UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Mathematical Sciences

Gravitational radiation and holography

by

Aaron David Poole

Thesis for the degree of Doctor of Philosophy

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ABSTRACT

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This thesis is concerned with the topic of gravitational radiation in asymptotically locally (anti)-de Sitter spacetimes and, in particular, how one can use the tools of holographic duality to provide deeper insights into the nature of radiation. The thesis can broadly be separated into two parts. The first part approaches the topic of gravitational radiation by studying gravity in Bondi-Sachs gauge, specifically solutions to the vacuum Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ in the presence of a cosmological constant $\Lambda \neq 0$. We solve these equations for an axisymmetric Bondi-Sachs metric and observe that these differential equations admit an *algebraic* re-writing based upon data at the conformal boundary, $\mathscr{I}$, of the space-time. Using the Fefferman-Graham coordinate expansion and tools of the AdS/CFT correspondence we are further able to analyse the solutions in the Bondi-Sachs gauge and comment upon the holographic interpretation of the Bondi Sachs data at $\mathscr{I}$. We examine the notion of Bondi mass in AdS and discuss whether or not the natural candidate for such a quantity obeys the monotonicity properties that one would expect due to outgoing gravitational radiation. We finally examine methods of 'breaking' the Bondi gauge in order to relax aspects of the gauge which appear overly restrictive.

The second part of the thesis turns attention to asymptotically locally dS_4 spacetimes ($\Lambda > 0$) and a discussion of how one can apply Bondi-Sachs gauge as well as other techniques in order to gain an understanding of gravitational radiation in this class of spacetimes. We give the analytic continuation of the Fefferman-Graham expansion from Bondi-AdS to Bondi-dS spacetimes as well as an analysis of the asymptotic gravitational charges using the covariant phase space formalism, together with holographic renormalisation techniques adapted to dS spacetime. We provide explicit examples of these charges by considering tensorial perturbations of dS_4 in the inflationary patch coordinates, before finally connecting this example to global coordinates via a Bogoliubov transformation of the tensorial mode coefficients.

Table of Contents

List of Figures

Authorship declaration

I, Aaron Poole, declare that this thesis entitled *Gravitational radiation and holography* and the work presented in it are my own and have been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:
   - "(A)dS_4 in Bondi gauge" A. Poole, K. Skenderis, M. Taylor 2019 *Class. Quantum Grav.* **36** 095005, `https://doi.org/10.1088/1361-6382/ab117c`
   - "Charges in asymptotically locally de Sitter spacetime" A. Poole, K. Skenderis, M. Taylor (to appear) 2020

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In memory of my Grandfather, Alexander Craven East. Things are still ok with Einstein.

CHAPTER 1

Introduction

In the 1960s the theoretical physicists Bondi, Metzner, Van der Burg, and Sachs (BMS) revolutionised the study of asymptotically flat spacetimes in a number of pioneering papers [1, 2, 3]. Using a metric based approach to asymptotic flatness, they were able to establish a notion of mass at future null infinity, $\mathscr{I}^+$, which characterises the mass loss of a spacetime due to outgoing gravitational radiation. In addition to this remarkable result, they also showed that the class of asymptotically flat spacetimes enjoys a larger symmetry group at $\mathscr{I}^+$ than the 10-dimensional Poincaré group of Minkowski spacetime, the infinite dimensional *BMS group*. This group consists of the Poincaré transformations as well as an infinite number of transformations named *supertranslations*[1].

Since the publication of the original papers, much work has been undertaken in various related areas. Connections have been established between the BMS group and the gravitational memory effect [4, 5, 6] as well as the scattering problem in general relativity [7]. There has also been an increased interest in further understanding the field of asymptotic symmetries in general, in both the algebra and associated charges of the BMS group [8, 9, 10, 11, 12, 13, 14, 15, 16, 17] as well as related symmetries in other theories, such as electromagnetism in $d \geq 4$ spacetime dimensions [18, 19]. Much of the recent work on the BMS group includes the consideration of an extended symmetry group by adding extra transformations known as *superrotations*.[2] Physically, these transformations

[1]So named as they extend the translations of the Poincaré group.
[2]So named as they extend the rotations of the Poincaré group.

are conjectured to be related to spacetimes with cosmic string singularities [20, 21, 22] (cosmic branes in higher spacetime dimension [23]) as well as (with supertranslations) the so-called 'soft hair' charges on black holes [24, 25, 26], an avenue which may shed light upon the black hole information paradox [27, 28, 29]. A broad programme of research has been initiated, primarily by Strominger and collaborators, into an 'infrared triangle' of physics relating soft theorems, gravitational memory, and asymptotic symmetries [30, 31].

Due to their rich geometric nature, aspects of the Bondi formalism are also of interest to the mathematical relativity community. The Bondi mass has been proven to obey a positive mass theorem [32, 33] in asymptotically flat spacetime, and the Bondi metric expansions have been generalised to include *polyhomogeneous* $\mathscr{I}^+$, all the while maintaining the asymptotic symmetry group as the BMS group [34, 35, 36, 37, 38, 39, 40].

An important feature of all the literature referenced thus far is that this work has been performed in the setting of asymptotically flat spacetime. The work described in this thesis attempts to fill a gap in the literature by investigating aspects of gravitational radiation, primarily by use of the Bondi-Sachs gauge, in asymptotically locally (anti)-de Sitter ((A)dS) spacetimes. These are spacetimes which solve the Einstein field equations in the presence of a cosmological constant $\Lambda \neq 0$ as opposed to $\Lambda = 0$ in the asymptotically flat case. Although a less studied case, the topic of gravitational radiation in such spacetimes has been discussed in the work of Ashtekar and collaborators, who pay particular attention to gravitational radiation in the $\Lambda > 0$ case [41, 42, 43, 44, 45], as well as work examining the energy and Bondi mass in $\Lambda \neq 0$ spacetimes [46, 47, 48, 49]. Also of note is the recent work [50, 51], which examines similar issues to published work written in part by the author of this thesis [52] and relaxes some of the extra spacetime symmetry that we enforced.

Part of the motivation for this change of asymptotics is the aim of gaining a *holographic* understanding of the Bondi-Sachs gauge, as well as extra physical intuition regarding the behaviour of radiation. Originally proposed by 't Hooft [53] and Susskind [54], the holographic principle conjectures that any quantum theory of gravity can be described in terms of a theory without gravity in one less dimension. The most famous example of this conjecture was discovered by Maldacena [55] in the context of string theory on anti-de Sitter space. Following soon after this work, [56, 57] fleshed out the description of a subclass of the holographic principle by conjecturing equivalence between an asymptotically locally AdS spacetime and a conformal field theory which lives on the conformal boundary of the spacetime, resulting in the so-called AdS/CFT correspondence. Early work [58] was able to verify with spectacular success the predictions of such a correspondence and this has become a highly active area of research. Of particular use when studying the gravitational side of the correspondence is the *Fefferman-Graham* coordinate system [59], a gauge choice for which all asymptotically locally AdS metrics can be written in, and one which provides a holographic interpretation of the spacetime via the AdS/CFT correspondence [60, 61, 62]. Part of the goal of this work is to compare the asymptotic description in this gauge with that of Bondi-Sachs and to provide an explicit map between the two choices.

Before we move on to more technical discussions, we will briefly summarise the structure of this thesis chapter by chapter. In chapter 2 we give a review of the important literature and technical work that we will need for the remainder of the thesis. This includes a review of the various spacetime asymptotics in 2.1; an introduction to the Bondi gauge and the BMS group in 2.2; the important results and techniques from holography in 2.3 and finally a discussion of the covariant phase space formalism for charges in the style of Wald et. al [63, 64, 65, 66, 67] in 2.4.

In chapter 3 we discuss the application of the Bondi gauge to asymptotically locally (anti)-de Sitter spacetimes. To save from repetition, we point the reader to the final paragraph of 3.1 which lists details of the contents of each section.

In chapter 4 we discuss a number of additional aspects of the Bondi gauge in asymptotically locally (A)dS spacetimes. In 4.1 we discuss the notion of Bondi mass in asymptotically locally AdS spacetime, and show that the naïve definition from the asymptotically flat literature no longer decreases monotonically in the presence of a small negative cosmological constant. In 4.2 we discuss the restrictions of the Bondi gauge, and give a proposal for a 'breaking' of the gauge in order to easily incorporate a larger phase space of metrics. A particular motivation for this is to apply a Bondi type gauge the AdS-Robinson-Trautman class of solutions [68, 69], which we also discuss. We summarise the main results and future directions of this chapter in 4.3.

In chapter 5 we switch focus from (predominantly) AdS spacetimes to asymptotically locally dS spacetimes, work which serves as a natural continuation of that performed in the preceding chapters. In section 5.1 we show that an understanding of the Bondi-Sachs gauge in asymptotically locally dS spacetimes can be gained via a suitable analytic continuation of our results in AdS, giving details of the transformations required. In 5.2 we give the prescription to define conserved quantities in AldS spacetime, work which adapts many of the techniques of the Hamiltonian approach to holographic renormalisation as described in [70, 71]. We will start by introducing the class of theories we consider and derive the equations of motion and conjugate momenta, paying close attention to the differences between AdS and dS. We will then discuss the computation of the Wald Hamiltonians for the theory and conclude that their existence is determined by the vanishing of the trace anomaly $\mathcal{A}$. In 5.3 we give an explicit example for the procedure that we have developed in the chapter, namely a computation of the Wald Hamiltonian for a tensorially perturbed dS_4 metric in the inflationary patch, an important example both for its relative simplicity as well as it being of considerable cosmological interest [72, 73, 74, 75, 76]. We end this chapter with an attempt to understand the global nature of our charges by performing the Bogoliubov transformation between the perturbation in the inflationary patch and global coordinates on dS_4, before drawing conclusions in 5.4

Finally, in chapter 6 we give our final summary of the body of work as a whole. We provide an outlook on possible future directions one could take, with speculation upon the development of a description of advanced gravitational wave phenomena (e.g. the memory effect) in $\Lambda \neq 0$ spacetimes, as well as holography for asymptotically flat spacetimes.

CHAPTER 2

Background literature review

2.1 Spacetime asymptotics

We will begin this thesis with a short review of spacetime asymptotics using the technique of *conformal compactification.* We will begin with the much discussed case of asymptotically flat space times, before moving to de Sitter and anti-de Sitter spacetimes, including the subtleties in their "local" asymptotics.

2.1.1 Asymptotic flatness

Asymptotic flatness may be viewed as the property that the spacetime tends to Minkowski spacetime as $r \to \infty$. This imprecise statement can be given a rigorous definition, which we will briefly touch upon referring to [77, 78] for a detailed discussion, before seeing how to implement asymptotic flatness in a coordinate dependent manner by imposing suitable fall-off conditions upon the metric components.

Let us first recall the notion of conformal compactification [79]. Consider a manifold with boundary $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$ where $\partial\mathcal{M}$ is the boundary. A metric $g_{\mu\nu}$ is *conformally* *compact* if there exists a defining function Ω which satisfies

$$\Omega(\partial\mathcal{M}) = 0, \qquad d\Omega(\partial\mathcal{M}) \neq 0, \qquad \Omega(\mathcal{M}) > 0. \tag{2.1.1}$$

and the metric $\bar{g}$ defined by

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \tag{2.1.2}$$

extends smoothly to $\partial\mathcal{M}$. Let us also consider another spacetime $(\tilde{\mathcal{M}}, \tilde{g})$, which we will refer to as the *unphysical spacetime*, and an embedding $f : \mathcal{M} \to \tilde{\mathcal{M}}$ such that f embeds $\mathcal{M}$ as a manifold with smooth boundary $\partial\mathcal{M}$ in $\tilde{\mathcal{M}}$ and such that

$$\bar{g} = f_*(\tilde{g}), \tag{2.1.3}$$

where $f_*(\tilde{g})$ denotes "push-forward" of the the metric $\tilde{g}$ with respect to the embedding function f. This embedding procedure is often referred to as the *conformal compactification* of the spacetime $(\mathcal{M}, g)$ and $\partial\mathcal{M}$ is the *conformal boundary* of the spacetime.

Asymptotic flatness is now defined by putting further conditions on the conformal compactification. Different definitions have been proposed through the years, see [77, 78] (and references therein). The precise details also depend on whether one would like to consider asymptotic flatness at spatial infinity, null infinity or both. We will not need these details here. For our purposes it suffices to say that we will consider cases with $R_{\mu\nu} = 0$ in an open neighbourhood of $\partial\mathcal{M}$ in $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$[1].

2.1.2 Anti-de Sitter and de Sitter asymptotics

Let us now consider spacetimes that satisfy the Einstein field equations with $\Lambda \neq 0$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2.1.4}$$

We will focus mainly on the case of anti-de Sitter asymptotics ($\Lambda < 0$) although the discussion generalises straightforwardly to the de Sitter case ($\Lambda > 0$). Throughout this thesis we will concentrate on vacuum spacetimes, i.e. $T_{\mu\nu} = 0$ (the generalisation to include matter is conceptually straightforward).

AdS_4 is the maximally symmetric solution to the vacuum Einstein equations with negative cosmological constant. The AdS_4 metric can be written in coordinates with an outgoing null time ,u, as

$$ds^2_{AdS} = -\left(1 + \frac{r^2}{l^2}\right) du^2 - 2du dr + r^2 d\Omega^2 \tag{2.1.5}$$

[1] Such conformal compactification is called *asymptotically empty*.

where $l^2 = -3/\Lambda$; l is the *AdS radius* or *curvature radius* of the spacetime as the Riemann tensor for AdS_4 takes the form

$$R_{\mu\nu\rho\sigma} = \frac{1}{l^2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}). \tag{2.1.6}$$

We define an *asymptotically locally AdS* metric to be a *conformally compact Einstein metric* of negative cosmological constant. In what follows we briefly review the key features relevant for this thesis, see [59, 80, 81, 62] for more details. Consider a manifold with boundary $\bar{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$, equipped with a conformally compact metric $g_{\mu\nu}$, as in (2.1.1)-(2.1.2). We further require that

$$g_{(0)} = \bar{g}|_{\partial\mathcal{M}} \tag{2.1.7}$$

is non-degenerate. Note that $g_{(0)}$ is not unique since the choice of defining function is non-unique: if Ω is a suitable defining function, then so is Ωe^w, where w is a function with no zeroes or poles on $\partial\mathcal{M}$. Thus the induced metric at $\partial\mathcal{M}$, $g_{(0)}$, is also non-unique. This procedure defines a conformal class of metric and $g_{(0)}$ is a representative of the conformal class of metrics.

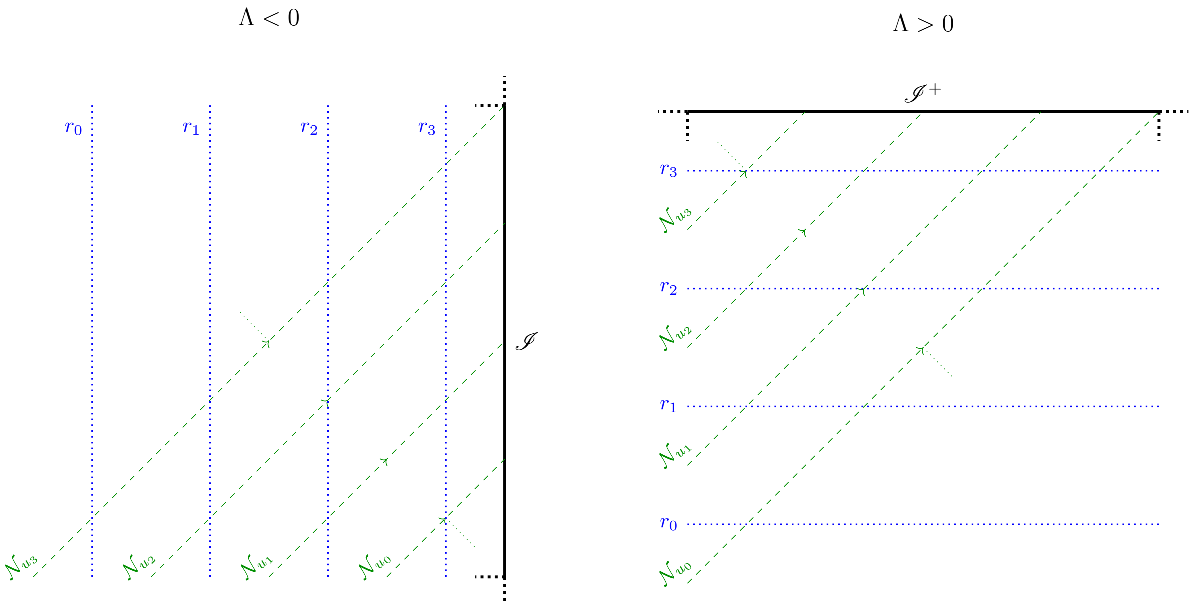

Figure2.1.1: *Left panel.* Penrose diagram of the asymptotic region of an asymptotically locally AdS spacetime, where the *timelike* boundary manifold $\partial\mathcal{M}$ is denoted $\mathscr{I}$. The dashed green curves represent null hypersurfaces $\mathcal{N}_{u_i} = \{u = u_i \,|\, u_i = \text{constant}\}$ and the dotted blue curves timelike surfaces of constant r. *Right panel.* Penrose diagram of the asymptotic region of an asymptotically locally dS space-time, where we have chosen to foliate the future spacelike boundary $\partial\mathcal{M} = \mathscr{I}^+$. The dashed green curves represent null hypersurfaces $\mathcal{N}_{u_i} = \{u = u_i \,|\, u_i = \text{constant}\}$ and the dotted blue curves spacelike surfaces of constant r. The difference in the properties of constant r surfaces between AdS (timelike) and dS (spacelike) is due to the presence of a cosmological horizon in asymptotically locally dS spacetimes.

Using (2.1.2) the Riemann tensor of $g_{\mu\nu}$ takes the form

$$R_{\alpha\beta\gamma\delta}[g] = |d\Omega|^2_{\bar{g}}(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}) + \mathcal{O}(\Omega^{-3}) \tag{2.1.8}$$

where the leading order term is $\mathcal{O}(\Omega^{-4})$. One can also define

$$|d\Omega|^2_{\bar{g}} = \bar{g}^{\mu\nu}(\partial_\mu\Omega)(\partial_\nu\Omega), \tag{2.1.9}$$

a quantity which smoothly extends to $\bar{\mathcal{M}}$; its restriction to $\partial\mathcal{M}$ is a conformal invariant [62]. The metric $g_{\mu\nu}$ should be Einstein, *i.e.* it should satisfy (2.1.4). As in the asymptotically flat case, $\Omega^{-1}T_{\mu\nu}$ should have a smooth conformal completion to $\partial\mathcal{M} = \{\Omega = 0\}$. Enforcing (2.1.4) upon (2.1.8) gives

$$3g_{\mu\nu}|d\Omega|^2_{\bar{g}} + \Lambda g_{\mu\nu} + \mathcal{O}(\Omega^{-1}) = 8\pi T_{\mu\nu} \tag{2.1.10}$$

and thus as $\Omega \to 0$ (after rearrangement)

$$|d\Omega|^2_{\bar{g}}|_{\partial\mathcal{M}} = \frac{1}{l^2}. \tag{2.1.11}$$

Thus near the boundary $\partial\mathcal{M}$, the Riemann curvature tensor of the metric $g_{\mu\nu}$ is to leading order the same as that of the AdS_4 metric.

We emphasise that this definition does not enforce any restriction on the topology of $\partial\mathcal{M}$ or the metric $g_{(0)}$ induced at $\partial\mathcal{M}$. For global AdS_4 the conformal boundary has the topology of $\mathbb{R} \times S^2$ and the metric $g_{(0)}$ is conformally flat. Asymptotically locally AdS spacetimes for which $g_{(0)}$ is conformally flat are called *asymptotically AdS spacetimes.* (Thus asymptotically AdS spacetimes are a subset of asymptotically locally AdS spacetimes). Holographically $g_{(0)}$ corresponds to conformal class of the background metric for the dual quantum field theory and it is thus essential to consider generic $g_{(0)}$, even if one is only interested in a CFT on a flat background. This is because $g_{(0)}$ acts as the source for the holographic stress tensor, and thus we will need to keep it generic in order to compute correlation functions. We will see more explicitly how this relationship works in section 2.3.4.2.

The discussion of asymptotically locally AdS metrics extends to the case of a positive cosmological constant in a very straightforward manner. The Einstein equations for asymptotically locally dS spacetimes are related to those of AdS via the simple transformation

$$l^2_{AdS} \to -l^2_{dS} \tag{2.1.12}$$

and thus to define dS asymptotics, one simply repeats (2.1.4)-(2.1.11) with every occurrence of l^2 being replaced by $-l^2$.

In preparation for the discussion in the next section we indicate in figure 2.1.1 how asymptotically locally $\Lambda \neq 0$ spacetimes are locally foliated by null hypersurfaces.

2.2 Bondi gauge metrics

We begin this section with an introduction to the Bondi gauge and explain its advantages in studying asymptotically flat spacetimes. We will then provide a short review of the asymptotic symmetry group of asymptotically flat spacetimes, the BMS group.

2.2.1 Null hypersurfaces

Bondi gauge metrics were introduced and studied in [1, 2] in the context of studying gravitational waves. The Bondi approach involves foliating the spacetime manifold by null hypersurfaces. Following [2], one chooses the coordinate system as follows. Consider a Lorentzian 4-manifold, $\mathcal{M}$, equipped with a metric $g_{\mu\nu}(x^\rho)$ of signature $(-+++)$ and assume the existence of a scalar field $F = F(x^\mu)$ such that the normal co-vector to F, $\partial_\mu F$, is null:

$$g^{\mu\nu}(\partial_\mu F)(\partial_\nu F) = 0. \tag{2.2.1}$$

This criterion means null hypersurfaces, $\mathcal{N}_a$, can be described in terms of the level sets of F i.e.

$$\mathcal{N}_a = \{x^\mu \in \mathcal{M} \,|\, F(x^\mu) = a\} \tag{2.2.2}$$

and the spacetime $(\mathcal{M}, g_{\mu\nu})$ can be foliated, at least locally, using the null hypersurfaces, namely

$$\mathcal{M} = \{\mathcal{N}_a \,|\, a \in \mathrm{Range}(F)\} \tag{2.2.3}$$

where Range(F) denotes all possible values of the function F.

The motivation for choosing null hypersurfaces can best be illustrated by looking at their interesting geometrical properties. Let us consider an arbitrary surface $\mathcal{N}_a \subset \mathcal{M}$ and the integral curves in the spacetime of the vector field $t^\mu = g^{\mu\nu}\partial_\nu F$; such curves are clearly null and normal to $\mathcal{N}_a$ and are commonly referred to as null rays. Null rays are also geodesic curves contained within $\mathcal{N}_a$:

$$t^\mu \nabla_\mu t^\nu = \lambda(x^\rho) t^\nu. \tag{2.2.4}$$

By choosing a suitable (affine) parametrisation we can set $\lambda = 0$ and thus the null rays are also null generators of $\mathcal{N}_a$. This outlines the overall picture of this procedure as being a way to work from space-time $\to$ null hypersurface $\to$ null ray $\to$ null geodesic.

An adapted coordinate system can be chosen to describe such a situation. Typically, one works in *retarded Bondi coordinates* $(u, r, \Theta^1, \Theta^2)$. The coordinate u is a retarded time coordinate which labels the null hypersurfaces $\mathcal{N}_a$ ($u = F$ from above equations); this coordinate is commonly referred to as the *Bondi time* and takes values in $\mathbb{R}$. The Θ^A are angular coordinates which are defined to be constant along null rays:

$$t^\mu \partial_\mu \Theta^1 = t^\mu \partial_\mu \Theta^2 = 0. \tag{2.2.5}$$

This condition means that rays take the form $c^\mu(\lambda) = (u_0, r(\lambda), \Theta_0^1, \Theta_0^2)$ and thus the coordinate r can be interpreted as a radial distance coordinate measuring the distance along a null ray. This setup is displayed on a Penrose diagram in figure 2.2.1.

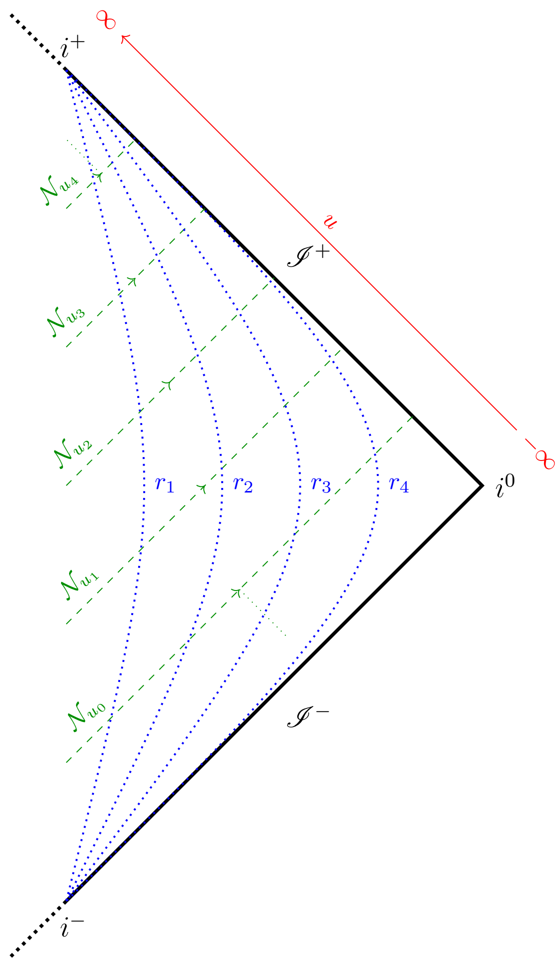

Figure2.2.1: Penrose diagram of null hypersurfaces, $\mathcal{N}_{u_i}$, foliating future null infinity, $\mathscr{I}^+$, of an asymptotically flat spacetime. As indicated by the solid red axis, the retarded time coordinate u ranges from $(-\infty, \infty)$ along $\mathscr{I}^+$ and thus the dashed green lines represent the $u =$ constant hypersurfaces. (The arrows show the direction of increasing radial coordinate r). The dotted blue curves represent timelike hypersurfaces of constant r.

2.2.2 The Bondi gauge

Following closely the notation of [30] (see also the reviews [82, 83]), the most general line element that satisfies the previously discussed coordinate conditions is

$$ds^2 = -Xdu^2 - 2e^{2\beta}dudr + h_{AB}\left(d\Theta^A + \frac{1}{2}U^A du\right)\left(d\Theta^B + \frac{1}{2}U^B du\right). \tag{2.2.6}$$

It is usual to impose in addition the following four gauge conditions:

$$\partial_r \det\left(\frac{h_{AB}}{r^2}\right) = 0, \qquad g_{rr} = g_{rA} = 0. \tag{2.2.7}$$

This metric together with the gauge conditions is known as the *Bondi gauge* (or Bondi-Sachs gauge) and any spacetime metric can be locally written in this form. It is the most commonly used approach to analyse foliations by null hypersurfaces, although there are alternative approaches based on the Newman-Penrose formalism [84] using a null tetrad instead of a metric e.g. [13].

The capital Roman indices A, B take values $\{1, 2\}$ which together with the symmetry of h_{AB}, gives seven functions in the line element: (X, β, h_{AB}, U^A), all of which depend upon the spacetime coordinates $(u, r, \Theta^1, \Theta^2)$. The gauge condition on the determinant of h_{AB} reduces the number of unknown functions in the metric to six. The latter are determined by the Einstein equations, subject to asymptotic data ($r \to \infty$).

One may choose to retain general covariance in the angular coordinates as in [12] but it is often useful to consider a local choice. In this thesis we will commonly utilise the usual (θ, ϕ) of the spherical coordinate system as well as complex coordinates $(\zeta, \bar{\zeta})$, related by

$$\zeta = e^{i\phi}\cot\left(\frac{\theta}{2}\right), \qquad \bar{\zeta} = e^{-i\phi}\cot\left(\frac{\theta}{2}\right). \tag{2.2.8}$$

2.2.3 Asymptotic flatness in the Bondi gauge

Let us now implement asymptotic flatness in a coordinate dependent manner. The Minkowski metric in *retarded* coordinates $(u, r, \zeta, \bar{\zeta})$ is given by

$$ds_M^2 = -du^2 - 2dudr + 2r^2\gamma_{\zeta\bar{\zeta}}d\zeta d\bar{\zeta} \tag{2.2.9}$$

where

$$u = t - r, \qquad \gamma_{\zeta\bar{\zeta}} = \frac{2}{(1+\zeta\bar{\zeta})^2}. \tag{2.2.10}$$

Here u is a retarded time coordinate and $\gamma_{\zeta\bar{\zeta}}$ is the round metric on S^2. This metric is in Bondi gauge with function choices $h_{\zeta\zeta} = h_{\bar{\zeta}\bar{\zeta}} = \beta = U^A = 0$, $X = 1$, $h_{\zeta\bar{\zeta}} = r^2\gamma_{\zeta\bar{\zeta}}$. Note that this choice of coordinates is suitable for analysis near $\mathscr{I}^+$. To analyse neighbourhoods of $\mathscr{I}^-$ the metric can be expressed in *advanced* coordinates $(v, r, \zeta, \bar{\zeta})$ where $v = t + r$ and thus

$$ds_M^2 = -dv^2 + 2dvdr + 2r^2\gamma_{\zeta\bar{\zeta}}d\zeta d\bar{\zeta}. \tag{2.2.11}$$

For this thesis, we will use retarded coordinates and thus restrict our attention to neighbourhoods of $\mathscr{I}^+$.

For a general asymptotically flat metric the metric functions admit power series expansions in $1/r$ with the leading order term being that of the Minkowski metric, as we will re-derive here. The review [30] discusses suitable fall-off conditions for the subleading terms in the series: the fall-off should include gravitational wave emitting solutions, as was the motivation in [1]. These criteria were imposed in [1, 2] and if we combine this with the following fall-off of the Weyl curvature tensor components at large r

$$C_{r\zeta r\zeta} \sim O(r^{-3}), \quad C_{rur\zeta} \sim O(r^{-3}), \quad C_{rur\bar{\zeta}} \sim O(r^{-3}) \tag{2.2.12}$$

as in [30] then we obtain the class of asymptotically flat metrics in Bondi gauge as

$$\begin{aligned} ds^2 = & ds_M^2 + \frac{2m_B}{r}du^2 + rC_{\zeta\zeta}d\zeta^2 + rC_{\bar{\zeta}\bar{\zeta}}d\bar{\zeta}^2 + D^{\zeta}C_{\zeta\zeta}dud\bar{\zeta} + D^{\bar{\zeta}}C_{\bar{\zeta}\bar{\zeta}}dud\bar{\zeta} \\ & + \frac{1}{r}\left(\frac{4}{3}(N_\zeta + u\partial_\zeta m_B - \frac{1}{4}\partial_\zeta(C_{\zeta\zeta}C^{\zeta\zeta})\right)dud\zeta + c.c. + \dots \end{aligned} \tag{2.2.13}$$

where D_A is the covariant derivative with respect to the metric of the round sphere γ_{AB} and the first term in the equation is just the Minkowski metric. The rest of the terms in the first line are the first order subleading terms in powers of r. Notice that although these terms have different powers of r preceding them, they are all subleading as $r \to \infty$ when compared to the Minkowski metric. The second line of the equation contains second order subleading terms, included here as these terms contain physically interesting functions.

At $\mathcal{O}(1/r)$ in g_{uu} is a function $m_B = m_B(u, \zeta, \bar{\zeta})$ is known as the *Bondi mass aspect.* One of the key results of [1] is that the Bondi mass aspect can be integreated over the unit S^2 to give the total *Bondi mass*[2] $\mathcal{M}_B$ of the system at time u

$$\mathcal{M}_B = \frac{1}{4\pi}\int_{S^2} m_B = \frac{1}{4\pi}\int d^2z\,\gamma_{\zeta\bar{\zeta}}\,m_B. \tag{2.2.14}$$

The Bondi mass is a natural way to define the mass of a system at $\mathscr{I}^+$, and is an alternative to the ADM mass which is defined as an integral at spatial infinity i^0.

[2] This quantity is sometimes referred to in the literature as the *Trautman-Bondi* mass, as it was also discussed by Trautman in [85], see also the lecture notes [86] for further comments and references.

Contained in the $1/r$ suppressed terms relative to the Minkowski metric is the shear tensor $C_{AB}(u,\zeta,\bar{\zeta})$; a symmetric and traceless tensor of type $[0,2]$. This tensor describes the gravitational waves in the spacetime (recall we wanted the fall-off conditions to include these solutions) and it motivates the definition of another key concept in the Bondi gauge, the *Bondi news tensor*, N_{AB},

$$N_{AB}(u,\zeta,\bar{\zeta}) = \partial_u C_{AB}(u,\zeta,\bar{\zeta}). \tag{2.2.15}$$

The news tensor is again a symmetric and traceless tensor of type $[0,2]$. The name "news" for this tensor can be best explained by imposing Einstein's equations (with $\Lambda = 0$) upon the metric

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \qquad \lim_{r\to\infty} T_{\mu\nu} = 0 \tag{2.2.16}$$

where the limit condition on the stress energy tensor is typically enforced such that $\Omega^{-1}T_{ab}$ has a smooth conformal completion to $\mathscr{I}^+ = \{\Omega = 0\}$ (where $\Omega \sim 1/r$ for the case at hand). This condition is a requirement for asymptotic flatness as it forces the asymptotically empty condition mentioned above. The authors of [1] solved the field equations by expanding in large r and solving the equations that arise at each order and we will streamline this derivation here. The leading order ($\mathcal{O}(r^{-2})$) of the (uu) component of the Einstein equations then reads (see discussions in [12, 30])

$$\partial_u m_B = \frac{1}{4}[D_\zeta^2 N^{\zeta\zeta} + D_{\bar{\zeta}}^2 N^{\bar{\zeta}\bar{\zeta}} - N_{\zeta\zeta}N^{\zeta\zeta}] - 4\pi \lim_{r\to\infty} r^2 T_{uu}. \tag{2.2.17}$$

Thus the news tensor, along with the stress tensor, governs the change in the Bondi mass aspect - it provides the "news" regarding the change in the mass aspect. If the spacetime under consideration is vacuum (as in [1]) then the news entirely governs the change in mass.

The final interesting term is the N^A which appears in the subleading terms in the second line of (2.2.13). This vector is named the *angular momentum aspect* and - in a similar fashion to the mass aspect - can be used to define the total angular momentum at $\mathscr{I}^+$ via a suitable integral. Both the mass aspect and angular momentum aspect arise as functions of integration in the full set of Einstein field equations, although the field equations do contain evolution equations for these [87, 7, 5, 12] which we will discuss in detail later.

Comparing the general Bondi gauge metric with the asymptotically flat metric, the fall-off conditions on the metric functions are

$$\begin{aligned} X &= 1 - \frac{2m_B}{r} + O(r^{-2}), \qquad \beta = O(r^{-2}), \\ g_{AB} &= r^2\gamma_{AB} + rC_{AB} + O(1), \qquad U_A = \frac{1}{r^2}D^B C_{AB} + O(r^{-3}). \end{aligned} \tag{2.2.18}$$

The infinite dimensional symmetry group of all coordinate transformations that preserves these conditions as well as the gauge itself is known as the BMS group [3].

2.2.4 Asymptotic symmetries

In this section we will derive the BMS group in the style most commonly seen in modern work concerning asymptotic symmetries and their applications (see for example the review articles [30, 82, 83]). This calculation will lead us to the result that the asymptotic symmetry group of asymptotically flat spacetimes is infinite dimensional and consists of the standard Lorentz transformations as well as an infinite number of "supertranslations", as was first discovered in [1, 2]. We will then briefly discuss how one can extend the group even further to include an infinite number of "superrotations" before commenting on some of the more modern work and applications of the BMS group.

As mentioned at the end of the previous section, we define the BMS group as the set of coordinate transformations which preserve both the Bondi gauge (2.2.6)-(2.2.7) as well as the $r \to \infty$ falloffs of (2.2.18). In order to identify these, we consider transformations generated by vector fields ξ^μ such that

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\alpha\nu}\partial_\mu \xi^\alpha \tag{2.2.19}$$

and now we must solve the differential equations which result from wanting to preserve the boundary conditions. In order to preserve Bondi gauge (2.2.7), we have the exact equations

$$\mathcal{L}_\xi g_{rr} = 0, \qquad \mathcal{L}_\xi g_{rA} = 0, \qquad g^{AB}\mathcal{L}_\xi g_{AB} = 4\omega(u, x^A) \tag{2.2.20}$$

where the final equation comes from the need to preserve the determinant condition in (2.2.7) and the factor of 4 is just for convenience [50]. We begin by solving these equations one by one for ξ^μ, explicitly we have

$$\mathcal{L}_\xi g_{rr} = 0 \implies 2g_{ru}\partial_r \xi^u = 0 \implies \xi^u = f(u, x^A) \tag{2.2.21}$$

and thus $\xi^\mu \partial_\mu = f\partial_u + \xi^r(u, r, x^B)\partial_r + \xi^A(u, r, x^B)\partial_A$. The next equation gives us

$$\mathcal{L}_\xi g_{rA} = 0 \implies -e^{2\beta}\partial_A f + g_{AB}\partial_r \xi^B = 0 \tag{2.2.22}$$

where we recall that $\beta(u, r, x^A)$ is one of the functions appearing in the Bondi gauge line element (2.2.6). Integrating this equation with respect to r gives us

$$\xi^A = Y^A(u, x^B) + I^A(u, r, x^B), \quad I^A = -\partial_B f \int_r^\infty dr'(e^{2\beta} g^{AB}) \tag{2.2.23}$$

and thus $\xi^\mu\partial_\mu = f\partial_u + \xi^r(u,r,x^B)\partial_r + (Y^A + I^A)\partial_A$. The third equation gives

$$\begin{aligned} & g^{AB}\mathcal{L}_\xi g_{AB} = 4\omega(u,x^A) \\ \implies & \xi^r g^{-1}\partial_r g + f g^{-1}\partial_u g + g^{AB}\mathcal{L}_{(Y^C+I^C)}g_{AB} + 2g^{AB}g_{Au}\partial_B f = 4\omega \\ \implies & \xi^r = -\frac{r}{2}\left(\mathcal{D}_A Y^A + \mathcal{D}_A I^A - 2\omega + \frac{1}{2}U^B\partial_B f + \frac{1}{2}fg^{-1}\partial_u g\right) \end{aligned} \tag{2.2.24}$$

where $g = \det(g_{AB})$ and $\mathcal{D}_A$ is the covariant derivative operator on g_{AB}. Putting this all together, we have derived the most generic coordinate transformations preserving Bondi gauge.

$$\begin{aligned} \xi^\mu\partial_\mu = f\partial_u + (Y^A + I^A)\partial_A - \\ \frac{r}{2}\left(\mathcal{D}_A Y^A + \mathcal{D}_A I^A - 2\omega + \frac{1}{2}U^B\partial_B f + \frac{1}{2}fg^{-1}\partial_u g\right)\partial_r \end{aligned} \tag{2.2.25}$$

notice that we have not had to enforce any asymptotic conditions yet. These vectors have 4 undetermined functions of u and x^A, (f, Y^A, ω), all of which are completely unconstrained at this point.

In order to complete our derivation of the BMS group, we must also solve the asymptotic Killing equation in order to find the vectors which preserve not only the Bondi gauge (2.2.6)-(2.2.7), but the asymptotic fall off conditions (2.2.18) as well. The fall off conditions give us the asymptotic Killing equations

$$\mathcal{L}_\xi g_{uu} = \mathcal{O}(r^{-1}), \quad \mathcal{L}_\xi g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_\xi g_{uA} = \mathcal{O}(1), \quad \mathcal{L}_\xi g_{AB} = \mathcal{O}(r) \tag{2.2.26}$$

which we will shortly solve for the components of the metric. Before we do this, we note that the fall off conditions on g_{AB} in (2.2.18) together with $\partial_r(g/r^4) = 0$ give us

$$g = \det(r^2\gamma_{AB} + rC_{AB} + \ldots) = r^4\det\left(\gamma_{AB} + \frac{C_{AB}}{r} + \ldots\right) = r^4\gamma \tag{2.2.27}$$

where $\gamma = \det(\gamma_{AB})$ is the determinant of the round unit S^2. We note that this determinant condition imposes constraints on the sub leading terms in the metric expansion on the angular part of the metric, the first of which is $C^A_A = 0$ (indices raised and lowered with γ_{AB}). For a list of the higher order constraints, we point the reader to [50].

As the leading order of g_{AB} is preserved, we thus necessarily have $\omega = \partial_u g = 0$. We also note

$$\mathcal{D}_A\xi^A = \frac{1}{\sqrt{g}}\partial_A(\sqrt{g}\xi^A) = \frac{1}{\sqrt{\gamma}}\partial_A(\sqrt{\gamma}\xi^A) = D_A\xi^A \tag{2.2.28}$$

which immediately simplifies the generic form of a BMS vector field to be

$$\xi^\mu\partial_\mu = f\partial_u + (Y^A + I^A)\partial_A - \frac{r}{2}\left(D_A Y^A + D_A I^A + \frac{1}{2}U^B\partial_B f\right)\partial_r. \tag{2.2.29}$$

Using this general form, we now solve the asymptotic Killing equations (2.2.26). The vanishing of the $\mathcal{O}(r)$ term of the first equation gives us

$$\partial_u D_A Y^A = 0 \implies D_A Y^A = h(x^B) \implies Y^A = Y^A(x^B) \tag{2.2.30}$$

and we will return to examine the $\mathcal{O}(1)$ equation shortly. We now move to the second equation of (2.2.26), whose $\mathcal{O}(1)$ term gives the equation

$$\partial_u f - \frac{D_A Y^A}{2} = 0 \implies f = T(x^A) + u\frac{D_A Y^A}{2} \tag{2.2.31}$$

which in itself is sufficent to solve the $\{ur\}$ component of the asymptotic Killing equation to the required order of $\mathcal{O}(r^{-2})$. This changes the form of the BMS vector field to be

$$\begin{aligned}\xi^\mu\partial_\mu = &\left(T + u\frac{D_A Y^A}{2} + \mathcal{O}(r^{-1})\right)\partial_u + \\ &\left(Y^A - \frac{1}{r}D^A(T + \frac{u}{2}D_B Y^B) + \mathcal{O}(r^{-2})\right)\partial_A - \\ &\frac{r}{2}\left(D_A Y^A - \frac{1}{r}D^2(T + \frac{u}{2}D_B Y^B) + \mathcal{O}(r^{-2})\right)\partial_r.\end{aligned} \tag{2.2.32}$$

The next step is to examine the fourth equation of (2.2.26). The vanishing of the $\mathcal{O}(r^2)$ term gives the equation

$$\mathcal{L}_Y \gamma_{AB} = \gamma_{AB} D_C Y^C \tag{2.2.33}$$

which is nothing other than the conformal Killing equation on S^2, enforcing the additional constraint that Y^A is a conformal Killing vector of S^2. We note the following identity for conformal Killing vectors on S^2

$$D^2 D_A Y^A \equiv -2 D_A Y^A \tag{2.2.34}$$

i.e. that $D_A Y^A$ are the $l = 1$ harmonics. To prove this identity, we first take the contracted covariant derivative of the conformal Killing equation (2.2.33) and rearrange to obtain

$$[D_A, D_B]Y^B = D^2 Y_A \tag{2.2.35}$$

then using the definition of the Riemann/Ricci tensor as well as the fact that $R_{AB} = \gamma_{AB}$ for the unit round S^2, we have

$$-Y_A = D^2 Y_A \tag{2.2.36}$$

and now taking another contracted covariant derivative and rearranging gives

$$\begin{aligned}-D_A Y^A &= D^A D^2 Y_A \\ &= [D^A, D^2]Y_A + D^2 D^A Y_A \\ &= D_A Y^A + D^2 D^A Y_A\end{aligned} \tag{2.2.37}$$

which completes the proof of (2.2.34). This result is necessary and sufficient to enforce the vanishing of the $\mathcal{O}(1)$ piece of $\mathcal{L}_\xi g_{uu}$ and thus we have solved all of the asymptotic Killing equations (2.2.26) to sufficient order. We write the asymptotic Killing vectors as

$$\begin{aligned}\xi^\mu\partial_\mu = &\left[T + u\frac{D_A Y^A}{2} + \mathcal{O}(r^{-1})\right]\partial_u + \\ &\left[Y^A - \frac{1}{r}D^A\left(T + \frac{u}{2}D_B Y^B\right) + \mathcal{O}(r^{-2})\right]\partial_A - \\ &\frac{r}{2}\left[D_A Y^A\left(1 + \frac{u}{r}\right) - \frac{1}{r}D^2 T + \mathcal{O}(r^{-2})\right]\partial_r.\end{aligned} \tag{2.2.38}$$

Now that we have derived the generic form of the asymptotic symmetry vector fields ξ, we will make a number of comments in order to illuminate the structure of the group.

- Notice that the function $T(x^A)$ is completely unconstrained by the asymptotic Killing equation (2.2.26), giving rise to infinitely many asymptotic Killing vectors ξ^μ defined by arbitrary angular functions $T(x^A)$. These are referred to as the *supertranslations* as they generalise the translations of the Poincaré group. If one restricts $T(x^A)$ to being the $l = 0, 1$ spherical harmonics then one reproduces the four Poincaré translations [82].

- The vector field $Y^A(x^B)$ is constrained by (2.2.33) to be a conformal Killing vector on S^2. If we write (2.2.33) in the coordinates $(\zeta, \bar{\zeta})$ as used in (2.2.9) then the equation reduces to the standard holomorphicity conditions

$$\begin{aligned}\partial_\zeta Y^{\bar{\zeta}} = 0 &\implies Y^{\bar{\zeta}} = \bar{Y}(\bar{\zeta}) = \sum_{k\in\mathbb{Z}} \bar{a}_k \bar{\zeta}^k \\ \partial_{\bar{\zeta}} Y^{\zeta} = 0 &\implies Y^{\zeta} = Y(\zeta) = \sum_{k\in\mathbb{Z}} a_k \zeta^k\end{aligned} \tag{2.2.39}$$

  and in order to restrict to vector fields with no singularities, one now sets $a_k = \bar{a}_k = 0$ for all $k \notin \{0, 1, 2\}$. This gives six globally well defined conformal Killing vectors of S^2, which generate exactly the Lorentz group $SO(3,1)$

- Putting these two sources of symmetry together, we have the BMS_4 group:

$$\mathrm{BMS}_4 = SO(3,1) \rtimes \text{supertranslations} \tag{2.2.40}$$

  where the semidirect product $\rtimes$ is due to the algebraic relations satisifed by the Lie bracket of vector fields in the group, [82]. This will not be important for our discussion.

- There is a common extension of the BMS_4 group, referred to as the *extended* BMS_4

group [82], which arises from dropping the restriction that the conformal Killing vectors Y^A have no singularities on S^2. Instead one allows for meromorphic vector fields, in this case those with singularities at the poles of the sphere [88, 87]. This is a natural thing to do from the point of view of two dimensional conformal field theory [89, 90, 30] and as in that case, one now finds an infinite number of conformal Killing vectors of the form $Y^\zeta = \zeta^k, \forall\, k \in \mathbb{Z}$ (and equivalently $Y^{\bar{\zeta}} = \bar{\zeta}^k$ for the anti-holomorphic component). These transformations typically gain the name *superrotations* as they extend the rotations of the Lorentz group.

- The structure of extended BMS_4 is:

$$\text{extended BMS}_4 = \text{superrotations} \rtimes \text{supertranslations} \tag{2.2.41}$$

which follows the style pioneered by Barnich & Troessaert. See [9, 87, 10, 11, 13] for a much deeper discussion of these transformations, as well as their charges and algebra. The singular nature of these transformations has been related to the presence of cosmic strings singularities in the spacetime [20, 21, 22] and thus they do appear to be physically motivated.

We will now conclude this subsection by pointing the reader to some of the literature related to the BMS group its physical applications (much of which is also pointed out in the review papers [30, 82, 91, 83]). We first point out that the asymptotic field expansions (2.2.18) that we used to derive the BMS group, namely analytic expansions in inverse powers of the Bondi radial coordinate r, are not the most general asymptotic solutions to the field equations. In particular, one can consider *polyhomogeneous* solutions to the field equations, where powers of $r \log r$ are also allowed in the asymptotic expansions of the metric coefficients. These solutions have been studied in depth in [34, 92, 93, 35, 36, 37, 94, 38, 39, 40] and one still finds the asymptotic symmetry structure of the BMS_4 group. In the next chapter we will show that polyhomogeneous solutions are absent in $\mathrm{AL(A)dS}_4$ spacetimes and comment upon our findings.

In addition to the discussion of enlarging the BMS_4 group to include the superrotations, an even larger extension has been proposed to extend the Y^A to include all diffeomorphisms of S^2 [95, 96]. Such an extension to the BMS group was motivated by the discovery of a new subleading soft graviton theorem of Cachazo and Strominger [97] and the extension of the BMS group to include $\mathrm{diff}(S^2)$ was necessary in order to match the Ward identities with the soft theorem. It remains an interesting problem to understand both the mathematics of this extension, as well as whether the associated charges can be obtained from covariant phase space methods [98, 99].

In our derivation of the BMS_4 group, we were explicitly considering 4-dimensional spacetime manifolds. Work has also been done to understand the structure of the BMS group in three spacetime dimensions [100, 101, 102] as well as five [103, 23] and higher [104]. Different dimensions have a number of subtleties which we will not discuss in this thesis as we will always be considering a four dimensional spacetime.

A major project in understanding the physical implications of the infinite number of transformations in the BMS group is underway, having been pioneered by Strominger et al. [7, 105, 97, 106, 4, 5]. These works aim to discover the relationship between the asymptotic symmetries (as we have discussed), gravitational wave memory effects [107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118] and soft theorems in quantum field theory [119]. This research programme has been nicknamed the “infrared triangle” due to the aim of connecting these different branches of IR physics. Much work has been done in this area by various groups, see for example [95, 96, 120, 121, 122, 123, 124, 125, 118], as well as works which attempt to extend the understanding of these relationships to higher dimensions [126, 127, 128, 129, 130, 6, 131].

Finally we comment that as the BMS group is an asymptotic symmetry group, one expects to be able to compute asymptotic charges corresponding to elements of the group and attempt to understand the meaning of such quantities. The charges of the extended BMS group have been computed [12] and the physics of such quantities is conjectured to be related to a possible resolution of the black hole information loss paradox [28]. This conjecture states that the “lost” information during the black hole evaporation is actually contained in BMS charges, named the *soft hair* of the black hole [24, 25, 132, 26]. With regard to how one computes these charges, we will introduce a method to compute conserved quantities in section 2.4. We will also show how this can be modified to derive the Bondi mass loss (and other BMS charges) in section 2.4.3.

2.3 Holography

In this section we will introduce another major topic which the work in this thesis will make great use of *the anti-de Sitter/conformal field theory correspondence*, which we will refer to as the AdS/CFT correspondence from here on. The main aspect of this vast topic that we will need is that of the procedure of *holographic renormalisation*, which we will introduce in this section and discuss some of the famous results that this procedure allows one to realise. We will give an adaptation of these techniques in the context of de Sitter spacetimes in chapter 5.

2.3.1 The holographic principle

Before discussing the AdS/CFT correspondence, we begin by giving some motivation for the correspondence and show how it has arisen as part of the search for a theory of quantum gravity. The guiding principle in establishing the AdS/CFT correspondence is the *holographic principle* as formulated by 't Hooft [53] and Susskind [54]. This principle is a conjecture upon the nature of quantum theories of gravity which states:

> Any quantum theory of gravity can be described in terms of a theory without gravity in one less (non-compact) dimension.

Such a conjecture arose from the observations made in the study of black hole physics [133, 134, 135, 136, 137, 138, 28], namely that the entropy of a black hole, S_{BH}, is directly proportional to the area, A, of the event horizon expressed in units of Planck area

$$S_{\mathrm{BH}} = \frac{A}{4}, \tag{2.3.1}$$

the much-celebrated Bekenstein-Hawking entropy formula. This formula is used to argue that black holes maximise the entropy in a given volume, going roughly as follows: A black hole is the densest object in nature, i.e. the most massive object in a fixed volume is single black hole (if one adds more mass to the system then the volume must necessarily increase, as the black hole mass, m, satisfies $m \propto r$, where r is the radius of the hole). The argument now proceeds by contradiction, where we assume existence of a system contained in the fixed volume with a greater entropy than the black hole. This system must necessarily contain less energy than the black hole and thus if we add more mass to the system, we would eventually create a black hole which fills the volume. However, in doing this, we would decrease the total entropy of the system, $\delta S < 0$. This goes against the second law of thermodynamics, $\delta S \geq 0$, and thus such a system cannot exist. The black hole is the object which maximises the entropy in a given volume.

The important result of this argument is that the maximal entropy of a system is related to the area, not the volume as one may naïvely expect. In a quantum system, there is a close relationship between the entropy and the dimension of the Hilbert space of the system, $\mathcal{N}$, namely

$$e^S = \mathcal{N} \tag{2.3.2}$$

and thus we may associate the maximal dimension of the Hilbert space with the area of the boundary of the region. The interpretation of this by 't Hooft [53], is that any theory of quantum gravity in the region should be entirely equivalent to a theory living at the boundary of the region, as all degrees of freedom are contained there. This tells us that the fundamental description of the physics in a given volume is described by a quantity of dimension one lower than the volume, a first hint at the “holographic” nature of gravitational theories. As quantum field theories typically have entropy scaling extensively with the spatial volume, the black hole argument suggests that any quantum theory of gravity can be described entirely in terms of a theory with one less spatial dimension.

This conjecture of the holographic principle has a concrete realisation in the class of AlAdS spacetimes (as discussed in section 2.1.2). We will now give a brief introduction to the more precise statements of this AdS/CFT correspondence, before moving on to a more technical discussion of the asymptotic analysis necessary to perform holographic renormalisation.

2.3.2 AdS/CFT correspondence

The first concrete realisation of the holographic principle was discovered by Maldacena in the seminal work [55]. In this work, Maldacena showed numerous examples of holographic duality on AlAdS spacetimes ($\times$ a compact space), inlcuding $\mathrm{AdS}_5 \times S^5$, $\mathrm{AdS}_4 \times S^7$, $\mathrm{AdS}_7 \times S^4$ and $\mathrm{AdS}_3 \times S^3$. In the case of $\mathrm{AdS}_5 \times S^5$, the gravitational theory was type IIB string theory on an $\mathrm{AdS}_5 \times S^5$ and the ‘holographically dual’ quantum theory a 4-dimensional $SU(N)$ gauge theory (Yang-Mills theory) with $\mathcal{N} = 4$ supersymmetry generators. The rough idea as to how this duality manifests itself is that the quantum theory arises as the worldvolume theory of N coincident D3-branes associated with the conformal boundary, $\partial\mathcal{M}$, of the AdS_5 factor of the background.

Many of the details regarding these theories will be unimportant for the work discussed in this thesis, and so we will not go into great detail on topics such as string theory, supersymmetry, D-branes etc. The important fact to note from this discovered duality is that the gravitational theory (ignoring for now the S^5 factor) is an *asymptotically locally AdS spacetime*, and the dual quantum theory is a *conformal field theory* (a theory invariant under the action of the conformal group of transformations). This (and other examples see e.g. [139]) lead to the conjecture of a special sub class of the holographic principle, namely the AdS/CFT correspondence:

> Quantum gravity in a $d+1$ dimensional asymptotically locally anti-de Sitter spacetime is equivalent to a d dimensional conformal field theory.

After the publication of [55], the conjectured correspondence was clarified in the works of Witten [57] and Gubser, Klebanov and Polyakov [56]. The more precise statement of the duality is as follows: Consider a (super)gravity theory on the product of an AlAdS_{d+1} manifold M and a compact manifold $\mathcal{C}$ (in Maldacena’s example that we discussed above $M = \mathrm{AdS}_5$ and $\mathcal{C} = S^5$). The CFT is defined on the conformal boundary ∂M of the

AlAdS manifold, and there is a 1-1 map between primary operators $\mathcal{O}$ of the CFT and fields ϕ in the gravitational theory. The *master formula* of the AdS/CFT correspondence is an conjectured equivalence between the partition function of the gravitational theory, Z_{grav}, and the generating functional of correlation functions in the CFT, Z_{CFT}:

$$Z_{\text{grav}}[\phi^{(0)}] = Z_{\text{CFT}}[\phi^{(0)}] \tag{2.3.3}$$

where $\phi^{(0)} = \phi|_{\partial M}$ constitutes boundary conditions for the fields ϕ. Explicitly one has

$$Z_{\text{grav}}[\phi^{(0)}] = \int_{\phi^{(0)}} \mathcal{D}\phi \, \exp(-S[\phi]) \tag{2.3.4}$$

where the path integral is taken over all field configurations ϕ which satisfy $\phi^{(0)} = \phi|_{\partial M}$ and $S[\phi]$ is the action functional for the gravitational theory. The gravitational theory is generically a quantum theory, thus making the partition function a quantum object (although for the work considered in this thesis we will consider the classical limit). On the right hand side of (2.3.3) we have

$$Z_{\text{CFT}} = \left\langle \exp\left(- \int_{\partial M} d^d x \, \mathcal{O}\phi^{(0)} \right) \right\rangle \tag{2.3.5}$$

and thus one can see that $\phi^{(0)}$ acts as a source for the operator $\mathcal{O}$.

In general, one finds that the full path integral in Z_{grav} is an extremely difficult object to calculate, and thus attention is typically restricted to the tree level term arising in the path integral, namely the classical solutions to the equations of motion. We denote these solutions as ϕ^{cl}, which satisfy the boundary condition $\phi^{\text{cl}}|_{\partial M} = \phi^{(0)}$. The tree level form of the gravitational is

$$Z^{\text{tree}}_{\text{grav}} = \exp(-S[\phi^{\text{cl}}]) \tag{2.3.6}$$

which is often sufficient to use as left hand side of (2.3.3) in order to verify equivalences between AlAdS spacetimes and CFTs, see for example [58].

We will work in this regime and thus we will only be considering classical gravitational theories which admit ALAdS solutions. In particular, we will consider the theory of general relativity in the presence of a cosmological constant, $\Lambda < 0$. The only dynamical field in the theory is the spacetime metric tensor i.e. $\phi = \{g_{\mu\nu}\}$ and the only operator which we will consider in the dual CFT is the stress tensor. The equations of motion for the spacetime metric are those given in (2.1.4), namely

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2.3.7}$$

Regarding boundary conditions for $g_{\mu\nu}$, we recall the discussion in and around equation (2.1.7) which shows that the boundary conditions $\phi^{(0)}$ for $g_{\mu\nu}$ will amount to specifying a given conformal structure $[g_{(0)}]$ at ∂M. This means that we do not have a unique metric specified at ∂M, but rather a metric unique up to conformal transformations. From the perspective of the CFT, the metric at the boundary, $g_{(0)}$, is the background and thus it makes perfect sense for this to only be specified up to a conformal transformation. This is because (by definition) the CFT is invariant under conformal transformations and thus will only have a background metric specified by the conformal class.

We will now discuss the solutions to (2.3.7) in the *asymptotic region* (near ∂M) of the spacetime, before moving on to discuss renormalisation of the classical action, $S[g_{\mu\nu}]$, for AlAdS spacetimes.

2.3.3 Fefferman-Graham gauge

We will begin this section by introducing the coordinates which we will use to study the asymptotic form of the metric. These are typically referred to as Fefferman-Graham coordinates due to the work of these authors [59], where a theorem was proved which states that on any AlAdS manifold there exists a coordinate system (ρ, x^a) such that the asymptotic metric can always be brought into the form

$$ds^2 = \frac{l^2}{\rho^2}\left(d\rho^2 + \hat{g}_{ab}(\rho, x^c)dx^a dx^b\right) \tag{2.3.8}$$

where $l = \sqrt{-d(d-1)/2\Lambda}$ is referred to as the AdS radius of the spacetime. This theorem was proven for metrics of general signature (p, q) where $p+q = d+1$, but we will consider the Riemannian case of signature $(d+1, 0)$ in order to retain consistency throughout section 2.3 (all previous formulae e.g. (2.3.6) have been written using Euclidean signature). We also note that for most of the work in this thesis we will put $d = 3$, but in this subsection we will keep d generic in order to highlight the differences between the odd and even cases.

Following the discussion of section 2.1.2, ρ is a coordinate which describes the location of the conformal boundary, specifically $\mathscr{I} = \{\rho = 0\}$. For this discussion we will take $\rho > 0$ and thus the range of ρ will be $(0, \infty)$ (we will later consider $\rho < 0$ in the context of AldS spacetimes). The lower case Roman indices (a, b) run over all other coordinates in the spacetime and for now we will leave these abstract indices in a covariant form. We will now solve the field equations, following the standard method as discussed in [59, 58, 81, 60].

In order to proceed with solving the field equations (2.3.7), we first compute the components of the equations, making a distinct split between $\{ab\}$, $\{\rho a\}$ and $\{\rho\rho\}$ components. For the $\{ab\}$ component of (2.3.7) we find

$$\rho\partial_\rho^2\hat{g}_{ab} + (1-d)\partial_\rho\hat{g}_{ab} - \hat{g}^{cd}(\partial_\rho\hat{g}_{cd})\hat{g}_{ab} - \rho(\partial_\rho\hat{g}_{ac})\hat{g}^{cd}(\partial_\rho\hat{g}_{db})$$

$$+\frac{\rho}{2}\hat{g}^{cd}(\partial_\rho\hat{g}_{cd})\partial_\rho\hat{g}_{ab} - 2\rho R_{ab}[\hat{g}] = 0 \quad (2.3.9)$$

which will be the 'main' equation that we will use in order to derive the asymptotic solutions to the field equations.[3] In the notation here, $\hat{g}^{ab}$ is the inverse of the metric $\hat{g}_{ab}$ and $R_{ab}[\hat{g}]$ is the Ricci tensor of $\hat{g}_{ab}$. The $\{\rho a\}$ and $\{\rho\rho\}$ components of (2.3.7) respectively take the forms

$$\begin{aligned} \nabla_a[\hat{g}^{bc}(\partial_\rho\hat{g}_{bc})] - \nabla^b(\partial_\rho\hat{g}_{ab}) = 0 \\ \rho\hat{g}^{ab}(\partial_\rho^2\hat{g}_{ab}) - \hat{g}^{ab}\partial_\rho\hat{g}_{ab} - \frac{\rho}{2}\hat{g}^{ab}(\partial_\rho\hat{g}_{bc})\hat{g}^{cd}(\partial_\rho\hat{g}_{da}) = 0. \end{aligned} \quad (2.3.10)$$

where ∇ is the covariant derivative operator associated with the metric at $\hat{g}_{ab}$ at a fixed value of ρ.

In order to solve the field equations, we first assume suitable regularity of the field $\hat{g}_{ab}$, where 'suitable regularity' will amount to requiring that we can take ρ-derivatives of (2.3.9) and evaluate at the resulting equations at $\rho = 0$ i.e. we will not a priori restrict the existence of $\partial_\rho^n\hat{g}_{ab}$ (we will discuss cases where this breaks down during our analysis). We then pick a metric $g_{ab}^{(0)} = \hat{g}_{ab}(\rho = 0)$ on ∂M which acts as a representative of the conformal equivalence class $[g_{(0)}]$. We can then solve the 'main equation' (2.3.9) order by order in ρ, first by differentiating the equation with respect to ρ the requisite number of times and then setting $\rho = 0$. In order to illuminate this process we will work through the first few orders explicitly.

At the first order we differentiate 0 times and set $\rho = 0$ in (2.3.9) to find

$$(1-d)\partial_\rho\hat{g}_{ab} - \hat{g}^{cd}(\partial_\rho\hat{g}_{cd})\hat{g}_{ab} \triangleq 0 \quad (2.3.11)$$

where we use the symbol $\triangleq$ to denote the fact that this equation only holds at $\rho = 0$. Taking the trace of this equation by contracting this with $\hat{g}^{ab}$ we get

$$\hat{g}^{cd}(\partial_\rho\hat{g}_{cd}) \triangleq 0 \quad (2.3.12)$$

which when put back into (2.3.11) gives us

$$\partial_\rho\hat{g}_{ab} \triangleq 0 \quad (2.3.13)$$

the statement that the first ρ derivative of $\hat{g}$ vanishes at the boundary. At the second order we differentiate once (2.3.9) with respect to ρ and then set $\rho = 0$. This gives us

$$\partial_\rho^2\hat{g}_{ab} + (1-d)\partial_\rho^2\hat{g}_{ab} - \hat{g}^{cd}(\partial_\rho^2\hat{g}_{cd})\hat{g}_{ab} - 2R_{ab}[\hat{g}] \triangleq 0 \quad (2.3.14)$$

[3] Our R_{ab} has the opposite sign to that in [60] due to our differences in curvature conventions. We use the conventions of [77].

upon which we perform a similar procedure as before by first taking the trace and then substituting the trace term back into this equation. This procedure results in

$$(2-d)\partial_\rho^2 \hat{g}_{ab} \triangleq \frac{R[\hat{g}]}{1-d}\hat{g}_{ab} + 2R_{ab}[\hat{g}] \tag{2.3.15}$$

and we thus see that (generically) the second ρ derivative of $\hat{g}_{ab}$ does not vanish at the boundary and is locally formally determined by $g^{(0)}_{ab}$. The procedure runs in this fashion order by order and one can compute precise relations for each number of ρ derivatives. We will now provide a general proof of the functional form of the higher order terms rather than evaluating them explicitly.

If we consider the order at which we differentiate (2.3.9) $\nu - 1$ times and set $\rho = 0$ we find

$$(\nu - d)\partial_\rho^\nu \hat{g}_{ab} - \hat{g}^{cd}(\partial_\rho^\nu \hat{g}_{cd})\hat{g}_{ab} \triangleq (\text{terms involving } \partial_\rho^\mu \hat{g}_{ab} \text{ where } \mu < \nu) \tag{2.3.16}$$

which is just a separation of the highest order derivatives on the left hand side and lower ones on the right. The first observation that we make from this (when combined with our inductive results above for $\nu = 1, 2$) is that $\partial_\rho^\nu \hat{g}_{ab}|_{\partial M}$ is locally formally determined by $g^{(0)}_{ab}$ as long as $\nu < d$.

The second important observation that we now make is that (2.3.9) respects the *parity* in ρ, by which we mean that the sum of powers of ρ and derivatives with respect to ρ in every term has the same parity (namely odd parity in (2.3.9)). In particular, this means that (2.3.16) also respects the parity in ρ, and thus all terms on the right hand side of (2.3.16) must have a total number of ρ-derivatives equal to ν. This observation is sufficient to prove by induction that $\partial_\rho^\nu \hat{g}_{ab} \triangleq 0$ for all odd $\nu < d$: The base case is the $\nu = 1$ example which we already showed to hold in (2.3.13) and the inductive step follows from the argument that all terms on the right hand side of (2.3.16) must vanish as they will all contain a factor of an odd number of ρ-derivatives. One then shows that $\partial_\rho^\nu \hat{g}_{ab} \triangleq 0$ using the trace and substitution procedure that we used in the $\nu = 1, 2$ cases.

This analysis breaks down at $\nu = d$ where the first term on the left hand side of (2.3.16) vanishes. In order to examine this order more carefully, we will now consider the cases of odd and even d separately and give the form of the general asymptotic solution for each in turn. We also note that this procedure of iterative differentiation can be carried in order to obtain solutions to the field equations with matter present, but the structure of the asymptotic expansions is now different. For a more technical discussion of the matter case, see section A.3.

Odd d:

In this case, (2.3.16) becomes

$$\hat{g}^{cd}(\partial_\rho^\nu \hat{g}_{cd})\hat{g}_{ab} \triangleq 0 \tag{2.3.17}$$

where all of the terms on the right hand side of (2.3.16) vanish by the parity argument. This equation tells us $\hat{g}^{cd}(\partial_\rho^\nu \hat{g}_{cd}) \triangleq 0$, i.e. that the trace of $(\partial_\rho^\nu \hat{g}_{cd}) \triangleq 0$, but now the trace free part is (at this point) completely unconstrained.

A constraint upon $(\partial_\rho^\nu \hat{g}_{cd})$ arises from the $\{\rho a\}$ component of the field equations, namely the first equation of (2.3.10). We observe this by first by taking $(\nu - 1)$ derivatives of this equation to obtain

$$\nabla_a[\hat{g}^{bc}(\partial_\rho^\nu \hat{g}_{bc})] + \nabla^b(\partial_\rho^\nu \hat{g}_{ab}) \triangleq (\text{terms involving } \partial_\rho^\mu \hat{g}_{ab},\ \mu < \nu) \tag{2.3.18}$$

and now noting that $\hat{g}^{cd}(\partial_\rho^\nu \hat{g}_{cd}) \triangleq 0$ as well as the vanishing of the terms on the right hand side by the parity argument we have

$$\nabla^b(\partial_\rho^\nu \hat{g}_{ab}) \triangleq 0 \tag{2.3.19}$$

which is our second constraint upon $(\partial_\rho^\nu \hat{g}_{ab})$, namely that it is conserved with respect to ∇, the covariant derivative associated to the metric $g_{ab}^{(0)}$.

Putting all of these results together, we can apply Taylor's theorem to conclude that the most general form of the asymptotic solution to the field equations for odd d takes the form

$$\hat{g}_{ab} = g_{ab}^{(0)} + \rho^2 g_{ab}^{(2)} + \ldots + \rho^{2k} g_{ab}^{(2k)} + \ldots + \rho^d g_{ab}^{(d)} + \ldots, \qquad k \in \mathbb{Z},\ 2k < d \tag{2.3.20}$$

where $g_{ab}^{(2k)}$ are all locally determined by $g_{ab}^{(0)}$. $g_{ab}^{(d)}$ is undetermined but is trace-free and conserved with respect to $g_{ab}^{(0)}$. In the next section we will discuss the importance of these terms in the context of AdS/CFT.

Even d:

Turning now to the case of even d, we again begin with the form of equation (2.3.16), observing that we now have

$$-\hat{g}^{cd}(\partial_\rho^\nu \hat{g}_{cd})\hat{g}_{ab} \triangleq (\text{terms involving } \partial_\rho^\mu \hat{g}_{ab},\ \mu < \nu) \tag{2.3.21}$$

where the main difference between this and (2.3.17) is that now the terms on the right hand side are not identically zero, but are locally determined by $g_{ab}^{(0)}$. In order to proceed we begin by taking the trace of (2.3.21) which leaves us with the result that $\hat{g}^{cd}(\partial_\rho^\nu \hat{g}_{cd})$ is locally determined by $g_{(0)}$.

This tracing procedure removed all trace-free terms on the right hand side of (2.3.21), meaning that the asymptotic solution given by a formal power series expansion is not the most general. In order to account for the possibility of a trace-free term, one has to introduce a term in the solution of the form $h_{ab}(x^a)\rho^d \log\rho$, where $g^{ab}_{(0)}h_{ab} = 0$. The inclusion of such a term modifies equation (2.3.21) to give

$$d!h_{ab} - \hat{g}^{cd}(\partial^\nu_\rho \hat{g}_{cd})\hat{g}_{ab} \triangleq (\text{terms involving } \partial^\mu_\rho \hat{g}_{ab},\ \mu < \nu) \tag{2.3.22}$$

and thus we see that h_{ab} is locally determined by $g^{(0)}_{ab}$ as it is identified with the trace-free part of the right hand side. There is one more constraint satisfied by h_{ab} which (as for $g^{(d)}$ in the odd d case) arises from studying the first equation of (2.3.10). Taking $(d-1)$ ρ-derivatives of this equation generates a logarithmic term of the form

$$(\log\rho)\nabla^b h_{ab} \tag{2.3.23}$$

which must vanish in order to allow us to take the $\rho \to 0$ limit. This immediately gives us that h_{ab} is conserved, i.e. $\nabla^b h_{ab} = 0$.

Putting these results together, we find that the general form of the asymptotic solution to the field equations for d even is given by

$$\hat{g}_{ab} = g^{(0)}_{ab} + \rho^2 g^{(2)}_{ab} + \ldots + \rho^{2k} g^{(2k)}_{ab} + \ldots + \rho^d(\log\rho)h_{ab} + \rho^d g^{(d)}_{ab} + \ldots, \quad k \in \mathbb{Z}, 2k < d \tag{2.3.24}$$

where again $g^{(2k)}_{ab}$ are all locally determined by $g^{(0)}_{ab}$. h_{ab} is locally determined by $g^{(0)}_{ab}$ and is trace-free and conserved. $g^{(d)}_{ab}$ is undetermined but $g^{ab}_{(0)}g^{(d)}_{ab}$ is locally determined.

We will now discuss holographic renormalisation and discuss how the terms in both the odd (2.3.20) and even (2.3.24) dimensional solutions (in particular $g^{(0)}$, $h_{(0)}$ and $g^{(d)}$) are interpreted in the context of the AdS/CFT correspondence [58, 140, 141, 60, 61, 142, 62].

2.3.4 Holographic renormalisation

So far we have discussed the asymptotics of AlAdS spacetimes, but we have not discussed how this allows one to realise results in the AdS/CFT correspondence via the master formula (2.3.3). In order to do this, we first note that the gravitational partition function (2.3.4) includes the action functional for the gravitational theory, $S[\phi]$, which will be our starting point for discussing how one can realise the AdS/CFT duality using the results of the previous section.

The object we would like to consider is the gravitational action for a spacetime (M, g)

with a boundary ∂M

$$S[g] = \frac{1}{16\pi G_N}\left[\int_M d^{d+1}x\,\sqrt{g}(R[g]-2\Lambda) + \int_{\partial M} d^d x\, 2\sqrt{\gamma}K\right] \tag{2.3.25}$$

where the first term is the Einstein-Hilbert action and the second is the Gibbons-Hawking-York term [143, 144], a necessary term to add when the spacetime contains a boundary. In order to explain the other notation here, we note that γ_{ij} is the metric induced on ∂M and $K = \nabla_a n^a$ is the trace of the extrinsic curvature of ∂M, where n^a is the *outward-pointing* unit normal vector.

