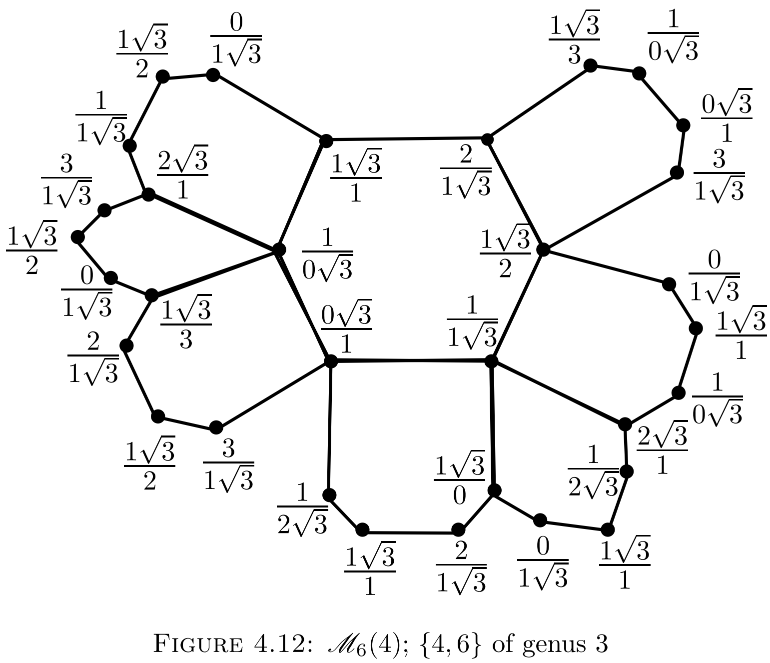


FIGURE 4.12: $\mathscr{M}_6(4)$; $\{4,6\}$ of genus 3

Example 4.15. The following table provides the vital statistics for some examples of maps $\mathscr{M}_6(n)$

n	$\mu_6(n)$	$\lvert E\rvert$	$\lvert V\rvert$	$\lvert F\rvert$	$g_6(n)$
2	12	6	6	2	0
3	18	9	6	3	1
4	48	24	12	8	3
5	120	60	24	20	9
6	108	54	18	18	37
7	336	168	48	56	33
8	384	192	48	64	41

TABLE 4.8: Vital statistics for some maps $\mathscr{M}_6(n)$

4.6 The star of a vertex

In a graph G the star of a vertex x consists of all vertices of G that are joined to x by an edge, including x itself.

Definition 4.16. An *even-star* is the star consisting of an even vertex of the map $\mathscr{M}_q(n)$ and all its neighbors (always odd vertices). Also an *odd-star* is the star consisting of an odd vertex of the map $\mathscr{M}_q(n)$ and all its neighbors (always even vertices).

Definition 4.17. An arithmetic progression consisting of successive neighbors of points $\frac{a}{c\sqrt{m}} \in \hat{\mathbb{Q}}(\sqrt{m})$ is a sequence $\frac{x_k\sqrt{m}}{y_k}$, where each of x_k and y_k is an arithmetic progression sequence of points in $\mathbb{Z}$ by itself. The first term is $\frac{x_0\sqrt{m}}{y_0}$. While an arithmetic progression consisting of successive neighbors of points $\frac{a\sqrt{m}}{c} \in \hat{\mathbb{Q}}(\sqrt{m})$ is a sequence $\frac{x_k}{y_k\sqrt{m}}$, where each of x_k and y_k is an arithmetic progression sequence of points in $\mathbb{Z}$ by itself. The first term is $\frac{x_0}{y_0\sqrt{m}}$.

If $T = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$ is a unuimodular matrix then $T(\frac{x_k\sqrt{m}}{y_k})$ is an arithmetic progression of points in $\hat{\mathbb{Q}}(\sqrt{m})$, meaning it is an arithmetic progression consisting of successive neighbors of the point $\frac{a}{c\sqrt{m}}$, where the first term is $T(\frac{b\sqrt{m}}{d})$ and the common difference is $T(\frac{a\sqrt{m}}{cm})$. Similarly if $T = \begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix}$, then $T(\frac{x_k}{y_k\sqrt{m}})$ is an arithmetic progression of neighbors of the point $\frac{a\sqrt{m}}{c}$, where the first term is $T(\frac{b}{d\sqrt{m}})$ and the common difference is $T(\frac{am}{c\sqrt{m}})$.

Theorem 4.18. *The even-star of the even vertex* $\frac{a}{\sqrt{m}c}$ *of the map* $\mathscr{M}_q(n)$ *consists of* $\frac{a}{\sqrt{m}c}$ *together with all odd vertices of the form*

$$\frac{(b+ak)\sqrt{m}}{d+mck} \tag{4.6.1}$$

where $ad - mbc \equiv 1 \bmod n$ *and* $k = 0, 1, ..., n-1$.

Proof. First we find the even-star of $\frac{1}{0\sqrt{m}}$ in $\mathbb{Q}(\sqrt{m})$. This consists of $\frac{1}{0\sqrt{m}}$ together with the odd vertices $\frac{0\sqrt{m}}{1}, \frac{1\sqrt{m}}{1}, ..., \frac{(n-1)\sqrt{m}}{1}$. Here $\frac{a}{\sqrt{m}c} = \frac{1}{0\sqrt{m}}$ so $\frac{\sqrt{m}b}{d} = \frac{0\sqrt{m}}{1}$, hence $a = d = 1$ and $b = c = 0$, therefore $\frac{(b+ak)\sqrt{m}}{d+mck} = \frac{k\sqrt{m}}{1}$, $k = 0, 1, ..., n-1$, as required.
More generally, let $T = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$ be a unimodular matrix, $U = \begin{pmatrix} 1 & 1\sqrt{m} \\ 0\sqrt{m} & 1 \end{pmatrix} \in H_q^e$.
Then $T(\frac{1}{0\sqrt{m}}) = \frac{a}{\sqrt{m}c}$. The stabilizer of $\frac{1}{0\sqrt{m}}$ is the cyclic group generated by U so the stabilizer of $\frac{a}{\sqrt{m}c}$ consists of elements of the form TU^kT^{-1}. Let

$$S = TU^kT^{-1} = \begin{pmatrix} 1-mkac & a^2k\sqrt{m} \\ -mkc^2\sqrt{m} & 1+mkac \end{pmatrix}.$$

Therefore if $\frac{a}{\sqrt{m}c} \longleftrightarrow \frac{\sqrt{m}b}{d}$, then $S(\frac{a}{\sqrt{m}c}) \longleftrightarrow S(\frac{\sqrt{m}b}{d})$ so that $\frac{a}{\sqrt{m}c} \longleftrightarrow S(\frac{\sqrt{m}b}{d})$ (where $\longleftrightarrow$ denotes an edge). Now

$$S(\frac{\sqrt{m}b}{d}) = \begin{pmatrix} 1-mkac & a^2k\sqrt{m} \\ -mkc^2\sqrt{m} & 1+mkac \end{pmatrix} \begin{pmatrix} \sqrt{m}b \\ d \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{m}(b - mkacb + ka^2d) \\ -m^2kbc^2 + mkacd + d \end{pmatrix}.$$

Using $ad - mbc = 1 \Longrightarrow ad = 1 + mbc$, this becomes

$$\frac{(b+ak)\sqrt{m}}{d+mck}.$$

This is true for $k = 0, 1, ..., n-1$ so that these are the n odd points in the even-star of $\frac{a}{\sqrt{m}c}$. □

Theorem 4.19. *The odd-star of the odd vertex* $\frac{\sqrt{m}a}{c}$ *of the map* $\mathscr{M}_q(n)$ *consists of* $\frac{\sqrt{m}a}{c}$ *together with all even vertices of the form*

$$\frac{b+mak}{(d+ck)\sqrt{m}} \tag{4.6.2}$$

where $mad - bc \equiv 1 \bmod n$ *and* $k = 0, 1, ..., n-1$.

The proof is similar to the proof of Theorem 4.18.

By Definition 4.17, the sequence (4.6.1) is an arithmetic progression consisting of successive neighbors of the point $\frac{a}{c\sqrt{m}}$, where the first term is $\frac{b\sqrt{m}}{d}$ and the common difference is $\frac{a\sqrt{m}}{cm}$. Similarly the sequence (4.6.2) is an arithmetic progression consisting of successive neighbors of the point $\frac{a\sqrt{m}}{c}$, where the first term is $\frac{b}{d\sqrt{m}}$ and the common difference is $\frac{am}{c\sqrt{m}}$.

Example 4.20. To find the even-star of $\frac{4}{1\sqrt{2}}$ in $\mathscr{M}_4(5)$, see Figure 4.7, we have $\frac{\sqrt{m}b}{d} = \frac{0\sqrt{2}}{1}$, then our unimodular matrix T is $\begin{pmatrix} 4 & 0\sqrt{2} \\ 1\sqrt{2} & 1 \end{pmatrix}$. The first term and the common difference of the arithmetic progression (4.6.1) are $\frac{0\sqrt{2}}{1}$ and $\frac{4\sqrt{2}}{2}$ respectively. Thus the successive neighbors of $\frac{4}{1\sqrt{2}}$ is given by the formula $\frac{4k\sqrt{2}}{1+2k}$. Therefore, the even-star of $\frac{4}{1\sqrt{2}}$ in $\mathscr{M}_4(5)$ is

$$\{\frac{4}{1\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{1\sqrt{2}}{2}, \frac{2\sqrt{2}}{0}, \frac{2\sqrt{2}}{2}, \frac{4\sqrt{2}}{1}\} = \{G1,\ A2,\ E2,\ H2,\ J2,\ G2\}.$$

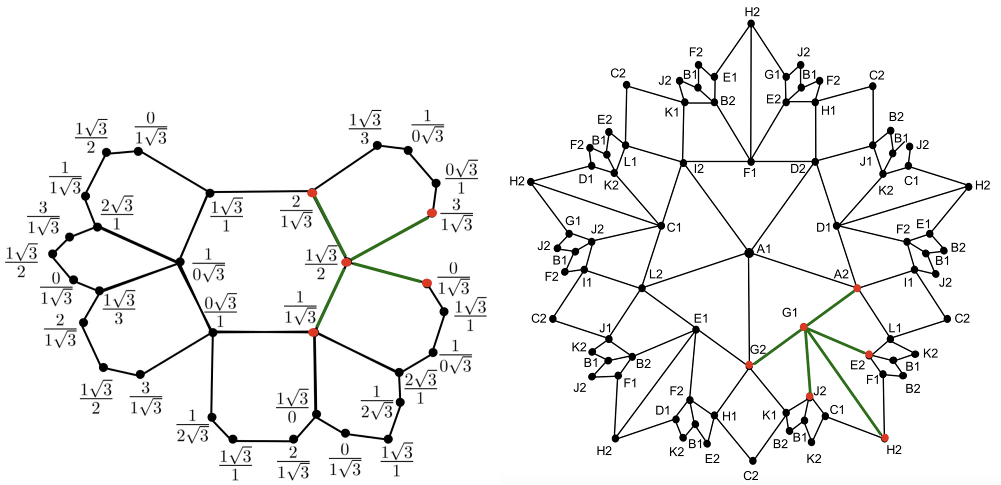


FIGURE 4.13: Left: The odd-star of $\frac{1\sqrt{3}}{2}$ in $\mathscr{M}_6(4)$. Right: The even-star of $\frac{4}{1\sqrt{2}}$ in $\mathscr{M}_4(5)$

Example 4.21. To find the odd-star of $\frac{1\sqrt{3}}{2}$ in $\mathscr{M}_6(4)$, see Figure 4.12, we have $\frac{b}{d\sqrt{m}} = \frac{0}{1\sqrt{3}}$, then our unimodular matrix T is $\begin{pmatrix} 1\sqrt{3} & 0 \\ 2 & 1\sqrt{3} \end{pmatrix}$. The first term and the common difference of the arithmetic progression (4.6.2) are $\frac{0}{1\sqrt{3}}$ and $\frac{3}{2\sqrt{3}}$ respectively. Thus the successive neighbors of $\frac{1\sqrt{3}}{2}$ is given by the formula $\frac{3k}{(1+2k)\sqrt{3}}$. Therefore, the odd-star of $\frac{1\sqrt{3}}{2}$ in $\mathscr{M}_6(4)$ is

$$\{\frac{1\sqrt{3}}{2}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}\}.$$

Theorem 4.22. *For a prime p the even-stars of $\frac{1}{0\sqrt{m}}, \frac{2}{0\sqrt{m}}, \ldots, \frac{(p-1)/2}{0\sqrt{m}}$, and the odd-stars of $\frac{1\sqrt{m}}{0}, \frac{2\sqrt{m}}{0}, \ldots, \frac{((p-1)/2)\sqrt{m}}{0}$ are disjoint and cover $\mathscr{M}_q(p)$.*

Proof. Consider an even vertex $\frac{a}{0\sqrt{m}}$. Let A be the inverse of a mod p. Then the even-star of $\frac{a}{0\sqrt{m}}$ is $\{\frac{a}{0\sqrt{m}}, \frac{0\sqrt{m}}{A}, \frac{1\sqrt{m}}{A}, ..., \frac{(p-1)\sqrt{m}}{A}\}$. Consider a distinct even-vertex $\frac{a'}{0\sqrt{m}}$, so that $a \not\equiv \pm a'$ mod p. Let A' be the inverse of a' mod p. Then $A \not\equiv \pm A'$ mod p so that the even-star of $\frac{a'}{0\sqrt{m}}$ is $\{\frac{a'}{0\sqrt{m}}, \frac{0\sqrt{m}}{A'}, \frac{1\sqrt{m}}{A'}, ..., \frac{(p-1)\sqrt{m}}{A'}\}$. Now let $\frac{b\sqrt{m}}{0}$ be an odd vertex. Let B be the inverse of b mod p. Then the odd-star of $\frac{b\sqrt{m}}{0}$ is $\{\frac{b\sqrt{m}}{0}, \frac{0}{B\sqrt{m}}, \frac{1}{B\sqrt{m}}, ..., \frac{(p-1)}{B\sqrt{m}}\}$. Let B' be the inverse of b' mod p. Then $B \not\equiv \pm B'$ mod p so that the odd-star of $\frac{b'\sqrt{m}}{0}$ is $\{\frac{b'\sqrt{m}}{0}, \frac{0}{B'\sqrt{m}}, \frac{1}{B'\sqrt{m}}, ..., \frac{(p-1)}{B'\sqrt{m}}\}$. As $A \not\equiv \pm A'$ mod p these even-stars are disjoint, also as $B \not\equiv \pm B'$ mod p, these odd-stars are disjoint. There are $p+1$ vertices in each even-star and odd-star and $(p-1)/2$ stars of each type. Thus there are p^2-1 vertices which is the total number of vertices of $\mathscr{M}_q(p)$. □

Example 4.23. $\mathscr{M}_6(5)$. Using Table 4.7 we find that the even-star of

$$\frac{1}{0\sqrt{3}} \text{ is } \{\frac{1}{0\sqrt{3}}, \frac{0\sqrt{3}}{4}, \frac{1\sqrt{3}}{4}, \frac{2\sqrt{3}}{4}, \frac{3\sqrt{3}}{4}, \frac{4\sqrt{3}}{4}\},$$

and that of

$$\frac{2}{0\sqrt{3}} \text{ is } \{\frac{2}{0\sqrt{3}}, \frac{0\sqrt{3}}{2}, \frac{1\sqrt{3}}{2}, \frac{2\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}, \frac{4\sqrt{3}}{2}\}.$$

The odd-star of

$$\frac{1\sqrt{3}}{0} \text{ is } \{\frac{1\sqrt{3}}{0}, \frac{0}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{4}{2\sqrt{3}}\},$$

and that of

$$\frac{2\sqrt{3}}{0} \text{ is } \{\frac{2\sqrt{3}}{0}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}\}.$$

Thus the even-stars of $\frac{1}{0\sqrt{3}}$ and $\frac{2}{0\sqrt{3}}$ and the odd-stars of $\frac{1\sqrt{3}}{0}$ and $\frac{2\sqrt{3}}{0}$ give us 24 vertices, as listed in Table 4.7, which is the number of vertices of $\mathscr{M}_6(5)$.

Example 4.24. $\mathscr{M}_4(5)$. Using Table 4.4 we find that the even-star of

$$A1 \text{ is } \{A1,\ D2,\ I2,\ L2,\ G2,\ A2\},$$

and that of

$$B1 \text{ is } \{B1,\ B2,\ J2,\ K2,\ F2,\ E2\},$$

The odd-star of

$$H2 \text{ is } \{H2,\ D1,\ C1,\ F1,\ G1,\ E1\},$$

and that of

$$C2 \text{ is } \{C2,\ K1,\ L1,\ I1,\ J1,\ H1\}$$

Thus the even-stars of $A1$ and $B1$ and the odd-stars of $H2$ and $C2$ give us 24 vertices, as listed in Table 4.4 and illustrated in Figure 4.14, which is the number of vertices of the map $\{5, 4\}$ of genus 4.

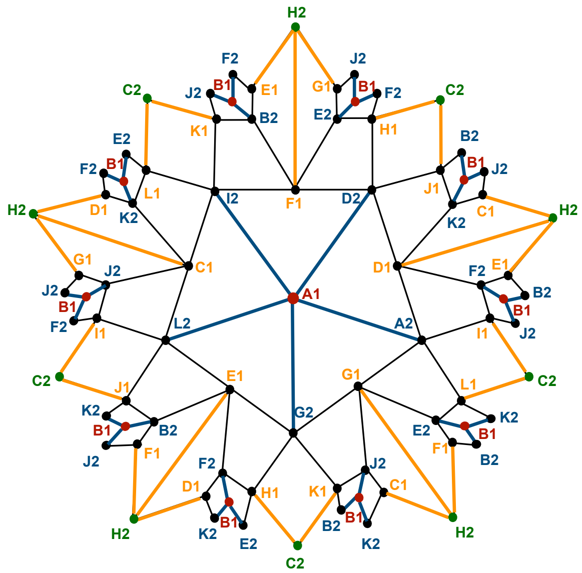


FIGURE 4.14: The even-stars of $A1$ and $B1$ and the odd-stars of $H2$ and $C2$ partition the vertices of the map $\{5, 4\}$ of genus 4

If we replace primes p by some composite numbers n, then the theorem does not apply as described in the following counter-example.

Example 4.25. $\mathscr{M}_6(4)$. Using Table 4.7 and Figure 4.12 we find that the even-star of

$$\frac{1}{0\sqrt{3}} \text{ is } \{\frac{1}{0\sqrt{3}}, \frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{1}, \frac{2\sqrt{3}}{1}, \frac{1\sqrt{3}}{3}\}.$$

The odd-star of

$$\frac{1\sqrt{3}}{0} \text{ is } \{\frac{1\sqrt{3}}{0}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}\}.$$

Thus the even-star of $\frac{1}{0\sqrt{3}}$ and the odd-star of $\frac{1\sqrt{3}}{0}$ give us nine vertices, which do not cover $\mathscr{M}_6(4)$.

4.7 Poles of $\mathscr{M}_q(n)$

We call the points of the form $\frac{a}{0\sqrt{m}}$ or $\frac{b\sqrt{m}}{0}$ the *even-poles* and *odd-poles* of $\mathscr{M}_q(n)$ respectively.

Lemma 4.26. *(i) If n is even then $\mathscr{M}_4(n)$ has no odd-poles.*
(ii) If $3|n$ then $\mathscr{M}_6(n)$ has no odd-poles.

Proof. (i) The odd elements of $\mathscr{M}_4(n)$ have the form $\frac{b\sqrt{2}}{d}$ with $(b, d, n) = 1$ and d is odd if n is even (Section 4.5.1). The odd-poles are those with $d = 0$. Since 0 is even, there can be no odd-poles if n is even. (ii) Similar to (i) but for $\mathscr{M}_6(n)$ the odd elements have the form $\frac{b\sqrt{3}}{d}$ with $(b, d, n) = 1$ and $3 \nmid d$ if $3|n$ (Section 4.5.2). As $3|0 = d$, then there can be no odd-poles if $3|n$. $\square$

Even-poles and odd-poles have the form $\frac{a}{0\sqrt{m}}$ and $\frac{b\sqrt{m}}{0}$ respectively, where $(a, 0, n) = 1$ and $(b, 0, n) = 1$ implies $(a, n) = 1$ and $(b, n) = 1$. Since the Euler function $\phi(n)$ counts all the integers that are relatively prime to n, and we identify a with $-a$ and b with $-b$ therefore the number of even-poles in $\mathscr{M}_q(n)$ is $\phi(n)/2$ for $n > 2$. Similarly the number of odd-poles, if they are exist, is $\phi(n)/2$. So the total number of poles is either $\phi(n)$ if both even and odd-poles exist or $\phi(n)/2$ if there are no odd-poles.

Example 4.27. From Table 4.3, for $n = 6$, the number of poles in $\mathscr{M}_4(6)$ is $\phi(6)/2 = 1$, namely $\frac{1}{0\sqrt{2}}$, and there are no odd-poles.
For $n = 7$, the number of poles in $\mathscr{M}_4(7)$ is $\phi(7) = 6$, and these are $\frac{1}{0\sqrt{2}}$, $\frac{2}{0\sqrt{2}}$, $\frac{3}{0\sqrt{2}}$, $\frac{1\sqrt{2}}{0}$, $\frac{2\sqrt{2}}{0}$, $\frac{3\sqrt{2}}{0}$.
From Table 4.7, for $n = 6$, the number of poles in $\mathscr{M}_6(6)$ is $\phi(6)/2 = 1$, namely $\frac{1}{0\sqrt{3}}$, and there are no odd-poles.
For $n = 5$, the number of poles in $\mathscr{M}_6(5)$ is $\phi(5) = 4$, and these are $\frac{1}{0\sqrt{3}}$, $\frac{2}{0\sqrt{3}}$, $\frac{1\sqrt{3}}{0}$, $\frac{2\sqrt{3}}{0}$.

In a connected graph the *distance* $\delta(x, y)$ between two vertices x, y is defined as the least number of edges in a path from x to y, provided at least one such path exists. The *diameter* of a graph or map is the maximum distance between two of its vertices.