This action is suitable in order to derive the equations of motion (2.3.7), however when evaluated on-shell it will diverge. These divergences are most easily seen by considering Fefferman-Graham gauge (2.3.8), where the double pole at $\rho = 0$ ensures that both the bulk volume term and the induced metric at $\partial M = \{\rho = 0\}$ (and thus the boundary term) diverge. Due to these divergences, we first need to regulate the theory. In order to do this we the restrict the action (2.3.25) to the spacetime region $\rho \geq \epsilon$ and evaluate the boundary term at $\rho = \epsilon$, where $\epsilon > 0$ is the *regulator*. This procedure allows us to define the *regulated action*, $S_{\text{reg}}[g]$, as

$$S_{\text{reg}}[g] = \frac{1}{16\pi G_N}\left[\int_{\rho\geq\epsilon} d^{d+1}x\,\sqrt{g}(R[g]-2\Lambda) + \int_{\rho=\epsilon} d^d x\, 2\sqrt{\gamma}K\right] \tag{2.3.26}$$

where now γ_{ij} is the metric induced upon the $\rho = \epsilon$ hypersurface (so in particular $\gamma_{ab} = (l^2 \hat{g}_{ab})/\epsilon^2$) and $K = \nabla_a n^a$ is the trace of the extrinsic curvature of the $\rho = \epsilon$ hypersurface.

We will begin our analysis of the action functional by evaluating it on-shell, using the Fefferman-Graham gauge (2.3.8) that we introduced in the previous section. We evaluate the action (2.3.26) on-shell, meaning we have

$$R[g] - 2\Lambda = -\frac{2d}{l^2} \tag{2.3.27}$$

and thus the on-shell regulated action takes the form

$$S_{\text{reg}}[g] = \frac{-l^{d-1}}{16\pi G_N}\int d^d x\left[\int_\epsilon d\rho\left(\frac{2d}{\rho^{1+d}}\sqrt{\hat{g}}\right) + \frac{2}{\rho^d}\left(\rho\partial_\rho\sqrt{\hat{g}} - d\sqrt{\hat{g}}\right)\Big|_{\rho=\epsilon}\right]. \tag{2.3.28}$$

This result prompts the observation that the regulated action diverges in the limit $\epsilon \to 0$. At a first pass, this seems to be a problem in how we should interpret the AdS/CFT correspondence, as both sides of (2.3.3) diverge. This is an IR divergence from the perspective of gravity and a UV divergence in the CFT and thus we can view (2.3.3) as representing an equivalence between divergent 'bare' quantities prior to renormalisation.

In order to obtain a relationship between finite quantities, one has to follow the procedure of *holographic renormalisation* as formulated in the seminal works [58, 145, 140, 60, 61, 62]. This procedure allows one to remove the divergent terms of (2.3.28) by subtraction of *covariant* counter terms. We will briefly review how to do this and state (without direct computation) some of the key results that this procedure allows one to realise (for the reader interested in further specifics, we point out the discussions given in [58, 60]).

The first step in the procedure is to use the asymptotic solution ((2.3.20) for d odd and (2.3.24) for d even) in order to express the on-shell regulated action (2.3.28) in terms of a divergent expansion in ϵ. For d odd we find

$$S_{\text{reg}} = \frac{-l^{d-1}}{16\pi G_N} \int d^d x \sqrt{g_{(0)}} \left(\epsilon^{-d} a_{(0)} + \epsilon^{-d+2} a_{(2)} + \ldots + \epsilon^{-1} a_{(d-1)} + \mathcal{O}(\epsilon) \right) \tag{2.3.29}$$

where each of the terms $a_{(i)}$ are local covariant expressions of the metric $g^{(0)}_{ab}$. This is important because it means that we will be able to renormalise the theory by addition of local counter terms. The infinities from the perspective of the dual QFT will be local infinities and thus we must also be able to cancel the IR infinites via addition of local counter terms. This local covariant form is due to the generic form of the Fefferman-Graham solution (2.3.20), where each of the $g^{(2k)}_{ab}$ (for $2k < d$) in the expansion were locally determined by $g^{(0)}_{ab}$.

For d even, we find

$$S_{\text{reg}} = \frac{-l^{d-1}}{16\pi G_N} \int d^d x \sqrt{g_{(0)}} \left(\epsilon^{-d} a_{(0)} + \epsilon^{-d+2} a_{(2)} + \ldots + \epsilon^{-2} a_{(d-2)} - (\log \epsilon) a_{(d)} + \mathcal{O}(\epsilon) \right) \tag{2.3.30}$$

where, again, all of the $a_{(i)}$ are local covariant expressions of the metric $g^{(0)}_{ab}$. This expression does have an important difference when compared with the odd d case of (2.3.29), firstly in that it contains a divergent piece of the form $\log \epsilon$. This term arises from the $\int d\rho$ term in (2.3.28) as $\sqrt{g}$ now admits terms of $\mathcal{O}(\rho^d)$ in the series expansion.

Now that we have seen explicitly how the on-shell action diverges, we are at the point at which we can renormalise the theory. In order to do this, one simply subtracts the divergent terms which arise in (2.3.28) from the action S_{reg} and then removes the regulator ϵ by taking the limit as it vanishes. Explicitly we have

$$S_{\text{ren}} = \lim_{\epsilon \to 0} (S_{\text{reg}} + S_{\text{ct}}) \tag{2.3.31}$$

where

$$S_{\text{ct}} = \frac{l^{d-1}}{16\pi G_N} \int d^d x \sqrt{g_{(0)}} \left(\epsilon^{-d} a_{(0)} + \epsilon^{-d+2} a_{(2)} + \ldots + \epsilon^{-1} a_{(d-1)} \right) \tag{2.3.32}$$

for d odd, and

$$S_{\text{ct}} = \frac{l^{d-1}}{16\pi G_N} \int d^d x \, \sqrt{g_{(0)}} \left(\epsilon^{-d} a_{(0)} + \epsilon^{-d+2} a_{(2)} + \ldots + \epsilon^{-2} a_{(d-2)} - (\log \epsilon) a_{(d)} \right) \quad (2.3.33)$$

for d even. We recall that each of the $a_{(i)}$ counterterms are covariant expressions in $g_{(0)}$, but in order to express their covariant nature at the $\rho = \epsilon$ hypersurface, they should be expressed as covariant expressions in terms of the induced metric γ_{ab}. In order to express $g^{(0)}_{ab}$ in terms of γ_{ab}, one has to formally invert the series expansion for $\hat{g}_{ab} = (\epsilon^2 \gamma_{ab})/l^2$ ((2.3.20) for odd d or (2.3.24) for even d). Once this procedure is completed to satisfactory order, one then places this expression into each $a_{(i)}$. For the explicit expressions of the $a_{(i)}$ coefficients in the cases of both odd and even d, we refer the reader to appendix B of [60]. We note that the series inversion is in general a challenging step in the procedure of holographic renormalisation. In the next section we will discuss an alternative scheme of holographic renormalisation based on [70], which will circumvent the series inversion and thus provide a computationally more efficient procedure.

Now that we have outlined the steps of how one performes this renormalisation, we will give details of two important results that this allows one to realise in the AdS/CFT correspondence.

2.3.4.1 The holographic Weyl anomaly

The first of these is the holographic Weyl anomaly, discovered in [58]. To discuss this anomaly, our starting point is the finite part of the action, S_{fin}, which is equivalent to the renormalised action S_{ren} before one takes the limit as $\epsilon \to 0$:

$$S_{\text{fin}} = S_{\text{reg}} + S_{\text{ct}}. \quad (2.3.34)$$

We will consider the variation of this expression under a Weyl transformation $\delta g^{(0)}_{ab} = 2\delta\sigma g^{(0)}_{ab}$. As we shall shortly explain, this necessarily takes the form

$$\delta S_{\text{fin}} = \delta S_{\text{ct}} = \frac{l^{d-1}}{16\pi G_N} \int d^d x \, \sqrt{g_{(0)}} \delta\sigma \mathcal{A} \quad (2.3.35)$$

where $\mathcal{A}$ is the Weyl anomaly which we will now calculate for both odd and even d.

In the odd d case we will have to consider the variation of (2.3.32). Following [58], we begin by considering a constant rescaling parameter $\delta\sigma$ and consider the combined transformation $\delta g^{(0)}_{ab} = 2\delta\sigma g^{(0)}_{ab}$, $\delta\epsilon = \delta\sigma\epsilon$, under which we have $\delta S_{\text{reg}} = 0$. This means that we have

$$\delta S_{\text{fin}} = \delta S_{\text{ct}} \quad (2.3.36)$$

and now $\delta S_{\rm ct}$ can be computed explicitly from (2.3.32). In order to evaluate this, we note that due to the covariant nature of the coefficients we have $\delta a_{(n)} = -n\delta\sigma a_{(n)}$, and using this we can check that each individual term in the series of (2.3.32) is invariant under the transformation. This gives us the result that $\delta S_{\rm ct}^{\rm odd} = 0$ which gives us

$$\mathcal{A}^{\rm odd} = 0. \tag{2.3.37}$$

and so in odd dimensions the anomaly vanishes.

In the d even case the argument remains the same up to the point where we need to compute $\delta S_{\rm ct}$, which is now computed by computing the variation of (2.3.33) instead. As before, we find that each of the power law terms are invariant under the transformation, but now we have the log term which transforms as

$$\delta(\sqrt{g_{(0)}}(\log\epsilon)a_{(d)}) = a_{(d)}\sqrt{g_{(0)}}\delta\sigma \tag{2.3.38}$$

which implies

$$\mathcal{A}^{\rm even} = -a_{(d)} \tag{2.3.39}$$

and thus in even dimensions the Weyl anomaly is generically non-zero and is given by the coefficient of the logarithmic counterterm, $a_{(d)}$.

The importance of this result is that it matches (or gives us new information) about the corresponding anomaly in the dual CFT. Indeed, in [58], the explicit form of $a_{(d)}$ was computed in the cases of $d = 2, 4, 6$ and in $d = 2, 4$ the result was shown to match with known results. For $d = 2$ the result agrees with the central charge computed from the asymptotic symmetry algebra of AdS_3 [146]. For $d = 4$ the anomaly matches with that of the $\mathcal{N} = 4$, $SU(N)$ SYM theory in the large N limit (see [147] for a further discussion of this anomaly). For $d = 6$ the procedure constructs the anomaly for the dual $(0, 2)$ theory, which was a new result at the time of publication [58].

2.3.4.2 The holographic energy momentum tensor

The second important result which we will make great use of in this thesis is that of the holographic energy momentum tensor, as first discussed in [60] (see also [140, 61, 62] for further discussion). In order to motivate this construction, we recall our master formula for AdS/CFT (2.3.3), which in our classical limit given by (2.3.6) now takes the form

$$S[\phi^{(0)}] = -\log(Z_{\rm CFT}[\phi^{(0)}]) = -W_{\rm CFT}[\phi^{(0)}] \tag{2.3.40}$$

where W_{CFT} is the generating function of connected graphs in the CFT. Following the standard techniques from quantum field theory, one is able to obtain the 1-point correlation function of the operator $\mathcal{O}$ by functional differentiation with respect to the source field $\phi^{(0)}$:

$$\left.\frac{\delta S}{\delta\phi^{(0)}(x)}\right|_{\phi^{(0)}=0} = \langle\mathcal{O}(x)\rangle. \tag{2.3.41}$$

Of course, the formula (2.3.41) is not useful without renormalisation as both sides diverge. We have already discussed how one renormalises the gravitational side via the process of holographic renormalisation and thus we state that the correct prescription in order to compute 1-point functions is to replace $S \to S_{\mathrm{ren}}$ in equation (2.3.41).

A particular 1-point function of interest to us is that of the expectation value of the *holographic energy momentum tensor*, T_{ab}. This is the operator sourced by our chosen representative of the conformal class on ∂M, $g^{(0)}_{ab}$, and thus its expectation value takes the form

$$\langle T_{ab}\rangle = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{\mathrm{ren}}}{\delta g^{ab}_{(0)}} \tag{2.3.42}$$

where the right hand side is evaluated by

$$\lim_{\epsilon\to0}\left(\frac{2}{\sqrt{\hat{g}(x,\epsilon)}}\frac{\delta S_{\mathrm{fin}}}{\delta\hat{g}^{ab}(x,\epsilon)}\right) = \lim_{\epsilon\to0}\left(\left(\frac{l}{\epsilon}\right)^{d-2}\frac{2}{\sqrt{\gamma}}\frac{\delta S_{\mathrm{fin}}}{\delta\gamma^{ab}}\right) = \lim_{\epsilon\to0}\left(\left(\frac{l}{\epsilon}\right)^{d-2}T_{ab}[\gamma]\right) \tag{2.3.43}$$

where the first equality arises from use of $\gamma_{ab} = (l^2\hat{g}_{ab})/\epsilon^2$ and the second is just to make it clear that $T_{ab}[\gamma]$ is the stress-energy tensor of the theory at $\rho = \epsilon$ which will be the key quantity to compute. We begin by recalling the split (2.3.34), which allows us to write the stress energy tensor as the sum of two components

$$T_{ab}[\gamma] = \frac{2}{\sqrt{\gamma}}\left(\frac{\delta S_{\mathrm{reg}}}{\delta\gamma^{ab}} + \frac{\delta S_{\mathrm{ct}}}{\delta\gamma^{ab}}\right) = T^{\mathrm{reg}}_{ab} + T^{\mathrm{ct}}_{ab} \tag{2.3.44}$$

where T^{reg}_{ab} can be computed from the regulated action (2.3.26) and T^{ct}_{ab} from the counter term action, (2.3.32) for d odd or (2.3.33) for d even. The expression for T^{reg} can be obtained from the standard Brown-York procedure as computed in [148], namely

$$T^{\mathrm{reg}}_{ab} = \frac{1}{8\pi G_N}(K_{ab} - K\gamma_{ab}). \tag{2.3.45}$$

where we also note that T^{reg}_{ab} is closely related to the ADM momentum, π_{ab}, conjugate to the induced metric γ_{ab} at the $\rho = \epsilon$ hypersurface

$$\pi_{ab} = \frac{\delta S_{\mathrm{reg}}}{\delta\gamma^{ab}} = \frac{1}{16\pi G_N}\sqrt{\gamma}(K_{ab} - K\gamma_{ab}). \tag{2.3.46}$$

For more on the ADM approach to holographic renormalisation, see section 2.3.5.

In the Fefferman-Graham gauge (2.3.8), T^{reg} can be expressed as

$$T^{\text{reg}}_{ab} = \frac{1}{8\pi G_N}\frac{l}{\epsilon^2}\left((1-d)\hat{g}_{ab} - \frac{\epsilon}{2}\partial_\epsilon \hat{g}_{ab} + \frac{\epsilon}{2}\hat{g}_{ab}(\hat{g}^{cd}\partial_\rho \hat{g}_{cd})\right). \tag{2.3.47}$$

In order to obtain the expression for T^{ct}_{ab}, one has to first express the counter term action (2.3.32)/(2.3.33) in terms of γ_{ab} before taking the variation. We again point the reader in the direction of appendix B of [60] for the relevant expressions in this case.

Before giving the results for the expectation of the energy momentum tensor, we begin with a comment regarding the properties it satisfies. Firstly, we note that T_{ab} is covariantly conserved with respect to $g^{(0)}_{ab}$. In order to see this, we first argue that both T^{reg}_{ab} and T^{ct}_{ab} are conserved with respect to the induced metric γ_{ab} on the $\rho = \epsilon$ hypersurface. In order to see the conservation of T^{reg}_{ab}, we take the contracted covariant derivative of (2.3.47) and obtain

$$\frac{1}{16\pi G_N}\frac{l}{\epsilon}\left(-\nabla^b(\partial_\epsilon \hat{g}_{ab}) + \nabla_a(\hat{g}^{bc}\partial_\epsilon \hat{g}_{bc})\right) = 0 \tag{2.3.48}$$

where the equality comes from applying the first equation in (2.3.10). Covariant conservation with respect to $\hat{g}_{ab}$ is equivalent to conservation with respect to γ_{ab}. In order to see that T^{ct}_{ab} is covariantly conserved with respect to γ, it suffices to consider the covariant nature of the counter term action (2.3.32)/(2.3.33), meaning that we necessarily have

$$\delta S_{\text{ct}} = -\int d^d x\, \sqrt{\gamma}\left(\frac{1}{2}T^{ab}_{\text{ct}}\delta\gamma_{ab}\right) \tag{2.3.49}$$

then using the residual diffeomorphism invariance of the covariant counter term action gives

$$0 = \int d^d x\, \sqrt{\gamma}\left(T^{ab}_{\text{ct}} D_a \xi_b\right) = \int d^d x\, \sqrt{\gamma}\left(D_a T^{ab}_{\text{ct}}\right)\xi_b \implies D_a T^{ab}_{\text{ct}} = 0 \tag{2.3.50}$$

where D_a is now explicitly the covariant derivative with respect to γ_{ab} and we move from the left to the right by integrating by parts and discarding the boundary term. This shows that T_{ab} is covariantly conserved with respect to γ_{ab} but recall that we wanted to argue that it was conserved with respect to the boundary metric $g^{(0)}_{ab}$. The final step in this argument is provided by observing that (by construction) all divergences cancel in (2.3.42) and thus the finite part of the expression is conserved with respect to $\lim_{\epsilon\to 0}\hat{g}_{ab} = g^{(0)}_{ab}$.

The second comment we have concerns the trace of $\langle T_{ab}\rangle$ with respect to the metric $g^{(0)}_{ab}$, where now the odd/even nature of d will play an important role. We consider the schematic form of the variation of the finite part of the action

$$\delta S_{\text{ren}} = \lim_{\epsilon\to 0}\delta S_{\text{fin}} = -\int d^d x\, \sqrt{g_{(0)}}\left(\frac{1}{2}\langle T^{ab}\rangle \delta g^{(0)}_{ab}\right) \tag{2.3.51}$$

and consider the variation as that of a Weyl rescaling $\delta g^{(0)}_{ab} = 2\delta\sigma g^{(0)}_{ab}$. In the d odd case of a vanishing Weyl anomaly (2.3.37), we have $\delta S_{\text{fin}} = 0$ and thus $\langle T^a_a \rangle = 0$. However, in the d even case of a non-zero anomaly (2.3.39) we now have

$$\langle T^a_a \rangle = \frac{l^{d-1}}{16\pi G_N} a_{(d)} \tag{2.3.52}$$

which can be shown by comparison with (2.3.35).

We end this section by giving the explicit formulae for $\langle T_{ab} \rangle$ in the case of d odd, as well as the generic form of such a quantity for d even. For d odd we have

$$\langle T_{ab} \rangle = -\frac{dl^{d-1}}{16\pi G_N} g^{(d)}_{ab} \tag{2.3.53}$$

which we note is manifestly conserved and traceless with respect to $g^{(0)}$ by comparison with (2.3.20). For the case of d even, we have

$$\langle T_{ab} \rangle = -\frac{dl^{d-1}}{16\pi G_N} g^{(d)}_{ab} + X_{ab}[g_{(0)}] \tag{2.3.54}$$

where X_{ab} is a tensor dependent upon $g^{(0)}_{ab}$ and the trace of X_{ab} is related to the conformal anomaly in the boundary conformal field theory. For explicit forms of the tensor X in the cases of $d = 2, 4, 6$, see [60]. For the majority of this thesis, we will consider the case of $d = 3$ (AlAdS$_4$ spacetimes) and thus we will be considering spacetimes with $X_{ab} = 0$.

2.3.5 Holographic renormalisation - Hamiltonian formalism

The method of renormalisation that we have discussed in the previous section is heavily reliant upon the computationally challenging step of inverting the series expression for the counter term action (2.3.32)/(2.3.33) (as discussed in some detail in the paragraph below equation (2.3.33)). We will now introduce an alternative method of holographic renormalisation developed by Papadimitriou & Skenderis [70, 149], based upon the *Hamiltonian* approach to general relativity. This method has an important advantage over the technique that we previously discussed, namely that the computational difficulty involved in inverting the series is now removed. This procedure can also be applied to nonrelativistic systems of importance in quantum gravity, see for example [150, 151]. We will later adapt this method to AldS$_4$ spacetimes in chapter 5 in order to renormalise the asymptotic charges of such solutions.

The starting point to discuss the Hamiltonian approach to holographic renormalisation is the original paper [70], of which we will follow closely in this section and summarise the useful results which we will use later. We note that there are some different conventions in that paper to the ones that we have applied thus far, namely the sign of the action

(2.3.26) carries an extra minus (such that it is positive definite when evaluated on-shell), and the convention is to set $l = 1$ throughout. We will retain our convention of the sign of the action (2.3.26) for the purpose of this subsection, but we will set $l = 1$ (this can always be reinstated via dimensional analysis).

2.3.5.1 The ADM formalism

The approach taken in this method is to analyse the gravitational field equations using the Hamiltonian (ADM) formalism [152, 153, 154] (see also [77] for an overview). In the typical setup of the ADM formalism, one chooses a global time function t and foliates the spacetime in question with a series of $t =$ constant hypersurfaces. The field equations are then solved by evolving in time after specifying conditions upon an initial hypersurface. In the case of a Riemannian manifold with boundary, no such time coordinate exists and thus the idea in [70] is to replace this “constant time” foliation by hypersurfaces whose normal vector is orthogonal to the conformal boundary ∂M of the manifold (in the language of the previous section these would be the $\rho =$ constant hypersurfaces). In the Lorentzian picture, this is a replacement of the “constant time” hypersurfaces by a foliation of the near boundary region by timelike hypersurfaces, and now the evolution is a radial evolution in the direction of the conformal boundary.

In order to implement this foliation geometrically, we decompose the metric $g_{\mu\nu}$, as follows: Pick a coordinate r as the radial coordinate emanating from ∂M, and consider the hypersurfaces Σ_r given by $r =$ constant. The normal covector to these hypersurfaces is

$$\Omega_\mu = \partial_\mu r \tag{2.3.55}$$

which has norm

$$||\Omega||_g^2 = g^{\mu\nu}\Omega_\mu\Omega_\nu = ||dr||_g^2 \tag{2.3.56}$$

and thus the *outward pointing unit normal* to the hypersurface Σ_r is

$$n^\mu = \frac{\Omega^\mu}{||\Omega||_g} = \frac{1}{||dr||_g} g^{\mu\nu}\partial_\nu r. \tag{2.3.57}$$

This allows us to express the metric on the Σ_r hypersurfaces, $\gamma_{\mu\nu}$, in a covariant way as

$$\hat{\gamma}_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu, \tag{2.3.58}$$

where the hat denotes a quantity which is transverse to n^μ (from this we can verify explicitly that $\hat{\gamma}_{\mu\nu}n^\mu = 0$). Finally, we decompose the metric as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \hat{\gamma}_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu + 2N_\mu d\hat{x}^\mu dr + (N^2 + N_\mu N^\mu)dr^2 \tag{2.3.59}$$

where N is the *lapse function* and N^μ is the *shift vector*. We will later gauge fix these to values $N = 1$, $N^\mu = 0$, which make the n^μ tangent to *geodesics* in M, orthogonal to Σ_r. For now, we will keep N and N^μ generic in order to highlight some of their features in the ADM formalism and we will define the acceleration vector as

$$a^\mu = n^\nu \nabla_\nu n^\mu \tag{2.3.60}$$

which of course vanishes in the case of geodesic n^μ.

The quantity which will be of primary importance to us is the extrinsic curvature of the hypersurfaces Σ_r

$$\hat{K}_{\mu\nu} = \hat{\gamma}^\rho_\mu \nabla_\rho n_\nu = \frac{1}{2}\mathcal{L}_n \hat{\gamma}_{\mu\nu} \tag{2.3.61}$$

where the second equality can be shown by explicitly expanding the Lie derivative of the induced metric and then using (2.3.57) to show that the totally antisymmetric part of $\hat{\gamma}^\rho_\mu \nabla_\rho n_\nu$ vanishes. With a mind to understanding this in relation to the Einstein equations, we now apply the *Gauss-Codacci* equations. These geometrical equations relate the intrinsic curvatures of $g_{\mu\nu}$ to the intrinsic and extrinsic curvatures of the hypersurfaces with induced metric $\hat{\gamma}_{\mu\nu}$. Explicitly these are given by

$$\begin{aligned} \hat{\gamma}^\alpha_\mu \hat{\gamma}^\beta_\nu \hat{\gamma}^\gamma_\rho \hat{\gamma}^\delta_\sigma R_{\alpha\beta\gamma\delta} &= \hat{R}_{\mu\nu\rho\sigma} + \hat{K}_{\mu\rho}\hat{K}_{\nu\sigma} - \hat{K}_{\mu\rho}\hat{K}_{\nu\sigma} \\ \hat{\gamma}^\rho_\nu n^\sigma R_{\rho\sigma} &= \hat{\nabla}_\mu \hat{K}^\mu_\nu - \hat{\nabla}_\nu \hat{K}^\mu_\mu \end{aligned} \tag{2.3.62}$$

where unhatted intrinsic curvature terms refer to those of $g_{\mu\nu}$ and hatted ones $\hat{\gamma}_{\mu\nu}$. In order to make explicit how one can apply the Einstein equations to these (purely geometric) equations, we want to replace terms in these equations with the Einstein tensor of $g_{\mu\nu}$, $G_{\mu\nu}$. Manipulation of the Gauss-Codacci equations allows us to rewrite them as

$$\begin{aligned} \hat{K}^2 - \hat{K}_{\mu\nu}\hat{K}^{\mu\nu} &= \hat{R} + 2G_{\mu\nu}n^\mu n^\nu \\ \hat{\nabla}_\mu \hat{K}^\mu_\nu - \hat{\nabla}_\nu \hat{K}^\mu_\mu &= G_{\rho\sigma}\hat{\gamma}^\rho_\nu n^\sigma \\ \mathcal{L}_n \hat{K}_{\mu\nu} + \hat{K}\hat{K}_{\mu\nu} - 2\hat{K}^\rho_\mu \hat{K}_{\rho\nu} &= \hat{R}_{\mu\nu} - \hat{\gamma}^\rho_\mu \hat{\gamma}^\sigma_\nu R_{\rho\sigma} + \hat{\nabla}_\mu a_\nu - a_\mu a_\nu \end{aligned} \tag{2.3.63}$$

where the first equation here arises from the full contraction ($\mu \leftrightarrow \rho$, $\nu \leftrightarrow \sigma$) of the first equation in (2.3.62) and the third from the partial contraction ($\mu \leftrightarrow \rho$ and a relabelling of free indices) of the same equation. The second equation above follows directly from substitution of the Einstein tensor in the second equation of (2.3.62). Upon substitution of the field equations (for vacuum gravity $G_{\mu\nu} = -\Lambda g_{\mu\nu}$) these equations become dynamical and not just purely geometric.

Another aspect of the ADM formalism is that it allows us to express the action (2.3.26) in terms of the quantities $(\hat{\gamma}_{\mu\nu}, N, N_\mu)$. The precise details of this are discussed in [77], where in particular the bulk term of (2.3.26) can be expressed as

$$S_{\text{bulk}} = \frac{1}{16\pi G_N} \int_M d^{d+1}x \sqrt{\hat{\gamma}} N(\hat{R} + \hat{K}^2 - \hat{K}_{\mu\nu}\hat{K}^{\mu\nu} - 2\Lambda) \tag{2.3.64}$$

from which we can define in the usual way the canonical momentum

$$\pi^{\mu\nu} = \frac{\delta L}{\delta \dot{\hat{\gamma}}_{\mu\nu}} = \frac{1}{16\pi G_N} \sqrt{\hat{\gamma}} (\hat{K}\hat{\gamma}^{\mu\nu} - \hat{K}^{\mu\nu}) \tag{2.3.65}$$

where here $S = \int dr\, L$ and the dot refers to the "radial derivative" of the induced metric, in particular $\dot{\hat{\gamma}}_{\mu\nu} = \hat{\gamma}^{\alpha}_{\mu}\hat{\gamma}^{\beta}_{\nu}\mathcal{L}_r\hat{\gamma}_{\alpha\beta}$ where the vector r^μ is given by

$$r^\mu = N n^\mu + N^\mu. \tag{2.3.66}$$

In order to derive the right hand side of (2.3.65), one first has to show explicitly how this radial derivative enters the terms in the action (2.3.64), a computation which is again performed in [77]. The momentum corresponding to the lapse and shift vanish identically.

2.3.5.2 Application of the ADM formalism to holographic renormalisation

Now that we have reviewed the application of the ADM formalism to radial evolution in AlAdS spaces, it remains to discuss how we can apply these methods to the procedure of holographic renormalisation. To begin this, we first recall that a key ingredient in the holographic renormalisation procedure that we have discussed so far was the on-shell action (2.3.28), a quantity whose bulk piece we can rewrite using the ADM formalism as

$$S^{\text{bulk}}_{\text{on-shell}} = \frac{1}{8\pi G_N} \int_{r=r_0}^{r=r_1} dr d^d x \sqrt{\hat{\gamma}} N(\hat{R} - 2\Lambda) \tag{2.3.67}$$

where now we have regulated by choosing the boundary ∂M to be located at $r = r_1$ ($r = r_0 < r_1$ is just an interior hypersurface which we can always choose in a manner such that it is sufficiently close to r_1). We can now define the momenta on the boundary Σ_{r_1} by functional differentiation

$$\pi^{\mu\nu}(r_1, x) = \frac{\delta S_{\text{on-shell}}}{\delta \hat{\gamma}_{\mu\nu}(r_1, x)} \tag{2.3.68}$$

where these momenta are equivalent to those of (2.3.65). The importance of rewriting these momenta in this way is apparent when considering holographic renormalisation, as the expression above is precisely that which we wanted to compute in order to define the regulated 1-point function ((2.3.43) before taking the limit as $\epsilon \to 0$). Of course, just as in the case of unrenormalised AlAdS spaces, the momenta as defined above will naïvely diverge as we take $r_1 \to \infty$. The main result of [70] is extending the renormalisation scheme to the momenta as defined above, the main points and advantages of which we will now summarise.

- The key focus in this approach to holographic renormalisation is the canonical momenta and not the on-shell action. We have already seen how one can relate these via equations (2.3.65) & (2.3.68).

- In order to implement this renormalisation, one first expresses the second order field equations (2.3.63) as *first order functional equations* by expressing the radial derivative as a functional derivative

$$\frac{d}{dr} = \int d^d x\, \dot{\hat{\gamma}}_{\mu\nu} \frac{\delta}{\delta \hat{\gamma}_{\mu\nu}} \sim \int d^d x\, 2\hat{K}_{\mu\nu} \frac{\delta}{\delta \hat{\gamma}_{\mu\nu}} \tag{2.3.69}$$

where $\sim$ indicates that one needs to fix the gauge in order to make this a strict equality (our choice of $N = 1$, $N^\mu = 0$ will show this).

- These first order equations are still very difficult to solve, but we can work asymptotically in order to solve them. To do this, we start by prescribing Dirichlet boundary conditions for the metric as $r \to \infty$

$$\hat{\gamma}_{\mu\nu} \sim e^{2r} \hat{\gamma}^{(0)}_{\mu\nu}. \tag{2.3.70}$$

Which we choose to prescribe because a bulk metric corresponds to a particular conformal class at the boundary, so a natural choice of boundary condition is to keep this conformal class fixed (but arbitrary). Any other choice of Dirichlet boundary condition would break the bulk diffeomorphisms associated with Weyl rescalings at the conformal boundary. We also note that in the context of AdS/CFT, specifying the conformal class corresponds to specifying the non-normalisable modes of the dual QFT, and thus the boundary conditions have an additional interpretation in this setting. For work discussing alternative boundary conditions, see [155]

Using these boundary conditions we find the asymptotic form of the radial derivative

$$\partial_r \sim \int d^d x\, 2\hat{\gamma}_{\mu\nu} \frac{\delta}{\delta \hat{\gamma}_{\mu\nu}} \tag{2.3.71}$$

where the right hand side is precisely the total dilatation operator, δ_D of the theory. The procedure of asymptotic expansion will now be to expand the momenta in terms of eigenfunctions of the operator δ_D. This is in contrast to the previous section, where the asymptotic expansion was in the small parameter ρ, the coordinate which measures distance from the conformal boundary. This choice explicitly broke bulk covariance, a feature not present in the Hamiltonian approach. We will explain below how preserving bulk covariance allows one to streamline the procedure of holographic renormalisation.

- As in the previous method, covariant counter terms are required in order to make

(2.3.68) finite as $r_1 \to \infty$. In order to construct these counter terms, one needs merely to identify the singular part of the momentum, a quantity which has already been expanded in eigenfunctions of δ_D. As it turns out, the singular pieces will be those with eigenvalue less than the scaling dimension of the dual operator in the CFT. In the case of the metric, the dual operator is of course the holographic energy momentum tensor, which has scaling dimension equal to d. This leads us to the result:

$$\langle \hat{T}_{\mu\nu} \rangle = \frac{1}{\kappa^2}(\hat{K}_{(d)\mu\nu} - \hat{K}_{(d)}\hat{\gamma}_{\mu\nu}) \tag{2.3.72}$$

where the subscript (d) indicates the eigenvalue under dilatation. As we discussed in (2.3.53) and (2.3.54), this expression is related to free data in the asymptotic solution.

- This method removes the difficult computational step from the previous method of computing the covariant counter terms for the on-shell action on the hypersurface Σ_r. Instead, one solves the equations of motion in order to generate counter terms for the momenta. These can then be used to write down the expressions for the 1-point functions. In principle, the need to compute $K_{(n)}$ as part of this procedure could be highly non-trivial, but as we will see in the case of general relativity, this can be done iteratively and is a simplification relative to the older method.

- Even though the covariant counter terms for the on-shell action are no longer necessary to write down 1-point functions, one can now compute them much more efficiently than before using this formalism. They will be computed recursively by solving the equations of motion and the step of series inversion is removed. In what follows, we will comment upon this in the case of general relativity.

- This procedure is totally equivalent to the methods of [60] as discussed in the previous section. This was shown explicitly in [70], where the map between $\hat{K}_{\mu\nu}[\hat{\gamma}]$ and the Fefferman-Graham coefficients $g^{(n)}_{\mu\nu}$ was derived.

We will conclude this section with an application of this method to the case of general relativity in the presence of a negative cosmological constant. This will serve as an example of the advantages of the newer method as we summarised above, as well as a prelude to chapter 5, where we will adapt these methods to the case of a positive cosmological constant in $d = 3$ (AldS$_4$).

We will follow the example of [70] by setting $l = 1$ and thus $\Lambda = d(d-1)/2$. As we mentioned beneath equation (2.3.59), we will begin by fixing the gauge such that $N^\mu = 0$ and $N = 1$, thus making the bulk metric

$$ds^2 = dr^2 + \gamma_{ij}(r,x)dx^i dx^j \tag{2.3.73}$$

where we drop the hats as indices $i, j = 1, \ldots, d$ are in directions along the hypersurface and are all automatically transverse to the r-direction. The extrinsic curvature becomes

$$K_{ij} = \frac{1}{2}\dot{\gamma}_{ij} \tag{2.3.74}$$

which we note now produces a precise equality in equation (2.3.69). The field equations are $G_{\mu\nu} = g_{\mu\nu} d(1-d)/2$ which in Gauss-Codacci form (2.3.63) now become

$$\begin{aligned} K^2 - K_{ij}K^{ij} &= R + d(d-1) \\ \nabla_i K^i_j - \nabla_j K &= 0 \\ \dot{K}^i_j + K K^i_j &= R^i_j + d\delta^i_j \end{aligned} \tag{2.3.75}$$

where we note that R and ∇ are now quantities with respect to γ_{ij} and $\dot{K}^i_j = \frac{d}{dr}(\gamma^{ik}K_{kj})$. Using the gauge (2.3.73), we can express the r-derivative of the on-shell action (2.3.67) as

$$\dot{S}_{\text{on-shell}} = L = \frac{1}{8\pi G_N}\int_{\Sigma_r} d^dx\,\sqrt{\gamma}(R + d(d-1)) \tag{2.3.76}$$

which means that we can obtain an expression for $S_{\text{on-shell}}$ if we are able to write the integrand on the right hand side as a total r-derivative of a covariant expression. In order to derive an expression for this expression, we introduce a covariant variable λ and write the on-shell action as

$$S_{\text{on-shell}} = \frac{1}{8\pi G_N}\int_{\Sigma_r} d^dx\,\sqrt{\gamma}(K - \lambda) \tag{2.3.77}$$

where the K term above can be seen to be the Gibbons-Hawking term in (2.3.26). Taking the radial derivative of the expression above and comparing with (2.3.76) we are able to establish that λ satisfies the first order differential equation

$$\dot{\lambda} + K\lambda = d \tag{2.3.78}$$

where we used the trace of the third equation of (2.3.75), namely $\dot{K} + K^2 = R + d^2$.

We will now define the asymptotic expansions that we will use to solve the field equations (2.3.75), which (as previously mentioned) are expansions of K and λ in eigenfunctions of the dilatation operator, δ_D

$$\begin{aligned} K^i_j[\gamma] &= K_{(0)}{}^i_j + K_{(2)}{}^i_j + \ldots + K_{(d)}{}^i_j - 2r\tilde{K}_{(d)}{}^i_j + \ldots \\ \lambda[\gamma] &= \lambda_{(0)} + \lambda_{(2)} + \ldots + \lambda_{(d)} - 2r\tilde{\lambda}_{(d)} + \ldots \end{aligned} \tag{2.3.79}$$

where the terms in the expansion transform as follows

$$\begin{aligned} \delta_D K_{(n)}{}^i_j &= -nK_{(n)}{}^i_j, \quad n < d, \\ \delta_D \tilde{K}_{(d)}{}^i_j &= -d\tilde{K}_{(d)}{}^i_j, \\ \delta_D K_{(d)}{}^i_j &= -dK_{(d)}{}^i_j - 2\tilde{K}_{(d)}{}^i_j \end{aligned} \tag{2.3.80}$$

and λ terms transform identically with K replaced by λ everywhere. We note this result follows from the identification between the radial derivative and the dilatation operator (2.3.71) and $K_{(d)}{}^i_j$ is not an eigenfunction of δ_D as it transforms inhomogeneously (due to the presence of the Weyl anomaly).

With this asymptotic expansion of the fields set up, we return to the consideration of the momenta, namely

$$\pi^{ij} = \frac{1}{16\pi G_N}\sqrt{\gamma}(K\gamma^{ij} - K^{ij}) = \frac{\delta S_{\text{on-shell}}}{\delta\gamma_{ij}} \tag{2.3.81}$$

which when compared with (2.3.77) becomes

$$K\gamma^{ij} - K^{ij} = \frac{2}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{ij}}\int_{\Sigma_r} d^dx\,\sqrt{\gamma}(K-\lambda), \tag{2.3.82}$$

into which we input the expansions in dilatation eigenfunctions (2.3.79). Comparing terms of the same dilatation weight then leads us to the equations

$$\begin{aligned} K_{(2n)}{}^i_j &= \lambda_{(2n)}\delta^i_j - \frac{2}{\sqrt{\gamma}}\int d^dx\,\sqrt{\gamma}\gamma_{kj}\frac{\delta}{\delta\gamma_{ik}}(K_{(2n)} - \lambda_{(2n)}), \quad 0 \le n \le \frac{d}{2} \\ \tilde{K}_{(d)}{}^i_j &= \tilde{\lambda}_{(d)}\delta^i_j - \frac{2}{\sqrt{\gamma}}\int d^dx\,\sqrt{\gamma}\gamma_{kj}\frac{\delta}{\delta\gamma_{ik}}(\tilde{K}_{(d)} - \tilde{\lambda}_{(d)}) \end{aligned} \tag{2.3.83}$$

which will serve as the main equations we use to recursively compute the extrinsic curvature terms $K_{(2n)}{}^i_j$. In order to do this, we will need to first find a way to specify all of the terms that appear on the right hand side of the equations. The first step in doing this is to take the trace of the equations above and use the relations (2.3.80) to get

$$\lambda_{(2n)} = \frac{2n-1}{2n-d}K_{(2n)}, \quad 0 \le n \le \frac{d}{2} - 1, \qquad \tilde{\lambda}_{(d)} = \frac{d-1}{2}K_{(d)}, \qquad \tilde{K}_{(d)} = 0. \tag{2.3.84}$$

These equations allow us to replace $\lambda_{(i)}$ terms in (2.3.83) with $K_{(i)}$ terms and also show us that $K_{(2n)}{}^i_j$ are only determined by this procedure up to $n < d/2$ (as $\lambda_{(d)}$ is not determined). In order to finish the discussion of the recursion relation, all we need now is an explicit expression for $K_{(2n)}$, which as it turns out is provided from the first equation of (2.3.75). Once again, we input the expansion of the extrinsic curvature in dilatation weights (2.3.79) and compare terms of the same dilatation weight. Using $K_{(0)}{}^i_j = \delta^i_j$ the next lowest order gives us

$$K_{(2)} = \frac{R}{2(d-1)} \tag{2.3.85}$$

which we can then use recursively to compute higher order terms by substitution in the first equation of (2.3.75), obtaining

$$K_{(2n)} = \frac{1}{2(d-1)} \sum_{m=1}^{n-1} [K_{(2m)ij} K_{(2n-2m)}{}^{ij} - K_{(2m)} K_{(2n-2m)}], \quad 2 \leq n \leq \frac{d}{2}. \tag{2.3.86}$$

The computations discussed above give us a procedure to compute the counter terms necessary in order to obtain the renormalised 1-point function (2.3.72). The counter terms that one needs to compute are precisely $K_{(2n)}{}^{i}_{j}$ for $2n < d$ as well as $\tilde{K}_{(d)}{}^{i}_{j}$. We can see this explicitly as the renormalised 1-point function is given by

$$\langle T_{ij} \rangle = \lim_{r \to \infty} \left[\pi_{ij} - \frac{1}{\kappa^2} \left(\sum_{m=0}^{d-1} (K_{(2m)ij} - K_{(2m)} \gamma_{ij}) - 2r \tilde{K}_{(d)ij} \right) \right] \tag{2.3.87}$$

and thus we observe that the equation (2.3.83) when utilised in tandem with equations (2.3.84), (2.3.85) and (2.3.86) allow us to generate all of the required counter terms, thus streamlining the procedure of holographic renormalisation!

We finish this section by expanding on the comment that this method is also more efficient at computing the counter terms at the level of the on-shell action, even though the renormalised action is no longer necessary in order to compute 1-point functions. By using (2.3.77) we observe that the counter terms are given by

$$S_{\text{ct}} = -\frac{1}{8\pi G_N} \int_{\Sigma_r} d^d x \sqrt{\gamma} \left[\sum_{m=0}^{\frac{d}{2}-1} (K_{(2m)} - \lambda_{(2m)}) - 2r(\tilde{K}_{(2m)} - \tilde{\lambda}_{(d)}) \right] \tag{2.3.88}$$

which upon application of (2.3.84) and using the coordinate transformation $r = -\frac{1}{2} \log \rho$ can be rewritten as

$$S_{\text{ct}} = \frac{d-1}{8\pi G_N} \int_{\rho=\epsilon} d^d x \sqrt{\gamma} \left[\sum_{m=0}^{\frac{d}{2}-1} \frac{1}{2m-d} K_{(2m)} + \frac{1}{2} K_{(d)} \log \epsilon \right]. \tag{2.3.89}$$

2.4 The covariant phase space formalism

In this section we will review the covariant phase space formalism, a formalism which allows one to define charges in diffeomorphism invariant theories, and in particular in asymptotically flat space times (we will later examine how one can adapt this formalism to AldS spacetimes). The seeds for this approach were developed in the pioneering work [156, 157, 158] but the formalism that we will discuss is principally due to Wald and collaborators, and was introduced and developed in [63, 64, 65, 66, 67]. These papers provide a detailed discussion of the mathematical structure of this topic of which we will summarise the useful information for the calculations to follow. We also note the existence of a formalism developed by Barnich and Brandt [159, 160] which suffices to provide a definition of conserved quantities for any gauge theory. For our purposes, the two prescriptions will be equivalent and we will stick to the language of Wald et al.

2.4.1 Preliminaries and definitions

We begin by considering theories which admit a diffeomorphism invariant Lagrangian description on a $D = (d+1)$-dimensional manifold $\mathcal{M}$. The dynamical fields for such theories will consist of a Lorentzian spacetime metric g_{ab} of signature $(- + \ldots +)$ as well as any other matter fields ϕ which one may want to add to the theory (this may include scalars, gauge fields etc). We will refer to the collective of both the metric and the extra matter fields as $\psi = (g_{ab}, \phi)$. More generally, we will follow [63, 64, 65, 66, 67] in referring to the space of allowed configurations of the fields as $\mathcal{F}$, and the "on-shell" subspace as $\bar{\mathcal{F}}$. This subspace is often named the covariant phase space, and thus will be our main object of consideration in this section.

To give the explicit form of the diffeomorphism covariant Lagrangian, we utilise the language of differential forms which we will mark with boldface lettering and often drop the indices. The Lagrangian form is a D-form defined on $\mathcal{M}$ which satisifies the following diffeomorphism invariance property: If we consider a diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ then we have

$$\mathbf{L}(f^*(\psi)) = f^*(\mathbf{L}(\psi)) \tag{2.4.1}$$

where f^* denotes the extension of f to arbitrary rank tensor fields [77]. Due to this diffeomorphism invariance property, the Lagrangian is forced to take the following form [65]

$$\mathbf{L} = \mathbf{L}(g_{ab}, \nabla_{a_1} R_{bcde}, \ldots, \nabla_{(a_1} \ldots \nabla_{a_k)} R_{bcde}, \phi, \nabla_{a_1}\phi, \ldots, \nabla_{(a_1} \ldots \nabla_{a_k)}\phi) \tag{2.4.2}$$

where R_{bcde} is the Riemann curvature tensor of the spacetime metric g_{ab}.

We now consider variations of $\mathbf{L}$ with respect to the dynamical fields ψ. The first variation of the Lagrangian always takes the form

$$\delta\mathbf{L} = \mathbf{E}\delta\psi + d\mathbf{\Theta} \tag{2.4.3}$$

where

$$\mathbf{E}\delta\psi = \mathbf{E}^{ab}\delta g_{ab} + \mathbf{E}_\phi \delta\phi \tag{2.4.4}$$

and in the second term on the right hand side a sum over all of the matter fields is understood. $\mathbf{E}$ is referred to as the equation of motion D-form and the equations of motion are given by $\mathbf{E} = 0$. $\mathbf{\Theta}$ is the symplectic potential $(D-1)$-form and is chosen to be covariant in the fields ψ and their first variations $\delta\psi$

$$\mathbf{\Theta} = \mathbf{\Theta}(\psi, \delta\psi). \tag{2.4.5}$$

At this point we provide a brief comment upon the structure of the equations that we have just introduced. Equation (2.4.3) takes the familiar form of the variation of a Lagrangian in that it consists of an equation of motion term ($\mathbf{E}\delta\psi$) and a total derivative ($d\mathbf{\Theta}$) and is thus the sort of expression that one would naturally expect to find. One may also notice that $\mathbf{\Theta}$ suffers an ambiguity as it is only defined up to addition of an exact differential form. If $\mathbf{\Theta}$ is a suitable symplectic potential form then so is $\mathbf{\Theta} + d\mathbf{T}$ where $\mathbf{T}$ is any $(D-2)$-form chosen such that $d\mathbf{T}$ is covariant in the fields and their variations

$$\mathbf{E}\delta\psi + d(\mathbf{\Theta} + d\mathbf{T}(\psi, \delta\psi)) = \mathbf{E}\delta\psi + d\mathbf{\Theta} + d^2\mathbf{T}(\psi, \delta\psi) = \mathbf{E}\delta\psi + d\mathbf{\Theta} = \delta\mathbf{L}. \tag{2.4.6}$$

This ambiguity is usually overcome by picking a $\mathbf{\Theta}$ which is particularly convenient for the problem at hand and will not enter into the calculations that we perform later.

Using the symplectic potential form we can define the symplectic current $\boldsymbol{\omega}$ which is given as the antisymmetrised variation of $\mathbf{\Theta}$

$$\boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) = \delta_1\mathbf{\Theta}(\psi, \delta_2\psi) - \delta_2\mathbf{\Theta}(\psi, \delta_1\psi) \tag{2.4.7}$$

and $\boldsymbol{\omega}$ is clearly a $(D-1)$-form. A particularly useful property of this form which we will use later is that $\boldsymbol{\omega}$ is closed when ψ satisfies the equations of motion and $\delta_1\psi$ and $\delta_2\psi$ the linearised equations of motion. i.e.

$$\mathbf{E} = \delta_1\mathbf{E} = \delta_2\mathbf{E} = 0 \quad \Longrightarrow \quad d\boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) = 0. \tag{2.4.8}$$

We will repeat the proof here which was originally given in [63, 67] (a nice recap is also given in [71]).

$$\begin{aligned} \delta_1\delta_2\mathbf{L} &= \delta_1(\mathbf{E}\delta_2\psi + d\mathbf{\Theta}(\psi, \delta_2\psi)) \\ &= \delta_1\mathbf{E}\delta_2\psi + \mathbf{E}\delta_1\delta_2\psi + \delta_1 d\mathbf{\Theta}(\psi, \delta_2\psi) \\ &= d\delta_1\mathbf{\Theta}(\psi, \delta_2\psi) \end{aligned} \tag{2.4.9}$$

where in going from the second to the third line we have used $\mathbf{E} = \delta_1\mathbf{E} = 0$ as well as the commutativity of the variation and the exterior derivative. By an almost identical line of argument we can write

$$\delta_2\delta_1\mathbf{L} = d\delta_2\mathbf{\Theta}(\psi, \delta_1\psi) \tag{2.4.10}$$

and now using the fact that the variation operators commute ($[\delta_1, \delta_2] = 0$) we are able to write

$$\begin{aligned} 0 &= d\delta_1\mathbf{\Theta}(\psi, \delta_2\psi) - d\delta_2\mathbf{\Theta}(\psi, \delta_1\psi) \\ &= d\boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) \end{aligned} \tag{2.4.11}$$

which completes the proof.

Another object associated with the symplectic current is the pre-symplectic form Ω_C which is defined as

$$\Omega_C(\psi, \delta_1\psi, \delta_2\psi) = \int_C \boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) \tag{2.4.12}$$

where C is typically taken to be a Cauchy surface in the spacetime. In the context of ALdS spacetimes we will revisit this property and argue that it is natural to define the integral over a timelike hypersurface instead of a Cauchy surface, more on this later. The name "form" given to Ω_C can at first seem confusing as it has no spacetime indices but is still considered a form due to its antisymmetry in the field variations $\delta_1\psi, \delta_2\psi$ and can thus be thought of as a form over the phase space of all possible field configurations.

Finally, we introduce the differential forms corresponding to the Noether currents and charges. To do this we first consider a diffeomorphism of the spacetime $\mathcal{M}$ generated a by smooth vector field ξ which is taken to be a fixed vector field. To such a diffeomorphism, we can associate a Noether current $(D-1)$-form $\mathbf{J}$ defined by

$$\mathbf{J}[\xi] = \mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - i_\xi\mathbf{L} \tag{2.4.13}$$

where $\mathcal{L}_\xi$ is the Lie derivative in the direction of ξ and $i_\xi\mathbf{L}$ denotes a contraction of ξ^a with the first index of $\mathbf{L}$. When we are on shell ($\mathbf{E} = 0$) we can relate the variation of the Noether current form, $\delta\mathbf{J}$, to the symplectic current form

$$\begin{aligned} \delta\mathbf{J}[\xi] &= \delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - \delta(i_\xi\mathbf{L}) \\ &= \delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - i_\xi\delta\mathbf{L} \\ &= \delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - i_\xi(d\mathbf{\Theta}(\psi, \delta\psi)) \\ &= \delta\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - \mathcal{L}_\xi\mathbf{\Theta}(\psi, \delta\psi) + d(i_\xi\mathbf{\Theta}(\psi, \delta\psi)) \\ &= \boldsymbol{\omega}(\psi, \delta\psi, \mathcal{L}_\xi\psi) + d(i_\xi\mathbf{\Theta}(\psi, \delta\psi)). \end{aligned} \tag{2.4.14}$$

In going from the first to the second line we have used the property of ξ being fixed and from the second to the third line we have used (2.4.3) together with $\mathbf{E} = 0$. In third to the fourth lines we have applied the useful identity $\mathcal{L}_\xi = i_\xi d + di_\xi$ for the Lie derivative acting on differential forms and finally we have applied (2.4.7) in order to reinstate $\boldsymbol{\omega}$. Following similar steps to this derivation, we are also able to show conservation of the Noether current form when on shell ($\mathbf{E} = 0$)

$$\begin{aligned} d\mathbf{J}[\xi] &= d\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - d(i_\xi \mathbf{L}) \\ &= \mathcal{L}_\xi \mathbf{L} - (\mathcal{L}_\xi \mathbf{L} - i_\xi d\mathbf{L}) \\ &= 0. \end{aligned} \tag{2.4.15}$$

On the third line we have used the fact that $d\mathbf{L} = 0$, which is of course true of the exterior derivative acting on any D-form. Since $\mathbf{J}$ is closed on shell, it is locally exact and thus (on shell) we can write

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] \tag{2.4.16}$$

where $\mathbf{Q}[\xi]$ is defined as the Noether charge $(D-2)$-form. Notice that in a similar vein to the symplectic potential form, this equation only defines $\mathbf{Q}$ up to the addition of an exact form, but again we will not be concerned with the specifics of this ambiguity for the purpose of this thesis. More generally, it has been shown in [66] that off-shell we can write

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] + \mathbf{C}_a \xi^a \tag{2.4.17}$$

where $\mathbf{C}_a$ is a $(D-1)$-form which vanishes on shell (physically this form vanishing is fulfillment of the constraint equations of the theory [63]). If we consider field configurations which solve both the equations of motion and the linearised equations of motion we can use (2.4.17) to simplify (2.4.14)

$$\begin{aligned} \boldsymbol{\omega}(\psi, \delta\psi, \mathcal{L}_\xi\psi) &= \delta(d\mathbf{Q}[\xi] + \mathbf{C}_a\xi^a) - d(i_\xi \mathbf{\Theta}(\psi, \delta\psi)) \\ &= d\delta\mathbf{Q}[\xi] - d(i_\xi \mathbf{\Theta}(\psi, \delta\psi)) \\ &= d(\delta\mathbf{Q}[\xi] - i_\xi \mathbf{\Theta}(\psi, \delta\psi)) \end{aligned} \tag{2.4.18}$$

where we used the linearised equations of motion ($\delta\mathbf{C}_a = 0$) in going to the second line. Notice that $\boldsymbol{\omega}$ is exact for these field configurations, thus providing us with an alternative way of proving (2.4.11).

With all of this technology established, we are finally at the stage where we can introduce the concept of Wald Hamiltonians. We will first give the basic definition of the Hamiltonian and then give two necessary and sufficient conditions for existence of such a Hamiltonian.

To define the Hamiltonian we restrict our configuration space to dynamical fields ψ which satisfy the equations of motion ($\mathbf{E} = 0$) but whose variations $\delta\psi$ do not necessarily satisfy the linearised equations of motion. Again, we consider a diffeomorphism of the spacetime $\mathcal{M}$ generated by a fixed vector field ξ and we define the Hamiltonian conjugate to the vector field ξ, H_ξ, as the function which satisfies the following equation

$$\delta H_\xi = \Omega_C(\psi, \delta\psi, \mathcal{L}_\xi\psi) = \int_C \boldsymbol{\omega}(\psi, \delta\psi, \mathcal{L}_\xi\psi) \tag{2.4.19}$$

when we take C to be a Cauchy surface (or a "slice" as in [67]) we can view H_ξ as giving us a natural definition of a conserved quantity associated with the diffeomorphism generated by ξ at the "instant of time" C. We will follow [67] in assuming that the integral on the right hand side of (2.4.19) converges for all ψ which solve the equations of motion and $\delta\psi$ which solve the linearised equations of motion. We will now restrict our consideration to this case.

In order to derive the existence conditions for the Hamiltonian we start by using (2.4.18) to write the variation of the Hamiltonian as

$$\delta H_\xi = \int_C d(\delta\mathbf{Q}[\xi] - i_\xi\boldsymbol{\Theta}(\psi, \delta\psi)) = \int_{\partial C} [\delta\mathbf{Q}[\xi] - i_\xi\boldsymbol{\Theta}(\psi, \delta\psi)] \tag{2.4.20}$$

where we have applied Stokes' theorem and the interpretation of the integral over ∂C is that one takes the integral over the co-dimension 2 manifold given by a cut of C in the limit to asymptotic infinity [67]. We will remain agnostic about the character (i.e. timelike, null or spacelike) of both the hypersurface C and the conformal boundary of the spacetime (which we will generically denote as $\mathscr{I}$) and we will write $\partial C = C \cap \mathscr{I}$.

The Hamiltonian, H_ξ, exists if (2.4.20) can be integrated to give H_ξ, i.e. if we can write the right hand side as a total variation. The first term on the right hand side of (2.4.20) is already a variation so it is only the second term which we need to worry about. By inspecting this term, we can quickly write a necessary and sufficient condition for existence of H_ξ, namely that there exists a $(D-1)$-form $\mathbf{B}$ such that

$$\int_{C\cap\mathscr{I}} i_\xi\boldsymbol{\Theta}(\psi, \delta\psi) = \delta\int_{C\cap\mathscr{I}} i_\xi\mathbf{B}(\psi, \delta\psi) \tag{2.4.21}$$

and thus

$$H_\xi = \int_{C\cap\mathscr{I}} [\mathbf{Q}[\xi] - i_\xi\mathbf{B}(\psi, \delta\psi)]. \tag{2.4.22}$$

An alternative condition for existence of H_ξ can be derived by considering $[\delta_1, \delta_2]H_\xi$, which of course must vanish by the commutativity of the variational derivatives.

$$\begin{aligned}
[\delta_1, \delta_2]H_\xi &= \delta_1\int_{C\cap\mathscr{I}} [\delta_2\mathbf{Q}[\xi] - i_\xi\boldsymbol{\Theta}(\psi, \delta_2\psi)] - \delta_2\int_{C\cap\mathscr{I}} [\delta_1\mathbf{Q}[\xi] - i_\xi\boldsymbol{\Theta}(\psi, \delta_1\psi)] \\
&= \int_{C\cap\mathscr{I}} i_\xi\delta_2\boldsymbol{\Theta}(\psi, \delta_1\psi) - \int_{C\cap\mathscr{I}} i_\xi\delta_1\boldsymbol{\Theta}(\psi, \delta_2\psi) \\
&= \int_{C\cap\mathscr{I}} i_\xi\boldsymbol{\omega}(\psi, \delta_2\psi, \delta_1\psi)
\end{aligned} \tag{2.4.23}$$

and thus the integrability condition becomes

$$\int_{C\cap\mathscr{I}} i_\xi \boldsymbol{\omega}(\psi, \delta_2\psi, \delta_1\psi) = 0. \tag{2.4.24}$$

It seems clear that this condition is necessary for existence but it is also sufficient (proved in [67]). We will now use this Hamiltonian description, along with these existence conditions to compute the Wald Hamiltonians for ALdS spacetimes.

2.4.2 Non existence of a Hamiltonian

In this section, we will briefly discuss the case when the integrability condition (2.4.24) fails, and how one can modify the definition of a ‘conserved quantity’ in this instance. This section will be a summary of the work [67], which originally answered these questions using the covariant phase space approach. We will also discuss an important physical case when this occurs, namely that of a BMS asymptotic symmetry $\mathscr{I}^+$ in asymptotically flat spacetime (as we introduced in section 2.2.4). This will allow us to derive a formula for the Bondi mass loss (2.2.17) from the covariant phase space approach.

With the example of BMS in mind, in this subsection we will always take the vector field ξ to be complete on $\mathcal{M}\cup\mathscr{I}$ (meaning ξ is tangent to $\mathscr{I}$ on $\mathscr{I}$) and an asymptotic symmetry. From the point of view of the covariant phase space, having ξ as an asymptotic symmetry means that the 1-parameter group of diffeomorphisms associated with ξ maps $\bar{\mathcal{F}}$ into itself, i.e. it will preserve any asymptotic conditions in the specification of $\bar{\mathcal{F}}$. An example would be the space of asymptotically flat spacetimes in the Bondi gauge (2.2.6), which are defined by the fall off conditions (2.2.18). In this case the asymptotic symmetries (the BMS group) are precisely the transformations which preserve both the gauge and the asymptotic falloffs, hence mapping $\bar{\mathcal{F}}$ into itself.

As discussed above, we will now consider the case when

$$\int_{C\cap\mathscr{I}} i_\xi \boldsymbol{\omega}(\psi, \delta_2\psi, \delta_1\psi) \neq 0 \tag{2.4.25}$$

and thus H_ξ will not exist. In order to construct a new notion of a ‘Hamiltonian’, we follow [67] and consider the pullback of $\boldsymbol{\omega}$ to $\mathscr{I}$, which we will denote $\bar{\boldsymbol{\omega}}$. On $\mathscr{I}$, we assume that there exists a symplectic potential $\bar{\boldsymbol{\Theta}}$ on $\mathscr{I}$ which generates $\bar{\boldsymbol{\omega}}$ via

$$\bar{\boldsymbol{\omega}}(\psi, \delta_1\psi, \delta_2\psi) = \delta_1\bar{\boldsymbol{\Theta}}(\psi, \delta_2\psi) - \delta_2\bar{\boldsymbol{\Theta}}(\psi, \delta_1\psi) \tag{2.4.26}$$

where we continue to take the all fields (ψ) and their variations ($\delta_1\psi, \delta_2\psi$) on-shell. $\bar{\boldsymbol{\Theta}}$ is required to be locally constructed from the fields ψ and their derivatives, as well as the “universal background structure” of $\mathcal{M}\cup\mathscr{I}$, which can be considered as non-dynamical quantities which enter into the definition of the configuration space (e.g. the conformal factor in (2.1.1)). Following [67], we will now define the *modified Hamiltonian*, $\mathcal{H}_\xi$, via

$$\delta\mathcal{H}_\xi = \int_{C\cap\mathscr{I}}[\delta\mathbf{Q} - i_\xi\mathbf{\Theta}] + \int_{C\cap\mathscr{I}} i_\xi\bar{\mathbf{\Theta}} \tag{2.4.27}$$

which we observe to be almost the same as (2.4.20), except now an additional term involving $\bar{\mathbf{\Theta}}$ makes an appearance on the right hand side. We can now repeat the calculation of the consistency check (2.4.23) and we find

$$[\delta_1,\delta_2]\mathcal{H}_\xi = -\int_{C\cap\mathscr{I}} i_\xi\boldsymbol{\omega}(\psi,\delta_1\psi,\delta_2\psi) + \int_{C\cap\mathscr{I}} i_\xi\bar{\boldsymbol{\omega}}(\psi,\delta_1\psi,\delta_2\psi) = 0 \tag{2.4.28}$$

where we used the property that ξ is tangent to $\mathscr{I}$ on $\mathscr{I}$. This shows that now the consistency check is satisfied and we have identified a candidate for a "conserved quantity" when the original prescription fails. However, we are not finished yet, due to an obvious ambiguity in the presymplectic potential on $\mathscr{I}$

$$\bar{\mathbf{\Theta}} \to \bar{\mathbf{\Theta}} + \delta\mathbf{W} \tag{2.4.29}$$

which clearly generates the same $\bar{\boldsymbol{\omega}}$ via (2.4.26). In order to remove this ambiguity, we will need to consider additional restrictions upon $\bar{\mathbf{\Theta}}$.

As a prelude to these restrictions, we begin by considering the difference between variations of $\mathcal{H}_\xi$ at different cuts of $\mathscr{I}$ (arising from distinct bulk hypersyrfaces C_1, C_2 where C_2 lies in the causal future of C_1)

$$\delta\mathcal{H}_\xi|_{\partial C_2} - \delta\mathcal{H}_\xi|_{\partial C_1} = -\int_{\mathscr{I}_{12}} (\bar{\boldsymbol{\omega}}(\psi,\delta\psi,\mathcal{L}_\xi\psi) + d[i_\xi\bar{\mathbf{\Theta}}(\psi,\delta\psi)]) \tag{2.4.30}$$

where $\mathscr{I}_{12}$ denotes the portion of $\mathscr{I}$ with boundary $\partial C_1 \sqcup \partial C_2$. In order to arrive the right hand side above, we have used (2.4.18) as well as Stokes' theorem in order to write the integrals over cuts as a boundary integral. The first observation that we make regarding the equation above is that it shows that for a general asymptotic symmetry ξ, our "conserved quantity" $\mathcal{H}_\xi$ will not actually be conserved as the right hand side of (2.4.30) does not vanish. Physically speaking, this means that there is a non-zero flux of charge through $\mathscr{I}_{12}$, to which we can associate a flux $(n-1)$-form $\boldsymbol{F}_\xi$, defined via

$$\delta\boldsymbol{F}_\xi = \bar{\boldsymbol{\omega}}(\psi,\delta\psi,\mathcal{L}_\xi\psi) + d[i_\xi\bar{\mathbf{\Theta}}(\psi,\delta\psi)] \tag{2.4.31}$$

upon which we can use the definition of the Lie derivative on differential forms together with equation (2.4.26) to write

$$\delta\boldsymbol{F}_\xi = \delta\bar{\mathbf{\Theta}}(\psi,\mathcal{L}_\xi\psi). \tag{2.4.32}$$

Finally we are at the stage where we can give the extra restriction upon $\bar{\mathbf{\Theta}}$, based upon physical considerations of the flux. Following [67], we require that the flux vanishes upon solutions where no radiation is present, i.e. stationary solutions. We also assume that there exists a "reference solution" ψ_0, where $\mathcal{H}_\xi$ vanishes, which is itself stationary. These assumptions translate to the result that $\bar{\mathbf{\Theta}}(\psi, \delta\psi)$ vanishes whenever ψ is stationary. We can use this, together with $\boldsymbol{F}_\xi|_{\psi_0} = 0$ to integrate (2.4.32) and obtain

$$\boldsymbol{F}_\xi = \bar{\mathbf{\Theta}}(\psi, \mathcal{L}_\xi \psi) \tag{2.4.33}$$

from which we note that we also have $\boldsymbol{F}_\xi = 0$ whenever $\mathcal{L}_\xi \psi = 0$ i.e. when ξ is an exact symmetry. For a further discussion on the requirements for the existence of a suitable ψ_0, we refer the reader to the original work [67]. We also note that this stationary assumption upon $\bar{\mathbf{\Theta}}(\psi, \delta\psi)$ does not a priori uniquely select $\boldsymbol{W}(\psi)$, but enforces the restriction that $\delta \boldsymbol{W}$ vanishes when ψ is stationary. In the case of null infinity, these requirements do uniquely select $\bar{\mathbf{\Theta}}$ [67].

To recap, (2.4.27) together with a zero-charge reference solution ψ_0 allow us to define a notion of a "charge with flux" for asymptotic symmetries ξ. Equation (2.4.33) is the flux formula which then allows us to describe how flux leaks through a portion of the conformal boundary $\mathscr{I}_{12}$. The most studied occurrence of such a phenomenon is that of null infinity in asymptotically flat spacetime [156, 161, 67, 87, 11] (as we will briefly recap) although there is a recent development [51] to try and study flux at the conformal boundary of $\Lambda \neq 0$ spacetimes. Such issues are intrinsically related to the boundary conditions imposed at the conformal boundary of the spacetime, as well as the presence (or lack of) a conformal anomaly. We will return to these issues in chapter 5, where we consider fluxes at the conformal boundary of AldS$_4$ spacetimes.

2.4.3 Fluxes at $\mathscr{I}^+$- Bondi mass loss

We will now give a summary of this procedure when applied to the theory of four dimensional general relativity ($\mathbf{L} = R\boldsymbol{\epsilon}/16\pi G_N$), in particular at null infinity in asymptotically flat spacetime. The first set of results we will briefly discuss are those of Wald & Zoupas [67] who used the conformal approach to asymptotic flatness at $\mathscr{I}^+$ as we discussed in section 2.1.1. We will then give the adaptation of these results when the spacetime is defined to be asymptotically flat using the Bondi gauge, as we also discussed in section 2.2.3, a procedure that was first developed by Barnich & Troessaert [11]. We will use these results to show how one can arrive at the Bondi mass loss formula from the covariant phase space formalism.

Using the procedure of the previous section, [67] give the following prescription for charges $\mathcal{H}_\xi$ where ξ is a BMS symmetry in the sense of (2.2.40)

$$\delta\mathcal{H}_\xi = \int_{C\cap\mathscr{I}^+}[\delta\mathbf{Q} - i_\xi\mathbf{\Theta}] - \frac{1}{32\pi G_N}\int_{C\cap\mathscr{I}^+}(N_{AB}\tau^{AB})i_\xi\bar{\boldsymbol{\epsilon}} \tag{2.4.34}$$

together with $\mathcal{H}_\xi = 0$ for all ξ and all cuts of $\mathscr{I}^+$ in Minkowski spacetime (in the language of the previous section, this is $\psi_0 = \{\eta_{\mu\nu}\}$). In the formula above we have

$$Q_{\mu\nu}[\xi] = -\frac{1}{16\pi G_N}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\xi^\sigma \tag{2.4.35}$$

and

$$\Theta_{\mu\nu\rho} = \frac{1}{16\pi G_N}\epsilon_{\mu\nu\rho\sigma}v^\sigma \tag{2.4.36}$$

where

$$v^\mu = g^{\mu\nu}g^{\rho\sigma}[\nabla_\rho\delta g_{\nu\sigma} - \nabla_\nu\delta g_{\rho\sigma}] \tag{2.4.37}$$

all of which can be verified starting from the definitions given in section 2.4.1. N_{AB} is the Bondi news tensor defined in (2.2.15) and τ_{AB} is given by

$$\tau_{AB} = \Omega\delta g_{AB} \tag{2.4.38}$$

where Ω is the conformal factor used in the definition of asymptotic flatness (2.1.1). Finally, we note that $\bar{\boldsymbol{\epsilon}}$ is the volume element at $\mathscr{I}^+$. Equation (2.4.34) also allows us to read off

$$\bar{\mathbf{\Theta}} = -\frac{1}{32\pi G_N}N_{AB}\tau^{AB}\bar{\boldsymbol{\epsilon}} \tag{2.4.39}$$

and defining $\chi_{AB} = \Omega\mathcal{L}_\xi g_{AB}$, we arrive at the flux formula

$$\boldsymbol{F}_\xi = \bar{\mathbf{\Theta}}(g_{\mu\nu}, \mathcal{L}_\xi g_{\mu\nu}) = -\frac{1}{32\pi G_N}N_{AB}\chi^{AB}\bar{\boldsymbol{\epsilon}}. \tag{2.4.40}$$

The analysis of [67] also includes a proof that that ambiguity of (2.4.29) does not play a role (the requirement that there is no flux for stationary spacetimes enforces $\boldsymbol{W} = 0$) and that Minkowski spacetime is the unique zero charge solution. These formulae are very elegant and give us a prescription to compute all supertranslation charges and their associated fluxes, although it is not immediately apparent how these formulae can be applied to the coordinate dependent definition of asymptotic flatness using the Bondi gauge as introduced section 2.2.3. This matching of the two definitions was performed in [11], where the authors found

$$\mathcal{H}_\xi = \frac{1}{16\pi G_N}\int_{C\cap\mathscr{I}^+} d^2\Omega\left[4fm_B + Y^A\left(2N_A + \frac{1}{16}\partial_A(C^{BC}C_{BC})\right)\right] \tag{2.4.41}$$

where $d^2\Omega$ is the area element on the unit round S^2 and we use the notation of section 2.2.4 in that f describes supertranslations, and Y^A superrotations. N_A is the angular momentum aspect and C_{AB} the Bondi shear tensor. [11] also gives the Bondi-gauge flux formula

$$\boldsymbol{F}_\xi = \frac{1}{32\pi G_N} N^{AB} \mathcal{L}_\xi C_{AB} \bar{\boldsymbol{\epsilon}} \tag{2.4.42}$$

which now gives us a direct connection between the flux at $\mathscr{I}^+$ and the Bondi news (N_{AB}) and shear (C_{AB}) tensors.

The most famous application of this formula is a computation of the flux formula for the Bondi mass, $\mathcal{M}_B$, which by comparison with (2.2.14) is clearly the modified Hamiltonian corresponding to $f = 1$, $Y^A = 0$ ($\xi = \partial_u$). Application of the flux formula (2.4.42) now yields

$$\boldsymbol{F}_{\partial_u} = \frac{1}{32\pi G_N} N_{AB} N^{AB} \bar{\boldsymbol{\epsilon}} \tag{2.4.43}$$

where we computed $\mathcal{L}_\xi C_{AB}$ using the action of a BMS symmetry on the metric. We also note that the coefficient of the flux form is strictly positive when evaluated upon a cut of $\mathscr{I}^+$ and gives the Bondi mass loss formula

$$\partial_u \mathcal{M}_B = -\int_{C\cap\mathscr{I}^+} \bar{\boldsymbol{F}}_{\partial_u} = -\frac{1}{32\pi G_N}\int_{C\cap\mathscr{I}^+} d^2\Omega\, N^{AB} N_{AB} \leq 0 \tag{2.4.44}$$

where $\bar{\boldsymbol{F}}$ is the pullback of $\boldsymbol{F}$ to $C \cap \mathscr{I}^+$.