Lemma 4.28. *Paths in a bipartite graph must be of even length if they are connecting two vertices in the same part and they must be of odd length if they are connecting vertices in different parts.*

Theorem 4.29. *If odd-poles exist, then the distance between any even-pole and any odd-pole of $\mathscr{M}_q(n)$ is equal to 3.*

Proof. For the existence of the odd-poles, we take $2 \nmid n$ and $3 \nmid n$ for $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$ respectively. By regularity of $\mathscr{M}_q(n)$ we may assume that one of the even-poles is $\frac{1}{0\sqrt{m}}$. As $\mathscr{M}_q(n)$ is a bipartite graph (Section 4.5.1, Section 4.5.2), then $\delta(\frac{1}{0\sqrt{m}}, \frac{a\sqrt{m}}{0})$ is odd, by Lemma 4.28. This even pole $\frac{1}{0\sqrt{m}}$ is not adjacent to an odd-pole $\frac{a\sqrt{m}}{0}$, so $\delta(\frac{1}{0\sqrt{m}}, \frac{a\sqrt{m}}{0}) \neq 1$. However, we can always find an odd vertex and even vertex to construct a path of length 3 between $\frac{1}{0\sqrt{m}}$ and $\frac{a\sqrt{m}}{0}$ of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{0\sqrt{m}}{1} \longleftrightarrow \frac{1}{y\sqrt{m}} \longleftrightarrow \frac{a\sqrt{m}}{0}, \tag{4.7.1}$$

where $y \in \mathbb{Z}_n$ such that $amy \equiv \pm 1 \mod n$ implies $y \equiv \pm(am)^{-1} \mod n$. As a^{-1} is the inverse of a modulo n exists because $(a, 0, n) = 1$, hence $(a, n) = 1$. Also m^{-1} is the inverse of m modulo n exists because $(m, n) = 1$ then y has a solution. $\square$

Example 4.30. 1- Using Figure 4.7 of the map $\mathscr{M}_4(5)$ and Table 4.3, when $n = 5$ and $x = 0$, then $\delta(\frac{1}{0\sqrt{2}}, \frac{1\sqrt{2}}{0}) = 3$, using (4.7.1) we have

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{1}{2\sqrt{2}} \longleftrightarrow \frac{1\sqrt{2}}{0} = A1 \longleftrightarrow A2 \longleftrightarrow I1 \longleftrightarrow C2$$

where $y = 2$. (See Figure 4.15.)

2- Using Figure 4.12 of the map $\mathscr{M}_6(4)$ and Table 4.7, when $n = 4$ and $x = 0$, then $\delta(\frac{1}{0\sqrt{3}}, \frac{1\sqrt{3}}{0}) = 3$, using (4.7.1) we have

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{1\sqrt{3}} \longleftrightarrow \frac{1\sqrt{3}}{0}$$

where $y = 1$. (See Figure 4.15.)

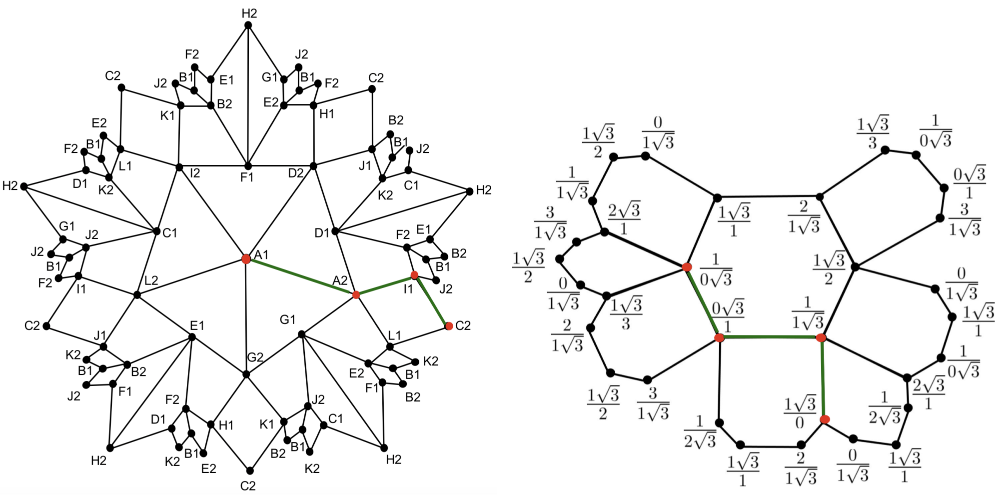


FIGURE 4.15: Left: $\delta(\frac{1}{0\sqrt{2}}, \frac{1\sqrt{2}}{0}) = d(A1, C2) = 3$. Right: $\delta(\frac{1}{0\sqrt{3}}, \frac{1\sqrt{3}}{0}) = 3$

Theorem 4.31. *The distance between any two distinct even-poles in $\mathscr{M}_q(n)$ is equal to 4, for $n > 6$.*

Proof. By regularity we may assume that one of the even-poles is $\frac{1}{0\sqrt{m}}$. As $\mathscr{M}_q(n)$ is a bipartite graph (Section 4.5.1, Section 4.5.2), then $\delta(\frac{1}{0\sqrt{m}}, \frac{a}{0\sqrt{m}})$ is even by Lemma 4.28, where $a \not\equiv \pm 1 \bmod n$. This immediately excludes $\delta(\frac{1}{0\sqrt{m}}, \frac{a}{0\sqrt{m}}) = 1$ and 3. Also there is no path of length 2 between $\frac{1}{0\sqrt{m}}$ and $\frac{a}{0\sqrt{m}}$, for otherwise there would be $x, y \in \mathbb{Z}_n$ such that we have the following path

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{x\sqrt{m}}{y} \longleftrightarrow \frac{a}{0\sqrt{m}}. \tag{4.7.2}$$

Then $y \equiv \pm 1 \bmod n$ and $ay \equiv \pm 1 \bmod n$. As $(a, n) = 1$ then a^{-1} exists, therefore y is the inverse of $a \bmod n$. Thus $ay \equiv a \equiv \pm 1 \bmod n$, implies $a \equiv \pm 1 \bmod n$, which is a contradiction. Thus $\delta(\frac{1}{0\sqrt{m}}, \frac{a}{0\sqrt{m}}) \neq 2$.

However, we can always construct a path of length 4 between $\frac{1}{0\sqrt{m}}$ and $\frac{a}{0\sqrt{m}}$, of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{0\sqrt{m}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{m}} \longleftrightarrow \frac{m^{-1}(a+1)\sqrt{m}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{m}}, \tag{4.7.3}$$

where a^{-1} is the inverse of a modulo n, and m^{-1} is the inverse of m modulo n. The inverse a^{-1} exists because $(a, 0, n) = 1$ and hence $(a, n) = 1$. The inverse m^{-1} exists if $(m, n) = 1$, meaning that whenever n is odd for $q = 4$, $m = 2$ and $3 \nmid n$ for $q = 6$, $m = 3$. If m^{-1} does not exist i.e. $(m, n) = m$, we have the following cases.
(i) For $q = 4, m = 2$ and n is an even integer, then a is odd, so we can construct a path of length 4 between $\frac{1}{0\sqrt{2}}$ and $\frac{a}{0\sqrt{2}}$, of the form

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{2}} \longleftrightarrow \frac{\frac{1}{2}(a+1)\sqrt{2}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{2}}. \tag{4.7.4}$$

(ii) For $q = 6, m = 3$ and $3|n$, then $3 \nmid a$. Let $a \equiv 1 \bmod 3$, so we can construct a path of length 4 between $\frac{1}{0\sqrt{3}}$ and $\frac{a}{0\sqrt{3}}$, of the form

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{3}} \longleftrightarrow \frac{\frac{1}{3}(a-1)\sqrt{3}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{3}}. \tag{4.7.5}$$

If $3 \nmid a$ and $a \equiv 2 \bmod 3$, we can construct a path of length 4 between $\frac{1}{0\sqrt{3}}$ and $\frac{a}{0\sqrt{3}}$, of the form

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{3}} \longleftrightarrow \frac{\frac{1}{3}(a+1)\sqrt{3}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{3}}. \tag{4.7.6}$$

Thus for all $n > 6$ the distance between two distinct even-poles in $\mathscr{M}_q(n)$ is equal to 4. $\square$

Theorem 4.32. *If odd-poles exist then the distance between any two distinct odd-poles in $\mathscr{M}_q(n)$ is equal to 4 for $n \geq 5$.*

Proof. For the existence of the odd-poles, we take $2 \nmid n$ and $3 \nmid n$ for $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$ respectively. By regularity we may assume that one of the odd-poles is $\frac{1\sqrt{m}}{0}$. As $\mathscr{M}_q(n)$ is a bipartite graph (Section 4.5.1, Section 4.5.2), then $\delta(\frac{1\sqrt{m}}{0}, \frac{a\sqrt{m}}{0})$ is even by Lemma 4.28, where $a \not\equiv \pm 1 \bmod n$. This immediately excludes $\delta(\frac{1\sqrt{m}}{0}, \frac{a\sqrt{m}}{0}) = 1$ and 3. Also there is no path of length 2 between $\frac{1\sqrt{m}}{0}$ and $\frac{a\sqrt{m}}{0}$, for otherwise there would be $x, y \in \mathbb{Z}_n$ such that we have the following path

$$\frac{1\sqrt{m}}{0} \longleftrightarrow \frac{x}{y\sqrt{m}} \longleftrightarrow \frac{a\sqrt{m}}{0}. \tag{4.7.7}$$

Then $my \equiv \pm 1 \bmod n$ and $amy \equiv \pm 1 \bmod n$. Thus my is the inverse of $a \bmod n$, so $amy \equiv a \equiv \pm 1 \bmod n$ implies $a \equiv \pm 1 \bmod n$, which is a contradiction. Therefore $\delta(\frac{1\sqrt{m}}{0}, \frac{a\sqrt{m}}{0}) \neq 2$. However, we can always construct a path of length 4 between $\frac{1\sqrt{m}}{0}$

and $\frac{a\sqrt{m}}{0}$, of the form

$$\frac{1\sqrt{m}}{0} \longleftrightarrow \frac{1}{m^{-1}\sqrt{m}} \longleftrightarrow \frac{a^{-1}\sqrt{m}}{a^{-1}+1} \longleftrightarrow \frac{a^{-1}-1}{a^{-1}m^{-1}\sqrt{m}} \longleftrightarrow \frac{a\sqrt{m}}{0}, \tag{4.7.8}$$

where a^{-1} is the inverse of a modulo n, and m^{-1} is the inverse of m modulo n. The inverse a^{-1} exists because $(a, 0, n) = 1$ and hence $(a, n) = 1$, also the inverse m^{-1} exist because $(m, n) = 1$. □

Example 4.33. 1- Consider Figure 4.7 of the map $\mathscr{M}_4(5)$ and Table 4.3, when $n = 5$. As $\phi(5)/2 = 2 > 1$, we have two distinct even-poles and odd-poles, thus (4.7.3) is applicable and we have $\delta(\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}) = 4$. Here we have $a^{-1} = 3$ and $m^{-1} = 3$.

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{4}{2\sqrt{2}} \longleftrightarrow \frac{1\sqrt{2}}{2} \longleftrightarrow \frac{2}{0\sqrt{2}} = A1 \longleftrightarrow A2 \longleftrightarrow L1 \longleftrightarrow E2 \longleftrightarrow B1.$$

(See Figure 4.16.)

2- Consider Table 4.3, when $n = 8$. As $\phi(8)/2 = 2 > 1$, we have two distinct even-poles in $\mathscr{M}_4(8)$, thus (4.7.4) is applicable and we have $\delta(\frac{1}{0\sqrt{2}}, \frac{3}{0\sqrt{2}}) = 4$. Here $a^{-1} = 3$, and we have the following path:

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{1}{3\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{3} \longleftrightarrow \frac{3}{0\sqrt{2}}.$$

3- When $q = 6$ and $n = 9$. As $\phi(9)/2 = 3 > 1$, we have three distinct even-poles in $\mathscr{M}_6(9)$, thus (4.7.6) is applicable and we have $\delta(\frac{1}{0\sqrt{3}}, \frac{2}{0\sqrt{3}}) = 4$. Here $a^{-1} = 5$ and $a = 2 \equiv 2 \bmod 3$. We have the following path:

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{5\sqrt{3}} \longleftrightarrow \frac{1\sqrt{3}}{5} \longleftrightarrow \frac{2}{0\sqrt{3}}.$$

Example 4.34. 1- Consider Figure 4.7 of the map $\mathscr{M}_4(5)$ and Table 4.3, when $n = 5$. As $\phi(5)/2 = 2 > 1$, we have two distinct odd-poles, thus (4.7.8) is applicable and $\delta(\frac{1\sqrt{2}}{0}, \frac{2\sqrt{2}}{0}) = 4$. Here $a^{-1} = 3$ and $m^{-1} = 3$. We have the following path:

$$\frac{1\sqrt{2}}{0} \longleftrightarrow \frac{4}{2\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{1} \longleftrightarrow \frac{3}{1\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{0} = C2 \longleftrightarrow L1 \longleftrightarrow I2 \longleftrightarrow F1 \longleftrightarrow H2.$$

(See Figure 4.16.)

2- Consider Table 4.7, when $3 \nmid n = 7$. As $\phi(7)/2 = 3 > 1$, we have three distinct odd-poles in $\mathscr{M}_6(7)$, thus (4.7.8) is applicable and we have $\delta(\frac{1\sqrt{3}}{0}, \frac{2\sqrt{3}}{0}) = 4$. Here $a^{-1} = 4$ and $m^{-1} = 5$. We have the following path:

$$\frac{1\sqrt{3}}{0} \longleftrightarrow \frac{6}{2\sqrt{3}} \longleftrightarrow \frac{3\sqrt{3}}{2} \longleftrightarrow \frac{4}{1\sqrt{3}} \longleftrightarrow \frac{2\sqrt{3}}{0}.$$

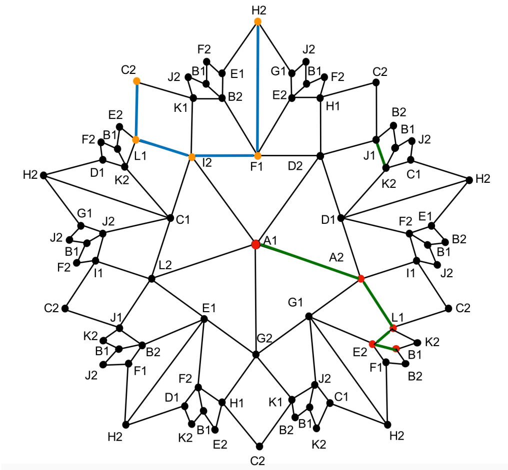


FIGURE 4.16: $\delta(\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}) = \delta(A1, B1) = 4$, $\delta(\frac{1\sqrt{2}}{0}, \frac{2\sqrt{2}}{0}) = \delta(C2, H2) = 4$ in $\mathscr{M}_4(5)$

Lemma 4.35 ([SS18, Lemma 10])**.** *Let b, d, n be integers such that $(b, d, n) = 1$. Then there exists an integer x so that $(b + dx, n) = 1$.*

Theorem 4.36. *Given an odd vertex $\frac{b\sqrt{m}}{d}$ in $\mathscr{M}_q(n)$, $n \geq 5$, then $\delta(\frac{1}{0\sqrt{m}}, \frac{b\sqrt{m}}{d}) = 3$ if and only if $d \not\equiv \pm 1$ mod n.*

Proof. By Lemma 4.28, the distance $\delta(\frac{1}{0\sqrt{m}}, \frac{b\sqrt{m}}{d})$ is odd. If $\delta(\frac{1}{0\sqrt{m}}, \frac{b\sqrt{m}}{d}) = 1$, then $\frac{b\sqrt{m}}{d}$ is adjacent to $\frac{1}{0\sqrt{m}}$ implies $d \equiv \pm 1$ mod n, which is a contradiction. If $d \equiv 0$ mod n then $\delta = 3$, by Theorem 4.26. Now for the case when $d \not\equiv 0$ mod n, there are no odd-poles, by Lemma 4.26 (i.e. n is even for $q = 4$ and $3|n$ for $q = 6$). We want to construct a path of length 3 between $\frac{1}{0\sqrt{m}}$ and $\frac{b\sqrt{m}}{d}$ of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{x\sqrt{m}}{1} \longleftrightarrow \frac{u}{v\sqrt{m}} \longleftrightarrow \frac{b\sqrt{m}}{d},$$

where x, u and $v \in \mathbb{Z}_n$. In terms of congruences this means that we have two simultaneous equations:

$$mxv - u \equiv \pm 1 \bmod n$$

$$ud - mvb \equiv \pm 1 \bmod n$$

$$\Longrightarrow \begin{cases} 2v(dx - b) \equiv d + 1 \bmod n & if\ q = 4 \\ 3v(dx - b) \equiv d - 1 \bmod n & if\ q = 6 \text{ and } d \equiv 1 \bmod 3 \\ 3v(dx - b) \equiv d + 1 \bmod n & if\ q = 6 \text{ and } d \equiv 2 \bmod 3. \end{cases} \tag{4.7.9}$$

By Lemma 4.7.12 we know that $(dx - b)$ is coprime to n and therefore has a multiplicative inverse modulo n. Hence the equations in 4.7.9 can be solved for v which, in turn, determines x and u. $\square$

Example 4.37. (i) The distance between $\frac{1}{0\sqrt{2}}$ and $\frac{2\sqrt{2}}{9}$ in $\mathscr{M}_4(18)$ is 3 and we have the following path

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{3\sqrt{2}}{1} \longleftrightarrow \frac{11}{2\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{9},$$

where $v = 2$, $u = 11$ and $x = 3$.
(ii) The distance between $\frac{1}{0\sqrt{3}}$ and $\frac{3\sqrt{3}}{7}$ in $\mathscr{M}_6(15)$ is 3 and we have the following path

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{2\sqrt{3}}{1} \longleftrightarrow \frac{11}{12\sqrt{3}} \longleftrightarrow \frac{3\sqrt{3}}{7},$$

where $v = 12$, $u = 11$ and $x = 2$.

Theorem 4.38. *The diameter of $\mathscr{M}_q(n)$ is equal to 4 for all $n > 6$.*

Proof. The distance between any odd vertex and even vertex in $\mathscr{M}_q(n)$ is equal 1, if they are adjacent. Otherwise, $\delta = 3$, by Theorems 4.29 and 4.36. Also, the distance between any two distinct vertices of the same type is ≤ 4, by Theorems 4.31 and 4.32. Thus the diameter of $\mathscr{M}_q(n)$ is equal to 4 for all $n > 6$. Refer to Table 4.9 for the diameters of $\mathscr{M}_q(n)$ for $n \leq 5$. $\square$

TABLE 4.9: The diameters of $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$ for $n \leq 6$

n	$q = 4$		$q = 6$	
	$\mathscr{M}_4(n)$	Diameter	$\mathscr{M}_6(n)$	Diameter
2	Square Figure 4.3	2	Hexagon Figure 4.10	3
3	Cube Figure 4.4	3	$\{3,6\}$ Figure 4.11	3
4	$\{4,4\}_{2,2}$ Figure 4.6	2	$\{4,6\}$ Figure 4.12	3
5	$\{5,4\}$ Figure 4.7	4	$\{5,6\}$	4
6	$\{6,4\}$	3	$\{6,6\}$	3

Definition 4.39. If $\frac{a}{c\sqrt{m}}$ and $\frac{b}{d\sqrt{m}}$ are two distinct even vertices and $\frac{a\sqrt{m}}{c}$ and $\frac{b\sqrt{m}}{d}$ are two distinct odd vertices in $\mathscr{M}_q(n)$, then $\Delta = (ad - bc)\sqrt{m}$.

Let us extend Definition 4.39 to include two vertices of different types, in this case $\Delta = \pm(ad - mbc)$.

Theorem 4.40. *Let $\frac{a}{c\sqrt{m}}$ be an even vertex and $\frac{b\sqrt{m}}{d}$ be an odd vertex in $\mathscr{M}_q(n)$. Then*

$$\delta(\frac{a}{c\sqrt{m}}, \frac{b\sqrt{m}}{d}) = \begin{cases} 1 & \textit{if and only if } |\Delta| \equiv 1 \bmod n, \\ 3 & \textit{otherwise.} \end{cases} \tag{4.7.10}$$

Proof. Let $\frac{a}{c\sqrt{m}}$, $\frac{b\sqrt{m}}{d}$ be an even and an odd vertices respectively. By Lemma 4.28, δ is odd. If $|\Delta| \not\equiv 1 \bmod n$, then the two vertices are not adjacent, thus we must have $\delta = 3$ by Theorem 4.38. Now to see if an even vertex $\frac{a}{c\sqrt{m}}$ and an odd vertex $\frac{b\sqrt{m}}{d}$ are distance three apart when $|\Delta| \not\equiv 1 \bmod n$, let $T \in H_q$ be the transformation where $T\big(\frac{a}{c\sqrt{m}}\big) = \big(\frac{1}{0\sqrt{m}}\big)$. Apply this transformation to the odd vertex, $T\big(\frac{b\sqrt{m}}{d}\big) = \big(\frac{x\sqrt{m}}{y}\big)$ and then we can apply Theorem 4.36. Otherwise, if $T\big(\frac{a}{c\sqrt{m}}\big) = \big(\frac{1}{0\sqrt{m}}\big)$ and $T\big(\frac{b\sqrt{m}}{d}\big) = \big(\frac{x\sqrt{m}}{0}\big)$, then we can apply Theorem 4.29.

$\square$

Example 4.41. (i) Let us check if $\delta(\frac{4}{1\sqrt{2}}, \frac{2\sqrt{2}}{3}) = \delta(G1, K2) = 3$ as in Figure 4.7. Let $T = \big(\begin{smallmatrix} 1 & 0\sqrt{2} \\ 1\sqrt{2} & 1 \end{smallmatrix}\big) \in H_q$, then $T\big(\frac{4}{1\sqrt{2}}\big) = \big(\frac{1}{0\sqrt{2}}\big) = A1$, and $T\big(\frac{2\sqrt{2}}{3}\big) = \big(\frac{2\sqrt{2}}{2}\big) = J2$. Then we can apply Theorem 4.36, to get $\delta(\frac{1}{0\sqrt{2}}, \frac{2\sqrt{2}}{2}) = \delta(A1, J2) = 3$.
(ii) Applying the same T we can check the distance between $\frac{4}{1\sqrt{2}} = G1$ and $\frac{2\sqrt{2}}{1} = I2$. We have $T\big(\frac{4}{1\sqrt{2}}\big) = \big(\frac{1}{0\sqrt{2}}\big) = A1$, and $T\big(\frac{2\sqrt{2}}{1}\big) = \big(\frac{2\sqrt{2}}{0}\big) = H2$, so $\delta(\frac{1}{0\sqrt{2}}, \frac{2\sqrt{2}}{0}) = \delta(A1, H2) = 3$ by applying Theorem 4.29.

Theorem 4.42. *If x and y are two distinct vertices that are either both odd or both even in $\mathscr{M}_q(p)$ where p is a prime, then*

$$\delta(x,y) = \begin{cases} 4 & \text{if and only if } \Delta = 0, \\ 2 & \text{otherwise.} \end{cases} \tag{4.7.11}$$

Proof. By Lemma 4.28, $\delta(x,y) = 2$ or 4 but not greater that 4, by Theorem 4.38. Let $x = \frac{a}{c\sqrt{m}} = \frac{1}{0\sqrt{m}}$ and $y = \frac{b}{d\sqrt{m}}$ be two distinct even vertices such that $\Delta \neq 0$. Then we can always construct a path of length 2 between them of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{(b\pm 1)m^{-1}d^{-1}\sqrt{m}}{1} \longleftrightarrow \frac{b}{d\sqrt{m}}, \tag{4.7.12}$$

where m^{-1} and c^{-1} are always exist because $(m,p) = 1$ and $(c,p) = 1$. Now if $\Delta = 0$, then x and y must be two distinct even-poles, thus $\delta(x,y) = 4$, by Theorem 4.31. Otherwise, $x = \frac{0}{1\sqrt{m}}$ and $y = \frac{0}{d\sqrt{m}}$ (i.e. $a \equiv b \equiv 0 \bmod p$). We can always construct a path of length 4 between $\frac{0}{1\sqrt{m}}$ and $\frac{0}{d\sqrt{m}}$ of the form

$$\frac{0}{1\sqrt{m}} \longleftrightarrow \frac{m^{-1}\sqrt{m}}{1} \longleftrightarrow \frac{1}{0\sqrt{m}} \longleftrightarrow \frac{d^{-1}m^{-1}\sqrt{m}}{1} \longleftrightarrow \frac{0}{d\sqrt{m}}, \tag{4.7.13}$$

where m^{-1} and d^{-1} are always exist because $(m,p) = 1$ and $(d,p) = 1$. Next we want to show that if $\delta(x,y) = 4$ then $\Delta = 0$. Apply the transformation $T \in H_q$, such that $T\big(\frac{a}{c\sqrt{m}}\big) = \big(\frac{1}{0\sqrt{m}}\big)$ and $T\big(\frac{b}{d\sqrt{m}}\big) = \big(\frac{e}{f\sqrt{m}}\big)$. Now consider these two new points, i.e. $\delta(\frac{1}{0\sqrt{m}}, \frac{e}{f\sqrt{m}}) = 4$ if $\Delta = 0$. As $\delta(x,y) = 4$ which therefore can happen when $f = 0$ then we can apply Theorem 4.31. The determinant of these two points $\frac{1}{0\sqrt{m}}$ and $\frac{e}{0\sqrt{m}}$, is equal to 0. As T preserves the determinants of points, thus the determinant of $\frac{a}{c\sqrt{m}}$

and $\frac{b}{d\sqrt{m}}$ is also equal 0.
Similarly, we can prove the theorem if given any two distinct odd vertices $\frac{a\sqrt{m}}{c}$ and $\frac{b\sqrt{m}}{d}$. $\square$

In the above theorem we have made this restriction from n to p as for the first case when $\delta(x, y) = 4$, the odd-poles and the even vertices of the form $\frac{0}{c\sqrt{m}}$ only exist if n is odd for $q = 4$ and $3 \nmid n$ for $q = 6$, by Lemma 4.26 and Sections 4.5.1, 4.5.1. Thus it is better to restrict n to p to satisfy that $(0, c, n) = 1$. Now for the second case if we replace p by some composite numbers n, then $\delta(x, y) = 2$ does not apply as described in the following counter-example.

Example 4.43. (i) The distance between $\frac{1}{0\sqrt{2}}$ and $\frac{2}{5\sqrt{2}}$ in $\mathscr{M}_4(25)$ is equal to 4 while $\Delta = 5\sqrt{2}$, as we can find a path of length 4 of the form

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{21\sqrt{2}}{1} \longleftrightarrow \frac{16}{1\sqrt{2}} \longleftrightarrow \frac{3\sqrt{2}}{2} \longleftrightarrow \frac{2}{5\sqrt{2}}.$$

(ii) The distance between $\frac{1}{0\sqrt{3}}$ and $\frac{5}{11\sqrt{3}}$ in $\mathscr{M}_6(24)$ is equal to 4 while $\Delta = 11\sqrt{3}$, as we can find a path of length 4 of the form

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{20\sqrt{3}}{1} \longleftrightarrow \frac{13}{1\sqrt{3}} \longleftrightarrow \frac{1\sqrt{3}}{2} \longleftrightarrow \frac{5}{11\sqrt{3}}.$$

Chapter 5

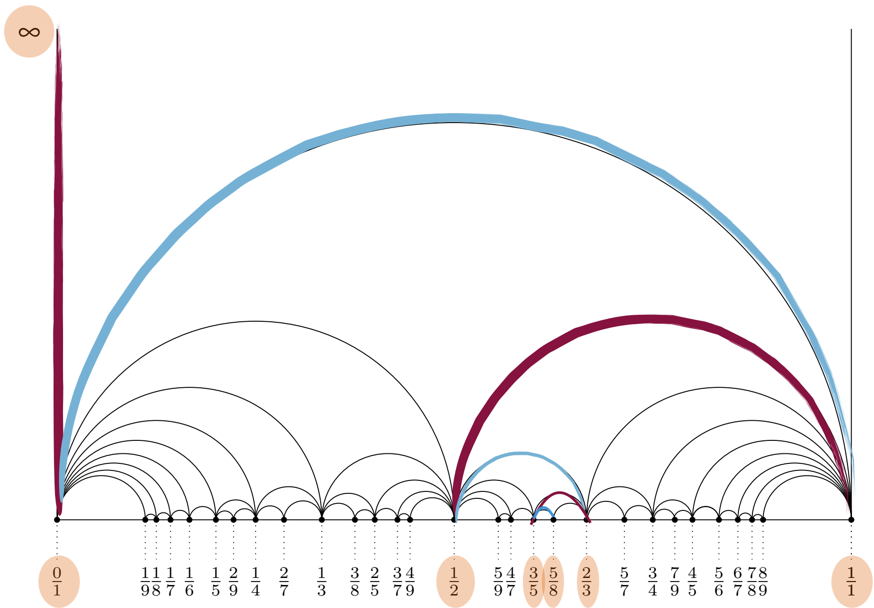
Petrie polygons

A Petrie path in a map $\mathscr{M}$ on an orientable surface is defined by the zig-zag path $V_1, V_2, V_3, ...$ through the map, where V_i are vertices of the map. We start at the vertex V_1, moving along an edge from V_1 to V_2, and then at V_2 we take the first left (moving anti-clockwise until we reach the next edge), reaching V_3 then turning right, etc, . There is then a path in which two consecutive edges belong to the same face, but in which no three consecutive edges belong to the same face. In a finite regular map we return to V_1 and then V_2 after a certain number of steps, independent of the initial edge; this number is called the *Petrie length* of the map, and in this situation the Petrie path will be referred to as a *Petrie polygon*.
Petrie paths for the Farey map $\hat{\mathscr{M}}_3$, were studied by Singerman and Strudwick in [SS16]. One of the results of this paper was an easy way to show that the Petrie polygons for the Klein quartic surface $\mathscr{M}_3(7)$ have length eight, giving " The Eightfold Way" [Lev99]. In this chapter we want to generalize some of these results for other values of q.

5.1 The Petrie polygons of the Farey map

Most of this section's material can be found in [SS16]. Consider a Petrie path $W_1, W_2, W_3, ...$ of $\hat{\mathscr{M}}_3$. By transitivity of the automorphism group on directed edges we may assume its first edge goes from $W_1 = \frac{1}{0}$ to $W_2 = \frac{0}{1}$. A left turn takes us to $W_3 = \frac{1}{1}$. Now a right turn takes us to $W_4 = \frac{1}{2}$ (see Figure 5.1).

FIGURE 5.1: Petrie path of Farey map

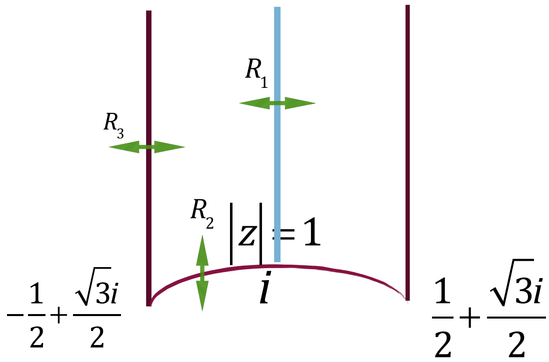


By applying a modular transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to the vertices ∞, 0 and 1 of the principal triangle, we find that generally the three consecutive vertices of a Petrie path are $\frac{a}{c}, \frac{b}{d}, \frac{a+b}{c+d}$, that is the third vertex is the *Farey median* of the previous two. Because the first two vertices of the Petrie path are $\frac{1}{0}$ and $\frac{0}{1}$, in [SS16], it was shown that the k-th vertex of the Petrie path is equal to $\frac{f_{k-1}}{f_k}$ where f_k is the k-th element of the Fibonacci sequence, as defined by $f_0 = 1, f_1 = 0, f_{k+1} = f_k + f_{k-1}$ for $k \geq 1$. Thus we define *the principal Petrie path* P_0^3 of $\hat{\mathscr{M}}_3$ as $\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \ldots$

In [CM57, Section 5.2], Coxeter and Moser showed that if the following three transformations $R_1(z) = -\bar{z}, \quad R_2(z) = \frac{1}{\bar{z}}, \quad R_3(z) = -\bar{z} - 1$, act on the fundamental region of Γ; as illustrated in Figure 5.2, then the composition of these three transformations is $R_1R_2R_3(z) = \frac{1}{\bar{z}+1}$, represents a transformation going one step along a Petrie path. The matrix corresponding to this composition is

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

FIGURE 5.2: Deriving the matrix P



Lemma 5.1. *The matrix* $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ *maps each vertex of* P_0^3 *of* $\hat{\mathscr{M}}_3$ *to the next one, and also* $P^k = \left(\begin{smallmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{smallmatrix}\right)$ *for all* $k \geq 1$.

Therefore all vertices of P_0^3 have the form $P^k(\infty)$ for $k = 0, 1, 2, \ldots$. For every k, the first column $\left(\begin{smallmatrix} f_k \\ f_{k+1} \end{smallmatrix}\right)$ of P^k is a vertex $\frac{f_k}{f_{k+1}} \in \mathbb{Q} \cup \{\infty\}$ of the principal Petrie path. The proof follows immediately from the definition of the Fibonacci sequence, and induction. Note that P has determinant -1, and thus it is not an element of Γ.
In this chapter we examine the Petrie polygons on $\mathscr{M}_q(n)$.

Definition 5.2. The *period* $\pi(n)$ of the Fibonacci sequence mod n is the least positive integer m, such that $f_{m-1} \equiv 1 \bmod n$, $f_m \equiv 0 \bmod n$. However, as $\frac{a}{b} = \frac{-a}{-b}$ we can consider the Fibonacci sequence mod n up to sign and so we define the *semi-period* $\sigma(n)$ of the Fibonacci sequence mod n to be the least positive integer m, with the property that $f_{m-1} \equiv \pm 1 \bmod n$, $f_m \equiv 0 \bmod n$.

Example 5.3. The Fibonacci sequence mod 5 is
$0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, \ldots$ so that $\pi(5) = 20$ and $\sigma(5) = 10$.
The Fibonacci sequence mod 7 is
$0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 4, 2, 6, 1, 0, 1, 1, \ldots$ so that $\pi(7) = 16$ and $\sigma(7) = 8$.
It is perhaps worth pointing out that it is possible that $\pi(n) = \sigma(n)$. For example the Fibonacci sequence mod 11 is
$0, 1, 1, 2, 3, 5, 8, 2, 10, 1, 0, 1, 1, 2 \ldots$ so that $\pi(11) = 10$ and $\sigma(11) = 10$.

Clearly $\sigma(n) = \pi(n)$ or $\pi(n)/2$ and it is an open question to know which of these occurs, see [SS16, Section 9].

5.2 The Petrie paths of the universal q-gonal map

In this section we generalize Lemma 5.1 to $\hat{\mathscr{M}}_q$. Applying the same ideas as Coxeter and Moser to the fundamental region of H_q , where $R_1(z) = -\bar{z}$, $\quad R_2(z) = \frac{1}{\bar{z}}$,

$R_3(z) = -\bar{z} - \lambda_q$, we get the matrix

$$Q = \begin{pmatrix} 0 & -1 \\ -1 & -\lambda_q \end{pmatrix},$$

which represents the same element as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & \lambda_q \end{smallmatrix}\right)$ in $PSL(2,\mathbb{R})$.
Recalling the extended triangle group defined in Section 1.3, and relating it to the Petrie paths, the extended triangle group $\Gamma^*(2,m,n)$ is a group generated by the reflections R_1, R_2, R_3. It has a presentation,

$$\langle R_1, R_2, R_3 \,|\, R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_2R_3)^m = (R_3R_1)^n = 1\rangle. \tag{5.2.1}$$

In [CM57, Section 8.6] it is shown that the transformation $R_1R_2R_3$ which is represented by the matrix Q goes one step along the Petrie path, i.e. this transformation has the effect of shifting a Petrie path one step along itself. If we have an extra relation of the form $(R_1R_2R_3)^r = I$, then this means that the Petrie path becomes a Petrie polygon having r edges. This will often be the case for the maps $\mathscr{M}_q(n)$.

As we deal with regular maps, the automorphism group acts transitively on Petrie paths, so they are all of the same length. Therefore it is sufficient to consider one of them and in particular *the principal Petrie path* P_0^q. By definition P_0^q's first arc goes from $V_1 = \frac{1}{0\lambda_q}$ to $V_2 = \frac{0\lambda_q}{1}$. A left turn then takes us to $V_3 = \frac{1}{1\lambda_q}$, (i.e. in case of $q = 4, 6$ it is alternating between even and odd vertices). The first two vertices of the principal Petrie path are $\frac{1}{0\lambda_q}$ and $\frac{0\lambda_q}{1}$.

Lemma 5.4. *The matrix $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & \lambda_q \end{smallmatrix}\right)$ maps each vertex of the principal Petrie path P_0^q of $\hat{\mathscr{M}}_q$ to the next one, and also*

$$Q^k = \begin{pmatrix} p_k(\lambda_q) & q_k(\lambda_q) \\ q_k(\lambda_q) & p_k(\lambda_q) + \lambda_q q_k(\lambda_q) \end{pmatrix}, \text{ for some } k \geq 1,\ q_k(\lambda_q) \text{ and } p_k(\lambda_q) \in \mathbb{Z}[\lambda_q],$$

where, for every $k \geq 1$, the first column $\left(\begin{smallmatrix} p_k(\lambda_q) \\ q_k(\lambda_q) \end{smallmatrix}\right)$ of Q^k is a vertex $\frac{p_k(\lambda_q)}{q_k(\lambda_q)} \in \mathbb{Q}(\lambda_q) \cup \{\infty\}$ of P_0^q.

Proof. Let $\Pi(k)$ be the proposition that

$$Q^k = \begin{pmatrix} p_k(\lambda_q) & q_k(\lambda_q) \\ q_k(\lambda_q) & p_k(\lambda_q) + \lambda_q q_k(\lambda_q) \end{pmatrix}.$$

Then, $\Pi(1)$ is true, as $Q^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & \lambda_q \end{smallmatrix}\right)$, where $p_1(\lambda_q) = 0$ and $q_1(\lambda_q) = 1$.
Assume that $\Pi(k)$ is true. Then,

$$Q^k = \begin{pmatrix} p_k(\lambda_q) & q_k(\lambda_q) \\ q_k(\lambda_q) & p_k(\lambda_q) + \lambda_q q_k(\lambda_q) \end{pmatrix}.$$

Hence

$$Q^{k+1} = Q^k Q = \begin{pmatrix} q_k(\lambda_q) & p_k(\lambda_q) + \lambda_q q_k(\lambda_q) \\ p_k(\lambda_q) + \lambda_q q_k(\lambda_q) & q_k(\lambda_q) + \lambda_q[p_k(\lambda_q) + \lambda_q q_k(\lambda_q)] \end{pmatrix}$$

Thus $\Pi(k+1)$ is true. Therefore, $\Pi(1)$ is true, and $\Pi(k) \implies \Pi(k+1)$; so by the principal of induction $\Pi(k)$ is true for all $k \geq 1$.

$\square$

Example 5.5. Letting $\lambda_q = \lambda$ and computing the first 16 powers of $Q = \left(\begin{smallmatrix} 0\lambda_q & 1 \\ 1 & \lambda_q \end{smallmatrix}\right)$ in general, to find the principal Petrie path's first 16 vertices, produce the following;

- $k = 1 \longrightarrow Q = \left(\begin{smallmatrix} 0\lambda & 1 \\ 1 & \lambda \end{smallmatrix}\right)$
- $k = 2 \longrightarrow Q^2 = \left(\begin{smallmatrix} 1 & \lambda \\ \lambda & \lambda^2+1 \end{smallmatrix}\right)$
- $k = 3 \longrightarrow Q^3 = \left(\begin{smallmatrix} \lambda & \lambda^2+1 \\ \lambda^2+1 & \lambda^3+2\lambda \end{smallmatrix}\right)$
- $k = 4 \longrightarrow Q^4 = \left(\begin{smallmatrix} \lambda^2+1 & \lambda^3+2\lambda \\ \lambda^3+2\lambda & \lambda^4+3\lambda^2+1 \end{smallmatrix}\right)$
- $k = 5 \longrightarrow Q^5 = \left(\begin{smallmatrix} \lambda^3+2\lambda & \lambda^4+3\lambda^2+1 \\ \lambda^4+3\lambda^2+1 & \lambda^5+4\lambda^3+3\lambda \end{smallmatrix}\right)$
- $k = 6 \longrightarrow Q^6 = \left(\begin{smallmatrix} \lambda^4+3\lambda^2+1 & \lambda^5+4\lambda^3+3\lambda \\ \lambda^5+4\lambda^3+3\lambda & \lambda^6+5\lambda^4+6\lambda^2+1 \end{smallmatrix}\right)$
- $k = 7 \longrightarrow Q^7 = \left(\begin{smallmatrix} \lambda^5+4\lambda^3+3\lambda & \lambda^6+5\lambda^4+6\lambda^2+1 \\ \lambda^6+5\lambda^4+6\lambda^2+1 & \lambda^7+6\lambda^5+10\lambda^3+4\lambda \end{smallmatrix}\right)$
- $k = 8 \longrightarrow Q^8 = \left(\begin{smallmatrix} \lambda^6+5\lambda^4+6\lambda^2+1 & \lambda^7+6\lambda^5+10\lambda^3+4\lambda \\ \lambda^7+6\lambda^5+10\lambda^3+4\lambda & \lambda^8+7\lambda^6+15\lambda^4+10\lambda^2+1 \end{smallmatrix}\right)$
- $k = 9 \longrightarrow Q^9 = \left(\begin{smallmatrix} \lambda^7+6\lambda^5+10\lambda^3+4\lambda & \lambda^8+7\lambda^6+15\lambda^4+10\lambda^2+1 \\ \lambda^8+7\lambda^6+15\lambda^4+10\lambda^2+1 & \lambda^9+8\lambda^7+21\lambda^5+20\lambda^3+5\lambda \end{smallmatrix}\right)$
- $k = 10 \longrightarrow Q^{10} = \left(\begin{smallmatrix} \lambda^8+7\lambda^6+15\lambda^4+10\lambda^2+1 & \lambda^9+8\lambda^7+21\lambda^5+20\lambda^3+5\lambda \\ \lambda^9+8\lambda^7+21\lambda^5+20\lambda^3+5\lambda & \lambda^{10}+9\lambda^8+28\lambda^6+35\lambda^4+15\lambda^2+1 \end{smallmatrix}\right)$
- $k = 11 \longrightarrow Q^{11} = \left(\begin{smallmatrix} \lambda^9+8\lambda^7+21\lambda^5+20\lambda^3+5\lambda & \lambda^{10}+9\lambda^8+28\lambda^6+35\lambda^4+15\lambda^2+1 \\ \lambda^{10}+9\lambda^8+28\lambda^6+35\lambda^4+15\lambda^2+1 & \lambda^{11}+10\lambda^9+36\lambda^7+56\lambda^5+35\lambda^3+6\lambda \end{smallmatrix}\right)$
- $k = 12 \longrightarrow Q^{12} = \left(\begin{smallmatrix} \lambda^{10}+9\lambda^8+28\lambda^6+35\lambda^4+15\lambda^2+1 & \lambda^{11}+10\lambda^9+36\lambda^7+56\lambda^5+35\lambda^3+6\lambda \\ \lambda^{11}+10\lambda^9+36\lambda^7+56\lambda^5+35\lambda^3+6\lambda & \lambda^{12}+11\lambda^{10}+45\lambda^8+84\lambda^6+70\lambda^4+21\lambda^2+1 \end{smallmatrix}\right)$
- $k = 13 \longrightarrow Q^{13} =$
$\left(\begin{smallmatrix} \lambda^{11}+10\lambda^9+36\lambda^7+56\lambda^5+35\lambda^3+6\lambda & \lambda^{12}+11\lambda^{10}+45\lambda^8+84\lambda^6+70\lambda^4+21\lambda^2+1 \\ \lambda^{12}+11\lambda^{10}+45\lambda^8+84\lambda^6+70\lambda^4+21\lambda^2+1 & \lambda^{13}+12\lambda^{11}+55\lambda^9+120\lambda^7+126\lambda^5+56\lambda^3+7\lambda \end{smallmatrix}\right)$
- $k = 14 \longrightarrow Q^{14} =$
$\left(\begin{smallmatrix} \lambda^{12}+11\lambda^{10}+45\lambda^8+84\lambda^6+70\lambda^4+21\lambda^2+1 & \lambda^{13}+12\lambda^{11}+55\lambda^9+120\lambda^7+126\lambda^5+56\lambda^3+7\lambda \\ \lambda^{13}+12\lambda^{11}+55\lambda^9+120\lambda^7+126\lambda^5+56\lambda^3+7\lambda & \lambda^{14}+13\lambda^{12}+66\lambda^{10}+165\lambda^8+210\lambda^6+126\lambda^4+28\lambda^2+1 \end{smallmatrix}\right)$
- $k = 15 \longrightarrow Q^{15} =$
$\left(\begin{smallmatrix} \lambda^{13}+12\lambda^{11}+55\lambda^9+120\lambda^7+126\lambda^5+56\lambda^3+7\lambda & \lambda^{14}+13\lambda^{12}+66\lambda^{10}+165\lambda^8+210\lambda^6+126\lambda^4+28\lambda^2+1 \\ \lambda^{14}+13\lambda^{12}+66\lambda^{10}+165\lambda^8+210\lambda^6+126\lambda^4+28\lambda^2+1 & \lambda^{15}+14\lambda^{13}+78\lambda^{11}+220\lambda^9+330\lambda^7+252\lambda^5+84\lambda^3+8\lambda \end{smallmatrix}\right)$
- $k = 16 \longrightarrow Q^{16} =$
$\left(\begin{smallmatrix} \lambda^{14}+13\lambda^{12}+66\lambda^{10}+165\lambda^8+210\lambda^6+126\lambda^4+28\lambda^2+1 & \lambda^{15}+14\lambda^{13}+78\lambda^{11}+220\lambda^9+330\lambda^7+252\lambda^5+84\lambda^3+8\lambda \\ \lambda^{15}+14\lambda^{13}+78\lambda^{11}+220\lambda^9+330\lambda^7+252\lambda^5+84\lambda^3+8\lambda & \lambda^{16}+15\lambda^{14}+91\lambda^{12}+286\lambda^{10}+495\lambda^8+462\lambda^6+210\lambda^4+36\lambda^2+1 \end{smallmatrix}\right)$

Calculating Q^{14} for $q = 3,\ 4,\ 5$ and 6, thus the 14^{th} vertex of the principal Petrie path for $q = 3, 4, 5, 6$ is given by the following table:

Q^{14} for $q = 3,\ 4,\ 5$ *and* 6				
λ_q	1	$\sqrt{2}$	$\frac{1+\sqrt{5}}{2}$	$\sqrt{3}$
$k = 14$	$\begin{pmatrix} 233 & 377 \\ 377 & 610 \end{pmatrix}$	$\begin{pmatrix} 2131 & 2911\sqrt{2} \\ 2911\sqrt{2} & 7953 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(5857+2597\sqrt{5}) & 6093+2740\sqrt{5} \\ 6093+2740\sqrt{5} & 12825+5715\sqrt{5} \end{pmatrix}$	$\begin{pmatrix} 10009 & 12649\sqrt{3} \\ 12649\sqrt{3} & 47956 \end{pmatrix}$
vertices	$\in \mathbb{Q} \cup \{\infty\}$	$\in \mathbb{Q}(\sqrt{2}) \cup \{\infty\}$	$\in \mathbb{Q}(\lambda_5) \cup \{\infty\}$	$\in \mathbb{Q}(\sqrt{3}) \cup \{\infty\}$

As we consider the principal Petrie path P_0^q in $\hat{\mathscr{M}}_q$ in which the first two vertices are $\frac{1}{0\lambda_q}$ and $\frac{0\lambda_q}{1}$. By Lemma 5.4 the k^{th} vertex of the principal Petrie path is equal to $\frac{p_k(\lambda_q)}{q_k(\lambda_q)}$, where

$$p_k(\lambda_q) = q_{k-1}(\lambda_q) \text{ and } q_k(\lambda_q) = p_{k-1}(\lambda_q) + \lambda_q q_{k-1}(\lambda_q). \tag{5.2.2}$$

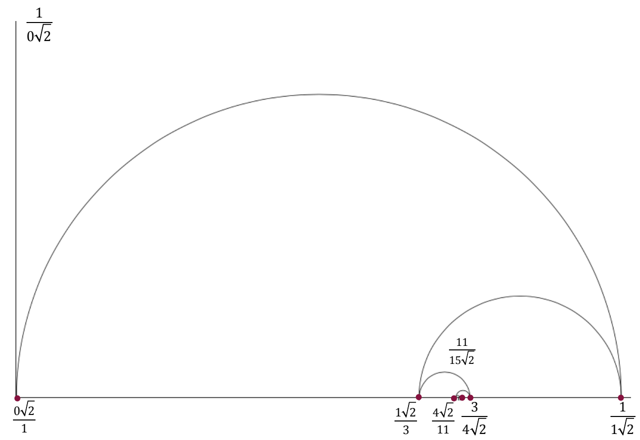
Also $p_k(\lambda_q)$ is the k^{th} element of the *Hecke-Fibonacci* sequence [ISg13] defined by

$$p_0(\lambda_q) = 1,\ \ p_1(\lambda_q) = 0\lambda_q,\ \ p_k(\lambda_q) = p_{k-2}(\lambda_q) + \lambda_q p_{k-1}(\lambda_q)\ \ for\ \ k \geq 1. \tag{5.2.3}$$

For $q = 3$ putting $\lambda_q = 1$ in (5.2.3), we get the Fibonacci Sequence and the vertices of the principal Petrie path for the Farey map as defined in Section 5.1.