This computation shows that one is able to recover the Bondi mass and mass loss from the covariant phase space formalism. In fact, the procedure given in [67] is sufficient to compute charges and fluxes for all supertranslations in the (unextended) BMS group. The work of Barnich & Troessaert [9, 87, 10, 11] goes beyond that of [67] in that it allows one to compute charges for the extended BMS group (2.2.41), i.e. to allow for meromorphic superrotations. We can see this from (2.4.41), where there is no obstruction to choosing a singular conformal Killing vector Y^A. We also comment that there is ongoing work to understand how one can compute charges for the proposed extension of the BMS group where Y^A is now allowed to represent any diffeomorphism of S^2 [96, 125, 99][4]. In this case, it seems from [98] that the symplectic structure for such an asymptotic symmetry group cannot be constructed from a local and covariant symplectic current at $\mathscr{I}^+$. Due to this, the attempts to define charges would necessarily include the feature of including terms non-local in the Bondi metric quantities.

[4]This is often to referred to as the generalised BMS group.

CHAPTER 3

(A)dS_4 in Bondi gauge

3.1 Introduction and summary of results

The Bondi metric was introduced in the seminal works of Bondi, Sachs and others on gravitational radiation [1, 2]. While all gauges are equivalent a convenient choice of a coordinate system may bring in simplifications and make the physical properties of spacetimes most transparent. In the case of gravitational radiation the objective was to examine the behaviour of the gravitational field far from the isolated object generating the radiation, and to obtain and use asymptotic solutions of Einstein equations to characterise radiating spacetime.

In asymptotically flat gravity, gravitational waves travel to future null infinity and the task becomes that of obtaining asymptotic solutions near future null infinity. It was shown in [1, 2] that in Bondi gauge the Einstein equations take a nested form and they can be readily integrated near null infinity. If one specifies initial data on an outgoing null hypersurface then the Einstein equations tell us how to propagate this data forward in time to a nearby outgoing null hypersurface. The asymptotic solution involves a number of data that are not determined by the asymptotic analysis alone: such data will be fixed in any given exact solution of the field equations. This undetermined data consists of a scalar function (the Bondi mass aspect); a vector (the angular momentum aspect) and a tensor (the Bondi news). The mass and angular momentum aspects integrated over a cut at null infinity define the total mass and total angular momentum[1] of the system at that time and the news tensor controls how these quantities change in time. In particular, one can show that if the news tensor vanishes (and the matter stress energy tensor goes to zero fast enough at future null infinity) the total mass is constant, while if the news tensor is non-vanishing the total Bondi mass monotonically decreases in time capturing the fact that the system loses mass by emitting gravitational waves.

In the presence of a cosmological constant the nature of infinity changes: with negative cosmological constant conformal infinity is timelike while with positive cosmological constant infinity is spacelike. As there is no null infinity in either case one may question whether analyzing Einstein equations with non-zero cosmological constant in Bondi gauge would be useful. There are however several reasons to do this. In the case of a negative cosmological constant, as we review below, asymptotic solutions in Fefferman-Graham gauge [59] have a clear holographic meaning [60] and one would like to understand the holographic meaning of the data in Bondi gauge. This may then be used to get insight into a possible holographic structure of asymptotically flat gravity. In addition, Bondi-like gauges where Einstein equations take a nested form have been in the used already in the holography literature (see [162] and references therein) and it would be desirable to understand how to extract the holographic data directly in this gauge. Furthermore, an analogue of Bondi mass with many interesting properties has already been defined for a class of asymptotically locally AdS spacetimes [69] and one would like to understand whether such a quantity exists more generally in asymptotically locally AdS spacetimes.

In the case of positive cosmological constant such results are needed even more urgently: current observations indicate that we live in a Universe with a positive cosmological constant and we have also observed gravitational waves. Yet a satisfactory discussion of gravitational waves in de Sitter spacetime is still missing. Recent works addressing these issues include [41, 163, 42, 43, 44, 164, 48, 46, 47, 49, 165].

With negative cosmological constant, the appropriate boundary conditions are to fix a conformal class of metrics on the conformal boundary, and a natural coordinate system to use is Gaussian normal coordinates centred at the conformal boundary, the Fefferman-Graham gauge [59]. One may then obtain the general asymptotic solution to Einstein equations by treating the radial coordinate as a small parameter. The Einstein equations become algebraic in this gauge (i.e. they are solved by algebraic manipulation rather than by integrating differential equations) and the pieces of data needed that are left undetermined by the asymptotic analysis are the conformal class and a covariantly conserved symmetric traceless tensor (in even dimensions, in odd dimensions the tensor has a trace). In holography, the boundary metric is the background for the dual CFT and the tensor

[1]This definition of angular momentum suffers from supertranslation ambiguities. This issue will not play a role here.

is (the quantum expectation value) of the energy momentum tensor [58, 60]. The same tensor can be used to obtain the bulk conserved changes when the spacetime possesses asymptotic Killing vectors [71].

With positive cosmological constant, one may similarly use Gaussian normal coordinates centred at future infinity and work out the asymptotic expansion [166] and the data are again a conformal class of metrics and a covariantly conserved symmetric traceless tensor. Actually, the asymptotic solutions for positive and negative cosmological are related by simple analytic continuation [62].

With non-zero cosmological constant, one may foliate infinity with null hypersurfaces, now ending either at timelike infinity (negative Λ) or spacelike infinity (positive Λ). The structure of the Einstein equations in Bondi gauge and in the presence of a cosmological constant is very similar to that with no cosmological constant. To explain the similarities and differences relative to the case of no cosmological constant we first briefly review the latter.

In this thesis for simplicity we restrict ourselves to $d = 4$ and axial and reflection symmetry. It would be straightforward but tedious to relax these conditions. The metric in Bondi gauge (for any cosmological constant) then takes the form

$$\begin{aligned} ds^2 = &-(Wr^2e^{2\beta} - U^2r^2e^{2\gamma})du^2 - 2e^{2\beta}dudr - \\ &2Ur^2e^{2\gamma}dud\theta + r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2). \end{aligned} \tag{3.1.1}$$

Here u is retarded Bondi time, r is a radial coordinate and θ, ϕ parametrise the transverse space (which is topologically an S^2) and W, U, β, γ are functions to be determined by solving Einstein equations.

We find it useful also to define the coordinate $z = 1/r$, which brings infinity to $z = 0$. Inserting (3.1.1) in the Einstein equations leads to four main equations and three supplementary conditions. One can then show that the coefficients appearing in these equations are regular as $z \to 0$. This means that they admit asymptotic solutions with W, U, β, γ being regular around $z = 0$ and one can obtain the asymptotic solutions by successively differentiating the equations w.r.t. z, setting $z = 0$ and solving the resulting equations (as was done for AdS gravity in Fefferman-Graham gauge in [60]). In all cases we solve the resulting equations in the most general way, so we obtain the most general asymptotic solutions of Einstein equations with the only assumption being that the functions W, U, β, γ are four times differentiable.

With no cosmological constant, one provides as initial condition the value of γ at a null hypersurface $u = u_0 = const.$ Imposing the "out-going gauge condition" $\gamma_{,zz} = 0$ [2] (as

[2]Indices after comma indicate differentiation, i.e. $\gamma_{,z} = \partial\gamma/\partial z$ etc.

in [1]) one finds that all functions admit regular Taylor expansions around $z = 0$, and one can iteratively solve for all coefficients, except that the coefficient $W_{,zzz}, U_{,zzz}, \gamma_{,z}$ are left undetermined by the asymptotic analysis (apart from two equations that link their u and θ derivatives). These three functions are essentially the Bondi mass aspect, the angular momentum aspect and the Bondi news mentioned earlier and the relation of their derivatives is linked to the monotonicity of the total Bondi mass. This data is then enough to determine $\gamma_{,u}$ which allows us to obtain γ at $u = u_0 + \delta u$ and thus continue the iterative construction of the solution. Note that if one is to relax the "out-going gauge condition" then the solution will also contain logarithmic terms in z [34].

In the presence of a cosmological constant (with any sign), three of the four main equations can be solved in exactly the same way as in the $\Lambda = 0$ case but the fourth equation couples the coefficients in such a way that the integration scheme we used for $\Lambda = 0$ does not work any more. We have found however two alternative integration schemes. First, we note that the "out-going gauge condition" $\gamma_{,zz} = 0$ is now implied by the field equations, so there is no possibility for logarithms in the case of the vacuum Einstein equations with cosmological constant (in four dimensions). In the presence of matter such terms can arise and they always have a meaning in the AdS/CFT correspondence: they are related to conformal anomalies of the dual CFT. The cases of $\Lambda > 0$ is related to $\Lambda < 0$ by analytic continuation. We will phrase our discussion using the AdS language, but the same integrations schemes also apply to the dS case (but one should note that ∂_u now becomes spacelike at future infinity).

The first integration scheme, which we call the "boundary scheme", requires as initial data the values of U, β, γ and $\gamma_{,zzz}, U_{,zzz}, W_{,zzz}$ at $z = 0$ (i.e. at the conformal boundary). One can understand the meaning of this data by transforming to the Fefferman-Graham gauge. Recall that in Fefferman-Graham gauge (l is the AdS radius)

$$ds^2 = l^2 \left[\frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2} (g_{(0)ab} + \rho^2 g_{(2)ab} + \rho^3 g_{(3)ab} + \ldots) dx^a dx^b \right], \tag{3.1.2}$$

where now the free data is $g_{(0)}$ and $g_{(3)}$ (with $g_{(3)}$ traceless and divergenceless), with $g_{(0)}$ being a representative of the conformal class and the background metric of the dual CFT and $g_{(3)}$ is related to the energy momentum tensor of the dual CFT. Now U, β, γ at $z = 0$ determine $g_{(0)}$, while $\gamma_{,zzz}, U_{,zzz}, W_{,zzz}$ at $z = 0$ determine $g_{(3)}$. So the analysis in Bondi gauge reproduces the salient features of the asymptotic solutions in Fefferman-Graham gauge.

As mentioned earlier, one can obtain the bulk conserved charges from $g_{(3)}$ and thus as in the asymptotically flat case $U_{,zzz}, W_{,zzz}$ are related to conserved charges, and so is $\gamma_{,zzz}$ which was not related to a conserved charge in the asymptotically flat case. In contrast to the asymptotically flat case $\gamma_{,z}$ is now fully determined in terms U, β, γ at $z = 0$, *i.e.* the analogue of the news is now fixed. If we further restrict to Asymptotically AdS solutions, $\gamma_{,z}$ actually vanishes and the Bondi mass is constant. Similar observations were made in [41, 164, 46, 47, 49] (mostly for the dS case). One can understand this result as follows. Since AdS and dS do not have a null infinity, any gravitational radiation will have to be absorbed at the conformal boundary and this would make the boundary metric time dependent. If we fix the boundary metric to be time independent as in the case of Asymptotically AdS solutions then there is no possibility for gravitational radiation. A class of radiating spacetimes in AdS, the Robinson-Trautman spacetimes are indeed asymptotically locally AdS and have a time dependent boundary metric [167, 69].

The second integration scheme is a hybrid version of the flat scheme and the previous one: one fixes now $\gamma, W_{,zzz}, U_{,zzz}$ at a null hypersurface $u = u_0 = const$ and U, β, γ at $z = 0$, for all times $u \geq u_0$. With this data one can recursively construct the solution to the future of the initial hypersurface.

The rest of this chapter is organised as follows. In section 3.2 we provide the detailed derivation of the asymptotic solutions and in section 3.3 we compare and contrast the different integration schemes used in section 3.2. In section 3.4 we derive the transformation from Bondi gauge to Fefferman-Graham gauge and discuss the holographic interpretation of the functions appearing in the asymptotic solution in Bondi gauge. In this section we also illustrate the discussion using AdS_4, Schwarzschild AdS_4 and AdS_4 black branes as examples and discuss the properties of Bondi mass for asymptotically AdS_4 solutions. We conclude in section 3.5. The chapter contains a number of appendices: in appendix A.1 we present the solution of the supplementary conditions for asymptotically locally (A)dS solutions, in appendix A.2 we provide technical details about the coordinate transformation from Bondi gauge to Fefferman-Graham gauge, in appendix A.3 we discuss the presence of logarithmic terms in the asymptotic solutions when appropriate matter is present and in appendix A.4 we show the equivalence of the Bondi and Abbott-Deser masses in asymptotically AdS spacetimes.

3.2 The Einstein field equations

In this section we will compute the vacuum Einstein equations in the presence of a cosmological constant for an axisymmetric, ϕ-reflection symmetric Bondi gauge metric. The techniques employed in doing this are very similar to those of [1] and many of the properties of the original method carry over.

3.2.1 General considerations

We first apply some simplifications to the general Bondi gauge metric. Working in coordinates (u, r, θ, ϕ), we enforce both axi-symmetry ($\partial/\partial\phi$ a Killing vector field) and reflection symmetry in ϕ (so the metric is invariant under $d\phi \to -d\phi$). In Bondi function notation, this means we set $h_{\theta\phi} = h_{\phi\theta} = U^{\phi} = 0$, reducing the number of unknown functions to four. These choices are made entirely for computational simplification in the calculations that follow.

Following [1], we now write the remaining functions in the form

$$X = Wr^2e^{2\beta}, \qquad h_{\theta\theta} = r^2e^{2\gamma}, \qquad h_{\phi\phi} = r^2\sin^2\theta e^{-2\gamma}, \qquad U^{\theta} = -2U \tag{3.2.1}$$

giving us the line element

$$\begin{aligned} ds^2 = &-(Wr^2e^{2\beta} - U^2r^2e^{2\gamma})du^2 - 2e^{2\beta}dudr- \\ &2Ur^2e^{2\gamma}dud\theta + r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2). \end{aligned} \tag{3.2.2}$$

This choice of metric has a restriction in the determinant along the sphere ($\det(h_{AB}/r^2) = \sin^2\theta$); r is a luminosity distance. The Einstein equations are expressed in terms of the four metric functions ($\gamma(u, r, \theta)$, $\beta(u, r, \theta)$, $U(u, r, \theta)$, $W(u, r, \theta)$).

In this chapter we will analyse the Einstein vacuum equations,

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{3.2.3}$$

The generalization to include matter would be straightforward. It is quite common in the relativity literature to solve Einstein's equations with "asymptotically vacuum" matter such that $\lim_{r\to\infty} T_{\mu\nu} = 0$; an example can be found in [12], involving an asymptotic power series expansion in negative powers of the radial coordinate. However, as is well known in holography, the presence of matter generically affects the powers arising in the asymptotic expansions and logarithmic terms can arise for matter of specific masses, see the discussion in appendix A.3 as well as the references [60, 62].

Following [1], we separate Einstein's equations into the four 'main equations'

$$R_{rr} = R_{r\theta} = 0, \quad R_{\theta\theta} = \Lambda r^2e^{2\gamma}, \quad R_{\phi\phi} = \Lambda r^2e^{-2\gamma}\sin^2\theta; \tag{3.2.4}$$

three 'trivial equations'

$$R_{u\phi} = R_{r\phi} = R_{\theta\phi} = 0 \tag{3.2.5}$$

and three 'supplementary equations'

$$R_{uu} = -\Lambda(Wr^2e^{2\beta} - U^2r^2e^{2\gamma}), \quad R_{u\theta} = -\Lambda Ur^2e^{2\gamma}, \quad R_{ur} = -\Lambda e^{2\beta}. \tag{3.2.6}$$

The main equations are so named because they must be solved in order to generate solutions to the field equations. The trivial equations are automatically satisfied because of the symmetries of the spacetime metric. The supplementary conditions will be shown to provide constraint equations for the functions of integration arising from the main equations. These will be discussed in section 3.2.4 but first we will focus our attention on the main equations:

$$0 = -R_{rr} = -4\left[\beta_{,r} - \frac{1}{2}r(\gamma_{,r})^2\right]r^{-1} \tag{3.2.7a}$$

$$\begin{aligned}0 = 2r^2R_{r\theta} = {}& [r^4e^{2(\gamma-\beta)}U_{,r}]_{,r} - \\ & 2r^2[\beta_{,r\theta} - \gamma_{,r\theta} + 2\gamma_{,r}\gamma_{,\theta} - 2\beta_{,\theta}r^{-1} - 2\gamma_{,r}\cot\theta]\end{aligned} \tag{3.2.7b}$$

$$\begin{aligned}-2\Lambda r^2e^{2\beta} = -R_{\theta\theta}e^{2(\beta-\gamma)} - r^2R^{\phi}_{\phi}e^{2\beta} = {}& 2(r^3W)_{,r} + \frac{1}{2}r^4e^{2(\gamma-\beta)}(U_{,r})^2 - r^2U_{,r\theta} - \\ & 4rU_{,\theta} - r^2U_{,r}\cot\theta - 4rU\cot\theta + \\ & 2e^{2(\beta-\gamma)}[-1 - (3\gamma_{,\theta} - \beta_{,\theta})\cot\theta - \\ & \gamma_{,\theta\theta} + \beta_{,\theta\theta} + (\beta_{,\theta})^2 + 2\gamma_{,\theta}(\gamma_{,\theta} - \beta_{,\theta})]\end{aligned} \tag{3.2.7c}$$

$$\begin{aligned}-\Lambda r^2e^{2\beta} = -r^2R^{\phi}_{\phi}e^{2\beta} = {}& 2r(r\gamma)_{,ur} + (1 - r\gamma_{,r})(r^3W)_{,r} - r^3(r\gamma_{,rr} + \gamma_{,r})W - \\ & r(1 - r\gamma_{,r})U_{,\theta} - r^2(\cot\theta - \gamma_{,\theta})U_{,r} + \\ & r(2r\gamma_{,r\theta} + 2\gamma_{,\theta} + r\gamma_{,r}\cot\theta - 3\cot\theta)U \\ & + e^{2(\beta-\gamma)}[-1 - (3\gamma_{,\theta} - 2\beta_{,\theta})\cot\theta - \\ & \gamma_{,\theta\theta} + 2\gamma_{,\theta}(\gamma_{,\theta} - \beta_{,\theta})].\end{aligned} \tag{3.2.7d}$$

Notice that the first two equations agree with the first two main equations in [1]. The second two are altered by the inclusion of the cosmological constant but they manifestly reduce to the original equations in the $\Lambda \to 0$ limit. We will now follow closely the integration scheme of [1] to see how this alters the solutions to the equations above.

We will first solve the main equations following the same approach as the original analysis [1]:

1) Specify $\gamma(u, r, \theta)$ on an initial null hypersurface $\mathcal{N}_{u_0}$ i.e. $\gamma(u_0, r, \theta)$.

2) Solve (3.2.7a) on the null hypersurface $\mathcal{N}_{u_0}$ to compute $\beta(u_0, r, \theta)$. This is possible as only $\gamma(u_0, r, \theta)$ appears in the equation,

3) Solve (3.2.7b) for $U(u_0, r, \theta)$. This is possible as only $\gamma(u_0, r, \theta)$ and $\beta(u_0, r, \theta)$ appear in the equation.

4) Solve (3.2.7c) for $W(u_0, r, \theta)$. Only $\gamma(u_0, r, \theta)$, $\beta(u_0, r, \theta)$ and $U(u_0, r, \theta)$ appear in the equation.

5) Solve equation (3.2.7d) for $\gamma_{,u}(u_0, r, \theta)$ i.e. to obtain γ on the next null hypersurface $\mathcal{N}_{u_0+\delta u}$.

6) Repeat from step 1 with the new Bondi time $u_0 + \delta u$. Iteration gives the Einstein solution for the future domain of dependence of $\mathcal{N}_{u_0}$, $D^+(\mathcal{N}_{u_0})$, see Fig. 3.2.1.

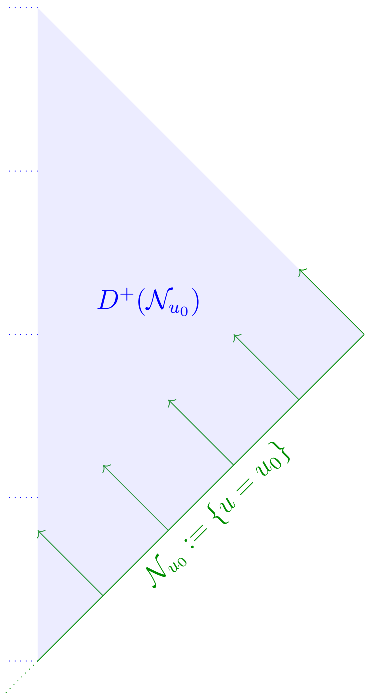

Figure3.2.1: Causal diagram illustrating how one applies the BMS scheme when given suitable initial data on a null hypersurface $\mathcal{N}_{u_0}$

Specialising briefly to the case of AdS, we observe that unlike asymptotically flat spacetime we have $D^+(\mathcal{N}_{u_0}) \neq J^+(\mathcal{N}_{u_0})$, where J^+ indicates the causal future. To solve the equations in $J^+(\mathcal{N}_{u_0})$ we would need to specify extra data on a new hypersurface (e.g. the conformal boundary $\mathscr{I}$). We will discuss in detail the integration scheme for AdS asymptotics in section 3.3.

In the case of asymptotically locally dS spacetimes the situation is slightly different, firstly because we now have two boundaries: future spacelike infinity, $\mathscr{I}^+$, and past spacelike infinity, $\mathscr{I}^-$. We will restrict consideration to a retarded null foliation of the future spacelike boundary, $\mathscr{I}^+$, when we discuss this case in greater detail in section 3.3.3. We will see that this has a number of different subtleties when compared with the flat and AdS cases.

3.2.2 Solving the main equations asymptotically

We are interested in solving the Einstein equations in the *asymptotic region* (large r) of the spacetime in the most general manner possible. It is convenient to implement an inversion map

$$r = \frac{1}{z} \tag{3.2.8}$$

so that solving the equations as $r \to \infty$ is reduced to the analytically simpler procedure of solving around $z = 0$. Carrying out this substitution in the main equations gives

$$0 = 2\beta_{,z} + z(\gamma_{,z})^2 \tag{3.2.9a}$$

$$\begin{aligned} 0 = {} & 4\beta_{,\theta} - 2e^{2\gamma-2\beta}U_{,z} - 2ze^{2\gamma-2\beta}U_{,z}\beta_{,z} - 4z\cot\theta\gamma_{,z} + 4z\gamma_{,\theta}\gamma_{,z} + \\ & 2ze^{2\gamma-2\beta}U_{,z}\gamma_{,z} + 2z\beta_{,z\theta} - 2z\gamma_{,z\theta} + ze^{2\gamma-2\beta}U_{,zz} \end{aligned} \tag{3.2.9b}$$

$$\begin{aligned} -2\Lambda e^{2\beta} = {} & 6W + 4z\cot\theta U - 2zW_{,z} - 4zU_{,\theta} - 2z^2e^{2\beta-2\gamma} + \\ & 2z^2e^{2\beta-2\gamma}\cot\theta\beta_{,\theta} + 2z^2e^{2\beta-2\gamma}(\beta_{,\theta})^2 - 6z^2e^{2\beta-2\gamma}\cot\theta\gamma_{,\theta} - \\ & 4z^2e^{2\beta-2\gamma}\beta_{,\theta}\gamma_{,\theta} + 4z^2e^{2\beta-2\gamma}(\gamma_{,\theta})^2 + 2z^2e^{2\beta-2\gamma}\beta_{,\theta\theta} - \\ & 2z^2e^{2\beta-2\gamma}\gamma_{,\theta\theta} + z^2\cot\theta U_{,z} + \frac{z^2}{2}e^{2\gamma-2\beta}(U_{,z})^2 + z^2U_{,z\theta} \end{aligned} \tag{3.2.9c}$$

$$\begin{aligned} -\Lambda e^{2\beta} = {} & 3W - 3z\cot\theta U - zU_{,\theta} + 2zU\gamma_{,\theta} - zW_{,z} + 2zW\gamma_{,z} + 2z\gamma_{,u} - \\ & z^2e^{2\beta-2\gamma} + 2z^2e^{2\beta-2\gamma}\cot\theta\beta_{,\theta} - 3z^2e^{2\beta-2\gamma}\cot\theta\gamma_{,\theta} - \\ & 2z^2e^{2\beta-2\gamma}\beta_{,\theta}\gamma_{,\theta} + 2z^2e^{2\beta-2\gamma}(\gamma_{,\theta})^2 - z^2e^{2\beta-2\gamma}\gamma_{,\theta\theta} + z^2\cot\theta U_{,z} - \\ & z^2\gamma_{,\theta}U_{,z} - z^2\cot\theta U\gamma_{,z} - z^2U_{,\theta}\gamma_{,z} - z^2W_{,z}\gamma_{,z} - \\ & 2z^2U\gamma_{,z\theta} - z^2W\gamma_{,zz} - 2z^2\gamma_{,uz}. \end{aligned} \tag{3.2.9d}$$

where we have multiplied the second equation through by z, and the last two through by z^2 in order to obtain expressions that are regular at $z = 0$. We have also rescaled the first equation by dividing through by $2z^3$; this is not necessary to make the equation regular at $z = 0$ but enables iterative differentiation of the field equation.

We assume that the metric functions (γ, β, U, W) are suitably differentiable (at least C^4) at $z = 0$ and derive asymptotic solutions to the field equations via the following procedure:

1) Evaluate the field equations at $z = 0$ and solve the resulting algebraic equations.

2) Differentiate the field equations with respect to z.

3) Return to step 1) for the differentiated field equation.

We will follow this procedure equation by equation, making use of the nested structure to move from one to the next.

3.2.2.1 The first equation

The first main equation is (3.2.9a). Applying the differentiation procedure and solving at each order we obtain

$$\begin{aligned}
&\beta_{,z}(u_0,0,\theta)=0\\
&\beta_{,zz}(u_0,0,\theta)=-\frac{1}{2}[\gamma_{,z}(u_0,0,\theta)]^2\\
&\beta_{,zzz}(u_0,0,\theta)=-2\gamma_{,z}(u_0,0,\theta)\gamma_{,zz}(u_0,0,\theta)\\
&\beta_{,zzzz}(u_0,0,\theta)=3\left[(\gamma_{,zz}(u_0,0,\theta))^2-\gamma_{,z}(u_0,0,\theta)\gamma_{,zzz}(u_0,0,\theta)\right].
\end{aligned} \tag{3.2.10}$$

This procedure can be continued to arbitrary order although the terms displayed above will be sufficient for our analysis. In solving the subsequent field equations, it will be left implicit that the equations are evaluated at $(u_0,0,\theta)$. Note that the iterative procedure does not produce an equation for $\beta(u_0,0,\theta)$; the latter is an integration function, which we denote as $\beta_0(u,\theta)=\beta(u,0,\theta)$.

3.2.2.2 The second equation

The second equation is (3.2.9b). Given that we know both γ and β from the first equation, we can now solve this equation via the recursive differentiation procedure. The first two iterations give

$$U_{,z}=2\beta_{0,\theta}e^{2(\beta_0-\gamma)} \tag{3.2.11a}$$

$$U_{,zz}=-2e^{2\beta_0-2\gamma}(2\beta_{0,\theta}\gamma_{,z}-2\gamma_{,\theta}\gamma_{,z}+\gamma_{,z\theta}+2\cot(\theta)\gamma_{,z}) \tag{3.2.11b}$$

where all the equations are implicitly evaluated at $z=0$ on the hypersurface $\mathcal{N}_{u_0}$. The procedure does not constrain $U(u_0,0,\theta)$ thus giving an integration function $U_0(u,\theta)=U(u,0,\theta)$.

The third iteration of the differentiation procedure does not give an equation for $U_{,zzz}$ but instead a constraint equation:

$$2\gamma_{,\theta}\gamma_{,zz}-\gamma_{,zz\theta}-2\cot(\theta)\gamma_{,zz}=0 \tag{3.2.12}$$

This equation was solved in the asymptotically flat case of [1] by setting $\gamma_{,zz}=0$, the 'outgoing wave condition'. It has since been argued, most notably in [34, 168, 35], that this equation implies the existence of *polyhomogeneous* asymptotic solutions for asymptotically flat spacetimes i.e. series involving terms of the form $z^i\log^j(z),\ i,j\in\mathbb{N}$.

We will leave this equation unsolved for now and return to discuss it after we solve the fourth of the main equations; we will argue that the solution to that equation forbids the possibility of a polyhomogeneous form of the solution for non-zero cosmological constant (in the absence of matter). For now, we merely note that this equation indicates the presence of another integration function, as $U_{,zzz}(u_0, 0, \theta)$ remains undetermined by the iterative procedure. We name this function $U_3(u, \theta) = U_{,zzz}(u, 0, \theta)/3!$ where the choice of normalisation will become clearer as we continue to solve the main equations. In the asymptotically flat literature, this function is related to the *Bondi angular momentum* of the spacetime [1, 169].

The fourth iteration of the differentiation procedure produces the following equation

$$\begin{aligned} U_{,zzzz} = &- 2e^{-2\gamma}(-16e^{2\beta_0}\beta_{0,\theta}\gamma_{,z}^3 + 30e^{2\beta_0}\gamma_{,\theta}\gamma_{,z}^3 - 15e^{2\beta_0}\gamma_{,z\theta}\gamma_{,z}^2 + 4e^{2\beta_0}\beta_{0,\theta}\gamma_{,zz}\gamma_{,z} + \\ & 10e^{2\beta_0}\gamma_{,zz}\gamma_{,\theta}\gamma_{,z} - 3e^{2\beta_0}\gamma_{,zz\theta}\gamma_{,z} + 2e^{2\beta_0}\beta_{0,\theta}\gamma_{,zzz} + 6e^{2\beta_0}\gamma_{,zzz}\gamma_{,\theta} - \\ & 4e^{2\beta_0}\gamma_{,zz}\gamma_{,z\theta} - 3e^{2\beta_0}\gamma_{,zzz\theta} - 30\cot(\theta)e^{2\beta_0}\gamma_{,z}^3 - \\ & 10\cot(\theta)e^{2\beta_0}\gamma_{,zz}\gamma_{,z} - 6\cot(\theta)e^{2\beta_0}\gamma_{,zzz} + 3e^{2\gamma}\gamma_{,z}U_{,zzz}) \end{aligned} \tag{3.2.13}$$

which is an algebraic equation for $U_{,zzzz}$ in terms of $U_{,zzz}$. The presence of this equation makes sense because of the structure of the integration functions for equation (3.2.9b). If we were to repeat the differentiation procedure we would see that $\partial_z^{(n+1)}U$ would be given algebraically in terms of $\partial_z^{(n)}U$ for $n \geq 3$ so we observe that knowledge of $U_{,zzz}(u, 0, \theta)$ would allow us to compute all higher derivatives at $z = 0$. We will later see via the supplementary conditions that one does arrive at an evolution equation for $U_{,zzz}$.

3.2.2.3 The third equation

The third equation is (3.2.9c); this is the first equation that explicitly includes the cosmological constant Λ and thus it will have different solutions from [1].

The equations are again solved by applying the iterative differentiation procedure:

$$W = -\frac{1}{3}e^{2\beta_0}\Lambda \tag{3.2.14a}$$

$$W_{,z} = \cot(\theta)U_0 + U_{0,\theta} \tag{3.2.14b}$$

$$\begin{aligned} W_{,zz} = e^{2(\beta_0-\gamma)}(2 + \Lambda e^{2\gamma}(\gamma_{,z})^2 + 4\cot(\theta)\beta_{0,\theta} + 8(\beta_{0,\theta})^2 + \\ 6\cot(\theta)\gamma_{,\theta} - 8\beta_{0,\theta}\gamma_{,\theta} - 4(\gamma_{,\theta})^2 + 4\beta_{0,\theta\theta} + 2\gamma_{,\theta\theta}). \end{aligned} \tag{3.2.14c}$$

The third equation does not give an algebraic equation for $W_{,zzz}$ but rather another constraint equation

$$\Lambda e^{2\beta_0}\gamma_{,z}\gamma_{,zz} = 0 \quad \Longrightarrow \quad \gamma_{,z}\gamma_{,zz} = 0. \tag{3.2.15}$$

Note that this equation is unique to $\Lambda \neq 0$. We will not yet solve this constraint: the solution is determined by the fourth main equation. This constraint equation again implies an integration function for the third equation as the differentiation procedure does not produce an equation for $W_{,zzz}$. We will name this integration function $W_3(u,\theta) = W_{,zzz}(u,0,\theta)/3!$. In the asymptotically flat case, this function is related to the *Bondi mass aspect* of the spacetime, a concept we will examine in more detail in the asymptotically *AdS* case in section 3.4.

The structure of the higher order equations in the recursive differentiation procedure is similar to that of the second equation. The result of the procedure is that $\partial_z^{(n+1)}W$ is determined algebraically in terms of $\partial_z^{(n)}W$. We again remark that once we know the integration function W_3 we can then compute all derivatives of third order and higher.

3.2.2.4 The fourth equation

The fourth and final main equation, which we consider as an equation for $\gamma_{,u}$, is (3.2.9d). We again apply the recursive scheme to solve for $\gamma_{,u}$. Using the solutions to the previous equations, the first non-trivial equation is

$$\Lambda\gamma_{,z} = -\frac{3}{2}e^{-2\beta_0}(\cot(\theta)U_0 - U_{0,\theta} - 2U_0\gamma_{,\theta} - 2\gamma_{,u}) \tag{3.2.16}$$

This equation is presented slightly differently to the previous main equations; we will discuss this further in section 3.3. The key point here is that the presence of the cosmological constant couples the equation for $\gamma_{,u}$ to $\gamma_{,z}$.

The next non-trivial equation is

$$\Lambda e^{2\beta_0}\gamma_{,zz} = 0 \implies \gamma_{,zz} = 0. \tag{3.2.17}$$

This constraint automatically solves the two previous constraint equations (3.2.12) and (3.2.15). This is precisely the *outgoing wave condition* that was enforced *a priori* in [1] and has since been understood in more generality in a Bondi type set up (see e.g. [170]). In the case of non-zero cosmological constant, $\gamma_{,zz} = 0$ is required by the field equations *i.e.* it is not an assumption.

At the next order of the recursive differentiation procedure, we find the equation

$$e^{2\gamma}\gamma_{,uzz}(u_0,0,\theta) = 0 \implies \gamma_{,uzz}(u_0,0,\theta) = 0 \tag{3.2.18}$$

which implies that the form of $\gamma_{,zz}(u_0,0,\theta)$ is preserved on hypersurfaces $\mathcal{N}_u$ for $u > u_0$. Since $\gamma_{,zz}(u_0,0,\theta) = 0$ from (3.2.17), the outgoing wave equation is propagated into

$D^+(\mathcal{N}_0)$.

When $\Lambda = 0$ equation (3.2.16) implies $\gamma(u_0, 0, \theta) = 0$ (as we will shortly discuss in detail in section 3.2.3), $\gamma_{,zz} = 0$ for $u > u_0$ as we just discussed, and we are left with one integration function $\gamma_{,z}(u, 0, \theta) = \gamma_1(u, \theta)$. (The u-derivative of) this integration function is essentially the Bondi news.

Returning to the $\Lambda \neq 0$ case we note that the procedure of differentiation did not produce an equation for $\gamma_{,uz}(u_0, 0, \theta)$, again implying the presence of an integration functions $\gamma_{,z}(u, 0, \theta) = \gamma_1(u, \theta)$ (as in the $\Lambda = 0$ case). Finally, to determine $\gamma_{,u}$ (so that we can move to the next null hypersurface $\mathcal{N}_{u_0+\delta u}$) we also need to know non-trivial integration functions (U_0, β_0). We will discuss in more detail (A)dS integration schemes in section 3.3.

As a final comment we note that the next non-trivial equation produced by the iterative procedure is

$$\begin{aligned}
\Lambda\gamma_{,zzzz} = -\frac{3}{2}e^{-2(\beta_0+\gamma)}(&48e^{2\beta_0}\beta_{0,\theta}^2(\gamma_{,z})^2 - 96e^{2\beta_0}\beta_{0,\theta}\gamma_{,\theta}(\gamma_{,z})^2 + 6e^{2\beta_0}\beta_{0,\theta\theta}(\gamma_{,z})^2 - \\
&24e^{2\beta_0}\gamma_{,\theta\theta}(\gamma_{,z})^2 + 108e^{2\beta_0}\beta_{0,\theta}\gamma_{,z\theta}\gamma_{,z} - 48e^{2\beta_0}\gamma_{,\theta}\gamma_{,z\theta}\gamma_{,z} + \\
&18e^{2\beta_0}\gamma_{,z\theta\theta}\gamma_{,z} + 18e^{2\beta_0}(\gamma_{,z\theta})^2 - 24\cot^2(\theta)e^{2\beta_0}(\gamma_{,z})^2 + \\
&90\cot(\theta)e^{2\beta_0}\beta_{0,\theta}(\gamma_{,z})^2 + 24\cot(\theta)e^{2\beta_0}\gamma_{,\theta}(\gamma_{,z})^2 + \\
&30\cot(\theta)e^{2\beta_0}\gamma_{,z\theta}\gamma_{,z} - 24\csc^2(\theta)e^{2\beta_0}(\gamma_{,z})^2 - 8e^{2\gamma}\gamma_{,uzzz} + \\
&e^{2\gamma}U_{,zzz}(-6\beta_{0,\theta} - 2\gamma_{,\theta} + \cot(\theta)) - 12e^{2\gamma}\gamma_{,zzz}U_{0,\theta} - \\
&e^{2\gamma}U_{,zzz\theta} - 8e^{2\gamma}\gamma_{,zzz\theta}U_0 - \\
&12\cot(\theta)e^{2\gamma}\gamma_{,zzz}U_0 - 2e^{2\gamma}\gamma_{,z}W_{,zzz})
\end{aligned} \tag{3.2.19}$$

which shows that the evolution equation for $\gamma_{,uzzz}$ is coupled to $\gamma_{,zzzz}$ via the cosmological constant Λ. This coupling is a general feature of this field equation at higher orders, namely the equation for $\partial_z^{(n)}\gamma_{,u}$ is given in terms of $\partial_z^{(n+1)}\gamma$. So if we provide a new integration function $\gamma_{,zzz}(u, 0, \theta)/3! = \gamma_3(u, \theta)$ then all higher order terms are determined. A more detailed discussion will be given in section 3.3.

3.2.3 General form of the asymptotic solutions

Using the procedure of recursive differentiation we have obtained a general form for the asymptotic solution to the field equations. The key to this structure is that $\gamma_{,zz}(u, 0, \theta) = 0$ which results in the vanishing of potential polyhomogeneous terms in the asymptotic solution (as discussed in [34]). Note that the vanishing of this term is forced by equation (3.2.17) rather than being assumed as it was in [1].

In previous literature it has been found that the metric function expansions can contain logarithmic terms of the form $\log^j(r)r^{-i}$, both for the asymptotically flat case in [168, 94] and for arbitrary Λ but with matter in [48]. These cases are qualitatively different. In the asymptotically flat case there is no analogue of (3.2.17). In the presence of a negative cosmological constant, logarithmic terms in asymptotic expansions arise whenever the coupled matter is of specific masses, see for example [60, 62]; such matter is associated with matter conformal anomalies in the dual CFT.

For pure cosmological constant, the most general form of the asymptotic solutions take the form of power series about $z = 0$, consistent with the boundary conditions of asymptotically locally AdS$_4$ and dS$_4$ in the absence of matter. Specifically, γ admits an expansion of the form

$$\gamma(u, r, \theta) = \sum_{n=0}^{\infty} \gamma_n(u, \theta) z^n = \sum_{n=0}^{\infty} \frac{\gamma_n(u, \theta)}{r^n}, \qquad \gamma_n = \left.\frac{\partial_z^{(n)}\gamma}{n!}\right|_{z=0} \tag{3.2.20}$$

and the other functions admit analogous expansions. These conditions ensure that the metric coefficients do not grow exponentially with r and that the metric has a pole of order two at the conformal boundary $\mathscr{I}$; this will be discussed in greater detail in section 3.4.

We also note how the presence of the cosmological constant in the Einstein equations modifies the solutions compared with the asymptotically flat case considered in [1], even though the asymptotic series form of the equations initially seems to be the same. At this point it will be helpful to consider the AdS and dS cases separately, as there are subtle differences in the two cases.

The key assumption made in [1] which results in this discrepancy is that the vector field ∂_u is everywhere timelike $\iff g_{uu} < 0$. Physically, this is a reasonable condition to impose on asymptotic solutions for $\Lambda \leq 0$, as the neighbourhood of the conformal boundary in these cases is exterior to any potential region where ∂_u ceases to be timelike (e.g inside a horizon). We note that the leading order terms (γ_0, β_0, U_0) are not present in the asymptotically flat case and are forced to vanish due to this condition.

These choices are overly restrictive in the AdS case as the cosmological constant allows for freedom in these functions. To see this, consider the limit

$$\lim_{r\to\infty} \frac{g_{uu}}{r^2} = -(W_0 e^{2\beta_0} - U_0^2 e^{2\gamma_0}) < 0 \tag{3.2.21}$$

where the inequality on the right hand side follows from the condition that ∂_u is timelike. In the flat case, $W_0 = 0$ and so the above equation reduces to $U_0 = 0$. It was then argued in [1] that $U_0 = 0$ implies $\gamma_{0,u} = 0$ (use (3.2.16) with $\Lambda = 0$) and this may be reduced further to $\gamma_0 = 0$ by using a coordinate transformation (for details see [1]).

In the AdS case $W_0 = -\Lambda e^{2\beta_0}/3$ so the inequality is different, namely

$$\frac{\Lambda e^{4\beta_0}}{3} + U_0^2 e^{2\gamma_0} < 0 \Rightarrow |U_0| < \sqrt{-\frac{\Lambda}{3} e^{4\beta_0 - 2\gamma_0}} = \frac{e^{2\beta_0 - \gamma_0}}{l} \qquad (3.2.22)$$

from which we see that U_0 can now clearly be non-zero, implying that generically $\gamma_0 \neq 0$ also.

The integration function β_0 is also set to zero in the flat case, using the freedom in the BMS group. Since the BMS group is the asymptotic symmetry group of flat space-time it would be premature to make the same choice before determining the *AdS* asymptotic structure. For the time being we will choose $\beta \neq 0$ to retain full generality.

Turning now to the dS case of $\Lambda > 0$, the previously imposed condition of ∂_u being timelike is unphysical in the asymptotic region. Using the Bondi gauge in a neighbourhood of $\mathscr{I}^+$, the cosmological horizon in the asymptotically locally dS spacetime must have been crossed, and thus the vector field ∂_u is spacelike in the region of interest, see discussion in [171]. Thus one should not impose this condition in the dS case, leaving (γ_0, β_0, U_0) generically unconstrained.

A second important difference to note, for any non-zero cosmological constant, is that the cosmological constant couples the fourth equation at each order in z. We find equations which give $\partial_z^{(n)}\gamma_{,u}$ in terms of $\partial_z^{(n+1)}\gamma$, *e.g.* (3.2.16) and (3.2.19). This coupling of orders together with the structure of the other main equations implies that if we are given suitable seed coefficients then we can obtain all the other expansion coefficients. The initial coefficients are (γ_0, β_0, U_0) together with (γ_3, U_3, W_3); from these the entire solution can be determined algebraically. We will see below that these coefficients have an important holographic interpretation but first we analyse the remaining Einstein equations, the so-called *supplementary conditions.*

3.2.4 The supplementary conditions

Although the main equations give equations for the four metric functions, they do not form the complete set of field equations. The remaining three supplementary equations are:

$$R_{uu} = \Lambda g_{uu} = -\Lambda(W r^2 e^{2\beta} - U^2 r^2 e^{2\gamma}); \qquad (3.2.23)$$

$$R_{u\theta} \quad = \quad \Lambda g_{u\theta} = -\Lambda U r^2 e^{2\gamma}, \quad R_{ur} = \Lambda g_{ur} = -\Lambda e^{2\beta}.$$

In the asymptotically flat case, these equations were denoted as supplementary conditions as they are automatically satisfied provided they hold on a particular hypersurface of constant radius and the main equations are satisfied [2]. In this section we will discuss how this property carries over to the $\Lambda \neq 0$ case.

Following the original work, the supplementary conditions are derived from the contracted Bianchi identity

$$\nabla^\nu G_{\nu\mu} = g^{\nu\sigma}\nabla_\sigma \left(R_{\nu\mu} - \frac{1}{2} g_{\nu\mu} R \right) = 0. \tag{3.2.24}$$

We can expand the Bianchi identity as

$$g^{\nu\sigma}\left(R_{\mu\nu,\sigma} - \Gamma^\beta_{\sigma\nu} R_{\beta\mu} \right) - g^{\nu\sigma}\Gamma^\beta_{\sigma\mu} R_{\beta\nu} - \frac{1}{2} R_{,\mu} = 0 \tag{3.2.25}$$

and using $R_{,\mu} = \nabla_\mu (g^{\nu\sigma} R_{\nu\sigma}) = g^{\nu\sigma}\nabla_\mu R_{\nu\sigma} = g^{\nu\sigma} R_{\nu\sigma,\mu} - 2g^{\nu\sigma}\Gamma^\beta_{\sigma\mu} R_{\beta\nu}$ allows us to write the contracted Bianchi identity as

$$g^{\nu\sigma}\left(R_{\mu\nu,\sigma} - \frac{1}{2} R_{\nu\sigma,\mu} - \Gamma^\beta_{\nu\sigma} R_{\beta\mu} \right) = 0. \tag{3.2.26}$$

To analyse the components of the contracted Bianchi identity we use the inverse metric

$$g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 & 0 \\ -e^{-2\beta} & W e^{-2\beta} r^2 & -U e^{-2\beta} & 0 \\ 0 & -U e^{-2\beta} & e^{-2\gamma} r^{-2} & 0 \\ 0 & 0 & 0 & e^{2\gamma} r^{-2} \sin^{-2}\theta \end{pmatrix} \tag{3.2.27}$$

where we use the coordinates (u, r, θ, ϕ). The following identity is also useful:

$$g^{\mu\nu}\Gamma^u_{\mu\nu} = 2e^{-2\beta} r^{-1} \tag{3.2.28}$$

This identity is computed using the inverse metric above and the metric (5.1.10); the same identity was given in [1], up to a sign change due to different signature conventions.

We will now examine the components of the contracted Bianchi identity (3.2.26) and show that they lead to the supplementary equations. When doing this, we enforce the main equations, expressed as

$$R_{rr} = R_{r\theta} = 0, \quad R_{\theta\theta} = \Lambda g_{\theta\theta}, \quad R_{\phi\phi} = \Lambda g_{\phi\phi} \tag{3.2.29}$$

as well as the trivial equations $R_{u\phi} = R_{r\phi} = R_{\theta\phi} = 0$.

Let us consider first the $\mu = r$ component of (3.2.26)

$$g^{\nu\sigma}\left(R_{r\nu,\sigma} - \frac{1}{2}R_{\nu\sigma,r} - \Gamma^{\beta}_{\nu\sigma}R_{\beta r}\right) = 0. \tag{3.2.30}$$

Using the main and trivial equations this reduces to

$$-\frac{1}{2}\left(g^{\theta\theta}\Lambda g_{\theta\theta,r} + g^{\phi\phi}\Lambda g_{\phi\phi,r}\right) - g^{\nu\sigma}\Gamma^{u}_{\nu\sigma}R_{ur} = 0. \tag{3.2.31}$$

The latter term can be processed using the identity (3.2.28) and after algebraic manipulation we obtain

$$R_{ur} = -\Lambda e^{2\beta} = \Lambda g_{ur}. \tag{3.2.32}$$

which is precisely the $\{ur\}$ component of the field equations. Thus we conclude that if the main equations hold then the $\{ur\}$ equation is automatically satisfied.

Next consider the $\mu = \theta$ component of (3.2.26)

$$g^{\nu\sigma}\left(R_{\theta\nu,\sigma} - \frac{1}{2}R_{\nu\sigma,\theta} - \Gamma^{\beta}_{\nu\sigma}R_{\beta\theta}\right) = 0 \tag{3.2.33}$$

Uising the main and trivial equations we obtain

$$\begin{aligned} &g^{ur}R_{u\theta,r} - g^{\nu\sigma}\Gamma^{u}_{\nu\sigma}R_{u\theta} + \\ &\Lambda\left(g^{r\theta}g_{\theta\theta,r} + \frac{1}{2}g^{\theta\theta}g_{\theta\theta,\theta} - g^{ur}g_{ur,\theta} - \frac{1}{2}g^{\phi\phi}g_{\phi\phi,\theta} - g^{\nu\sigma}\Gamma^{\theta}_{\nu\sigma}g_{\theta\theta}\right) = 0. \end{aligned} \tag{3.2.34}$$

Applying equation (3.2.28) to the second term on the first line and using equations (5.1.10) and (3.2.27) to write the second line in terms of metric functions we obtain

$$-r^{-2}e^{-2\beta}\frac{\partial}{\partial r}(r^2 R_{u\theta}) = \Lambda e^{2\gamma-2\beta}r(rU_{,r} + 2U(2+\gamma_{,r})) \tag{3.2.35}$$

which can be integrated to give

$$r^2 R_{u\theta} = -\Lambda U r^4 e^{2\gamma} + f(u,\theta) \tag{3.2.36}$$

where $f(u,\theta)$ is an integration function. Dividing through by r^2 gives

$$R_{u\theta} = -\Lambda U r^2 e^{2\gamma} + \frac{f(u,\theta)}{r^2} = \Lambda g_{u\theta} + \frac{f(u,\theta)}{r^2}. \tag{3.2.37}$$

which implies that the $\{u\theta\}$ component of the Einstein equations is only satisfied if $f(u,\theta) = 0$; this is our first supplementary condition.

Finally consider the $\mu = u$ component of the contracted Bianchi identity

$$g^{\nu\sigma}\left(R_{u\nu,\sigma} - \frac{1}{2}R_{\nu\sigma,u} - \Gamma^{\beta}_{\nu\sigma}R_{\beta u}\right) = 0. \tag{3.2.38}$$

Applying the field equations (including $f(u,\theta)=0$), we obtain

$$\begin{aligned}&g^{ur}R_{uu,r}-g^{\nu\sigma}\Gamma^u_{\nu\sigma}R_{uu}+\\&\Lambda\Big[g^{rr}g_{ur,r}+g^{r\theta}(g_{ur,\theta}+g_{u\theta,r})+g^{\theta\theta}\left(g_{u\theta,\theta}-\frac{1}{2}g_{\theta\theta,u}\right)-\\&\frac{1}{2}g^{\phi\phi}g_{\phi\phi,u}-g^{\nu\sigma}\Gamma^r_{\nu\sigma}g_{ur}-g^{\nu\sigma}\Gamma^\theta_{\nu\sigma}g_{u\theta}\Big]=0.\end{aligned}\tag{3.2.39}$$

The structure of this equation is similar to that of (3.2.34): the first line contains the Ricci tensor terms of interest and all other terms can be written explicitly using (5.1.10) and (3.2.27). Doing this gives

$$-r^{-2}e^{-2\beta}\frac{\partial}{\partial r}(r^2R_{uu})=\Lambda r[2W(2+r\beta_{,r})+rW_{,r}]-2\Lambda e^{2\gamma-2\beta}rU[2U+rU_{,r}+rU\gamma_{,r}]\tag{3.2.40}$$

which can be integrated to give

$$r^2R_{uu}=\Lambda r^4(-We^{2\beta}+U^2e^{2\gamma})+g(u,\theta)\tag{3.2.41}$$

with $g(u,\theta)$ an integration function. Thus

$$R_{uu}=\Lambda r^2(-We^{2\beta}+U^2e^{2\gamma})+\frac{g(u,\theta)}{r^2}=\Lambda g_{uu}+\frac{g(u,\theta)}{r^2}\tag{3.2.42}$$

implying that the second supplementary condition is $g(u,\theta)=0$.

Explicit expressions for the supplementary conditions may be derived using the solutions to the main equations up to $\mathcal{O}(1/r^4)$ for (γ,β,U,W) and then inputting these into the above equations to derive expressions for (f,g). The resulting equations take the form of evolution equations for U_3 and W_3 and they will be discussed further in section 3.3.2.2. The explicit expressions for these equations can be found in appendix A.1. Here we present the much simpler expressions for asymptotically (A)dS and flat spacetimes.

Asymptotically (A)dS spacetimes in Bondi coordinates have $\gamma_0=\beta_0=U_0=0$ (this will be shown explicitly in section 3.4) which gives $\gamma_1=0$ by equation (3.2.16). Setting these values in the supplementary equations gives us

$$U_{3,u}=\frac{1}{3}(4\Lambda\cot(\theta)\gamma_3+W_{3,\theta}+2\Lambda\gamma_{3,\theta})\tag{3.2.43a}$$

$$W_{3,u}=-\frac{1}{2}\Lambda(\cot(\theta)U_3+U_{3,\theta})\tag{3.2.43b}$$

For the asymptotically flat supplementary conditions, we again have $\gamma_0=\beta_0=U_0=0$ as well as $\Lambda=0$ but now $\gamma_1\neq0$. Then

$$U_{3,u}=\frac{1}{3}\left(7\gamma_{1,\theta}\gamma_{1,u}+\gamma_1\left(3\gamma_{1,u\theta}+16\cot(\theta)\gamma_{1,u}\right)+W_{3,\theta}\right)\tag{3.2.44a}$$

$$W_{3,u} = 2\left(\gamma_{1,u}\right)^2 + 2\gamma_{1,u} - \gamma_{1,u\theta\theta} - 3\cot(\theta)\gamma_{1,u\theta}. \tag{3.2.44b}$$

in agreement with the expressions given in [1].

3.3 Integration scheme

In this section we will discuss the integration scheme used in the previous section in order to solve the Einstein equations. We will begin with a reminder of the Bondi integration scheme in asymptotically flat spacetime before focusing specifically on the $\Lambda < 0$ case of asymptotically locally AdS spacetime. We will propose two modified integration schemes for the AdS case which will be compared and contrasted to the flat scheme. Much of what we will discuss for the AdS case has a corresponding description in the $\Lambda > 0$ case of asymptotically locally dS spacetime, a topic we will discuss briefly here and elaborate upon in future work.

3.3.1 The flat scheme

Let us briefly review the integration scheme in the asymptotically flat case as presented in [1]. The basic quantity necessary to solve the field equations for all u was the knowledge of γ on some initial null hypersurface $\mathcal{N}_{u_0}$; this allows us to solve the main equations up to the undetermined integration functions. In the Ricci flat case we can reapply the field equations (3.2.7a-3.2.7d) although we now set $\Lambda = 0$ in those equations. For the remainder of this subsection we have $\Lambda = 0$.

Knowledge of $\gamma|_{\mathcal{N}_{u_0}}$ allows us to solve for the other functions. Disregarding integration functions, (3.2.7a) determines $\beta|_{\mathcal{N}_{u_0}}$; (3.2.7b) determines $U|_{\mathcal{N}_{u_0}}$, (3.2.7c) gives $W|_{\mathcal{N}_{u_0}}$ and (3.2.7d) allows us to compute our γ at the next time step i.e $\gamma|_{\mathcal{N}_{u_0+\delta}}$. Iterating this process allows us to determine all metric functions at time $u > u_0$, i.e. the functions in the future domain of dependence of $\mathcal{N}_{u_0}$, $D^+(\mathcal{N}_{u_0})$, as shown below in figure 3.3.1.

Turning to the integration functions, we recall that the main equations in the flat case admit five such functions; $(\beta_0, U_0, U_3, \gamma_1, W_3)$. The original argument of [1] was that U_0 and β_0 could be set to zero. U_0 is set to zero to preserve the condition that the vector field ∂_u is everywhere timelike and β_0 can be fixed to zero using the freedom of the BMS group. These restrictions also give $\gamma_{0,u} = 0$ and thus we can also set $\gamma_0 = 0$ by a suitable BMS transformation.

Such considerations reduce the number of unknown functions to three: (γ_1, U_3, W_3),

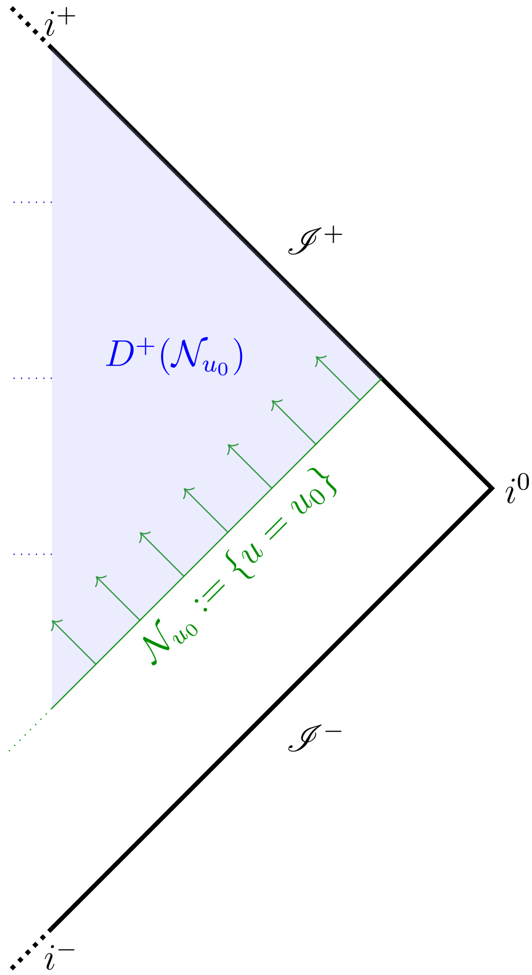

Figure3.3.1: Penrose diagram illustrating the integration scheme for asymptotically flat space time

all of which are functions of u and θ. These integration functions have well understood physical meaning: γ_1 plays the role of the Bondi news function, U_3 the Bondi angular momentum aspect and W_3 the Bondi mass aspect [1]. If we know the values of these three functions and we know $\gamma|_{\mathcal{N}_{u_0}}$, then from the main equations we can obtain the full solution to the Einstein equations in the region $D^+(\mathcal{N}_{u_0})$. The integration scheme runs as follows (for the no-log case with $\gamma_2 = 0$)

$$\gamma_1(u,\theta) \xrightarrow{(3.2.7a)} \beta_1, \beta_2, \beta_3 \xrightarrow{(3.2.7b)} U_1, U_2 \xrightarrow{(3.2.7c)} W_0, W_1, W_2, W_4 \tag{4.1a}$$

so γ_1 gives us these functions. The rest of the scheme is

$$\gamma_n(u_0) \xrightarrow{(3.7a)} \beta_{n+1}(u_0) \xrightarrow{(3.7b)} U_{n+1}(u_0) \xrightarrow{(3.7c)} W_{n+2}(u_0) \xrightarrow{(3.7d)} \gamma_{n,u}(u_0) \tag{4.1b}$$

where the subscript $n > 2$. The final arrow going back to the original function indicates that we are solving for γ at the next instant of time i.e. $u_0 + \delta u_0$, so iteration gives us the future evolution. We note that if the functions U_3, W_3 are not specified for all u a priori, this scheme treats them as integration functions which are constrained by the supplementary conditions (A.1.1), (A.2) respectively. We will return to discuss these equations in the context of the AdS integration schemes to follow but for now these steps outline the procedure of the integration scheme in the asymptotically flat case, using some of the simplifications that BMS originally applied (namely $\gamma_2 = 0$).

3.3.2 The AdS integration schemes

In order to understand how one needs to modify the specified data in the case of asymptotically locally AdS spacetimes it is convenient to first observe the results when one naïvely applies the flat scheme as described in the previous section to asymptotically locally AdS spacetime.

To repeat the steps of the flat scheme we again specify γ on an initial null hypersurface $\mathcal{N}_{u_0}$ as well as γ_1, U_3, W_3 over the whole spacetime. The issue with applying this procedure to an asymptotically AdS spacetime is that we now have three additional integration functions ($\gamma_0(u,\theta), \beta_0(u,\theta), U_0(u,\theta)$) and in particular β_0, U_0 will not be determined using the Einstein equations (3.2.7a-3.2.7d) and the specified data (γ_0 would be determined using γ_0 on $\mathcal{N}_{u_0}$ and equation (3.2.16)). These functions will also appear in the expressions for the higher order coefficients (e.g (3.2.11a)) and can be seen in the evolution equation for γ_0 (3.2.16). Clearly we will need an alternative integration scheme which specifies these functions and thus generates a fully determined solution to the field equations.

This issue can also be framed in terms of a causal picture as in figure 3.2.1. In this figure we see that specifying γ on an initial null hypersurface $\mathcal{N}_{u_0}$ and γ_1, U_3, W_3 for $u \geq u_0$ and following the flat scheme will give us the solution in $D^+(\mathcal{N}_{u_0})$. In the AdS case (unlike the flat case) this region is not equivalent to the causal future of the null hypersurface, $J^+(\mathcal{N}_{u_0})$ (as shown in figure 3.3.2 below). In order to solve the Einstein equations for $J^+(\mathcal{N}_{u_0})$ in asymptotically locally AdS space-time, one either has to specify extra data on an additional hypersurface or different data to that of γ on the null slice $\mathcal{N}_{u_0}$. We will now present two different integration scheme for asymptotically locally AdS spacetimes which will allow one to solve the field equations in $J^+(\mathcal{N}_{u_0})$.

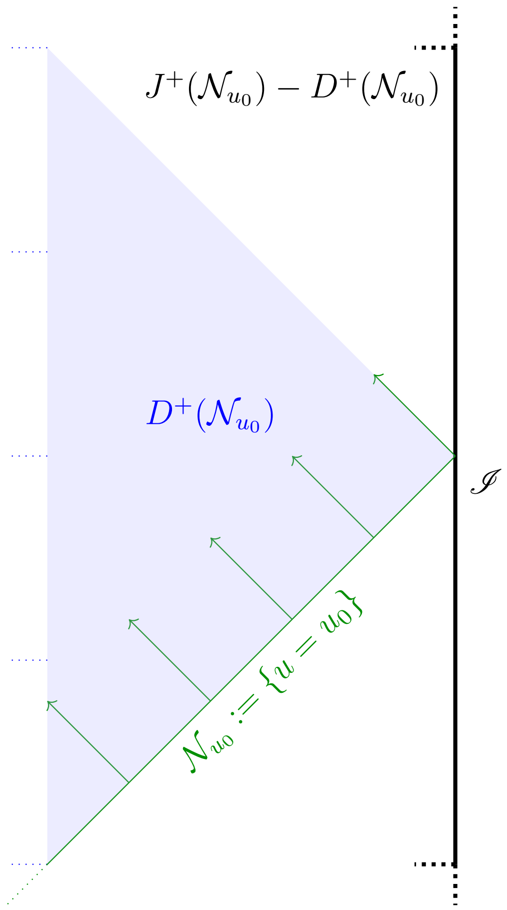

Figure3.3.2: Penrose diagram illustrating the difference between $D^+(\mathcal{N}_{u_0})$ and $J^+(\mathcal{N}_{u_0})$ in asymptotically locally AdS spacetime.

3.3.2.1 The "boundary" scheme

The first scheme we present is one which we will refer to as the "boundary" scheme where instead of specifying all coefficients γ_i on an initial null hypersurface, one should specify certain coefficients (of our metric functions) for all available Bondi time, and use these coefficients in order to make the equations algebraic. The coefficients that should be specified are

$$\gamma_0, \quad \beta_0, \quad U_0, \quad \gamma_3, \quad U_3, \quad W_3. \tag{4.2}$$

We will see later that these particular coefficients admit a natural holographic interpretation. Even before relating them to coefficients in the Fefferman-Graham expansion, one can note that the coefficients (γ_0, β_0, U_0) clearly specify the values of the metric functions

(γ, β, U) at the conformal boundary $\mathscr{I}$;

$$\lim_{r\to\infty} \gamma(u,r,\theta) = \gamma_0(u,\theta), \quad \lim_{r\to\infty} \beta(u,r,\theta) = \beta_0(u,\theta), \quad \lim_{r\to\infty} U(u,r,\theta) = U_0(u,\theta) \quad (4.3)$$

and thus define the boundary metric for the dual conformal field theory. We will understand the precise physical meaning of the components (γ_3, U_3, W_3) in section 3.4, when we discuss the relation to the asymptotic expansion in Fefferman-Graham gauge. In particular, we will see that the (γ_0, β_0, U_0) and (γ_3, U_3, W_3) are conjugate variables in a radial Hamiltonian formalism, thus explaining why they provide a good set of initial data.

The scheme works in two parts. Given the boundary data (γ_0, β_0, U_0) we see that the first part of the integration scheme is

$$\begin{aligned} &\gamma_0, \beta_0, U_0 \xrightarrow{(3.2.7a)} \beta_1 \xrightarrow{(3.2.7b)} U_1 \xrightarrow{(3.2.7c)} W_0, W_1 \xrightarrow{(3.2.7d)} \gamma_1 \dots \\ &\dots \xrightarrow{(3.2.7a)} \beta_2 \xrightarrow{(3.2.7b)} U_2 \xrightarrow{(3.2.7c)} W_2 \xrightarrow{(3.2.7d)} \gamma_2 \dots \\ &\dots \xrightarrow{(3.2.7a)} \beta_3. \end{aligned} \quad (4.4)$$

In words: we specify the data (γ_0, β_0, U_0) at $\mathscr{I}$; (shown in figure 3.3.3 below) at the 2-surface where a particular null hypersurface $\mathcal{N}_{u_0}$ meets the conformal boundary. We can solve equations (3.2.7a)-(3.2.7c) algebraically for the coefficients β_1, U_1, W_0, W_1. This is indicated in the figure by the solid green arrow in the diagram which points from $\mathscr{I}$ to the timelike surface $r = r_1$.

In order to continue the scheme, we need to know $\gamma_{0,u}$ as this function will allow us to algebraically solve equation (3.2.7d) at the lowest non-trivial order for γ_1 (equation (3.2.16)). Since we know all values of γ_0 on $\mathscr{I}$ and we know $\gamma_{0,u}$. The knowledge of this derivative is indicated in the diagram by the dotted red arrow which points into the bulk spacetime, again ending on the timelike surface $r = r_1$. In order to implement this step in a numerical scheme, one would want to know $\gamma_0(u_0)$ and $\gamma_0(u_0 - \delta u_0)$ and construct a backward difference. This explains why the dotted red arrow starts at a different cut of $\mathscr{I}$, simply to indicate that we have used the extra information of $\gamma_0(u_0 - \delta u_0)$ (and thus $\gamma_{0,u}$ discretely) in order to solve (3.2.7d).

The arrows point towards smaller values of r as we solve the Einstein equations. The purpose of this is to show that as we solve the Einstein equations, we obtain the values of higher order coefficients in the metric functions γ, β, U, W. Obtaining these higher order coefficients extends the series expansions (5.1.13) to higher powers of $1/r$, hence our solution includes contributions from smaller (but still asymptotic) values of r.

After these first steps have been performed, we solve (3.2.7a-3.2.7d) algebraically for $\beta_2, U_2, W_2, \gamma_2, \beta_3$ (no extra evolution equation is needed as the field equation imply $\gamma_2 = 0$,

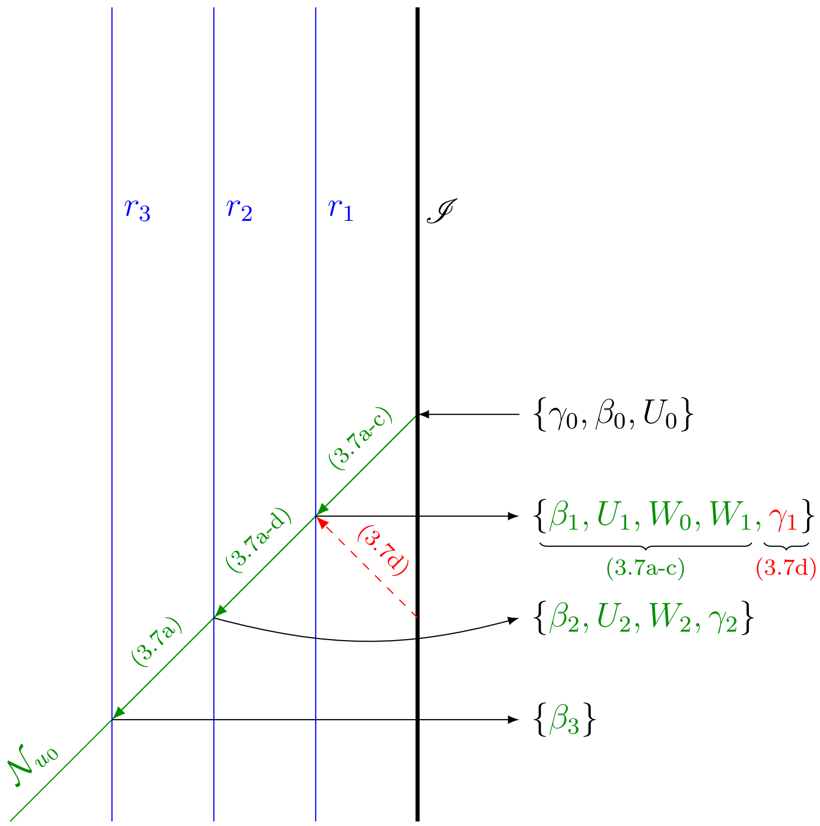

Figure3.3.3: Penrose diagram for AdS indicating discretely how the first part of the scheme is solved. This figure only includes one hypersurface, $\mathcal{N}_{u_0}$, for clarity; when solving the equations explicitly we would consider all null surfaces $\mathcal{N}_i$ in the foliation.

as noted earlier). Knowledge of these functions is not enough to continue the integration scheme as the next unknown function in the field equations is U_3, an integration function which cannot be determined by the iteration process.

We now give the second piece of the scheme: now the functions γ_3, U_3, W_3 are specified for all Bondi time u. This allows us to compute the higher order metric function coefficients via the following application of the Einstein equations

$$\begin{aligned}
\gamma_3, U_3, W_3 &\xrightarrow{(3.2.7a)} \beta_4 \xrightarrow{(3.2.7b)} U_4 \xrightarrow{(3.2.7c)} W_4 \xrightarrow{(3.2.7d)} \gamma_4 \dots \\
\dots &\xrightarrow{(3.2.7a)} \beta_5 \xrightarrow{(3.2.7b)} U_5 \xrightarrow{(3.2.7c)} W_5 \xrightarrow{(3.2.7d)} \gamma_5 \dots \\
&\vdots \\
\dots &\xrightarrow{(3.2.7a)} \beta_n \xrightarrow{(3.2.7b)} U_n \xrightarrow{(3.2.7c)} W_n \xrightarrow{(3.2.7d)} \gamma_n
\end{aligned} \tag{4.5}$$

as shown in figure 3.3.4.

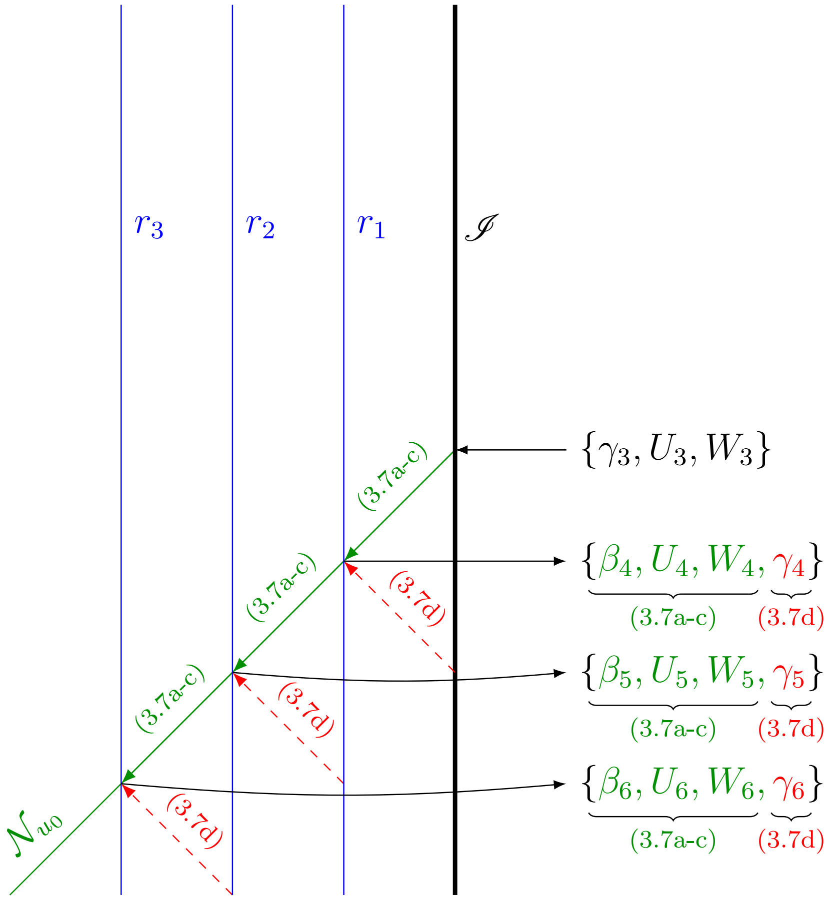

Figure3.3.4: Penrose diagram for AdS indicating how the second part of the scheme is implemented. The logic for this scheme is much the same as the one presented on the original diagram of figure 3.3.3.

Putting the two parts of the integration scheme together, we observe that knowledge of the six functions $\gamma_0, \beta_0, U_0, U_3, \gamma_3, W_3$ is sufficient to algebraically solve the Einstein equations for all other coefficients.

Finally, we recall the functions U_3 and W_3 have close analogies to the angular momentum and mass aspect functions and may be thought of as representatives for these functions. We will discuss the holographic interpretation of these functions and gain an extra understanding using the AdS/CFT correspondence in section 3.4.

As a final comment upon this procedure, we note that this alternative scheme includes no evolution from one null hypersurface to the next. This algebraic procedure may be somewhat preferable when applied to a numerical scheme as one does not have to worry about errors accumulating in a discretisation scheme when evolving from one null hypersurface to the next. We will now present another new scheme which is based both on null evolution and boundary data.