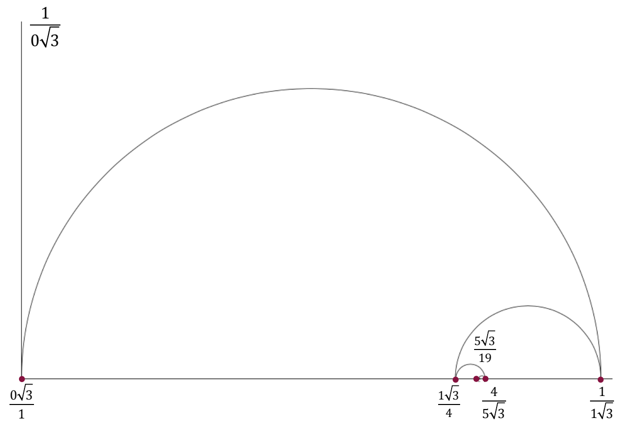
Example 5.6. Applying Example 5.5 for $\lambda_4 = \sqrt{2}$ and $k = 0, ..., 6$, we get the first seven vertices of P_0^4 of $\hat{\mathscr{M}}_4$ as follows;

$k = 0$	$Q^0 = \begin{pmatrix} 1 & 0\sqrt{2} \\ 0\sqrt{2} & 1 \end{pmatrix}$	$k = 1$	$Q = \begin{pmatrix} 0\sqrt{2} & 1 \\ 1 & 1\sqrt{2} \end{pmatrix}$	$k = 2$	$Q^2 = \begin{pmatrix} 1 & 1\sqrt{2} \\ 1\sqrt{2} & 3 \end{pmatrix}$
$k = 3$	$Q^3 = \begin{pmatrix} 1\sqrt{2} & 3 \\ 3 & 4\sqrt{2} \end{pmatrix}$	$k = 4$	$Q^4 = \begin{pmatrix} 3 & 4\sqrt{2} \\ 4\sqrt{2} & 11 \end{pmatrix}$	$k = 5$	$Q^5 = \begin{pmatrix} 4\sqrt{2} & 11 \\ 11 & 15\sqrt{2} \end{pmatrix}$
$k = 6$	$Q^6 = \begin{pmatrix} 11 & 15\sqrt{2} \\ 15\sqrt{2} & 41 \end{pmatrix}$				



These are $\frac{1}{0\sqrt{2}}$, $\frac{0\sqrt{2}}{1}$, $\frac{1}{1\sqrt{2}}$, $\frac{1\sqrt{2}}{3}$, $\frac{3}{4\sqrt{2}}$, $\frac{4\sqrt{2}}{11}$, $\frac{11}{15\sqrt{2}}$.
For $\lambda_6 = \sqrt{3}$, calculating the first seven powers of Q to get the first seven vertices of P_0^6 of $\hat{\mathscr{M}}_6$ as follows;

$k=0$	$Q^0 = \begin{pmatrix} 1 & 0\sqrt{3} \\ 0\sqrt{3} & 1 \end{pmatrix}$	$k=1$	$Q = \begin{pmatrix} 0\sqrt{3} & 1 \\ 1 & 1\sqrt{3} \end{pmatrix}$	$k=2$	$Q^2 = \begin{pmatrix} 1 & 1\sqrt{3} \\ 1\sqrt{3} & 4 \end{pmatrix}$
$k=3$	$Q^3 = \begin{pmatrix} 1\sqrt{3} & 4 \\ 4 & 5\sqrt{3} \end{pmatrix}$	$k=4$	$Q^4 = \begin{pmatrix} 4 & 5\sqrt{3} \\ 5\sqrt{3} & 19 \end{pmatrix}$	$k=5$	$Q^5 = \begin{pmatrix} 5\sqrt{3} & 19 \\ 19 & 24\sqrt{3} \end{pmatrix}$
$k=6$	$Q^6 = \begin{pmatrix} 19 & 24\sqrt{3} \\ 24\sqrt{3} & 91 \end{pmatrix}$				



These are $\frac{1}{0\sqrt{3}}$, $\frac{0\sqrt{3}}{1}$, $\frac{1}{1\sqrt{3}}$, $\frac{1\sqrt{3}}{4}$, $\frac{4}{5\sqrt{3}}$, $\frac{5\sqrt{3}}{19}$, $\frac{19}{24\sqrt{3}}$.

Now for $\lambda_5 = \frac{1+\sqrt{5}}{2}$, we have to use $\lambda_5^2 = \lambda_5 + 1$ as mentioned in Table 2.1;

$k=0$	$Q^0 = \begin{pmatrix} 1 & 0\lambda_5 \\ 0\lambda_5 & 1 \end{pmatrix}$	$k=1$	$Q = \begin{pmatrix} 0\lambda_5 & 1 \\ 1 & 1\lambda_5 \end{pmatrix}$	$k=2$	$Q^2 = \begin{pmatrix} 1 & 1\lambda_5 \\ 1\lambda_5 & 1\lambda_5+2 \end{pmatrix}$
$k=3$	$Q^3 = \begin{pmatrix} 1\lambda_5 & 1\lambda_5+2 \\ 1\lambda_5+2 & 4\lambda_5+1 \end{pmatrix}$	$k=4$	$Q^4 = \begin{pmatrix} \lambda_5+2 & 4\lambda_5+1 \\ 4\lambda_5+1 & 6\lambda_5+6 \end{pmatrix}$	$k=5$	$Q^5 = \begin{pmatrix} 4\lambda_5+1 & 6\lambda_5+6 \\ 6\lambda_5+6 & 16\lambda_5+7 \end{pmatrix}$
$k=6$	$Q^6 = \begin{pmatrix} 6\lambda_5+6 & 16\lambda_5+7 \\ 16\lambda_5+7 & 29\lambda_5+22 \end{pmatrix}$				

Therefore the first seven vertices of P_0^5 of $\hat{\mathscr{M}}_5$ are
$\frac{1}{0\lambda_5}$, $\frac{0\lambda_5}{1}$, $\frac{1}{1\lambda_5}$, $\frac{1\lambda_5}{\lambda_5+2}$, $\frac{\lambda_5+2}{4\lambda_5+1}$, $\frac{4\lambda_5+1}{6\lambda_5+6}$, $\frac{6\lambda_5+6}{16\lambda_5+7}$.

5.2.1 The even and odd vertices of the principal Petrie paths

As the Hecke groups H_4, H_6 have even and odd elements, we can define even and odd vertices of the principal Petrie paths of $\hat{\mathscr{M}}_4$ and $\hat{\mathscr{M}}_6$, the even vertices have the form $\frac{a}{c\sqrt{m}}$ while the odd vertices have the form $\frac{b\sqrt{m}}{d}$. Lemma 5.4 shows that the k^{th}

vertex of the principal Petrie path which is the first column of Q^k is equal to $\frac{p_k(\lambda_q)}{q_k(\lambda_q)}$. By (5.2.2) $\frac{p_k(\lambda_q)}{q_k(\lambda_q)} = \frac{p_k(\lambda_q)}{p_{k+1}(\lambda_q)}$, where $p_k(\lambda_q)$ is the k^{th} element of the Hecke-Fibonacci sequence defined in (5.2.3) and given by

$$p_0(\lambda_q) = 1,\ \ p_1(\lambda_q) = 0\lambda_q,\ \ p_k(\lambda_q) = p_{k-2}(\lambda_q) + \lambda_q p_{k-1}(\lambda_q)\ \ for\ \ k \geq 1.$$

Start our principal Petrie path with $\frac{1}{0\lambda_q}$ and call this vertex v_0. The next vertex v_1 is $\frac{0\lambda_q}{1}$. Then we can now generalize the idea of even and odd vertices of these principal Petrie paths for all values of q of $\hat{\mathscr{M}}_q$, as stated in the following definition.

Definition 5.7. The vertices $v_0, v_2, v_4, ...$ are defined to be the even-numbered vertices of the principal Petrie path of $\hat{\mathscr{M}}_q$, and $v_1, v_3, v_5, ...$ are defined to be the odd-numbered vertices.

As in our work we consider only the principal Petrie paths starting with even-numbered vertex $v_0 = \frac{1}{0\lambda_q}$, hence in the case of $\hat{\mathscr{M}}_4$ and $\hat{\mathscr{M}}_6$ Definition 5.7 agrees with the definition of even and odd vertices when $q = 4$ or 6: the even/odd-numbered vertices of P_0^q are even/odd vertices respectively. Now we introduce the following formulas to find these vertices. Letting $M = Q^2 = \left(\begin{smallmatrix} 1 & \lambda_q \\ \lambda_q & \lambda_q^2+1 \end{smallmatrix}\right)$. We use the minimal polynomials of λ_q, as listed in Table 2.1 to compute the powers of λ_q.

Lemma 5.8. *The matrix* $M = Q^2 = \left(\begin{smallmatrix} 1 & \lambda_q \\ \lambda_q & \lambda_q^2+1 \end{smallmatrix}\right)$ *maps each even-numbered and odd-numbered vertex of* P_0^q *of* $\hat{\mathscr{M}}_q$ *to the next even-numbered and odd-numbered vertex respectively and also* M^k *has the form* $M^k = \left(\begin{smallmatrix} a_k(\lambda_q) & b_k(\lambda_q) \\ b_k(\lambda_q) & a_k(\lambda_q)+\lambda_q b_k(\lambda_q) \end{smallmatrix}\right)$, *where* $a_k,\ b_k \in \mathbb{Z}[\lambda_q]$. *Here the first columns* $\frac{a_k(\lambda_q)}{b_k(\lambda_q)}$ *of the powers of* M *are the even-numbered vertices of* P_0^q *of* $\hat{\mathscr{M}}_q$ *while the second columns* $\frac{b_k(\lambda_q)}{a_k(\lambda_q)+\lambda_q b_k(\lambda_q)}$ *are the odd-numbered vertices.*

Proof. This follows from Lemma 5.4. The formula for M^k follows by induction; $M^1 = \left(\begin{smallmatrix} 1 & \lambda_q \\ \lambda_q & \lambda_q^2+1 \end{smallmatrix}\right)$ has the required form, where $a_1(\lambda_q) = 1,\ b_1(\lambda_q) = 1\lambda_q$ and $a_2(\lambda_q) = \lambda_q^2+1$. Assume the formula for M^k is true, so we have $a_{k+1}(\lambda_q) = a_k(\lambda_q) + \lambda_q b_k(\lambda_q)$. Then

$$M^{k+1} = \begin{pmatrix} a_k(\lambda_q) + \lambda_q b_k(\lambda_q) & \lambda_q a_k(\lambda_q) + (\lambda_q^2+1)b_k(\lambda_q) \\ \lambda_q a_k(\lambda_q) + (\lambda_q^2+1)b_k(\lambda_q) & (\lambda_q^2+1)a_k(\lambda_q) + (\lambda_q^3+2\lambda_q)b_k(\lambda_q) \end{pmatrix},$$

which has the required form. Thus by induction M^k has the required form for all $k \geq 1$. Also $a_{k+2}(\lambda_q) = (\lambda_q^2+1)a_k(\lambda_q) + (\lambda_q^3+2\lambda_q)b_k(\lambda_q)$.

□

Theorem 5.9. *If* $a_k(\lambda_q)$ *is the numerator of the* k^{th} *even-numbered vertex of* P_0^q *of* $\hat{\mathscr{M}}_q$ *starting from* $a_0(\lambda_q)$, *then it satisfies the following recurrence relation*

$$a_0(\lambda_q) = 1,\ a_1(\lambda_q) = 1,\ a_k(\lambda_q) = (\lambda_q^2+2)a_{k-1}(\lambda_q) - a_{k-2}(\lambda_q)\ \ for\ \ k \geq 1. \tag{5.2.4}$$

Proof. As $p_k(\lambda_q) = p_{k-2}(\lambda_q) + \lambda_q p_{k-1}(\lambda_q)$ by (5.2.3) and $a_k(\lambda_q) = p_{2k}(\lambda_q)$ by (5.2.9). Considering (5.2.3) for the indices $2k$, $2k-1$ and $2k-2$ we get the following equations

$$p_{2k}(\lambda_q) = p_{2k-2}(\lambda_q) + \lambda_q p_{2k-1}(\lambda_q), \tag{5.2.5}$$

$$p_{2k-1}(\lambda_q) = p_{2k-3}(\lambda_q) + \lambda_q p_{2k-2}(\lambda_q), \tag{5.2.6}$$

and

$$p_{2k-2}(\lambda_q) = p_{2k-4}(\lambda_q) + \lambda_q p_{2k-3}(\lambda_q). \tag{5.2.7}$$

Now the RHS of (5.2.3) is

$$\begin{aligned}
(\lambda_q^2+2)a_{k-1}(\lambda_q) - a_{k-2}(\lambda_q) &= (\lambda_q^2+2)p_{2k-2}(\lambda_q) - p_{2k-4}(\lambda_q) \\
&= (\lambda_q^2+2)p_{2k-2}(\lambda_q) - [p_{2k-2}(\lambda_q) - (\lambda_q)p_{2k-3}(\lambda_q)] \text{ by } (5.2.7) \\
&= (\lambda_q^2+2)p_{2k-2}(\lambda_q) - [p_{2k-2}(\lambda_q) - \lambda_q(p_{2k-1}(\lambda_q) - \lambda_q p_{2k-2}(\lambda_q))] \text{ by } (5.2.6) \\
&= p_{2k-2}(\lambda_q) + \lambda_q p_{2k-1}(\lambda_q) = p_{2k}(\lambda_q) \text{ by } (5.2.7) \\
&= a_k(\lambda_q) = \text{the LHS.}
\end{aligned}$$

□

Similarly we can prove the following theorem.

Theorem 5.10. *If $c_k(\lambda_q)$ is the numerator of the k^{th} odd-numbered vertex of P_0^q of $\hat{\mathscr{M}}_q$ starting from $c_0(\lambda_q)$, then it satisfies the following recurrence relation*

$$c_0(\lambda_q) = 0\lambda_q,\ c_1(\lambda_q) = 1\lambda_q,\ c_k(\lambda_q) = (\lambda_q^2+2)c_{k-1}(\lambda_q) - c_{k-2}(\lambda_q) \ \ for\ \ k \geq 1. \tag{5.2.8}$$

This means that these terms of (5.2.4) relate to the matrix Q^k as follows,

$$\begin{pmatrix} a_k(\lambda_q) \\ b_k(\lambda_q) \end{pmatrix} = \begin{pmatrix} p_{2k}(\lambda_q) \\ q_{2k}(\lambda_q) \end{pmatrix} = \begin{pmatrix} p_{2k}(\lambda_q) \\ p_{2k+1}(\lambda_q) \end{pmatrix}, \tag{5.2.9}$$

while these of (5.2.8) relate to the matrix Q^k as follows,

$$\begin{pmatrix} c_k(\lambda_q) \\ d_k(\lambda_q) \end{pmatrix} = \begin{pmatrix} p_{2k+1}(\lambda_q) \\ q_{2k+1}(\lambda_q) \end{pmatrix} = \begin{pmatrix} p_{2k+1}(\lambda_q) \\ p_{2k+2}(\lambda_q) \end{pmatrix} = \begin{pmatrix} b_k(\lambda_q) \\ a_{k+1}(\lambda_q) \end{pmatrix}.$$

The Hecke groups H_4, H_6 are much simpler than the other Hecke groups as we know all the elements of these groups. For this reason the maps $\hat{\mathscr{M}}_4$ and $\hat{\mathscr{M}}_6$ are easy to describe as we know precisely what their vertices, faces and edges are. So we now consider Theorems 5.9 and 5.10 for those simple cases.

For $q = 4$, the sequence in (5.2.4) happens to appear on the Online Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are
1, 1, 3, 11, 41, 153, 571, 2131, 7953, 29681. Those are the first few numerators of the even vertices of P_0^4 of $\hat{\mathscr{M}}_4$.
For $q = 6$, the first few terms of this sequence are
1, 1, 4, 19, 91, 436, 2089, 10009, 47956, 229771. Again those are the first few numerators of the even vertices of P_0^6 of $\hat{\mathscr{M}}_6$.

This means that these terms of (5.2.4) relate to the matrix Q^k as follows,

$$\begin{pmatrix} a_k(\sqrt{m}) \\ b_k(\sqrt{m}) \end{pmatrix} = \begin{pmatrix} p_{2k}(\sqrt{m}) \\ q_{2k}(\sqrt{m}) \end{pmatrix} = \begin{pmatrix} p_{2k}(\sqrt{m})) \\ p_{2k+1}(\sqrt{m}) \end{pmatrix}. \tag{5.2.10}$$

Example 5.11. As illustrated in Example 5.5, for $k = 3$ and $q = 4$,

$$\begin{pmatrix} a_3(\sqrt{2}) \\ b_3(\sqrt{2}) \end{pmatrix} = \begin{pmatrix} p_6(\sqrt{2}) \\ q_6(\sqrt{2}) \end{pmatrix} = \begin{pmatrix} 11 \\ 15\sqrt{2} \end{pmatrix}$$

where a_3 is the numerator of the fourth even vertex of P_0^4 of $\hat{\mathscr{M}}_4$, taking consideration that the first even vertex of P_0^4 of $\hat{\mathscr{M}}_4$ is $\left(\begin{smallmatrix} a_0 \\ b_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p_0(\sqrt{2}) \\ q_0(\sqrt{2}) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0\sqrt{2} \end{smallmatrix}\right)$ as shown in Exmaple 5.6.
Also for $k = 3$ and $q = 6$,

$$\begin{pmatrix} a_3(\sqrt{3}) \\ b_3(\sqrt{3}) \end{pmatrix} = \begin{pmatrix} p_6(\sqrt{3}) \\ q_6(\sqrt{3}) \end{pmatrix} = \begin{pmatrix} 19 \\ 24\sqrt{3} \end{pmatrix}$$

where a_3 is the numerator of the fourth even vertex of P_0^6 of $\hat{\mathscr{M}}_6$.

For $q = 4$, the first few terms of the sequence (5.2.8) are
$0\sqrt{2}$, $1\sqrt{2}$, $4\sqrt{2}$, $15\sqrt{2}$, $56\sqrt{2}$, $209\sqrt{2}$, $780\sqrt{2}$, $2911\sqrt{2}$. Those are the first few numerators of the odd vertices of P_0^4 of $\hat{\mathscr{M}}_4$.
For $q = 6$, the first few terms of this sequence are
$0\sqrt{3}$, $1\sqrt{3}$, $5\sqrt{3}$, $24\sqrt{3}$, $115\sqrt{3}$, $551\sqrt{3}$, $2640\sqrt{3}$, $12649\sqrt{3}$. Again those are the first few numerators of the odd vertices of P_0^6 of $\hat{\mathscr{M}}_6$.

This means that these terms of (5.2.8) relate to the matrix Q^k as follows,

$$\begin{pmatrix} c_k(\sqrt{m}) \\ d_k(\sqrt{m}) \end{pmatrix} = \begin{pmatrix} p_{2k+1}(\sqrt{m}) \\ q_{2k+1}(\sqrt{m}) \end{pmatrix} = \begin{pmatrix} p_{2k+1}(\sqrt{m})) \\ p_{2k+2}(\sqrt{m}) \end{pmatrix} = \begin{pmatrix} b_k(\sqrt{m})) \\ a_{k+1}(\sqrt{m}) \end{pmatrix}.$$

Example 5.12. As illustrated in Example 5.5, for $k = 3$ and $q = 4$,

$$\begin{pmatrix} c_3(\sqrt{2}) \\ d_3(\sqrt{2}) \end{pmatrix} = \begin{pmatrix} p_7(\sqrt{2}) \\ q_7(\sqrt{2}) \end{pmatrix} = \begin{pmatrix} 15\sqrt{2} \\ 41 \end{pmatrix}$$

where a_3 is the numerator of the fourth odd vertex of P_0^4 of $\hat{\mathscr{M}}_4$. Also for $k = 3$ and $q = 6$,

$$\begin{pmatrix} c_3(\sqrt{3}) \\ d_3(\sqrt{3}) \end{pmatrix} = \begin{pmatrix} p_7(\sqrt{3}) \\ q_7(\sqrt{3}) \end{pmatrix} = \begin{pmatrix} 24\sqrt{3} \\ 91 \end{pmatrix}$$

where a_3 is the numerator of the fourth odd vertex of P_0^6 of $\hat{\mathscr{M}}_6$.

What happens if $q = 5$ or 7?

For $q = 5$ we have the following recurrence relation

$$a_0(\lambda_5) = 1,\ a_1(\lambda_5) = 1,\ a_k(\lambda_5) = (\lambda_5 + 3)a_{k-1}(\lambda_5) - a_{k-2}(\lambda_5)\ \ for\ \ k \geq 1. \tag{5.2.11}$$

The first few terms of this sequence are

$1,\ 1,\ \lambda_5 + 2,\ 6\lambda_5 + 6,\ 29\lambda_5 + 22,\ 132\lambda_5 + 89.$

For $q = 7$, the recurrence relation defined as follow

$$a_0(\lambda_7) = 1,\ a_1(\lambda_7) = 1,\ a_k(\lambda_7) = (\lambda_7^2 + 2)a_{k-1}(\lambda_7) - a_{k-2}(\lambda_7)\ \ for\ \ k \geq 1. \tag{5.2.12}$$

The first few terms of this sequence are

$1,\ 1,\ \lambda_7^2 + 1,\ 6\lambda_7^2 + \lambda_7,\ 30\lambda_7^2 + 8\lambda_7 - 8,\ 144\lambda_7^2 + 61\lambda_7 - 38.$

Example 5.13. Using (5.2.4) for $q = 5$, and $k = 4$, we have

$$\begin{pmatrix} a_4(\lambda_5) \\ b_4(\lambda_5) \end{pmatrix} = \begin{pmatrix} p_8(\lambda_5) \\ q_8(\lambda_5) \end{pmatrix}.$$

$$\begin{aligned} M^4 = \begin{pmatrix} a_4(\lambda_5) & b_4(\lambda_5) \\ b_k(\lambda_5) & a_4(\lambda_5) + \lambda_5 b_k(\lambda_5) \end{pmatrix} &= Q^8 = \begin{pmatrix} p_8(\lambda_5) & q_8(\lambda_5) \\ q_8(\lambda_5) & p_8(\lambda_5) + \lambda_5 q_8(\lambda_5) \end{pmatrix} \\ &= \begin{pmatrix} 29\lambda_5 + 22 & 67\lambda_5 + 36 \\ 67\lambda_5 + 36 & 144\lambda_5 + 82 \end{pmatrix}, \end{aligned}$$

so $a_4(\lambda_5)$ is the numerator of the ninth vertex of the principal Petrie path of $\hat{\mathscr{M}}_5$ and it is also the numerator of the fifth even-numbered vertex of (5.2.11) as shown below:

$$\begin{array}{ccccccccccccccccccc}
p_0 & & p_1 & & p_2 & & p_3 & & p_4 & & p_5 & & p_6 & & p_7 & & p_8 & & p_9 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
a_0 & & c_0 & & a_1 & & c_1 & & a_2 & & c_2 & & a_3 & & c_3 & & a_4 & & c_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{0\lambda_5} & \to & \frac{0\lambda_5}{1} & \to & \frac{1}{1\lambda_5} & \to & \frac{1\lambda_5}{1\lambda_5+2} & \to & \frac{1\lambda_5+2}{4\lambda_5+1} & \to & \frac{4\lambda_5+1}{6\lambda_5+6} & \to & \frac{6\lambda_5+6}{16\lambda_5+7} & \to & \frac{16\lambda_5+7}{29\lambda_5+22} & \to & \frac{29\lambda_5+22}{67\lambda_5+36} & \to & \frac{67\lambda_5+36}{132\lambda_5+89}
\end{array}$$

5.3 The principal Petrie polygons of $\mathscr{M}_q(n)$

In general, in this section we study the semi-period $\hat{\sigma}_q(n)$ of the Hecke-Fibonacci sequence mod n because this will enable us to determine the sizes of the principal Petrie polygons $Pe_q(n)$ of $\mathscr{M}_q(n)$. We follow the ideas of [SS16]. The vertices of $\mathscr{M}_q(n)$ are equivalence classes of vertices of $\hat{\mathscr{M}}_q$. To clarify matters we denote the equivalence class of $\frac{1}{0\lambda_q}, \frac{0\lambda_q}{1}, ..., \frac{p_{k-1}(\lambda_q)}{p_k(\lambda_q)}, \frac{p_k(\lambda_q)}{p_{k+1}(\lambda_q)}$ by $[\frac{1}{0\lambda_q}], [\frac{0\lambda_q}{1}], ..., [\frac{p_{k-1}(\lambda_q)}{p_k(\lambda_q)}], [\frac{p_k(\lambda_q)}{p_{k+1}(\lambda_q)}]$ respectively in $\mathscr{M}_q(n)$. As these equivalence classes are consecutive vertices in $Pe_q(n)$ then they are joined by edges in $\mathscr{M}_q(n)$.

Then we consider some special cases for $q = 4, 5$ and 6.

Recalling (5.2.3), we introduce the following definition.

Definition 5.14. The *semi-period* $\hat{\sigma}_q(n)$ of the Hecke-Fibonacci sequence mod n defined by

$$p_0(\lambda_q) = 1, \;\; p_1(\lambda_q) = 0\lambda_q, \;\; p_k(\lambda_q) = p_{k-2}(\lambda_q) + \lambda_q p_{k-1}(\lambda_q) \;\; for \;\; k \geq 1$$

is the least positive integer i with the property that $p_i(\lambda_q) \equiv \pm 1 \bmod n$ and $p_{i+1}(\lambda_q) \equiv 0\lambda_q \bmod n$.

The term $p_{i-1}(\lambda_q)$, is the last term of the Hecke-Fibonacci sequence mod n (5.2.3) before we get the repetition, and $p_{i-1}(\lambda_q) \equiv \pm 1\lambda_q \bmod n$. Using Definition 5.14 and (5.2.3), we get $p_{i-1}(\lambda_q) = p_{i+1}(\lambda_q) - \lambda_q p_i(\lambda_q) \equiv 0\lambda_q - \pm 1\lambda_q \equiv \mp 1\lambda_q \bmod n$.

Definition 5.15. A principal Petrie polygon on $\mathscr{M}_q(n)$, has the points $[\frac{1}{0\lambda_q}], [\frac{0\lambda_q}{1}], ...,$ $[\frac{p_k(\lambda_q)}{p_{k+1}(\lambda_q)}]$ where $p_k(\lambda_q) \in \mathbb{Z}[\lambda_q]$ and it closes up when the two successive points $[\frac{1}{0\lambda_q}], [\frac{0\lambda_q}{1}]$ repeat. These points form the vertices of a principal Petrie polygon which we call $Pe_q(n)$. The vertices, which are the Farey fractions mod n, can be obtained using the Hecke-Fibonacci sequence mod n, by dividing each term of the sequence by its next term.

Lemma 5.16. *The semi-period $\hat{\sigma}_q(n)$ of the Hecke-Fibonacci sequence* mod n *is the order of the matrix*

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & \lambda_q \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_q]/(n)).$$

Proof. In Lemma 5.4 we defined the k^{th} power of Q to be

$$Q^k = \begin{pmatrix} p_k(\lambda_q) & p_{k+1}(\lambda_q) \\ p_{k+1}(\lambda_q) & p_{k+2}(\lambda_q) \end{pmatrix}.$$

Substituting k by $\hat{\sigma}_q(n)$ we get

$$Q^{\hat{\sigma}_q(n)} = \begin{pmatrix} p_{\hat{\sigma}_q(n)}(\lambda_q) & p_{\hat{\sigma}_q(n)+1}(\lambda_q) \\ p_{\hat{\sigma}_q(n)+1}(\lambda_q) & p_{\hat{\sigma}_q(n)+2}(\lambda_q) \end{pmatrix}.$$

Now $p_{\hat{\sigma}_q(n)+2}(\lambda_q) \equiv \pm 1 \bmod n$, $p_{\hat{\sigma}_q(n)+1}(\lambda_q) \equiv 0\lambda_q \bmod n$ and $p_{\hat{\sigma}_q(n)}(\lambda_q) \equiv \pm 1 \bmod n$, by Definition 5.14. Thus $Q^{\hat{\sigma}_q(n)} \equiv \pm I \bmod n$, so the order of Q divides $\hat{\sigma}_q(n)$. Therefore $\hat{\sigma}_q(n) = k$ since $\hat{\sigma}_q(n)$ is the least positive integer k with $Q^k \equiv \pm I \bmod n$. □

Theorem 5.17. *For all positive integers $n > 2$, $\hat{\sigma}_q(n)$ is even.*

Proof. Recall Lemma 5.16, we have

$$Q^{\hat{\sigma}_q(n)} = \begin{pmatrix} p_{\hat{\sigma}_q(n)}(\lambda_q) & p_{\hat{\sigma}_q(n)+1}(\lambda_q) \\ p_{\hat{\sigma}_q(n)+1}(\lambda_q) & p_{\hat{\sigma}_q(n)+2}(\lambda_q) \end{pmatrix} \equiv \pm I \bmod n.$$

From the proof of Lemma 5.16 $\det(Q^{\hat{\sigma}_q(n)}) \equiv 1 \bmod n$. Also

$$\det Q = \begin{pmatrix} 0 & 1 \\ 1 & \lambda_q \end{pmatrix} = 1.$$

As $\det(Q^{\hat{\sigma}_q(n)}) = \det(Q)^{\hat{\sigma}_q(n)}$ thus $(-1)^{\hat{\sigma}_q(n)} \equiv$
$1 \bmod n$, hence $\hat{\sigma}_q(n)$ is even. □

Theorem 5.18. *The Petrie length of $Pe_q(n)$ is equal to $\hat{\sigma}_q(n)$.*

Proof. By Lemma 5.16 the order of the matrix Q is $\hat{\sigma}_q(n)$. Also the length of the principal Petrie polygon is the order of the transformation $(R_1R_2R_3)$ that is represented by the matrix Q as mentioned in Section 5.2. Thus the Petrie length of $Pe_q(n)$ is equal to $\hat{\sigma}_q(n)$. □

In the finite group $H_q/H_q(n)$, we have a relation $(R_1R_2R_3)^{\hat{\sigma}_q(n)} = 1$. In some cases the group with presentation

$$< R_1, R_2, R_3 \,|\, R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^2 = (R_2R_3)^m = (R_3R_1)^n = (R_1R_2R_3)^{\hat{\sigma}_q(n)} = 1 > \tag{5.3.1}$$

is enough to define the group, (e.g. $n = 7, q = 3$), but usually this is not the case, see [CM57, Section 8.6].

5.3.1 Some special cases when $q = 4, 5, 6$

The vertices of $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$ are equivalence classes of vertices of $\hat{\mathscr{M}}_4$ and $\hat{\mathscr{M}}_6$. To clarify matters we denote the equivalence class of $\frac{a}{c\sqrt{m}}$ and $\frac{b\sqrt{m}}{d}$ by $[\frac{a}{c\sqrt{m}}]$ and $[\frac{b\sqrt{m}}{d}]$ respectively in $\mathscr{M}_q(n)$. These equivalence classes $[\frac{a}{c\sqrt{m}}]$ and $[\frac{b\sqrt{m}}{d}]$ are joined by an edge in $\mathscr{M}_q(n)$ if and only if $ad - mbc \equiv \pm 1$ or $mad - bc \equiv \pm 1 \bmod n$.

As we will consider the principal Petrie polygon modulo n and as $\frac{a}{c\sqrt{m}} = \frac{-a}{-c\sqrt{m}}$, we list sequences (5.2.4) modulo n for $n = 1, 2, ..., 10$ for the cases $q = 4, 6$.

Table 5.1: The sequence (5.2.4) modulo $n = 2, 3, ..., 10$ for $q = 4$

n	(5.2.4) modulo n
2	$1, 1, 1, 1$
3	$1, 1, 0, 2, 2, 0, 1, 1, 0$
4	$1, 1, 3, 3, 1, 1, 3, 3$
5	$1, 1, 3, 1, 1, 3$
6	$1, 1, 3, 5, 5, 3, 1, 1, 3$
7	$1, 1, 3, 4, 6, 6, 4, 3, 1, 1, 3$
8	$1, 1, 3, 3, 1, 1, 3$
9	$1, 1, 3, 2, 5, 0, 4, 7, 6, 8, 8, 6, 7, 4, 0, 5, 2, 3, 1, 1$
10	$1, 1, 3, 1, 1, 3$

Table 5.2: The sequence (5.2.4) modulo $n = 2, 3, ..., 10$ for $q = 6$

n	(5.2.4) modulo n
2	$1, 1, 0, 1, 1, 0$
3	$1, 1, 1, 1$
4	$1, 1, 0, 3, 3, 0, 1, 1, 0$
5	$1, 1, 4, 4, 1, 1, 4, 4$
6	$1, 1, 4, 1, 1, 4$
7	$1, 1, 4, 5, 0, 2, 3, 6, 6, 3, 2, 0, 5, 4, 1, 1, 4, 5, 0$
8	$1, 1, 4, 3, 3, , 4, 1, 1, 4$
9	$1, 1, 4, 1, 1, 4$
10	$1, 1, 4, 9, 1, 6, 9, 9, 6, 1, 9, 4, 1, 1, 4$

Recalling Theorem 5.9, we introduce the following definition.

Definition 5.19. The *period* $\pi_q^*(n)$ of the sequence (5.2.4) mod n is the least positive integer i, such that $a_i(\sqrt{m}) \equiv a_{i+1}(\sqrt{m}) \equiv 1 \bmod n$. We call the the least positive integer i with the property that $a_i(\sqrt{m}) \equiv a_{i+1}(\sqrt{m}) \equiv \pm 1 \bmod n$ the *semi-period* $\sigma_q^*(n)$ of the sequence (5.2.4) mod n.

Recalling Definition 5.14, and applying it for $q = 4, 6$, the semi-period $\hat{\sigma}_q(n)$ of the Hecke-Fibonacci sequence mod n defined by

$$p_0(\sqrt{m}) = 1, \quad p_1(\sqrt{m}) = 0\sqrt{m}, \quad p_k(\sqrt{m}) = p_{k-2}(\sqrt{m}) + \sqrt{m}p_{k-1}(\sqrt{m}) \quad for \quad k \geq 1$$

is the least positive integer i with the property that $p_i(\sqrt{m}) \equiv \pm 1 \bmod n$ and $p_{i+1}(\sqrt{m}) \equiv 0\sqrt{m}$ mod n.

Example 5.20. The principal Petrie polygon $Pe_4(7)$ has vertices

$$[\frac{1}{0\sqrt{2}}],[\frac{0\sqrt{2}}{1}],[\frac{1}{1\sqrt{2}}],[\frac{1\sqrt{2}}{3}],[\frac{3}{4\sqrt{2}}],[\frac{4\sqrt{2}}{4}],[\frac{4}{1\sqrt{2}}],[\frac{1\sqrt{2}}{6}].$$

The next two vertices are $[\frac{1}{0\sqrt{2}}],[\frac{0\sqrt{2}}{1}]$ so we have closed up our polygon, which has eight vertices. These vertices are obtained by dividing each term of the Hecke-Fibonacci sequence mod n in Table 5.5 by its next term for $n = 7$.

TABLE 5.3: The period and semi-period of the sequence (5.2.4) modulo low values of n for $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$

n	$\mathscr{M}_4(n)$		$\mathscr{M}_6(n)$	
	$\pi_q^*(n)$	$\sigma_q^*(n)$	$\pi_q^*(n)$	$\sigma_q^*(n)$
2	2	2	3	3
3	3	3	3	3
4	4	2	6	3
5	3	3	4	2
6	6	3	3	3
7	8	4	14	7
8	4	4	6	6
9	18	9	3	3
10	3	3	12	6
11	10	5	12	6
12	12	12	6	6
13	12	6	14	7
14	8	4	42	21
15	6	6	4	4
16	8	8	12	12
17	18	9	16	8
18	18	9	3	3
19	5	5	10	5
20	12	12	12	12
21	24	24	14	14
22	10	5	12	6

Table 5.4: The period and semi-period of the sequence (5.2.4) modulo low values of n for $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$

n	$\mathscr{M}_4(n)$		$\mathscr{M}_6(n)$	
	$\pi_q^*(n)$	$\sigma_q^*(n)$	$\pi_q^*(n)$	$\sigma_q^*(n)$
23	11	11	8	4
24	12	12	6	6
25	15	15	20	10
26	12	6	42	21
27	54	27	9	9
28	8	8	42	21
29	15	15	5	5
30	6	6	12	12

Table 5.5: The semi-period $\hat{\sigma}_4(n)$ of the Hecke-Fibonacci sequence mod n for $q = 4$ for low values of n

n	$\hat{\sigma}_4(n)$	$Pe_4(n)$	Hecke-Fibonacci sequence mod n (repeated block)
2	4	4	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 1, 0\sqrt{2}$
3	6	6	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 0, 1\sqrt{2}, 2, 0\sqrt{2}$
4	4	4	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 0\sqrt{2}$
5	6	6	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 1, 0\sqrt{2}$
6	12	12	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 5, 3\sqrt{2}, 5, 2\sqrt{2}, 3, 5\sqrt{2}, 1, 0\sqrt{2}$
7	8	8	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 4, 1\sqrt{2}, 6, 0\sqrt{2}$
8	8	8	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 3, 7\sqrt{2}, 1, 0\sqrt{2}$
9	18	18	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 2, 6\sqrt{2}, 5, 2\sqrt{2}, 0, 2\sqrt{2}, 4, 6\sqrt{2}, 7, 4\sqrt{2}, 6, 1\sqrt{2}, 8, 0\sqrt{2}$
10	12	12	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 1, 5\sqrt{2}, 1, 6\sqrt{2}, 3, 9\sqrt{2}, 1, 0\sqrt{2}$
11	10	10	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 0, 4\sqrt{2}, 8, 1\sqrt{2}, 10, 0\sqrt{2}$
12	24	24	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 3\sqrt{2}, 5, 8\sqrt{2}, 9, 5\sqrt{2}, 7, 0\sqrt{2}, 7, 7\sqrt{2}, 9,$ $4\sqrt{2}, 5, 9\sqrt{2}, 11, 8\sqrt{2}, 3, 11\sqrt{2}, 1, 0\sqrt{2}$
13	12	12	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 2\sqrt{2}, 2, 4\sqrt{2}, 10, 1\sqrt{2}, 12, 0\sqrt{2}$
14	8	8	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 1\sqrt{2}, 13, 0\sqrt{2}$
15	12	12	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 0\sqrt{2}, 11, 11\sqrt{2}, 3, 14\sqrt{2}, 1, 0\sqrt{2}$
16	16	16	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 9, 8\sqrt{2}, 9, 1\sqrt{2}, 11, 12\sqrt{2}, 3, 15\sqrt{2},$ $1, 0\sqrt{2}$
17	18	18	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 7, 5\sqrt{2}, 0, 5\sqrt{2}, 10, 15\sqrt{2}, 6, 4\sqrt{2},$ $14, 1\sqrt{2}, 16, 0\sqrt{2}$

TABLE 5.6: The semi-period $\hat{\sigma}_4(n)$ of the Hecke-Fibonacci sequence mod n for $q = 4$ for low values of n

n	$\hat{\sigma}_4(n)$	$Pe_4(n)$	Hecke-Fibonacci sequence mod n (repeated block)
18	36	36	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 5, 2\sqrt{2}, 9, 11\sqrt{2}, 13, 6\sqrt{2}, 7, 13\sqrt{2}, 15, 10\sqrt{2}, 17, 9\sqrt{2}, 17, 8\sqrt{2}, 15, 5\sqrt{2}, 7, 12\sqrt{2}, 13, 7\sqrt{2}, 9, 16\sqrt{2}, 5, 3\sqrt{2}, 11, 14\sqrt{2}, 3, 17\sqrt{2}, 1, 0\sqrt{2}$
19	10	10	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 3, 18\sqrt{2}, 1, 0\sqrt{2}$
20	24	24	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 1, 16\sqrt{2}, 13, 9\sqrt{2}, 11, 0\sqrt{2}, 11, 11\sqrt{2}, 13, 4\sqrt{2}, 1, 5\sqrt{2}, 11, 16\sqrt{2}, 3, 19\sqrt{2}, 1, 0\sqrt{2}$
21	48	48	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 20, 14\sqrt{2}, 6, 20\sqrt{2}, 4, 3\sqrt{2}, 10, 13\sqrt{2}, 15, 7\sqrt{2}, 8, 15\sqrt{2}, 17, 11\sqrt{2}, 18, 8\sqrt{2}, 13, 0\sqrt{2}, 13, 13\sqrt{2}, 18, 10\sqrt{2}, 17, 6\sqrt{2}, 8, 14\sqrt{2}, 15, 8\sqrt{2}, 10, 18\sqrt{2}, 4, 1\sqrt{2}, 6, 7\sqrt{2}, 20, 6\sqrt{2}, 11, 17\sqrt{2}, 3, 20\sqrt{2}, 1, 0\sqrt{2}$
22	20	20	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 19, 12\sqrt{2}, 21, 11\sqrt{2}, 21, 10\sqrt{2}, 19, 7\sqrt{2}, 11, 18\sqrt{2}, 3, 21\sqrt{2}, 1, 0\sqrt{2}$
23	22	22	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 18, 10\sqrt{2}, 15, 2\sqrt{2}, 19, 21\sqrt{2}, 15, 13\sqrt{2}, 18, 8\sqrt{2}, 11, 19\sqrt{2}, 3, 22\sqrt{2}, 1, 0\sqrt{2}$
24	24	24	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 17, 8\sqrt{2}, 9, 17\sqrt{2}, 19, 12\sqrt{2}, 19, 7\sqrt{2}, 9, 16\sqrt{2}, 17, 9\sqrt{2}, 11, 20\sqrt{2}, 3, 23\sqrt{2}, 1, 0\sqrt{2}$
25	30	30	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 16, 6\sqrt{2}, 3, 9\sqrt{2}, 21, 5\sqrt{2}, 6, 11\sqrt{2}, 3, 14\sqrt{2}, 6, 20\sqrt{2}, 21, 16\sqrt{2}, 3, 19\sqrt{2}, 16, 10\sqrt{2}, 11, 21\sqrt{2}, 3, 24\sqrt{2}, 1, 0\sqrt{2}$
26	12	12	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 15, 4\sqrt{2}, 23, 1\sqrt{2}, 25, 0\sqrt{2}$
27	54	54	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 14, 2\sqrt{2}, 18, 20\sqrt{2}, 4, 24\sqrt{2}, 25, 22\sqrt{2}, 15, 10\sqrt{2}, 8, 18\sqrt{2}, 17, 8\sqrt{2}, 6, 14\sqrt{2}, 7, 2\sqrt{2}, 22, 16\sqrt{2}, 0, 16\sqrt{2}, 5, 21\sqrt{2}, 20, 14\sqrt{2}, 21, 8\sqrt{2}, 10, 18\sqrt{2}, 19, 10\sqrt{2}, 12, 22\sqrt{2}, 2, 24\sqrt{2}, 23, 20\sqrt{2}, 9, 2\sqrt{2}, 13, 15\sqrt{2}, 16, 4\sqrt{2}, 24, 1\sqrt{2}, 26, 0\sqrt{2}$
28	16	16	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 13, 0\sqrt{2}, 13, 13\sqrt{2}, 11, 24\sqrt{2}, 3, 27\sqrt{2}, 1, 0\sqrt{2}$
29	30	30	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 12, 27\sqrt{2}, 8, 6\sqrt{2}, 20, 26\sqrt{2}, 14, 11\sqrt{2}, 7, 18\sqrt{2}, 14, 3\sqrt{2}, 20, 23\sqrt{2}, 8, 2\sqrt{2}, 12, 14\sqrt{2}, 11, 25\sqrt{2}, 3, 28\sqrt{2}, 1, 0\sqrt{2}$
30	12	12	$1, 0\sqrt{2}, 1, 1\sqrt{2}, 3, 4\sqrt{2}, 11, 15\sqrt{2}, 11, 26\sqrt{2}, 3, 29\sqrt{2}, 1, 0\sqrt{2}$