3.3.2.2 The "hybrid" scheme

It has been shown that asymptotically locally AdS spacetimes admit an integration scheme where one specifies data at the conformal boundary as opposed to an initial null hypersurface (as in asymptotically flat spacetime). We will now present a "hybrid" scheme for asymptotically locally AdS spacetimes, where one specifies a mixture of data on the conformal boundary $\mathscr{I}$ and on an initial null hypersurface $\mathcal{N}_{u_0}$.

This scheme consists of the following data which one must specify before solving the field equations: $\{\gamma\}$ on $\mathcal{N}_{u_0}$, $\{\gamma_0, U_0, \beta_0\}$ on $\mathscr{I}$ $\forall$ $u \geq u_0$ and $\{U_3, W_3\}$ at the corner $\mathscr{I}_{u_0}$. This is illustrated in the asymptotic Penrose diagram below. As we will see in the section 3.4, $\{\gamma_0, U_0, \beta_0\}$ are related to positions and $\{\gamma_3, W_3, U_3\}$ to (radial) canonical momenta in the covariant phase space of the theory, thus we effectively specify momenta on $\mathscr{I}_{u_0}$ and positions at the conformal boundary, as shown in figure 3.3.5.

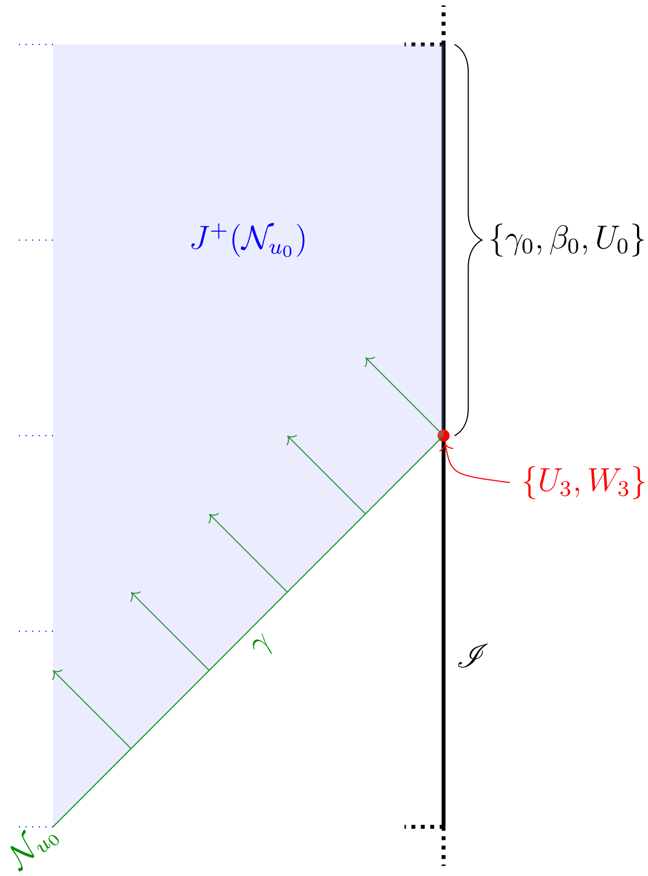

Figure3.3.5: Penrose diagram for the "hybrid scheme". The data which we specify is indicated on the hypersurfaces $\mathcal{N}_{u_0}$ and $\mathscr{I}$ for $u \geq u_0$. Specifying these coefficients allows one to solve the field equations in the causal future of $\mathcal{N}_{u_0}$, $J^+(\mathcal{N}_{u_0})$.

It remains to explain how this data is sufficient to solve for all coefficients of the series expansions of the metric functions in $J^+(\mathcal{N}_{u_0})$. We will first show that one is able to obtain all coefficients of γ and then use this to show that one can obtain all coefficients of the other metric functions, including U_3 and W_3.

Specifying $\{\gamma_0, U_0, \beta_0\}$ at $\mathscr{I}$ gives us these functions as well as all spatial and time derivatives of these functions $\forall\ u \geq u_0$. This information gives γ_1 and all spatial and time derivatives of γ_1 $\forall\ u \geq u_0$ via the Einstein equation (3.2.16). The higher order coefficients of γ are obtained by evolving from the initial null hypersurface instead. As we saw in equation (3.2.17), $\gamma_2 = 0$, so the first coefficient to consider is γ_3. In order to do this we apply equation (3.2.19) to obtain $\gamma_{3,u}$ on the null hypersurface $\mathcal{N}_{u_0}$ (this can be done because the scheme specifies γ, U_3, W_3 on $\mathcal{N}_{u_0}$). As was the case in asymptotically flat spacetime, when applied to a numerical scheme this will correspond to knowing γ_3 on the next null hypersurface $\mathcal{N}_{u_0+\delta u}$. This procedure will repeat for the higher order coefficients in γ in that the Einstein equations will produce expressions for $\gamma_{n,u}$ on $\mathcal{N}_{u_0}$ and thus will determine γ_n on $\mathcal{N}_{u_0+\delta u}$ $\forall\ n \geq 3$.

Using the main equations (3.2.7a-3.2.7d) we know that knowledge of $\gamma(u_0 + \delta u)$ is of course sufficient to give us $\beta(u_0 + \delta u)$, as well as $U(u_0 + \delta u)$ and $W(u_0 + \delta u)$ up to the coefficients W_3 and U_3 which are of course not determined by the main equations (higher coefficients are also determined by these). To solve for these coefficients we will need to consider the supplementary conditions (A.1.1), (A.2) which take the schematic form

$$U_{3,u} = \mathcal{F}(\tilde{\gamma}_0, \tilde{\beta}_0, \tilde{U}_0, \tilde{\gamma}_{0,u}, \tilde{\beta}_{0,u}, \tilde{\gamma}_1, \tilde{\gamma}_{1,u}, \tilde{\gamma}_3, \tilde{\gamma}_{3,u}, \tilde{U}_3, \tilde{W}_3, \tilde{\gamma}_4) \tag{4.6a}$$

$$W_{3,u} = \mathcal{H}(\tilde{\gamma}_0, \tilde{\beta}_0, \tilde{U}_0, \tilde{\gamma}_{0,u}, \tilde{\beta}_{0,u}, \tilde{\gamma}_1, \tilde{\gamma}_{1,u}, \tilde{\gamma}_3, \tilde{\gamma}_{3,u}, \tilde{U}_3, \tilde{W}_3, \tilde{\gamma}_4, \tilde{U}_{3,u}) \tag{4.6b}$$

where the tildes indicate that spatial derivatives of these functions may also be present.

These are u-evolution equations for the functions W_3 and U_3. Note that all of the functions on the right hand side of each equation are known on $\mathcal{N}_{u_0}$. Starting with equation (4.6a): $\tilde{\gamma}_0, \tilde{\beta}_0, \tilde{U}_0, \tilde{\gamma}_{0,u}, \tilde{\beta}_{0,u}, \tilde{\gamma}_3, \tilde{U}_3, \tilde{W}_3, \tilde{\gamma}_4$ are all given on $\mathcal{N}_{u_0}$ as part of the specified data and the remaining functions $\tilde{\gamma}_1, \tilde{\gamma}_{1,u}, \tilde{\gamma}_{3,u}$ can all be determined on $\mathcal{N}_{u_0}$ by using the Einstein equations (3.2.16) and (3.2.19) as discussed above. This means that we are able to obtain $U_{3,u}$ on $\mathcal{N}_{u_0}$ and thus U_3 on the next hypersurface $\mathcal{N}_{u_0+\delta u}$. An identical argument holds for (4.6b), although now there is the extra requirement of knowing $U_{3,u}$ on $\mathcal{N}_{u_0}$, which is of course obtained from (4.6a).

Putting all of this together, we conclude that the specified data, along with iteration of both the main equations and supplementary conditions is an alternative way of constructing solutions to the field equations for asymptotically locally AdS metrics in the Bondi gauge for $J^+(\mathcal{N}_{u_0})$.

3.3.3 dS schemes

Much of the previous discussion for asymptotically locally AdS spacetimes has a parallel discussion in the case of asymptotically dS spacetimes. The two new integration schemes that we have introduced are only dependent upon $\Lambda \neq 0$ in the field equations (3.2.7a-3.2.7d), and are insensitive to the sign of Λ. Due to this, we will now provide a brief description of the Bondi scheme applied to asymptotically locally dS spacetimes, as well as an analogue of the two AdS schemes that we have introduced.

Firstly, we must mention that will restrict our attention to a retarded null foliation of $\mathscr{I}^+$ when discussing the Bondi approach to dS. If we consider applying the asymptotically flat integration scheme of specifying γ on $\mathcal{N}_{u_0}$ as well as (γ_1, U_3, W_3) for all u and θ, then in a similar fashion to the AdS case we will not be able to construct a fully determined solution to the field equations in a neighbourhood of $\mathscr{I}^+$. In the dS case (as in AdS) we will still have the undetermined functions (β_0, U_0) which will propagate into solutions at later retarded times via the null hypersurface evolution.

In order to remedy this problem we can adjust the two AdS integration schemes that we introduced in the previous section in order to describe asymptotically locally *dS* spacetimes and solve the field equations in precisely the same order as before. The "boundary" scheme now consists of specifying the data $\{\gamma_0, \beta_0, U_0, \gamma_3, U_3, W_3\}$ on $\mathscr{I}^+$ and then solving the field equations in the same order as described in section 3.3.2.1. This scheme is displayed pictorially in figure 3.3.6.

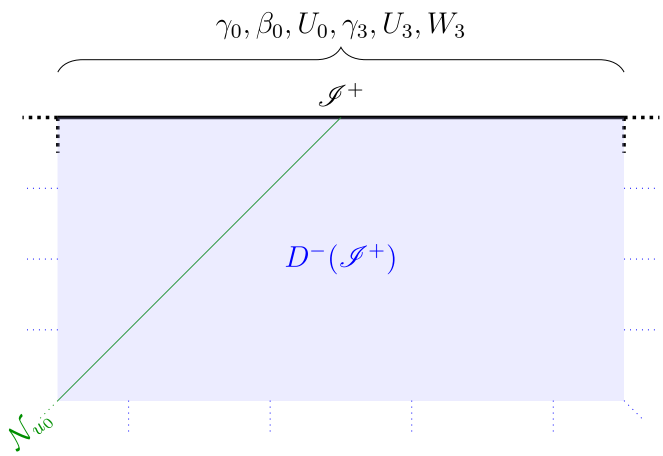

Figure3.3.6: Penrose diagram for the boundary scheme applied to an asymptotically locally dS spacetime. Notice that giving the data over the whole boundary $\mathscr{I}^+$ gives us the solution in $D^-(\mathscr{I}^+)$

The "hybrid" scheme is again a scheme which involves specifying data on $\mathscr{I}^+$ and $\mathcal{N}_{u_0}$.

As in the AdS hybrid scheme we specify (γ_0, U_0, β_0) on $\mathscr{I}^+$ for $u \geq u_0$ and (γ, W_3, U_3) on $\mathcal{N}_{u_0}$, solving the field equations in the same manner as described in section 3.3.2.2 (see figure 3.3.7 below).

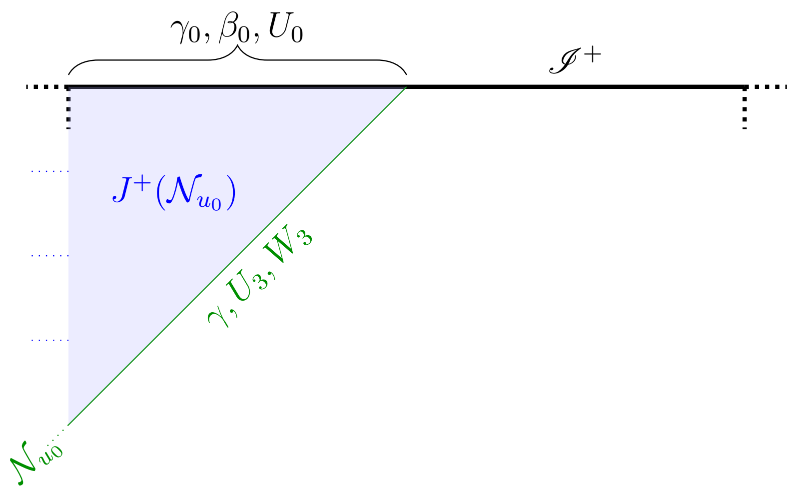

Figure3.3.7: Penrose diagram for the hybrid scheme applied to an asymptotically locally dS spacetime. This scheme generates solutions to the field equations in $J^+(\mathcal{N}_{u_0})$

It seems that the hybrid scheme applied to dS generates a smaller portion of the spacetime when compared with the scheme applied to AdS as shown in figure 3.3.5. This discrepancy is simply due to the causal differences between the respective cases and not an issue with either class of spacetimes. We note that in both cases the hybrid scheme generates the solutions to the field equations in $J^+(\mathcal{N}_{u_0})$ and thus the solutions in the neighbourhood of the conformal boundary to the future of $\mathcal{N}_{u_0}$. This method of specifying data agrees with similar Bondi type integration schemes for asymptotically dS spacetimes as discussed in [172, 171].

3.4 Holographic interpretation

In this section we will study the Bondi gauge metric from the perspective of holography, connecting with [55, 56, 57, 58, 145, 140, 60, 61, 62, 71]. We begin with a review of the Fefferman-Graham coordinate system before deriving the coordinate transformation from Bondi gauge to Fefferman-Graham gauge. This would allow us to give a holographic interpretation to the metric functions used in the integration scheme of section 3.3.

3.4.1 Fefferman-Graham gauge

Asymptotically locally AdS spacetimes can be described in Fefferman-Graham gauge in the neighbourhood of the conformal boundary $\partial\mathcal{M} = \mathscr{I}$; see the review [62]. In this gauge the metric can be expressed as

$$ds^2 = l^2 \left[\frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(g_{(0)ab} + \rho^2 g_{(2)ab} + \rho^3 g_{(3)ab} + \dots)dx^a dx^b \right], \tag{3.4.1}$$

where $l = \sqrt{-3/\Lambda}$ is the AdS radius. Following the discussion of section 2.1.2, ρ is a coordinate which describes the location of the conformal boundary, specifically $\mathscr{I} = \{\rho = 0\}$. The lower case Roman indices (a, b) run from 1 to 3 for asymptotically locally AdS$_4$ spacetimes.

Comparing with (2.1.2) and choosing ρ as the defining function, we see that the term $g_{(0)}$ in the FG expansion is a representative of the conformal class of metrics induced on $\mathscr{I}$. If the metric $g_{(0)ab}$ is conformally flat i.e. the Cotton tensor vanishes, then the spacetime is *Asymptotically* AdS; otherwise it is *Asymptotically locally* AdS.

Holographically, $g_{(0)}$ is viewed as the background metric for the 3-dimensional conformal field theory dual to the 4-dimensional spacetime. The coefficients of even powers of ρ in the asymptotic expansion are determined locally in terms of derivatives of $g_{(0)}$; see [60] for explicit expressions. The coefficient $g_{(3)}$ is constrained to be divergenceless and traceless with respect to $g_{(0)}$, but is otherwise undetermined. This coefficient corresponds to the energy momentum tensor in the dual 3-dimensional field theory, which is defined as [58, 140, 60, 62]

$$\langle T_{ab} \rangle = \frac{2}{\sqrt{-\det g_{(0)}}} \frac{\delta S_r}{\delta g^{ab}_{(0)}} \tag{3.4.2}$$

where S_r is the renormalised on-shell gravitational action. For asympotically locally AdS$_4$ spacetimes

$$\langle T_{ab} \rangle = -\frac{3l^2}{2\kappa^2} g_{(3)ab} \tag{3.4.3}$$

where $2\kappa^2 = 16\pi G$ and G is Newton's constant. This energy momentum tensor satisfies tracelessness and conservation properties with respect to $g_{(0)}$

$$g^{ab}_{(0)} \langle T_{ab} \rangle = 0, \qquad \nabla^a_{(0)} \langle T_{ab} \rangle = 0. \tag{3.4.4}$$

Finally, we note that the pair $(g_{(0)}, T_{ij})$ or equivalently $(g_{(0)}, g_{(3)})$ provide local coordinates on the covariant phase space [158, 63] of the theory; in a radial Hamiltonian formalism, where the radial coordinate plays the role of time, $g_{(0)}$ is the position and $g_{(3)}$ the corresponding canonical momentum [71].

3.4.2 Coordinate transformations

In order to extract holographic data from spacetimes expressed in Bondi gauge, we need to determine the coordinate transformation from our asymptotically locally AdS metric in Bondi gauge

$$\begin{aligned} ds^2 =& -(Wr^2e^{2\beta} - U^2r^2e^{2\gamma})du^2 - 2e^{2\beta}dudr - \\ & 2Ur^2e^{2\gamma}dud\theta + r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2) \end{aligned} \tag{3.4.5}$$

to the Fefferman-Graham form of (5.1.1). We will derive the transformation up to the coefficient $g_{(3)}$, as this is the highest order term of holographic interest.

3.4.2.1 Global AdS_4

A useful first step in performing this computation is to recall the transformation of global AdS_4 spacetime into Fefferman-Graham form. We begin with the metric of AdS_4 in Bondi gauge

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right)du^2 - 2dudr + r^2d\Omega^2, \tag{3.4.6}$$

where we have reinstated the factors of l for clarity. In Bondi coordinates, the metric for AdS_4 corresponds to choosing functions

$$\beta = \gamma = U = 0; \qquad W = \frac{1}{l^2} + \frac{1}{r^2}, \tag{3.4.7}$$

so in the notation of section 3.2 this corresponds to $W_0 = 1/l^2$, $W_2 = 1$, with all other coefficients zero.

We begin by transforming from the retarded time coordinate u into the usual time coordinate t. This is achieved by

$$t = u + r_* \tag{3.4.8}$$

where the tortoise coordinate r_* is defined by

$$dr_* = \frac{dr}{f(r)} = \frac{dr}{1 + (r/l)^2} \implies r_* = l \arctan\left(\frac{r}{l}\right) + c, \tag{3.4.9}$$

with c is an integration constant whose value will be fixed later. Applying equations (3.4.8) and (3.4.9) transforms (3.4.6) into the standard AdS metric of

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{3.4.10}$$

The next step is to transform from our radial distance coordinate r into the tortoise coordinate r_*. The motivation for doing this is that we can fix the the conformal boundary to be located at $r_* = 0$, providing an immediate comparison with the FG coordinate ρ as the conformal boundary in those coordinates is also located at $\rho = 0$. Choosing the integration constant in (3.4.9) to be $c = -l\pi/2$ allows us locate the conformal boundary $\mathscr{I}$ at $r_* = 0$. We implement this part of the transformation by only including the leading order term in the large r approximation of r_*

$$r_* = -\frac{l^2}{r} + \mathcal{O}(r^{-3}) \tag{3.4.11}$$

which brings the line element (3.4.10) into the form

$$ds^2 = \frac{l^2}{r_*^2}\left[-\left(1+\frac{r_*^2}{l^2}\right)dt^2 + \left(1+\frac{r_*^2}{l^2}\right)^{-1} dr_*^2 + l^2 d\Omega^2\right]. \tag{3.4.12}$$

This metric has similarities with (5.1.1); the gauge conditions of $g_{\rho t} = g_{\rho\theta} = g_{\rho\phi} = 0$ are all satisfied automatically if $r_* = f(\rho)$ for any function $f(\rho)$. We hence need to solve for f such that $g_{\rho\rho} = l^2/\rho^2$. Carrying out this procedure we derive the defining equation for $f(\rho)$

$$\frac{l^2 f'^2}{f^2[l^2+f^2]} = \frac{1}{\rho^2} \tag{3.4.13}$$

which admits two solutions

$$f_1 = \frac{2kl\rho}{1-(k\rho)^2}, \qquad f_2 = \frac{2kl\rho}{\rho^2 - k^2} \tag{3.4.14}$$

where in both cases k is an integration constant. These two solutions are related via the map $k \to -1/k$ so it is unimportant which is chosen to be f.

Picking $f = f_1$ we observe that in a neighbourhood of $\mathscr{I}$ we have

$$r_* = \frac{2k\rho l}{1-k^2\rho^2} \approx 2k\rho l \tag{3.4.15}$$

The metric (3.4.12) transforms to

$$ds^2 = \frac{l^2}{\rho^2}d\rho^2 - \frac{(1+k^2\rho^2)^2}{4k^2\rho^2}dt^2 + \frac{l^2(k^2\rho^2-1)^2}{4k^2\rho^2}d\Omega^2. \tag{3.4.16}$$

We can now read off $g_{(0)}$, which is conformally equivalent to the Einstein metric on $\mathbb{R}\times S^2$

$$ds^2_{(0)} = -dt^2 + d\Omega^2. \tag{3.4.17}$$

Notice that the leading order truncations of the Taylor series for our transformations (3.4.11), (3.4.15) allow us to compute only $g_{(0)}$ correctly. To compute higher order $g_{(i)}$ we need to include higher order terms in the transformation, giving

$$ds^2_{(2)} = \frac{1}{2}(-dt^2 - d\theta^2 - \sin^2\theta d\phi^2) \tag{3.4.18}$$

as well as $g_{(3)ab} = 0$. The latter is the expected result for the energy momentum tensor of the CFT state dual to global AdS_4.

In generalising this procedure to asymptotically locally AdS_4 spacetimes we repeat the steps of this procedure, namely using series expansions to transform the coordinates and truncating at the necessary point to compute each $g_{(i)}$ coefficient.

3.4.2.2 Computing $\mathbf{g_{(0)ab}}$

For computational and notational simplicity we will from here onwards fix the AdS radius $l = 1$ ($\Lambda = -3$). Factors of the radius may be reinstated using the following dimensional considerations. The Fefferman-Graham coordinates (t, ρ, θ, ϕ) are dimensionless coordinates, and thus the only dimensions are those of the Bondi metric functions (γ, β, U, W). Working with dimensional conventions of [length] = +1 we first compute the dimensions of the functions in the Bondi gauge metric (3.4.5). Using the standard definitions of the Bondi coordinates we have

$$[u] = 1, \quad [r] = 1, \quad [\theta] = 0, \quad [\phi] = 0 \tag{3.4.19}$$

and the line element has dimension $[ds^2] = 2$. Using the length dimensions of the coordinates, the dimensions of the Bondi functions are

$$[\gamma] = 0, \quad [\beta] = 0, \quad [U] = -1, \quad [W] = -2. \tag{3.4.20}$$

Each of these functions is expanded in negative powers of r in the asymptotic region of the spacetime (5.1.13). Using this, we can determine the dimension of each of the coefficients in the asymptotic expansions as

$$[\gamma_i] = i, \quad [\beta_i] = i, \quad [U_i] = i - 1, \quad [W_i] = i - 2. \tag{3.4.21}$$

To reinstate all the factors of l in the transformation formulae one simply needs to match the dimensions of each side of the equations by multiplying the Bondi functions by suitable powers of l as determined by (3.4.21).

To compute $g_{(0)}$ we need to impose the vacuum Einstein equations to leading order; this corresponds to switching on the leading coefficients in the metric functions β_0, γ_0, U_0 and imposing $W_0 = e^{2\beta_0}$). The leading order line element (3.4.5) takes the form

$$\begin{aligned} ds^2 =& -\left(e^{4\beta_0}r^2 - U_0^2 r^2 e^{2\gamma_0}\right) du^2 - 2e^{2\beta_0} du dr - \\ & 2U_0 r^2 e^{2\gamma_0} du d\theta + r^2 (e^{2\gamma_0} d\theta^2 + e^{-2\gamma_0} \sin^2\theta d\phi^2). \end{aligned} \tag{3.4.22}$$

We now carry out the coordinate transformations (3.4.8) → (3.4.11) → (3.4.15) using the form $r_* = \rho$ as we are for now only concerned about computing $g_{(0)}$. This sequence of transformations gives the metric components at order $1/\rho^2$ as

$$g_{\rho\rho} = \frac{1}{\rho^2}(2e^{2\beta_0} - e^{4\beta_0} + e^{2\gamma_0}U_0^2) \tag{3.4.23a}$$

$$g_{\rho t} = \frac{1}{\rho^2}(e^{4\beta_0} - e^{2\beta_0} - e^{2\gamma_0}U_0^2) \tag{3.4.23b}$$

$$g_{\rho\theta} = \frac{e^{2\gamma_0}U_0}{\rho^2} \tag{3.4.23c}$$

$$g_{tt} = \frac{1}{\rho^2}(e^{2\gamma_0}U_0^2 - e^{4\beta_0}) \tag{3.4.23d}$$

$$g_{t\theta} = -\frac{e^{2\gamma_0}U_0}{\rho^2} \tag{3.4.23e}$$

$$g_{\theta\theta} = \frac{e^{2\gamma_0}}{\rho^2} \tag{3.4.23f}$$

$$g_{\phi\phi} = \frac{e^{-2\gamma_0}\sin^2(\theta)}{\rho^2}. \tag{3.4.23g}$$

The resulting coefficients (3.4.23a-3.4.23c) are clearly incompatible with the Fefferman-Graham gauge. We thus carry out further transformations in θ and t, namely

$$t \to t + \alpha_1(t,\theta)\rho, \qquad \theta \to \theta + \alpha_2(t,\theta)\rho. \tag{3.4.24}$$

where $\alpha_{1,2}$ are functions which are fixed by setting $g_{\rho\rho} = 1/\rho^2$, $g_{\rho t} = g_{\rho\theta} = 0$. When considering the $\mathcal{O}(1/\rho^2)$ pieces of the metric it suffices to transform the forms as

$$dt \to dt + \alpha_1(t,\theta)d\rho + \cdots, \qquad d\theta \to d\theta + \alpha_2(t,\theta)d\rho + \cdots \tag{3.4.25}$$

as terms involving derivatives of $\alpha_{1,2}$ are subleading in the radial expansion.

Under this transformation $g_{\rho\rho}$ is given by

$$\begin{aligned} g_{\rho\rho} =& \frac{1}{\rho^2}[(-e^{4\hat\beta_0} + e^{2\hat\gamma_0}\hat U_0^2)\alpha_1^2 - \alpha_1(2(e^{2\hat\beta_0} - e^{4\hat\beta_0} + e^{2\hat\gamma_0}\hat U_0^2) + 2e^{2\hat\gamma_0}\hat U_0\alpha_2) + \\ & (2e^{2\hat\beta_0} - e^{4\hat\beta_0} + e^{2\hat\gamma_0}\hat U_0^2 + e^{2\hat\gamma_0}\hat U_0\alpha_2 + e^{2\hat\gamma_0}\alpha_2^2)] \end{aligned} \tag{3.4.26}$$

where the hat symbol over metric functions signifies the boundary value e.g..

$$\hat\gamma_0(t,\theta) = \lim_{r_*\to 0}\gamma_0(u,\theta). \tag{3.4.27}$$

Let us now solve the equation $g_{\rho\rho} = l^2/\rho^2$, which is regarded as a quadratic equation for α_1 (or equivalently α_2). Solving this equation gives us two roots:

$$\alpha_1^+ = \frac{1 - e^{2\hat{\beta}_0} + e^{\hat{\gamma}_0}\hat{U}_0 + e^{\hat{\gamma}_0}\alpha_2}{e^{\gamma_0}\hat{U}_0 - e^{2\hat{\beta}_0}} \tag{3.4.28a}$$

$$\alpha_1^- = \frac{-1 + e^{2\hat{\beta}_0} + e^{\hat{\gamma}_0}\hat{U}_0 + e^{\hat{\gamma}_0}\alpha_2}{e^{\hat{\gamma}_0}\hat{U}_0 + e^{2\hat{\beta}_0}}. \tag{3.4.28b}$$

There seems to be no particular motivation to choose one or the other so we will proceed by choosing α_1^+; we will show below that either root could have been chosen. Notice that (3.4.28) gives α_1 in terms of α_2, which is viewed as a free function. Examining the transformations of the $g_{\rho t}, g_{\rho\theta}$ coefficients fixes α_2 and thus α_1 also.

Using the transformation with $\alpha_1 = \alpha_1^+$, $g_{\rho t}$ reduces to

$$g_{\rho t} = \frac{e^{\hat{\gamma}_0}(\hat{U}_0 + e^{2\hat{\beta}_0}\alpha_2)}{\rho^2} \tag{3.4.29}$$

so we can set $g_{\rho t} = 0$ by choosing $\alpha_2 = -\hat{U}_0 e^{-2\hat{\beta}_0}$. We thus conclude that the coordinate transformations are given by

$$t \to t + (1 - e^{-2\hat{\beta}_0})\rho, \qquad \theta \to \theta - \hat{U}_0 e^{-2\hat{\beta}_0}\rho. \tag{3.4.30}$$

Note that this value of α_2 automatically sets $\alpha_1^+ = \alpha_1^-$. We could have alternatively started by choosing $\alpha_1 = \alpha_1^-$; this would have resulted in the same value for α_2, showing that the freedom in choosing α_1 was actually trivial. As a final check for this part of the transformation, we can show that $g_{\rho\theta} = 0$, verifying that the Fefferman-Graham gauge has been reached.

This transformation illustrates the leading order part of the general procedure to transform from Bondi to FG gauge. Using our solutions of the vacuum Einstein equations, we first transform from the Bondi coordinates (u, r, θ, ϕ) into coordinates (t, r_*, θ, ϕ) and then use transformations of the form

$$r_* \to \sum_{j=1}^{i+1} r_{*j}(t,\theta)\rho^j, \quad t \to t + \sum_{j=1}^{i+1} t_j(t,\theta)\rho^j, \quad \theta \to \theta + \sum_{j=1}^{i+1} \theta_j(t,\theta)\rho^j \tag{3.4.31}$$

where the limit of the sum $i+1$ indicates the order necessary to compute the coefficient $g_{(i)}$ (thus we will only be concerned about summing to an upper limit of four). At each order we need to solve for the coefficients r_{*j}, t_j, θ_j to preserve the FG gauge conditions $g_{\rho\rho} = 1/\rho^2$, $g_{t\rho} = g_{\theta\rho} = 0$ ($g_{\phi\rho} = 0$ will be satisfied automatically due to axisymmetry and trivial $\phi \to \phi$ transformation). More detail and computation of the higher order coefficients is given in appendix A.2.

3.4.3 Background metric

The transformation (3.4.24) gives the following results for $g_{(0)ab}$:

$$ds^2_{(0)} = (e^{2\hat{\gamma}_0}\hat{U}_0^2 - e^{4\hat{\beta}_0})dt^2 - 2e^{2\hat{\gamma}_0}\hat{U}_0 dt d\theta + e^{2\hat{\gamma}_0}d\theta^2 + e^{-2\hat{\gamma}_0}\sin^2(\theta)d\phi^2. \tag{3.4.32}$$

Note that the boundary is not necessarily topologically equivalent to $\mathbb{R}\times S^2$ in general; the spacetimes are asymptotically locally AdS rather than asymptotically AdS.

When $\hat{U}_0$ vanishes, the boundary metric is topologically $\mathbb{R}\times S^2$ but the metric on the S^2 is deformed by non-trivial $\hat{\gamma}_0$. The boundary metric retains the determinant condition on the angular part of the metric

$$d\Omega^2 = e^{2\hat{\gamma}_0}d\theta^2 + e^{-2\hat{\gamma}_0}\sin^2(\theta)d\phi^2 \implies |\Omega| = \sin^2\theta, \tag{3.4.33}$$

which was part of the definition of the Bondi gauge. This is an unusual restriction on the boundary metric: it is somewhat unnatural to impose a fixed determinant for the metric on the sphere. It would thus be interesting to revisit the Bondi gauge analysis, dropping the determinant condition on the spherical part of the metric.

3.4.4 The energy-momentum tensor

The final term of physical interest in the Fefferman-Graham expansion is $g_{(3)}$ as this describes the energy-momentum tensor of the dual conformal field theory (3.4.3). To compute $g_{(3)ab}$ we have to include terms up to $\mathcal{O}(r^{-3})$ in the metric functions

$$\gamma(u,r,\theta) = \gamma_0 + \frac{\gamma_1}{r} + \frac{\gamma_3}{r^3} \tag{3.4.34a}$$

$$\beta(u,r,\theta) = \beta_0 - \frac{\gamma_1^2}{4r^2} \tag{3.4.34b}$$

$$\begin{aligned}U(u,r,\theta) = U_0 + \frac{2}{r}\beta_{0,\theta}e^{2(\beta_0-\gamma_0)} - \\ \frac{1}{r^2}e^{2(\beta_0-\gamma_0)}(2\beta_{0,\theta}\gamma_1 - 2\gamma_{0,\theta}\gamma_1 + \gamma_{1,\theta} + 2\cot(\theta)\gamma_1) + \frac{U_3}{r^3}\end{aligned} \tag{3.4.34c}$$

$$\begin{aligned}W(u,r,\theta) = e^{2\beta_0} + \frac{1}{r}[\cot(\theta)U_0 + U_{0,\theta}] + \frac{1}{2r^2}e^{2(\beta_0-\gamma_0)}[2 - 3e^{2\gamma_0}\gamma_1^2 + 4\cot(\theta)\beta_{0,\theta} + \\ 8(\beta_{0,\theta})^2 + 6\cot(\theta)\gamma_{0,\theta} - 8\beta_{0,\theta}\gamma_{0,\theta} - 4(\gamma_{0,\theta})^2 + 4\beta_{0,\theta\theta} + 2\gamma_{0,\theta\theta}] + \frac{W_3}{r^3}.\end{aligned} \tag{3.4.34d}$$

As a brief aside, we observe that the integration functions U_3 and W_3 enter the metric at this order. Recall that W_3 has the interpretation in asymptotically flat spacetime as the Bondi mass aspect, $W_3 = -2m_B$ [1]. If we follow [48] in defining the mass aspect function as the $\mathcal{O}(1/r)$ term in the Bondi metric component g_{uu} then we obtain

$$\begin{aligned}2m_B =& - e^{-2(\beta_0+\gamma_0)}(2\gamma_{0,u} - U_0(\cot(\theta) - 2\gamma_{0,\theta}) + U_{0,\theta})(4e^{4\beta_0}(\beta_{0,\theta})^2 - \\ & e^{2\gamma_0}U_0(-2\gamma_{0,u\theta} + 4\gamma_{0,u}(\gamma_{0,\theta} - \cot(\theta)) + U_0(4(\gamma_{0,\theta})^2 - \\ & 2\gamma_{0,\theta\theta} - 6\cot(\theta)\gamma_{0,\theta} + \cot^2(\theta) - 1) - U_{0,\theta\theta} - \cot(\theta)U_{0,\theta})) + \\ & e^{-2\gamma_0}(2e^{4\gamma_0}U_0U_3 - 2e^{2\beta_0}\beta_{0,\theta}(-2\gamma_{0,u\theta} + 4\gamma_{0,u}(\gamma_{0,\theta} - \cot(\theta)) + \\ & U_0(4(\gamma_{0,\theta})^2 - 2\gamma_{0,\theta\theta} - 6\cot(\theta)\gamma_{0,\theta} + \cot^2(\theta) - 1) - U_{0,\theta\theta} - \cot(\theta)U_{0,\theta})) + \\ & \frac{1}{3}e^{2\gamma_0}U_0^2\left[6\gamma_3 + \frac{1}{2}e^{-6\beta_0}(-2\gamma_{0,u} + U_0(\cot(\theta) - 2\gamma_{0,\theta}) - U_{0,\theta})^3\right] + \\ & 2e^{-2\beta_0}\beta_{0,\theta}U_0(2\gamma_{0,u} - U_0(\cot(\theta) - 2\gamma_{0,\theta}) + U_{0,\theta})^2 + \\ & \frac{1}{8}e^{-2\beta_0}(U_{0,\theta} + \cot(\theta)U_0)(2\gamma_{0,u} - U_0(\cot(\theta) - 2\gamma_{0,\theta}) + U_{0,\theta})^2 - e^{2\beta_0}W_3.\end{aligned} \tag{3.4.35}$$

Here we have used the Einstein equation (3.2.16) to express contributions in terms of (γ_0, U_0, β_0) wherever possible. In the asymptotically AdS case $\gamma_0 = \beta_0 = U_0 = 0$ we obtain the same definition of the mass aspect, $2m_B = -W_3$, as in the asymptotically flat case [1].

In the asymptotically flat case, the Bondi mass at time $u = u_0$ is obtained by integrating the mass aspect over the u_0 cut of $\mathscr{I}^+$ (2.2.14). It is natural to suggest that an extension should exist for the AdS case whereby one could obtain the analogue of the Bondi mass in asymptotically locally AdS spacetime by integrating over a cut of $\mathscr{I}$ instead. We will discuss this definition in asymptotically AdS spacetimes in section 3.4.5.5 while the more general case of asymptotically locally AdS remains ongoing work.

Returning to the discussion of the coordinate transformation in order to obtain $g_{(3)}$, we note that when performing the series transformation into the Fefferman-Graham form we also need to extend our transformation in the coordinates to $\mathcal{O}(\rho^4)$

$$\begin{aligned} r_* &\to \rho + b_1(t,\theta)\rho^2 + c_1(t,\theta)\rho^3 + d_1(t,\theta)\rho^4 \\ t &\to t + \alpha_1(t,\theta)\rho + b_2(t,\theta)\rho^2 + c_2(t,\theta)\rho^3 + d_2(t,\theta)\rho^4 \\ \theta &\to \theta + \alpha_2(t,\theta)\rho + b_3(t,\theta)\rho^2 + c_3(t,\theta)\rho^3 + d_3(t,\theta)\rho^4. \end{aligned} \tag{3.4.36}$$

where α_i, b_i, c_i are the functions already obtained from previous orders (see appendix A.2 for b_i and c_i). To obtain $g_{(3)ab}$ we will need to choose $d_{1,2,3}$ suitably in order to force the $d\rho$ terms to vanish at $\mathcal{O}(1/\rho)$.

Once we have performed this transformation we have to check equations (3.4.4) are satisfied. First we use the $g_{(0)}$ of equation (3.4.32) to check tracelessness

$$g^{ab}_{(0)}g_{(3)ab} = g^{tt}_{(0)}g_{(3)tt} + 2g^{t\theta}_{(0)}g_{(3)t\theta} + g^{\theta\theta}_{(0)}g_{(3)\theta\theta} + g^{\phi\phi}_{(0)}g_{(3)\phi\phi} = 0, \tag{3.4.37}$$

which is automatically satisfied by $g_{(3)ab}$ without having to apply either the supplementary conditions or the higher order main equations.

In order to present expressions for the $g_{(3)}$ coefficients, we give formulae for (U_3, γ_3, W_3) which have been obtained via rearrangement of the expressions for $(g_{(3)tt}, g_{(3)t\theta}, g_{(3)\theta\theta})$. Although there are four non-zero components of the energy-momentum tensor, the three functions below suffice to read off all components due to the tracelessness equation (3.4.37).

$$\begin{aligned}\hat{U}_3 &= e^{-2\hat{\gamma}_0}(g_{(3)\theta\theta}\hat{U}_0 + g_{(3)t\theta}) + \mathcal{U}_3(\hat{\gamma}_0, \hat{\beta}_0, U_0);\\ \hat{W}_3 &= \frac{3}{2}e^{-2\hat{\beta}_0}\left(g_{(3)\theta\theta}\hat{U}_0^2 + 2g_{(3)t\theta}\hat{U}_0 + g_{(3)tt}\right) + \mathcal{W}_3(\hat{\gamma}_0, \hat{\beta}_0, \hat{U}_0);\\ \hat{\gamma}_3 &= \frac{1}{4}\left(e^{-4\hat{\beta}_0}(g_{(3)\theta\theta}\hat{U}_0^2 + 2g_{(3)t\theta}\hat{U}_0 + g_{(3)tt}) - 2e^{-2\hat{\gamma}_0}g_{(3)\theta\theta}\right) + \mathcal{G}_3(\hat{\gamma}_0, \hat{\beta}_0, \hat{U}_0),\end{aligned} \tag{3.4.38}$$

where all of the metric coefficients are functions of (t, θ), defined at $\mathscr{I}$, and explicit expressions for $(\mathcal{U}_3, \mathcal{W}_3, \mathcal{G}_3)$ can be found in appendix A.2.3.

Verification of the conservation condition (3.4.4) is less straightforward than checking tracelessness. The simplest component to check is the ϕ component, for which the required result is obtained using the equations (A.2.20-A.2.22) above and the tracelessness property (3.4.37)

$$\nabla^a_{(0)}g_{(3)a\phi} = g^{ac}_{(0)}\nabla_{(0)c}g_{(3)a\phi} = -g^{ca}_{(0)}\Gamma^{\phi}_{ca}g_{(3)\phi\phi} - g^{ca}_{(0)}\Gamma^{d}_{c\phi}g_{(3)ad} = 0 \tag{3.4.39}$$

where the Christoffel symbols Γ^a_{bc} are those associated with the metric $g_{(0)ab}$.

The remaining conservation equations are harder to verify. The Einstein equations (3.2.16), (3.2.19) for $\hat{\gamma}_1$ and $\hat{\gamma}_{3,t}$ and the supplementary conditions (A.1.1-A.2) are required, the latter giving expressions for the functions $\hat{U}_{3,t}$ and $\hat{W}_{3,t}$. These equations, combined with the relations (A.2.20-A.2.22), are sufficient to show that the t and θ components of the conservation conditions (3.4.4) are satisfied.

3.4.5 Asymptotically AdS$_4$ examples

The first interesting example to look at is the class of asymptotically AdS$_4$ Bondi gauge pacetimes. Recall that we defined asymptotically AdS$_4$ spacetimes as asymptotically locally AdS$_4$ spacetimes for which $g_{(0)}$ is conformally flat. We can choose the representative of this conformal class to be

$$\hat{\gamma}_0 = \hat{\beta}_0 = \hat{U}_0 = 0, \tag{3.4.40}$$

so that the metric $g_{(0)}$ is the standard metric on the Einstein universe.

Applying these values to (A.2.16a-A.2.16d) and (A.2.20-A.2.22) to compute $g_{(3)}$ we obtain

$$ds^2_{(2)} = -\frac{1}{2}[dt^2 + d\Omega^2] \tag{3.4.41}$$

$$ds^2_{(3)} = \frac{2}{3}\hat{W}_3 dt^2 + 2\hat{U}_3 dt d\theta + \left(\frac{1}{3}\hat{W}_3 - 2\hat{\gamma}_3\right) d\theta^2 + \left(\frac{1}{3}\sin^2\theta \hat{W}_3 + 2\sin^2\theta\hat{\gamma}_3\right) d\phi^2 \tag{3.4.42}$$

Notice that $g_{(2)}$ can also be obtained from (3.4.17) using the curvature formula (A.2.12)).

The second of these two formulae gives us the energy-momentum tensor for an asymptotically AdS_4 spacetime in terms of Bondi gauge functions. From (3.4.42) we note that

$$g_{(3)tt} = \frac{2\hat{W}_3}{3} = -\frac{4\hat{m}_B}{3} \tag{3.4.43}$$

which arises from the formula (3.4.35) for the Bondi mass aspect, m_B, now restricted to the boundary, $\hat{m}_B = m_B|_{\mathscr{I}}$. Thus, the $g_{(3)tt}$ component of the energy-momentum tensor is determined entirely by the mass aspect function. This implies in particular that the Bondi mass for asymptotically $AdS_{\mathbf{4}}$ spacetimes is equal to the mass computed using the holographic energy momentum tensor. Indeed,

$$\mathcal{M} = \int_{S^2} dS_\mu \langle T^\mu{}_\nu\rangle \xi^\nu = -\frac{3}{16\pi}\int_{S^2} g_{(3)tt} = \frac{1}{4\pi}\int_{S^2}\hat{m}_B = \mathcal{M}_B \tag{3.4.44}$$

where in the first equality ξ^μ is an asymptotic timelike killing vector, which we take to be $\xi^\mu = -\left(\frac{\partial}{\partial t}\right)^\mu$ and we set $l = G = 1$. This also implies that the Bondi mass for asymptotically $AdS_{\mathbf{4}}$ spacetimes is equal with all other definitions of mass for asymptotically $AdS_{\mathbf{4}}$ spacetimes as all of them are known to agree with the holographic mass (as they had to since [71] provided a first principles derivation that the conserved charges for general AlAdS spacetimes are the holographic charges). In appendix A.4 we demonstrate the equality between the Bondi mass and the Abbott-Deser mass.

We will now discuss interesting examples of asymptotically AdS_4 spacetimes.

3.4.5.1 Global AdS_4

An obvious example of an asymptotically AdS_4 spacetime is the case of global AdS_4 itself. Using the usual normalisation of $l = 1$, the line-element in retarded Bondi coordinates reads

$$ds^2 = -(1+r^2)du^2 - 2dudr + r^2 d\Omega^2. \tag{3.4.45}$$

Clearly $W_3 = U_3 = \gamma_3 = 0$. Applying this to (3.4.42) we see that $g_{(3)}$ vanishes and thus the energy-momentum tensor of the CFT state (the vacuum state) dual to global AdS_4 is zero.

3.4.5.2 AdS$_4$ Schwarzschild

We now consider the AdS$_4$-Schwarzschild black hole solution whose metric in retarded Bondi coordinates reads

$$ds^2 = -\left(1 + r^2 - \frac{2m}{r}\right) du^2 - 2dudr + r^2 d\Omega^2. \tag{3.4.46}$$

This solution is an example of an asymptotically AdS$_4$ metric and thus it automatically has the same values for $g_{(0)}$ and $g_{(2)}$ as presented above.

This solution has metric functions $\beta = \gamma = U = 0$ and matching (3.4.46) with the general Bondi gauge metric (5.1.10) gives $W = 1 + 1/r^2 - 2m/r^3$ i.e. $W_3 = -2m$. Using the relation (3.4.42) we obtain

$$g_{(3)ab} = -\frac{2m}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta \end{pmatrix} \tag{3.4.47}$$

which reduces to the case of global AdS$_4$ when $m = 0$.

3.4.5.3 Flat $\mathbf{g_{(0)}}$

Let us now consider the case where the metric $g_{(0)}$ is flat. One can show explicitly that the metric on the Einstein universe is conformally flat using the coordinate transformation

$$\tau \pm y = \tan\left[\frac{1}{2}(t \pm \theta)\right]. \tag{3.4.48}$$

to obtain

$$ds^2_{(0)} = 4\cos^2\left[\frac{1}{2}(t+\theta)\right]\cos^2\left[\frac{1}{2}(t-\theta)\right](-d\tau^2 + dy^2 + y^2 d\phi^2) \tag{3.4.49}$$

which is clearly conformal to the flat metric on $\mathbb{R}^{2,1}$ in polar coordinates.

Under a conformal transformation $g_{(0)} \to e^{2\sigma} g_{(0)}$ the coefficients of the Fefferman-Graham expansion transform as (see discussion in [61])

$$\begin{aligned} g'_{(0)ab} &= e^{2\sigma} g_{(0)ab} \\ g'_{(2)ab} &= g_{(2)ab} + \nabla_a\nabla_b\sigma - \nabla_a\sigma\nabla_b\sigma + \frac{1}{2}(\nabla\sigma)^2 g_{(0)ab} \\ g'_{(3)ab} &= e^{-\sigma} g_{(3)ab} \end{aligned} \tag{3.4.50}$$

and therefore

$$
\begin{aligned}
g'_{(0)ab} &= \eta_{ab}, \qquad ds'^2_{(0)} = -d\tau^2 + dy^2 + y^2 d\phi^2 \\
g'_{(2)ab} &= 0 \\
g'_{(3)ab} &= 2\cos\left[\frac{1}{2}(t+\theta)\right]\cos\left[\frac{1}{2}(t-\theta)\right]\begin{pmatrix} \frac{2}{3}\hat{W}_3 & \hat{U}_3 & 0 \\ \hat{U}_3 & \frac{1}{3}\hat{W}_3 - 2\hat{\gamma}_3 & 0 \\ 0 & 0 & \sin^2\theta\left(\frac{1}{3}\hat{W}_3 + 2\hat{\gamma}_3\right) \end{pmatrix} \\
ds'^2_{(3)} &= \frac{2}{3}\left[(1+(\tau+y)^2)(1+(\tau-y)^2)\right]^{-5/2} \\
&\quad \times \{[-48y\tau(1+y^2+\tau^2)\hat{U}_3 + 8(y^4+(1+\tau^2)^2+y^2(1+4\tau^2))\hat{W}_3 - \\
&\quad 96y^2\tau^2\hat{\gamma}_3]d\tau^2 + [24(y^4+(1+\tau^2)^2+y^2(2+6\tau^2))\hat{U}_3 - \\
&\quad 48y\tau(1+y^2+\tau^2)\hat{W}_3 + 96y\tau(1+y^2+\tau^2)\hat{\gamma}_3]dyd\tau + \\
&\quad [-48y\tau(1+y^2+\tau^2)\hat{U}_3 + 4(y^4+(1+\tau^2)^2+2y^2(1+5\tau^2))\hat{W}_3 - \\
&\quad 24(1+y^2+\tau^2)^2\hat{\gamma}_3]dy^2 + \\
&\quad [4y^2((1+(\tau+y)^2)(1+(\tau-y)^2))(\hat{W}_3+6\hat{\gamma}_3)]d\phi^2\}.
\end{aligned}
\tag{3.4.51}
$$

The equation for $g'_{(0)}$ is presented in the flat coordinates (τ, y, ϕ) and both $g_{(2)}$ and $g_{(3)}$ have been presented in both the old (t, θ, ϕ) coordinates as well as the new coordinates (τ, y, ϕ) ($g_{(2)}$ trivially so).

We observe that $g'_{(3)}$ is merely (3.4.42) multiplied by a conformal factor. (3.4.51) presents the specific factor when we have a flat metric at the boundary $g_{(0)ab} = \eta_{ab}$. We also remark that one could immediately deduce that $g'_{(2)ab}$ vanishes by applying (A.2.12) to the flat metric.

3.4.5.4 AdS$_4$ black brane

An example of a vacuum solution with a flat $g_{(0)}$ is the AdS black brane solution. The black brane is an asymptotically AdS solution to the vacuum Einstein equations with planar horizon topology,

$$
\begin{aligned}
ds^2 &= \frac{d\rho^2}{4\rho^2 f_b(\rho)} + \frac{-f_b(\rho)dt^2 + dx_1^2 + dx_2^2}{\rho}, \\
f_b(\rho) &= 1 - \frac{\rho^{3/2}}{b^3}
\end{aligned}
\tag{3.4.52}
$$

where b is related to the temperature T of the brane via $b = 3/(4\pi T)$.

It is straightforward to transform the black brane solution into the Fefferman-Graham form using a redefintion of the radial coordinate ρ (see for example [173]), resulting in Fefferman-Graham expansion coefficients:

$$\begin{aligned} g_{(0)ab} &= \eta_{ab}; \quad ds^2_{(0)} = -d\tau^2 + dy^2 + y^2 d\phi^2; \\ g_{(2)ab} &= 0; \\ g_{(3)ab} &= -\frac{1}{3}\left(\frac{4\pi T}{3}\right)^3 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y^2 \end{pmatrix}, \end{aligned} \tag{3.4.53}$$

where we use the flat coordinates (τ, y, ϕ) of (3.4.51).

We can calculate the relevant Bondi quantities for the AdS black brane from (3.4.51) and (3.4.53):

$$\begin{aligned} \hat{\gamma}_3 &= \frac{1}{8}\left(\frac{4\pi T}{3}\right)^3 \tau^2 y^2 \sqrt{((y-\tau)^2+1)((\tau+y)^2+1)} \\ \hat{U}_3 &= -\frac{1}{4}\left(\frac{4\pi T}{3}\right)^3 \tau y \left(\tau^2+y^2+1\right) \sqrt{((y-\tau)^2+1)((\tau+y)^2+1)} \\ \hat{W}_3 &= -\frac{1}{8}\left(\frac{4\pi T}{3}\right)^3 \sqrt{((y-\tau)^2+1)((\tau+y)^2+1)} \\ &\quad \times \left(\left(\tau^2+1\right)^2 + y^4 + \left(4\tau^2+2\right) y^2\right). \end{aligned} \tag{3.4.54}$$

Note that $\hat{W}_3$ will be related to the mass aspect if we use (3.4.50) to transform the solution so that to boundary metric is $\mathbb{R} \times S^2$. The corresponding mass will then be the conserved charge associated with time translations. However, as the coordinate transformation (3.4.48) transforms t to τ and y, what was a mass aspect on $\mathbb{R} \times S^2$ is not a mass aspect on $\mathbb{R}^{1,2}$. Indeed, it was shown in [61] that

$$\partial_t = \frac{1}{2}(P_\tau + K_\tau) \tag{3.4.55}$$

where $P_\tau = \partial_\tau$ is the generator of τ-translations and $K_i = x^2\partial_i - 2x_i x^j \partial_j$ the generator of special conformal transformations (see also the discussion in [174]) . Thus, $\hat{W}_3$ is related to a linear combination of the mass and the "special conformal" aspects on $\mathbb{R}^{1,2}$. [3]

3.4.5.5 Bondi mass

In our gauge the Bondi mass (2.2.14) reduces to

$$\mathcal{M}_B = \frac{1}{4\pi}\int_{S^2} m_B = \frac{1}{2}\int_0^\pi m_B \sin(\theta)\, d\theta \tag{3.4.56}$$

[3]One can explicitly confirm this using (3.4.50), (3.4.48) and $K_\tau = (y^2+\tau^2)\partial_\tau + 2\tau y \partial_y$.

where m_B is the mass aspect function defined in (3.4.35).

We would like to examine whether or not the Bondi mass in asymptotically locally AdS spacetimes maintains the monotonicity property of the mass in asymptotically flat spacetime [1, 2], namely

$$\frac{\partial \mathcal{M}_B}{\partial u} \leq 0. \tag{3.4.57}$$

Note that for asymptotically flat spacetimes saturation of the bound corresponds to the absence of gravitational radiation.

To examine the AdS analogue of this result, we begin by examining the case of asymptotically AdS space-times for which $\gamma_0 = \beta_0 = U_0 = 0$. In this case the mass aspect coincides with the original definition, $2m_B = -W_3$ and

$$\frac{\partial \mathcal{M}_B}{\partial u} = \frac{1}{2}\int_0^\pi \frac{\partial m_B}{\partial u}\sin(\theta)\, d\theta = -\frac{1}{4}\int_0^\pi \frac{\partial W_3}{\partial u}\sin(\theta)\, d\theta. \tag{3.4.58}$$

To analyse this, we use the supplementary condition (A.2) (evolution equation for W_3), which reduces to

$$\begin{aligned} W_{3,u} =& \frac{1}{2}[6\gamma_1^4 - \gamma_{1,\theta}^2 + 4\gamma_{1,u}^2 + 4\gamma_{1,u} - 2\gamma_{1,u\theta\theta} - 8\cot^2(\theta)\gamma_1^2 + \\ & \gamma_1(-12\gamma_3 + \gamma_{1,\theta\theta} - 15\cot(\theta)\gamma_{1,\theta}) - 6\cot(\theta)\gamma_{1,u\theta} + 3U_{3,\theta} + 3\cot(\theta)U_3] \end{aligned} \tag{3.4.59}$$

in this case.

To simplify this relation further we use the Einstein equation (3.2.16), which implies

$$\gamma_1 = 0 \tag{3.4.60}$$

and thus

$$W_{3,u} = \frac{3}{2}\left(U_{3,\theta} + \cot(\theta)U_3\right) \tag{3.4.61}$$

and thus substituting into equation (3.4.57) gives

$$\frac{\partial \mathcal{M}_B}{\partial u} = -\frac{3}{8}\int_0^\pi \left(U_{3,\theta}\sin(\theta) + \cos(\theta)U_3\right)\, d\theta = -\frac{3}{8}\left[U_3\sin(\theta)\right]_0^\pi . \tag{3.4.62}$$

To evaluate the limits of this integral we use the same regularity conditions as in [1]. At the poles of the 2-sphere,

$$\frac{U_3}{\sin(\theta)} = f(\cos(\theta)) \tag{3.4.63}$$

where the function f is regular at the poles. Applying this condition in (3.4.62) gives

$$\frac{\partial \mathcal{M}_B}{\partial u} = -\frac{3}{8}[\sin^2(\theta)f(\cos(\theta))]_0^\pi = 0 \tag{3.4.64}$$

using the regularity of f.

Thus for asymptotically *AdS* spacetimes the Bondi mass is constant and does not vary with respect to the Bondi time, u. This confirms earlier results in [41, 164, 46, 47, 49] (mostly for the dS case). The result is striking and is what would be expected on physical grounds, as we will now explain.

Firstly, let us recall the interpretation of equation (3.4.60) in the language of the original work by BMS. Vanishing of γ_1 implies there is no news and thus (in the asymptotically flat case) the mass is automatically conserved. This interpretation carries over to the asymptotically AdS case. Note however that it seems less likely that this result will extend trivially to the broader class of asympotically locally AdS spacetimes, as it is possible to have vanishing γ_1 but non-trivial (γ_0, β_0, U_0). The latter would play a role in the equation (A.2) for the evolution of the mass aspect and could alter the monotonicity properties of the mass.

Another way to understand why the Bondi mass remains constant for asymptotically AdS space-times is that the boundary metric is unchanging, indicating a lack of gravitational radiation to perturb it. Any outgoing radiation would effect the boundary metric and as the metric is unchanging with time there is no gravitational radiation. The original motivation of BMS was to define a mass which captured radiation escaping at (null) infinity and thus our conclusion is consistent with their approach.

3.4.6 Integration scheme

In this section we summarise the relation between the Fefferman-Graham integration scheme, which effectively allows the spacetime to be reconstructed in the neighbourhood of the conformal boundary in terms of CFT data, and the integration scheme in Bondi gauge discussed in section 3.3.2. In the latter, one specifies the data

$$\{\hat{\gamma}_0(t,\theta), \hat{\beta}_0(t,\theta), \hat{U}_0(t,\theta), \hat{\gamma}_3(t,\theta), \hat{U}_3(t,\theta), \hat{W}_3(t,\theta) \,|\, t \in \mathbb{R},\, \theta \in (0, 2\pi)\} \tag{3.4.65}$$

which has the effect of reducing the Einstein equations to *algebraic* equations from which one construct fully the asymptotic solutions to the Einstein equations without having to evolve betweeen null hypersurfaces.

The holographic interpretation of $(\hat{\gamma}_0, \hat{\beta}_0, \hat{U}_0)$ is given by equation (3.4.32): these functions define the metric at the conformal boundary, $g_{(0)ab}$. The commonly imposed determinant constraint on the spherical part of the metric in Bondi gauge translates into a determinant constraint on the spherical part of the boundary metric, a constraint which is unnatural from a CFT perspective.

The data $(\hat{\gamma}_3, \hat{U}_3, \hat{W}_3)$ defines the energy momentum tensor of the dual theory, T_{ab}. More precisely, equation (3.4.38) gives the relation between $g_{(0)ab}$ and T_{ab} ($\sim g_{(3)ab}$) and the coefficients $(\hat{\gamma}_3, \hat{U}_3, \hat{W}_3)$. With this holographic interpretation we can rephrase the Bondi integration scheme in the following form:

Knowledge of the metric $g_{(0)ab}$ at $\mathscr{I}$ and the energy momentum tensor $T_{ab} \sim g_{(3)ab}$ for the CFT dual of the Bondi gauge spacetime is sufficient to algebraically solve the vacuum Einstein equations in the asymptotic region.

3.5 Conclusions

The main result of this chapter is the general asymptotic solution of asymptotically local AdS and dS spacetimes in Bondi gauge. We saw that we can use two different integration schemes: in the boundary scheme we fix data on the conformal boundary only, while in the hybrid scheme we give data on a null hypersurface and a portion of the conformal boundary. We also presented the coordinate transformation to Fefferman-Graham coordinates and identified how to extract the holographic data/conserved quantities directly in Bondi gauge.

The analysis was done for vacuum Einstein gravity in four dimensions and for solutions that are axially and reflection symmetric. It would be straightforward to relax these conditions, *i.e.* to consider solutions with no axial and reflection symmetry, add matter and generalise to higher dimensions. In odd dimensions the asymptotic expansion will involve logarithmic terms, and so it will in any dimension with specific types of matter (as discussed for $d = 4$ in appendix A.3). These logarithms are related to logarithmic divergences in the on-shell value of the gravitational action [58, 60].

One undesirable feature of the Bondi gauge is the determinant condition on the angular part of the metric (2.2.7). In the context of (A)dS this implies that the angular part of the boundary metric satisfies a similar condition (3.4.33). Via gauge/gravity duality however the boundary metric also has the interpretation of a source for the energy momentum tensor of the dual QFT and in QFT the sources should be unconstrained. This issue is due to the way that the Bondi gauge is defined, and should not restrict the phase space in any way. There are a number of known approaches to relax this restriction e.g. [87] suggests keeping the original Bondi gauge determinant condition more general by requiring $r^4 \det(g_{AB}) = b(u, \theta)$ where $b(u, \theta)$ is an arbitrary *but known* function. This allows for a more general conformal factor multiplying the angular part of the metric in (3.4.22). Another approach would be to use gauge preserving diffeomorphisms in FG gauge in order to attempt to remove this constraint (for related ideas, see [50]). In both cases, the physical picture will become muddied, as the Bondi metric functions will lose their immediate physical significance, a significance which could be recovered by use of the linkage formulas of [175].

We have seen that the Bondi mass is constant for asymptotically (A)dS metrics, reflecting the fact that these boundary conditions do not allow for radiating spacetimes. To accommodate radiating spacetimes one needs to consider asymptotically locally (A)dS spacetimes with a time dependent boundary metric. While we now know the general asymptotic solution for such spacetimes, we do not know yet what is the correct identification of the appropriate notion of mass that accounts for the radiation (but see [163, 164, 48, 46, 47, 49, 165]). Physically, we expect that the mass of a compact object radiating outgoing gravitational waves should decrease monotonically, regardless of asymptotics. It is not yet clear whether the Bondi mass defined using (3.4.38) has this quality. It would be interesting to investigate this candidate as well as the more general issue regarding the required boundary conditions that one need impose for the existence of a monotonically decreasing quantity (i.e. to exclude incoming radiation). A radiating spacetime which is asymptotically locally AdS and possesses a "Bondi mass" with the required properties [69] is the AdS Robinson-Trautman solution. It would thus be useful to bring this solution to Bondi gauge and use it as a playground.

In this chapter we only touched upon the case of positive cosmological constant, only discussing properties that can be directly inferred from those of negative Λ. There are however important global differences between the two cases and it would be interesting to completely analyse the case of positive cosmological constant in detail, especially given its phenomenological importance. We return to this and related issues in chapter 5.

The direct analogue of the asymptotically flat case when $\Lambda \neq 0$ is the case of asymptotically (A)dS spacetimes. When $\Lambda \neq 0$, however, we have seen that we can obtain asymptotic solutions more generally for asymptotically locally (A)dS spacetimes. It would be interesting to revisit the case of no cosmological constant and determine the most general boundary conditions allowed by Einstein's equations (and the variational problem) at null infinity and find the corresponding asymptotic solutions. This may be relevant in understanding how holography works in asymptotically flat gravity.

CHAPTER 4

Physical applications of the Bondi gauge in Al(A)dS spacetimes

4.1 The Bondi mass in asymptotically locally AdS spacetime

As part of the pioneering work of [1], BMS proved that the Bondi mass in axisymmetric, asymptotically flat (AF) spacetime was monotonically decreasing. It would be interesting to see if such notions of monotonicity carry over to the AdS asymptotics and thus this section of the thesis attempts to replicate the steps of the original proof but now in axisymmetric, asymptotically locally AdS (AlAdS) spacetime.

By performing this anaylsis, we hope to find either a proof of monotonicity or an explicit counter-example to monotonicity. If the monotonicity carries over, then the Bondi mass in AdS would be a natural candidate (as in the AF case) to describe the mass loss due gravitational radiation and thus it may help us understand gravitational waves in AdS spacetime. If we are able to find a counter example, then we would (somewhat less satisfactorily) be able to conclude that the Bondi mass is an unsuitable candidate for describing the mass loss due to gravitational waves in AdS, although the definition could potentially be modified to regain monotonicity.

Before we begin the computation, we comment that this work can be viewed as a small contribution to of a larger problem of understanding mass in asymptotically locally AdS spacetimes [176, 177, 178, 179]. Here we are of course considering the specific case of the Bondi mass, which is well understood in flat spacetime [32, 33] and has recently been considered with generic $\Lambda \neq 0$ in [48].

4.1.1 Defintion and Basics

The first step in following the original monotonicity proof is to recall the definition of the Bondi mass, $\mathcal{M}_B$, as given in [1]

$$\mathcal{M}_B(u,\theta) = \frac{1}{4\pi}\int_{S^2} m_B(u,\theta) \tag{4.1.1}$$

where the integrand m_B is free data known as the Bondi mass aspect, and it can be read off from the $\mathcal{O}(1/r)$ coefficient in the asymptotic series expansion for g_{uu}, the $\{uu\}$ component of the metric in the Bondi-Sachs gauge. The geometric interpretation of the S^2 which we integrate over is that it is the manifold induced at a given cut of $\mathscr{I}^+$.

We will now re-use this definition for a cut of $\mathscr{I}$, where $\mathscr{I}$ is the timelike boundary of AlAdS spacetime. In equation (3.4.35) we presented the mass aspect in axisymmetric AlAdS spacetime, m_B^{AdS}. Here we re-write that formula in a slightly different form, the advantages of which will become clearer as we move through the attempted proof.

$$\begin{aligned}2m_B(u,\theta) =&\frac{1}{2}e^{2\beta_0}\gamma_1^2(U_{0,\theta}+\cot(\theta)U_0)+\frac{2}{3}e^{-2\gamma_0}(12e^{4\beta_0}\beta_{0,\theta}\gamma_{0,\theta}\gamma_1-6e^{4\beta_0}\beta_{0,\theta}\gamma_{1,\theta}-\\ &12\cot(\theta)e^{4\beta_0}\beta_{0,\theta}\gamma_1+12\gamma_{0,\theta}\gamma_1^2U_0e^{2\beta_0+2\gamma_0}-6\gamma_{1,\theta}\gamma_1U_0e^{2\beta_0+2\gamma_0}-\\ &12\cot(\theta)\gamma_1^2U_0e^{2\beta_0+2\gamma_0}+2e^{4\gamma_0}\gamma_1^3U_0^2+3e^{4\gamma_0}U_0U_3+3e^{4\gamma_0}\gamma_3U_0^2)-e^{2\beta_0}W_3\end{aligned} \tag{4.1.2}$$

To get to this form of the mass aspect from (3.4.35), first reinstate Λ in the previous formula ((3.4.35) used $\Lambda = -3$) and then reinstate γ_1 using the following Einstein equation

$$\gamma_{0,u} = \frac{1}{6}(2\Lambda e^{2\beta_0}\gamma_1 - 6\gamma_{0,\theta}U_0 - 3U_{0,\theta} + 3\cot(\theta)U_0). \tag{4.1.3}$$

Part of the purpose of writing down a formula which includes γ_1 terms is that $\gamma_{1,u}$ had the interpretation in the original work by BMS as being the 'news function' which governed the rate of mass loss. It will be of interest to see if it plays a similar role in the AlAdS case.

To make a direct comparison between (4.1.2) and the formula for the mass aspect presented in [1], we note that in the AF case we had

$$\gamma_0 = \beta_0 = U_0 = 0 \tag{4.1.4}$$

and so if we were to switch back to Minkowskian asymptotics we would find

$$2m_B = -W_3 \tag{4.1.5}$$

exactly as BMS originally had.

Now that we have the mass aspect for AlAdS spacetime, we need to define the Bondi mass using a suitable integral. Following the argument of BMS, we want to define the mass at a given Bondi time u as the integral of the mass aspect over the u cut of $\mathscr{I}$. (note that future null infinity $\mathscr{I}^+$ in the AF case has here been replaced by timelike infinity $\mathscr{I}$). Denoting this cut $\mathscr{I}_u$, we write the Bondi mass as

$$\mathcal{M}_B(u,\theta) = \frac{1}{4\pi}\int_{\mathscr{I}_u} m_B(u,\theta). \tag{4.1.6}$$

We can simplify this further by considering the metric on $\mathscr{I}_u$. In (3.4.32), we showed that the metric induced at the conformal boundary $\mathscr{I}$ was

$$ds^2_{0} = (e^{2\hat{\gamma}_0}\hat{U}_0^2 - e^{4\hat{\beta}_0})dt^2 - 2e^{2\hat{\gamma}_0}\hat{U}_0 dt d\theta + e^{2\hat{\gamma}_0}d\theta^2 + e^{-2\hat{\gamma}_0}\sin^2(\theta)d\phi^2 \tag{4.1.7}$$

where $t = u|_{\mathscr{I}}$ so these coordinates can be freely interchanged here and the $\hat{}$ notation has been used to denote these functions as those evaluated on $\mathscr{I}$ (again this makes no mathematical difference here, but serves as a useful way to track the functions). We want to integrate over given cuts of $\mathscr{I}$ i.e. $u =$ constant slices of this 3-manifold, upon which the metric takes the form

$$ds^2_{\mathscr{I}_u} = e^{2\hat{\gamma}_0}d\theta^2 + e^{-2\hat{\gamma}_0}\sin^2(\theta)d\phi^2. \tag{4.1.8}$$

This is clearly not the metric on the round S^2 unless $\gamma_0 = 0$, an overly restrictive assumption in the AlAdS case. This difference may lead one to think that the integral one performs differs from that in the AF case but this turns out to not be true. In a similar fashion to the metric on S^2, we have written the metric on $\mathscr{I}_u$ in coordinates $\theta \in (0,\pi)$, $\phi \in [0, 2\pi)$ which we assume to form a one chart atlas for the manifold $\mathscr{I}_u$ up to a set of zero measure (in a similar way to the the coordinates (θ,ϕ) on the round S^2 not covering the poles). This assumption means that the only potential difference in an integral over S^2 and over $\mathscr{I}_u$ could arise from the metric determinant, fortunately the BMS gauge choice means

$$\det(g_{\mathscr{I}_u}) = \sin^2\theta = \det(g_{S^2}) \tag{4.1.9}$$

which one can easily see from (4.1.8). Using this we are able to conclude that

$$\mathcal{M}_B(u,\theta) = \frac{1}{4\pi}\int_{\mathscr{I}_u} m_B(u,\theta) = \frac{1}{4\pi}\int_{S^2} m_B(u,\theta) = \frac{1}{2}\int_0^\pi m_B(u,\theta)\sin\theta\, d\theta. \tag{4.1.10}$$

where in the final line we have used the axisymmetry of the spacetime to perform the ϕ

integral.

Having set up the mass integral in AlAdS spacetime, we are now ready to investigate it's monotonicity properties. First we take the derivative of $\mathcal{M}_B$ with respect to the Bondi time u

$$\frac{\partial \mathcal{M}_B}{\partial u} = \frac{1}{2}\int_0^\pi \frac{\partial m_B}{\partial u}\sin\theta\, d\theta. \tag{4.1.11}$$

and now we should explicitly apply (4.1.2) in the above formula. This generates a long printout which we leave for now in '`AdS_Bondi_Mass_Monotonicity.nb`'

One thing we can notice without any explicit calculation is that the application of this derivative will produce terms of the form $\gamma_{0,u}, \gamma_{3,u}, U_{3,u}, W_{3,u}$. These are terms that we already have expressions for using the Einstein equations. We derived these previously (also included in the MATHEMATICA file for convenience) and as we are attempting to follow the logic of the original monotonicity proof by BMS, it seems sensible to attempt to apply Einstein's equations in order to remove these terms. In the AF case, the only term which arose from taking the u-derivative of the mass aspect formula was $W_{3,u}$ which upon application of the 'supplementary' Einstein equation ((35) in [1]) was shown to give a monotonically decreasing quantity and thus a monotonically decreasing mass.

Performing these substitutions is conceptually fairly straightforward although due to the sheer length of the expressions we leave these steps in the MATHEMATICA file. Once this substitution has been performed we can rearrange the resulting expression by collecting in powers of Λ to write the u-derivative of the mass aspect as

$$\frac{\partial m_B}{\partial u} = A_0(u,\theta) + A_1(u,\theta)\Lambda + A_2(u,\theta)\Lambda^2 \tag{4.1.12}$$

where

$$A_2 = -\frac{1}{6}e^{6\beta_0}\gamma_1(\gamma_1^3 - 2\gamma_3) \tag{4.1.13a}$$

$$\begin{aligned} A_1 =& \frac{1}{36}e^{2\beta_0-2\gamma_0}(-3e^{2\beta_0}\gamma_1^2(8e^{2\beta_0}\gamma_{0,\theta}^2 - 16\cot(\theta)e^{2\beta_0}\gamma_{0,\theta}+ \\ & 7e^{2\beta_0}\beta_{0,\theta}(\cot(\theta) - 2\gamma_{0,\theta}) + 4e^{2\beta_0}\beta_{0,\theta}^2 + 7e^{2\beta_0}\beta_{0,\theta\theta} + 8\cot^2(\theta)e^{2\beta_0}+ \\ & 12e^{2\gamma_0}\gamma_{1,\theta}U_0) - 24\gamma_1^3U_0e^{2(\beta_0+\gamma_0)}(\beta_{0,\theta} - \gamma_{0,\theta} + \cot(\theta))+ \\ & 3\gamma_1(e^{4\beta_0}(\gamma_{1,\theta}(14\beta_{0,\theta} + 14\gamma_{0,\theta} - 15\cot(\theta)) + \gamma_{1,\theta\theta}) + 8e^{4\gamma_0}\gamma_3U_0^2)+ \\ & 9U_3e^{2(\beta_0+\gamma_0)}(2\beta_{0,\theta} + \cot(\theta)) + 3(-e^{4\beta_0}\gamma_{1,\theta}^2 + 3U_{3,\theta}e^{2(\beta_0+\gamma_0)}+ \\ & 8U_0e^{2(\beta_0+\gamma_0)}(2\gamma_3(\beta_{0,\theta} - \gamma_{0,\theta} + \cot(\theta)) + \gamma_{3,\theta}) + 4e^{4\gamma_0}\gamma_4U_0^2) + 16e^{4\gamma_0}\gamma_1^4U_0^2) \end{aligned} \tag{4.1.13b}$$

while the expression for A_0 is very long and is omitted here (details are again in the

MATHEMATICA file).