TABLE 5.7: The semi-period $\hat{\sigma}_6(n)$ of the Hecke-Fibonacci sequence mod n for $q = 6$ for low values of n

n	$\hat{\sigma}_6(n)$	$Pe_6(n)$	Hecke-Fibonacci sequence mod n (repeated block)
2	6	6	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 0, 1\sqrt{3}, 1, 0\sqrt{3}$
3	6	6	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 1, 2\sqrt{3}, 1, 0\sqrt{3}$
4	6	6	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 0, 1\sqrt{3}, 3, 0\sqrt{3}$
5	4	4	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 0\sqrt{3}$

TABLE 5.8: The semi-period $\hat{\sigma}_6(n)$ of the Hecke-Fibonacci sequence mod n for $q = 6$ for low values of n

n	$\hat{\sigma}_6(n)$	$Pe_6(n)$	Hecke-Fibonacci sequence mod n (repeated block)
6	6	6	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 1, 0\sqrt{3}$
7	14	14	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 5, 3\sqrt{3}, 0, 3\sqrt{3}, 2, 5\sqrt{3}, 3, 1\sqrt{3}, 6, 0\sqrt{3}$
8	12	12	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 3, 0\sqrt{3}, 3, 3\sqrt{3}, 4, 7\sqrt{3}, 1, 0\sqrt{3}$
9	18	18	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 1, 6\sqrt{3}, 1, 7\sqrt{3}, 4, 2\sqrt{3}, 1, 3\sqrt{3}, 1, 4\sqrt{3}, 4, 8\sqrt{3}, 1, 0\sqrt{3}$
10	12	12	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 9, 4\sqrt{3}, 1, 5\sqrt{3}, 6, 1\sqrt{3}, 9, 0\sqrt{3}$
11	12	12	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 8, 2\sqrt{3}, 3, 5\sqrt{3}, 7, 1\sqrt{3}, 10, 0\sqrt{3}$
12	12	12	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 7, 0\sqrt{3}, 7, 7\sqrt{3}, 4, 11\sqrt{3}, 1, 0\sqrt{3}$
13	14	14	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 6, 11\sqrt{3}, 0, 11\sqrt{3}, 7, 5\sqrt{3}, 9, 1\sqrt{3}, 12, 0\sqrt{3}$
14	42	42	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 5, 10\sqrt{3}, 7, 3\sqrt{3}, 2, 5\sqrt{3}, 3, 8\sqrt{3}, 13, 7\sqrt{3}, 6, 13\sqrt{3},$ $3, 2\sqrt{3}, 9, 11\sqrt{3}, 0, 11\sqrt{3}, 5, 16\sqrt{3}, 11, 13\sqrt{3}, 8, 7\sqrt{3}, 1, 8\sqrt{3}, 11, 5\sqrt{3}, 12,$ $3\sqrt{3}, 7, 10\sqrt{3}, 9, 5\sqrt{3}, 10, 1\sqrt{3}, 13, 0\sqrt{3}$
15	24	24	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 4, 9\sqrt{3}, 1, 10\sqrt{3}, 1, 11\sqrt{3}, 4, 0\sqrt{3}, 4, 4\sqrt{3}, 1, 5\sqrt{3},$ $1, 6\sqrt{3}, 4, 10\sqrt{3}, 4, 14\sqrt{3}, 1, 0\sqrt{3}$
16	24	24	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 3, 8\sqrt{3}, 11, 3\sqrt{3}, 4, 7\sqrt{3}, 9, 0\sqrt{3}, 9, 9\sqrt{3}, 4, 13\sqrt{3},$ $11, 8\sqrt{3}, 3, 11\sqrt{3}, 4, 15\sqrt{3}, 1, 0\sqrt{3}$
17	16	16	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 2, 7\sqrt{3}, 6, 13\sqrt{3}, 11, 7\sqrt{3}, 15, 5\sqrt{3}, 13, 1\sqrt{3}, 16, 0\sqrt{3}$
18	18	18	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 1, 6\sqrt{3}, 1, 7\sqrt{3}, 4, 11\sqrt{3}, 1, 12\sqrt{3}, 1, 13\sqrt{3}, 4, 17\sqrt{3}, 1, 0\sqrt{3}$
19	10	10	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 0, 5\sqrt{3}, 15, 1\sqrt{3}, 18, 0\sqrt{3}$
20	24	24	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 4\sqrt{3}, 11, 15\sqrt{3}, 16, 11\sqrt{3}, 9, 0\sqrt{3}, 9, 9\sqrt{3}, 16,$ $5\sqrt{3}, 11, 16\sqrt{3}, 19, 15\sqrt{3}, 4, 19\sqrt{3}, 1, 0\sqrt{3}$
21	84	84	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 3\sqrt{3}, 7, 10\sqrt{3}, 16, 5\sqrt{3}, 10, 15\sqrt{3}, 13, 7\sqrt{3}, 13,$ $20\sqrt{3}, 10, 9\sqrt{3}, 16, 4\sqrt{3}, 7, 11\sqrt{3}, 19, 9\sqrt{3}, 4, 13\sqrt{3}, 1, 14\sqrt{3}, 1, 15\sqrt{3}, 4,$ $19\sqrt{3}, 19, 17\sqrt{3}, 7, 3\sqrt{3}, 16, 19\sqrt{3}, 10, 8\sqrt{3}, 13, 0\sqrt{3}, 13, 13\sqrt{3}, 10, 2\sqrt{3},$ $16, 18\sqrt{3}, 7, 4\sqrt{3}, 19, 2\sqrt{3}, 4, 6\sqrt{3}, 1, 7\sqrt{3}, 1, 8\sqrt{3}, 4, 12\sqrt{3}, 19, 10\sqrt{3}, 7,$ $17\sqrt{3}, 16, 12\sqrt{3}, 10, 1\sqrt{3}, 13, 14\sqrt{3}, 13, 6\sqrt{3}, 10, 16\sqrt{3}, 16, 11\sqrt{3}, 7,$ $18\sqrt{3}, 19, 16\sqrt{3}, 4, 20\sqrt{3}, 1, 0\sqrt{3}$
22	12	12	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 2\sqrt{3}, 3, 5\sqrt{3}, 18, 1\sqrt{3}, 21, 0\sqrt{3}$
23	8	8	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 1\sqrt{3}, 22, 0\sqrt{3}$
24	12	12	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 0\sqrt{3}, 19, 19\sqrt{3}, 4, 23\sqrt{3}, 1, 0\sqrt{3}$
25	20	20	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 24\sqrt{3}, 16, 15\sqrt{3}, 11, 1\sqrt{3}, 14, 15\sqrt{3}, 9,$ $24\sqrt{3}, 6, 5\sqrt{3}, 21, 1\sqrt{3}, 24, 0\sqrt{3}$
26	42	42	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 24\sqrt{3}, 13, 11\sqrt{3}, 20, 5\sqrt{3}, 9, 14\sqrt{3}, 25,$ $13\sqrt{3}, 12, 25\sqrt{3}, 9, 8\sqrt{3}, 7, 15\sqrt{3}, 0, 15\sqrt{3}, 19, 8\sqrt{3}, 17, 25\sqrt{3}, 14,$ $13\sqrt{3}, 1, 14\sqrt{3}, 17, 5\sqrt{3}, 6, 11\sqrt{3}, 13, 24\sqrt{3}, 7, 5\sqrt{3}, 22, 1\sqrt{3}, 25, 0\sqrt{3}$

Table 5.9: The semi-period $\hat{\sigma}_6(n)$ of the Hecke-Fibonacci sequence mod n for $q = 6$ for low values of n

n	$\hat{\sigma}_6(n)$	$Pe_6(n)$	Hecke-Fibonacci sequence mod n (repeated block)
27	54	54	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 24\sqrt{3}, 10, 7\sqrt{3}, 4, 11\sqrt{3}, 10, 21\sqrt{3}, 19, 13\sqrt{3}, 4, 17\sqrt{3}, 1, 18\sqrt{3}, 1, 19\sqrt{3}, 4, 23\sqrt{3}, 19, 15\sqrt{3}, 10, 25\sqrt{3}, 4, 2\sqrt{3}, 10, 12\sqrt{3}, 19, 4\sqrt{3}, 4, 8\sqrt{3}, 1, 9\sqrt{3}, 1, 10\sqrt{3}, 4, 14\sqrt{3}, 19, 6\sqrt{3}, 10, 16\sqrt{3}, 4, 20\sqrt{3}, 10, 3\sqrt{3}, 19, 22\sqrt{3}, 4, 26\sqrt{3}, 1, 0\sqrt{3}$
28	42	42	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 24\sqrt{3}, 7, 3\sqrt{3}, 16, 19\sqrt{3}, 17, 8\sqrt{3}, 13, 21\sqrt{3}, 20, 13\sqrt{3}, 3, 16\sqrt{3}, 23, 11\sqrt{3}, 0, 11\sqrt{3}, 5, 16\sqrt{3}, 25, 13\sqrt{3}, 8, 21\sqrt{3}, 15, 8\sqrt{3}, 11, 19\sqrt{3}, 12, 3\sqrt{3}, 21, 24\sqrt{3}, 9, 5\sqrt{3}, 24, 1\sqrt{3}, 27, 0\sqrt{3}$
29	10	10	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 24\sqrt{3}, 4, 28\sqrt{3}, 1, 0\sqrt{3}$
30	24	24	$1, 0\sqrt{3}, 1, 1\sqrt{3}, 4, 5\sqrt{3}, 19, 24\sqrt{3}, 1, 25\sqrt{3}, 16, 11\sqrt{3}, 19, 0\sqrt{3}, 19, 19\sqrt{3}, 16, 5\sqrt{3}, 1, 6\sqrt{3}, 19, 25\sqrt{3}, 4, 29\sqrt{3}, 1, 0\sqrt{3}$

Example 5.21. The sequence (5.2.4) mod 5 for $\mathscr{M}_4(5)$ is $1, 1, 3, 1, 1, 3, 1, 1, 3, \ldots$ hence $\pi_4^*(5) = 3$ and $\sigma_4^*(5) = 3$. Therefore there are three even vertices of the principal Petrie polygon and $Pe_4(5)$ has vertices

$$[\frac{1}{0\sqrt{2}}], [\frac{0\sqrt{2}}{1}], [\frac{1}{1\sqrt{2}}], [\frac{1\sqrt{2}}{3}], [\frac{3}{4\sqrt{2}}], [\frac{4\sqrt{2}}{1}].$$

The next two vertices are $[\frac{1}{0\sqrt{2}}], [\frac{0\sqrt{2}}{1}]$ so we have closed up our polygon, which has six vertices. The length of $Pe_4(5)$ is equal to $2\sigma_4^*(5) = 6$.

Example 5.22. The sequence (5.2.4) mod 6 for $\mathscr{M}_4(6)$ is $1, 1, 3, 5, 5, 3, 1, 1, 3, 5, \ldots$ hence $\pi_4^*(6) = 6$ and $\sigma_4^*(6) = 3$. Therefore there are six even vertices of the principal Petrie polygon and $Pe_4(6)$ has vertices

$$[\frac{1}{0\sqrt{2}}], [\frac{0\sqrt{2}}{1}], [\frac{1}{1\sqrt{2}}], [\frac{1\sqrt{2}}{3}], [\frac{3}{4\sqrt{2}}], [\frac{4\sqrt{2}}{5}], [\frac{5}{3\sqrt{2}}], [\frac{3\sqrt{2}}{5}], [\frac{5}{2\sqrt{2}}], [\frac{2\sqrt{2}}{3}], [\frac{3}{5\sqrt{2}}], [\frac{5\sqrt{2}}{1}].$$

The next two vertices are $[\frac{1}{0\sqrt{2}}], [\frac{0\sqrt{2}}{1}]$ so we have closed up our polygon, which has twelve vertices. The length of $Pe_4(6)$ is equal to $4\sigma_4^*(6) = 12$.

Example 5.23. The sequence (5.2.4) mod 6 for $\mathscr{M}_6(6)$ is $1, 1, 4, 1, 1, 4, \ldots$ hence $\pi_6^*(6) = 3$ and $\sigma_6^*(6) = 3$. Therefore there are three even vertices of the principal Petrie polygon and $Pe_6(6)$ has vertices

$$[\frac{1}{0\sqrt{3}}], [\frac{0\sqrt{3}}{1}], [\frac{1}{1\sqrt{3}}], [\frac{1\sqrt{3}}{4}], [\frac{4}{5\sqrt{3}}], [\frac{5\sqrt{3}}{1}].$$

The next two vertices are $[\frac{1}{0\sqrt{3}}], [\frac{0\sqrt{3}}{1}]$ so we have closed up our polygon, which has six vertices. The length of $Pe_6(6)$ is equal to $2\sigma_6^*(3) = 6$.

Example 5.24. The sequence (5.2.4) mod 4 for $\mathscr{M}_6(4)$ is $1, 1, 0, 3, 3, 0, 1, 1, 0, \ldots$ hence $\pi_6^*(4) = 6$ and $\sigma_6^*(4) = 3$. Therefore there are three even vertices of the principal Petrie polygon and $Pe_6(4)$ has vertices

$$[\frac{1}{0\sqrt{3}}], [\frac{0\sqrt{3}}{1}], [\frac{1}{1\sqrt{3}}], [\frac{1\sqrt{3}}{0}], [\frac{0}{1\sqrt{3}}], [\frac{1\sqrt{3}}{3}].$$

The next two vertices are $[\frac{1}{0\sqrt{3}}], [\frac{0\sqrt{3}}{1}]$ so we have closed up our polygon, which has six vertices. The length of $Pe_6(4)$ is equal to $2\sigma_6^*(4) = 6$.

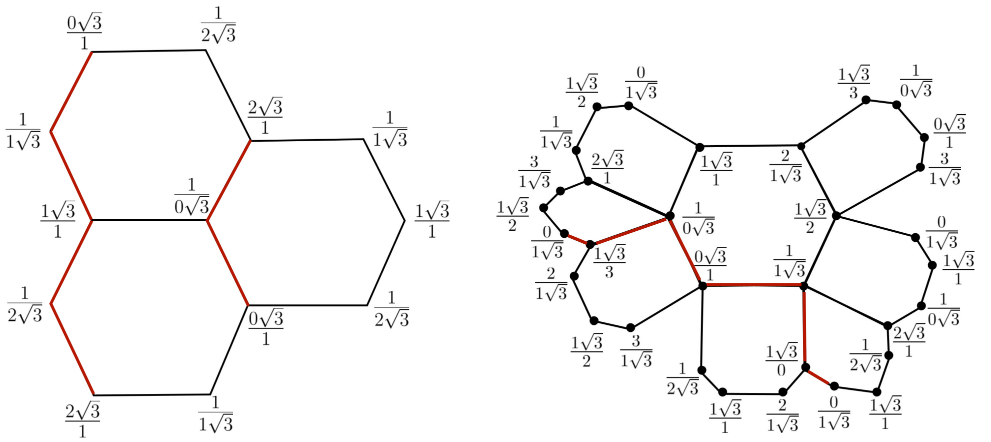


FIGURE 5.3: Left: $Pe_6(3)$. Right: $Pe_6(4)$

Lemma 5.25. *$\pi_q^*(n) = \sigma_q^*(n)$ if and only if $a_{\sigma_q^*(n)}(\sqrt{m}) \equiv 1 \bmod n$, and $\pi_q^*(n) = 2\sigma_q^*(n)$ if and only if $a_{\sigma_q^*(n)}(\sqrt{m}) \equiv -1 \bmod n$, for $n > 2$.*

Proof. Let k be the least positive integer such that $a_k(\sqrt{m}) \equiv a_{k+1}(\sqrt{m}) \equiv -1 \bmod n$. Then $k = \sigma_q^*(n)$ and $a_{2\sigma_q^*(n)+1}(\sqrt{m}) \equiv -a_{\sigma_q^*(n)+1}(\sqrt{m}) \bmod n$, so that $a_{2\sigma_q^*(n)+1}(\sqrt{m}) \equiv -a_{\sigma_q^*(n)+1}(\sqrt{m}) \equiv 1 \bmod n$, $a_{2\sigma_q^*(n)}(\sqrt{m}) \equiv 1 \bmod n$ and $\pi_q^*(n) = 2\sigma_q^*(n)$. Alternatively, $a_k(\sqrt{m}) \equiv 1 \bmod n$ and then $\pi_q^*(n) = \sigma_q^*(n)$. □

Tables 5.5 and 5.7 reflect Theorem 5.18 as we can observe that the lengths of $Pe_q(n)$'s column for $q = 4, 6$ matches the column of $\hat{\sigma}_q(n)$'s values.

Lemma 5.26. *If $a_{\sigma_q^*(n)} \equiv \pm 1 \bmod n$ and $b_{\sigma_q^*(n)}(\sqrt{m}) \equiv 0\sqrt{m} \bmod n$, then $\hat{\sigma}_q(n) = 2\sigma_q^*(n)$. Otherwise, $\hat{\sigma}_q(n) = 2m\sigma_q^*(n)$, where $m = 2, 3$ for $q = 4, 6$ respectively.*

Proof. If

$$\begin{pmatrix} a_{\sigma_q^*(n)} \\ b_{\sigma_q^*(n)}(\sqrt{m}) \end{pmatrix} \equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{m} \end{pmatrix} \mod n,$$

then by (5.2.10)

$$\begin{pmatrix} a_{\sigma_q^*(n)} \\ b_{\sigma_q^*(n)}(\sqrt{m}) \end{pmatrix} = \begin{pmatrix} p_{2\sigma_q^*(n)}(\sqrt{m})) \\ p_{2\sigma_q^*(n)+1}(\sqrt{m}) \end{pmatrix} \equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{m} \end{pmatrix} \mod n.$$

Thus $\hat{\sigma}_q(n) = 2\sigma_q^*(n)$ by Definition 5.14.
Now if

$$\begin{pmatrix} a_{\sigma_q^*(n)} \\ b_{\sigma_q^*(n)}(\sqrt{m}) \end{pmatrix} \not\equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{m} \end{pmatrix} \mod n,$$

then $\hat{\sigma}_q(n) \neq 2\sigma_q^*(n)$.

Therefore there is a least positive integer z such that

$$\begin{pmatrix} a_{z\sigma_q^*(n)} \\ b_{z\sigma_q^*(n)}(\sqrt{m}) \end{pmatrix} = \begin{pmatrix} p_{2z\sigma_q^*(n)}(\sqrt{m})) \\ p_{2z\sigma_q^*(n)+1}(\sqrt{m}) \end{pmatrix} \equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{m} \end{pmatrix} \mod n,$$

here $\hat{\sigma}_q(n) = 2z\sigma_q^*(n)$, implies $z = \hat{\sigma}_q(n)/2\sigma_q^*(n)$.

For $q = 4$, if $a_{\sigma_4^*(n)} \equiv \pm 1 \mod n$ then $a_{z\sigma_4^*(n)} \equiv \pm 1 \mod n$. As $b_{\sigma_4^*(n)}(\sqrt{2}) \not\equiv 0\sqrt{2} \mod n$, then $n \nmid b_{\sigma_4^*(n)}(\sqrt{2})$. Now suppose that n is even, thus $b_{\sigma_4^*(n)}(\sqrt{2})$ is odd such that $(n, b_{\sigma_4^*(n)}) = b_{\sigma_4^*(n)}$. We need the least positive z such that $b_{z\sigma_4^*(n)}(\sqrt{2}) \equiv 0(\sqrt{2}) \mod n$. Choose $z = 2$ such that $n | b_{2\sigma_4^*(n)}(\sqrt{2})$ implies $b_{2\sigma_4^*(n)}(\sqrt{2}) = 2b_{\sigma_4^*(n)} \equiv 0\sqrt{2} \mod n$. Therefore

$$\begin{pmatrix} a_{2\sigma_4^*(n)} \\ b_{2\sigma_4^*(n)}(\sqrt{2}) \end{pmatrix} = \begin{pmatrix} p_{4\sigma_4^*(n)}(\sqrt{2})) \\ p_{4\sigma_4^*(n)+1}(\sqrt{2}) \end{pmatrix} \equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{2} \end{pmatrix} \mod n,$$

$\hat{\sigma}_4(n) = 2(2\sigma_4^*(n)) = 4\sigma_4^*(n)$.

For $q = 6$, if $a_{\sigma_6^*(n)} \equiv \pm 1 \mod n$ and $b_{\sigma_6^*(n)}(\sqrt{3}) \not\equiv 0\sqrt{3} \mod n$, then $n \nmid b_{\sigma_6^*(n)}(\sqrt{3})$. Now suppose that $3|n$ and $n \nmid b_{\sigma_6^*(n)}(\sqrt{3})$. We need the least positive z such that $b_{z\sigma_6^*(n)}(\sqrt{3}) \equiv 0\sqrt{3} \mod n$. Since $n \nmid b_{\sigma_6^*(n)}(\sqrt{3})$, then $n \nmid 2b_{\sigma_6^*(n)}(\sqrt{3}) \equiv b_{2\sigma_6^*(n)} \equiv -b_{\sigma_6^*(n)}(\sqrt{3}) \mod n$, hence $z = 2$ is excluded. Choose $z = 3$ such that $n | b_{3\sigma_6^*(n)}(\sqrt{3})$, we have $b_{3\sigma_6^*(n)}(\sqrt{3}) = 3b_{\sigma_6^*(n)}(\sqrt{3}) \equiv 0(\sqrt{3}) \mod n$. Therefore

$$\begin{pmatrix} a_{3\sigma_6^*(n)} \\ b_{3\sigma_6^*(n)}(\sqrt{3}) \end{pmatrix} = \begin{pmatrix} p_{6\sigma_6^*(n)}(\sqrt{3})) \\ p_{6\sigma_6^*(n)+1}(\sqrt{3}) \end{pmatrix} \equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{3} \end{pmatrix} \mod n,$$

$\hat{\sigma}_6(n) = 2(3\sigma_6^*(n)) = 6\sigma_6^*(n)$. Thus the value of z is 2 for $q = 4$ and 3 for $q = 6$ which are the values of m.