4.1.2 Infinitesimal Λ 'counter example'

The full expression for $m_{B,u}$ is very long, and it seems impossible to make any comments about the monotonicity of the integral (4.1.11) from simply looking at these terms. To make this problem more tractable, we will first restrict our attention to the case of the *asymptotically flat limit*, i.e we look at *infinitesimal* Λ and ignore all terms in (4.1.12) of $\mathcal{O}(\Lambda^n)$ for $n \geq 2$. The motivation for taking this approach is twofold: Firstly, we hope that this will simplify the expression for $\partial m_B/\partial u$ into something more manageable. Secondly, this should relate the case we are studying here of $\Lambda \neq 0$ to that of the AF case which is of course the building block for the calculation we are performing here.

So in this limit we have

$$\frac{\partial m_B}{\partial u} = A_0(u,\theta) + A_1(u,\theta)\Lambda + \mathcal{O}(\Lambda^2) \tag{4.1.14}$$

which at first doesn't seem to simplify the expression much as A_2 was by far the shortest expression of the three coefficients. However, we have so far neglected to include the scaling of the metric functions in the asymptotically flat limit, recall the fall-off conditions in the AF case were

$$\gamma_0 = \beta_0 = U_0 = 0 \tag{4.1.15}$$

so we would expect these functions to also be small in the asymptotically flat limit. This agrees with the bound on U_0 that we previously computed

$$|U_0| < \sqrt{-\frac{\Lambda}{3}e^{4\beta_0 - 2\gamma_0}} \tag{4.1.16}$$

so we expect U_0 to vanish as we take $\Lambda \to 0$.

From the perspective of the Einstein equations the functions γ_0, β_0, U_0 were all free quantities that had to be specified on $\mathscr{I}$ in order to *algebraically* generate solutions to the Einstein equations in the asymptotic interior of the spacetime (see section 3.3). The fact that these quantities are free means that we can re-scale them by a constant and still generate solutions corresponding to the new expressions. In order to look at the asymptotically flat limit, we re-scale the functions accordingly

$$\tilde{\gamma}_0 = \epsilon\gamma_0, \qquad \tilde{\beta}_0 = \epsilon\beta_0, \qquad \tilde{U}_0 = \epsilon U_0 \tag{4.1.17}$$

where ϵ is a small parameter of $\mathcal{O}(\Lambda)$. These rescalings mean that we can also treat γ_1 as a function of $\mathcal{O}(1)$ via the Einstein equation (4.1.3) and as such we can examine the AF limit by applying these rescaled functions directly to (4.1.14). When doing this we will ignore any terms of $\mathcal{O}(\Lambda^n)$ for $n \geq 2$ which of course includes terms of the form $\epsilon^2, \epsilon\Lambda$ etc.

Applying this rescaling gives us the following formula

$$2\frac{\partial m_B}{\partial u} = \tilde{A}_0 + \Lambda \tilde{A}_\Lambda + \epsilon \tilde{A}_\epsilon + \mathcal{O}(\Lambda^2) \tag{4.1.18}$$

where $\tilde{A}_i$ are terms which we will now discuss in more detail. Firstly we have

$$\tilde{A}_0 = -2\gamma_{1,u}^2 + \gamma_{1,u\theta\theta} - 2\gamma_{1,u} + 3\cot\theta\gamma_{1,u\theta} \tag{4.1.19}$$

which is precisely the same term that BMS found in the AF case. The fact that this term makes an appearance in our AF limit of AlAdS spacetimes is a reassuring sanity check.

The deviations from the AF case are encoded in the terms $\tilde{A}_\Lambda$ and $\tilde{A}_\epsilon$

$$\tilde{A}_\Lambda = \frac{1}{6}(\gamma_{1,\theta\theta}\gamma_1 - \gamma_{1,\theta}^2 - 8\cot^2(\theta)\gamma_1^2 - 15\cot(\theta)\gamma_{1,\theta}\gamma_1 + 3U_{3,\theta} + 3\cot(\theta)U_3) \tag{4.1.20}$$

$$\begin{aligned}
\tilde{A}_\epsilon =& \frac{1}{6}(6U_0\gamma_1\cot^3(\theta) + 24\gamma_1 U_{0,\theta}\cot^2(\theta) + 21U_0\gamma_{1,\theta}\cot^2(\theta) + 6\beta_{0,\theta\theta}\cot^2(\theta) + \\
& 9\gamma_{0,\theta\theta}\cot^2(\theta) + 48\beta_0\gamma_{1,u}\cot^2(\theta) - 24\gamma_0\gamma_{1,u}\cot^2(\theta) + 9U_0W_3\cot(\theta) - \\
& 12\csc^2(\theta)U_0\gamma_1\cot(\theta) + 6\csc^2(\theta)\beta_{0,\theta}\cot(\theta) + 12\beta_{0,\theta}\cot(\theta) + \\
& 9\csc^2(\theta)\gamma_{0,\theta}\cot(\theta) - 6\gamma_{0,\theta}\cot(\theta) + 39U_{0,\theta}\gamma_{1,\theta}\cot(\theta) + 12\gamma_1 U_{0,\theta\theta}\cot(\theta) + \\
& 21U_0\gamma_{1,\theta\theta}\cot(\theta) + 12\beta_{0,\theta\theta\theta}\cot(\theta) + 12\gamma_{0,\theta\theta\theta}\cot(\theta) - 45\gamma_1^2 U_{0,u}\cot(\theta) - \\
& 38U_0\gamma_1\gamma_{1,u}\cot(\theta) + 12\beta_{0,\theta}\gamma_{1,u}\cot(\theta) - 36\gamma_{0,\theta}\gamma_{1,u}\cot(\theta) - 12\gamma_1\beta_{0,u\theta}\cot(\theta) - \\
& 12\gamma_1\gamma_{0,u\theta}\cot(\theta) + 72\beta_0\gamma_{1,u\theta}\cot(\theta) - 36\gamma_0\gamma_{1,u\theta}\cot(\theta) - 24\beta_0\gamma_{1,u}^2 + 9W_3U_{0,\theta} - \\
& 18\csc^2(\theta)\gamma_1 U_{0,\theta} + 10U_0W_{3,\theta} - 24\csc^2(\theta)U_0\gamma_{1,\theta} + 6\gamma_{1,\theta}U_{0,\theta\theta} - 12\csc^2(\theta)\beta_{0,\theta\theta} + \\
& 12\beta_{0,\theta\theta} - 18\csc^2(\theta)\gamma_{0,\theta\theta} - 6\gamma_{0,\theta\theta} + 15U_{0,\theta}\gamma_{1,\theta\theta} - 6\gamma_1 U_{0,\theta\theta\theta} + 6U_0\gamma_{1,\theta\theta\theta} + \\
& 6\beta_{0,\theta\theta\theta\theta} + 3\gamma_{0,\theta\theta\theta\theta} + 12U_3U_{0,u} - 24\gamma_1\gamma_{1,\theta}U_{0,u} - 24W_3\beta_{0,u} - 48\csc^2(\theta)\beta_0\gamma_{1,u} + \\
& 24\csc^2(\theta)\gamma_0\gamma_{1,u} - 6\gamma_1 U_{0,\theta}\gamma_{1,u} - 20U_0\gamma_{1,\theta}\gamma_{1,u} - 12\beta_{0,\theta\theta}\gamma_{1,u} - 12\gamma_{0,\theta\theta}\gamma_{1,u} + \\
& 3\gamma_1^2 U_{0,u\theta} - 12\gamma_{1,\theta}\gamma_{0,u\theta} - 12U_0\gamma_1\gamma_{1,u\theta} - 24\gamma_{0,\theta}\gamma_{1,u\theta} + 12\gamma_1\beta_{0,u\theta\theta} - \\
& 12\gamma_1\gamma_{0,u\theta\theta} + 24\beta_0\gamma_{1,u\theta\theta} - 12\gamma_0\gamma_{1,u\theta\theta}).
\end{aligned} \tag{4.1.21}$$

The $\tilde{A}_\Lambda$ term comes from the $\mathcal{O}(\epsilon^0)$ term in $A_1(u,\theta)$ and the $\tilde{A}_\epsilon$ term comes from the $\mathcal{O}(\epsilon)$ term in $A_0(u,\theta)$. We have managed to write the integrand in a (somewhat) more tractable form and we are now ready to begin removing terms by writing them as total derivatives.

Before we begin this analysis, we first recall the regularity conditions that BMS originally imposed upon the metric functions at the coordinate values $\theta = 0, \pi$. We will carry these conditions over for the purpose of our analysis.

$$\begin{aligned} &\lim_{\sin\theta\to 0} W(u,r,\theta) = f_W(u,r,\cos\theta), &&\lim_{\sin\theta\to 0} \beta(u,r,\theta) = f_\beta(u,r,\cos\theta)\\ &\lim_{\sin\theta\to 0} \frac{U(u,r,\theta)}{\sin\theta} = f_U(u,r,\cos\theta), &&\lim_{\sin\theta\to 0} \frac{\gamma(u,r,\theta)}{\sin^2\theta} = f_\gamma(u,r,\cos\theta)\end{aligned} \tag{4.1.22}$$

where each of the f_g are functions regular at $\cos(\theta) = \pm 1$. These regularity conditions allow us to explicitly evaluate the integrals of the total derivative terms, for example the $\tilde{A}_0$ term of (4.1.11) is

$$\frac{1}{4}\int_0^\pi \tilde{A}_0 \sin\theta\, d\theta = \frac{1}{4}\int_0^\pi \left[-2\sin(\theta)\gamma_{1,u}^2 + \partial_\theta(\gamma_{1,u}\cos\theta) + \frac{1}{2}\partial_\theta(\gamma_{1,u\theta}\sin\theta)\right] d\theta. \tag{4.1.23}$$

Now we focus on the total derivative pieces. Here and from now on we will use the notation f_g^i to refer to the $\mathcal{O}(r^{-i})$ component of the metric function g i.e. we have

$$f_g(u,r,\cos\theta) = \sum_{i=0}^{\infty} \frac{f_g^i(u,\cos\theta)}{r^i}. \tag{4.1.24}$$

Using this notation we can write the first total derivative term as

$$\int_0^\pi \partial_\theta(\gamma_{1,u}\cos\theta)\, d\theta = \gamma_{1,u}\cos\theta\,\big|_0^\pi = \sin^2\theta\cos\theta\,\partial_u[f_\gamma^1(u,\cos\theta)]\big|_0^\pi = 0 \tag{4.1.25}$$

where the final equality comes from the regularity conditions and the $\sin^2\theta$ factor. We can apply a similar treatment to the second total derivative term

$$\begin{aligned}\int_0^\pi \partial_\theta(\gamma_{1,u\theta}\sin\theta) &= \gamma_{1,u\theta}\sin\theta\big|_0^\pi = \partial_{u,\theta}[\sin^2\theta f_\gamma^1(u,\cos\theta)]\sin\theta\big|_0^\pi\\ &= \{2\sin^2\theta\cos\theta\,\partial_u[f_\gamma^1(u,\cos\theta)] - \sin^4\theta\,\partial_{u,\cos\theta}[f_\gamma^1(u,\cos\theta)]\}\big|_0^\pi = 0\end{aligned} \tag{4.1.26}$$

where in the final line we have again used the regularity of f_γ^1. All that remains of equation (4.1.23) is

$$-\frac{1}{2}\int_0^\pi \gamma_{1,u}^2 \sin\theta\, d\theta \tag{4.1.27}$$

which is precisely the monotonically decreasing term that BMS found in their analysis of the asymptotically flat case. We will now apply similar considerations to the integrals of the terms $\tilde{A}_\Lambda$ and $\tilde{A}_\epsilon$

We can re-write the $\tilde{A}_\Lambda$ piece the integral (4.1.11) as

$$\begin{aligned}\frac{\Lambda}{24}\int_0^\pi [&-2\sin(\theta)\gamma_{1,\theta}^2 - 8\csc\theta\gamma_1^2 +\\ &\partial_\theta(\sin(\theta)\gamma_1\gamma_{1,\theta}) - 8\partial_\theta(\cos(\theta)\gamma_1^2) + 3\partial_\theta(\sin\theta U_3)]d\theta\end{aligned} \tag{4.1.28}$$

where the terms on the second line are clearly all total derivatives. By applying the regularity conditions (4.1.22), one can easily show that these all vanish. This means that we are left with

$$-\frac{\Lambda}{12}\int_0^\pi (\sin\theta\gamma_{1,\theta}^2 + 4\csc\theta\gamma_1^2)\, d\theta \tag{4.1.29}$$

which is positive for the AlAdS case of $\Lambda < 0$. Although this is clearly not the whole expression for the u-derivative of the mass aspect, we can already begin to see how the presence of positive terms like this one in the AlAdS case may be used to construct counter examples to the monotonically decreasing property that BMS found in the AF setting. We will return to this point once we have looked at the $\mathcal{O}(\epsilon)$ terms in the mass integral.

To look at the $\mathcal{O}(\epsilon)$ terms, we need to consider $\tilde{A}_\epsilon$ terms in the integral (4.1.11). By attempting to write as many terms as possible as total derivatives we can re-write the expression as

$$\begin{aligned}
&\frac{\epsilon}{24}\int_0^\pi [\sin\theta U_0 W_{3,\theta} - 24\sin\theta W_3\beta_{0,u} + 12\sin\theta U_3 U_{0,u} + \\
&48\sin(\theta)\gamma_1^2 U_{0,u\theta} + 66\sin(\theta)\gamma_{1,\theta}\gamma_1 U_{0,u} + 32\sin(\theta)\gamma_{1,u}\gamma_1 U_{0,\theta} + \\
&26\sin(\theta)\gamma_{1,u\theta}\gamma_1 U_0 + \sin(\theta)\gamma_{1,\theta}U_{0,\theta\theta} + 3\sin(\theta)\gamma_{1,\theta\theta}U_{0,\theta} + 18\sin(\theta)\gamma_{1,\theta}\gamma_{1,u}U_0 - \\
&9\csc(\theta)\gamma_{1,\theta}U_0 + 24\sin(\theta)\beta_{0,u\theta\theta}\gamma_1 - 24\sin(\theta)\beta_0\gamma_{1,u}^2 + 24\sin(\theta)\beta_{0,\theta\theta}\gamma_{1,u} + \\
&12\sin(\theta)\beta_{0,u\theta}\gamma_{1,\theta} + 12\sin(\theta)\beta_{0,\theta}\gamma_{1,u\theta} + 9\partial_\theta(\sin\theta U_0 W_3) - 36\partial_\theta(\sin(\theta)\beta_{0,\theta}\gamma_{1,u}) - \\
&12\partial_\theta(\sin(\theta)\beta_{0,u\theta}\gamma_1) + 24\partial_\theta(\sin(\theta)\beta_0\gamma_{1,u\theta}) + 48\partial_\theta(\cos(\theta)\beta_0\gamma_{1,u}) + \\
&6\partial_\theta(\sin(\theta)\beta_{0,\theta\theta\theta}) + 6\partial_\theta(\cos(\theta)\beta_{0,\theta\theta}) + 6\partial_\theta((2\sin(\theta) - \csc(\theta))\beta_{0,\theta}) + \\
&\partial_\theta(\sin(\theta)\gamma_1\gamma_{1,\theta}) + 3\partial_\theta(\sin(\theta)\gamma_{0,\theta\theta\theta}) - 12\partial_\theta(\sin(\theta)\gamma_{0,\theta}\gamma_{1,u}) - \\
&12\partial_\theta(\sin(\theta)\gamma_1\gamma_{0,u\theta}) - 12\partial_\theta(\sin(\theta)\gamma_0\gamma_{1,u\theta}) - 8\partial_\theta(\cos(\theta)\gamma_1^2) + \\
&9\partial_\theta(\cos(\theta)\gamma_{0,\theta\theta}) - 24\partial_\theta(\cos(\theta)\gamma_0\gamma_{1,u}) + 3\partial_\theta((\cos(2\theta) - 4)\csc(\theta)\gamma_{0,\theta}) + \\
&6\partial_\theta(\sin(\theta)\gamma_{1,\theta}U_{0,\theta}) - 6\partial_\theta(\sin(\theta)\gamma_1 U_{0,\theta\theta}) + 6\partial_\theta(\sin(\theta)\gamma_{1,\theta\theta}U_0) - \\
&45\partial_\theta(\sin(\theta)\gamma_1^2 U_{0,u}) - 38\partial_\theta(\sin(\theta)\gamma_1\gamma_{1,u}U_0) + 18\partial_\theta(\cos(\theta)\gamma_1 U_{0,\theta}) + \\
&15\partial_\theta(\cos(\theta)\gamma_{1,\theta}U_0) + 6\partial_\theta(\cos(\theta)\cot(\theta)\gamma_1 U_0)]\, d\theta
\end{aligned} \tag{4.1.30}$$

which we can simplify further by considering the total derivative terms and the regularity conditions (4.1.22). The first step is to eliminate all total derivative piece with a $\sin\theta$

factor, as these will vanish automatically

$$\frac{\epsilon}{24}\int_0^\pi [\sin\theta U_0 W_{3,\theta} - 24\sin\theta W_3\beta_{0,u} + 12\sin\theta U_3 U_{0,u} + 48\sin(\theta)\gamma_1^2 U_{0,u\theta} + 66\sin(\theta)\gamma_{1,\theta}\gamma_1 U_{0,u} + 32\sin(\theta)\gamma_{1,u}\gamma_1 U_{0,\theta} + 26\sin(\theta)\gamma_{1,u\theta}\gamma_1 U_0 + \sin(\theta)\gamma_{1,\theta}U_{0,\theta\theta} + 3\sin(\theta)\gamma_{1,\theta\theta}U_{0,\theta} + 18\sin(\theta)\gamma_{1,\theta}\gamma_{1,u}U_0 - 9\csc(\theta)\gamma_{1,\theta}U_0 + 24\sin(\theta)\beta_{0,u\theta\theta}\gamma_1 - 24\sin(\theta)\beta_0\gamma_{1,u}^2 + 24\sin(\theta)\beta_{0,\theta\theta}\gamma_{1,u} + 12\sin(\theta)\beta_{0,u\theta}\gamma_{1,\theta} + 12\sin(\theta)\beta_{0,\theta}\gamma_{1,u\theta} + 48\partial_\theta(\cos(\theta)\beta_0\gamma_{1,u}) + 6\partial_\theta(\cos(\theta)\beta_{0,\theta\theta}) - 6\partial_\theta(\csc\theta\beta_{0,\theta}) - 8\partial_\theta(\cos(\theta)\gamma_1^2) + 9\partial_\theta(\cos(\theta)\gamma_{0,\theta\theta}) - 24\partial_\theta(\cos(\theta)\gamma_0\gamma_{1,u}) + 3\partial_\theta((\cos(2\theta)-4)\csc(\theta)\gamma_{0,\theta}) + 18\partial_\theta(\cos(\theta)\gamma_1 U_{0,\theta}) + 15\partial_\theta(\cos(\theta)\gamma_{1,\theta}U_0) + 6\partial_\theta(\cos(\theta)\cot(\theta)\gamma_1 U_0)]\,d\theta \tag{4.1.31}$$

now we can also use the regularity conditions for γ, β, U to quickly eliminate some more of the total derivatives

$$\frac{\epsilon}{24}\int_0^\pi [\sin\theta U_0 W_{3,\theta} - 24\sin\theta W_3\beta_{0,u} + 12\sin\theta U_3 U_{0,u} + 48\sin(\theta)\gamma_1^2 U_{0,u\theta} + 66\sin(\theta)\gamma_{1,\theta}\gamma_1 U_{0,u} + 32\sin(\theta)\gamma_{1,u}\gamma_1 U_{0,\theta} + 26\sin(\theta)\gamma_{1,u\theta}\gamma_1 U_0 + \sin(\theta)\gamma_{1,\theta}U_{0,\theta\theta} + 3\sin(\theta)\gamma_{1,\theta\theta}U_{0,\theta} + 18\sin(\theta)\gamma_{1,\theta}\gamma_{1,u}U_0 - 9\csc(\theta)\gamma_{1,\theta}U_0 + 24\sin(\theta)\beta_{0,u\theta\theta}\gamma_1 - 24\sin(\theta)\beta_0\gamma_{1,u}^2 + 24\sin(\theta)\beta_{0,\theta\theta}\gamma_{1,u} + 12\sin(\theta)\beta_{0,u\theta}\gamma_{1,\theta} + 12\sin(\theta)\beta_{0,\theta}\gamma_{1,u\theta} + 6\partial_\theta(\cos(\theta)\beta_{0,\theta\theta}) - 6\partial_\theta(\csc\theta\beta_{0,\theta}) + 9\partial_\theta(\cos(\theta)\gamma_{0,\theta\theta}) + 3\partial_\theta((\cos(2\theta)-4)\csc(\theta)\gamma_{0,\theta})]\,d\theta. \tag{4.1.32}$$

The final total derivative terms must be considered a little more carefully. First we start with the β_0 terms

$$\frac{\epsilon}{4}\int_0^\pi [\partial_\theta(\cos\theta\beta_{0,\theta\theta}) - \partial_\theta(\csc\theta\beta_{0,\theta})]\,d\theta = \frac{\epsilon}{4}[\cos\theta\beta_{0,\theta\theta} - \csc\theta\beta_{0,\theta}]\Big|_0^\pi \tag{4.1.33}$$

applying the regularity conditions to the term on the RHS

$$\frac{\epsilon}{4}[-\cos^2\theta\partial_{\cos\theta}(f^0_\beta(u,\cos\theta)) + \sin^2\theta\cos\theta\partial^2_{\cos\theta}(f^0_\beta(u,\cos\theta)) + \partial_{\cos\theta}(f^0_\beta(u,\cos\theta))]\Big|_0^\pi \tag{4.1.34}$$

$$= \frac{\epsilon}{4}[\sin^2\theta\partial_{\cos\theta}(f^0_\beta(u,\cos\theta)) + \sin^2\theta\cos\theta\partial^2_{\cos\theta}(f^0_\beta(u,\cos\theta))]\Big|_0^\pi = 0 \tag{4.1.35}$$

where the final equality comes from the $\sin^2\theta$ factor and the regularity of the functions

f^0_β. We perform a similar analysis on the γ_0 terms

$$
\begin{aligned}
&\frac{\epsilon}{24}\int_0^\pi [9\partial_\theta(\cos(\theta)\gamma_{0,\theta\theta}) + 3\partial_\theta((\cos(2\theta)-4)\csc(\theta)\gamma_{0,\theta})]\,d\theta \\
=&\frac{\epsilon}{24}[9\cos\theta\gamma_{0,\theta\theta} + 3\cos(2\theta)\csc\theta\gamma_{0,\theta} - 12\csc\theta\gamma_{0,\theta}]\Big|_0^\pi \\
=&\frac{\epsilon}{24}[18\cos^3\theta f^0_\gamma(u,\cos\theta) - 18\cos\theta\sin^2\theta f^0_\gamma(u,\cos\theta) + \\
&\quad 9\sin^2\theta\cos^2\theta\partial_{\cos\theta}(f^0_\gamma(u,\cos\theta)) + 9\cos\theta\sin^4\theta\partial^2_{\cos\theta}(f^0_\gamma(u,\cos\theta)) + \\
&\quad 6\cos^3\theta f^0_\gamma(u,\cos\theta) - 3\cos^2\theta\sin^2\theta\partial_{\cos\theta}(f^0_\gamma(u,\cos\theta)) - 6\sin^2\theta\cos\theta f^0_\gamma(u,\cos\theta) + \\
&\quad 3\sin^4\theta\partial_{\cos\theta}f(u,\cos\theta) - 24\cos\theta f^0_\gamma(u,\cos\theta) + 12\sin^2\theta\partial_{\cos\theta}(f^0_\gamma(u,\cos\theta))]\Big|_0^\pi \\
=&\epsilon[\cos^3\theta f^0_\gamma(u,\cos\theta) - \cos\theta f^0_\gamma(u,\cos\theta)]\Big|_0^\pi = -\epsilon[\cos\theta\sin^2\theta f^0_\gamma(u,\cos\theta)]\Big|_0^\pi = 0.
\end{aligned}
\tag{4.1.36}
$$

So we again find that the total derivative terms cancel. We are now able to write our expression for the u-derivative of the Bondi mass in the AF limit as

$$
\begin{aligned}
\frac{\partial\mathcal{M}_B}{\partial u} =& -\frac{1}{2}\int_0^\pi \gamma^2_{1,u}\sin\theta\,d\theta - \frac{\Lambda}{12}\int_0^\pi(\sin\theta\gamma^2_{1,\theta} + 4\csc\theta\gamma_1^2)\,d\theta + \\
&\frac{\epsilon}{24}\int_0^\pi[\sin\theta U_0W_{3,\theta} - 24\sin\theta W_3\beta_{0,u} + 12\sin\theta U_3U_{0,u} + \\
&\quad 48\sin(\theta)\gamma_1^2U_{0,u\theta} + 66\sin(\theta)\gamma_{1,\theta}\gamma_1U_{0,u} + 32\sin(\theta)\gamma_{1,u}\gamma_1U_{0,\theta} + \\
&\quad 26\sin(\theta)\gamma_{1,u\theta}\gamma_1U_0 + \sin(\theta)\gamma_{1,\theta}U_{0,\theta\theta} + 3\sin(\theta)\gamma_{1,\theta\theta}U_{0,\theta} + \\
&\quad 18\sin(\theta)\gamma_{1,\theta}\gamma_{1,u}U_0 - 9\csc(\theta)\gamma_{1,\theta}U_0 + 24\sin(\theta)\beta_{0,u\theta\theta}\gamma_1 - \\
&\quad 24\sin(\theta)\beta_0\gamma^2_{1,u} + 24\sin(\theta)\beta_{0,\theta\theta}\gamma_{1,u} + \\
&\quad 12\sin(\theta)\beta_{0,u\theta}\gamma_{1,\theta} + 12\sin(\theta)\beta_{0,\theta}\gamma_{1,u\theta}]\,d\theta.
\end{aligned}
\tag{4.1.37}
$$

The first term is the standard 'news' term from the AF set up, the second term is proportional to the cosmological constant Λ and the third term is due to the rescaling of the functions β_0, γ_0, U_0 in the AF limit.

It still seems difficult from simply studying these terms to make a direct conclusion about the monotonicity of $\mathcal{M}_B$ although in the case of $\Lambda < 0$ (AlAdS), we are able to use this formula to construct an explicit counter-example. To construct this, we consider the 'newsless' case

$$\gamma_{1,u} = 0. \tag{4.1.38}$$

The motivation for doing this when trying to construct a counter-example is that we do not want the first term of (4.1.37) to be non-zero. It is negative and it has great size relative to the other two terms in the AF limit. In this case equation (4.1.37) reduces to

$$\begin{aligned}\frac{\partial \mathcal{M}_B}{\partial u} =& -\frac{\Lambda}{12}\int_0^\pi (\sin\theta\gamma_{1,\theta}^2 + 4\csc\theta\gamma_1^2)\,d\theta + \\ & \frac{\epsilon}{24}\int_0^\pi [\sin\theta U_0 W_{3,\theta} - 24\sin\theta W_3\beta_{0,u} + 12\sin\theta U_3 U_{0,u} + 48\sin(\theta)\gamma_1^2 U_{0,u\theta} + \\ & 66\sin(\theta)\gamma_{1,\theta}\gamma_1 U_{0,u} + \sin(\theta)\gamma_{1,\theta}U_{0,\theta\theta} + 3\sin(\theta)\gamma_{1,\theta\theta}U_{0,\theta} - \\ & 9\csc(\theta)\gamma_{1,\theta}U_0 + 24\sin(\theta)\beta_{0,u\theta\theta}\gamma_1 + 12\sin(\theta)\beta_{0,u\theta}\gamma_{1,\theta}]\,d\theta.\end{aligned} \tag{4.1.39}$$

We can now finalise the construction of an increasing mass by considering the special case of having metric functions

$$\beta_0 = \beta_0(\theta), \qquad U_0 = 0, \qquad \gamma_0 \neq 0. \tag{4.1.40}$$

The first two conditions force all of the $\mathcal{O}(\epsilon)$ terms in (4.1.39) to vanish and the third equation is essential in forcing that the $\mathcal{O}(\Lambda)$ term does not. If we were to have $\gamma_0 = \beta_0 = 0$ then our spacetime would be *asymptotically AdS* [62, 69] and $\mathcal{M}_B$ would not vary with the Bondi time u. Here we still need to have $\gamma_0 \neq 0$ as otherwise the Einstein equation (4.1.3) would give us $\gamma_1 = 0$ and again $\partial\mathcal{M}_B/\partial u = 0$.

Using the conditions for the metric functions above, equation (4.1.39) reduces to

$$\frac{\partial \mathcal{M}_B}{\partial u} = -\frac{\Lambda}{12}\int_0^\pi (\sin\theta\gamma_{1,\theta}^2 + 4\csc\theta\gamma_1^2)\,d\theta \tag{4.1.41}$$

an expression which is clearly positive for $\Lambda < 0$ as the integrand is positive over the domain of integration. This is an explicit counter example to the AF case of the Bondi mass being monotonically decreasing, in the case of an infinitesimal cosmological constant.

There are still issues with this construction which allow us to question the existence of a monotonically decreasing quantity which could play the role of 'Bondi mass' in AlAdS spacetimes. The main issue being that this construction is only defined in the infinitesimal Λ case and enforcing this involved rescaling the metric functions by a somewhat arbitrary factor of ϵ.

As a piece of speculation, an approach one could use to construct a monotonically decreasing mass is via the subtraction of terms, similar in spirit to the ideas considered in the analysis of [46]. To do this, the first steps one may want to take would be to carry the total u-derivative terms over the the LHS of equation (4.1.37). For example we could redefine a 'mass' $\tilde{\mathcal{M}}_B$

$$\tilde{\mathcal{M}}_B = \mathcal{M}_B - \frac{\epsilon}{4}\int_0^\pi \left[4\sin\theta\beta_{0,\theta\theta}\gamma_1 + 2\sin\theta\beta_{0,\theta}\gamma_{1,\theta} + 11\sin\theta\gamma_1\gamma_{1,\theta}U_0 + 8\sin\theta\gamma_1^2\right]d\theta \tag{4.1.42}$$

which would then give the following expression for the Bondi time derivative

$$\begin{aligned}\frac{\partial \tilde{\mathcal{M}}_B}{\partial u} =& -\frac{1}{2}\int_0^\pi \gamma_{1,u}^2 \sin\theta \, d\theta - \frac{\Lambda}{12}\int_0^\pi (\sin\theta\gamma_{1,\theta}^2 + 4\csc\theta\gamma_1^2)\, d\theta + \\ & \frac{\epsilon}{24}\int_0^\pi [\sin\theta U_0 W_{3,\theta} - 24\sin\theta W_3\beta_{0,u} + 12\sin\theta U_3 U_{0,u} + 3\sin\theta U_{0,\theta}\gamma_{1,\theta\theta} + \\ & 6\sin\theta\gamma_{1,\theta}U_{0,\theta\theta} - 9\csc\theta\gamma_{1,\theta}U_0 - 48\sin\theta U_0\gamma_{1,\theta}\gamma_{1,u} - \\ & 40\sin\theta U_0\gamma_{1,u\theta}\gamma_1 - 64\sin\theta U_{0,\theta}\gamma_1\gamma_{1,u} - 24\sin\theta\beta_0\gamma_{1,u}^2]\, d\theta.\end{aligned} \tag{4.1.43}$$

While this expression is somewhat simplified when compared to (4.1.37), it still doesn't seem clear at this point how all of these terms may be manipulated further in order to generate a monotonically decreasing quantity. We also note that the subtraction procedure of the type that we consider in equation (4.1.42) could be problematic as the subtracted terms are non-local expressions on the cut of $\mathscr{I}$ (they involve γ_1 terms, which depend upon $\gamma_{0,u}$ through the field equation (4.1.3)). Understanding the required properties for monotonicity, as well as whether a monotonic quantity can be constructed via addition of local counter terms is ongoing work.

4.1.3 Progress with general Λ case

The existence of a monotonically increasing Bondi mass in the infinitesimal Λ case is encouragement that a similar example exists in the general Λ regime. We shall now outline the strategy for the analysis in search of a counter example.

The first step we perform is to make explicit all Λ dependence in (4.1.12). This is performed by applying (4.1.3) in order to remove all γ_1 terms. The result of this substitution is

$$\frac{\partial m_B}{\partial u} = \frac{I_{-3}}{\Lambda^3} + \frac{I_{-2}}{\Lambda^2} + \frac{I_{-1}}{\Lambda} + I_0 + I_1\Lambda \tag{4.1.44}$$

where the coefficients I_i are functions of the metric functions $\gamma_0, \beta_0, U_0, \gamma_3, U_3, W_3$. The explicit forms of these are, as expected, excessively long and instead of printing them here we confine them to the supplementary notebook '`Mass_Integrand_Zero_Functions.nb`'.

Motivated by the form of the counter example that we found before we try what would appear to be the next most straightforward case of the Bondi metric functions after the asymptotically AdS case, namely

$$\beta_0 = \beta_0(\theta), \qquad U_0 = \gamma_0 = \gamma_3 = 0 \tag{4.1.45}$$

with U_3 and W_3 as yet undetermined but still forced to satisfy the supplementary field equations (A.1.1, A.2) respectively. With this choice of functions, (4.1.44) reduces to

$$\begin{aligned}\frac{\partial m_B}{\partial u} = &\frac{\Lambda}{2}e^{4\beta_0}\left(\frac{1}{2}U_3\cot(\theta)+\frac{1}{2}U_{3,\theta}+U_3\beta_{0,\theta}\right)- \\ &\frac{1}{2}e^{6\beta_0}[-16\cot(\theta)\beta_{0,\theta}^3+4\beta_{0,\theta}^2(\cot^2(\theta)-4\beta_{0,\theta\theta})+\cot^2(\theta)\beta_{0,\theta\theta}-4\beta_{0,\theta\theta}^2- \\ &\quad 2\cot(\theta)\beta_{0,\theta\theta\theta}-\beta_{0,\theta}(\cot(\theta)\{2+\csc^2(\theta)\}+16\cot(\theta)\beta_{0,\theta\theta}+8\beta_{0,\theta\theta\theta})- \\ &\quad \beta_{0,\theta\theta\theta\theta}]\end{aligned} \tag{4.1.46}$$

and the supplementary equations (A.1.1, A.2) reduce nicely to

$$U_{3,u} = \frac{1}{3}e^{2\beta_0}(W_{3,\theta}+4W_3\beta_{0,\theta}) \tag{4.1.47a}$$

$$\begin{aligned}W_{3,u} = &-\Lambda e^{2\beta_0}\left(\frac{1}{2}U_3\cot(\theta)+\frac{1}{2}U_{3,\theta}+U_3\beta_{0,\theta}\right)+ \\ &e^{4\beta_0}[-16\cot(\theta)\beta_{0,\theta}^3+4\beta_{0,\theta}^2(\cot^2(\theta)-4\beta_{0,\theta\theta})+\cot^2(\theta)\beta_{0,\theta\theta}-4\beta_{0,\theta\theta}^2- \\ &\quad 2\cot(\theta)\beta_{0,\theta\theta\theta}-\beta_{0,\theta}(\cot(\theta)\{2+\csc^2(\theta)\}+16\cot(\theta)\beta_{0,\theta\theta}+8\beta_{0,\theta\theta\theta})- \\ &\quad \beta_{0,\theta\theta\theta\theta}]\end{aligned} \tag{4.1.47b}$$

the second of which is equivalent to

$$W_{3,u} = -2e^{-2\beta_0}m_{B,u} \tag{4.1.48}$$

which we would find automatically when applying (4.1.45) to the definition of the mass aspect in (4.1.2).

4.2 Modified Bondi gauge

In order to motivate this new direction of work we first recount some of the basic results of the previous sections. In [1], the Bondi-Metzner-Sachs (BMS) gauge is presented as the class of metrics with the following line-element

$$ds^2 = -(Vr^{-1}e^{2\beta}-U^2r^2e^{2\gamma})du^2-2e^{2\beta}dudr-2Ur^2e^{2\gamma}dud\theta+r^2(e^{2\gamma}d\theta^2+e^{-2\gamma}\sin^2\theta d\phi^2). \tag{4.2.1}$$

The four functions W, β, U, γ which appear in the line-element are all assumed to be functions of the coordinates (u, r, θ), thus making this metric axisymmetric.

The Bondi gauge conditions are $g_{rr} = g_{r\theta} = 0$ (clear from the form of the line element above) as well as a determinant condition on the angular part of the metric

$$\det\left(\frac{g_{AB}}{r^2}\right) = \sin^2\theta \tag{4.2.2}$$

where the indices A, B run over the values $\{2,3\} = \{\theta, \phi\}$. This condition defines the radial coordinate r as a luminosity parameter and is also clearly implemented in equation (4.2.1).

For the purpose of this section we will focus on the second of these two gauge conditions with the goal of relaxing this condition in order to write a more general form of the angular part of the metric. This gauge choice has been used in much work concerning the Bondi gauge in asymptotically flat (AF) [2, 9, 10, 11, 7, 4, 5, 12, 24, 13, 25, 30] spacetime and the author of this thesis has also began a programme of generalising the use of the Bondi-gauge to asymptotically locally AdS (AlAdS) spacetimes [52], including gaining a holographic understanding of the gauge via the AdS/CFT correspondence (as discussed in the previous chapter).

Before we discuss the specifics of the how we will modify the Bondi gauge in order to accommodate a larger class of spacetimes, we also note the existence of a similar gauge often used in asymptotically flat literature: The Newman Unti (NU) gauge [180], in which the radial coordinate r is chosen to be an affine parameter for the generators of the null hypersurfaces. Our modified Bondi gauge will present some similarities to this gauge, although we will not apply the NU radial coordinate definition in general. For some recent work on asymptotic symmetries in the NU gauge, see [181].

One of the key stages in gaining a holographic understanding of a gravitational theory in the presence of a cosmological constant $\Lambda < 0$ is to transform the metric of the theory into Fefferman-Graham (FG) coordinates [59] which take the following form in spacetime dimension $d = 3 + 1$

$$ds^2 = l^2\left[\frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(g_{0ab} + \rho^2 g_{2ab} + \rho^3 g_{3ab} + \ldots)dx^a dx^b\right] \tag{4.2.3}$$

where $l = \sqrt{-3/\Lambda}$ is the AdS radius of the spacetime and the indices a, b run over all coordiantes other than ρ. Using this particular coordinate system, it has been shown [60, 61] that one is able to extract holographic data for the conformal field theory (CFT) dual to the gravitational theory under question. The coordinate ρ is analogous to a distance coordinate in the bulk spacetime and $\rho = 0$ is the location of the 3-dimensional conformal boundary $\mathscr{I}$, upon which the CFT_3 lives.

The background metric for the CFT_3 is given by the g_{0} component of the FG metric and the expectation value of the energy-momentum tensor of the CFT_3 is proportional to

g_3. In [52], the transformation from an AlAdS Bondi-gauged spacetime to FG coordinates was performed, with the background metric on the CFT_3 being given by

$$ds_0^2 = (e^{2\hat{\gamma}_0}\hat{U}_0^2 - e^{4\hat{\beta}_0})dt^2 - 2e^{2\hat{\gamma}_0}\hat{U}_0 dt d\theta + e^{2\hat{\gamma}_0}d\theta^2 + e^{-2\hat{\gamma}_0}\sin^2(\theta)d\phi^2. \tag{4.2.4}$$

Notice that this line-element seems to roughly maintain the Bondi gauge condition (4.2.2). Stated more precisely, we found that enforcing (4.2.2) upon the original Bondi metric results in a restriction on the angular part of the background metric for our dual theory. This is a restrictive and somewhat unnatural property from a CFT perspective and thus we would like to find a method of removing this property, with a natural place to start being to remove the condition (4.2.2) on the original Bondi metric.

This section is organised as follows: First we will discuss how we break the gauge condition upon the Bondi-Sachs spacetime and solve the vacuum Einstein equations (in the presence of a cosmological constant $\Lambda < 0$) in the new gauge. We will then compute the transformation of our solutions from the Bondi to the FG gauge before discussing how this procedure can allow us to apply our results to the Robinson-Trautman class of spacetimes. Finally, we will consider how to transform from our new gauge back into the Bondi gauge, giving equations for this procedure.

4.2.1 Vacuum Einstein Equations

In this section we will discuss a method of breaking the Bondi gauge condition, before solving the vacuum Einstein equations with the new gauge.

4.2.1.1 Breaking the Bondi Gauge

Before attempting to solve the vacuum Einstein equations we must first decide how we wish to break the Bondi gauge presented in (4.2.1). To do this generically, we introduce a 5^{th} unknown function $\delta(u, r, \theta)$ into the metric

$$ds^2 = -(Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma})du^2 - 2e^{2\beta}dudr - 2Ur^2e^{2\gamma}dud\theta + r^2(e^{2\gamma}d\theta^2 + e^{-2\delta}\sin^2\theta d\phi^2). \tag{4.2.5}$$

which clearly breaks the Bondi gauge condition (4.2.2).

We begin somewhat naïvely in constructing the Einstein field equations with an arbitrary δ and we will then pick special choices depending upon their capacity to solve the field equations themselves. To attempt to solve the field equations we follow the scheme of [1] which involves writing the field equations as four 'main equations'

$$R_{rr} = R_{r\theta} = 0, \qquad R_{\theta\theta} = \Lambda g_{\theta\theta} = \Lambda r^2 e^{2\gamma}, \qquad R_{\phi\phi} = \Lambda g_{\phi\phi} = \Lambda r^2 \sin^2\theta e^{-2\delta} \tag{4.2.6}$$

which we use to write the functions $E_i(u, r, \theta)$

$$E_1 = -R_{rr} = 0, \tag{4.2.7a}$$

$$E_2 = 2r^2 R_{r\theta} = 0, \tag{4.2.7b}$$

$$E_3 = 2\Lambda r^2 e^{2\beta} - R_{\theta\theta} e^{2(\beta-\gamma)} - r^2 R^{\phi}_{\phi} e^{2\beta} = 0, \tag{4.2.7c}$$

$$E_4 = \Lambda r^2 e^{2\beta} - r^2 R^{\phi}_{\phi} e^{2\beta} = 0 \tag{4.2.7d}$$

and thus solving the Einstein equations boils down to solving the four equations above. Explicit computation of the component of the curvature tensor gives us the equations

$$E_1 = \frac{r\gamma_r^2 + 2\gamma_r + r\delta_r^2 - 2\delta_r - 2\beta_r(r\gamma_r - r\delta_r + 2) + r\gamma_{rr} - r\delta_{rr}}{r} \tag{4.2.8a}$$

$$\begin{aligned} E_2 = {}& e^{-2\beta} r(2e^{2\beta}\beta_\theta(r\gamma_r - r\delta_r + 2) - r(-e^{2\gamma}U_{rr}r^2 + e^{2\gamma}U_r(2r\beta_r - 3r\gamma_r + r\delta_r - 4)r- \\ & 2e^{2\beta}(\cot(\theta) - \delta_\theta)\gamma_r - 2e^{2\beta}\cot(\theta)\delta_r + 2e^{2\beta}\delta_\theta\delta_r + 2e^{2\beta}\beta_{r\theta} - 2e^{2\beta}\delta_{r\theta}))) \end{aligned} \tag{4.2.8b}$$

$$\begin{aligned} E_3 = {}& \frac{1}{2}e^{2\gamma-2\beta}U_r^2 r^4 + 2e^{2\beta}\Lambda r^2 - \cot(\theta)U_r r^2 - \gamma_\theta U_r r^2 + \delta_\theta U_r r^2 - U_{r\theta}r^2 - 2\gamma_r\gamma_u r^2 + \\ & 2\delta_r\gamma_u r^2 + 2\gamma_r\delta_u r^2 - 2\delta_r\delta_u r^2 - 2\gamma_{ur}r^2 + 2\delta_{ur}r^2 + V\gamma_r^2 r + V\delta_r^2 r + V_r\gamma_r r- \\ & V_r\delta_r r - 2V\gamma_r\delta_r r - 2U_\theta(r\gamma_r - r\delta_r + 2)r - 2\csc(\theta)U(r\gamma_r\cos(\theta) - r\delta_r\cos(\theta)+ \\ & 2\cos(\theta) + \sin(\theta)\gamma_\theta(r\gamma_r - r\delta_r + 2) - \sin(\theta)\delta_\theta(r\gamma_r - r\delta_r + 2) + r\sin(\theta)\gamma_{r\theta}- \\ & r\sin(\theta)\delta_{r\theta})r + V\gamma_{rr}r - V\delta_{rr}r - 4\gamma_u r + 4\delta_u r - 2e^{2\beta-2\gamma} + 2e^{2\beta-2\gamma}\beta_\theta^2 + \\ & 2e^{2\beta-2\gamma}\delta_\theta^2 + 2e^{2\beta-2\gamma}\cot(\theta)\beta_\theta - 2e^{2\beta-2\gamma}\cot(\theta)\gamma_\theta - 2e^{2\beta-2\gamma}\beta_\theta\gamma_\theta- \\ & 4e^{2\beta-2\gamma}\cot(\theta)\delta_\theta - 2e^{2\beta-2\gamma}\beta_\theta\delta_\theta + 2e^{2\beta-2\gamma}\gamma_\theta\delta_\theta + 2e^{2\beta-2\gamma}\beta_{\theta\theta}- \\ & 2e^{2\beta-2\gamma}\delta_{\theta\theta} + 2V_r + 3V\gamma_r - 3V\delta_r) \end{aligned} \tag{4.2.8c}$$

$$\begin{aligned} E_4 = {}& e^{2\beta}\Lambda r^2 - \cot(\theta)U_r r^2 + \delta_\theta U_r r^2 + \delta_r\gamma_u r^2 + \gamma_r\delta_u r^2 - 2\delta_r\delta_u r^2 + 2\delta_{ur}r^2 + \\ & V\delta_r^2 r - V_r\delta_r r - V\gamma_r\delta_r r + U_\theta(r\delta_r - 1)r - U(r\gamma_r\cot(\theta) - 2r\delta_r\cot(\theta)+ \\ & 3\cot(\theta) + \gamma_\theta(1 - r\delta_r) - \delta_\theta(r\gamma_r - 2r\delta_r + 3) - 2r\delta_{r\theta})r - V\delta_{rr}r - \gamma_u r + 3\delta_u r- \\ & e^{2\beta-2\gamma} + e^{2\beta-2\gamma}\delta_\theta^2 + 2e^{2\beta-2\gamma}\cot(\theta)\beta_\theta - e^{2\beta-2\gamma}\cot(\theta)\gamma_\theta- \\ & 2e^{2\beta-2\gamma}\cot(\theta)\delta_\theta - 2e^{2\beta-2\gamma}\beta_\theta\delta_\theta + e^{2\beta-2\gamma}\gamma_\theta\delta_\theta - e^{2\beta-2\gamma}\delta_{\theta\theta} + V_r + V\gamma_r - 2V\delta_r \end{aligned} \tag{4.2.8d}$$

In order to solve these equations we will use the original integration scheme due to BMS, although we will modify it slightly to include the 5^{th} function δ. This scheme proceeds as follows: Starting with the values of δ, γ on some null hypersurface, $\mathcal{N}_{u_0} = \{u = u_0\}$, we solve $E_1 = 0$ on $\mathcal{N}_{u_0}$ to obtain $\beta|_{\mathcal{N}_{u_0}}$. We then solve $E_2 = 0$ for $U|_{\mathcal{N}_{u_0}}$, $E_3 = 0$ for $V|_{\mathcal{N}_{u_0}}$

and finally $E_4 = 0$ should give us a differential equation containing γ_u and δ_u.

In the original four function scheme we could solve E_4 in order to give γ at "the next time-step" i.e. $\gamma|_{\mathcal{N}_{u_0+\tilde{\delta}u_0}}$ in terms of known quantities on $\mathcal{N}_{u_0}$. The scheme could then be iterated to generate solutions in the future domain of dependence of the initial null hypersurface $D^+(\mathcal{N}_{u_0})$. With five functions it seems the most we can hope for from E_4 is to obtain a relationship between $\gamma|_{\mathcal{N}_{u_0+\tilde{\delta}u_0}}$ and $\delta|_{\mathcal{N}_{u_0+\tilde{\delta}u_0}}$, nonetheless we will proceed with this scheme and we will show that picking particular values of δ will allow us to solve all of the main equations directly.

As before, we are looking to solve the vacuum equations in the asymptotic region of spacetime, thus invoking the assumption [1] that the functions γ, β, U, δ all admit power series in negative powers of the radial coordinate r

$$\begin{aligned} \gamma(u,r,\theta) &= \sum_{n=0}^{\infty} \frac{\gamma_n(u,\theta)}{r^n}, \quad \beta(u,r,\theta) = \sum_{n=0}^{\infty} \frac{\beta_n(u,\theta)}{r^n}, \\ U(u,r,\theta) &= \sum_{n=0}^{\infty} \frac{U_n(u,\theta)}{r^n}, \quad \delta(u,r,\theta) = \sum_{n=0}^{\infty} \frac{\delta_n(u,\theta)}{r^n} \end{aligned} \tag{4.2.9}$$

and V admits an expansion of the form

$$V(u,r,\theta) = \sum_{n=0}^{\infty} \frac{V_n(u,\theta)}{r^{n-3}}. \tag{4.2.10}$$

In order to keep the V series consistent with the others we define the new function

$$W(u,r,\theta) = \frac{V(u,r,\theta)}{r^3} = \sum_{n=0}^{\infty} \frac{V_n(u,\theta)}{r^n} \tag{4.2.11}$$

We can now solve the Einstein equations as algebraic equations order by order in powers of r, giving us the coefficients of the power series'.

We could now give details of how to solve equations order by order and obtain solutions but much of this procedure has already been covered in [52]. Instead, we will discuss the two new constraint equations (equations that force dependence between γ and δ) that arise as a consequence of our new function δ

- *Constraint Equation 1*

  This equation arises at $\mathcal{O}(1/r)$ when solving $E_2 = 0$. This constraint reads

$$\begin{aligned} 0 = &- \gamma_1^2\delta_{0,\theta} - 2\gamma_2\delta_{0,\theta} + \gamma_{1,\theta}\gamma_1 + \gamma_{2,\theta} + \cot(\theta)\gamma_1^2 + 2\cot(\theta)\gamma_2 + \delta_1^2\delta_{0,\theta} - \\ &2\delta_2\delta_{0,\theta} - \delta_1\delta_{1,\theta} + \delta_{2,\theta} - \cot(\theta)\delta_1^2 + 2\cot(\theta)\delta_2 \end{aligned} \tag{4.2.12a}$$

- *Constraint Equation 2*

This equation arises at $\mathcal{O}(1/r)$ when solving $E_3 = 0$

$$-\frac{1}{3}\Lambda e^{2\beta_0}(\gamma_1+\delta_1)\left(\gamma_1^2+2\gamma_2-\delta_1^2+2\delta_2\right)=0. \tag{4.2.12b}$$

We focus first on (4.2.12b), finding a solution to be $\delta_1 = -\gamma_1$. Applying this result to (4.2.12a) simplifies the equation to

$$-2\gamma_2\delta_{0,\theta}+\gamma_{2,\theta}+2\cot(\theta)\gamma_2-2\delta_2\delta_{0,\theta}+\delta_{2,\theta}+2\cot(\theta)\delta_2=0 \tag{4.2.13}$$

which we observe to be solved when we pick $\delta_2 = -\gamma_2$.

These simple solutions lead us to conjecture that the choice of $\delta = -\gamma$ would be an interesting function choice to make in this problem. Not only would this solve the constraint equations but it would force the angular part of the metric to take the form $e^{2\gamma}d\Omega_2^2$, the round S^2 metric multiplied by a conformal factor. This form of the metric would allow us to apply our new Bondi-style gauge to various metrics for which the original one didn't easily apply, such as the AdS Robinson-Trautman metrics [167, 69, 182]. We note that in breaking the Bondi gauge in the manner that we have followed, one loses the geometrical definition of r as a luminosity parameter. In fact, the radial coordinate is yet unspecified (one can perform a transformation $r \to \tilde{r}(r,u,\theta)$ without spoiling any of the gauge conditions in (4.2.5)). We will preserve this condition for now, as at a later stage a particular choice of definition for r may prove to be most convenient.

A final motivation for this choice is that it begins to simplify E_4. At $\mathcal{O}(r)$ we find the equation

$$-\Lambda e^{2\beta_0}\gamma_1-\Lambda e^{2\beta_0}\delta_1+3\gamma_{0,u}+3\delta_{0,u}+3\gamma_{0,\theta}U_0+3\delta_{0,\theta}U_0+3U_{0,\theta}-3\cot(\theta)U_0=0 \tag{4.2.14}$$

which is first simplified using our special solution to the constraint equations ($\delta_1 = -\gamma_1$)

$$3\gamma_{0,u}+3\delta_{0,u}+3\gamma_{0,\theta}U_0+3\delta_{0,\theta}U_0+3U_{0,\theta}-3\cot(\theta)U_0=0 \tag{4.2.15}$$

now notice that picking $\gamma_0 = -\delta_0$ simplifies this equation to an easily solvable differential equation for U_0

$$U_{0,\theta}-\cot(\theta)U_0=0 \implies U_0=c(u)\sin(\theta) \tag{4.2.16}$$

where $c(u)$ is the function of integration.

Due to these promising first choices we will now solve in full the Einstein equations using the choice $\delta(u,r,\theta) = -\gamma(u,r,\theta)$.

4.2.1.2 Solutions

In this subsection we will present solutions to the main equations up to 4^{th} order. This means that we will present the solutions as the coefficients of the power series (4.2.9),(4.2.11) up to the $\mathcal{O}(1/r^4)$ coefficients. The equations are straightforward enough to solve to higher order although we omit printing these solutions due to their sheer length.

We begin by solving $E_1 = 0$ for β in terms of γ which we assume to know at some given retarded time u_0. Using the asymptotic expansions of (4.2.9) we obtain coefficients

$$\begin{aligned}
\beta_1 &= 0 \\
\beta_2 &= \frac{1}{4}(\gamma_1^2 - 2\gamma_2) \\
\beta_3 &= \frac{1}{6}(-\gamma_1^3 - 6\gamma_1\gamma_2 - 6\gamma_3) \\
\beta_4 &= \frac{1}{8}(-\gamma_1^4 - 8\gamma_1^2\gamma_2 - 8\gamma_2^2 - 12\gamma_1\gamma_3 - 12\gamma_4).
\end{aligned} \tag{4.2.17}$$

with β_0 being an undetermined function of integration. Armed with these solutions, we solve $E_2 = 0$

$$\begin{aligned}
U_1 &= 2e^{2\beta_0 - 2\gamma_0}\beta_{0,\theta} \\
U_2 &= -e^{2\beta_0 - 2\gamma_0}(2\gamma_1\beta_{0,\theta} - \gamma_{1,\theta}) \\
U_4 &= -\frac{1}{6}(-16e^{2\beta_0}\beta_{0,\theta}\gamma_1^3 + 15e^{2\beta_0}\gamma_{1,\theta}\gamma_1^2 + 24e^{2\beta_0}\beta_{0,\theta}\gamma_2\gamma_1 - 12e^{2\beta_0}\gamma_{2,\theta}\gamma_1 + \\
&\qquad 12e^{2\beta_0}\beta_{0,\theta}\gamma_3 + 6e^{2\beta_0}\gamma_2\gamma_{1,\theta} - 6e^{2\beta_0}\gamma_{3,\theta} + 18e^{2\gamma_0}\gamma_1 U_3).
\end{aligned} \tag{4.2.18}$$

Notice that solving this equation does not determine U_0 or U_3. This setup provides no equation for U_0 and instead of an equation for U_3, we would arrive at constraint equation (4.2.12a) (automatically solved due to our choice of $\delta = -\gamma$). In the original work [1] U_1 and U_3 were left undetermined by the main equations, and considered as functions of integration. Here we will see instead that both will be determined by enforcing $E_4 = 0$.

First we need to consider $E_3 = 0$. As the equations are nicely nested, we can solve this using the functions γ, β, U which we have obtained from the previous equations. The solutions to this equation are

$$
\begin{aligned}
W_0 &= -\frac{1}{3}e^{2\beta_0}\Lambda \\
W_1 &= \frac{1}{3}(3\cot\theta U_0 - 2e^{2\beta_0}\Lambda\gamma_1 + 3U_{0,\theta} + 6U_0\gamma_{0,\theta} + 6\gamma_{0,u}) \\
W_2 &= -\frac{1}{6}e^{-2\gamma_0}(5\Lambda\gamma_1^2 e^{2\beta_0+2\gamma_0} + 6\Lambda\gamma_2 e^{2\beta_0+2\gamma_0} + 6e^{2\beta_0}\gamma_{0,\theta\theta} + 6\cot(\theta)e^{2\beta_0}\gamma_{0,\theta} - \\
&\quad 6e^{2\beta_0} - 24e^{2\beta_0}\beta_{0,\theta}^2 - 12e^{2\beta_0}\beta_{0,\theta\theta} - 12\cot(\theta)e^{2\beta_0}\beta_{0,\theta} - \\
&\quad 12e^{2\gamma_0}\gamma_{0,u}\gamma_1 - 12e^{2\gamma_0}\gamma_{1,u} - 6e^{2\gamma_0}\gamma_1 U_{0,\theta} - 12e^{2\gamma_0}\gamma_{0,\theta}\gamma_1 U_0 - \\
&\quad 12e^{2\gamma_0}\gamma_{1,\theta}U_0 - 6\cot(\theta)e^{2\gamma_0}\gamma_1 U_0) \\
W_4 &= \frac{1}{24}e^{-2\gamma_0}(-43e^{2\beta_0+2\gamma_0}\Lambda\gamma_1^4 + 48e^{2\gamma_0}\cot(\theta)U_0\gamma_1^3 + 48e^{2\gamma_0}U_{0,\theta}\gamma_1^3 + \\
&\quad 96e^{2\gamma_0}U_0\gamma_{0,\theta}\gamma_1^3 + 96e^{2\gamma_0}\gamma_{0,u}\gamma_1^3 + 12e^{2\beta_0}\gamma_1^2 - 172e^{2\beta_0+2\gamma_0}\Lambda\gamma_2\gamma_1^2 + \\
&\quad 36e^{2\beta_0}\cot(\theta)\beta_{0,\theta}\gamma_1^2 - 12e^{2\beta_0}\cot(\theta)\gamma_{0,\theta}\gamma_1^2 + 96e^{2\gamma_0}U_0\gamma_{1,\theta}\gamma_1^2 + \\
&\quad 36e^{2\beta_0}\beta_{0,\theta\theta}\gamma_1^2 - 12e^{2\beta_0}\gamma_{0,\theta\theta}\gamma_1^2 + 96e^{2\gamma_0}\gamma_{1,u}\gamma_1^2 - 24e^{2\gamma_0}W_3\gamma_1 + \\
&\quad 144e^{2\gamma_0}\cot(\theta)U_0\gamma_2\gamma_1 - 120e^{2\beta_0+2\gamma_0}\Lambda\gamma_3\gamma_1 + 144e^{2\gamma_0}\gamma_2 U_{0,\theta}\gamma_1 + \\
&\quad 288e^{2\gamma_0}U_0\gamma_2\gamma_{0,\theta}\gamma_1 - 12e^{2\beta_0}\cot(\theta)\gamma_{1,\theta}\gamma_1 + 168e^{2\beta_0}\beta_{0,\theta}\gamma_{1,\theta}\gamma_1 + \\
&\quad 96e^{2\gamma_0}U_0\gamma_{2,\theta}\gamma_1 - 12e^{2\beta_0}\gamma_{1,\theta\theta}\gamma_1 + 288e^{2\gamma_0}\gamma_2\gamma_{0,u}\gamma_1 + 96e^{2\gamma_0}\gamma_{2,u}\gamma_1 - \\
&\quad 52e^{2\beta_0+2\gamma_0}\Lambda\gamma_2^2 + 192e^{2\beta_0}\gamma_2\beta_{0,\theta}^2 + 12e^{2\beta_0}\gamma_{1,\theta}^2 - 12e^{2\gamma_0}\cot(\theta)U_3 + \\
&\quad 24e^{2\beta_0}\gamma_2 + 72e^{2\gamma_0}\cot(\theta)U_0\gamma_3 - 40e^{2\beta_0+2\gamma_0}\Lambda\gamma_4 + 72e^{2\gamma_0}\gamma_3 U_{0,\theta} - \\
&\quad 12e^{2\gamma_0}U_{3,\theta} + 72e^{2\gamma_0}U_3\beta_{0,\theta} + 24e^{2\beta_0}\cot(\theta)\gamma_2\beta_{0,\theta} - 24e^{2\gamma_0}U_3\gamma_{0,\theta} - \\
&\quad 24e^{2\beta_0}\cot(\theta)\gamma_2\gamma_{0,\theta} + 144e^{2\gamma_0}U_0\gamma_3\gamma_{0,\theta} + 96e^{2\gamma_0}U_0\gamma_2\gamma_{1,\theta} + \\
&\quad 12e^{2\beta_0}\cot(\theta)\gamma_{2,\theta} + 24e^{2\beta_0}\beta_{0,\theta}\gamma_{2,\theta} + 48e^{2\gamma_0}U_0\gamma_{3,\theta} + 24e^{2\beta_0}\gamma_2\beta_{0,\theta\theta} - \\
&\quad 24e^{2\beta_0}\gamma_2\gamma_{0,\theta\theta} + 12e^{2\beta_0}\gamma_{2,\theta\theta} + 144e^{2\gamma_0}\gamma_3\gamma_{0,u} + \\
&\quad 96e^{2\gamma_0}\gamma_2\gamma_{1,u} + 48e^{2\gamma_0}\gamma_{3,u})
\end{aligned}
\tag{4.2.19}
$$

where we note that no equation for W_3 has been presented. The equation in it's place is the constraint equation (4.2.12b), again automatically solved.

The final main equation, $E_4 = 0$, is all that now remains of the main system of equations. As was discussed at the end of the previous subsection, the $\mathcal{O}(r)$ coefficient gives us an equation which forces

$$U_0 = c(u)\sin(\theta). \tag{4.2.20}$$

As it turns out, this also forces the $\mathcal{O}_1$ and $\mathcal{O}(1/r)$ terms in the expansion of E_4 to vanish.

The next non-vanishing term is at $\mathcal{O}(1/r^2)$. This term reads

$$\begin{aligned}
&4\beta_{0,\theta}^2\gamma_1^2 e^{2\beta_0-2\gamma_0} - \beta_{0,\theta}\gamma_{0,\theta}\gamma_1^2 e^{2\beta_0-2\gamma_0} + \frac{1}{2}\beta_{0,\theta\theta}\gamma_1^2 e^{2\beta_0-2\gamma_0} - 3\beta_{0,\theta}\gamma_{1,\theta}\gamma_1 e^{2\beta_0-2\gamma_0} + \\
&\gamma_{0,\theta}\gamma_{1,\theta}\gamma_1 e^{2\beta_0-2\gamma_0} - \frac{1}{2}\gamma_{1,\theta\theta}\gamma_1 e^{2\beta_0-2\gamma_0} - 8\beta_{0,\theta}^2\gamma_2 e^{2\beta_0-2\gamma_0} - \frac{1}{2}\gamma_{1,\theta}^2 e^{2\beta_0-2\gamma_0} + \\
&2\beta_{0,\theta}\gamma_2\gamma_{0,\theta} e^{2\beta_0-2\gamma_0} + 3\beta_{0,\theta}\gamma_{2,\theta} e^{2\beta_0-2\gamma_0} - \gamma_{0,\theta}\gamma_{2,\theta} e^{2\beta_0-2\gamma_0} - \beta_{0,\theta\theta}\gamma_2 e^{2\beta_0-2\gamma_0} + \\
&\frac{1}{2}\gamma_{2,\theta\theta} e^{2\beta_0-2\gamma_0} - \frac{1}{2}\cot(\theta)\beta_{0,\theta}\gamma_1^2 e^{2\beta_0-2\gamma_0} + \frac{1}{2}\cot(\theta)\gamma_{1,\theta}\gamma_1 e^{2\beta_0-2\gamma_0} + \\
&\cot(\theta)\beta_{0,\theta}\gamma_2 e^{2\beta_0-2\gamma_0} - \frac{1}{2}\cot(\theta)\gamma_{2,\theta} e^{2\beta_0-2\gamma_0} - 3\beta_{0,\theta}U_3 - \frac{1}{2}U_{3,\theta} + \frac{1}{2}\cot(\theta)U_3,
\end{aligned} \tag{4.2.21}$$

the vanishing of which we can treat as a PDE for U_3. As it turns out, this PDE is easily integrable, giving the solution

$$U_3 = h(u)\sin(\theta)e^{-6\beta_0} + e^{2\beta_0-2\gamma_0}\left(\beta_{0,\theta}\gamma_1^2 - 2\beta_{0,\theta}\gamma_2 - \gamma_{1,\theta}\gamma_1 + \gamma_{2,\theta}\right) \tag{4.2.22}$$

where $h(u)$ is the function of integration.

This solution is enough to satisfy the vanishing of E_4 at orders $\mathcal{O}(1/r^3)$ and $\mathcal{O}(1/r^4)$ but not at $\mathcal{O}(1/r^5)$. This order is only satisfied if we enforce

$$h(u) = 0. \tag{4.2.23}$$

Showing this requires solving the Einstein equations to higher order than we have displayed in this thesis. Due to the long expressions involved in these computations, we relegate this to the MATHEMATICA file `'delta=-gamma_solutions.nb'` (all references to MATHEMATICA files in this section refer to this notebook).

4.2.1.3 Supplementary Conditions

The four 'main equations' that we have solved above do not constitute a full set of the Einstein field equations. As in the AF case [1] as well as the AlAdS case [52], there will also be extra 'supplementary conditions' corresponding to the $\{uu\}$ and $\{u\theta\}$ components of the field equations (all other components automatically satisfy the field equations).

Using an identical procedure to that employed in [1, 52], one can study the contracted Bianchi identities for the spacetime (after enforcing the main equations hold) to show that the equations admit the following forms

$$R_{u\theta} = \Lambda g_{u\theta} + \frac{f(u,\theta)e^{-2\gamma}}{r^2} \tag{4.2.24a}$$

$$R_{uu} = \Lambda g_{uu} + \frac{g(u,\theta)e^{-2\gamma}}{r^2} \tag{4.2.24b}$$

where $f(u,\theta), g(u,\theta)$ are functions of integration which have to vanish if the vacuum field equations are to hold. To explicitly compute f and g, we use the solutions from the previous subsection (up to and including $\mathcal{O}(1/r^4)$ terms is sufficient) and put them back into the $\{u\theta\}$ and $\{uu\}$ components of the field equations. The surviving $\mathcal{O}(1/r^2)$ terms will now allow us to read off the functions and thus the equations we need to enforce the full set of field equations. As was the case in the Bondi-AdS case studied previously in this thesis, these supplementary conditions produce exceedingly long printouts for f and g, which we leave in the MATHEMATICA file.

4.2.2 Fefferman Graham Coordinate Transformation

Now that we have solved the Einstein equations, we want to transform our AlAdS solution to the vacuum Einstein equations into the Fefferman-Graham coordinate system

$$ds^2 = l^2\left[\frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(g_{0ab}(x^c) + \rho^2 g_{(2)ab}(x^c) + \rho^3 g_{(3)ab}(x^c) + \ldots)dx^a dx^b\right]. \tag{4.2.25}$$

This coordinate system is a choice of gauge for which all asymptotically locally AdS spacetimes can be written in near the conformal boundary $\partial\mathcal{X} = \mathscr{I} = \{\rho = 0\}$ [62]. $l = \sqrt{-3/\Lambda}$ is the AdS radius of the spacetime, and the indicies a, b, c run over all coordinates other than ρ. The reason we want to transform into this gauge is it allows us to gain a holographic understanding of our "broken Bondi" spacetime via the AdS/CFT correspondence [58, 140, 60, 62].

Here we will give a sketch of the coordinate transformation from our metric

$$ds^2 = -(Vr^{-1}e^{2\beta} - U^2r^2e^{2\gamma})du^2 - 2e^{2\beta}dudr - 2Ur^2e^{2\gamma}dud\theta + r^2e^{2\gamma}d\Omega_2^2 \tag{4.2.26}$$

to (4.2.25). To simplify the computation we will use the normalisation of $l = 1$ ($\Lambda = -3$) throughout.

Conceptually, this transformation is identical to that performed in [52]. Starting with (4.2.26) (with the metric functions being our solutions to the vacuum Einstein equations from the previous section) we first transform into real time t and tortoise radial coordinate r_*. The real time t is defined by the relation

$$t = u - r_* \tag{4.2.27}$$

and r_* by

$$dr_* = \frac{dr}{f(r)}. \tag{4.2.28}$$

Here we choose profile function $f(r) = 1 + r^2$, giving

$$r_* = \arctan(r) + c \tag{4.2.29}$$

where c is the constant of integration. To motivate our choice of the value of this constant of integration, we use the fact that the conformal boundary of the AlAdS spacetime in Fefferman-Graham coordinates is located at $\rho = 0$. When transforming into Fefferman-Graham coordinates we want to roughly identify the coordinate ρ with r_*, so it makes sense to choose a constant of integration corresponding to the boundary condition $\lim_{r\to\infty} r_* = 0$. This forces $c = -\pi/2$ and thus

$$r = \tan\left(r_* + \frac{\pi}{2}\right) = -\cot(r_*). \tag{4.2.30}$$

Using these explicit coordinate transformations, we can transform (4.2.26) into coordinates (t, r_*, θ, ϕ) and from these coordinates we are now ready to move into Fefferman-Graham coordinates $(\tilde{t}, \rho, \tilde{\theta}, \tilde{\phi})$. The first transformation which we note is

$$\phi = \tilde{\phi} \tag{4.2.31}$$

which follows directly from the from the axisymmetry of the spacetime. The other coordinates are all obtained by power series expansions in ρ

$$\begin{aligned} r_* &= \rho + a_2(\tilde{t},\tilde{\theta})\rho^2 + a_3(\tilde{t},\tilde{\theta})\rho^3 + a_4(\tilde{t},\tilde{\theta})\rho^4 + \mathcal{O}(\rho^5) \\ t &= \tilde{t} + b_1(\tilde{t},\tilde{\theta})\rho + b_2(\tilde{t},\tilde{\theta})\rho^2 + b_3(\tilde{t},\tilde{\theta})\rho^3 + b_4(\tilde{t},\tilde{\theta})\rho^4 + \mathcal{O}(\rho^5) \\ \theta &= \tilde{\theta} + c_1(\tilde{t},\tilde{\theta})\rho + c_2(\tilde{t},\tilde{\theta})\rho^2 + c_3(\tilde{t},\tilde{\theta})\rho^3 + c_4(\tilde{t},\tilde{\theta})\rho^4 + \mathcal{O}(\rho^5) \end{aligned} \tag{4.2.32}$$

where the coefficients have yet to be determined. In order to determine these coefficients, one must apply the transformation and force the resulting line-element to take the form of (4.2.25). This procedure involves taking the series of any functions in the metric about $\rho = 0$ and then forcing the terms proportional to $d\rho^2$, $d\rho d\tilde{t}$, $d\rho d\tilde{\theta}$ to vanish at all orders of ρ (excluding, of course the $d\rho^2/\rho^2$ piece). Note that it is also common practice to drop the tildes on the Fefferman-Graham coordinates after transformation, so from here on we will refer to the Fefferman-Graham system as (t, ρ, θ, ϕ).

- $\mathcal{O}(1/\rho^2)$:

  Requiring that the transformed system is in Fefferman-Graham coordinates at this order in ρ forces us to pick coefficients

$$b_1(t,\theta) = 1 - e^{-2\beta_0(t,\theta)} \tag{4.2.33a}$$

$$c_1(t,\theta) = -e^{-2\beta_0(t,\theta)}c(t)\sin(\theta), \tag{4.2.33b}$$

where the function $c(t)$ is the function of integration that we saw in the metric function U_0. The metric at this order in the Fefferman-Graham expansion is

$$ds^2_{{}_0} = \frac{1}{\rho^2}\left[d\rho^2 - (1 + e^{2\gamma_0}c(t)^2\sin^2(\theta))dt^2 - 2c(t)\sin(\theta)e^{2\gamma_0}dtd\theta + e^{2\gamma_0}d\Omega^2\right] \tag{4.2.34}$$

which allows us to read off $g_{{}_0}$, the metric induced at the conformal boundary

$$ds^2 = -(1 + e^{2\gamma_0}c(t)^2\sin^2(\theta))dt^2 - 2c(t)\sin(\theta)e^{2\gamma_0}dtd\theta + e^{2\gamma_0}d\Omega^2 \tag{4.2.35}$$

This is the first interesting piece of holographic data, as in the AdS/CFT correspondence, this is the background metric for the CFT_3 dual to our gravitational theory.