□

Example 5.27. Recall Example 5.22 and Tables 5.1, 5.3 and 5.5.
As $\sigma_4^*(6) = 3$, then

$$\begin{pmatrix} a_{\sigma_4^*(6)} \\ b_{\sigma_4^*(6)}(\sqrt{2}) \end{pmatrix} \not\equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{2} \end{pmatrix} \mod 6,$$

but

$$\begin{pmatrix} a_{2\sigma_4^*(6)} \\ b_{2\sigma_4^*(6)}(\sqrt{2}) \end{pmatrix} \equiv \begin{pmatrix} \pm 1 \\ 0\sqrt{2} \end{pmatrix} \mod 6.$$

The length of the principal Petrie polygon $Pe_4(6) = \hat{\sigma}_4(6) = 4\sigma_4^*(6) = 12$. There are six even vertices of the principal Petrie polygon $Pe_4(6)$, these are

$$[\frac{1}{0\sqrt{2}}],[\frac{0\sqrt{2}}{1}],[\frac{1}{1\sqrt{2}}],[\frac{1\sqrt{2}}{3}],[\frac{3}{4\sqrt{2}}],[\frac{4\sqrt{2}}{5}],[\frac{5}{3\sqrt{2}}],[\frac{3\sqrt{2}}{5}],[\frac{5}{2\sqrt{2}}],[\frac{2\sqrt{2}}{3}],[\frac{3}{5\sqrt{2}}],[\frac{5\sqrt{2}}{1}].$$

There are many examples, for $q = 4$ when $n = 6, 10, 18, 22$ as shown in Table 5.5 where $\hat{\sigma}_4(n) = 4\sigma_4^*(n)$, also for $q = 6$ where $n = 9, 15, 18, 21, 27$ as shown in Table 5.7 where $\hat{\sigma}_6(n) = 6\sigma_6^*(n)$.

Extending my calculations to include $q = 5$, the following table shows some facts about Theorem 5.18 for values of n up to 15.

TABLE 5.10: The semi-period $\hat{\sigma}_5(n)$ of the Hecke-Fibonacci sequence mod n for $q = 5$ for $n = 2, ..., 15$

n	$\hat{\sigma}_5(n)$	$Pe_5(n)$
2	4	4
3	10	10
4	10	10
5	30	30
6	10	10
7	12	12
8	20	20
9	30	30
10	30	30
11	40	40
12	20	20
13	84	84
14	60	60
15	30	30

Example 5.28. The Hecke-Fibonacci sequence (5.2.3) mod 3 for $\mathscr{M}_5(3)$ is $1, 0\lambda_5, 1, \lambda_5, \lambda_5+2, \lambda_5+1, 0, \lambda_5+1, 2\lambda_5+1, \lambda_5, 2, 0\lambda_5, 2, ...$ hence $\hat{\sigma}_5(3) = 10$. $Pe_5(3)$ has vertices

$$[\frac{1}{0\lambda_5}],[\frac{0\lambda_5}{1}],[\frac{1}{1\lambda_5}],[\frac{\lambda_5}{\lambda_5+2}],[\frac{\lambda_5+2}{\lambda_5+1}],[\frac{\lambda_5+1}{0}],[\frac{0}{\lambda_5+1}],[\frac{\lambda_5+1}{2\lambda_5+1}],[\frac{2\lambda_5+1}{\lambda_5}],[\frac{\lambda_5}{2}].$$

The next two vertices are $[\frac{2}{0\lambda_5}],[\frac{0\lambda_5}{2}]$ so we have closed up our polygon, which has ten vertices. The length of $Pe_5(3)$ is equal to $\hat{\sigma}_5(3)$.

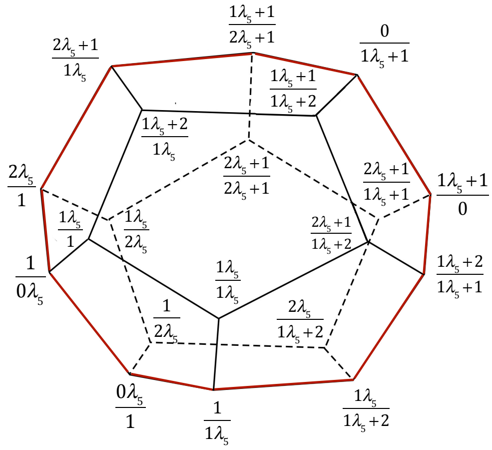


FIGURE 5.4: $Pe_5(3)$

It is a general fact that for regular maps, the map and its dual always have the same Petrie path lengths. The map $\mathscr{M}_3(5)$ (the icosahedron) has a Petrie polygon $Pe_3(5)$ of length ten as shown in [SS18, Figure 2], and its dual $\mathscr{M}_5(3)$ (the dodecahedron) has the same length as shown in Figure 5.4.

It is very interesting to extend this to $q = 7$ because the non-identity finite quotients of $\Gamma(2,3,7)$ are the Hurwitz groups of $84(g-1)$ automorphisms of a compact Riemann surface of genus $g \geq 2$. Using the minimal polynomial $\lambda_7^3 - \lambda_7^2 - 2\lambda_7 + 1$ of λ_7 we can draw $\mathscr{M}_7(3)$.
Using the Hecke-Fibonacci sequence (5.2.3) mod 3 for $q = 7$ we have the following $Pe_7(3)$

$$\frac{1}{0\lambda_7} \longleftrightarrow \frac{0\lambda_7}{1} \longleftrightarrow \frac{1}{\lambda_7} \longleftrightarrow \frac{\lambda_7}{\lambda_7^2+1} \longleftrightarrow \frac{\lambda_7^2+1}{\lambda_7^2+\lambda_7+2} \longleftrightarrow \frac{\lambda_7^2+\lambda_7+2}{\lambda_7} \longleftrightarrow \frac{\lambda_7}{2\lambda_7^2+\lambda_7+2}$$

$$\longleftrightarrow \frac{2\lambda_7^2+\lambda_7+2}{\lambda_7+1} \longleftrightarrow \frac{\lambda_7+1}{2\lambda_7+2} \longleftrightarrow \frac{2\lambda_7+2}{2\lambda_7^2+1} \longleftrightarrow \frac{2\lambda_7^2+1}{2\lambda_7^2+\lambda_7} \longleftrightarrow \frac{2\lambda_7^2+\lambda_7}{2\lambda_7^2+\lambda_7+2} \longleftrightarrow \frac{2\lambda_7^2+\lambda_7+2}{2\lambda_7^2+\lambda_7+1}$$

$$\longleftrightarrow \frac{2\lambda_7^2+\lambda_7+1}{2\lambda_7^2} \longleftrightarrow \frac{2\lambda_7^2}{\lambda_7^2+2\lambda_7+2} \longleftrightarrow \frac{\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+\lambda_7+2} \longleftrightarrow \frac{2\lambda_7^2+\lambda_7+2}{\lambda_7^2+2\lambda_7} \longleftrightarrow \frac{\lambda_7^2+2\lambda_7}{2\lambda_7^2+1}$$

$$\longleftrightarrow \frac{2\lambda_7^2+1}{\lambda_7+1} \longleftrightarrow \frac{\lambda_7+1}{\lambda_7+1} \longleftrightarrow \frac{\lambda_7+1}{\lambda_7^2+2\lambda_7+1} \longleftrightarrow \frac{\lambda_7^2+2\lambda_7+1}{\lambda_7} \longleftrightarrow \frac{\lambda_7}{2\lambda_7^2+2\lambda_7+1}$$

$$\longleftrightarrow \frac{2\lambda_7^2+2\lambda_7+1}{\lambda_7^2+1} \longleftrightarrow \frac{\lambda_7^2+1}{2\lambda_7} \longleftrightarrow \frac{2\lambda_7}{1} \longleftrightarrow \frac{1}{0\lambda_7} \longleftrightarrow \frac{0\lambda_7}{1}.$$

We find that the length of the principal Petrie polygon is 26 which corresponds to the Hurwitz group of the form $PSL_2(13)$. In [Mac99] Macbeath shows that there are three Hurwitz surfaces associated with $PSL_2(13)$. These three surfaces correspond to choosing a generator of order 7 from one of the three conjugacy classes of such elements in this group. These elements have traces ± 3, ± 5 and ± 6, and the corresponding maps have Petrie lengths $14, 26$ and 12 [JW16, Example 5.4].

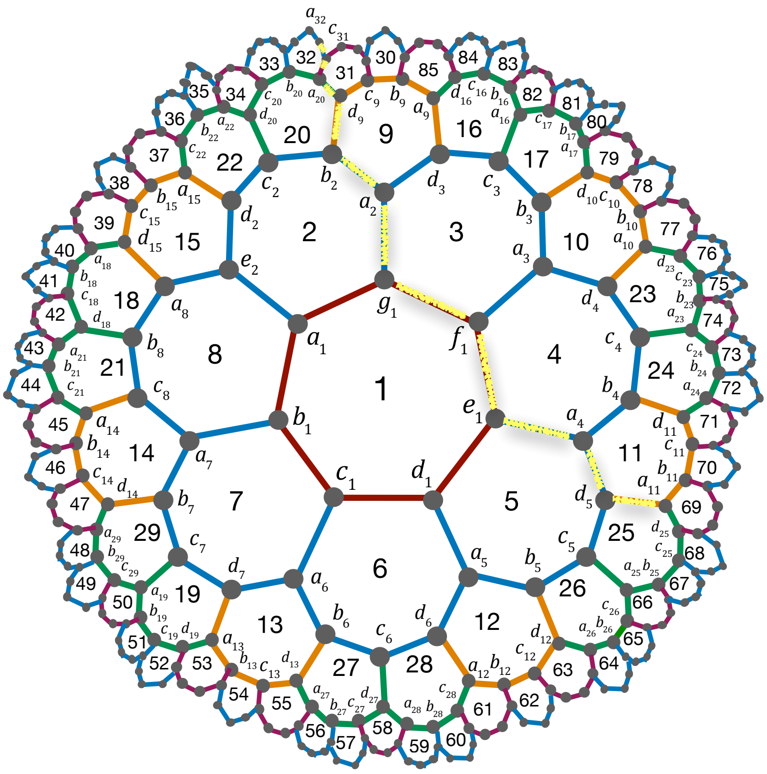


FIGURE 5.5: A partial picture of the map $\mathscr{M}_7(3)$ showing a highlighted part of $Pe_7(3)$.

For a list of vertices of some of the numbered faces in Figure 5.5, see Appendix B.

Appendices

Appendix A

Drawing Technique for $\hat{\mathscr{M}}_q$

This Appendix illustrates the technique that I use and follow to draw the universal q-gonal tessellation $\hat{\mathscr{M}}_q$ for $q = 3,\ 4,\ 5,\ 6,\ 7$ as shown in Figures 3.5, 3.11, 3.12, 3.13, 3.15. One of the drawing examples in Chapter 3, is the universal 4-gonal tessellation $\hat{\mathscr{M}}_4$ (Figure 3.11), which we draw by means of the following LaTeX code using TikZ package. TikZ package is one of the most powerful and complex tool that allows us to program and create graphic elements and figures into LaTeX documents. We can create images easily by defining some of their key properties. Furthermore TikZ is a recursive acronym for 'TikZ ist kein Zeichenprogramm' and is a part of a large package called PGF 'Portable Graphics Format'. In this drawing technique the coordinates were found by hand and calculator.

```
\documentclass[tikz]{standalone}
\begin{document}
\begin{tikzpicture}[scale=30]
\draw  [dotted] (0,0) -- (1.414,0);
    \filldraw (0,0) circle (0.09pt);
    \filldraw (1.414,0) circle (0.09pt);

\draw (0,0) -- (0,.618);
\draw (1.414,0) -- (1.414,.618);
\draw (0.707,0) arc (0:180:.353);
    \filldraw (0.707,0) circle (0.09pt);

\draw (1.414,0) arc (0:180:.353);

\draw [dotted] (0,0) -- (0,-.03) node[below]
{\Huge$\frac{0\sqrt(2)}{1}$};
\draw [dotted] (1.414,0) -- (1.414,-.03) node[below]
{\Huge$\frac{\sqrt{2}}{1}$};
\draw [dotted] (0.17678,0) -- (0.17678,-.03) node
[below]{\Large$a_{1}$};
\draw (0.17678,0) arc (0:180:.0883);
    \filldraw (0.17678,0) circle (0.06pt);
\draw [dotted] (0.20203,0) -- (0.20203,-.03) node[below]
{\Large$b_{1}$};
\draw (0.20203,0) arc (0:180:.0125);
```

```
    \filldraw (0.20203,0) circle (0.06pt);

\draw [dotted] (0.23570,0) -- (0.23570,-.03) node[below]
{\Large$c_{1}$};
\draw (0.23570,0) arc (0:180:.0166);
\draw (0.23570,0) arc (0:180:.118);
    \filldraw (0.23570,0) circle (0.06pt);


\draw [dotted] (0.257129,0) -- (0.257129,-.03) node[below]
{\large$d_{1}$};
\draw (0.257129,0) arc (0:180:.0105);
    \filldraw (0.257129,0) circle (0.06pt);
\draw [dotted] (0.265165,0) -- (0.265165,-.02) node[below]
{\large$e_{1}$};
\draw (0.265165,0) arc (0:180:.004);
    \filldraw (0.265165,0) circle (0.06pt);
\draw (0.28284,0) arc (0:180:.009);

\draw [dotted] (0.28284,0) -- (0.28284,-.03) node[below]
{\Large$f_{1}$};
\draw (0.28284,0) arc (0:180:.0235);
     \filldraw (0.28284,0) circle (0.06pt);

\draw [dotted] (0.35355,0) -- (0.35355,-.03) node[below]
{\huge$\frac{1}{2\sqrt{2}}$};
    \filldraw (0.35355,0) circle (0.09pt);
\draw (0.35355,0) arc (0:180:.0355);
\draw (0.35355,0) arc (0:180:.177);

\draw [dotted] (0.40406,0) -- (0.40406,-.03) node[below]
{\Large$g_{1}$};
\draw (0.40406,0) arc (0:180:.025);
    \filldraw (0.40406,0) circle (0.06pt);


\draw [dotted] (0.42426,0) -- (0.42426,-.03) node[below]
{\Large$j_{1}$};
\draw (0.42426,0) arc (0:180:.004);
\draw (0.42426,0) arc (0:180:.01);
    \filldraw (0.42426,0) circle (0.06pt);

\draw [dotted] (0.415945,0) -- (0.415945,-.01) node[below
]{\large$i_{1}$};
\draw (0.415945,0) arc (0:180:.0015);
    \filldraw (0.415945,0) circle (0.03pt);

\draw [dotted] (0.4124789,0) -- (0.4124789,-.02) node[below]
{\large$h_{1}$};
\draw (0.4124789,0) arc (0:180:.004);
    \filldraw (0.4124789,0) circle (0.03pt);


\draw [dotted] (0.47140,0) -- (0.47140,-.03) node[below]
{\huge$\frac{\sqrt{2}}{3}$};
\draw (0.47140,0) arc (0:180:.0235);
\draw (0.47140,0) arc (0:180:.059);
    \filldraw (0.47140,0) circle (0.09pt);
```

```
\draw [dotted] (0.56569,0) -- (0.56569,-.03) node[below]
{\Large$n_{1}$};
\draw (0.565693,0) arc (0:180:.0175);
\draw (0.565693,0) arc (0:180:.008);

     \filldraw (0.565693,0) circle (0.06pt);


\draw [dotted] (0.606091,0) -- (0.606091,-.03) node[below]
{\Large$p_{1}$};
\draw (0.606091,0) arc (0:180:.0084);

   \filldraw (0.606091,0) circle (0.06pt);
\draw [dotted] (0.58925,0) -- (0.58925,-.03) node[below]
{\Large$o_{1}$};
\draw (0.58925,0) arc (0:180:.0115);

   \filldraw (0.58925,0) circle (0.06pt);


\draw [dotted] (0.70711,0) -- (0.70711,-.03) node[below]
    {\Huge$\frac{1}{\sqrt{2}}$};
\draw (0.70711,0) arc (0:180:.0705);
\draw (0.70711,0) arc (0:180:.11755);
\draw (0.70711,0) arc (0:180:.0505);

       \filldraw (0.70711,0) circle (0.06pt);
\draw [dotted] (0.53033,0) -- (0.53033,-.03) node[below]
    {\Large$k_{1}$};
        \filldraw (0.53033,0) circle (0.06pt);
\draw (0.53033,0) arc (0:180:.029);

\draw [dotted] (0.543928,0) -- (0.543928,-.02) node[below
             ]{\large$l_{1}$};
         \filldraw (0.543928,0) circle (0.03pt);
\draw (0.543928,0) arc (0:180:.0065);
\draw [dotted] (0.549971,0) -- (0.549971,-.01) node[below]
             {\large$m_{1}$};
        \filldraw (0.549971,0) circle (0.03pt);
\draw (0.549971,0) arc (0:180:.0029);





 \draw [dotted] (1.06066,0) -- (1.06066,-.03) node[below]
    {\huge$\frac{3}{2\sqrt{2}}$};
\draw (1.41421,0) arc (0:180:.1175);
\draw (1.41421,0) arc (0:180:.088);
        \filldraw (1.06066,0) circle (0.09pt);

\draw (1.41421,0) arc (0:180:.177);

\draw [dotted] (0.98994,0) -- (0.98994,-.03) node[below]
    {\Large$g_{2}$};
\draw [dotted] (1.01015,0) -- (1.01015,-.03) node[below]
    {\Large$j_{2}$};
\draw [dotted] (0.94281,0) -- (0.94281,-.03) node[below]
```

```
    {\huge$\frac{2\sqrt{2}}{3}$};
       \filldraw (0.99826,0) circle (0.03pt);
\draw [dotted] (0.99826,0) -- (0.99826,-.02) node[below]
           {\large$h_{2}$};

\draw [dotted] (1.00173,0) -- (1.00173,-.01) node[below]
     {\large$i_{2}$};
    \filldraw (1.00173,0) circle (0.03pt);
\draw (1.00173,0) arc (0:180:.0015);
\draw (0.99826,0) arc (0:180:.004);
\draw (1.01015,0) arc (0:180:.004);
\draw (1.06066,0) arc (0:180:.059);
\draw (1.06066,0) arc (0:180:.025);
\draw (0.98994,0) arc (0:180:.0235);
\draw (1.01015,0) arc (0:180:.0107);
    \filldraw (0.98994,0) circle (0.09pt);
    \filldraw (1.01015,0) circle (0.09pt);
    \filldraw (0.94281,0) circle (0.09pt);


\draw [dotted] (0.88388,0) -- (0.88388,-.03) node[below]
    {\Large$f_{2}$};
\draw [dotted] (0.84853,0) -- (0.84853,-.03) node[below]
        {\Large$c_{2}$};
\draw [dotted] (0.808122,0) -- (0.808122,-.03) node[below]
                {\Large$a_{2}$};
\draw [dotted] (0.824957,0) -- (0.824957,-.03) node[below]
    {\Large$b_{2}$};
    \filldraw (0.824957,0) circle (0.06pt);
    \filldraw (0.808122,0) circle (0.06pt);
\draw (0.84853,0) arc (0:180:.0119);
\draw (0.824957,0) arc (0:180:.0085);
\draw (0.808122,0) arc (0:180:.05);

\draw (0.84853,0) arc (0:180:.07);

\draw (0.94281,0) arc (0:180:.0295);
\draw (0.88388,0) arc (0:180:.0175);
\draw (0.88388,0) arc (0:180:.007);

\draw (0.94281,0) arc (0:180:.117);
    \filldraw (0.88388,0) circle (0.06pt);
    \filldraw (0.84853,0) circle (0.06pt);



 \draw [dotted] (0.86424,0) -- (0.86424,-.02) node[below]
     {\large$d_{2}$};
        \filldraw (0.86424,0) circle (0.03pt);
\draw (0.86424,0) arc (0:180:.0075);


 \draw [dotted] (0.870285,0) -- (0.870285,-.01) node[below]
  {\large$e_{2}$};
        \filldraw (0.870285,0) circle (0.03pt);
\draw (0.870285,0) arc (0:180:.003);
```

```
 \draw [dotted] (1.13137,0) -- (1.13137,-.03) node[below]
 {\Large$k_{2}$};
       \filldraw (1.13137,0) circle (0.06pt);
\draw (1.13137,0) arc (0:180:.0355);


 \draw [dotted] (1.178511,0) -- (1.178511,-.03) node[below]
 {\Large$n_{2}$};
         \filldraw (1.178511,0) circle (0.06pt);
\draw (1.178511,0) arc (0:180:.0235);
\draw (1.178511,0) arc (0:180:.0105);


\draw [dotted] (1.21218,0) -- (1.21218,-.03) node[below]
{\Large$o_{2}$};
         \filldraw (1.21218,0) circle (0.06pt);
\draw (1.21218,0) arc (0:180:.0165);




\draw [dotted] (1.23743,0) -- (1.23743,-.03) node[below]
    {\Large$p_{2}$};
        \filldraw (1.23743,0) circle (0.06pt);
\draw (1.23743,0) arc (0:180:.0126);


\draw [dotted] (1.149048,0) -- (1.149048,-.02) node[below]
{\large$l_{2}$};
       \filldraw (1.149048,0) circle (0.03pt);
\draw (1.149048,0) arc (0:180:.0085);


\draw [dotted] (1.15708,0) -- (1.15708,-.01) node[below]
{\large$m_{2}$};
     \filldraw (1.15708,0) circle (0.03pt);
\draw (1.15708,0) arc (0:180:.004);

 \end{tikzpicture}
 \end{document}
```

Appendix B

Faces of the map $\mathscr{M}_7(3)$

In this appendix, we list a table showing vertices of some of the numbered faces of the map $\mathscr{M}_7(3)$ in anticlockwise order as shown in Figure 5.5.