- $\mathcal{O}(1/\rho)$:

This order gives us some more of the coefficients

$$a_2(t,\theta) = -e^{2\beta_0}(c(t)\cos\theta + e^{2\beta_0}\gamma_1 + c(t)\sin\theta\gamma_{0,\theta} + \gamma_{0,t}) \tag{4.2.36a}$$

$$\begin{aligned} b_2(t,\theta) = -e^{-4\beta_0}(&e^{2\beta_0}c(t)\cos\theta + e^{4\beta_0}\gamma_1 + c(t)\sin\theta\beta_{0,\theta} + \\ &e^{2\beta}c(t)\sin\theta\gamma_{0,\theta} + \beta_{0,t} + e^{2\beta_0}\gamma_{0,t}) \end{aligned} \tag{4.2.36b}$$

$$\begin{aligned} c_2(t,\theta) = \frac{1}{2}e^{-4\beta_0-2\gamma_0}(&c(t)^2e^{2\gamma_0}\cos\theta\sin\theta + c'(t)e^{2\gamma_0}\sin\theta + 2e^{4\beta_0}\beta_{0,\theta} - \\ &2c(t)^2e^{2\gamma_0}\sin^2(\theta)\beta_{0,\theta} - 2c(t)e^{2\gamma_0}\sin\theta\beta_{0,t}) \end{aligned} \tag{4.2.36c}$$

and the Fefferman-Graham metric at this order vanishes (as expected)

$$ds^2_{(1)} = 0 \tag{4.2.37}$$

- $\mathcal{O}(1)$:

This order gives us the coefficients a_3, b_3, c_3 as well as the $g_{(2)}$ piece of the metric (left out here due to length). This order also provides us with a consistency check as $g_{(2)}$ has to satisfy the equation

$$g_{(2)} = -R_{{}_0ab} + \frac{1}{4}R_{{}_0}g_{{}_0ab} \tag{4.2.38}$$

where $g_{{}_0}$ is the metric induced at the boundary (4.2.35) and $R_{{}_0ab}, R_{{}_0}$ are respectively the Ricci tensor and scalar of $g_{{}_0}$. This constraint has been checked in MATHEMATICA with the $g_{(2)}$ from the series expansion and these two expressions agree.

- $\mathcal{O}(\rho)$:

This order gives us the coefficients a_4, b_4, c_4 and the $g_{(3)}$ piece of the metric (again these are exceedingly long formulae). The $g_{(3)}$ term has the holographic interpretation as being the energy momentum tensor of the dual CFT. Explicitly we have

$$T_{ab} = -\frac{3}{2\kappa^2}\left(-\frac{3}{\Lambda}\right) g_{(3)ab} \tag{4.2.39}$$

where $\kappa = 8\pi G/c^2$ is Einstein's constant. T_{ab} is called the holographic energy-momentum tensor as it gives the expectation value of the energy momentum tensor in the dual CFT

$$\langle T_{ab}\rangle = \frac{2}{\sqrt{-\det g_0}}\frac{\delta S_{\text{ren}}}{\delta g_0^{ab}}. \tag{4.2.40}$$

T_{ab} is both conserved and traceless with respect to the metric g_{0ab}

$$g_0^{ab} T_{ab} = 0, \qquad \nabla_0^a T_{ab} = 0 \tag{4.2.41}$$

which means that $g_{(3)ab}$ should also satisfy these equations.

This has again been checked in MATHEMATICA. $g_{(3)ab}$ has been verified to be traceless and conserved as all components of the divergence vanish.

4.2.3 Application to Robinson-Trautman Metrics

As well as breaking the unnatural restriction on the angular part of the metric in the CFT, another important motivation for modifying the Bondi gauge was to be able to apply the new gauge to spacetimes for which the original gauge seemed unnatural or difficult to transform into.

A particular example of a spacetime for which we would like to examine in our new gauge is the Robinson-Trautman (RT) class of metrics [68, 183, 184, 185, 186, 187, 188]. The RT spacetimes are the unique class of solutions to the vacuum Einstein equations which admit a null geodesic congruence with zero shear and twist and non-vanishing divergence. These spacetimes have been a topic of interest in both the AF [189, 190] and AlAdS cases [69, 182] although the latter case has not yet been adapted to the Bondi gauge, a task we hope to begin to give a resolution to now.

For the purpose of this thesis we will use the coordinate system of [182] and restrict consideration to AlAdS axisymmetric RT spacetimes. These are described by the spacetime metric

$$\begin{aligned} ds^2 &= -F(u,r,\theta)du^2 - 2dudr + r^2 g_{ab}dx^a dx^b \\ g_{ab}dx^a dx^b &= \frac{1}{\sigma(u,\theta)^2}d\Sigma_k^2 \\ F(u,r,\theta) &= -\frac{\Lambda}{3}r^2 - 2r\frac{\partial_u \sigma}{\sigma} + \frac{R_g}{2} - \frac{2m}{r} \end{aligned} \tag{4.2.42}$$

where u is a retarded time coordinate and r is the radial distance. The indices a, b run over the angular coordinates θ, ϕ (we will drop the covariant form in order to impose the axisymmetry) The constant m which appears in the final term of F is a constant of integration, associated with the physical mass of the system.

$d\Sigma_k^2$ describes the metric on a Riemannian 2-manifold of constant scalar curvature $2k$, $k = -1$ describes $\mathbb{H}^2$, $k = 0$ is either T^2 or $\mathbb{R}^2$ and $k = 1$ is S^2. In order to connect the RT metric with our gauge choice of (4.2.26), we will from now on set $k = 1$. R_g is the Ricci scalar of g_{ab} and so choosing $k = 1$ means we have

$$R_g = 2\left[\sigma^2 - (\partial_\theta\sigma)^2 + \sigma(\cot(\theta)\partial_\theta\sigma + \partial_\theta^2\sigma)\right]. \tag{4.2.43}$$

This form of the metric alone is not enough to solve the vacuum Einstein equations. We also have an equation for $\sigma(u,\theta)$ which is often referred to as the *Robinson-Trautman equation*. We express this equation as a Calabi-flow equation for the metric g_{ab} on a topological 2-sphere

$$\partial_u g_{ab} = \frac{1}{12m}(\nabla_g^2 R_g)g_{ab} \tag{4.2.44}$$

in coordinates where $d\Sigma_1^2 = d\theta^2 + \sin^2(\theta)d\phi^2$, the 4 components of the Calabi-flow equation only give us one equation

$$\begin{aligned} \partial_u\sigma = -\frac{\sigma^2}{12m}&[(\partial_\theta\sigma)^2(\cot(\theta)\partial_\theta\sigma - \partial_\theta^2\sigma) + \\ &\sigma\{(4 - 3\mathrm{cosec}^2(\theta))(\partial_\theta\sigma)^2 - (\partial_\theta^2\sigma)^2 + \partial_\theta\sigma(4\cot(\theta)\partial_\theta^2\sigma + \partial_\theta^3\sigma)\} + \\ &\sigma^2(2\cot(\theta)\mathrm{cosec}^2(\theta)\partial_\theta\sigma - 2\cot^2(\theta)\partial_\theta^2\sigma + \cot(\theta)\partial_\theta^3\sigma + \partial_\theta^4\sigma)] \end{aligned} \tag{4.2.45}$$

which we can now explicitly see to be a 4$^{\text{th}}$ order differential equation for $\sigma(u,\theta)$. The general properties of this equation and it's solutions have been analysed, most notably in [191]. Here we will focus on selecting the functions in our broken Bondi gauge s.t. they solve this equation.

We want to compare the RT spacetime with our "Broken Bondi" gauge. It is a helpful first step to write both spacetimes metrics side by side

$$ds^2_{RT} = -\left(-\frac{\Lambda}{3}r^2 - 2r\frac{\partial_u\sigma}{\sigma} + (\sigma^2 - (\partial_\theta\sigma)^2 + \sigma(\cot(\theta)\partial_\theta\sigma + \partial^2_\theta\sigma)) - \frac{2m}{r}\right)du^2 - 2dudr + \frac{r^2}{\sigma^2}\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{4.2.46a}$$

$$ds^2_{BB} = -\left(Wr^2e^{2\beta} - U^2r^2e^{2\gamma}\right)du^2 - 2e^{2\beta}dudr - 2Ur^2e^{2\gamma}dud\theta + r^2e^{2\gamma}\left(d\theta^2 + \sin^2(\theta)d\phi^2\right). \tag{4.2.46b}$$

By direct comparison of the angular parts of the metrics, we conclude that $e^{2\gamma_{RT}} = 1/\sigma^2 \iff \gamma_{RT} = -\log(\sigma)$. Comparing the $\{u\theta\}$ components of the metrics now gives us $U_{RT} = 0$ and $\beta_{RT} = 0$ follows from the $\{ur\}$ components. Finally we can compare the $\{uu\}$ components to read off W_{RT}

$$\begin{aligned} W_{RT}(u,r,\theta) &= -\frac{\Lambda}{3} - \frac{2}{r}\frac{\partial_u\sigma}{\sigma} + \frac{1}{r^2}(\sigma^2 - (\partial_\theta\sigma)^2 + \sigma(\cot(\theta)\partial_\theta\sigma + \partial^2_\theta\sigma)) - \frac{2m}{r^3} \\ &= -\frac{\Lambda}{3} + \frac{2}{r}\partial_u\gamma_{RT} + \frac{1}{r^2}e^{-2\gamma_{RT}}(1 - \partial_\theta\gamma_{RT}\cot\theta - \partial^2_\theta\gamma_{RT}) - \frac{2m}{r^3} \end{aligned} \tag{4.2.47}$$

where the second expression is consistent with the solutions to the Einstein equations that we generated in (4.2.19).

So the RT metrics are written in the broken Bondi gauge. Due to this we can also read off the FG expansion for axisymmetric RT metrics from the transformation that we performed.

$$ds^2_0 = -dt^2 + e^{2\gamma_0}(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{4.2.48a}$$

$$\begin{aligned} ds^2_{(2)} = &\frac{1}{2}e^{-2\gamma_0}\left(e^{2\gamma_0}\gamma^2_{0,u} + \gamma_{0,\theta\theta} + 2e^{2\gamma_0}\gamma_{0,uu} + \cot(\theta)\gamma_{0,\theta} - 1\right)dt^2 + 2\gamma_{0,u\theta}dtd\theta + \\ &\frac{1}{2}\left(-e^{2\gamma_0}\gamma^2_{0,u} - 1 + \gamma_{0,\theta\theta} + \cot(\theta)\gamma_{0,\theta}\right)[d\theta^2 + d\phi^2\sin^2(\theta)] \end{aligned} \tag{4.2.48b}$$

$$\begin{aligned} ds^2_{(3)} = &-\frac{4m}{3}dt^2 + \\ &\frac{2}{3}e^{-2\gamma_0}\left(\gamma_{0,\theta\theta\theta} - 2\cot(\theta)\gamma^2_{0,\theta} + \cot(\theta)\gamma_{0,\theta\theta} - \gamma_{0,\theta}\left(2\gamma_{0,\theta\theta} + \csc^2(\theta) - 2\right)\right)dtd\theta + \\ &\frac{1}{3}\left(-2me^{2\gamma_0} + \gamma_{0,u\theta\theta} - \gamma_{0,u\theta}\left(2\gamma_{0,\theta} + \cot(\theta)\right)\right)d\theta^2 + \\ &\frac{1}{3}\sin^2(\theta)\left(\gamma_{0,u\theta}\left(2\gamma_{0,\theta} + \cot(\theta)\right) - \left(2me^{2\gamma_0} + \gamma_{0,u\theta\theta}\right)\right)d\phi^2 \end{aligned} \tag{4.2.48c}$$

where again we've used $\Lambda = -3$. This result agrees with the energy-monetum tensor for the AdS-RT spacetime as presented in [69]. To see this explicitly one simply needs to transform from our choice of angular coordinates, (θ, ϕ), into the Kähler coordinates,

$(z, \bar{z})$, used in [69]

$$z = \sqrt{2} \cot\left(\frac{\theta}{2}\right) e^{i\phi}, \qquad \bar{z} = \sqrt{2} \cot\left(\frac{\theta}{2}\right) e^{-i\phi} \tag{4.2.49}$$

4.2.4 Idea for transformation from Bondi to "Broken Bondi" gauge

We still wish to find a way of connecting our new choice of "broken Bondi" gauge to the original gauge. The principal motivation for such an undertaking is that we already understand the physical aspects of the old gauge (mass, news etc.) and we would like to apply these new considerations to the new gauge.

In order to do this, we need to find a coordinate transformation from the Bondi gauge (4.2.1) into the broken gauge (4.2.26). We immediately notice that the only difference between these two gauges is in the angular part of the metric and thus to simplify the problem we will consider maps between the 2-metrics. It is of interest to the author of this thesis to return to this in the near future in order to explicitly perform this map. This would allow us to further develop the notion of Bondi mass in AlAdS_4 spacetime by having a concrete in the class of Robinson-Trautman spacetimes.

4.3 Conclusions and outlook

We have shown that the Bondi-Sachs gauge admits various interesting properties within the framework of asymptotically locally AdS spacetimes, both via a careful consideration of the asymptotics in the Fefferman-Graham gauge and utilisation of the AdS/CFT dictionary.

With regards to future progression, the first aim is to complete the ongoing work comprising sections 4.1 and 4.2. A greater understanding of the Bondi mass in AdS would potentially allow us to model the effects of gravitational radiation in AdS and help to bolster the understanding of mass in asymptotically hyperbolic spacetimes. We also have the motivation to better understand the Robinson-Trautman spacetime and it's Bondi mass when $\Lambda < 0$. In [69], results were derived for Robinson-Trautman metrics using the Bondi mass (as defined in asymptotically flat spacetime), and it would be interesting to examine whether the same carries over using a well understood notion of 'Bondi mass' in AdS. The technique of breaking the Bondi gauge that we are beginning to implement should hopefully provide us with an avenue to do this, as well as the necessary tools to look at the physical aspects of Bondi-Sachs spacetimes without the constraint on the angular part of the metric.

Finally, it is also of interest to understand the conformal field theory implications of any result that one finds in the gravitational setup. Understanding mass loss and the new gauge seems to have a direct meaning in the dual CFT (related to the energy momentum tensor and the background metric respectively) and once the current work has been completed we hope to be able to understand these equivalences more precisely. Speculatively, one could even begin to use this work as a launch-pad in establishing a holographic understanding of flat spacetime, as we now have both the elegant properties of the Bondi-Sachs gauge in flat spacetime, as well as a holographic understanding via AdS/CFT. This idea remains in it's infancy here, but shows another direction that this research may eventually follow.

CHAPTER 5

Charges in asymptotically locally de Sitter spacetimes

5.1 Analytic continuation to asymptotically de Sitter spacetime

Now that we have seen how to extend the use of the Bondi-Sachs gauge to that of asymptotically locally anti-de Sitter space time, it is a natural question to ask whether this analysis extends to asymptotically de Sitter (dS) spacetimes. These spacetimes are of interest for many reasons, principally because it has been experimentally verified that our universe has a cosmological constant $\Lambda > 0$ [192] and although it is still debated whether this solution is static or running, any new understanding of de Sitter physics may help us to understand our universe. There is also interest in a holographic understanding of quantum gravity in de Sitter spacetime, mainly via the so-called "dS/CFT" correspondence [193, 194] as well as more recent work on holographic cosmology [72, 73, 74, 75, 195, 196, 197, 198].

For the purpose of extending our results, it was shown in an appendix of [62] that the analysis of the near-boundary (Fefferman-Graham) expansion is remarkably similar in the AdS and dS cases, and one can use the tools of analytic continuation to transform from one to the other. In this section we will explain how to perform this analytic continuation and interpret our results from the previous sections in the context of asymptotically de Sitter spacetime.

5.1.1 Triple Wick rotation

To move from the *AdS* to *dS* form of the Fefferman-Graham expansion, one has to Wick rotate three of the variables, a procedure we will refer to as a triple Wick rotation. We will explain this first through the straightforward example of Lorentzian *AdS* in Poincaré coordinates (as in [62], although there the starting point was Euclidean *AdS*) before moving to the more general Fefferman-Graham form

$$ds^2 = -\frac{3}{\Lambda}\left[\frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(g_{(0)ab} + \rho^2 g_{(2)ab} + \rho^3 g_{(3)ab} + \ldots)dx^a dx^b\right] \tag{5.1.1}$$

and applying the rotation using our Bondi-Sachs expansion as computed in [52].

Before studying metrics explicitly, it will help to gain intuition about how to take the rotation. We write the vacuum Einstein equations as

$$R_{ab} = \Lambda g_{ab} \tag{5.1.2}$$

where $\Lambda < 0$ for *AdS* and $\Lambda > 0$ for *dS*. We recall the characteristic length scales for each

$$l_{AdS}^2 = -\frac{3}{\Lambda}, \qquad l_{dS}^2 = \frac{3}{\Lambda} \tag{5.1.3}$$

and thus the Einstein equations for an *AdS* and *dS* spacetime respectively read

$$\begin{aligned} R_{ab} &= -\frac{3}{l_{AdS}^2} \qquad (AdS) \\ R_{ab} &= \frac{3}{l_{dS}^2} \qquad (dS) \end{aligned} \tag{5.1.4}$$

from which we see that we can map between the two equations by taking $l_{AdS}^2 \leftrightarrow -l_{dS}^2$. Thus this is one of the Wick rotations we perform when analytically continuing from *AdS* to *dS*.

To illustrate the rotations of the other variables, it will help to consider the specific example of Lorentzian *AdS*. We write the metric for Lorentzian *AdS* in dimensionless Poincaré coordinates (ρ, t, x_1, x_2)

$$ds^2 = \frac{l_{AdS}^2}{\rho^2}(d\rho^2 - dt^2 + dx_1^2 + dx_2^2) \tag{5.1.5}$$

where we note by comparison with (5.1.1) that this metric has $g_{(0)ab} = \eta_{ab}$ with all other $g_{(i)}$ vanishing. We now perform the expected rotation upon l_{AdS}^2 as well as the following rotations in the other coordinates

$$\rho^2 \to -\tilde{\rho}^2, \qquad t^2 \to -\tilde{t}^2 \tag{5.1.6}$$

which brings the line element into the form

$$ds^2 = \frac{l_{dS}^2}{\tilde{\rho}^2}(-d\tilde{\rho}^2 + d\tilde{t}^2 + dx_1^2 + dx_2^2). \tag{5.1.7}$$

This metric is a line element for de Sitter spacetime which we will follow the conventions of [62] in naming this the "big bang" metric. $\tilde{\rho}$ is now a time coordinate (the conformal time) and the surfaces of $\tilde{\rho} =$ constant are spacelike 3-planes. These coordinates cover half of the global geometry of dS [171] and depending upon the ranges of the coordinates one can choose which of the two spacelike boundaries of de Sitter space is covered: Choosing $\tilde{\rho} \in (-\infty, 0)$ covers $\mathscr{I}^+$ and $\tilde{\rho} \in (0, \infty)$ $\mathscr{I}^-$ where in both cases the covered boundary is located at $\tilde{\rho} = 0$. The case of covering $\mathscr{I}^+$ is displayed in the Penrose diagram of Figure 5.1.1.

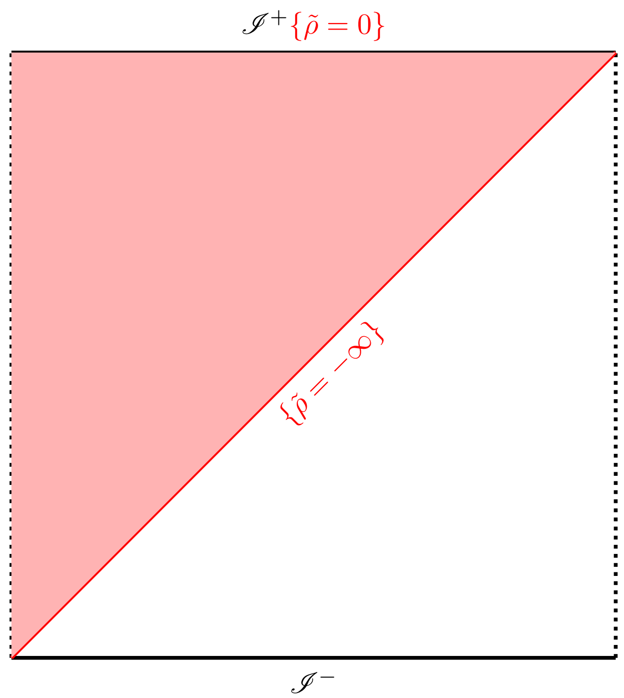

Figure5.1.1: Penrose diagram for global de Sitter spacetime showing the region covered by the "big bang" metric. The dotted line on the left hand side is a coordinate singularity where $\tilde{t} = x_1 = x_2 = 0$

This example shows that we can transform from AdS to dS by performing the triple Wick rotation

$$(l_{AdS}^2, \rho^2, t^2) \to -(l_{dS}^2, \tilde{\rho}^2, \tilde{t}^2) \tag{5.1.8}$$

and the structure of the equation (5.1.7) hints at a more general relation between the asymptotic expansion of the AdS and dS metrics. It turns out that the line element for an asymptotically locally dS metric can be brought into the Fefferman-Graham form [59, 62]

$$ds^2 = l_{dS}^2 \left[-\frac{d\tilde{\rho}^2}{\tilde{\rho}^2} + \frac{1}{\tilde{\rho}^2}(\tilde{g}_{(0)ij} + \tilde{\rho}^2 \tilde{g}_{(2)ij} + \tilde{\rho}^3 \tilde{g}_{(3)ij} + \dots) d\tilde{x}^i d\tilde{x}^j \right] \tag{5.1.9}$$

where one is able to bring a metric into this form by performing the transformation (5.1.8) upon the line element of an asymptotically AdS metric in Fefferman-Graham gauge (5.1.1). As in the Poincaré example, the new coordinate $\tilde{\rho}$ is now a time coordinate and thus the metric induced at the conformal boundary $\tilde{g}_{(0)ij}$ will be Riemannian (our convention for the boundary will be to take the coordinate patch such that this is $\mathscr{I}^+$).

5.1.2 Rotating the Bondi-Sachs-Fefferman-Graham metric

Now that the general machinery of the triple Wick rotation has been set up, we want to perform this rotation upon our Bondi-Sachs-Fefferman-Graham expansion as computed in [52]. In order to do this we must first perform dimensional analysis in order to reintroduce the factors of l_{AdS} (we previously used $l_{AdS} = 1$).

Working in the dimensional conventions of [length] $= +1$ we first compute the dimensions of the functions in the Bondi-Sachs metric

$$\begin{aligned} ds^2 = &-\left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 - 2e^{2\beta} du dr - \\ &2Ur^2 e^{2\gamma} du d\theta + r^2(e^{2\gamma} d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2). \end{aligned} \tag{5.1.10}$$

(5.1.10). Using the standard definitions of the Bondi-Sachs coordinates we have

$$[u] = 1, \quad [r] = 1, \quad [\theta] = 0, \quad [\phi] = 0 \tag{5.1.11}$$

and of course we want to enforce $[ds^2] = 2$. Using the length dimensions of the coordinates as given above we find the dimensions of the Bondi functions are

$$[\gamma] = 0, \quad [\beta] = 0, \quad [U] = -1, \quad [V] = 1, \quad [W] = \left[\frac{V}{r^3}\right] = -2. \tag{5.1.12}$$

We can elaborate further on these dimensions by recalling that we derived the result that these functions admit a series expansion in negative powers of r in the asymptotic region of the spacetime.

$$\gamma(u, r, \theta) = \sum_{n=0}^{\infty} \gamma_n(u, \theta) z^n = \sum_{n=0}^{\infty} \frac{\gamma_n(u, \theta)}{r^n}, \qquad \gamma_n = \left.\frac{\partial_z^{(n)} \gamma}{n!}\right|_{z=0} \tag{5.1.13}$$

Using this, we can compute the dimension of each of the functions which act as the coefficients in the power series

$$[\gamma_i] = i, \quad [\beta_i] = i, \quad [U_i] = i-1, \quad [W_i] = i-2. \tag{5.1.14}$$

With these length dimensions in mind we now want to Wick rotate the AdS Fefferman-Graham expansion of the metric. To do this we choose the Fefferman-Graham coordinates (t, ρ, θ, ϕ) to be dimensionless so that we have $[g_{(i)ab}] = 0$. Enforcing this for $i = 0, 2, 3$ will involve rescaling the Bondi functions in order to make the terms containing them dimensionless (using (5.1.14)). Once this rescaling has been performed we apply the triple Wick rotation using the following prescription

$$l_{AdS} \to i a l_{dS}, \quad t \to i b \tilde{r}, \quad \rho \to i c \tilde{\rho} \tag{5.1.15}$$

where a, b, c are all undetermined constants which satisfy $a^2 = b^2 = c^2 = 1$. These constants have been included in the Wick rotation as choosing their sign allows us to specify whether the rotations are being performed in the clockwise or anti-clockwise direction. For now we do not know if one direction has any advantages over the other so choosing this ansatz allows us to perform the rotation in full generality.

5.1.2.1 Rotation of $g_{(0)}$

We now give the explicit computation of the triple Wick rotation when applied to our the $g_{(0)}$ term in the AdS expansion (or more precisely, the $\mathcal{O}(1/\rho^2)$ terms). The $\mathcal{O}(1/\rho^2)$ piece of the expansion reads

$$ds^2 = \frac{l_{AdS}^2}{\rho^2}[d\rho^2 + g_{(0)ab}dx^a dx^b] + \mathcal{O}(\rho^0) \tag{5.1.16}$$

where

$$g_{(0)ab}dx^a dx^b = -(e^{4\beta_0} - e^{2\gamma_0}l_{AdS}^2 U_0^2)dt^2 - 2e^{2\gamma_0}l_{AdS}U_0 dt d\theta + e^{2\gamma_0}d\theta^2 + e^{-2\gamma_0}\sin^2(\theta)d\phi^2 \tag{5.1.17}$$

which is the same equation as given in [52] for the metric induced at the conformal boundary although now the factors of l_{AdS} have been reinstated. We now perform the rotation (5.1.15) and obtain

$$d\tilde{s}^2 = \frac{l_{dS}^2}{\tilde{\rho}^2}[-d\tilde{\rho}^2 + \tilde{g}_{(0)ab}d\tilde{x}^a d\tilde{x}^b] + \mathcal{O}(\tilde{\rho}^0) \tag{5.1.18}$$

where the rotation of $g_{(0)}$, $\tilde{g}_{(0)}$, is given by

$$\tilde{g}_{(0)ab}d\tilde{x}^a d\tilde{x}^b = (e^{4\beta_0} + e^{2\gamma_0}l_{dS}^2 U_0^2)d\tilde{r}^2 + 2abe^{2\gamma_0}l_{dS}U_0 d\tilde{r}d\theta + e^{2\gamma_0}d\theta^2 + e^{-2\gamma_0}\sin^2(\theta)d\phi^2. \tag{5.1.19}$$

This Riemannian metric has the interpretation as being the induced metric on the spacelike boundary of an asymptotically locally dS spacetime. We observe that the coefficient $g_{(0)\tilde{t}\theta}$ is only determined up to a sign after performing the Wick rotation and it seems reasonable at this point to ask whether one sign is preferred to the other.

In order to explore this further we first note that if we rotate l_{AdS} and t in the same direction ($a = b$) then we receive a positive sign. If instead the opposite direction is chosen ($a = -b$) then the sign is negative. At this stage the only cause for concern would be if one of the choices of sign resulted in a metric with a non-Riemannian signature, an issue which neither of the sign choices face. We can show this first by computing the determinant of the induced metric

$$|\tilde{g}_{(0)}| = e^{4\beta_0}\sin^2(\theta) > 0, \qquad \beta \in \mathbb{R} \tag{5.1.20}$$

so the signature is either the desired $(+++)$ or the undesired $(+--)$. To determine which, we compute the eigenvalues of $g_{(0)}$

$$\begin{aligned}
\lambda_1 &= e^{-2\gamma_0}\sin^2(\theta)\\
\lambda_2 &= \frac{(e^{4\beta_0} + e^{2\gamma_0} + e^{2\gamma_0} l_{dS}^2 U_0^2) + \sqrt{(e^{4\beta_0} - e^{2\gamma_0})^2 + e^{4\gamma_0} l_{dS}^2 U_0^2 (2e^{4\beta_0 - 2\gamma_0} + 2 + l_{dS}^2 U_0^2)}}{2}\\
\lambda_3 &= \frac{(e^{4\beta_0} + e^{2\gamma_0} + e^{2\gamma_0} l_{dS}^2 U_0^2) - \sqrt{(e^{4\beta_0} - e^{2\gamma_0})^2 + e^{4\gamma_0} l_{dS}^2 U_0^2 (2e^{4\beta_0 - 2\gamma_0} + 2 + l_{dS}^2 U_0^2)}}{2}
\end{aligned} \tag{5.1.21}$$

where we notice that all of the eigenvalues are independent of a, b (a strong indication that one rotation will not be preferred to the other). $\lambda_{1,2}$ are manifestly positive and thus λ_3 is also positive due to the positivity of the determinant of the metric. Thus the signature is a Riemannian $(+++)$ for both choices of rotation, indicating that both directions are fine so far and as of yet neither one is preferred.[1]

5.1.2.2 Rotations of $g_{(2)}, g_{(3)}$

In order to compute the rest of the Fefferman-Graham expansion in asymptotically dS spacetime we simply follow the procedure that we applied above for the leading order term of the expansion $g_{(0)}$. As the next terms in the expansion ($\tilde{g}_{(2)}$ and $\tilde{g}_{(3)}$) give much longer printouts we leave their explicit form to appendix B.1 as well as the supplementary MATHEMATICA file (`'BS_AdS_dS_continuation_FG.nb'`) and we will instead make some basic comments about their structure here.

[1] The signature $(+--)$ can be obtained if $\gamma_0 \in \mathbb{C}$

Recall that in the *AdS* case, one could immediately derive $g_{(2)}$ from $g_{(0)}$ using

$$g_{(2)ab} = -R_{(0)ab} + \frac{1}{4} R_{(0)} g_{(0)ab}. \tag{5.1.22}$$

In the *dS* case a very similar relation exists, although with a subtle difference. In asymptotically locally *dS* spacetime we have

$$\tilde{g}_{(2)ab} = \tilde{R}_{(0)ab} - \frac{1}{4} \tilde{R}_{(0)} \tilde{g}_{(0)ab} \tag{5.1.23}$$

where $\tilde{R}_{(0)ab}$ and $\tilde{R}$ are respectively the Ricci tensor and scalar of $\tilde{g}_{(0)ab}$. The RHS of this equation has an extra minus sign when compared with (A.2.12) which arises from combining the rotations of l_{AdS} and ρ on the *AdS* Fefferman-Graham expansion (more detail on this is provided in the appendix of [62]). We have checked that the $\tilde{g}_{(2)ab}$ obtained from the Wick rotation of the *AdS* metric agrees with (5.1.23).

With regards to the $\tilde{g}_{(3)ab}$ term in the expansion, we comment that this term obeys equivalent constraints to $g_{(3)ab}$, namely that it is traceless and conserved with respect to the boundary metric $\tilde{g}_{(0)ab}$

$$\tilde{g}_{(3)ab} \tilde{g}^{ab}_{(0)} = 0, \qquad \tilde{\nabla}^a_{(0)} g_{(3)ab} = 0 \tag{5.1.24}$$

where once again, these properties have been sucessfully checked for the $\tilde{g}_{(3)ab}$ obtained from the triple Wick rotation. The procedure for doing this is very similar to the *AdS* case discussed earlier, and these constraints are satisfied by imposing the vacuum Einstein equations (with cosmological constant).

5.1.3 Comment on cosmological horizon and holographic interpretation

Now that we have sucessfully computed the Fefferman-Graham expansion for an asymptotically locally *dS* Bondi-Sachs metric, we wish to interpret our results. We will do this primarily by noting the contrasts and similarities with the *AdS* case.

An initial observation for the asymptotically *dS* solution is the apparent breakdown of the metric restrictions that BMS themselves enforced in [1]. In particular, they wanted to enforce that $\partial/\partial u$ be an everywhere timelike vector field and thus $g_{uu} < 0$. When this property is considered in the presence of a cosmological constant $\Lambda \neq 0$ we can consider the following limit

$$\lim_{r\to\infty} \frac{g_{uu}}{r^2} = \frac{\Lambda e^{4\beta_0}}{3} + U_0^2 e^{2\gamma_0} \tag{5.1.25}$$

which of course must be negative (or zero) when applying the BMS condition. As previously discussed, considering this inequality in the *AdS* case gave us a bound upon U_0 but in the dS case of $\Lambda > 0$ the first term is manifestly positive and the second manifestly non-negative (as U_0 can be zero). This gives us $g_{uu} > 0$ as $r \to \infty$, a violation of the original metric set-up.

The explanation for the change of sign in dS is due to the presence of a *cosmological horizon* in the spacetime [171]. In the Bondi-Sachs spacetime this horizon is located at some value of $r = r_c \in (0, \infty)$, past which $g_{uu} > 0$ and thus $\partial/\partial u$ is spacelike. The Bondi coordinates are chosen s.t. they are smooth across the horizon and as such they cover both regions of the spacetime.

It seems at this point there are two choices in how we can proceed with the Bondi-Sachs gauge given the knowledge of this horizon. The first would be to keep the condition that $g_{uu} < 0$ and thus the coordinates only cover a horizon-free region of spacetime. This means that we would be restricting our coordinates to only cover the exterior region of the cosmological horizon and thus we would not be able to analyse the expansion in the asymptotic region near the conformal boundary.

The other choice would be simply to relax the requirement that $g_{uu} < 0$ for the dS asymptotics, allowing us to apply our Fefferman-Graham expansion (5.1.9) and examine the spacetime near $\mathscr{I}^{\pm}$. This choice seems preferable when it comes to understanding the solution holographically, as many of the results that one understands via the AdS/CFT correspondence are mapped over via the analytic continuation that we performed in our triple Wick rotation. It is in fact an active area of research to understand the cosmological dS spacetime of our universe using similar analytic continuation techniques in tandem with the AdS/CFT correspondence [193, 199, 200, 72, 73, 74, 201, 202, 75, 203, 195, 204, 196, 197, 198, 76]

We finish this discussion with a comment on some aspects of the analysis that transfer directly over from the AdS case. Via the analytic continuation, $\tilde{g}_{(0)ab}$ acts as the background metric for the dual QFT and $\tilde{g}_{(3)ab}$ the energy momentum tensor. We can perform the same algebraic re-writing of the Einstein field equations as we did for AdS in [52], where this time the necessary data to specify at the conformal boundary is $\{\tilde{g}_{(0)ab}, \tilde{g}_{(3)ab}\}$.

5.2 Asymptotic analysis and charge prescription

Much of this section can be read as an analytic continuation of the work in [71] in that we will follow closely the structure of that paper but with the suitable modifications to account for the change from ALAdS to ALdS spacetimes. We will begin with an introduction to our theory of interest and the coordinates for the ALdS spacetimes which we will study.

We will then give a number of our results which correspond to the ALdS equivalent of ALAdS quantities computed in [71]. We will give the Hamiltonian equations of motion, expressions for the momenta conjugate to the fields of the theory, an injectivity proof for the spaces of asymptotic conformal killing vectors and asymptotic bulk killing vectors, and finally the existence and value of the Wald Hamiltonians for the ALdS theory.

5.2.1 General setup

We will use the methods developed in [70, 71, 205] and apply a time Hamiltonian evolution to ALdS spacetimes, specifically we will consider a theory with the following Lagrangian D-form

$$\mathbf{L} = \left(\frac{1}{2\kappa^2}(R - 2\Lambda) - V(\Phi)\right) \star \mathbf{1} - \frac{1}{2}G_{IJ}(\Phi)d\Phi^I \wedge \star d\Phi^J - \frac{1}{2}U(\Phi)\mathbf{F} \wedge \star\mathbf{F} \tag{5.2.1}$$

which is that same as that of [71]. The only difference in the way that we write the Lagrangian is that we have now separated the Λ (cosmological constant) term from that of the scalar potential $V(\Phi)$ in order to show explicitly that we are considering a theory with $\Lambda \neq 0$. As in [71], the Φ^I are scalars and $G_{IJ}(\Phi)$ acts as the metric over the space of the scalars. $\mathbf{F} = d\mathbf{A}$ is the field strength spacetime 2-form and $U(\Phi)$ couples this field to the scalars.

We will look for solutions to the equations of motion for this theory where the spacetime metric g_{ab} admits ALdS solutions and thus we will consider solutions where $\Lambda < 0$. We want to choose coordinates which are most suitable to be used in a neighbourhood of the conformal boundary $\mathscr{I}^+$. It is well known from [59] (see also an appendix of [62]) that in a neighbourhood of the conformal boundary, one can write the metric of an ALdS spacetime in the Fefferman-Graham coordinate system

$$ds^2 = l_{dS}^2 \left(-\frac{d\rho^2}{\rho^2} + \frac{g_{ij}(x,\rho)}{\rho^2}dx^i dx^j\right) \tag{5.2.2}$$

where l_{dS} is the dS radius of the spacetime, given by

$$l_{dS}^2 = \frac{(D-1)(D-2)}{2\Lambda} \tag{5.2.3}$$

and $\rho = 0$ is the location of the conformal boundary. As ALdS spacetimes have two boundaries (future spacelike infinity, $\mathscr{I}^+$, and past spacelike infinity, $\mathscr{I}^-$) one will need two sets of such coordinates, one to cover the neighbourhood of each boundary. We will fix l_{dS} (and Λ via (5.2.3)) by choosing the normalisation $l_{dS} = 1$ (the length factors can always be reinstated via dimensional analysis) and consider the coordinate transformation

$$\rho = e^{-t} \tag{5.2.4}$$

where $t = \infty$ corresponds to $\mathscr{I}^+$, the boundary which we will restrict our attention to. This transformation brings the line element into the form

$$ds^2 = -dt^2 + \gamma_{ij}(x,t)dx^i dx^j \tag{5.2.5}$$

where $\gamma_{ij}(x,t) = e^{2t} g_{ij}(x, e^{-t})$ and $\gamma_{ij}(x, t_0)$ is the induced metric on the hypersurface Σ_{t_0} of constant time t_0. This is the line element for which we now want to perform a time evolution Hamiltonian analysis on, a contrast to [71] where a similar analysis was performed but for a radial evolution in ALAdS spacetime.

We return to our theory (5.2.1) and write down the equations of motion by considering variations of the Lagrangian. The general form of such variations takes the form given by (2.4.3), (2.4.4) which for our theory is

$$\delta \mathbf{L} = \mathbf{E}^{ab}_{(1)} \delta g_{ab} + \mathbf{E}^{a}_{(2)} \delta A_a + \mathbf{E}^{(3)}_I \delta \Phi^I + d\mathbf{\Theta}(\psi, \delta\psi). \tag{5.2.6}$$

As in [71] the forms are given by

$$\begin{aligned}
\mathbf{E}^{ab}_{(1)} &= -\frac{1}{2\kappa^2}\left(R^{ab} - \frac{1}{2}Rg^{ab} + \Lambda g^{ab} - \kappa^2 \tilde{T}^{ab}\right) \star \mathbf{1} \\
\mathbf{E}^{a}_{(2)} &= \nabla_b (U(\Phi) F^{ba}) \star \mathbf{1} \\
\mathbf{E}^{(3)}_I &= \left(\nabla^a (G_{IJ}(\Phi)\partial_a \Phi^J) - \frac{1}{2}\frac{\partial G_{JK}}{\partial \Phi^I}\partial_a \Phi^J \partial^a \Phi^K - \frac{\partial V}{\partial \Phi^I} - \frac{1}{4}\frac{\partial U}{\partial \Phi^I} F_{ab}F^{ab}\right)
\end{aligned} \tag{5.2.7}$$

where again we have the matter stress energy tensor given by

$$\tilde{T}_{ab} = G_{IJ}(\Phi)\partial_a \Phi^I \partial_b \Phi^J + U(\Phi) F_{ac} F_b{}^c - g_{ab}\mathcal{L}_m \tag{5.2.8}$$

where $\mathcal{L}_m$ accounts for the the matter part of the Lagrangian. Explicitly, we have

$$\mathcal{L}_m = V(\Phi) + \frac{1}{2}G_{IJ}(\Phi)\partial^a \Phi^J \partial_a \Phi^I + \frac{1}{4}U(\Phi)F^{ab}F_{ab}. \tag{5.2.9}$$

These results are all equivalent to those of [71] although we have highlighted the Λ term as a method of explicitly keeping track of the cosmological constant. Also computed in [71] was the symplectic potential $(D-1)$-form, given by[2]

$$\mathbf{\Theta}(\psi, \delta\psi) = (-1)^D \star v(\psi, \delta\psi) \tag{5.2.10}$$

[2]This expression differs from that of [71] by the factor of $(-1)^D$, due to a difference in convention regarding the definition of the Hodge dual. In this work we use the convention that for a p-form X, $(\star X)_{a_1 \ldots a_{D-p}} = \frac{1}{p!}\nu_{a_1 \ldots a_{D-p} b_1 \ldots b_p} X^{b_1 \ldots b_p}$

where

$$v^a = -\frac{1}{2\kappa^2}(g^{ab}\nabla^c\delta g_{bc} - g^{bc}\nabla^a\delta g_{bc}) + G_{IJ}(\Phi)\delta\Phi^I\nabla^a\Phi^J + U(\Phi)F^{ab}\delta A_b \qquad (5.2.11)$$

5.2.2 Equations of motion

We will perform the asymptotic analysis in the same style as [71], namely by applying a Hamiltonian evolution. The key difference between the analysis in ALdS and ALAdS is in the character of the evolution, i.e. time vs radial evolution.

In the ALdS case we evolve with respect to time and thus we can consider the equations of motion as equations relating data on a spacelike constant t hypersurface, Σ_{t_i}, to data on the next constant t hypersurface, $\Sigma_{t_{i+1}}$. This is the more familiar Hamiltonian evolution from the perspective of classical relativity [77], but differs from the standard technique in ALAdS spacetimes [70, 71] which evolves data from one timelike hypersurface of constant radius, Σ_{r_i}, to the next, $\Sigma_{r_{i+1}}$ [141, 206, 207]. The important mathematical difference between these approaches is that the induced metric, γ_{ab}, on the Σ hypersurfaces is given by

$$\gamma_{ab} = g_{ab} \pm n_a n_b \qquad (5.2.12)$$

where the + corresponds to time evolution (ALdS) and the $-$ to radial evolution (ALdS). In both cases n^a is an outward pointing unit normal vector to Σ so we have $n^a n_a = \mp 1$ and the sign in (5.2.12) is chosen such that $\gamma_{ab}n^a = 0$.

In order to implement the time evolution we will use the Gauss-Codacci equations to write the equations of motion (5.2.7) in terms of quantities defined on the hypersurfaces Σ_t. We will use the gauge fixing that we introduced in (5.2.5) for the gravity, thus inducing the metric γ_{ij} on the Σ_t hypersurfaces. We also follow [71] in choosing the gauge for the gauge field as $A_t = 0$ and we will now give the equations of motion for the gravity, γ_{ij}, the gauge field, A_i, and the scalars, Φ^I

Gravity:

$$\begin{aligned} K_{ij}K^{ij} - K^2 &= \hat{R} - 2\kappa^2\tilde{T}_{tt} - 2\Lambda \\ D_iK^i_j - D_jK &= \kappa^2\tilde{T}_{jt} \\ -\dot{K}^i_j - KK^i_j &= \hat{R}^i_j - \kappa^2\left(\tilde{T}^i_j + \frac{1}{1-d}\tilde{T}^\alpha_\alpha\delta^i_j\right) + \delta^i_j\left(\frac{2\Lambda}{1-d}\right) \end{aligned} \qquad (5.2.13)$$

where $K_{ij} = \frac{1}{2}\mathcal{L}_n\gamma_{ij} = \frac{1}{2}\frac{d}{dt}\gamma_{ij} = \frac{1}{2}\dot{\gamma}_{ij}$ and $\hat{R}$ are respectively the extrinsic curvature and scalar curvature of the hypersurfaces Σ_t. Note that these formulae differ in certain signs when compared with the Gauss-Codacci equations in [71], this is entirely due to the change in Hamiltonian evolution as described by equation (5.2.12). In order to give the equations for the gauge field and scalars we first present the Christoffel symbols of the spacetime metric

$$\Gamma^t_{ij} = K_{ij}, \qquad \Gamma^i_{tj} = K^i_j, \qquad \Gamma^i_{jk}[g] = \Gamma^i_{jk}[\gamma] \tag{5.2.14}$$

with all other symbols vanishing. We again note that Γ^t_{ij} has the opposite sign to [71].

Gauge field:

$$\begin{aligned} &D_i(U(\Phi)F^{ti}) = 0 \\ &\partial_t(U(\Phi)F^{tj}) + KU(\Phi)F^{tj} + D_i(U(\Phi)F^{ij}) = 0 \end{aligned} \tag{5.2.15}$$

where D_i is the covariant derivative operator associated with the hypersurface Σ_t (these equations agree with [71]).

Scalar:

$$\begin{aligned} &-\partial_t(G_{IJ}(\Phi)\dot{\Phi}^J) - KG_{IJ}(\Phi)\dot{\Phi}^J + D^i(G_{IJ}(\Phi)\partial_i\Phi^J) + \\ &\frac{1}{2}\frac{\partial G_{JK}}{\partial \Phi^I}(\dot{\Phi}^J\dot{\Phi}^K - \partial_i\Phi^J\partial^i\Phi^K) - \frac{\partial V}{\partial \Phi^I} + \frac{1}{4}\frac{\partial U}{\partial \Phi^I}(2\gamma^{ij}\dot{A}_i\dot{A}_j - F_{ij}F^{ij}) = 0. \end{aligned} \tag{5.2.16}$$

This equation specifically uses the gauge fixing condition of $A_t = 0$ and again takes a very similar form to [71], with suitable sign changes which account for the new evolution scheme.

5.2.3 Momenta

Now that we have computed the equations of motion in terms of quantities defined on the hypersurfaces Σ_t, we note that these can equally be written as Hamiltonian equations of motion for the 'time canonical momenta'. These momenta are defined by the usual equations

$$\pi_{ij} = \frac{\partial L}{\partial \dot{\gamma}^{ij}}, \qquad \pi_i = \frac{\partial L}{\partial \dot{A}^i}, \qquad \pi_I = \frac{\partial L}{\partial \dot{\Phi}^I} \tag{5.2.17}$$

where $L = \mathbf{L}_{01\ldots(D-1)}$. It is a straightforward calculation to take our Lagrangian (5.2.1) and compute these momenta

$$\pi_{ij} = \frac{1}{2\kappa^2}\sqrt{\gamma}(K_{ij} - K\gamma_{ij}), \quad \pi_i = \sqrt{\gamma}U(\Phi)\dot{A}^i, \quad \pi_I = \sqrt{\gamma}G_{IJ}(\Phi)\dot{\Phi}^J \tag{5.2.18}$$

which can now be reinserted into the equations of motion (5.2.13)-(5.2.16) in order to write the equations of motion in the explicit Hamiltonian form

Gravity:

$$\dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{ij}}, \qquad \dot{\pi}_{ij} = -\frac{\delta H}{\delta \gamma^{ij}} \tag{5.2.19}$$

(note that this is not all of (5.2.13) as the first two equations of (5.2.13) are constraint equations arising from the variation of H with respect to the lapse and shift).

Gauge field:

$$\dot{A}_i = \frac{\delta H}{\delta \pi^{i}}, \qquad \dot{\pi}_{i} = -\frac{\delta H}{\delta A^{i}} \tag{5.2.20}$$

Scalar:

$$\dot{\Phi}_I = \frac{\delta H}{\delta \pi^{I}}, \qquad \dot{\pi}_{I} = -\frac{\delta H}{\delta \Phi^{I}} \tag{5.2.21}$$

where the Hamiltonian H is given by

$$H = \int_{\Sigma_t} \left(\pi^{ij}\dot{\gamma}_{ij} + \pi^i \dot{A}_i + \sum_I \pi^I \dot{\Phi}_I - L \right). \tag{5.2.22}$$

For now, this use of Hamiltonian language is simply an alternative method of writing the equations of motion for the fields. We will return later to look at the Hamiltonian of timelike hypersurfaces using the covariant phase space language that we introduced in section 2.4.

5.2.4 Asymptotic symmetries

In [71], one of the significant steps in defining the Wald Hamiltonians in ALAdS spacetime was to prove injectivity between asymptotic conformal Killing vectors of the spacetime and asymptotic bulk killing vectors of the spacetime. We will now recreate the main steps of this proof for ALdS spacetime, showing that the main features of the proof carry over, with some small differences which we will note.

First we give asymptotic scaling properties of the fields from [71] but adapted to ALdS spacetimes. The key fact to note is that the linearised supergravity equations of motion admit two linearly independent solutions, the *normalisable* and *non-normalisable* modes, given by $e^{-s_+ t}$ and $e^{-s_- t}$ respectively. The values of $s_\pm$ for each field are as follows

$$\begin{aligned} \gamma_{ij} :& \quad s_+ = D-3, \quad s_- = -2 \\ A_i :& \quad s_+ = D-3, \quad s_- = 0 \\ \Phi^I :& \quad s_+ = \Delta_I, \quad s_- = D-1-\Delta_I \end{aligned} \tag{5.2.23}$$

Where Δ_I can be thought of as the analytic continuation of the scaling dimension a corresponding scalar field in AdS. Explicitly

$$\Delta_I = \frac{D-1}{2} + \sqrt{\frac{(D-1)^2}{4} - m_I^2 l_{dS}^2} \tag{5.2.24}$$

where m_I is the mass of the scalar field Φ_I. This result follows from the usual analytic continuation $l_{dS}^2 = -l_{AdS}^2$ (for a review of the scalar in an AdS background, see e.g. [208]). We note that this function is only real when the mass lies in the range

$$-\frac{(D-1)}{2l_{dS}} \leq m_I \leq \frac{(D-1)}{2l_{dS}} \tag{5.2.25}$$

which we will enforce in order to ensure reality of Δ_I. We also note that (5.2.24) ensures that $s_+ \geq s_-$ for the case of the scalar fields Φ^I. The falloff conditions given in (5.2.23) will be used in the result we are about to prove.

Definition: In ALdS spacetime, an *asymptotic conformal Killing vector*, ξ, is a bulk vector field which satisfies the following properties

$$\xi^t = \mathcal{O}(e^{(1-D)t}), \qquad \xi^i(x,t) = \zeta^i(x)(1 + \mathcal{O}(e^{-(D+1)t})) \tag{5.2.26}$$

where ζ^i is a conformal Killing vector of the metric induced at the conformal boundary $\mathscr{I}^+$, $g_{(0)ij} = \lim_{t\to\infty}(e^{-2t}\gamma_{ij})$.

Theorem: If ξ is an asymptotic conformal killing vector, then it is in 1-1 correspondence with an asymptotic bulk killing vector. i.e. there exists vector fields $\hat{\xi}, \hat{\alpha}$ such that

$$\mathcal{L}_{\xi-\hat{\xi}}\psi = \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+ t}) \tag{5.2.27}$$

which is precisely the equation for an asymptotic bulk killing vector, up to the gauge transformation given by $\delta_{\hat{\alpha}}$. We can use the linearity of the Lie derivative to rewrite this equation as

$$\mathcal{L}_{\xi}\psi = \mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi + \mathcal{O}(e^{-s_+ t}). \tag{5.2.28}$$

Proof: In order to prove this, we will attempt to find the values of the vectors $\hat{\xi}$ and $\hat{\alpha}$ which satisfy the equation above. We follow the guidance of [71] by noting that both $\mathcal{L}_{\xi}\psi$ and $\mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi$ will solve the linearised equations of motion as they are on shell perturbations to solutions to the background field equations. This means that we can invoke the asymptotic scaling properties as given in the paragraph above equation (5.2.23) and note that they must be formed of a linear combination of normalisable and non-normalisable modes. As (5.2.28) only requires equality up to the normalisable mode, all we will need to show is that the non-normalisable mode terms (i.e. the leading order) between the $\mathcal{L}_\xi \psi$ and $\mathcal{L}_{\hat{\xi}} \psi + \delta_{\hat{\alpha}} \psi$ agree.

To show equality of the leading order terms we work in the usual gauge

$$ds^2 = -dt^2 + \gamma_{ij}(t,x) dx^i dx^j, \qquad A_t = 0 \tag{5.2.29}$$

which allows us to compute the Lie derivatives of the various fields

$$\begin{aligned} \mathcal{L}_\xi g_{tt} &= -\dot{\xi}^t \\ \mathcal{L}_\xi g_{ti} &= \gamma_{ij}(\dot{\xi}^j - \partial^j \xi^t) \\ \mathcal{L}_\xi g_{ij} &= L_\xi \gamma_{ij} + 2K_{ij}\xi^t \end{aligned} \tag{5.2.30}$$

$$\begin{aligned} \mathcal{L}_\xi A_t &= A_i \dot{\xi}^i \\ \mathcal{L}_\xi A_i &= L_\xi A_i + \xi^t \dot{A}_i \end{aligned} \tag{5.2.31}$$

$$\mathcal{L}_\xi \Phi^I = L_\xi \Phi^I + \xi^t \dot{\Phi}^I \tag{5.2.32}$$

where L_ξ represents the Lie derivative in the direction of ξ^i (i.e. it is just the ordinary Lie derivative with ξ^t set to zero). We recall that ξ is an asymptotic conformal Killing vector and now use the first property of (5.2.26) together with the scaling properties of the fields in (5.2.23) in order to write these Lie derivatives together as

$$\mathcal{L}_\xi \psi = L_\xi \psi + \mathcal{O}(e^{-s_+ t}). \tag{5.2.33}$$

Which is a simple power counting exercise for all fields except the scalar which requires use of a slightly nuanced consideration. When we apply the power counting to (5.2.32) we find

$$\mathcal{L}_\xi \Phi^I = L_\xi \Phi^I + \xi^t \dot{\Phi}^I = L_\xi \Phi^I + \mathcal{O}(e^{(2-2D+\Delta_I)t}) \tag{5.2.34}$$

so for this field to satisfy (5.2.33) we need to satisfy the inequality

$$2 - 2D + \Delta_I \leq -\Delta_I \implies D - 1 \geq \Delta_I \tag{5.2.35}$$

and such a consideration of operator weights to those precisely restricts to those which are relevant or marginal [70]. This is something which is automatically enforced by the fact that Δ_I is real (see (5.2.24)) and hence equation (5.2.33) holds for all of the fields.

We now want to use the second condition of (5.2.26) to arrive at an expression for $L_\xi \psi$.

We write

$$L_\xi \psi = L_{\zeta(1+\mathcal{O}(e^{-(D+1)t}))}\psi = L_\zeta \psi (1+\mathcal{O}(e^{-(D+1)t})) \tag{5.2.36}$$

and we also recall that ζ is a boundary conformal killing vector. To illustrate how this acts on the fields we will consider the metric

$$\gamma_{ij} = e^{2t} g_{(0)ij} + \mathcal{O}(e^t) \tag{5.2.37}$$

and use the fact that ζ^i is a conformal Killing vector of $g_{(0)ij}$:

$$L_\zeta \gamma_{ij} = e^{2t}\frac{2}{D-1}(D_k\zeta^k) g_{(0)ij} + \mathcal{O}(e^t) = \frac{2}{D-1}(D_k\zeta^k)\partial_t \gamma_{ij}(1+\mathcal{O}(e^{-t})). \tag{5.2.38}$$

We will now show that one can identify the time derivative operator ∂_t with an operator which we will refer to as the 'time dilatation' operator δ_D. To see this, we consider the following variational chain rule

$$\partial_t = \int d^{D-1}x \left(\dot{\gamma}_{ij}\frac{\delta}{\delta \gamma_{ij}} + \dot{A}_i \frac{\delta}{\delta A_i} + \dot{\Phi}^I \frac{\delta}{\delta \Phi^I}\right) \tag{5.2.39}$$

and apply the asymptotic behaviour of the fields $(\gamma_{ij}, A_i, \Phi^I)$ as given in (5.2.23)

$$\partial_t \sim \int d^{D-1}x \left(2\gamma_{ij}\frac{\delta}{\delta \gamma_{ij}} + (\Delta_I - D + 1)\Phi^I \frac{\delta}{\delta \Phi^I}\right) = \delta_D \tag{5.2.40}$$

where we have used $\sim$ to indicate that we only kept the leading order terms in the asymptotic behaviours of the fields. Having provided the specific example of how $L_\zeta \gamma_{ij}$ is computed, we note that due to the fact that ζ is a conformal Killing vector of $g_{(0)}$

$$L_\zeta \psi = \frac{1}{D-1}(D_k\zeta^k)\delta_D \psi. \tag{5.2.41}$$

In order to complete this proof we now need to pick a vector field $\hat{\xi}$ and a gauge vector $\hat{\alpha}$ s.t.

$$\mathcal{L}_{\hat{\xi}}\psi + \delta_{\hat{\alpha}}\psi \sim \frac{1}{D-1}(D_k\zeta^k)\delta_D\psi \tag{5.2.42}$$

as we recall that we want to match the leading order terms in (5.2.28). As it turns out, such a transformation is readily available to us, namely the 'PBH transformation' [80, 146, 209, 71]. This transformation induces a Weyl transformation on the spacetime and in ALdS spacetime this takes the form

$$\begin{aligned} \hat{\xi}^t &= \delta\sigma(x) + \mathcal{O}(e^{-(D-1)t}) \\ \hat{\xi}^i &= -\partial_j \delta\sigma(x) \int_t^\infty dt' \gamma^{ji}(t',x) + \mathcal{O}(e^{-(D+1)t}) \\ \hat{\alpha} &= \partial_i \delta\sigma(x) \int_t^\infty dt' A^i(t',x) + \mathcal{O}(e^{-(D+1)t}) \end{aligned} \tag{5.2.43}$$

where we pick

$$\delta\sigma(x) = \frac{1}{D-1} D_i \xi^i \tag{5.2.44}$$

in order to match the leading order behaviour with (5.2.42). It is now a simple exercise to check this, using the identification of ∂_t and δ_D as outlined in (5.2.40)

We finish this section with a comment upon the falloff in the second condition of (5.2.26). In order to preserve the gauge given in equation (5.2.29) we must have $\mathcal{L}_\xi g_{ti} = 0$ from the middle equation in (5.2.30) this gives us

$$\dot{\xi}^j - \partial^j \xi^t = 0 \implies \dot{\xi}^j = \mathcal{O}(e^{-(D+1)t}) \tag{5.2.45}$$

where the fall-off condition simply arises from the power counting of the ξ^t falloff given in (5.2.26) as well as the scaling behaviour of the metric given in (5.2.23).

5.2.5 Wald Hamiltonians

In this subsection we will first discuss the differences in definition of the pre-symplectic form (and thus Wald Hamiltonians) in ALdS and ALAdS spacetimes, providing an alternative definition to that of [210]. We will then write down the Wald Hamiltonians H_α, H_ξ for ALdS spacetime corresponding to the symmetries of an asymptotically constant gauge transformation, α, and an asymptotic conformal Killing vector, ξ. We will show that for the asymptotic CKV Hamiltonian, the existence criterion (2.4.24) is equivalent to having a pre-symplectic form Ω_C which is independent of the slice C. As before many of these results follow from the ALAdS setup of [71] and we will comment upon their similarities.

5.2.5.1 Pre-symplectic form in AldS spacetime

Before explicitly writing down the Wald Hamiltonians for AldS spacetimes we will return to the pre-symplectic form, Ω_C, which we first defined for generic spacetime in (2.4.12). As was mentioned earlier, the "slice" of spacetime C is typically taken to be a Cauchy surface in the spacetime (or at least a spacelike hypersurface in the ALAdS case) but in the near-boundary region of ALdS spacetime it will make more sense to treat C as a timelike hypersurface due to the time-evolution Hamiltonian procedure that we have employed, as well as the spacelike character of $\mathscr{I}^+$. We will argue our case with an example taken from [71] which we modify for ALdS spacetime.

First we consider an asymptotic region as shown below.

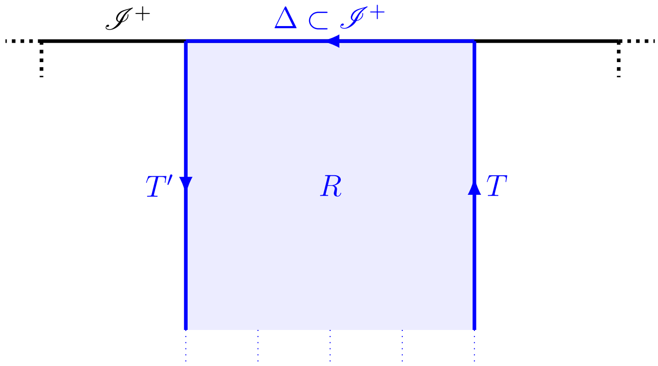

Figure5.2.1: Penrose diagram showing the asymptotic region R of an ALdS spacetime bounded by $\Delta \subset \mathscr{I}^+$ and two timelike hypersurfaces T and T'

We now consider $d\boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi)$ where ψ satisfies the equations of motion and $\delta_1\psi, \delta_2\psi$ satisfy the linearised equations of motion. As this quantity is identically zero by (2.4.11) obviously the integral of this quantity over the spacetime region R also vanishes

$$\int_R d\boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) = 0. \tag{5.2.46}$$

This seems like a trivial equation until we realise that we can apply Stokes' theorem to the integral to rewrite the equation as

$$\int_T \boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) - \int_{\Delta\subset\mathscr{I}^+} \boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) - \int_{T'} \boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) = 0 \tag{5.2.47}$$

which we see is

$$\Omega_T(\psi, \delta_1\psi, \delta_2\psi) - \Omega_{T'}(\psi, \delta_1\psi, \delta_2\psi) = \Omega_\Delta(\psi, \delta_1\psi, \delta_2\psi) \tag{5.2.48}$$

and thus we see that the pre-symplectic form is independent of the surface T iff Ω_Δ vanishes.

To compute Ω_Δ, we first need to compute the pullback of $\boldsymbol{\Theta}$ to a constant t hypersurface Σ_t ($\mathscr{I}^+$ can then be thought of as Σ_∞). This pull back is relatively straightforward as the map between the manifolds is simply the inclusion map $i : \Sigma_t \hookrightarrow \mathcal{M}$. This means that the pullback, $i^*\boldsymbol{\Theta}$, is given by

$$(i^*\boldsymbol{\Theta})_{i_1\ldots i_{D-1}} = \boldsymbol{\Theta}_{i_1\ldots i_{D-1}} \tag{5.2.49}$$

where the indices are now restricted to run over the spatial values only. We apply (5.2.10) and write

$$\boldsymbol{\Theta}_{i_1\ldots i_{D-1}} = -\nu_{ti_1\ldots i_{D-1}}v^t = -\tilde{\nu}_{i_1\ldots i_{D-1}}v^t = n_\mu v^\mu \tilde{\nu}_{i_1\ldots i_{D-1}} = n_\mu v^\mu (\star_{\Sigma_t}\mathbf{1}) \tag{5.2.50}$$

where $\tilde{\nu}$ and $\star_{\Sigma_t}$ are respectively the volume form and Hodge star of Σ_t. n^μ is the future-pointing unit normal to Σ_t and hence we have $n_\mu = (-1, 0, \ldots, 0)$.

We will now write down the pullback in terms of the canonical momenta that we derived in section 5.2.3. The first step in doing this is to consider the regulated on shell action (with Gibbons-Hawking term) as in [70, 71]. Note that the sign of L_{GHY} has been adapted in order to accommodate the spacelike character of the boundary Σ_{t_0}

$$S_{t_0} = \int_{\mathcal{M}_{t_0}} \mathbf{L}_{\text{on shell}} - \frac{1}{\kappa^2} \int_{\Sigma_{t_0}} K \star_{\Sigma_t} \mathbf{1} \tag{5.2.51}$$

now we of course have $\delta \mathbf{L}_{\text{on shell}} = d\boldsymbol{\Theta}$ so taking the variation of the above equation gives

$$\delta S_{t_0} = \int_{\mathcal{M}_{t_0}} d\boldsymbol{\Theta} - \frac{1}{\kappa^2} \int_{\Sigma_{t_0}} \delta(K \star_{\Sigma_t} \mathbf{1}). \tag{5.2.52}$$

At this point we will introduce an alternative definition of the time canonical momenta (see [70, 71]). To derive this condition we consider a generic action

$$\begin{aligned} S &= \int L(\phi, \dot{\phi})\, dt \\ \implies \delta S &= \int \left(\frac{\partial L}{\partial \phi^I} \delta \phi^I + \frac{\partial L}{\partial \dot{\phi}^I} \delta \dot{\phi}^I \right) dt \\ &= \int \left(\frac{\partial L}{\partial \phi^I} \delta \phi^I + \frac{\partial L}{\partial \dot{\phi}^I} \partial_t (\delta \phi^I) \right) dt \\ &= \int \left[\frac{\partial L}{\partial \phi^I} \delta \phi^I + \partial_t \left(\frac{\partial L}{\partial \dot{\phi}^I} \delta \phi^I \right) - \delta \phi^I \partial_t \left(\frac{\partial L}{\partial \dot{\phi}^I} \right) \right] \\ &= \left(\frac{\partial L}{\partial \dot{\phi}^I} \right) \delta \phi^I \bigg|^t + \int \delta \phi^I \left(\frac{\partial L}{\partial \phi^I} - \partial_t \left(\frac{\partial L}{\partial \dot{\phi}^I} \right) \right) \end{aligned} \tag{5.2.53}$$

but the second term on the right hand side is precisely the Euler-Lagrange equation, and thus on shell we find

$$\frac{\delta S_{\text{on shell}}}{\delta \phi^I} = \frac{\partial L}{\partial \dot{\phi}^I} = \pi_I. \tag{5.2.54}$$

This proof carries through identically for all fields (not just scalars) and thus we also use the following definition for the canonical momenta

$$\pi^{ij} = \frac{\delta S_{t_0}}{\delta \gamma_{ij}}, \qquad \pi_I = \frac{\delta S_{t_0}}{\delta \Phi^I}, \qquad \pi^i = \frac{\delta S_{t_0}}{\delta A_i} \tag{5.2.55}$$

and thus by the variational chain rule we have

$$\delta S_{t_0} = \int_{\Sigma_{t_0}} d^{D-1}x \left[\pi^{ij} \delta \gamma_{ij} + \pi_I \delta \Phi^I + \pi^i \delta A_i \right]. \tag{5.2.56}$$

These formulae allow us to rearrange (5.2.52) and apply Stokes theorem in order to write

$$\int_{\Sigma_{t_0}} \boldsymbol{\Theta} = \int_{\Sigma_{t_0}} d^{D-1}x \left[\pi^{ij}\delta\gamma_{ij} + \pi_I\delta\Phi^I + \pi^i\delta A_i\right] + \frac{1}{\kappa^2}\int_{\Sigma_{t_0}} \delta(K \star_{\Sigma_t} \mathbf{1}) \tag{5.2.57}$$

which gives

$$i^*\boldsymbol{\Theta} = \left[\pi^{ij}\delta\gamma_{ij} + \pi_I\delta\Phi^I + \pi^i\delta A_i + \frac{1}{\kappa^2}\delta(K\sqrt{\gamma})\right] d\mu \tag{5.2.58}$$

where we have followed [70, 71] in writing $\sqrt{\gamma}d\mu = \star_\Sigma\mathbf{1}$. As in [71], we remark that the form of the pullback gives us the expected property [146] that the Gibbons-Hawking term is all one needs to have a well-defined variational problem when the fields at the boundary are kept fixed

$$\delta\gamma_{ij} = 0, \qquad \delta A_i = 0, \qquad \delta\Phi^I = 0 \quad \text{on } \Sigma_{t_0} \tag{5.2.59}$$

However, these boundary conditions will only be suitable when the boundary Σ_{t_0} is located at a finite proper distance (i.e. t_0 finite). Since we want to look at the conformal boundary $\mathscr{I}^+$ which is located $t_0 = \infty$ (in contrast to a t_0 being finite for a physical boundary) of an ALAdS spacetime, we are restricted to the weaker boundary conditions

$$\delta\gamma_{ij} = 2\gamma_{ij}\delta\sigma, \qquad \delta A_i = 0, \qquad \delta\Phi^I = (\Delta_I - d)\Phi^I\delta\sigma \quad \text{on } \mathscr{I}^+ \tag{5.2.60}$$

which only determine to boundary values of the fields up to a Weyl transformation. As an extra remark, we note that comparison of (5.2.58) and (5.2.50) allows us to read off v^t, namely

$$v^t = -\frac{1}{\sqrt{\gamma}}\left[\pi^{ij}\delta\gamma_{ij} + \pi_I\delta\Phi^I + \pi^i\delta A_i + \frac{1}{\kappa^2}\delta(K\sqrt{\gamma})\right]. \tag{5.2.61}$$

Now that we have obtained the pullback of $\boldsymbol{\Theta}$ on Σ_t, we are easily able to obtain the pullback of the symplectic current, $\boldsymbol{\omega}$, using the definition (2.4.7). Explicitly we have

$$\begin{aligned} i^*\boldsymbol{\omega}(\psi, \delta_1\psi, \delta_2\psi) &= \delta_1(i^*\boldsymbol{\Theta}(\psi, \delta_2\psi)) - (1 \leftrightarrow 2) \\ &= \{\delta_1\pi^{ij}\delta_2\gamma_{ij} + \delta_1\pi_I\delta_2\Phi^I + \delta_1\pi^i\delta_2 A_i - (1 \leftrightarrow 2)\} \end{aligned} \tag{5.2.62}$$

where in the second line we have used the fact that the field variations commute: $\delta_1\delta_2 - \delta_2\delta_1 = 0$. We will now focus on the momenta and following [71], expand them in eigenfunctions of the dilatation operator δ_D

$$\begin{aligned} \pi^{ij} &= \sqrt{\gamma}(\pi^{ij}_{(0)} + \pi^{ij}_{(2)} + \ldots + \pi^{ij}_{(d)} + \tilde{\pi}^{ij}_{(d)} \log e^{-2t} + \ldots), \\ \pi^i &= \sqrt{\gamma}(\pi^i_{(3)} + \pi^i_{(4)} + \ldots + \pi^i_{(d)} + \tilde{\pi}^i_{(d)} \log e^{-2t} + \ldots), \\ \pi^I &= \sqrt{\gamma}(\sum_{d-\Delta_I \le s < \Delta_I} \pi_{(s)I} + \pi_{(\Delta_I)} + \tilde{\pi}_{(\Delta_I)I} \log e^{-2t} + \ldots) \end{aligned} \tag{5.2.63}$$

where the non-normalisable terms in the expansions will turn out to be related to the counter term action required to remormalise the action [62]. It was shown explicitly in

[71] that these expansions allow us to write the pullback of the symplectic current form as

$$i^*\boldsymbol{\omega}(\psi,\delta_1\psi,\delta_2\psi) = \{\delta_1(\sqrt{\gamma}\pi^{ij}_{(d)})\delta_2\gamma_{ij} + \delta_1(\sqrt{\gamma}\pi_{(\Delta_I)I})\delta_2\Phi^I + \delta_1(\sqrt{\gamma}\pi^i_{(d)})\delta_2 A_i - (1\leftrightarrow 2)\}. \tag{5.2.64}$$

We can now substitute the boundary conditions (5.2.60) in order to write this as

$$\begin{aligned} i^*\boldsymbol{\omega}(\psi,\delta_1\psi,\delta_2\psi) &= \{\delta_1(\sqrt{\gamma}\pi^{ij}_{(d)})2\gamma_{ij}\delta_2\sigma + \delta_1(\sqrt{\gamma}\pi_{(\Delta_I)I})(\Delta_I - d)\Phi^I\delta_2\sigma - (1\leftrightarrow 2)\} \\ &= \{(\delta_1(\sqrt{\gamma}[2\pi^i_{i(d)} + \pi_{(\Delta_I)I}(\Delta_I - d)\Phi^I])\delta_2\sigma - (1\leftrightarrow 2)\} \end{aligned} \tag{5.2.65}$$

which at first looks like an unremarkable rearrangement until one uses the result for the trace anomaly, $\mathcal{A}$ [58, 71]

$$\mathcal{A} = 2\pi^i_{i(d)} + \pi_{(\Delta_I)I}(\Delta_I - d)\Phi^I \tag{5.2.66}$$

which allows us to write the symplectic current as

$$i^*\boldsymbol{\omega}(\psi,\delta_1\psi,\delta_2\psi) = \{\delta_1(\sqrt{\gamma}\mathcal{A})\delta_2\sigma - (1\leftrightarrow 2)\} \tag{5.2.67}$$

and thus we observe that if the trace anomaly vanishes then so does the pullback of the symplectic current form to the conformal boundary $\mathscr{I}^+$. This in turn implies that $\Omega_\Delta = 0$ and finally gives us the desired result that the presymplectic 2-form is independent of the 'slice' upon which it is integrated over. If the trace anomaly is non-vanishing, then one has to impose the stronger boundary conditions (5.2.59) which again enforce the independence of Ω_C upon C.

5.2.5.2 Computation of H_α

We will now explicitly compute the Hamiltonians on the covariant phase space [158, 211, 63, 67] for the theory (5.2.1) and show that their existence is guaranteed by the fact that the pre-symplectic 2-form, Ω_C, is independent of the slice C in ALdS spacetime. We will begin by computing H_α, the Hamiltonian associated with asymptotically constant gauge transformations (without loss of generality, we will take $\alpha \to 1$ asymptotically). These gauge transformations also satisfy

$$\delta_\alpha\psi = \mathcal{O}(e^{-s_+ t}) \tag{5.2.68}$$

where s_+ is given in equation (5.2.23). The $U(1)$ current associated with these gauge transformations is given by

$$\mathbf{J}_\alpha = \boldsymbol{\Theta}(\psi,\delta_\alpha\psi) \tag{5.2.69}$$

which is closed on shell and thus we can locally write

$$d\mathbf{Q}_\alpha = \mathbf{J}_\alpha = \mathbf{\Theta}(\psi, \delta_\alpha\psi) \tag{5.2.70}$$

where $\mathbf{Q}_\alpha$ is the $U(1)$ Noether charge $(D-2)$-form, which integrates over a $(D-2)$ manifold in order to give the Noether charge.

In order to associate the Hamiltonian with the Noether charges, we recall equation (2.4.19) which gave the definition of the Hamiltonian, and adapt it suitably for the asymptotically constant gauge transformation α

$$\delta H_\alpha = \Omega_C(\psi, \delta\psi, \delta_\alpha\psi) = \int_C \omega(\psi, \delta\psi, \delta_\alpha\psi) \tag{5.2.71}$$

where as before we are restricting our consideration to on shell fields ψ ($\mathbf{E} = 0$). In order to explicitly compute H_α, we first write

$$\omega(\psi, \delta\psi, \delta_\alpha\psi) = \delta\mathbf{\Theta}(\psi, \delta_\alpha\psi) - \delta_\alpha\mathbf{\Theta}(\psi, \delta\psi) \tag{5.2.72}$$

and then we can use the gauge invariance of the Lagrangian ($\delta_\alpha\mathbf{L} = 0$) in order to conclude that $\mathbf{\Theta}$ is also gauge invariant ($\delta_\alpha\mathbf{\Theta} = 0$). This means that the second term on the right hand side vanishes and on shell we can write

$$\omega(\psi, \delta\psi, \delta_\alpha\psi) = \delta\mathbf{\Theta}(\psi, \delta_\alpha\psi) = d\delta\mathbf{Q}_\alpha \tag{5.2.73}$$

and thus we have

$$\delta H_\alpha = \int_C d\delta\mathbf{Q}_\alpha = \int_{C\cap\mathscr{I}^+} \delta\mathbf{Q}_\alpha \tag{5.2.74}$$

and so up to addition of a constant we have

$$H_\alpha = \int_{C\cap\mathscr{I}^+} \mathbf{Q}_\alpha \tag{5.2.75}$$

which is precisely the Noether charge.

We can derive the explicit expression for the Noether charge using the equations (5.2.10) and (5.2.11) as well as identities for differential forms. We start by noticing

$$v_a(\psi, \delta_\alpha\psi) = U(\Phi)F_{ab}\nabla^b\alpha \tag{5.2.76}$$

and then applying the equation of motion $\nabla_b(U(\Phi)F^{ba}) = 0$ gives

$$v_a(\psi, \delta_\alpha\psi) = \nabla^b(U(\Phi)F_{ab}\alpha) = -(\star\, d \star (U(\Phi)F\alpha))_a \tag{5.2.77}$$

where we have used the identity $(\star\, d \star X)_a = -\nabla^b X_{ab}$ for any two-form X. We now find

$$(-1)^{D-1}\boldsymbol{\Theta} = -\star v = \star(\star\, d \star (U(\Phi)F\alpha)) = -(-1)^{D-1} d \star (U(\Phi)F\alpha) \tag{5.2.78}$$

where again we have used an identity for differential forms, this time $\star(\star Y) = -(-1)^{p(D-p)}Y$ for any p form on a D-dimensional Lorentzian manifold. With these rearrangements performed, we conclude that the Noether charge form is given by

$$\mathbf{Q}_\alpha = -\alpha \star \mathcal{F} \tag{5.2.79}$$

where $\mathcal{F}_{ab} = U(\Phi)F_{ab}$.

5.2.5.3 Computation and existence of H_ξ

We will now turn our attention to the more interesting Wald Hamiltonian, namely that of H_ξ, the Hamiltonian associated with diffeomorphisms induced by asymptotically conformal Killing vectors ξ (as defined in (5.2.26)). As before, Hamilton's equations of motion read

$$\delta H_\xi = \Omega_C(\psi, \delta\psi, \mathcal{L}_\xi \psi) = \int_{C\cap\mathscr{I}^+} (\delta \mathbf{Q}[\xi] - i_\xi \boldsymbol{\Theta}) \tag{5.2.80}$$

where we have used the identity (2.4.18) which we proved earlier. We will derive a necessary and sufficient condition for existence of H_ξ by considering the integrability condition described in (2.4.24)

$$\int_{C\cap\mathscr{I}^+} i_\xi \boldsymbol{\omega}(\psi, \delta_2\psi, \delta_1\psi) = 0 \tag{5.2.81}$$

now recall that ξ is asymptotically tangent to the constant time hypersurfaces and thus we have $\xi^t|_{\mathscr{I}^+} = 0$. We can expand the integrand as

$$i_\xi \omega\big|_{\mathscr{I}^+} = \xi^a \omega_{a\mu_1\ldots\mu_{D-2}}\big|_{\mathscr{I}^+} = \xi^r \omega_{r\mu_1\ldots\mu_{D-2}} + \xi^i \omega_{i\mu_1\ldots\mu_{D-2}} \tag{5.2.82}$$

and pulling this form back to the constant r slice C gives

$$i_\xi \omega\big|_{C\cap\mathscr{I}^+} = \xi^r \omega_{r\mu_1\ldots\mu_{D-2}} \tag{5.2.83}$$

where all of the terms on the far right of equation (5.2.82) vanish as they will all necessarily have two of their antisymmetric form indices taking the same value. Using the explicit form of the pullback of a tensor allows us to write the integrability condition as

$$\int_{C\cap\mathscr{I}^+} d^{D-2}x\, \xi^r \{\delta_1(\sqrt{\gamma}\mathcal{A})\delta_2\sigma - 1 \leftrightarrow 2\} = 0 \tag{5.2.84}$$

and we see that the condition is satisfied in two possible cases:

1. The trace anomaly vanishes: $\mathcal{A} = 0$

2. ξ is an asymptotic Killing vector: $\delta\sigma = 0$

We note that these are the very same conditions which were required for Ω_C to be independent of the slice C. We conclude, as in the ALAdS case, that the existence of a H_ξ is equivalent to Ω_C being independent of C.

As this integrability condition is both necessary and sufficient for existence of H_ξ, we are able to write the Hamiltonian as

$$H_\xi = \int_{C\cap\mathscr{I}^+} (\mathbf{Q}[\xi] - i_\xi \mathbf{B}) \tag{5.2.85}$$

where

$$\int_{C\cap\mathscr{I}^+} i_\xi \mathbf{\Theta} = \delta \int_{C\cap\mathscr{I}^+} i_\xi \mathbf{B}. \tag{5.2.86}$$

All that now remains is to explicitly compute $\mathbf{Q}[\xi]$ and $\mathbf{B}$ for the class of theories given by (5.2.1) which we will do now.

First we compute $\mathbf{Q}[\xi]$ using $d\mathbf{Q}[\xi] = \mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) - i_\xi\mathbf{L}$ and we use a similar method to that which we applied for the $U(1)$ charge: First we want to compute $\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi)$ which will involve analysis of the vector field $v_a(\psi, \mathcal{L}_\xi\psi)$ which was first given in equation (5.2.10). After some rearrangement and applying the equations of motion (5.2.7) we obtain

$$\begin{aligned} v_a(\psi, \mathcal{L}_\xi\psi) = -\frac{1}{2\kappa^2}\big[-2\nabla^b\nabla_{[a}\xi_{b]} + (R-2\Lambda)\xi_a + 2\kappa^2\tilde{T}_a^b\xi_b\big] + \\ G_{IJ}(\Phi)\xi^b\nabla_b\Phi^I\nabla_a\Phi^J + U(\Phi)F_{ab}F^{cb}\xi_c + \nabla^b(U(\Phi)F_{ab}A_c\xi^c) \end{aligned} \tag{5.2.87}$$

which is useful but not quite in the required form for us to extract $\mathbf{\Theta}$ from. In order to compute $\mathbf{\Theta}$ we first introduce the two form $\boldsymbol{E}$ which satisfies $E_{ab} = \nabla_{[a}\xi_{b]}$ and use equation (5.2.87) to write

$$\begin{aligned} \frac{\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi)}{(-1)^{D-1}} = -\star v(\psi, \mathcal{L}_\xi\psi) = & \frac{1}{2\kappa^2}\big[2(-1)^D(d\star\boldsymbol{E}) + \star((R-2\Lambda)\xi_a + 2\kappa^2\tilde{T}_a^b\xi_b)\big] \\ & - G_{IJ}(\nabla^a\Phi^I)\xi_a(i_{\nabla\Phi^J}\nu) + (-1)^D d\star(U(\Phi)\boldsymbol{F}A_a\xi^a) \\ & - \star(U(\Phi)F_{ab}F^{cb}\xi_c) \end{aligned} \tag{5.2.88}$$

which upon applying equation for (5.2.8) for $\tilde{T}_a^b$ becomes

$$\mathbf{\Theta}(\psi, \mathcal{L}_\xi\psi) = -\frac{1}{\kappa^2}d\star(\boldsymbol{E} + \kappa^2 U(\Phi)\boldsymbol{F}A_a\xi^a) + i_\xi\boldsymbol{L} \tag{5.2.89}$$

and so we can conclude

$$\mathbf{Q}[\xi] = -\frac{1}{\kappa^2}\star\mathbf{\Xi}[\xi] \tag{5.2.90}$$

where $\boldsymbol{\Xi}$ is a 2-form whose components are

$$\Xi_{ab} = \nabla_{[a}\xi_{b]} + \kappa^2 U(\Phi) F_{ab} A_c \xi^c \tag{5.2.91}$$

Computing $\mathbf{B}$ is in many ways more straightforward than $\mathbf{Q}[\xi]$. First we follow the steps of equations (5.2.81)-(5.2.83) in order to write

$$i_\xi \boldsymbol{\Theta}\big|_{C\cap\mathscr{I}^+} = \xi^r i^*(\boldsymbol{\Theta}_{r\mu_1\ldots\mu_{D-2}}) \tag{5.2.92}$$

where i^* denotes the pullback onto the hypersurface Σ_t. To explicitly evaluate this term, we follow [71] and write the pullback of the symplectic potential form on Σ_t as

$$\begin{aligned} i^*\boldsymbol{\Theta} = \Big\{ \delta\left(\frac{1}{\kappa^2}\sqrt{\gamma}[K - (K+\lambda)_{\text{ct}}]\right) + \\ \sqrt{\gamma}(\pi^{ij}_{(d)}\delta\gamma_{ij} + \pi^i_{(d)}\delta A_i + \pi_{(\Delta_I)I}\delta\Psi^I) + \ldots\Big\} d\mu \end{aligned} \tag{5.2.93}$$

a procedure which involves substituting the mode expansion (5.2.63) for the momenta and then identifying the modes which contribute to the counterterms required to renormalise the theory. The differences in sign relative to [71] are purely due to the now spacelike character of $\mathscr{I}^+$, which suitably adjusts the signs of terms involving the extrinsic curvature K. In this expression, λ is defined as the Σ_t-covariant variable which satisfies

$$\int_{\mathcal{M}_{t_0}} \mathbf{L}_{\text{on-shell}} = -\frac{1}{\kappa^2}\int_{\Sigma_{t_0}} d^d x \sqrt{\gamma}\lambda. \tag{5.2.94}$$

which again admits an expansion in eigenfunctions of the dilatation operator

$$\lambda = \lambda_{(0)} + \lambda_{(2)} + \ldots + \lambda_{(d)} + \tilde{\lambda}_{(d)} \log e^{-2t} + \ldots \tag{5.2.95}$$

and the notation "$(f)_{\text{ct}}$" indicates the "counter term" part of the argument when expanded in dilatation weight, i.e.

$$(K+\lambda)_{\text{ct}} = \sum_{n=0}^{d-1}(K_{(n)} + \lambda_{(n)}) + (\tilde{K}_{(d)} + \tilde{\lambda}_{(d)}) \log e^{-2t_0} \tag{5.2.96}$$

To expression (5.2.93) we then apply the usual boundary conditions (5.2.60) which allows us to write

$$\begin{aligned} i^*\boldsymbol{\Theta} = \Big\{ \delta\left(\frac{1}{\kappa^2}\sqrt{\gamma}[K - (K+\lambda)_{\text{ct}}]\right) + \\ \sqrt{\gamma}(\pi^{ij}_{(d)}(2\gamma_{ij}\delta\sigma) + \pi_{(\Delta_I)I}(\Delta_I - d)\Psi^I\delta\sigma) + \ldots\Big\} d\mu. \end{aligned} \tag{5.2.97}$$

We now apply the formula (5.2.66) for the trace anomaly and find

$$i^*\mathbf{\Theta} = \left\{\delta\left(\frac{1}{\kappa^2}\sqrt{\gamma}[K-(K+\lambda)_{\text{ct}}]\right) + \sqrt{\gamma}\delta\sigma\mathcal{A} + \dots\right\}d\mu \tag{5.2.98}$$

which together with (5.2.92) allows us to write the overall integral as

$$\int_{C\cap\mathscr{I}^+} i_\xi\mathbf{\Theta} = \int_{C\cap\mathscr{I}^+} d\sigma_r\xi^r\mathcal{A}\delta\sigma + \frac{1}{\kappa^2}\delta\int_{C\cap\mathscr{I}^+} d\sigma_r\xi^r[K-(K+\lambda)_{\text{ct}}] \tag{5.2.99}$$

where the volume element $d\sigma_i$ is defined via $\xi^i d\sigma_i = \sqrt{\gamma}i_\xi\epsilon$. ϵ is an $(d-1)$-form on the hypersurface Σ_t, defined as to have orientation $\epsilon_{ri_2\dots i_{d-1}} = 1$. This choice of orientation is opposite to that of [71] and thus our integral expressions are modified accordingly.

Comparison of equations (5.2.86) and (5.2.99) shows us again that the Hamiltonian exists when either the trace anomaly vanishes or the vector ξ is an asymptotic Killing vector, in perfect agreement with our conclusion from earlier. This comparison also allows us to read off $\mathbf{B}$ as

$$\mathbf{B} = \frac{1}{\kappa^2}[K-(K+\lambda)_{\text{ct}}]\star_{\Sigma_t}\mathbf{1} \tag{5.2.100}$$

5.2.5.4 An explicit expression for H_ξ

Now that we have proven existence of H_ξ under suitable conditions, we will manipulate the generic formula of (5.2.85) in order to give us an expression from which we can easily start to compute charges (see section 5.3). We will do this for the case of pure gravity ($A_a = \Phi^I = 0$) in even spacetime dimension ($D \in 2\mathbb{Z}$) as this will be the case for the example we describe in detail in section 5.3. The reimplementation of the extra fields is straightforward and is covered in [71].

We start with the generic expression for the Wald Hamiltonians

$$\begin{aligned} H_\xi &= \int_{C\cap\mathscr{I}^+}(\mathbf{Q}[\xi] - i_\xi\mathbf{B}) \\ &= -\frac{1}{\kappa^2}\int_{C\cap\mathscr{I}^+}\left(\star\mathbf{\Xi}[\xi] + \left[\sum_{n=0}^{d}K_{(n)} - \sum_{n=0}^{d-1}(K_{(n)}+\lambda_{(n)})\right]i_\xi(\star_{\Sigma_t}\mathbf{1})\right) \end{aligned} \tag{5.2.101}$$

where in the second line we have used (5.2.90) and (5.2.100), as well as expanding the coefficient of $\mathbf{B}$ in terms of eigenfunctions of the time dilatation operator δ_D. Explicit evaluation of the first term (in the gauge (5.2.29)) gives

$$\star\mathbf{\Xi}[\xi] = \nu_{j_1\dots j_{d-1}ti}\Xi^{ti} = \sqrt{\gamma}\epsilon_{tij_1\dots j_{d-1}}\nabla^{[t}\xi^{i]} \tag{5.2.102}$$

now recall that ξ^i is an asymptotic conformal Killing vector and thus it satisfies (5.2.28), which in this case reduces to

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_{\hat{\xi}} g_{\mu\nu} + \mathcal{O}(e^{-s_+ t}). \tag{5.2.103}$$

where $\hat{\xi}$ is given in (5.2.43). Using this and the fact that ξ preserves the gauge (5.2.29), we can write

$$\begin{aligned} \nabla^{[t}\xi^{i]} &= \nabla^t \xi^i \\ &= g^{tt}(\partial_t \xi^i + \Gamma^i_{t\mu}\xi^\mu) \\ &= -(\dot{\xi}^i + \Gamma^i_{tj}\xi^j) \\ &= -K^i_j \xi^j + \mathcal{O}(e^{-(d+2)t}) \end{aligned} \tag{5.2.104}$$

which allows us to rewrite (5.2.101) as

$$H_\xi = -\frac{1}{\kappa^2}\int_{C\cap\mathscr{I}^+} d\sigma_i \left(-K^i_j \xi^j + \left[K_{(d)} - \sum_{n=0}^{d-1} \lambda_{(n)} \right] \xi^i \right) \tag{5.2.105}$$

where the $\mathcal{O}(e^{-(d+2)t})\sqrt{\gamma} \sim \mathcal{O}(e^{-2t})$ term gets removed as $t \to \infty$. We now recall the definition of the momenta in equation (5.2.18) as well as their expansion in eigenfunctions of the time dilatation operator (5.2.63), allowing us to write

$$\pi^{ij}_{(d)} = \frac{1}{2\kappa^2}\left(K^{ij}_{(d)} - K_{(d)}\gamma^{ij} \right) \tag{5.2.106}$$

and thus

$$H_\xi = 2\int_{C\cap\mathscr{I}^+} d\sigma_i \pi^i_{j(d)}\xi^j + \frac{1}{\kappa^2}\int_{C\cap\mathscr{I}^+} d\sigma_i \left[\sum_{n=0}^{d-1} (K^i_{j(n)} + \lambda_{(n)}\delta^i_j) \right] \xi^j \tag{5.2.107}$$

where we have now separated the expression for the Hamiltonian into two explicit terms, one with the integrand of dilatation weight d and the other of all weights $n < d$. We will now show that the second term is identically zero and thus only the highest order term contributes to the overall expression for the charge.

The first step is to recall the on-shell expression

$$d\mathbf{Q}[\xi] + i_\xi \mathbf{L}_{\text{on-shell}} = \mathbf{\Theta}(g, \mathcal{L}_\xi g) \tag{5.2.108}$$

which for our case of pure gravity becomes

$$-\frac{1}{\kappa^2} d\star\mathbf{\Xi}[\xi] + \left(\frac{4\Lambda}{d-1}\right) i_\xi \star \mathbf{1} = (-1)^D \star v(g, \mathcal{L}_\xi g) \tag{5.2.109}$$

and upon taking the Hodge star we obtain

$$\nabla_\mu \Xi^{\mu\nu} = \kappa^2 \xi^\nu \left(\frac{4\Lambda}{1-d}\right) - \kappa^2 v^\nu(g, \mathcal{L}_\xi g). \tag{5.2.110}$$

The next thing that we want to do is to find a way to introduce λ in the above equation, as this will bring us closer to the second term in (5.2.107). In order to do this, we recall the definition of λ as given in (5.2.94) and take the t-derivative of each side, giving us

$$\int_{\Sigma_{t_0}} d^d x \sqrt{\gamma} \left(\frac{4\Lambda}{d-1}\right) = -\frac{1}{\kappa^2} \int_{\Sigma_{t_0}} d^d x \sqrt{\gamma} \left(\dot{\lambda} + K\lambda\right) \tag{5.2.111}$$

where we used $\partial_t(\sqrt{\gamma}) = \sqrt{\gamma} K$. Comparing the integrands above and applying this to (5.2.110) gives

$$\nabla_\mu \Xi^{\mu\nu} = \xi^\nu \left(\dot{\lambda} + K\lambda\right) - \kappa^2 v^\nu(g, \mathcal{L}_\xi g). \tag{5.2.112}$$

Now we take the $\nu = i$ component of this equation and use the result that $\nabla_\mu \Xi^{\mu i} = \partial_\nu(\sqrt{-g}\Xi^{\nu i})/\sqrt{-g}$ (as $\Xi^{\mu\nu}$ is antisymmetric) to write

$$\frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}\Xi^{\mu i}) = \xi^i \frac{1}{\sqrt{\gamma}} \partial_t(\sqrt{\gamma}\lambda) - \kappa^2 v^i(g, \mathcal{L}_\xi g) \tag{5.2.113}$$

and of course using the gauge (5.2.29) we have $\sqrt{-g} = \sqrt{\gamma}$. This fact allows us to rearrange the expression above as

$$\partial_t[\sqrt{\gamma}(\Xi^{ti} - \xi^i \lambda)] = -\partial_j(\sqrt{\gamma}\Xi^{ij}) - \kappa^2 \sqrt{\gamma} v^i(g, \mathcal{L}_\xi g) + \mathcal{O}(e^{-2t}) \tag{5.2.114}$$

where we also used $\sqrt{\gamma}\lambda\partial_t \xi^i = \mathcal{O}(e^{-2t})$. Recalling (5.2.104), we can write this as

$$\partial_t\{\sqrt{\gamma}(-K^i_j - \lambda\delta^i_j)\xi^j\} = -\partial_j(\sqrt{\gamma}\Xi^{ij}) - \kappa^2 \sqrt{\gamma} v^i(g, \mathcal{L}_\xi g) + \mathcal{O}(e^{-2t}) \tag{5.2.115}$$

and now we note that the term under the LHS above is closely related to the integrand of the second term on the RHS of (5.2.107). We will shortly make this relation more precise but first we will show that v^r falls off as $\mathcal{O}(e^{-2t})$ as $t \to \infty$ (under mild symmetry assumptions).

First we have

$$v^r(g, \mathcal{L}_\xi g) = v^r(g, \mathcal{L}_{\hat{\xi}} g + \mathcal{O}(e^{-2t})) = v^r(g, \mathcal{L}_{\hat{\xi}} g) + \mathcal{O}(e^{-(d-2)t}) \tag{5.2.116}$$

by using (5.2.103) together with $s_+ = D-3 = d-2$ for the metric γ_{ij}. Using the explicit form of v as given in (5.2.11) and the gauge (5.2.29), we now find

$$v^r = -\frac{1}{\kappa^2}(\gamma^{rj}\gamma^{ki} - \gamma^{ri}\gamma^{jk})\nabla_i \nabla_{(j}\hat{\xi}_{k)} = -\frac{1}{\kappa^2}(\gamma^{rj}\gamma^{ki} - \gamma^{ri}\gamma^{jk}) D_i(D_{(j}\hat{\xi}_{k)} - 2K_{jk}\delta\sigma) \tag{5.2.117}$$

where we have used (5.2.14) and $\delta\sigma$ is given in (5.2.44). Now we use the second equation of motion for the metric in (5.2.13) in order to write

$$v^r = -\frac{1}{\kappa^2}(\gamma^{rj}\gamma^{ki} - \gamma^{ri}\gamma^{jk})(D_i D_{(j}\hat{\xi}_{k)} - 2K_{jk}D_i\delta\sigma). \tag{5.2.118}$$

The expression above will vanish if we take an assumption from [71] and modify it for AldS spacetime. The assumption is that our spacetime contains an asymptotic killing vector ∂_r as well as some other commuting isometries ∂_{ϕ^α} (hence the asymptotic metric is independent of $x^a = \{r, \phi^\alpha\}$) and thus we can write the asymptotic form of the metric as

$$ds^2 = \gamma_{ij}dx^i dx^j = \tau_{ab}(\tilde{x}, t)dx^a dx^b + \sigma_{\imath\jmath}(\tilde{x}, t)d\tilde{x}^\imath d\tilde{x}^\jmath. \tag{5.2.119}$$

This equation is important for a number of reasons: Since we want our conformal rescalings of the metric to preserve this generic form we must have $\delta\sigma(x) = \delta\sigma(\tilde{x})$, and thus $\hat{\xi}^\mu(x) = \hat{\xi}^\mu(\tilde{x})$, as well as $\xi^\imath = 0$ from (5.2.43). The Christoffel symbols and extrinsic curvature for the metric must satisfy

$$\Gamma^a_{bc} = \Gamma^a_{\imath\jmath} = \Gamma^\imath_{a\jmath} = K_{a\imath} = 0 \tag{5.2.120}$$

and applying all of these to (5.2.118), we see that the right hand side now vanishes and we are left with

$$v^r(g, \mathcal{L}_\xi g) = \mathcal{O}(e^{-(d-2)t}) \tag{5.2.121}$$

which gives us

$$\partial_t\{\sqrt{\gamma}(K^r_j + \lambda\delta^r_j)\xi^j\} = \partial_j(\sqrt{\gamma}\Xi^{rj}) \tag{5.2.122}$$

as we take the limit $t \to \infty$.

Finally, we have to use this equation to show that the second term on the right hand side of (5.2.107) vanishes. In order to do this, we will expand each side of the equation in eigenfunctions of δ_D, the time dilatation operator. We have already seen that the time derivative is very closely related to this operator (5.2.40) and following [70, 71] we can formally expand this operator as

$$\partial_t = \delta_D + \sum_{i=1} \delta_{(i)} \tag{5.2.123}$$

where $\delta_{(i)}$ are covariant operators of time dilatation weight i which satisfy $[\delta_D, \delta_{(i)}] = 0$. To shorten notation we will write $T^r = (K^r_j + \lambda\delta^r_j)\xi^j$ and using our previous expansion (5.2.95) as well as the expansion for K^i_j as given in [70] we have

$$T = T^r_{(0)} + T^r_{(2)} + \ldots + T^r_{(d)} + \ldots \tag{5.2.124}$$

as well as the generic expansion

$$\Xi^{ij} = \Xi^{ij}_{(2)} + \Xi^{ij}_{(3)} + \ldots \tag{5.2.125}$$

Using these expansions and matching terms of the same dilatation weight we find

$$
\begin{aligned}
T_{(0)} &= 0 \\
\sqrt{\gamma} T_{(2)} &= \frac{1}{d-2} \partial_j (\sqrt{\gamma} \Xi^{rj}_{(2)}) \\
\sqrt{\gamma} T_{(3)} &= \frac{1}{d-3} \partial_j \left(\sqrt{\gamma} \Xi^{rj}_{(3)} - \frac{1}{d-2} \delta_{(1)} \sqrt{\gamma} \Xi^{rj}_{(2)} \right) \\
&\vdots \\
\sqrt{\gamma} T_{(d-1)} &= \partial_j (\sqrt{\gamma} \Xi^{rj}_{(d-1)} + \ldots)
\end{aligned}
\tag{5.2.126}
$$

which is finally evidence that the second integral on the right hand side of (5.2.107) vanishes. We observe that the integrand there can be rewritten as $\sum_{n=0}^{d-1} T_{(n)}$ and thus every term is a total derivative which vanishes when integrated over $C \cap \mathscr{I}^+$.

This gives us a final expression for the charges in an AldS spacetime, namely

$$
H_\xi = 2 \int_{C \cap \mathscr{I}^+} d\sigma_i \pi^i_{j(d)} \xi^j \tag{5.2.127}
$$

and noting that $2\pi_{ij} = T_{ij}$ [70], we can alternatively write this equation as

$$
H_\xi = \int_{C \cap \mathscr{I}^+} d^{d-1}x \sqrt{g^{(0)}} g^{ri}_{(0)} T_{ij} \xi^j_{(0)} \tag{5.2.128}
$$

which makes apparent the extent to which the "locally dS" boundary conditions ($g_{(0)}$) enter the expression for the Hamiltonian, as well as providing an agreement with the recent work [51]. We will now make a number of comments upon this important result

- This expression can be considered to be the $\Lambda > 0$ version of the corresponding $\Lambda < 0$ formula in [71].

- This expression is finite, due to the cancellation of the potentially divergent terms in (5.2.107) under mild symmetry assumptions. Such a formula can also be obtained from a renormalisation at the level of the symplectic structure as in [51].

- As first observed in [71], these Hamiltonians are precisely the Noether charges of the theory. They are integrable due to our imposition of the boundary conditions (5.2.60) and they are conserved with respect to *radial* translations (as opposed to time translations in the $\Lambda < 0$ case).

- Due to the integrability of these charges, note that our Hamiltonians do not contain the "heat term", $\Xi_\xi[\delta\phi; \phi]$, in [51]. This is due to our boundary conditions (5.2.60) as well our choice of field independent asymptotic conformal Killing vectors, $\delta\xi_{(0)} = 0$.

In order to get a better understanding of these charges, we will now show that one can construct a variety of non-trivial charges in the simple case of linearised perturbations of dS_4.

5.3 An example: Charges of perturbed dS_4

5.3.1 General setup

In this section, we will consider an example of an ALdS spacetime and compute the Wald Hamiltonian corresponding to the asymptotic conformal Killing vectors of the spacetime in question. We will study the case of perturbed dS_4 ($D = 4$) spacetime in the inflationary patch coupled to a single minimally coupled scalar field, following the style of [73]. We choose the following ansatz for the metric and scalar

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \qquad \Phi = \varphi(t) \tag{5.3.1}$$

where the indices i, j run from 1 to 3. This metric ansatz is closely related to the picture of dS_4 in the inflationary patch. $a(t) = \exp(t/l_{dS})$ reproduces the metric for dS_4 in coordinates which cover half of the global spacetime [171]. These coordinates are related to the "big-bang" coordinates that we gave earlier in equation (5.1.7) by the coordinate transformation

$$\tilde{\rho} = -\exp\left(-\frac{t}{l_{dS}}\right), \quad x_1 \to \frac{x_1}{l_{dS}}, \quad x_2 \to \frac{x_2}{l_{dS}}, \quad \tilde{t} \to \frac{x_3}{l_{dS}} \tag{5.3.2}$$

and thus we see that they cover the same region of the global spacetime as displayed in figure 5.1.1.

More generally, we will consider the action

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}[(R - 2\Lambda) - 2\kappa^2 V(\Phi) - (\partial_a\Phi)(\partial^a\Phi)] \tag{5.3.3}$$

which is clearly an example of the class of theories described by (5.2.1), with the only (cosmetic) difference here being that the scalar field Φ is chosen to be dimensionless (instead of having mass dimension +1). If we first restrict to our ansatz (5.3.1), the equations of motion for the metric and the scalar field take the following form

$$\frac{\dot{a}}{a} = -\frac{1}{2}W; \qquad \dot{\varphi} = W_{,\varphi}; \qquad -2\kappa^2 V - 2\Lambda = (W_{,\varphi})^2 - \frac{3}{2}W^2, \tag{5.3.4}$$

where $W(\varphi)$ is a function known as the 'fake superpotential' [73]. The name *superpotential* comes from supergravity, namely in that if the theory (5.3.3) may be obtained from a consistent truncation of supergravity, such a supergravity theory will contain a superpotential, $W(\varphi)$, which determines the potential $V(\varphi)$ (via the third equation of (5.3.4) above). However, we cannot generally expect that the theory (5.3.3) is obtained from any consistent truncation of supergravity, and thus we refer to the superpotential $W(\varphi)$ as *fake*. For a greater discussion of fake supersymmetry, including applications to domain wall and cosmological solutions see [212, 213, 214]. These equations of motion can be derived directly via the variational principle (varying S with respect to g^{ab} and Φ) or we can read them off by comparison with (5.2.7).

In order to compute non trivial values for the Wald Hamiltonians, we want to apply linear perturbations to the background given by the solutions to these equations. We will follow the generic perturbation ansatz for cosmology [215] and write the linearly perturbed metric as

$$\begin{aligned} ds^2 = &-[1+2\phi(t,x^i)]dt^2 + 2a^2(t)[\partial_i\nu(t,x^j) + \nu_i(t,x^j)]dtdx^i + \\ &a^2(t)[\delta_{ij} - 2\psi(t,x^k)\delta_{ij} + 2\partial_i\partial_j\chi(t,x^k) + 2\partial_{(i}w_{j)}(t,x^k) + X_{ij}(t,x^k)]dx^idx^j \end{aligned} \tag{5.3.5}$$

and the linearly perturbed scalar

$$\Phi = \varphi + \delta\varphi(t,x^i) \tag{5.3.6}$$

This metric perturbation ansatz is the standard cosmological choice in that it is parametrised by four scalars: ϕ, ν, ψ, χ, two transverse vectors: ν_i, w_i, and a transverse-traceless rank-2 tensor: X_{ij}. When considering the equations of motion for the perturbations, it is helpful to express the perturbations as gauge invariant combinations as in [73]

$$\begin{aligned} \zeta &= \psi + \frac{H}{\dot\varphi}\delta\varphi \\ \hat\phi &= \phi - \left(\frac{\delta\varphi}{\dot\phi}\right)^{\cdot} \\ \hat\nu &= \nu - \dot\chi + \left(\frac{\delta\varphi}{a^2\dot\varphi}\right) \\ \hat\nu_i &= \nu_i - \dot w_i \end{aligned} \tag{5.3.7}$$

where in writing these equations we have introduced the Hubble rate $H = \dot a/a$. These combinations of perturbations can be checked to be invariant under simultaneous small changes in the metric ($\delta_\xi g_{ab} = \mathcal{L}_\xi g_{ab}$) and scalar fields ($\delta_\xi\Phi = \mathcal{L}_\xi\Phi = \xi^t\dot\varphi$).

The equations of motion for the perturbations are also given in [215] and they decouple nicely into Hamiltonian and momentum constraints

$$\hat\phi = -\frac{\dot\zeta}{H}, \qquad \hat\nu = \frac{\zeta}{a^2H} + \frac{\epsilon\dot\zeta}{q^2}, \qquad \hat\nu_i = 0 \tag{5.3.8}$$

where $\epsilon(z)$ is given by

$$\epsilon = -\frac{\dot{H}}{H^2} = 2\left(\frac{W_{,\varphi}}{W}\right)^2 \tag{5.3.9}$$

and q_i is the comoving wavevector of the perturbations which in momentum space acts by multiplication and we use the notation $q^2 = \delta_{ij}q^iq^j$. On the note of momentum space, we will also consider the remaining equations of motion in momentum space as it will make them easier to solve explicity, at which point we are free to Fourier transform back into position space. The remaining equations of motion are for the independent perturbations $\zeta(t, q^k)$ and $X_{ij}(t, q^k)$:

$$0 = \ddot{\zeta} + \left(3H + \frac{\dot{\epsilon}}{\epsilon}\right)\dot{\zeta} + a^{-2}q^2\zeta \tag{5.3.10a}$$

$$0 = \ddot{X}_{ij} + 3H\dot{X}_{ij} + a^{-2}q^2X_{ij} \tag{5.3.10b}$$

where we notice that these equations are linear second order ODEs with non-constant coefficients.

5.3.2 Solving the equations of motion

We have now introduced all of the important points of the theory under consideration and now we want to compute the charges for a specific example. First we will restrict our attention to a simple example of this theory: ALdS solutions with no coupled scalar.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R - 2\Lambda) \tag{5.3.11}$$

First we solve the background field equations (5.3.4), the second of which gives us $W = c$ with c a constant. We then look at the third equation which gives

$$-2\Lambda = -\frac{3}{2}c^2 \implies c = -2\sqrt{\frac{\Lambda}{3}} = -\frac{2}{l_{dS}} \tag{5.3.12}$$

where we choose the minus sign for reason which will become clear shortly. Finally the first equation in (5.3.4) gives

$$\frac{\dot{a}}{a} = \frac{1}{l_{dS}} \implies a = \exp\left(\frac{t}{l_{dS}}\right) \tag{5.3.13}$$

which agrees precisely with inflationary patch coordinates of dS_4 spacetime. We now look to solve the equations of motion for perturbations of this spacetime (5.3.8) and (5.3.10b) although we first restrict our consideration to perturbations which do not perturb either g_{tt} or the off-diagonal terms in the metric, i.e. those with $\phi = \nu = \nu_i = 0$. This restriction immediately forces the final relation of (5.3.7) to become

$$\dot{w}_i = 0 \tag{5.3.14}$$

and we will further restrict this by choosing the solution $w_i = 0$. We have now switched off six of the degrees of freedom in the perturbations, although we will still show that one is able to compute interesting gravitational charges. The remaining equations for the gauge invariant perturbations are

$$\zeta = \psi, \qquad \hat{\phi} = \phi = 0, \qquad \hat{\nu} = -\dot{\chi} \tag{5.3.15}$$

so we can move freely between gauge invariant perturbations $(\zeta, \hat{\phi}, \hat{\nu})$ and physical perturbations (ψ, χ, X_{ij}). Applying first the 'constraint' equations of motion (5.3.8) we find

$$\phi = -l_{dS}\dot{\psi} \implies \dot{\psi} = 0, \qquad -\dot{\chi} = l_{dS}\psi \exp\left(-\frac{2t}{l_{dS}}\right) \tag{5.3.16}$$

and thus solving for ψ will give us the perturbations ϕ and χ (up to a function of integration) directly. The remaining three degrees of freedom can all be solved for by looking at the same ODEs, namely equation (5.3.10b), which now becomes the same equation for both ψ and X_{ij}

$$0 = \ddot{\psi} + \left(\frac{3}{l_{dS}}\right)\dot{\psi} + \exp\left(-\frac{2t}{l_{dS}}\right)q^2\psi \tag{5.3.17a}$$

$$0 = \ddot{X}_{ij} + \left(\frac{3}{l_{dS}}\right)\dot{X}_{ij} + \exp\left(-\frac{2t}{l_{dS}}\right)q^2 X_{ij}. \tag{5.3.17b}$$

Equation (5.3.17a) is simplified by using the first equation of (5.3.16), upon which it reduces to $\psi(t, \vec{q}) = 0$. Now the second equation of (5.3.16) gives us $\dot{\chi} = 0 \implies \chi = \chi(\vec{q})$. In this work we will restrict our attention to pure tensor perturbations and thus we will only consider the solution $\chi = 0$.

No such simplifications are available for equation (5.3.17b) so we will consider the general solution for such an equation

$$\begin{aligned} X_{ij} = A_{ij}(q^k)\exp\left(il_{dS}qe^{-\frac{t}{l_{dS}}}\right)\left(1 - il_{dS}qe^{-\frac{t}{l_{dS}}}\right) - \\ B_{ij}(q^k)\frac{\exp\left(-il_{dS}qe^{-\frac{t}{l_{dS}}}\right)\left(l_{dS}qe^{-\frac{t}{l_{dS}}} - i\right)}{l_{dS}^3 q^3} \end{aligned} \tag{5.3.18}$$

which produces the following expression for the line element of the perturbed metric

$$ds^2 = -dt^2 + e^{2t/l_{dS}}(\delta_{ij} + X_{ij}(t, x))dx^i dx^j \tag{5.3.19}$$

where the x dependence in the perturbation terms is to be understood as the inverse

Fourier transform of our solutions to the problem in momentum space. The tensors of integration A_{ij}, B_{ij} will soon be seen to be related to the gravitational charges of the setup. We note that although this expression is not manifestly real in momentum space, it will reduce to a real function in position space (each of the q's provide an imaginary contribution). Now that we have a solution to the equations of motion, we are ready to compute the Wald Hamiltonians of such a solution.

5.3.3 Charge integrand

We will compute the Wald Hamiltonians, H_ξ, as given by (5.2.85) for our perturbed inflationary patch metric of (5.3.19). The vectors ξ which we will compute the charges for will be the Killing vectors of the unperturbed inflationary patch spacetime. Following the notation of [41, 44] we recall that the inflationary patch of dS_4 has seven Killing vector fields given by three translations, T_i

$$T_i = \partial_i \tag{5.3.20}$$

three rotations, R_{ij}

$$R_{ij} = x_i\partial_j - x_j\partial_i \tag{5.3.21}$$

and a scaling transformation

$$D = -\frac{1}{l_{dS}}(x^a\partial_a) \tag{5.3.22}$$

which we note is referred to as a 'time translation' in [44] because it is the limit of the Schwarzchild-de Sitter spacetime as the Schwarzchild mass goes to zero. We will denote a member of this set of Killing fields generically as ξ until we refer to the individual fields later.

We will now evaluate (5.2.128) for our example. The first step of this will be to transform the coordinates of the perturbed metric into Fefferman-Graham coordinates, as these will be most convenient to examine the divergences of the metric near $\mathscr{I}^+$. The transformation is

$$\rho = -e^{-t/l_{dS}} \tag{5.3.23}$$

where the sign above is chosen so that ρ increases from $-\infty$ (at the past coordinate horizon) to $\rho = 0$ (at $\mathscr{I}^+$). We use this convention as it ensures that ρ increases in the forward time direction. At this point, we will also introduce the notation Σ_ρ to denote $\rho =$ constant hypersurfaces in the spacetime and of course we have $\Sigma_0 = \mathscr{I}^+$.

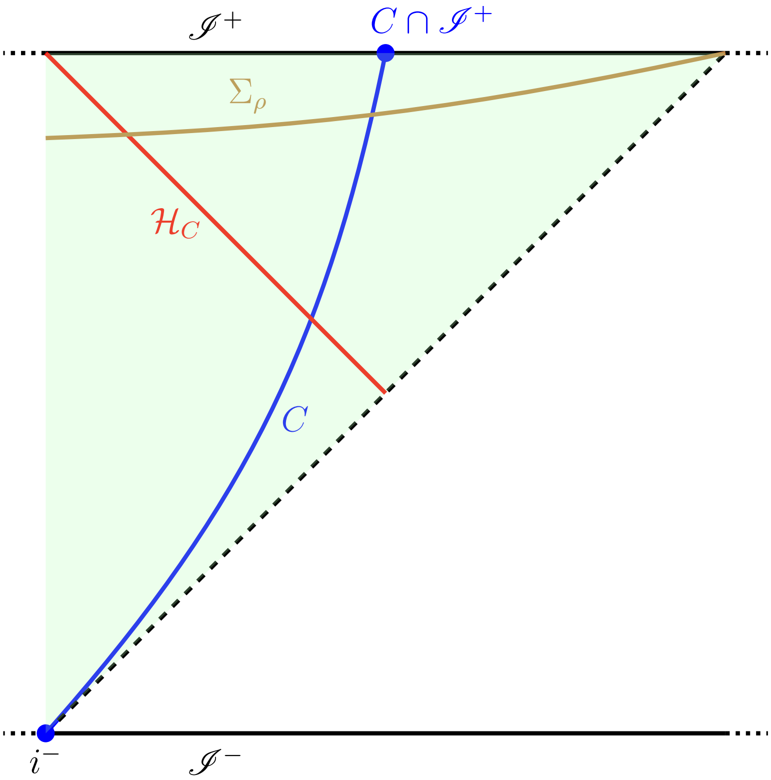

Figure5.3.1: Penrose diagram for a perturbation of the inflationary patch. The shaded region represents the patch covered by the coordinates of (5.3.19). The blue timelike hypersurface C is a 'slice' of the spacetime which intersects $\mathscr{I}^+$ as shown. The red null hypersurface $\mathcal{H}_C$ is the cosmological horizon present in the spacetime. The brown spacelike hypersurfaces Σ_ρ is a $\rho = \text{constant}\ (< 0)$ hypersurface.

The transformation (5.3.23) brings the metric into the form

$$ds^2 = -\frac{l_{dS}^2}{\rho^2} d\rho^2 + \frac{1}{\rho^2}\tilde{g}_{ij} dx^i dx^j \tag{5.3.24}$$

where

$$\tilde{g}_{ij} = \delta_{ij} + h_{ij} = \delta_{ij} + X_{ij} = g_{(0)ij} + \rho^2 g_{(2)ij} + \rho^3 g_{(3)ij} + \dots \tag{5.3.25}$$

and recalling the relationship [69, 60]

$$T_{ij} = \frac{3}{2\kappa^2} g_{(3)ij} \tag{5.3.26}$$

we can write (5.2.128) as

$$H_\xi = \frac{3}{2\kappa^2} \int_{C\cap\mathscr{I}^+} d\sigma_k \sqrt{g_{(0)}} g_{(0)}^{ki} g_{(3)ij} \xi_{(0)}^j \tag{5.3.27}$$

where we have used the notation $d\sigma_k = d^2x\, n_k$ where n^k is a spacelike unit normal vector to $C \cap \mathscr{I}^+$.

5.3.4 Inflationary patch charges

Now that we have successfully implemented the holographic renormalisation scheme in order to derive the charges of perturbed dS_4, it remains to provide the explicit expressions for the charges when we consider the Killing vectors given in equations (5.3.20)-(5.3.22). The first quantity to identify is $g_{(3)ij}$, which we will need in coordinate space. In order to compute this, we first write our momentum space expression for X_{ij} (5.3.18) in Fefferman-Graham coordinates by using transformation (5.3.23) with $l_{dS} = 1$[3]

$$X_{ij} = A_{ij}(q^k)(1 + iq\rho)[\cos(q\rho) - i\sin(q\rho)] + B_{ij}(q^k)\frac{(i + q\rho)[\cos(q\rho) + i\sin(q\rho)]}{q^3} \quad (5.3.28)$$

and perform the following change of basis on the functions of integration

$$A_{ij}(q^k) = \frac{1}{2}\left[F_{ij}(q^k) - i\frac{G_{ij}(q^k)}{q^3}\right], \quad B_{ij}(q^k) = \frac{1}{2}[G_{ij}(q^k) - iq^3F_{ij}(q^k)], \quad (5.3.29)$$

upon which (5.3.28) becomes

$$X_{ij} = F_{ij}[\cos(q\rho) + q\rho\sin(q\rho)] + \frac{G_{ij}}{q^3}[q\rho\cos(q\rho) - \sin(q\rho)] \quad (5.3.30)$$

and the expansion of the perturbation about $\mathscr{I}^+ = \{\rho = 0\}$ is

$$X_{ij} = F_{ij} + \frac{1}{2}\rho^2q^2F_{ij} - \frac{1}{3}\rho^3G_{ij} + \mathcal{O}(\rho^4). \quad (5.3.31)$$

This rewriting shows explicitly the Fefferman-Graham result that the metric is asymptotically constructed from two pieces of independent data $\{g_{(0)}, g_{(3)}\} \sim \{F, G\}$ [59, 60].

Aside from this, the main purpose of writing (5.3.31) is to make the inverse Fourier transform easier to compute. Defining

$$\begin{aligned}\mathcal{F}^{-1}(F_{ij}) &= \int_{\mathbb{R}^3}\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{x}}F_{ij}(q^k) = \tilde{F}_{ij}(x^k)\\ \mathcal{F}^{-1}(G_{ij}) &= \int_{\mathbb{R}^3}\frac{d^3q}{(2\pi)^3}e^{i\vec{q}\cdot\vec{x}}G_{ij}(q^k) = \tilde{G}_{ij}(x^k)\end{aligned} \quad (5.3.32)$$

we can compute the position space form of the perturbation, $\tilde{X}_{ij}(x^k, \rho)$, given by

$$\tilde{X}_{ij} = \tilde{F}_{ij} - \frac{1}{2}\rho^2\partial_k\partial^k\tilde{F}_{ij} - \frac{1}{3}\rho^3\tilde{G}_{ij} + \mathcal{O}(\rho^4). \quad (5.3.33)$$

Using this expansion, we can read off the terms in the Fefferman-Graham coefficients,

[3] we use the notation $q = \vec{q} = \sqrt{q_1^2 + q_2^2 + q_3^2}$

namely

$$
\begin{aligned}
g_{(0)ij} &= \delta_{ij} + \tilde{F}_{ij} \\
g_{(2)ij} &= -\frac{1}{2}\partial_k\partial^k g_{(0)ij} = -\frac{1}{2}\partial_k\partial^k \tilde{F}_{ij} \\
g_{(3)ij} &= -\frac{1}{3}\tilde{G}_{ij}.
\end{aligned}
\tag{5.3.34}
$$

where $g_{(3)ij}$ is the object we will use in order to compute the charges for our spacetime. Applying this to the generic expression for the charges (5.3.27) and only keeping terms up to linear order we find

$$H = -\frac{1}{2\kappa^2}\int_{C\cap\mathscr{I}^+} \tilde{G}_{ij}\xi^i n^j \tag{5.3.35}$$

where n^j is a spacelike unit normal vector to the surface $C \cap \mathscr{I}^+$. If we pick some coordinates on $C \cap \mathscr{I}^+$ such that $n^i = (1, 0, 0)$ then the charges take the following forms

Spatial translations:

$$H_{\xi_T} = -\frac{1}{2\kappa^2}\int \tilde{G}_{i1}\, dx_2 dx_3, \qquad i = 1, 2, 3 \tag{5.3.36}$$

Spatial rotations:

$$H_{\xi_R} = -\frac{1}{2\kappa^2}\int (\tilde{G}_{j1}x^i - \tilde{G}_{i1}x^j)\, dx_2 dx_3 \qquad i, j = 1, 2, 3 \qquad i \neq j \tag{5.3.37}$$

'Time dilatations':

$$H_{\xi_D} = \frac{1}{2\kappa^2}\int \tilde{G}_{i1}x^i\, dx_2 dx_3 \tag{5.3.38}$$

where here summation over i is now implied.

5.3.4.1 Independence of the slice

Before we proceed in computing the charges for some explicit perturbations, we will first show that the charges are conserved i.e. that they are independent of the 'slice' of spacetime characterised by the x_1 coordinate. Our starting point is the generic expression for the charges given by (5.3.35) which we write as

$$H = -\frac{1}{2\kappa^2}\int \tilde{G}_{i1}(x)\xi^i(x)\, dx_2 dx_3 \tag{5.3.39}$$

in order to make the functional dependencies explicit. To show that this expression is slice-independent, we differentiate with respect to x_1

$$\begin{aligned} -\partial_1 H &= \frac{1}{2\kappa^2}\int (\xi^i \partial_1 \tilde{G}_{i1} + \tilde{G}_{i1}\partial_1 \xi^i)\, dx_2 dx_3 \\ &= \frac{1}{2\kappa^2}\int [\tilde{G}_{i1}\partial_1 \xi^i - \xi^i(\partial_2 \tilde{G}_{i2} + \partial_3 \tilde{G}_{i3})]\, dx_2 dx_3 \\ &= \frac{1}{2\kappa^2}\int \tilde{G}_{ij}\partial^j \xi^i\, dx_2 dx_3 \\ &= \frac{1}{6\kappa^2}\int \tilde{G}_{ij}\delta^{ij}\partial_k \xi^k\, dx_2 dx_3 = 0 \end{aligned} \tag{5.3.40}$$

where in moving to the second line we used the conservation of $\tilde{G}_{ij}$. In moving to the third we have used integration by parts and discarded the total derivative term (this requires assuming suitably fast fall offs of the perturbation). In the final line we have used the fact that ξ is a conformal Killing vector of our flat $\mathscr{I}^+$ as well as the fact that $\tilde{G}_{ij}$ is traceless. This shows that the Hamiltonians are independent of the slice and thus are explicitly conserved quantities.

5.3.5 Examples of finite charges

Now that we have computed the generic expression for the charges, we would like to find some examples of examples of spacetimes which have finite, non-trivial charges. To do this, we will consider which values of $\tilde{G}_{ij}$ give rise to finite integrals (5.3.35). We will begin by considering the spatial translation charges (5.3.36) which we can write as

$$H_{\xi_T} = -\frac{1}{2\kappa^2}\int \tilde{G}_{i1}\, dx_2 dx_3, \qquad i = 1, 2, 3 \tag{5.3.41}$$

where the independence of the slice is now manifest purely due to the conservation of $\tilde{G}_{ij}$.

At this point, we can start to guess choices for the integration tensors which give a finite charge. An example of which would be

$$\tilde{G}_{i1} = P_i(x_2, x_3) \tag{5.3.42}$$

where P_i is a conserved covector. We will now discuss a number of examples of this form which possess different conserved charges.

5.3.5.1 Rotational charges

In order to construct finite charges, we can consider a covector with a Gaussian-type component, for example

$$P_i = \left(\frac{1}{2\pi} x_2 \exp\left(-\frac{1}{2}(x_2^2 + x_3^2)\right), 0, 0\right) \tag{5.3.43}$$

which is clearly conserved and the first component integrates to zero over $C \cap \mathscr{I}^+$. As a result all of the translational charges vanish for this perturbation.

For completeness, we will now obtain the other components of the perturbation $\tilde{G}_{ij}$. For the Gaussian-type component (5.3.43), we have

$$\tilde{G}_{ij} = \begin{pmatrix} \frac{x_2}{2\pi} e^{-\frac{1}{2}(x_2^2+x_3^2)} & 0 & 0 \\ 0 & \tilde{G}_{22} & \tilde{G}_{23} \\ 0 & \tilde{G}_{23} & \tilde{G}_{33} \end{pmatrix} \tag{5.3.44}$$

and we will assume, as in the case of $\tilde{G}_{i1}$, that all components are independent of x_1. The traceless and remaining conservation constraints are

$$\begin{aligned} \tilde{G}_{22} + \tilde{G}_{33} &= -\frac{x_2}{2\pi} e^{-\frac{1}{2}(x_2^2+x_3^2)} \\ \partial_2 \tilde{G}_{22} + \partial_3 \tilde{G}_{23} &= 0 \\ \partial_2 \tilde{G}_{23} + \partial_3 \tilde{G}_{33} &= 0 \end{aligned} \tag{5.3.45}$$

a system which admits an example solution of

$$\begin{aligned} \tilde{G}_{22} &= -\frac{x_2 e^{-\frac{1}{2}\left(x_2^2+x_3^2\right)} \left(x_2^4 \left(x_3^2 - 1\right) + 2x_2^2 \left(x_3^4 + x_3^2 - 1\right) + x_3^2 \left(x_3^4 + 3x_3^2 + 6\right)\right)}{2\pi \left(x_2^2 + x_3^2\right)^3} \\ \tilde{G}_{23} &= \frac{x_3 e^{-\frac{1}{2}\left(x_2^2+x_3^2\right)} \left(x_2^6 + x_2^4 \left(2x_3^2 + 3\right) + x_2^2 \left(x_3^4 + 2x_3^2 + 6\right) - x_3^2 \left(x_3^2 + 2\right)\right)}{2\pi \left(x_2^2 + x_3^2\right)^3} \\ \tilde{G}_{33} &= -\frac{x_2 e^{-\frac{1}{2}\left(x_2^2+x_3^2\right)} \left(x_2^6 + x_2^4 \left(2x_3^2 + 1\right) + x_2^2 \left(x_3^4 - 2x_3^2 + 2\right) - 3x_3^2 \left(x_3^2 + 2\right)\right)}{2\pi \left(x_2^2 + x_3^2\right)^3} \end{aligned} \tag{5.3.46}$$

where we have set all of the integration functions arising from solving this system to zero, a choice motivated by the fact that we are just looking for one solution with non-vanishing charges, not necessarily the most general solution.

In order to show that this solution does admit non-zero charges, we study the rotational charges corresponding to (5.3.43)

$$H_{\xi_R} = -\frac{1}{2\kappa^2} \int (\tilde{G}_{j1} x^i - \tilde{G}_{i1} x^j)\, dx_2 dx_3 \qquad i, j = 1, 2, 3 \qquad i < j. \tag{5.3.47}$$

Clearly the $i = 2, j = 3$ charge vanishes, but the others require a little more examination. For $i = 1, j = 2$ we find

$$(H_{\xi_R})_{12} = \frac{1}{4\pi\kappa^2} \int x_2^2 \exp\left(-\frac{1}{2}(x_2^2 + x_3^2)\right) dx_2 dx_3 = \frac{1}{2\kappa^2} \tag{5.3.48}$$

as well as

$$(H_{\xi_R})_{13} = \frac{1}{4\pi\kappa^2} \int x_2 x_3 \exp\left(-\frac{1}{2}(x_2^2 + x_3^2)\right) dx_2 dx_3 = 0 \tag{5.3.49}$$

for $i = 1, j = 3$. Although two of these charges vanish, the third is non-trivial. We have thus found a non-zero, finite charge corresponding to perturbations of the inflationary patch. One can also check that the time dilatation charge vanishes for this perturbation, which follows directly from applying the perturbation (5.3.43) to equation (5.3.38).

5.3.5.2 Distributional solutions

It is reasonable to ask why we chose P_1 in (5.3.43) to take the "expectation" of a Gaussian, rather than just a pure Gaussian of the form

$$P_1 \sim \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_2^2 + x_3^2)\right). \tag{5.3.50}$$

The reasoning behind this is due to the subtleties involved in the slice-independence for this perturbation, which makes it more difficult to compute the charges. If one naïvely computes the time dilatation charge using (5.3.38), then the charge is

$$H_{\xi_D} = \frac{1}{4\pi\kappa^2} \int x_1 \exp\left(-\frac{1}{2}(x_2^2 + x_3^2)\right) dx_2 dx_3 = \frac{x_1}{2\kappa^2} \tag{5.3.51}$$

which seems to be in contradiction with the result proven in subsection 5.3.4.1.

As it turns out, the issue is not with the slice independence proof but rather the fact that a solution of the form of (5.3.50) needs a distributional correction in order to satisfy the conservation and tracelessness equations. Indeed, if we begin with the aim of solving the system of partial differential equations

$$\begin{aligned} \tilde{G}_{22} + \tilde{G}_{33} &= -\frac{1}{2\pi} e^{-\frac{1}{2}(x_2^2 + x_3^2)} \\ \partial_2 \tilde{G}_{22} + \partial_3 \tilde{G}_{23} &= 0 \\ \partial_2 \tilde{G}_{23} + \partial_3 \tilde{G}_{33} &= 0 \end{aligned} \tag{5.3.52}$$

then using the usual methods for solving first order systems we find

$$\begin{aligned}\tilde{G}_{22} &= -\frac{e^{-\frac{1}{2}\left(x_2^2+x_3^2\right)}\left(x_2^2\left(x_3^2-1\right)+x_3^4+x_3^2\right)}{2\pi\left(x_2^2+x_3^2\right)^2} \\ \tilde{G}_{23} &= \frac{x_2x_3e^{-\frac{x_2^2}{2}-\frac{x_3^2}{2}}\left(x_2^2+x_3^2+2\right)}{2\pi\left(x_2^2+x_3^2\right)^2}\end{aligned} \tag{5.3.53}$$

where $\tilde{G}_{33}$ can just be read off from the tracelessness constraint. Upon careful checking of the solution, we find that the third equation of (5.3.52) gives

$$\partial_2\tilde{G}_{23}+\partial_3\tilde{G}_{33} = -\partial_3\left[\delta^2(x_2,x_3)e^{-\frac{1}{2}\left(x_2^2+x_3^2\right)}\right], \tag{5.3.54}$$

a result which can be derived by recalling the following Green's function equation for the two dimensional Laplacian $\Delta = \partial_2^2+\partial_3^2$

$$\Delta\log(x_2^2+x_3^2) = 4\pi\delta^2(x_2,x_3). \tag{5.3.55}$$

In order to correct this inconsistency it is natural to absorb the distributional term into $\tilde{G}_{33}$ as both are exact derivatives of x_3. This then forces a change in $\tilde{G}_{11}$ in order to preserve tracelessness ($\tilde{G}_{22}$ is left alone in order to preserve the middle equation of (5.3.52)). Putting this together, we find a suitable solution of the form

$$\tilde{G}_{i1} = P_i = \left(\left(\frac{1}{2\pi}-\delta^2(x_2,x_3)\right)\exp\left(-\frac{1}{2}(x_2^2+x_3^2)\right),0,0\right) \tag{5.3.56}$$

which now has a zero "time dilatation" charge (in fact all seven charges are zero). We note that the distributional correction was not necessary in the "expectation" case because of the different source term. In that example we would have found a correction of the form $\sim x_2\delta^2(x_2,x_3)\exp(\ldots)$, which would not have contributed anything under integration due to the extra factor of x_2.

Even though we found vanishing charges for our second example above, the distributional perturbations remain an interesting class of solutions, principally because they are related to the gravitational memory effect in cosmological spacetimes [216]. If we choose the solution

$$\tilde{G}_{i1} = Q_i(x_2,x_3) \tag{5.3.57}$$

where

$$Q_i(x_2,x_3) = \left(x_2\delta^2(x_2,x_3),0,0\right) \tag{5.3.58}$$

then we find distributional solutions with all zero charges. It is not clear whether these solutions have a non-trivial memory by the definition of [216] because there the memory effect was defined by the δ-function singularities which appeared in the $\{g_{(0)},g_{(2)}\}$ terms in the metric expansion. Our finite charges have singular $g_{(3)}$ but other terms are not specified in this prescription. We also note that we can combine the solutions of (5.3.42) and (5.3.57) in the solution given by

$$\tilde{G}_{11} = f(x_2, x_3; \nu) = \frac{x_2}{2\pi\nu} \exp\left(-\frac{1}{2\nu}(x_2^2 + x_3^2)\right), \qquad \nu > 0 \tag{5.3.59}$$

where we note that

$$\begin{aligned} &\lim_{\nu \to 0} f = x_2 \delta^2(x_2, x_3) \\ &\lim_{\nu \to 1} f = \frac{1}{2\pi} x_2 \exp\left(-\frac{1}{2}(x_2^2 + x_3^2)\right). \end{aligned} \tag{5.3.60}$$

Also note that there is also a discontinuity in the charge as we take this limit. The non-zero rotational charge given by (5.3.48) is a finite constant of the Gaussian solution ($\nu > 0$) but vanishes for the distributional perturbation.

We conclude this section by noting that this example is simply one example of a perturbation which gives a finite charge. There may be an interesting mathematical structure in underlying tensors of integration $\tilde{G}_{ij}$ which give finite, non-zero charges. It would also be of interest to consider more general solutions to the equations of motion (i.e. those with non-zero scalar and vector perturbations) and see if they also admit finite charges, as well as considering which choices of integration functions $\tilde{F}_{ij}, \tilde{G}_{ij}$ give rise to regular solutions in the bulk spacetime.

5.3.5.3 'Time dilatation' charge

We will finish our list of examples with a perturbation which gives a non-zero time dilatation charge as given by (5.3.38). An example of such a perturbation is given by

$$P_i = \left(0, -e^{-x_3^2}(1 - 2x_3^2) \int_0^{x_2} e^{-t^2}\, dt, x_3 e^{-x_3^2 - x_2^2}\right) \tag{5.3.61}$$

where we have chosen to set the component $P_1 = 0$ in order to avoid having to modify the solution due to the contribution of distributional terms.

Given (5.3.61), we can compute all charges using (5.3.36)-(5.3.38) and in particular we find for the dilatation charge

$$\begin{aligned} H_{\xi_D} &= \frac{1}{2\kappa^2}\int_{C\cap\mathscr{I}^+}(P_2x_2+P_3x_3) \\ &= \frac{1}{2\kappa^2}\lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a}\left[\left(-x_2e^{-x_3^2}(1-2x_3^2)\int_0^{x_2}e^{-t^2}\,dt\right)+x_3^2e^{-x_3^2-x_2^2}\right]dx_2dx_3 \\ &= \frac{1}{8\kappa^2}\lim_{a\to\infty}\left(-4\left(2a^2+1\right)e^{-a^2}a\int_0^a e^{-t^2}\,dt-4e^{-2a^2}a^2+8\left(\int_0^a e^{-t^2}\right)^2\right) \\ &= \frac{\pi}{4\kappa^2} \end{aligned} \tag{5.3.62}$$

which is clearly non-zero. One finds all other charges vanish for this perturbation.

In order to extend this to a full solution, we have to fill in the components $\tilde{G}_{AB}$, $A,B\in\{2,3\}$ which are subject to the usual tracelessness and conservation constraints

$$\begin{aligned} \tilde{G}_A^A &= 0 \\ \partial_A\tilde{G}_B^A &= 0 \end{aligned} \tag{5.3.63}$$

where indices have been raised and lowered with the two-dimensional flat metric δ_{AB}. Any solution to these equations will do as these components do not affect the charges of the spacetime and thus we may choose our perturbation

$$\tilde{G}_{ij} = \begin{pmatrix} 0 & P_2 & P_3 \\ P_2 & 0 & 0 \\ P_3 & 0 & 0 \end{pmatrix} \tag{5.3.64}$$

with P_i given in (5.3.61).

5.3.6 Connection with global coordinates

All of the analysis of the current section has been performed with the background metric being the dS inflationary patch metric (again, we set $l_{dS}=1$ for convenience)

$$ds^2 = -dt^2 + e^{2t}(dx^2+dy^2+dz^2), \tag{5.3.65}$$

the coordinates of which only cover half of the total space time (as shown in figure 5.1.1).

We would like to understand the implications of our charges in a global dS_4 background spacetime. In order to do this, we will extend our solutions (5.3.30) across the coordinate horizon ($\{t=-\infty\}$) of the inflationary patch, a procedure which will involve performing the Bogoliubov transformation of the tensor perturbation from the Fefferman-Graham coordinates into global coordinates on dS_4 spacetime.

Before giving the details of the transformation, we will begin with a review of solving the wave equation on dS backgrounds in different coordinates, closely following [217]. Our starting point is the dS_4 metric in global coordinates $(\tau, \psi, \theta, \phi)$, where the line element takes the form

$$ds^2 = -d\tau^2 + \cosh^2(\tau)[d\psi^2 + \sin^2(\psi)(d\theta^2 + \sin^2(\theta)d\phi^2)] \tag{5.3.66}$$

and the coordinate ranges are $-\infty < \tau < \infty$, $0 \le \psi \le \pi$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. We now want to solve the equation of motion for the tensor perturbation X_{ij} in these global coordinates, which can be written covariantly as

$$(\Box X)_{ij} = 0. \tag{5.3.67}$$

This equation tells us that each component of the tensor X_{ij} solves the wave equation with respect to the background metric and due to this we can consider the scalar wave equation $\Box X = 0$. We have already solved the equation for the background metric being (5.3.65) but now we want replace this with (5.3.66) and look for solutions to this equation.

Following [217], we first transform the global time coordinate τ into the conformal time coordinate η, given by

$$\eta = 2\arctan(e^\tau) \tag{5.3.68}$$

upon which the line element (5.3.66) becomes

$$ds^2 = \frac{1}{\sin^2(\eta)}[-d\eta^2 + d\psi^2 + \sin^2(\psi)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{5.3.69}$$

which is a Robertson-Walker spacetime with a closed slicing, a type of spacetime for which the solutions to the wave equation are well known. Quoting [217], and following their notation of $\boldsymbol{x} = (\psi, \theta, \phi)$, $\boldsymbol{k} = (k, J, M)$, we find the solution is given by [218]

$$X_G = \sum_{k=1}^{\infty}\sum_{J=0}^{k-1}\sum_{M=-J}^{J} A_{\boldsymbol{k}} w_{\boldsymbol{k}}(x) + A^*_{\boldsymbol{k}} w^*_{\boldsymbol{k}}(x) \tag{5.3.70}$$

where

$$w_{\boldsymbol{k}}(x) = \sin^{3/2}(\eta)\left(\frac{1}{2}P^{\frac{3}{2}}_{k-\frac{1}{2}}(-\cos\eta) - \frac{i}{k(k^2-1)}Q^{\frac{3}{2}}_{k-\frac{1}{2}}(-\cos\eta)\right)\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x}) \quad \text{for } k \ge 2 \tag{5.3.71}$$

where P, Q are Legendre functions and $\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x})$ are eigenfunctions of the Laplacian on the unit round S^3:

$$\Box_{S^3}\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x}) = (1-k^2)\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x}), \tag{5.3.72}$$

which are given by

$$\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x}) = \Pi^{(+)}_{kJ}(\psi) Y^M_J(\theta,\phi) \tag{5.3.73}$$

where

$$\Pi^{(+)}_{kJ}(\psi) = \left[-\frac{\pi}{2}k^2(1-k^2)\dots(J^2-k^2)\right]^{-1/2}(-i)^J \sin^J(\psi)\left(\frac{d}{d\cos(\psi)}\right)^{1+J}\cos(k\psi) \tag{5.3.74}$$

and $Y^M_J(\theta,\phi)$ are the usual spherical harmonics. We note [219, 217] that one can also expand $\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x})$ in terms of Gegenbauer functions C^A_B, using the relationship

$$\Pi^{(+)}_{kJ}(\psi) = i(-1)^J C^{J+1}_{k-J-1}(\cos(\psi))\sin^J(\psi)\sqrt{\frac{2^{1+2J}(k-J-1)!k[\Gamma(J+1)]^2}{\pi\Gamma(k+J+1)}}. \tag{5.3.75}$$

Finally, we note that the normalisation of (5.3.71) is not suitable for the choice of $k = 1\,(\implies J = M = 0)$. For this case, the suitably normalised solution to (5.3.67) is

$$w_{(1,0,0)} = \frac{-2(i+\eta)+\sin(2\eta)}{4\sqrt{2}\pi} \tag{5.3.76}$$

an example we will refer to later when we compute Bogoliubov coefficients.

In order to expand the inflationary patch solution in terms of these modes, we need to define an inner product on the global coordinates. Again, we take the standard result for inner products of modes on curved backgrounds from [217], which in this case reduces to

$$\begin{aligned}(\phi_1,\phi_2)_G &= -\frac{i}{\sin^2\eta}\int_{S^3} W(\phi_1,\phi_2^*)\\ &= -\frac{i}{\sin^2\eta}\int_0^\pi\int_0^\pi\int_0^{2\pi}(\phi_1\partial_\eta\phi_2^* - \phi_2^*\partial_\eta\phi_1)\sin^2(\psi)\sin(\theta)\,d\psi d\theta d\phi\end{aligned} \tag{5.3.77}$$

with $W(\cdot\,,\cdot)$ denoting the Wronskian with respect to η. We note that the modes $w_{\boldsymbol{k}}$ as defined in (5.3.71), (5.3.76) satisfy the normalisation conditions

$$(w_{\boldsymbol{k}}, w_{\boldsymbol{k'}})_G = \delta(\boldsymbol{k},\boldsymbol{k'}), \quad (w^*_{\boldsymbol{k}}, w^*_{\boldsymbol{k'}})_G = -\delta(\boldsymbol{k},\boldsymbol{k'}), \quad (w_{\boldsymbol{k}}, w^*_{\boldsymbol{k'}})_G = 0 \tag{5.3.78}$$

where $\delta(\boldsymbol{k},\boldsymbol{k'})$ is the Dirac delta function for the measure given in (5.3.70) over which we sum the modes, i.e.

$$\sum_{k=1}^{\infty}\sum_{J=0}^{k-1}\sum_{M=-J}^{J}\delta(\boldsymbol{k},\boldsymbol{k'})f(\boldsymbol{k}) = f(\boldsymbol{k'}). \tag{5.3.79}$$

This normalisation was ensured by our choice of coefficients for the Legendre P and Q functions in (5.3.71). We also note that these normalisation conditions can be recast as normalising the Wronskian between solutions, for more details on this see [217].

Now that we have the generic expansion for the components of our perturbation in global coordinates, we wish to compare this with the Fefferman-Graham coordinates of

(5.3.30) and find a relationship between the global mode coefficients $A_{\boldsymbol{k}}, A^*_{\boldsymbol{k}}$ and the mode coefficients for the inflationary patch expansion. To make a direct comparison with the global mode expansion of (5.3.70), we first express our FG solution in position space as the Fourier transform of our solution (5.3.30)

$$X_{FG} = \int \frac{d^3q}{(2\pi)^3} \left(F_{\boldsymbol{q}}[\cos(q\rho) + q\rho\sin(q\rho)] + \frac{G_{\boldsymbol{q}}[q\rho\cos(q\rho) - \sin(q\rho)]}{q^3} \right) e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}} \tag{5.3.80}$$

where in order to avoid confusing notation, we write $\boldsymbol{q} = (q_1, q_2, q_3)$, $\tilde{\boldsymbol{x}} = (x_1, x_2, x_3)$ and $q = ||\boldsymbol{q}||$. We also follow [42] in noting that since X_{FG} is real, the coefficients $F_{\boldsymbol{q}}, G_{\boldsymbol{q}}$ obey the following equations

$$(F_{\boldsymbol{q}})^* = F_{-\boldsymbol{q}}, \qquad (G_{\boldsymbol{q}})^* = G_{-\boldsymbol{q}} \tag{5.3.81}$$

In order to obtain expressions for the global coefficients $A_{\boldsymbol{k}}, A^*_{\boldsymbol{k}}$ in terms of $F_{\boldsymbol{q}}, G_{\boldsymbol{q}}$, we use the inner product (5.3.77) and note

$$(X_{FG}, w_{\boldsymbol{k}})_G = (X_G, w_{\boldsymbol{k}})_G = A_{\boldsymbol{k}}, \quad (X_{FG}, w^*_{\boldsymbol{k}})_G = (X_G, w^*_{\boldsymbol{k}})_G = -A^*_{\boldsymbol{k}}, \tag{5.3.82}$$

hence we will need to compute the integral corresponding to $(X_{FG}, w_{\boldsymbol{k}})_G$. In order to do this, we first need transform the coordinates of X_{FG} into the 'conformal global coordinates' of (5.3.69), with the transformation being

$$\begin{aligned} \rho &= -\frac{\sin(\eta)}{\cos(\psi) - \cos(\eta)}, \quad x_1 = \frac{\cos(\theta)\sin(\psi)}{\cos(\psi) - \cos(\eta)}, \\ x_2 &= \frac{\cos(\phi)\sin(\theta)\sin(\psi)}{\cos(\psi) - \cos(\eta)}, \quad x_3 = \frac{\sin(\phi)\sin(\theta)\sin(\psi)}{\cos(\psi) - \cos(\eta)}. \end{aligned} \tag{5.3.83}$$

Working in these coordinates, we are now able to write down the integral expression corresponding to the global mode coefficients

$$A_{\boldsymbol{k}} = -\frac{i}{\sin^2(\eta)} \int_{S^3} W(X_{FG}, w^*_{\boldsymbol{k}}) \tag{5.3.84}$$

which allows one to read off the expansion coefficients in the global coordinates given the Fefferman-Graham data.

5.3.6.1 Computing $A_{(1,0,0)}$

As an example, we will now compute the coefficient $A_{(1,0,0)}$ for an asymptotically dS solution with $g_{(3)}$ perturbations (i.e. (5.3.80) with $F_{\boldsymbol{q}} = 0$, $G_{\boldsymbol{q}} = 1$). We note that we will need to pick a suitable tensorial factor in order to ensure that $G_{ij}(q^k)$ is transverse traceless. The building block for such a tensor will be

$$\pi_{ij} = \delta_{ij} - \frac{q_i q_j}{q^2} \tag{5.3.85}$$

which is clearly symmetric ($\pi_{ij} = \pi_{ji}$) and transverse ($q^i\pi_{ij} = 0$) but not traceless ($\pi = \pi^i_i = 2$). We also note that the Fourier transform of π is

$$\pi_{ij} = (\delta_{ij} - \Delta^{-1}\partial_i\partial_j)\delta^3(\tilde{x}) = \delta_{ij}\delta^3(\tilde{x}) + \partial_i\partial_j\left(\frac{1}{4\pi\tilde{x}}\right). \tag{5.3.86}$$

In order to construct a traceless tensor, we introduce the three dimensional transverse traceless projection operator [73]

$$\Pi_{ijkl} = \frac{1}{2}(\pi_{ik}\pi_{lj} + \pi_{il}\pi_{kj} - \pi_{ij}\pi_{kl}) \tag{5.3.87}$$

and use this to construct transverse traceless symmetric polarisation tensors, see [220, 195] for the relevant expressions. We will focus for now on the transformation of the scalar factor, as the tensorial contribution can always be accounted for via convolution of Fourier transforms as discussed below.