TABLE B.1: Table of Correspondence for $\mathscr{M}_7(3)$

Face	Correspondent vertices
1	$a_1 : \frac{1\lambda_7}{1} \leftrightarrow b_1 : \frac{\lambda_7^2+2}{\lambda_7} \leftrightarrow c_1 : \frac{\lambda_7^2+2}{\lambda_7^2+2} \leftrightarrow d_1 : \frac{\lambda_7}{\lambda_7^2+2} \leftrightarrow e_1 : \frac{1}{\lambda_7} \leftrightarrow f_1 : \frac{0\lambda_7}{1} \leftrightarrow g_1 : \frac{1}{0\lambda_7}$
2	$a_2 : \frac{2\lambda_7}{1} \leftrightarrow b_2 : \frac{2\lambda_7^2+2}{\lambda_7} \leftrightarrow c_2 : \frac{\lambda_7^2+\lambda_7+1}{\lambda_7^2+2} \leftrightarrow d_2 : \frac{\lambda_7^2+2\lambda_7+2}{\lambda_7^2+2} \leftrightarrow e_2 : \frac{\lambda_7^2+1}{\lambda_7} \leftrightarrow a_1 \leftrightarrow g_1$
3	$f_1 \leftrightarrow a_3 : \frac{2}{\lambda_7} \leftrightarrow b_3 : \frac{2\lambda_7}{\lambda_7^2+2} \leftrightarrow c_3 : \frac{2\lambda_7^2+1}{\lambda_7^2+2} \leftrightarrow d_3 : \frac{2\lambda_7^2+1}{\lambda_7} \leftrightarrow a_2 \leftrightarrow g_1$
4	$e_1 \leftrightarrow a_4 : \frac{2\lambda_7}{2\lambda_7^2+2} \leftrightarrow b_4 : \frac{2\lambda_7+1}{2\lambda_7^2+\lambda_7+1} \leftrightarrow c_4 : \frac{2\lambda_7^2+1}{\lambda_7^2+2\lambda_7+2} \leftrightarrow d_4 : \frac{2\lambda_7}{\lambda_7^2+1} \leftrightarrow a_3 \leftrightarrow f_1$
5	$a_5 : \frac{\lambda_7^2+1}{\lambda_7^2+2\lambda_7+2} \leftrightarrow b_5 : \frac{\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+\lambda_7} \leftrightarrow c_5 : \frac{2\lambda_7^2+\lambda_7+1}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow d_5 : \frac{2\lambda_7^2+2}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow a_4 \leftrightarrow e_1 \leftrightarrow d_1$
6	$a_6 : \frac{\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+\lambda_7+1} \leftrightarrow b_6 : \frac{2\lambda_7^2+\lambda_7}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow c_6 : \frac{2\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+2\lambda_7} \leftrightarrow d_6 : \frac{2\lambda_7^2+2\lambda_7+1}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow a_5 \leftrightarrow d_1 \leftrightarrow c_1$
7	$a_7 : \frac{2\lambda_7^2+\lambda_7+1}{2\lambda_7^2+2} \leftrightarrow b_7 : \frac{2\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow c_7 : \frac{2\lambda_7^2+2\lambda_7}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow d_7 : \frac{2\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+\lambda_7} \leftrightarrow a_6 \leftrightarrow c_1 \leftrightarrow b_1$
8	$e_2 \leftrightarrow a_8 : \frac{2\lambda_7^2+2\lambda_7+1}{2\lambda_7^2+2} \leftrightarrow b_8 : \frac{2\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+\lambda_7+1} \leftrightarrow c_8 : \frac{2\lambda_7^2+\lambda_7}{\lambda_7^2+2\lambda_7+2} \leftrightarrow a_7 \leftrightarrow b_1 \leftrightarrow a_1$
9	$d_3 \leftrightarrow a_9 : \frac{\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+2} \leftrightarrow b_9 : \frac{2\lambda_7^2+\lambda_7}{2\lambda_7^2+\lambda_7+1} \leftrightarrow c_9 : \frac{2\lambda_7^2+2\lambda_7+2}{\lambda_7^2+2\lambda_7+2} \leftrightarrow d_9 : \frac{2\lambda_7^2+2\lambda_7+1}{\lambda_7^2+1} \leftrightarrow b_2 \leftrightarrow a_2$
10	$d_4 \leftrightarrow a_{10} : \frac{\lambda_7^2+1}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow b_{10} : \frac{\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow c_{10} : \frac{2\lambda_7^2+\lambda_7+1}{2\lambda_7^2+\lambda_7} \leftrightarrow d_{10} : \frac{2\lambda_7^2+2}{\lambda_7^2+2\lambda_7+2} \leftrightarrow b_3 \leftrightarrow a_3$
11	$d_5 \leftrightarrow a_{11} : \frac{2\lambda_7^2+2\lambda_7+1}{2\lambda_7} \leftrightarrow b_{11} : \frac{2\lambda_7^2+2\lambda_7+2}{\lambda_7+2} \leftrightarrow c_{11} : \frac{2\lambda_7^2+\lambda_7}{\lambda_7^2} \leftrightarrow d_{11} : \frac{\lambda_7^2+2\lambda_7+2}{\lambda_7^2+\lambda_7} \leftrightarrow b_4 \leftrightarrow a_4$
12	$d_6 \leftrightarrow a_{12} : \frac{2\lambda_7}{\lambda_7+2} \leftrightarrow b_{12} : \frac{\lambda_7+2}{2\lambda_7^2+1} \leftrightarrow c_{12} : \frac{\lambda_7^2}{2\lambda_7^2+\lambda_7+2} \leftrightarrow d_{12} : \frac{\lambda_7^2+\lambda_7}{\lambda_7^2} \leftrightarrow b_5 \leftrightarrow a_5$
13	$d_7 \leftrightarrow a_{13} : \frac{\lambda_7+2}{\lambda_7^2} \leftrightarrow b_{13} : \frac{2\lambda_7^2+1}{2\lambda_7^2+\lambda_7+2} \leftrightarrow c_{13} : \frac{2\lambda_7^2+\lambda_7+2}{2\lambda_7^2+1} \leftrightarrow d_{13} : \frac{\lambda_7^2}{\lambda_7+2} \leftrightarrow b_6 \leftrightarrow a_6$
14	$c_8 \leftrightarrow a_{14} : \frac{\lambda_7^2}{\lambda_7^2+\lambda_7} \leftrightarrow b_{14} : \frac{2\lambda_7^2+\lambda_7+2}{\lambda_7^2} \leftrightarrow c_{14} : \frac{2\lambda_7^2+1}{\lambda_7+2} \leftrightarrow d_{14} : \frac{\lambda_7+2}{2\lambda_7} \leftrightarrow b_7 \leftrightarrow a_7$
15	$d_2 \leftrightarrow a_{15} : \frac{\lambda_7^2+\lambda_7}{\lambda_7^2+2\lambda_7+2} \leftrightarrow b_{15} : \frac{\lambda_7^2}{2\lambda_7^2+\lambda_7} \leftrightarrow c_{15} : \frac{\lambda_7+2}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow d_{15} : \frac{2\lambda_7}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow a_8 \leftrightarrow e_2$

Table B.2: Table of Correspondence for $\mathscr{M}_7(3)$

Face	Correspondent vertices
16	$e_3 \leftrightarrow a_{16} : \frac{\lambda_7^2+2\lambda_7+2}{\lambda_7^2+2\lambda_7+2} \leftrightarrow b_{16} : \frac{\lambda_7^2+\lambda_7+1}{2\lambda_7^2+\lambda_7} \leftrightarrow c_{16} : \frac{\lambda_7^2+\lambda_7}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow d_{16} : \frac{\lambda_7^2+\lambda_7+1}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow a_9 \leftrightarrow d_3$
17	$d_{10} \leftrightarrow a_{17} : \frac{2\lambda_7^2+2\lambda_7+1}{\lambda_7^2+\lambda_7+1} \leftrightarrow b_{17} : \frac{2\lambda_7^2+2\lambda_7+2}{\lambda_7^2+\lambda_7} \leftrightarrow c_{17} : \frac{2\lambda_7^2+\lambda_7}{\lambda_7^2+\lambda_7+1} \leftrightarrow a_{16} \leftrightarrow e_3 \leftrightarrow b_3$
18	$d_{15} \leftrightarrow a_{18} : \frac{\lambda_7^2+2\lambda_7+1}{2\lambda_7} \leftrightarrow b_{18} : \frac{\lambda_7+2}{\lambda_7+2} \leftrightarrow c_{18} : \frac{2}{\lambda_7^2} \leftrightarrow d_{18} : \frac{\lambda_7+1}{\lambda_7^2+\lambda_7} \leftrightarrow b_8 \leftrightarrow a_8$
19	$a_{19} : \frac{0}{\lambda_7+1} \leftrightarrow b_{19} : \frac{\lambda_7^2+\lambda_7}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow c_{19} : \frac{2\lambda_7^2+2\lambda_7+2}{\lambda_7^2+\lambda_7} \leftrightarrow d_{19} : \frac{\lambda_7^2+\lambda_7+1}{1} \leftrightarrow a_{13} \leftrightarrow d_7 \leftrightarrow c_7$
20	$d_9 \leftrightarrow a_{20} : \frac{\lambda_7}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow b_{20} : \frac{2\lambda_7+1}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow c_{20} : \frac{2\lambda_7^2}{2\lambda_7^2+\lambda_7} \leftrightarrow d_{20} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7^2+2\lambda_7+2} \leftrightarrow c_2 \leftrightarrow b_2$
21	$d_{18} \leftrightarrow a_{21} : \frac{2\lambda_7^2+2\lambda_7+1}{\lambda_7+1} \leftrightarrow b_{21} : \frac{\lambda_7^2+\lambda_7}{0} \leftrightarrow c_{21} : \frac{1}{2\lambda_7+2} \leftrightarrow a_{14} \leftrightarrow c_8 \leftrightarrow b_8$
22	$d_{20} \leftrightarrow a_{22} : \frac{\lambda_7+1}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow b_{22} : \frac{0}{\lambda_7^2+\lambda_7} \leftrightarrow c_{22} : \frac{2\lambda_7+2}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow a_{15} \leftrightarrow d_2 \leftrightarrow c_2$
23	$a_{23} : \frac{2\lambda_7^2+\lambda_7+1}{\lambda_7^2+\lambda_7} \leftrightarrow b_{23} : \frac{2\lambda_7^2+\lambda_7}{2\lambda_7^2} \leftrightarrow c_{23} : \frac{2\lambda_7^2+2\lambda_7+2}{2\lambda_7+1} \leftrightarrow d_{23} : \frac{2\lambda_7+1}{\lambda_7^2+\lambda_7} \leftrightarrow a_{10} \leftrightarrow d_4 \leftrightarrow c_4$
24	$d_{11} \leftrightarrow a_{24} : \frac{\lambda_7^2+\lambda_7+1}{\lambda_7+1} \leftrightarrow b_{21} \leftrightarrow c_{24} : \frac{\lambda_7^2+\lambda_7+1}{2\lambda_7+2} \leftrightarrow a_{23} \leftrightarrow c_4 \leftrightarrow b_4$
25	$a_{25} : \frac{\lambda_7^2+\lambda_7}{\lambda_7+1} \leftrightarrow b_{25} : \frac{2\lambda_7^2}{1} \leftrightarrow b_{18} \leftrightarrow d_{25} : \frac{\lambda_7^2+\lambda_7}{2\lambda_7^2+1} \leftrightarrow a_{11} \leftrightarrow d_5 \leftrightarrow c_5$
26	$d_{12} \leftrightarrow a_{26} : \frac{\lambda_7+1}{2} \leftrightarrow b_{22} \leftrightarrow c_{26} : \frac{2\lambda_7+2}{\lambda_7^2+\lambda_7+2} \leftrightarrow a_{25} \leftrightarrow c_5 \leftrightarrow b_5$
27	$d_{13} \leftrightarrow a_{27} : \frac{2}{\lambda_7+2} \leftrightarrow b_{27} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7^2+\lambda_7+1} \leftrightarrow c_{27} : \frac{\lambda_7^2+\lambda_7+2}{2\lambda_7^2+2\lambda_7} \leftrightarrow d_{27} : \frac{\lambda_7+1}{0} \leftrightarrow c_6 \leftrightarrow b_6$
28	$d_{27} \leftrightarrow a_{28} : \frac{1}{\lambda_7^2+\lambda_7} \leftrightarrow b_{28} : \frac{2\lambda_7+1}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow c_{28} : \frac{2\lambda_7^2+1}{\lambda_7^2+\lambda_7+1} \leftrightarrow a_{12} \leftrightarrow d_6 \leftrightarrow c_6$
29	$d_{14} \leftrightarrow a_{29} : \frac{\lambda_7+2}{\lambda_7^2+2\lambda_7+1} \leftrightarrow b_{29} : \frac{\lambda_7^2+\lambda_7+1}{\lambda_7+2} \leftrightarrow c_{29} : \frac{\lambda_7^2+\lambda_7}{1} \leftrightarrow a_{19} \leftrightarrow c_7 \leftrightarrow b_7$
30	$a_{30} : \frac{2\lambda_7^2}{\lambda_7^2+\lambda_7} \leftrightarrow b_{30} : \frac{1}{\lambda_7+1} \leftrightarrow b_{21} \leftrightarrow c_{30} : \frac{2\lambda_7^2+2}{2\lambda_7^2+2} \leftrightarrow d_{30} : \frac{\lambda_7+1}{2\lambda_7^2+2\lambda_7} \leftrightarrow c_9 \leftrightarrow b_9$
31	$c_9 \leftrightarrow d_{30} \leftrightarrow a_{31} : \frac{1}{\lambda_7^2} \leftrightarrow b_{31} : \frac{2\lambda_7+1}{\lambda_7+2} \leftrightarrow c_{31} : \frac{2\lambda_7^2+\lambda_7+2}{2\lambda_7} \leftrightarrow a_{20} \leftrightarrow d_9$
32	$c_{31} \leftrightarrow a_{32} : \frac{\lambda_7+1}{\lambda_7^2+2\lambda_7+1} \leftrightarrow b_{32} : \frac{2\lambda_7^2+1}{\lambda_7+2} \leftrightarrow c_{32} : \frac{2\lambda_7^2+\lambda_7}{2} \leftrightarrow a_{25} \leftrightarrow b_{20} \leftrightarrow a_{20}$
33	$a_{25} \leftrightarrow a_{33} : \frac{2\lambda_7^2+1}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow c_6 \leftrightarrow b_{33} : \frac{\lambda_7^2+2\lambda_7+1}{1} \leftrightarrow c_{33} : \frac{2\lambda_7^2+2}{2\lambda_7^2} \leftrightarrow c_{20} \leftrightarrow a_{20}$
34	$c_{20} \leftrightarrow c_{33} \leftrightarrow a_{34} : \frac{2\lambda_7^2+2}{2\lambda_7^2+\lambda_7+2} \leftrightarrow b_{34} : \frac{\lambda_7^2}{2\lambda_7^2+1} \leftrightarrow c_{34} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7+2} \leftrightarrow a_{22} \leftrightarrow d_{20}$
35	$c_{34} \leftrightarrow a_{35} : \frac{\lambda_7^2+2\lambda_7+2}{\lambda_7+2} \leftrightarrow b_{35} : \frac{\lambda_7^2+2\lambda_7+2}{\lambda_7^2+\lambda_7+1} \leftrightarrow c_{35} : \frac{2\lambda_7^2+2\lambda_7}{2\lambda_7^2+2\lambda_7} \leftrightarrow d_{27} \leftrightarrow b_{22} \leftrightarrow a_{22}$
36	$d_{27} \leftrightarrow a_{36} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7^2+\lambda_7} \leftrightarrow b_{10} \leftrightarrow c_{36} : \frac{\lambda_7^2}{\lambda_7^2+\lambda_7+1} \leftrightarrow d_{36} : \frac{2\lambda_7^2+2\lambda_7}{2\lambda_7+1} \leftrightarrow c_{22} \leftrightarrow b_{22}$
37	$c_{22} \leftrightarrow d_{36} \leftrightarrow a_{37} : \frac{2\lambda_7^2}{2\lambda_7^2+1} \leftrightarrow b_{37} : \frac{\lambda_7^2+1}{2\lambda_7^2+\lambda_7+2} \leftrightarrow c_{37} : \frac{2\lambda_7^2+2}{\lambda_7^2} \leftrightarrow b_{15} \leftrightarrow a_{15}$
38	$c_{37} \leftrightarrow a_{38} : \frac{1}{2} \leftrightarrow b_{38} : \frac{\lambda_7^2+\lambda_7+1}{2\lambda_7^2+2\lambda_7} \leftrightarrow c_{38} : \frac{2\lambda_7^2+1}{\lambda_7^2+\lambda_7+2} \leftrightarrow d_{38} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7+1} \leftrightarrow c_{15} \leftrightarrow b_{15}$
39	$c_{15} \leftrightarrow d_{38} \leftrightarrow a_{39} : \frac{2\lambda_7^2+\lambda_7}{1} \leftrightarrow b_{39} : \frac{2\lambda_7^2+1}{2\lambda_7+1} \leftrightarrow c_{39} : \frac{\lambda_7+1}{2\lambda_7^2+\lambda_7+2} \leftrightarrow a_{18} \leftrightarrow d_{15}$
40	$c_{39} \leftrightarrow a_{40} : \frac{2\lambda_7+2}{\lambda_7+1} \leftrightarrow b_{40} : \frac{2\lambda_7^2+\lambda_7+2}{2\lambda_7^2+1} \leftrightarrow c_{40} : \frac{\lambda_7+2}{2\lambda_7^2+\lambda_7} \leftrightarrow d_{40} : \frac{2\lambda_7^2+\lambda_7+1}{\lambda_7^2+\lambda_7} \leftrightarrow b_{18} \leftrightarrow a_{18}$

Table B.3: Table of Correspondence for $\mathscr{M}_7(3)$

Face	Correspondent vertices
41	$d_{40} \leftrightarrow a_{41} : \frac{0}{2\lambda_7^2+1} \leftrightarrow b_{41} : \frac{\lambda_7^2+2\lambda_7+2}{\lambda_7^2+\lambda_7+1} \leftrightarrow c_{41} : \frac{1}{\lambda_7^2+2\lambda_7+1} \leftrightarrow d_{41} : \frac{1}{2\lambda_7^2+2} \leftrightarrow c_{18} \leftrightarrow b_{18}$
42	$c_{18} \leftrightarrow d_{41} \leftrightarrow a_{42} : \frac{2\lambda_7+2}{2\lambda_7^2+2} \leftrightarrow b_{42} : \frac{2\lambda_7^2+2\lambda_7+1}{\lambda_7^2} \leftrightarrow c_{42} : \frac{\lambda_7^2+2}{2\lambda_7^2+2\lambda_7} \leftrightarrow a_{21} \leftrightarrow d_{18}$
43	$c_{42} \leftrightarrow a_{43} : \frac{0}{\lambda_7^2+2\lambda_7+2} \leftrightarrow b_{43} : \frac{2\lambda_7^2+1}{\lambda_7^2+2\lambda_7+2} \leftrightarrow c_{43} : \frac{2\lambda_7^2+2\lambda_7+1}{2\lambda_7^2+2\lambda} \leftrightarrow a_{25} \leftrightarrow b_{21} \leftrightarrow a_{21}$
44	$a_{25} \leftrightarrow a_{44} : \frac{1}{2\lambda_7^2+2\lambda_7} \leftrightarrow b_{44} : \frac{\lambda_7^2+2\lambda_7}{\lambda_7^2+2\lambda_7+2} \leftrightarrow c_{44} : \frac{\lambda_7+1}{\lambda_7^2} \leftrightarrow d_{44} : \frac{\lambda_7^2+2\lambda_7}{2\lambda_7^2+2\lambda} \leftrightarrow c_{21} \leftrightarrow b_{21}$
45	$c_{21} \leftrightarrow d_{44} \leftrightarrow a_{45} : \frac{\lambda_7}{2\lambda_7^2} \leftrightarrow b_{45} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7^2+1} \leftrightarrow c_{45} : \frac{\lambda_7^2+1}{2\lambda_7^2+2} \leftrightarrow b_{14} \leftrightarrow a_{14}$
46	$c_{45} \leftrightarrow a_{46} : \frac{\lambda_7+1}{1} \leftrightarrow b_{46} : \frac{\lambda_7+2}{\lambda_7^2+\lambda_7+1} \leftrightarrow c_{46} : \frac{\lambda_7^2+\lambda_7+2}{2\lambda_7^2+1} \leftrightarrow d_{46} : \frac{\lambda_7^2}{2\lambda_7^2+2\lambda_7} \leftrightarrow c_{14} \leftrightarrow b_{14}$
78	$a_{78} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7+1} \leftrightarrow b_{78} : \frac{2\lambda_7+2}{2\lambda_7^2+2\lambda_7+1} \leftrightarrow b_{22} \leftrightarrow c_{78} : \frac{\lambda_7^2+1}{1} \leftrightarrow a_{79} \leftrightarrow c_{10} \leftrightarrow b_{10}$
79	$c_{10} \leftrightarrow a_{79} : \frac{2\lambda_7^2+2\lambda_7}{\lambda_7^2} \leftrightarrow b_{79} : \frac{2\lambda_7^2}{2\lambda_7^2+\lambda_7+2} \leftrightarrow c_{79} : \frac{2\lambda_7+1}{2\lambda_7^2+1} \leftrightarrow d_{79} : \frac{\lambda_7}{\lambda_7+2} \leftrightarrow a_{17} \leftrightarrow d_{16}$
80	$d_{79} \leftrightarrow a_{80} : \frac{2\lambda_7^2+\lambda_7+2}{\lambda_7+2} \leftrightarrow b_{80} : \frac{2\lambda_7+1}{\lambda_7^2+\lambda_7+1} \leftrightarrow c_{80} : \frac{1}{2\lambda_7^2+2\lambda_7} \leftrightarrow d_{27} \leftrightarrow b_{17} \leftrightarrow a_{17}$
82	$a_{82} : \frac{\lambda_7^2+2\lambda_7}{2\lambda_7^2+2\lambda_7+2} \leftrightarrow b_{82} : \frac{2\lambda_7^2}{\lambda_7+2} \leftrightarrow c_{82} : \frac{\lambda_7^2+2\lambda_7+1}{2\lambda_7^2+1} \leftrightarrow d_{82} : \frac{\lambda_7^2+2}{2\lambda_7^2+\lambda_7+2} \leftrightarrow a_{83} \leftrightarrow b_{16} \leftrightarrow a_{16}$
83	$a_{83} : \frac{2\lambda_7+1}{\lambda_7^2} \leftrightarrow a_1 \leftrightarrow b_{83} : \frac{2\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+2\lambda_7} \leftrightarrow c_{83} : \frac{\lambda_7^2+\lambda_7}{\lambda_7^2+\lambda_7+2} \leftrightarrow a_{19} \leftrightarrow c_{16} \leftrightarrow b_{16}$
84	$a_{19} \leftrightarrow a_{84} : \frac{2\lambda_7^2+2\lambda_7}{1} \leftrightarrow b_{84} : \frac{\lambda_7^2+\lambda_7+1}{2\lambda_7+1} \leftrightarrow c_{84} : \frac{2\lambda_7^2+2\lambda_7+2}{2\lambda_7^2+1} \leftrightarrow a_{85} \leftrightarrow d_{16} \leftrightarrow c_{16}$
85	$d_{16} \leftrightarrow a_{85} : \frac{2\lambda_7+1}{2\lambda_7} \leftrightarrow b_{85} : \frac{\lambda_7^2+2}{\lambda_7+2} \leftrightarrow c_{85} : \frac{\lambda_7^2+2\lambda_7+1}{\lambda_7^2} \leftrightarrow a_{30} \leftrightarrow b_9 \leftrightarrow a_9$

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