This construction gives us a momentum space solution of the form

$$(X_{FG})_{ij} = P_{ij}\left(\frac{q\rho\cos(q\rho) - \sin(q\rho)}{q^3}\right) \tag{5.3.88}$$

which we now need to Fourier transform into position space before computing the global expansion coefficients $A_{\boldsymbol{k}}$. In order to compute this Fourier transform we use the convolution theorem

$$\begin{aligned}&\int\frac{d^3q}{(2\pi)^3}P_{ij}\left(\frac{q\rho\cos(q\rho) - \sin(q\rho)}{q^3}\right)e^{i\boldsymbol{y}\cdot\boldsymbol{q}} = \\ &\int d^3x\left\{\left[\int\frac{d^3q}{(2\pi)^3}\left(\frac{q\rho\cos(q\rho) - \sin(q\rho)}{q^3}\right)e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\right]\left[\int\frac{d^3p}{(2\pi)^3}P_{ij}e^{i(\boldsymbol{y}-\boldsymbol{x})\cdot\boldsymbol{p}}\right]\right\}\end{aligned} \tag{5.3.89}$$

the transform of the first term is

$$\begin{aligned}&\int\frac{d^3q}{(2\pi)^3}\,\frac{q\rho\cos(q\rho) - \sin(q\rho)}{q^3}e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\\ &\quad = \frac{1}{2\pi^2}\int_0^\infty\frac{1}{q}\mathrm{sinc}(q\tilde{x})(q\rho\cos(q\rho) - \sin(q\rho))\,dq\\ &\quad = \frac{1}{4\pi}\Theta(-(\rho+\tilde{x})) = \frac{1}{4\pi}\Theta\left(\cot\left(\frac{\eta+\psi}{2}\right)\right)\end{aligned} \tag{5.3.90}$$

where Θ denotes the Heaviside step function. We note that the location of the cosmological horizon, $\mathcal{H}_C$, in the inflationary patch is at $\tilde{x}^2 = \rho^2$ and thus the physical interpretation of this solution is of a perturbation which vanishes outside the cosmological horizon,(i.e. when $\tilde{x}^2 > \rho^2 \implies \rho+\tilde{x} > 0$) and takes a constant value inside ($\tilde{x}^2 < \rho^2 \implies \rho+\tilde{x} < 0$). We also have

$$w^*_{(1,0,0)} = \frac{2(i-\eta) + \sin(2\eta)}{4\sqrt{2}\pi}. \tag{5.3.91}$$

Which is all we need in order to compute the integrand in equation (5.3.84).

We will now perform the naïve computation of the mode coefficients, treating X_{FG} as the scalar solution, without for now including the contribution from the tensorial term P_{ij}. This will serve as a computation of the mode coefficients for the Bogoliubov transformation of a scalar field minimally coupled to de Sitter space, and will give us a hint as to the solution for the tensorial perturbation (whose components satisfy the same equation as the scalar field).

$$\begin{aligned} W(X_{FG}, w^*_{(1,0,0)}) = & -\frac{\sin^2\eta}{4\sqrt{2}\pi^2}\Theta\left(\cot\left(\frac{\eta+\psi}{2}\right)\right) + \\ & \frac{2(i-\eta)+\sin(2\eta)}{32\sqrt{2}\pi^2}\csc^2\left(\frac{\eta+\psi}{2}\right)\delta\left(\cot\left(\frac{\eta+\psi}{2}\right)\right) \end{aligned} \tag{5.3.92}$$

which needs to be integrated over a unit S^3. We first perform the θ and ϕ integrals trivially and then the ψ integral as follows

$$\begin{aligned} A_{(1,0,0)} = & \frac{i}{\sqrt{2}\pi}\int_0^\pi \Theta\left(\cot\left(\frac{\eta+\psi}{2}\right)\right)\sin^2\psi\, d\psi - \\ & \frac{i}{\sin^2\eta}\frac{2(i-\eta)+\sin(2\eta)}{8\sqrt{2}\pi}\int_0^\pi \csc^2\left(\frac{\eta+\psi}{2}\right)\delta\left(\cot\left(\frac{\eta+\psi}{2}\right)\right)\sin^2\psi\, d\psi \\ = & \frac{i}{\sqrt{2}\pi}\int_0^{\pi-\eta}\sin^2\psi\, d\psi + \\ & \frac{i}{\sin^2\eta}\frac{2(i-\eta)+\sin(2\eta)}{4\sqrt{2}\pi}\int_{\cot\left(\frac{\eta}{2}\right)}^{\cot\left(\frac{\eta+\pi}{2}\right)}\sin^2\psi\,\delta(u)\, du \\ = & \frac{1}{2\sqrt{2}}(1+\pi i). \end{aligned} \tag{5.3.93}$$

This computation also gives us

$$A^*_{(1,0,0)} = \frac{1}{2\sqrt{2}}(1-\pi i) \tag{5.3.94}$$

which can also be checked by performing the inner product $(X_{FG}, w^*_{\boldsymbol{k}})_G$.

5.3.6.2 Computing $A_{\boldsymbol{k}}$

The previously performed calculation allowed us to identify the first coefficient of the global mode expansion for a perturbation which was previously specified in terms of Fefferman-Graham data. It has also given us a strategy to compute generic coefficients $A_{\boldsymbol{k}}$ for a perturbation which only transformed $g_{(3)}$. The strategy would be to replace the expression (5.3.76) for the lowest mode in (5.3.92) with the expression (5.3.71) for a generic mode. We would also like to extend the computation of these coefficients to encompass generic perturbations in Fefferman-Graham coordinates, the main issue with such a computation is the difficulty in performing the Fourier transform for the $F_{\boldsymbol{k}}$ term.

In order to extend our computation of (5.3.84) to all $\boldsymbol{k}$, we first note that we can simplify the problem by setting η to a constant value (because $A_{\boldsymbol{k}}$ is η independent). A particularly convenient choice is $\eta = \pi/2$, and this is the value we will use from here on. We note the following values of the functions in the integrand of (5.3.84) at $\eta = \pi/2$:

$$\begin{aligned} w^*_{\boldsymbol{k}}|_{\eta=\frac{\pi}{2}} &= -\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x})\left[\frac{k}{\sqrt{2\pi}}\sin\left(\frac{k\pi}{2}\right) + \frac{i}{k^2-1}\sqrt{\frac{\pi}{2}}\cos\left(\frac{k\pi}{2}\right)\right] \\ \partial_\eta w^*_{\boldsymbol{k}}|_{\eta=\frac{\pi}{2}} &= \mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x})\frac{1}{\sqrt{2\pi}k}\left[k(k^2-1)\cos\left(\frac{k\pi}{2}\right) - i\pi\sin\left(\frac{k\pi}{2}\right)\right] \end{aligned} \tag{5.3.95}$$

where the only pieces of these which contribute non-trivially to the S^3 integral in (5.3.84) are the harmonics $\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x})$. We also observe that the k dependence in (5.3.95) results in a neat bifurcation of the cases when k is an even or odd integer. For $k = 2n\,(n \in \mathbb{Z})$ we find

$$\begin{aligned} \left(w^*_{\boldsymbol{k}}|_{\eta=\frac{\pi}{2}}\right)^{\text{even}} &= -\frac{i\sqrt{2\pi}(-1)^n}{8n^2-2}\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) \\ \left(\partial_\eta w^*_{\boldsymbol{k}}|_{\eta=\frac{\pi}{2}}\right)^{\text{even}} &= \frac{(-1)^n\left(4n^2-1\right)}{\sqrt{2\pi}}\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) \end{aligned} \tag{5.3.96}$$

and for $k = 2n+1$ $(n \in \mathbb{Z})$

$$\begin{aligned} \left(w^*_{\boldsymbol{k}}|_{\eta=\frac{\pi}{2}}\right)^{\text{odd}} &= -\frac{(-1)^n(2n+1)}{\sqrt{2\pi}}\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) \\ \left(\partial_\eta w^*_{\boldsymbol{k}}|_{\eta=\frac{\pi}{2}}\right)^{\text{odd}} &= -\frac{i\sqrt{\frac{\pi}{2}}(-1)^n}{2n+1}\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) \end{aligned} \tag{5.3.97}$$

We also need to understand the contribution of X_{FG} term to the S^3 integral. In order to avoid printing out the cumbersome expressions for $X_{FG}(\theta,\phi,\psi)|_{\eta=\frac{\pi}{2}}$, we will simply list the terms which will contribute non-trivially to the S^3 integral.

$$\begin{aligned} X_{FG}|_{\eta=\frac{\pi}{2}} &: e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\sin(\rho q),\, e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\cos(\rho q),\, e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\rho\sin(\rho q),\, e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\rho\cos(\rho q) \\ \partial_\eta X_{FG}|_{\eta=\frac{\pi}{2}} &: e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\rho^3\sin(\rho q),\, e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\rho^3\cos(\rho q),\, \frac{\partial\tilde{x}^i}{\partial\breve{a}\eta}(\text{“terms from } X_{FG}|_{\eta=\frac{\pi}{2}}\text{”}) \end{aligned} \tag{5.3.98}$$

and thus in order to fully compute the integral over the S^3 we need to compute the integrals of $\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x})$ multiplied by the terms in (5.3.98) e.g.

$$\int_{S^3}\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x})e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\sin(\rho q) = f_{\boldsymbol{k}} \tag{5.3.99}$$

where

$$e^{i\boldsymbol{q}\cdot\tilde{\boldsymbol{x}}}\sin(\rho q) = \sum_{k=1}^{\infty}\sum_{J=0}^{k-1}\sum_{M=-J}^{J} f_{\boldsymbol{k}}\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x}) \tag{5.3.100}$$

i.e. computing these integrals is equivalent to computing the expansion coefficients for the integrand when expanded in S^3 harmonics. In coordinates this integral is explicitly

$$\int_{S^3} \mathcal{Y}^*_{\boldsymbol{k}}(\theta,\phi,\psi) e^{i(\cos(\theta)q_1+\sin(\theta)[\cos(\phi)q_2+\sin(\phi)q_3])\tan(\psi)} \sin(\sec(\psi)q)\sqrt{g_{S^3}}\, d\theta d\phi d\psi \quad (5.3.101)$$

and one finds similar integrals in order to compute the other terms in (5.3.98).

As a first example of computing the coefficients $A_{\boldsymbol{k}}$, we will return to the class of conditions that we discussed in 5.3.6.1, namely $F_{\boldsymbol{q}} = 0$, $G_{\boldsymbol{q}} = 1$ and we will again consider only the solution to the scalar wave equation. These again seem reasonable conditions to consider for linear perturbations because all of the information regarding the conserved quantities is contained in the $G_{\boldsymbol{q}}$ term (see equation (5.3.35)). This means that even though the resulting coefficients will not be related to the expansion of the most general solution in global coordinates, they will contain all of the important information regarding the charges. Once again, we have X_{FG} taking the same value as (5.3.90) and thus

$$\begin{aligned} X_{FG}|_{\eta=\frac{\pi}{2}} &= \frac{1}{4\pi}\Theta\left(\cot\left(\frac{\pi+2\psi}{4}\right)\right) \\ \partial_\eta X_{FG}|_{\eta=\frac{\pi}{2}} &= -\frac{1}{8\pi}\csc^2\left(\frac{\pi+2\psi}{4}\right)\delta\left(\cot\left(\frac{\pi+2\psi}{4}\right)\right) \end{aligned} \quad (5.3.102)$$

and now performing the integral (5.3.84) becomes a case of evaluating

$$A_{\boldsymbol{k}} = -i\int_{S^3} W(X_{FG}, w^*_{\boldsymbol{k}})|_{\eta=\frac{\pi}{2}} = -i\int_{S^3}(X_{FG}\partial_\eta w^*_{\boldsymbol{k}} - \partial_\eta X_{FG} w^*_{\boldsymbol{k}})|_{\eta=\frac{\pi}{2}} \quad (5.3.103)$$

where we have all of the necessary information in equations (5.3.95) and (5.3.102) in order to compute these integrals. We will now go through some of the steps of computing these integrals and discuss the resulting expressions.

The first integral we want to compute is

$$-i\frac{1}{\sqrt{2\pi k}}\left[k(k^2-1)\cos\left(\frac{k\pi}{2}\right) - i\pi\sin\left(\frac{k\pi}{2}\right)\right]\frac{1}{4\pi}\int_{S^3}\Theta\left(\cot\left(\frac{\pi+2\psi}{4}\right)\right)\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) \quad (5.3.104)$$

ignoring (for now) the prefactor we have

$$I_1 = \int_{S^3}\Theta\left(\cot\left(\frac{\pi+2\psi}{4}\right)\right)\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) = \int_{S^2} Y^*_{MJ}(\theta,\phi)\int_0^{\frac{\pi}{2}}\Pi^{(+)*}_{kJ}(\psi)\sin^2\psi\, d\psi \quad (5.3.105)$$

and evaluating the S^2 integral we obtain

$$I_1 = 2\sqrt{\pi}\delta_{M,0}\delta_{J,0}\int_0^{\frac{\pi}{2}}\Pi^{(+)*}_{kJ}(\psi)\sin^2\psi\, d\psi = 2\sqrt{\pi}\delta_{M,0}\int_0^{\frac{\pi}{2}}\Pi^{(+)*}_{k0}(\psi)\sin^2\psi\, d\psi. \quad (5.3.106)$$

In order to perform this Π-function integral we use the Gegenbauer representation (5.3.75) and we obtain

$$I_1 = -2\sqrt{2}i\delta_{M,0}\int_0^{\frac{\pi}{2}} C^1_{k-1}(\cos\psi)\sin^2\psi\, d\psi = -2\sqrt{2}i\delta_{M,0}\frac{k}{k^2-1}\cos\left[\frac{\pi}{2}(k-2)\right] \tag{5.3.107}$$

where we used the integral identity

$$\int (1-z^2)^{\lambda-\frac{1}{2}} C^\lambda_n(z)\, dz = -\frac{2\lambda(1-z^2)^{\lambda+\frac{1}{2}}}{n(n+2\lambda)} C^{\lambda+1}_{n-1}(z) \tag{5.3.108}$$

to reach the right hand side of (5.3.107). Putting everything together, and noting that only even k terms contribute to the integral, we find the contribution of (5.3.104) is the remarkably simple

$$\frac{k}{2\pi^{\frac{3}{2}}}\delta_{M,0} \tag{5.3.109}$$

where $k \in 2\mathbb{N}$.

Now for the second term we have to evaluate

$$\begin{aligned} i\left(\frac{1}{8\pi}\right)&\left[\frac{k}{\sqrt{2\pi}}\sin\left(\frac{k\pi}{2}\right) + \frac{i}{k^2-1}\sqrt{\frac{\pi}{2}}\cos\left(\frac{k\pi}{2}\right)\right] \\ &\times \int_{S^3} \csc^2\left(\frac{\pi+2\psi}{4}\right)\delta\left(\cot\left(\frac{\pi+2\psi}{4}\right)\right)\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) \end{aligned} \tag{5.3.110}$$

and (again) ignoring the prefactor for now, the integral we want to compute is

$$I_2 = \int_{S^3} \csc^2\left(\frac{\pi+2\psi}{4}\right)\delta\left(\cot\left(\frac{\pi+2\psi}{4}\right)\right)\mathcal{Y}^*_{\boldsymbol{k}}(\boldsymbol{x}) \tag{5.3.111}$$

$$= 4\sqrt{\pi}\delta_{M,0}\int_{-1}^{1} \delta(u)\sin^2(\psi(u))\,\Pi^{(+)*}_{k0}(\psi(u))\, du \tag{5.3.112}$$

where we have used the substitution $u = \cot((2\psi+\pi)/4)$, which means that when we evaluate this integral we will need to use $\psi(u=0) = \pi/2$. Using this we find

$$I_2 = 4\sqrt{\pi}\delta_{M,0}\Pi^{(+)*}_{k0}\left(\frac{\pi}{2}\right) = -4i\sqrt{2}\delta_{M,0}\cos\left(\frac{\pi}{2}(k-1)\right) \tag{5.3.113}$$

and thus this term is only non-vanishing for odd k. Putting the prefactor back in, we find the contribution from the second term is

$$\frac{k}{2\pi^{\frac{3}{2}}}\delta_{M,0} \tag{5.3.114}$$

now $k \in 2\mathbb{N}-1$.

This gives us a remarkably short expression for the coefficients of the expansion for

$k > 1$ in global coordinates, namely

$$A_{\boldsymbol{k}} = \frac{k}{2\pi^{\frac{3}{2}}}\delta_{M,0}\delta_{J,0}. \tag{5.3.115}$$

which is valid $\forall\ k \in \mathbb{N}$.

The next step in this computation is to compute the explicit form of P_{ij} in (5.3.88) and then include this in the computation of mode coefficients (5.3.90). This will give us Bogoliubov transformation from FG to global coordinates for the full tensorial perturbation, as opposed to that of the background scalar (which we expect to act as a proxy). This is ongoing work and we hope to complete this soon.

5.4 Conclusions

In this chapter we have further developed the understanding of a number of asymptotic aspects of AldS spacetimes, most often using analytic continuation of the already well-understood results in AlAdS spacetimes. This leads to results which, while mathematically consistent, require a careful approach to be understood physically.

The first main result is the map between the Starobinksky/Fefferman-Graham [166, 59] and Bondi coordinates in dS, a result which is obtained quickly via an analytic continuation of the AdS result discussed in chapter 3. A potential shortcoming of the Bondi coordinates in the (future) near boundary region of an AldS spacetime is that the Bondi time coordinate becomes spacelike due to the presence of a cosmological horizon in the spacetime. This issue leads us to consider how best to extract the physical interpretation of the Bondi coordinates in AldS spacetime. One promising proposal to approach this procedure [42, 45] is to perform the Bondi analysis (no-ingoing radiation condition etc) upon $\mathcal{H}_C$ rather than $\mathscr{I}^+$, preserving the timelike property of the Bondi time. From our perspective of comparing the asymptotic structure of the Bondi and Fefferman-Graham gauges, this is somewhat unsatisfactory as the FG gauge will not generally be applicable in the near $\mathcal{H}_C$ region of the spacetime. It seems that the best way to directly compare these gauges is by relaxing the restriction upon the Bondi time coordinate, now the radiation will be encoded in the asymptotic structure due to the $\tilde{r}$ dependence of the boundary metric.

This "radial" dependence seems to indicate the property that radiation in AldS spacetime is now described by a conservation law in space rather than time. This is confirmed by our charge prescription for AldS spacetimes, the second main result of the chapter. The important distinction between this prescription and previous discussions of charges in dS (e.g. [221, 210]) is that the slices of spacetime are now timelike hypersurfaces, and thus the charges are conserved with respect to translations in a spacelike direction. We found that for an asymptotic symmetry group consisting of asymptotic conformal Killing vectors of $\mathscr{I}^+$, the Wald Hamiltonians of AldS_4 spacetimes with no matter always exist and thus one finds no symplectic flux at $\mathscr{I}^+$. We note that the recent work [51] does consider the possibility of flux at $\mathscr{I}^+$ by considering asymptotic symmetries consisting of diffeomorphisms of $\mathscr{I}^+$. However, these boundary conditions are somewhat difficult to interpret, both for physical and mathematical reasons: Physically, the intuition from AdS/CFT of only being able to specify the conformal class of a boundary field given the bulk field is now lost. Mathematically, as was shown in [71], the conformal boundary conditions give rise to a well-posed variational problem, something the analysis of [51] gives up. It would be interesting to further examine the classes of boundary conditions which give rise to flux terms at $\mathscr{I}^+$.

Our final results in this chapter discuss the charge prescription applied to the straightforward example of a linear perturbation of dS_4. We chose this basic example with the aim of gaining a greater physical insight into our charges, in particular the unfamiliar process of spatial conservation. This lead to a number of examples of non-trivial charges, with a possible interesting future direction being a more detailed analysis of the relationship between the singular solutions and the gravitational memory effect in cosmological backgrounds [216]. The perturbations diverge as one moves towards the coordinate horizon of the inflationary patch of pure dS_4, and thus in order to obtain a solution suitably regular at the horizon we need to transform the mode coefficients in the style of Unruh [222, 217] (see also [223, 224] for applications to AdS). This computation is ongoing, but we would expect to find a regular solution consisting of a specific combination of singular modes. We hope to gain a greater insight into the global nature of the charges when armed with this solution.

CHAPTER 6

Summary and outlook

We conclude this thesis with an overall summary of the work performed and some outlook on work which could be performed as next steps of the material discussed in this thesis. In order to avoid repetition of previous chapters in the thesis, we point the reader to the more detailed concluding sections 3.5, 4.3, 5.4 at the end of their respective chapters.

The main purpose of this thesis was to investigate the physics of gravitational radiation in Al(A)dS spacetimes using a marriage of techniques from the literature on asymptotically flat spacetimes and holographic results from the AdS/CFT correspondence. In chapter 3 we have obtained a comprehensive understanding of the Bondi gauge when applied to Al(A)dS spacetimes, including the most general form of the asymptotic solution to the field equations, various integration schemes in order to solve such equations as well as a holographic understanding of all Bondi quantities via an explicit map from Bondi to Fefferman-Graham gauges. This analysis leads to a time-dependent conformal class of of metrics at $\mathscr{I}$ which capture the radiative nature of spacetimes which are merely asymptotically locally (A)dS.

In chapter 4 we extend these ideas to consider how the physics of gravitational radiation in AlAdS spacetime is encoded in the metric quantities, in particular when we attempt to understand the nature of Bondi mass and news. We were able to find an explicit counter example to the Bondi mass loss property in AlAdS spacetimes, showing that the energy associated with $\Lambda < 0$ is important, and thus the definition of a radiative mass may require modification. We also work on a method of modifying the Bondi gauge by removing the restrictive determinant condition of the original choice (unnatural from the perspective of AdS/CFT). This procedure should allow us to apply the gauge to additional metrics, in particular the Robinson-Trautman class of solutions which should act as a testing ground for gravitational radiation in AlAdS spacetimes.

In chapter 5 we have honed the focus on de Sitter spacetime in order to obtain a new asymptotic charge prescription, much of which is based upon analytic continuation of results in AlAdS spacetimes. These charges have the unusual feature of being conserved with spatial rather than time translations, and as such we provide a number of simple examples in order to understand the physics of such a prescription. A computation in order to understand the global mode expansion of the perturbations is under way and we hope that this will illuminate the global nature of the charge carried by the perturbation.

This work has a number of interesting potential future directions, some of which simply require continuation of computations that have been discussed in this thesis. In order to better understand the monotonicity properties of Bondi mass in AlAdS spacetimes, one needs to work beyond the small Λ limit, a procedure which may require a modification of the definition of mass aspect as one encounters Λ-dependent terms which break monotonicity. If this procedure is able to yield interesting results, then an illustrative example should be that of the Robinson-Trautman class of spacetimes. Understanding this example fully would first require a further understanding of the computation that we started in attempting to relate the RT spacetimes with the Bondi gauge. We also note that it would be of interest to attempt to derive such a mass from covariant phase space techniques as discussed in this thesis, although at this point in time there is a less obvious route to performing this computation. Finally, the global understanding of the charges in perturbed dS_4 spacetime is not yet obtained, we hope to complete this shortly in order to give a more complete picture of charges in AldS spacetimes.

In addition to these concrete computations, there are also some new topics in which this work could be applied. The first of these would be to use the new understanding of AlAdS spacetimes in Bondi gauge order to describe more advanced gravitational wave phenomena such as the gravitational memory effect. The effort to understand how memory can be described in curved backgrounds is an active research topic (see e.g. [216]), although none of the current work takes into account the subtle details of the “asymptotically locally” structure of the spacetime. By using the asymptotic comparison of the Bondi and Fefferman-Graham gauges, one could hope to analyse the geodesic deviation equations and understand how the distinction between radiation and background curvature play a role in the deviation of test particles. This work could also help illuminate the concept of Bondi news (and thus mass) in AlAdS spacetimes, as well as asymptotic symmetries and physical motivation for certain choices of boundary conditions [50, 51].

Finally, there is the potential for this work to be applied in the other direction, namely in using the holographic understanding of the Bondi gauge as presented in this thesis in order to gain an insight into the holography in asymptotically flat spacetimes. This is a procedure which would require use of the comparison between the Bondi and Fefferman-Graham gauges as presented here, together with a suitable flat limit of $\Lambda \to 0$. There are many subtleties associated with this limit which have not yet been fully addressed, although it seems that a further level of renormalisation may be required in order for this limit to exist [51]. We note that this is a somewhat different approach to understanding holography in asymptotically flat spacetimes than the 'celestial holography' approach of [88, 8, 87, 7, 97, 95, 127, 225, 226, 227] which conjectures that holography in asymptotically flat spacetime is described via an identification of a 2d CFT living on a cut of $\mathscr{I}^+$ of an asymptotically flat spacetime. The approach that we have sketched in terms of taking the flat limit would be an attempt to gain an understanding of flat holography directly from a 3d CFT instead, although it is expected that these CFTs of different dimensions would be related [228]. In any case, it is useful to have multiple approaches to the important unsolved problem of asymptotically flat holography.

APPENDIX A

Appendices for chapter 3

A.1 Supplementary conditions

In section 3.2.4, we explained how the $\{u\theta\}$ and $\{uu\}$ Einstein equations can be reduced to $f = 0$ and $g = 0$ where f, g are functions of (u, θ). Here we present these equations in the $\Lambda < 0$ case as constraints upon the derivatives of the functions $U_{3,u}$ and $W_{3,u}$. We have used the normalisation of $l = \sqrt{-3/\Lambda} = 1$. The formulae for $\Lambda > 0$ can be obtained by using dimensional analysis (see section 3.4.2.2) to reinstate l and then Λ.

$f = 0$**:**

$$\begin{aligned}
U_{3,u} = &\frac{4}{3}\gamma_3((2\gamma_{0,\theta} - 4\cot(\theta))(U_0)^2 + 2(e^{2\beta_0}\gamma_1 - U_{0,\theta} + \gamma_{0,u})U_0 - \\
&3e^{4\beta_0-2\gamma_0}(\cot(\theta) + \beta_{0,\theta} - \gamma_{0,\theta})) + \frac{1}{9}(28e^{2\beta_0}U_0(\gamma_1)^4 - \\
&30e^{4\beta_0-2\gamma_0}(\cot(\theta) - \beta_{0,\theta} - \gamma_{0,\theta})(\gamma_1)^3 - 14(U_0)^2(\cot(\theta) - 2\gamma_{0,\theta})(\gamma_1)^3 + \\
&14U_0(U_{0,\theta} + 2\gamma_{0,u})(\gamma_1)^3 + 3e^{2\beta_0-2\gamma_0}(U_{0,\theta}(7\cot(\theta) + 8\beta_{0,\theta} - 8\gamma_{0,\theta}) + \\
&e^{2\beta_0}\gamma_{1,\theta} - U_{0,\theta\theta} + 4(-4\cot(\theta) + \beta_{0,\theta} + 4\gamma_{0,\theta})\gamma_{0,u} + 3\beta_{0,u\theta} - 8\gamma_{0,u\theta})(\gamma_1)^2 - \\
&6U_0W_3\gamma_1 + 3e^{2\beta_0-4\gamma_0}(2e^{2\beta_0}(-4(\gamma_{0,\theta})^3 + 6\cot(\theta)(\gamma_{0,\theta})^2 + \\
&(3\csc^2(\theta) + 4\beta_{0,\theta\theta} + 6\gamma_{0,\theta\theta} + 2)\gamma_{0,\theta} + 8(\beta_{0,\theta})^2(2\cot(\theta) - \gamma_{0,\theta}) + \\
&2\cot(\theta)\beta_{0,\theta\theta} - 3\cot(\theta)\gamma_{0,\theta\theta} - 2\beta_{0,\theta}(\csc^2(\theta) - 4(\gamma_{0,\theta})^2 + 8\cot(\theta)\gamma_{0,\theta} + \\
&4\beta_{0,\theta\theta} + 2\gamma_{0,\theta\theta} + 2(-2\beta_{0,\theta\theta\theta} - \gamma_{0,\theta\theta\theta}) + e^{2\gamma_0}(4\gamma_{1,\theta}(U_{0,\theta} - 2\gamma_{0,u}) + \\
&2(8\cot(\theta) + 3\beta_{0,\theta} - 8\gamma_{0,\theta})\gamma_{1,u} + 3\gamma_{1,u\theta}))\gamma_1 - 24e^{2\beta_0}U_0\gamma_4 - 12U_0U_{3,\theta} + \\
&3e^{2\beta_0-2\gamma_0}W_{3,\theta} + 12e^{2\beta_0-2\gamma_0}W_3\beta_{0,\theta} - 12U_0U_3(\cot(\theta) + \gamma_{0,\theta}) - \\
&18e^{4\beta_0-2\gamma_0}\gamma_{3,\theta} - 24(U_0)^2\gamma_{3,\theta} + \\
&12e^{4\beta_0-4\gamma_0}(\gamma_{1,\theta}(2\beta_{0,\theta}(\cot(\theta) + 3\beta_{0,\theta} - 2\gamma_{0,\theta}) + \beta_{0,\theta\theta}) + \beta_{0,\theta}\gamma_{1,\theta\theta}) + \\
&3e^{2\beta_0-2\gamma_0}U_0((\csc^2(\theta) + 16(\gamma_{0,\theta})^2 + 8\beta_{0,\theta}(2\cot(\theta) + \beta_{0,\theta}) - \\
&4(7\cot(\theta) + 3\beta_{0,\theta})\gamma_{0,\theta} + 4\beta_{0,\theta\theta} - 12(\gamma_{0,\theta\theta} + 1))(\gamma_1)^2 + \\
&2((15\cot(\theta) + 12\beta_{0,\theta} - 16\gamma_{0,\theta})\gamma_{1,\theta} + 3\gamma_{1,\theta\theta})\gamma_1 + 10(\gamma_{1,\theta})^2) + \\
&6U_3(2e^{2\beta_0}\gamma_1 - 2U_{0,\theta} + 3\beta_{0,u} - \gamma_{0,u}) + 21e^{2\beta_0-2\gamma_0}\gamma_{1,\theta}\gamma_{1,u} - 24U_0\gamma_{3,u})
\end{aligned} \tag{A.1.1}$$

$g = 0$**:**

$$\begin{aligned}
W_{3,u} = &3e^{4\beta_0}\gamma_1^4 + \frac{1}{2}e^{-2\gamma_0}(e^{2\gamma_0}U_{0,\theta}^2 - e^{4\beta_0}(8\mathrm{ct}^2(\theta) - 16\gamma_{0,\theta}\mathrm{ct}(\theta) + 4\beta_{0,\theta}^2 + 8\gamma_{0,\theta}^2 + \\
&7\beta_{0,\theta}(\mathrm{ct}(\theta) - 2\gamma_{0,\theta}) + 7\beta_{0,\theta\theta}))(\gamma_1)^2 + \frac{1}{2}e^{-2\gamma_0}(-12e^{4\beta_0+2\gamma_0}\gamma_3 - \\
&2U_{0,\theta}(4e^{2\beta_0}\mathrm{ct}^2(\theta) - 9e^{2\beta_0}\gamma_{0,\theta}\mathrm{ct}(\theta) - 3e^{2\beta_0}\csc^2(\theta) + 6e^{2\beta_0}\beta_{0,\theta}^2 + 6e^{2\beta_0}\gamma_{0,\theta}^2 + \\
&e^{2\beta_0}\beta_{0,\theta}(11\mathrm{ct}(\theta) - 16\gamma_{0,\theta}) + 3e^{2\beta_0}\beta_{0,\theta\theta} - 5e^{2\beta_0}\gamma_{0,\theta\theta} - 2e^{2\gamma_0}\gamma_{1,u}) + \\
&e^{2\beta_0}(-e^{2\beta_0}(15\mathrm{ct}(\theta) - 14\beta_{0,\theta} - 14\gamma_{0,\theta})\gamma_{1,\theta} + 4(-\mathrm{ct}(\theta) + \beta_{0,\theta} + \gamma_{0,\theta})U_{0,\theta\theta} + \\
&e^{2\beta_0}\gamma_{1,\theta\theta} + 2U_{0,\theta\theta\theta} + 32\mathrm{ct}(\theta)\beta_{0,\theta}\gamma_{0,u} - 32\beta_{0,\theta}\gamma_{0,\theta}\gamma_{0,u} - 12\mathrm{ct}(\theta)\beta_{0,u\theta} - \\
&16\beta_{0,\theta}\beta_{0,u\theta} + 16\gamma_{0,\theta}\beta_{0,u\theta} + 4\mathrm{ct}(\theta)\gamma_{0,u\theta} + 24\beta_{0,\theta}\gamma_{0,u\theta} - 8\gamma_{0,\theta}\gamma_{0,u\theta} - 4\beta_{0,u\theta\theta} + \\
&4\gamma_{0,u\theta\theta}))\gamma_1 + 8e^{4\beta_0-4\gamma_0}(\gamma_{0,\theta})^4 - 16e^{4\beta_0-4\gamma_0}\mathrm{ct}(\theta)(\beta_{0,\theta})^3 + (-14e^{4\beta_0-4\gamma_0}\mathrm{ct}(\theta) - \\
&8e^{4\beta_0-4\gamma_0}\beta_{0,\theta})(\gamma_{0,\theta})^3 - \frac{1}{2}e^{4\beta_0-2\gamma_0}(\gamma_{1,\theta})^2 + (U_0)^3(\frac{7}{3}e^{2\gamma_0-2\beta_0}(\mathrm{ct}(\theta) - 2\gamma_{0,\theta})(\gamma_1)^3 + \\
&4e^{2\gamma_0-2\beta_0}(\gamma_3(2\mathrm{ct}(\theta) - \gamma_{0,\theta}) + \gamma_{3,\theta})) + 4e^{2\beta_0-4\gamma_0}\beta_{0,\theta}^2(-e^{2\beta_0}\mathrm{ct}^2(\theta) - 2e^{2\beta_0} + \\
&2e^{2\beta_0}\csc^2(\theta) - 4e^{2\beta_0}\beta_{0,\theta\theta} + 2e^{2\beta_0}\gamma_{0,\theta\theta} + e^{2\gamma_0}\gamma_{1,u}) + \gamma_{0,\theta}^2(-32e^{4\beta_0-4\gamma_0}\beta_{0,\theta}^2 + \\
&28e^{4\beta_0-4\gamma_0}\mathrm{ct}(\theta)\beta_{0,\theta} - e^{2\beta_0-4\gamma_0}(-3e^{2\beta_0}\mathrm{ct}^2(\theta) + 4e^{2\beta_0} + 9e^{2\beta_0}\csc^2(\theta) +
\end{aligned}$$

$$
\begin{aligned}
&16e^{2\beta_0}\beta_{0,\theta\theta}+18e^{2\beta_0}\gamma_{0,\theta\theta}+4e^{2\gamma_0}\gamma_{1,u}))+(U_0)^2(\frac{1}{3}(-14)e^{2\gamma_0}\gamma_1^4-\\
&\frac{7}{3}e^{2\gamma_0-2\beta_0}(U_{0,\theta}+2\gamma_{0,u})(\gamma_1)^3+\frac{1}{2}(-19\mathrm{ct}^2(\theta)+60\gamma_{0,\theta}\mathrm{ct}(\theta)+10\csc^2(\theta)-\\
&8\beta_{0,\theta}^2-32\gamma_{0,\theta}^2+\beta_{0,\theta}(8\gamma_{0,\theta}-17\mathrm{ct}(\theta))-7\beta_{0,\theta\theta}+20\gamma_{0,\theta\theta}+8)(\gamma_1)^2+\\
&\frac{1}{2}e^{-2\beta_0}(2e^{2\gamma_0}W_3-e^{2\beta_0}(8e^{2\gamma_0}\gamma_3+(51\mathrm{ct}(\theta)+30\beta_{0,\theta}-56\gamma_{0,\theta})\gamma_{1,\theta}+9\gamma_{1,\theta\theta}))\gamma_1+\\
&\frac{1}{2}e^{-2\beta_0}(-13e^{2\beta_0}(\gamma_{1,\theta})^2+8e^{2(\beta_0+\gamma_0)}\gamma_4)+8e^{2\gamma_0}\gamma_3U_{0,\theta}+7e^{2\gamma_0}U_{3,\theta}+\\
&3e^{2\gamma_0}U_3(3\mathrm{ct}(\theta)-2\beta_{0,\theta}+2\gamma_{0,\theta})-8e^{2\gamma_0}\gamma_3\gamma_{0,u}+8e^{2\gamma_0}\gamma_{3,u}))+\\
&\gamma_{1,\theta}(-2e^{2\beta_0-2\gamma_0}\beta_{0,\theta}(3U_{0,\theta}-4\gamma_{0,u})-\frac{1}{2}e^{2\beta_0-2\gamma_0}(13\mathrm{ct}(\theta)U_{0,\theta}+2U_{0,\theta\theta}+\\
&8\beta_{0,u\theta}-4\gamma_{0,u\theta}))+e^{2\beta_0-4\gamma_0}\beta_{0,\theta}(-e^{2\beta_0}\mathrm{ct}(\theta)\csc^2(\theta)-2e^{2\beta_0}\mathrm{ct}(\theta)+3e^{4\gamma_0}U_3-\\
&16e^{2\beta_0}\mathrm{ct}(\theta)\beta_{0,\theta\theta}-10e^{2\beta_0}\mathrm{ct}(\theta)\gamma_{0,\theta\theta}-8e^{2\beta_0}\beta_{0,\theta\theta\theta}-2e^{2\beta_0}\gamma_{0,\theta\theta\theta}- \qquad \text{(A.2)}\\
&10e^{2\gamma_0}\mathrm{ct}(\theta)\gamma_{1,u}-4e^{2\gamma_0}\gamma_{1,u\theta})+\gamma_{0,\theta}(32e^{4\beta_0-4\gamma_0}(\beta_{0,\theta})^3+8e^{4\beta_0-4\gamma_0}\mathrm{ct}(\theta)\beta_{0,\theta}^2+\\
&2e^{2\beta_0-4\gamma_0}(-2e^{2\beta_0}\mathrm{ct}^2(\theta)+6e^{2\beta_0}+3e^{2\beta_0}\csc^2(\theta)+24e^{2\beta_0}\beta_{0,\theta\theta}+6e^{2\beta_0}\gamma_{0,\theta\theta}+\\
&4e^{2\gamma_0}\gamma_{1,u})\beta_{0,\theta}+8e^{2\beta_0-2\gamma_0}U_{0,\theta}\gamma_{1,\theta}+\frac{1}{2}e^{2\beta_0-4\gamma_0}(-3e^{2\beta_0}\mathrm{ct}(\theta)\csc^2(\theta)+2e^{2\beta_0}\mathrm{ct}(\theta)+\\
&8e^{2\beta_0}\mathrm{ct}(\theta)\beta_{0,\theta\theta}+30e^{2\beta_0}\mathrm{ct}(\theta)\gamma_{0,\theta\theta}+16e^{2\beta_0}\beta_{0,\theta\theta\theta}+10e^{2\beta_0}\gamma_{0,\theta\theta\theta}+\\
&12e^{2\gamma_0}\mathrm{ct}(\theta)\gamma_{1,u}+8e^{2\gamma_0}\gamma_{1,u\theta}))+U_0(10e^{2\beta_0}(\mathrm{ct}(\theta)-\beta_{0,\theta}-\gamma_{0,\theta})(\gamma_1)^3+\\
&(U_{0,\theta}(-6\mathrm{ct}(\theta)-8\beta_{0,\theta}+8\gamma_{0,\theta})-e^{2\beta_0}\gamma_{1,\theta}+U_{0,\theta\theta}+16\mathrm{ct}(\theta)\gamma_{0,u}-4\beta_{0,\theta}\gamma_{0,u}-\\
&16\gamma_{0,\theta}\gamma_{0,u}-3\beta_{0,u\theta}+8\gamma_{0,u\theta})(\gamma_1)^2+e^{-2\gamma_0}(-e^{2\beta_0}\mathrm{ct}^3(\theta)+3e^{2\beta_0}\gamma_{0,\theta}\mathrm{ct}^2(\theta)+\\
&2e^{2\beta_0}\csc^2(\theta)\mathrm{ct}(\theta)-14e^{2\beta_0}\gamma_{0,\theta}^2\mathrm{ct}(\theta)-9e^{2\beta_0}\beta_{0,\theta\theta}\mathrm{ct}(\theta)+9e^{2\beta_0}\gamma_{0,\theta\theta}\mathrm{ct}(\theta)-\\
&14e^{2\gamma_0}\gamma_{1,u}\mathrm{ct}(\theta)+8e^{2\beta_0}(\gamma_{0,\theta})^3-4e^{4\gamma_0}U_3-4e^{2\beta_0}\gamma_{0,\theta}-10e^{2\beta_0}\csc^2(\theta)\gamma_{0,\theta}-\\
&2e^{2\beta_0}\beta_{0,\theta}^2(15\mathrm{ct}(\theta)-8\gamma_{0,\theta})-2e^{2\gamma_0}U_{0,\theta}\gamma_{1,\theta}-16e^{2\beta_0}\gamma_{0,\theta}\gamma_{0,\theta\theta}+2e^{2\beta_0}\beta_{0,\theta\theta\theta}+\\
&4e^{2\beta_0}\gamma_{0,\theta\theta\theta}+8e^{2\gamma_0}\gamma_{1,\theta}\gamma_{0,u}+16e^{2\gamma_0}\gamma_{0,\theta}\gamma_{1,u}+\beta_{0,\theta}(-13e^{2\beta_0}\mathrm{ct}^2(\theta)+\\
&60e^{2\beta_0}\gamma_{0,\theta}\mathrm{ct}(\theta)+8e^{2\beta_0}+12e^{2\beta_0}\csc^2(\theta)-32e^{2\beta_0}\gamma_{0,\theta}^2+8e^{2\beta_0}\beta_{0,\theta\theta}+\\
&20e^{2\beta_0}\gamma_{0,\theta\theta}-6e^{2\gamma_0}\gamma_{1,u})-3e^{2\gamma_0}\gamma_{1,u\theta})\gamma_1+\frac{1}{2}e^{-2(\beta_0+\gamma_0)}(-7e^{4\beta_0}\gamma_{1,\theta}\mathrm{ct}^2(\theta)-\\
&48e^{4\beta_0}\beta_{0,\theta}\gamma_{1,\theta}\mathrm{ct}(\theta)+16e^{4\beta_0}\gamma_{0,\theta}\gamma_{1,\theta}\mathrm{ct}(\theta)-7e^{4\beta_0}\gamma_{1,\theta\theta}\mathrm{ct}(\theta)+8e^{4\gamma_0}U_3U_{0,\theta}-\\
&4e^{2(\beta_0+\gamma_0)}W_{3,\theta}-e^{2(\beta_0+\gamma_0)}W_3(3\mathrm{ct}(\theta)+4\beta_{0,\theta})+\\
&24e^{4\beta_0+2\gamma_0}\gamma_3(\mathrm{ct}(\theta)+\beta_{0,\theta}-\gamma_{0,\theta})+8e^{4\beta_0}\csc^2(\theta)\gamma_{1,\theta}-40e^{4\beta_0}\beta_{0,\theta}^2\gamma_{1,\theta}-\\
&8e^{4\beta_0}\gamma_{0,\theta}^2\gamma_{1,\theta}+64e^{4\beta_0}\beta_{0,\theta}\gamma_{0,\theta}\gamma_{1,\theta}+12e^{4\beta_0+2\gamma_0}\gamma_{3,\theta}-12e^{4\beta_0}\gamma_{1,\theta}\beta_{0,\theta\theta}+\\
&8e^{4\beta_0}\gamma_{1,\theta}\gamma_{0,\theta\theta}-16e^{4\beta_0}\beta_{0,\theta}\gamma_{1,\theta\theta}+8e^{4\beta_0}\gamma_{0,\theta}\gamma_{1,\theta\theta}-2e^{4\beta_0}\gamma_{1,\theta\theta\theta}+6e^{4\gamma_0}U_{3,u}-\\
&12e^{4\gamma_0}U_3\beta_{0,u}+4e^{4\gamma_0}U_3\gamma_{0,u}-6e^{2(\beta_0+\gamma_0)}\gamma_{1,\theta}\gamma_{1,u}))+\frac{1}{2}e^{-4\gamma_0}(-2e^{4\beta_0}\beta_{0,\theta\theta}\mathrm{ct}^2(\theta)-\\
&3e^{4\beta_0}\gamma_{0,\theta\theta}\mathrm{ct}^2(\theta)-4e^{2(\beta_0+\gamma_0)}\gamma_{1,u}\mathrm{ct}^2(\theta)+3e^{2\beta_0+4\gamma_0}U_3\mathrm{ct}(\theta)-4e^{4\beta_0}\beta_{0,\theta\theta\theta}\mathrm{ct}(\theta)-\\
&4e^{4\beta_0}\gamma_{0,\theta\theta\theta}\mathrm{ct}(\theta)-6e^{2(\beta_0+\gamma_0)}\gamma_{1,u\theta}\mathrm{ct}(\theta)-8e^{4\beta_0}(\beta_{0,\theta\theta})^2+6e^{4\beta_0}(\gamma_{0,\theta\theta})^2+\\
&4e^{4\gamma_0}(\gamma_{1,u})^2+3e^{2\beta_0+4\gamma_0}U_{3,\theta}-4e^{4\beta_0}\beta_{0,\theta\theta}+4e^{4\beta_0}\csc^2(\theta)\beta_{0,\theta\theta}+2e^{4\beta_0}\gamma_{0,\theta\theta}+\\
&6e^{4\beta_0}\csc^2(\theta)\gamma_{0,\theta\theta}+8e^{4\beta_0}\beta_{0,\theta\theta}\gamma_{0,\theta\theta}-5e^{2(\beta_0+\gamma_0)}U_{0,\theta}\gamma_{1,\theta\theta}-2e^{4\beta_0}\beta_{0,\theta\theta\theta\theta}-
\end{aligned}
$$

$$e^{4\beta_0}\gamma_{0,\theta\theta\theta\theta} - e^{4\gamma_0}W_3(3U_{0,\theta} - 4\beta_{0,u}) + 4e^{2(\beta_0+\gamma_0)}\csc^2(\theta)\gamma_{1,u}+$$
$$4e^{2(\beta_0+\gamma_0)}\beta_{0,\theta\theta}\gamma_{1,u} + 4e^{2(\beta_0+\gamma_0)}\gamma_{0,\theta\theta}\gamma_{1,u} - 2e^{2(\beta_0+\gamma_0)}\gamma_{1,u\theta\theta})$$

where we have used the abbreviations 'ct(θ)' to refer to the cotangent function and 'csc(θ)' for cosecant. These two equations are essential to check that $g_{(3)ab}$ satisfies the conservation property (3.4.4).

The supplementary conditions are enormously complicated by the presence of $\Lambda \neq 0$ with non-trivial coefficients γ_0, β_0, U_0. These formulae simplify significantly in the asymptotically (A)dS and asymptotically flat cases.

Asymptotically (A)dS spacetimes in Bondi coordinates have $\gamma_0 = \beta_0 = U_0 = 0$ which gives $\gamma_1 = 0$ by equation (3.2.16). Setting these values in the supplementary equations above gives us

$$U_{3,u} = \frac{1}{3}(4\Lambda\cot(\theta)\gamma_3 + W_{3,\theta} + 2\Lambda\gamma_{3,\theta}) \tag{A.3a}$$

$$W_{3,u} = -\frac{1}{2}\Lambda(\cot(\theta)U_3 + U_{3,\theta}) \tag{A.3b}$$

where we have reinstated the factors of Λ using dimensional analysis.

For the asymptotically flat supplementary conditions, we again have $\gamma_0 = \beta_0 = U_0 = 0$ as well as $\Lambda = 0$ but now $\gamma_1 \neq 0$. As given in [1], the asymptotically flat supplementary conditions are

$$U_{3,u} = \frac{1}{3}\left(7\gamma_{1,\theta}\gamma_{1,u} + \gamma_1\left(3\gamma_{1,u\theta} + 16\cot(\theta)\gamma_{1,u}\right) + W_{3,\theta}\right) \tag{A.4a}$$

$$W_{3,u} = 2\left(\gamma_{1,u}\right)^2 + 2\gamma_{1,u} - \gamma_{1,u\theta\theta} - 3\cot(\theta)\gamma_{1,u\theta}. \tag{A.4b}$$

A.2 Intermediate pieces of the Fefferman-Graham transformation

In this appendix we provide formulae for transforming the Bondi gauge metric into the Fefferman-Graham form. Expressions for the intermediate metric tensors are omitted for brevity.

A.2.1 Vanishing of $g_{(1)}$

In this section we demonstrate explicitly that $g_{(1)}$ vanishes. Note first that the Bondi metric (3.4.22) used to compute $g_{(0)}$ is insufficient for computing $g_{(1)}$: it only includes the solution to Einstein's equations at leading order but for $g_{(1)}$ we require $1/r \sim \rho$ contributions to the metric. To compute $g_{(1)}$ we therefore need to retain the following contributions to the metric functions

$$\gamma(u,r,\theta) = \gamma_0(u,\theta) + \frac{\gamma_1(u,\theta)}{r} \tag{A.2.1a}$$

$$\beta(u,r,\theta) = \beta_0(u,\theta) \tag{A.2.1b}$$

$$U(u,r,\theta) = U_0(u,\theta) + \frac{2}{r}e^{2(\beta_0(u,\theta)-\gamma_0(u,\theta))}\beta_{0,\theta}(u,\theta) \tag{A.2.1c}$$

$$W(u,r,\theta) = e^{2\beta_0(u,\theta)} + \frac{\cot(\theta)U_0(u,\theta) + U_{0,\theta}}{r}. \tag{A.2.1d}$$

which are the solutions to the field equations (3.2.7a-3.2.7d) up to $\mathcal{O}(1/r)$. Note also that we use the normalisation $l = 1$.

As before, we begin with the Bondi metric in the form (3.4.5) and transform into the coordinates (t, r_*, θ, ϕ) using transformations (3.4.8) and (3.4.11)

$$u = t - r_* \tag{A.2.2a}$$

$$r = \tan\left(r_* + \frac{\pi}{2}\right) \tag{A.2.2b}$$

where we have written the transformation (3.4.11) in exact form.

Next we extend the transformations (3.4.15, 3.4.24) to one order higher in ρ

$$r_* \to \rho + b_1(t,\theta)\rho^2 \tag{A.2.3a}$$

$$t \to t + \alpha_1(t,\theta)\rho + b_2(t,\theta)\rho^2 \tag{A.2.3b}$$

$$\theta \to \theta + \alpha_2(t,\theta)\rho + b_3(t,\theta)\rho^2 \tag{A.2.3c}$$

where $\alpha_{1,2}$ are given in (3.4.30) and $b_{1,2,3}$ are to be determined. When considering how the differentials transform it will again be sufficient to consider the pieces which contribute to the metric at $\mathcal{O}(1/\rho)$

$$dr_* \to d\rho + 2b_1\rho d\rho \tag{A.2.4a}$$

$$dt \to dt + \alpha_1 d\rho + (\partial_t\alpha_1)\rho dt + (\partial_\theta\alpha_1)\rho d\theta + 2\rho b_2 d\rho \tag{A.2.4b}$$

$$d\theta \to d\theta + \alpha_2 d\rho + (\partial_t\alpha_2)\rho dt + (\partial_\theta\alpha_2)\rho d\theta + 2\rho b_3 d\rho. \tag{A.2.4c}$$

i.e. we do not need to include terms of $\mathcal{O}(\rho^2)$ or higher.

The final subtlety when applying this procedure is to take into account that the metric functions (γ, β, U, W) are all functions of $(t - r_*, \theta)$ prior to applying these transformations.

Terms up to $\mathcal{O}(\rho)$ need to be included in these arguments, i.e.

$$t - r_* \to t + \rho\alpha_1 - \rho + \mathcal{O}(\rho^2) = t + \rho(\alpha_1 - 1) + \mathcal{O}(\rho^2) \tag{A.2.5}$$

and

$$\theta \to \theta + \alpha_2\rho + \mathcal{O}(\rho^2), \tag{A.2.6}$$

to calculate all terms contributing at $\mathcal{O}(1/\rho)$.

At order $1/\rho$ we are initially left with a seemingly non-zero term with dependence upon our three undetermined transformation coefficients $b_{1,2,3}$. To fix $b_{1,2,3}$ we enforce the following

$$g_{(1)\rho\rho}(b_1, b_2, b_3) = g_{(1)\rho t}(b_1, b_2, b_3) = g_{(1)\rho\theta}(b_1, b_2, b_3) = 0 \tag{A.2.7}$$

which gives us three equations for the three unknowns $b_{1,2,3}$ ($g_{(1)\rho\phi}$ vanishes automatically by the axi and reflection symmetry). It turns out that the $g_{\rho\rho}$ term is given by

$$g_{(1)\rho\rho} = g_{(1)\rho\rho}(b_1) = 2b_1 + e^{-2\hat{\beta}_0}\cot(\theta)\hat{U}_0 + e^{-2\hat{\beta}_0}\hat{U}_{0,\theta} \tag{A.2.8}$$

so we can solve $g_{(1)\rho\rho} = 0$ for b_1 and then we will be left with two equations for the other two unknowns $b_{2,3}$. Solving $g_{(1)\rho\rho} = 0$ gives us

$$b_1 = -\frac{1}{2}e^{-2\hat{\beta}_0}(\hat{U}_{0,\theta} + \cot(\theta)\hat{U}_0) \tag{A.2.9a}$$

using b_1 it is straightforward to now solve the remaining equations of (A.2.7), with solutions

$$b_2 = -\frac{1}{2}e^{-4\hat{\beta}_0}(e^{2\hat{\beta}_0}(\hat{U}_{0,\theta} + \cot(\theta)\hat{U}_0) + 2(\hat{\beta}_{0,t} + \hat{\beta}_{0,\theta}\hat{U}_0)) \tag{A.2.9b}$$

$$b_3 = e^{-2\hat{\gamma}_0}\hat{\beta}_{0,\theta} + \frac{1}{2}e^{-4\hat{\beta}_0}(\hat{U}_{0,t} + \hat{U}_0(\hat{U}_{0,\theta} - 2(\hat{\beta}_{0,t} + \hat{\beta}_{0,\theta}\hat{U}_0))). \tag{A.2.9c}$$

where all function arguments are (t, θ).

Enforcing equations (A.2.9a-A.2.9c) in the transformation should make all other coefficients at $\mathcal{O}(1/\rho)$ vanish. To check this we input the values of $b_{1,2,3}$ in (A.2.9a-A.2.9c). At $\mathcal{O}(1/\rho)$, the line element reduces to

$$\begin{aligned} ds^2_{(1)} = -\frac{1}{2}e^{-2(\hat{\beta}_0+\hat{\gamma}_0)}&(2dt^2e^{4\hat{\gamma}_0}\hat{U}_0^2 - 4dtd\theta e^{4\hat{\gamma}_0}\hat{U}_0 + 2d\theta^2e^{4\hat{\gamma}_0} - 2\sin^2(\theta)d\phi^2)\times \\ &(\hat{U}_{0,\theta} + 2\hat{\gamma}_{0,t} - \cot(\theta)\hat{U}_0 + 2\hat{\gamma}_{0,\theta}\hat{U}_0 + 2e^{2\hat{\beta}_0}\hat{\gamma}_1). \end{aligned} \tag{A.2.10}$$

From equation (3.2.16)

$$\gamma_1 = \frac{1}{2}e^{-2\beta_0}(\cot(\theta)U_0 - U_{0,\theta} - 2U_0\gamma_{0,\theta} - 2\gamma_{0,u}) \tag{A.2.11}$$

and thus the second line of (A.2.10) is precisely this Einstein equation (at the boundary), forcing equation (A.2.10) to vanish, as required.

A.2.2 Checking $g_{(2)}$

The $g_{(2)}$ term in the Fefferman-Graham expansion is a useful consistency check as it must take the form [145, 60]

$$g_{(2)ab} = -R_{(0)ab} + \frac{1}{4} R_{(0)} g_{(0)ab} \tag{A.2.12}$$

where $R_{(0)ab}$ and $R_{(0)}$ are respectively the Ricci tensor and scalar of the boundary metric tensor $g_{(0)ab}$.

We now proceed to compute $g_{(2)}$ from the Fefferman-Graham expansion and check it via use of the formula above. The procedure for this step is the same as before with a term of one order higher added in each step. We impose the solutions to the Einstein equations as terms up to $\mathcal{O}(1/r^2)$. This procedure gives us the functions

$$\gamma(u,r,\theta) = \gamma_0 + \frac{\gamma_1}{r} \tag{A.2.13a}$$

$$\beta(u,r,\theta) = \beta_0 - \frac{\gamma_1^2}{4r^2} \tag{A.2.13b}$$

$$\begin{aligned} U(u,r,\theta) = U_0 + \frac{2}{r}\beta_{0,\theta} e^{2(\beta_0-\gamma_0)} - \\ \frac{1}{r^2} e^{2\beta_0 - 2\gamma_0}(2\beta_{0,\theta}\gamma_1 - 2\gamma_{0,\theta}\gamma_1 + \gamma_{1,\theta} + 2\cot(\theta)\gamma_1) \end{aligned} \tag{A.2.13c}$$

$$\begin{aligned} W(u,r,\theta) = e^{2\beta_0} + \frac{1}{r}[\cot(\theta)U_0 + U_{0,\theta}] + \\ \frac{1}{2r^2} e^{2(\beta_0-\gamma_0)}[2 - 3e^{2\gamma_0}\gamma_1^2 + 4\cot(\theta)\beta_{0,\theta} + 8(\beta_{0,\theta})^2 + 6\cot(\theta)\gamma_{0,\theta} - \\ 8\beta_{0,\theta}\gamma_{0,\theta} - 4(\gamma_{0,\theta})^2 + 4\beta_{0,\theta\theta} + 2\gamma_{0,\theta\theta}] \end{aligned} \tag{A.2.13d}$$

where, as usual, all of the coefficient functions are taken to be functions of (u,θ). We will also make use of (3.2.16) throughout.

The full transformation is again performed by first using the transformations of (A.2.2) to move into real time t and tortoise coordinate r_* before expanding our coordinates (r_*, t, θ) in a series in powers of ρ. In order to correctly compute $g_{(2)}$ these power series will include terms up to $\mathcal{O}(\rho^3)$. We use the choices of α_i and β_i as before and introduce new unknown coefficients $c_i(t,\theta)$ at the next order

$$\begin{aligned} r_* &\to \rho + b_1(t,\theta)\rho^2 + c_1(t,\theta)\rho^3 \\ t &\to t + \alpha_1(t,\theta)\rho + b_2(t,\theta)\rho^2 + c_2(t,\theta)\rho^3 \\ \theta &\to \theta + \alpha_2(t,\theta)\rho + b_3(t,\theta)\rho^2 + c_3(t,\theta)\rho^3. \end{aligned} \tag{A.2.14}$$

The procedure for obtaining the c_i is very similar to that for b_i: we fix them by setting $g_{(2)\rho\rho} = g_{(2)\rho t} = g_{(2)\rho\theta} = 0$. This gives

$$
\begin{aligned}
8c_1(t,\theta) = &\frac{1}{3}e^{-4\hat{\beta}_0-2\hat{\gamma}_0}(-8e^{4\hat{\beta}_0+2\hat{\gamma}_0} - 12e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2 - 24e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta}+\\
&6e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta} + 18\cot(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta} + 6e^{4\hat{\beta}_0} + 24e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2 + 12e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+\\
&12\cot(\theta)e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta} - 6e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2 - 12\hat{\beta}_{0,\theta}e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{U}_0 - 12\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}-\\
&12\cot(\theta)\hat{\beta}_{0,\theta}e^{2\hat{\gamma}_0}(\hat{U}_0)^2 - 12\cot(\theta)\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{U}_0 - 6e^{2\hat{\gamma}_0}(\hat{U}_0)^2-\\
&6e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,\theta})^2(\hat{U}_0)^2 - 6e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,\theta}\hat{U}_0 + 6e^{2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}\hat{U}_0-\\
&12e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}\hat{U}_0 + 3e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2 - 6e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}\hat{U}_{0,\theta} + 6e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}-\\
&3\cot^2(\theta)e^{2\hat{\gamma}_0}(\hat{U}_0)^2 + 6\cot(\theta)e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}(\hat{U}_0)^2 + 18\cot(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{U}_0+\\
&6\cot(\theta)e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}\hat{U}_0 + 6\cot(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t})
\end{aligned}
\tag{A.2.15a}
$$

$$
\begin{aligned}
8c_2(t,\theta) = &-\frac{1}{3}e^{-6\hat{\beta}_0-2\hat{\gamma}_0}(2e^{2\hat{\gamma}_0}(\hat{U}_0)^2 + 6e^{2\hat{\beta}_0+2\hat{\gamma}_0}(\hat{U}_0)^2 + 4e^{2\hat{\gamma}_0}\cot^2(\theta)(\hat{U}_0)^2+\\
&3e^{2\hat{\beta}_0+2\hat{\gamma}_0}\cot^2(\theta)(\hat{U}_0)^2 - 4e^{2\hat{\gamma}_0}\csc^2(\theta)(\hat{U}_0)^2 + 32e^{2\hat{\gamma}_0}(\hat{\beta}_{0,\theta})^2(\hat{U}_0)^2+\\
&6e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,\theta})^2(\hat{U}_0)^2 + 6e^{2\hat{\beta}_0+2\hat{\gamma}_0}(\hat{\gamma}_{0,\theta})^2(\hat{U}_0)^2 - 4e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,\theta}(\hat{U}_0)^2+\\
&12e^{2\hat{\beta}_0+2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,\theta}(\hat{U}_0)^2 - 6e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,\theta}(\hat{U}_0)^2-\\
&6e^{2\hat{\beta}_0+2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,\theta}(\hat{U}_0)^2 - 8e^{2\hat{\gamma}_0}\hat{\beta}_{0,\theta\theta}(\hat{U}_0)^2-\\
&18e^{2\hat{\beta}_0+2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,\theta}\hat{U}_0 - 12e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\beta}_{0,\theta}\hat{U}_0 + 12e^{2\hat{\beta}_0+2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\beta}_{0,\theta}\hat{U}_0+\\
&6e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\gamma}_{0,\theta}\hat{U}_0 + 6e^{2\hat{\beta}_0+2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\gamma}_{0,\theta}\hat{U}_0 + 2e^{2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}\hat{U}_0-\\
&6e^{2\hat{\beta}_0+2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}\hat{U}_0 - 4e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,t}\hat{U}_0 + 12e^{2\hat{\beta}_0+2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,t}\hat{U}_0+\\
&64e^{2\hat{\gamma}_0}\hat{\beta}_{0,\theta}\hat{\beta}_{0,t}\hat{U}_0 - 6e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,t}\hat{U}_0 - 6e^{2\hat{\beta}_0+2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,t}\hat{U}_0+\\
&12e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}\hat{U}_0 + 12e^{2\hat{\beta}_0+2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}\hat{U}_0 - 16e^{2\hat{\gamma}_0}\hat{\beta}_{0,t\theta}\hat{U}_0 - 2e^{4\hat{\beta}_0}-\\
&6e^{6\hat{\beta}_0} + 8e^{6\hat{\beta}_0+2\hat{\gamma}_0} + 2e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2 - 3e^{2\hat{\beta}_0+2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2 - 24e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-\\
&24e^{6\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2 + 4e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2 + 12e^{6\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2 + 32e^{2\hat{\gamma}_0}(\hat{\beta}_{0,t})^2+\\
&6e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2 + 6e^{2\hat{\beta}_0+2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2 - 4e^{4\hat{\beta}_0}\cot(\theta)\hat{\beta}_{0,\theta} - 12e^{6\hat{\beta}_0}\cot(\theta)\hat{\beta}_{0,\theta}-\\
&6e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta} - 18e^{6\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta} + 8e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta} + 24e^{6\hat{\beta}_0}\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta}-\\
&4e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta} - 12e^{6\hat{\beta}_0}\hat{\beta}_{0,\theta\theta} - 2e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta} - 6e^{6\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta} + 2e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t}-\\
&6e^{2\hat{\beta}_0+2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t} - 8e^{2\hat{\gamma}_0}\hat{\beta}_{0,\theta}\hat{U}_{0,t} - 4e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\beta}_{0,t}+\\
&12e^{2\hat{\beta}_0+2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\beta}_{0,t} + 6e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\gamma}_{0,t} + 6e^{2\hat{\beta}_0+2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\gamma}_{0,t} + 2e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}-\\
&6e^{2\hat{\beta}_0+2\hat{\gamma}_0}\hat{U}_{0,t\theta} - 8e^{2\hat{\gamma}_0}\hat{\beta}_{0,tt})
\end{aligned}
\tag{A.2.15b}
$$

$$\begin{aligned}8c_3(t,\theta) = &-\frac{2}{3}e^{-6\hat{\beta}_0-2\hat{\gamma}_0}(e^{2\hat{\gamma}_0}(\hat{U}_0)^3 + 2e^{2\hat{\gamma}_0}\cot^2(\theta)(\hat{U}_0)^3 - 2e^{2\hat{\gamma}_0}\csc^2(\theta)(\hat{U}_0)^3+\\
&16e^{2\hat{\gamma}_0}(\hat{\beta}_{0,\theta})^2(\hat{U}_0)^3 + 3e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,\theta})^2(\hat{U}_0)^3 - 2e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,\theta}(\hat{U}_0)^3-\\
&3e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,\theta}(\hat{U}_0)^3 - 4e^{2\hat{\gamma}_0}\hat{\beta}_{0,\theta\theta}(\hat{U}_0)^3 - 18e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\beta}_{0,\theta}(\hat{U}_0)^2+\\
&3e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\gamma}_{0,\theta}(\hat{U}_0)^2 + 3e^{2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}(\hat{U}_0)^2 - 2e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,t}(\hat{U}_0)^2+\\
&32e^{2\hat{\gamma}_0}\hat{\beta}_{0,\theta}\hat{\beta}_{0,t}(\hat{U}_0)^2 - 3e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,t}(\hat{U}_0)^2 + 6e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}(\hat{U}_0)^2-\\
&8e^{2\hat{\gamma}_0}\hat{\beta}_{0,t\theta}(\hat{U}_0)^2 - 4e^{4\hat{\beta}_0}\cot^2(\theta)\hat{U}_0 + 4e^{4\hat{\beta}_0}\csc^2(\theta)\hat{U}_0 + 3e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2\hat{U}_0-\\
&12e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2\hat{U}_0 - 6e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2\hat{U}_0 + 16e^{2\hat{\gamma}_0}(\hat{\beta}_{0,t})^2\hat{U}_0 + 3e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2\hat{U}_0-\\
&e^{4\hat{\beta}_0}\hat{U}_0 + 6e^{4\hat{\beta}_0}\cot(\theta)\hat{\beta}_{0,\theta}\hat{U}_0 + 9e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta}\hat{U}_0 - 12e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta}\hat{U}_0+\\
&6e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}\hat{U}_0 + 3e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}\hat{U}_0 + e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t}\hat{U}_0 - 16e^{2\hat{\gamma}_0}\hat{\beta}_{0,\theta}\hat{U}_{0,t}\hat{U}_0-\\
&14e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\beta}_{0,t}\hat{U}_0 + 3e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{\gamma}_{0,t}\hat{U}_0 + 5e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}\hat{U}_0 - 4e^{2\hat{\gamma}_0}\hat{\beta}_{0,tt}\hat{U}_0+\\
&12e^{4\hat{\beta}_0}\hat{U}_{0,\theta}\hat{\beta}_{0,\theta} + 2e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}\hat{U}_{0,t} - 12e^{2\hat{\gamma}_0}\hat{U}_{0,t}\hat{\beta}_{0,t} + 8e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,t}-\\
&16e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t} - 8e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t} + 8e^{4\hat{\beta}_0}\hat{\beta}_{0,t\theta} + 4e^{4\hat{\beta}_0}\hat{\gamma}_{0,t\theta} + 2e^{2\hat{\gamma}_0}\hat{U}_{0,tt})\end{aligned} \tag{A.2.15c}$$

Using the c_i coefficients above, we obtain the following components for $g_{(2)}$:

$$\begin{aligned}g_{(2)tt} = &\frac{1}{2}e^{2\hat{\gamma}_0-4\hat{\beta}_0}((\hat{\gamma}_{0,\theta})^2 - 3\cot(\theta)\hat{\gamma}_{0,\theta} - 2\hat{\beta}_{0,\theta}(\cot(\theta) - 2\hat{\gamma}_{0,\theta}) - 2\hat{\gamma}_{0,\theta\theta}-\\
&1)(\hat{U}_0)^4 + \frac{1}{2}e^{2\hat{\gamma}_0-4\hat{\beta}_0}(\hat{U}_{0,\theta}(2\hat{\beta}_{0,\theta} - 3\hat{\gamma}_{0,\theta}) - \hat{U}_{0,\theta\theta} - 2\cot(\theta)\hat{\beta}_{0,t}+\\
&4\hat{\gamma}_{0,\theta}\hat{\beta}_{0,t} - 3\cot(\theta)\hat{\gamma}_{0,t} + 4\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t} + 2\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t} - 4\hat{\gamma}_{0,t\theta})(\hat{U}_0)^3+\\
&\frac{1}{2}e^{-4\hat{\beta}_0}(-e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2 + e^{2\hat{\gamma}_0}(2\hat{\beta}_{0,t} - \hat{\gamma}_{0,t})\hat{U}_{0,\theta} + 2e^{4\hat{\beta}_0} + 4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-\\
&3e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2 + e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2 + 6e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta} + 4e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}(\cot(\theta) - 2\hat{\gamma}_{0,\theta})+\\
&2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta} + 3e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta} + e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t} - 2e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t} + 4e^{2\hat{\gamma}_0}\hat{\beta}_{0,t}\hat{\gamma}_{0,t}-\\
&e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta} - 2e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt})(\hat{U}_0)^2 + \frac{1}{2}(\hat{U}_{0,\theta}(3\hat{\gamma}_{0,\theta} - 2\hat{\beta}_{0,\theta}) + \hat{U}_{0,\theta\theta}-\\
&2\cot(\theta)\hat{\beta}_{0,t} + 5\cot(\theta)\hat{\gamma}_{0,t} - 8\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t} - 2\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t} + 4\hat{\gamma}_{0,t\theta})\hat{U}_0+\\
&\frac{1}{2}e^{-2\hat{\gamma}_0}(e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2 - e^{2\hat{\gamma}_0}(2\hat{\beta}_{0,t} - 3\hat{\gamma}_{0,t})\hat{U}_{0,\theta} - e^{4\hat{\beta}_0} - 4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2+\\
&2e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2 + 3e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2 - 3e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta} - 2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}(\cot(\theta) - 2\hat{\gamma}_{0,\theta})-\\
&2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta} - e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta} + e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t} + e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta})\end{aligned} \tag{A.2.16a}$$

$$
\begin{aligned}
g_{(2)t\theta} =& 2\hat{\gamma}_{0,t}(\hat{\beta}_{0,\theta}+\hat{\gamma}_{0,\theta}-\cot(\theta))-\hat{\gamma}_{0,t\theta}+\frac{1}{2}(\hat{U}_0)^3 e^{2\hat{\gamma}_0-4\hat{\beta}_0}(2\hat{\beta}_{0,\theta}(\cot(\theta)-\\
& 2\hat{\gamma}_{0,\theta})-(\hat{\gamma}_{0,\theta})^2+2\hat{\gamma}_{0,\theta\theta}+3\cot(\theta)\hat{\gamma}_{0,\theta}+1)+\frac{1}{2}(\hat{U}_0)^2 e^{2\hat{\gamma}_0-4\hat{\beta}_0}(-4\hat{\beta}_{0,t}\hat{\gamma}_{0,\theta}-\\
& 4\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}+2\cot(\theta)\hat{\beta}_{0,t}-2\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}+4\hat{\gamma}_{0,t\theta}+3\cot(\theta)\hat{\gamma}_{0,t}+\hat{U}_{0,\theta}(3\hat{\gamma}_{0,\theta}-\\
& 2\hat{\beta}_{0,\theta})+\hat{U}_{0,\theta\theta})+\frac{1}{2}e^{-4\hat{\beta}_0}\hat{U}_0(2e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2-e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}-4\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}-\\
& 3\cot(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}-2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}(\cot(\theta)-2\hat{\gamma}_{0,\theta})-e^{4\hat{\beta}_0}-4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-\\
& 2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}-e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2+2e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt}-e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(2\hat{\beta}_{0,t}-\hat{\gamma}_{0,t})+e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2+\\
& 2e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t}+e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}-\cot(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t})
\end{aligned}
\tag{A.2.16b}
$$

$$
\begin{aligned}
g_{(2)\theta\theta} =& \frac{1}{2}(\hat{U}_0)^2 e^{2\hat{\gamma}_0-4\hat{\beta}_0}(-2\hat{\beta}_{0,\theta}(\cot(\theta)-2\hat{\gamma}_{0,\theta})+(\hat{\gamma}_{0,\theta})^2-2\hat{\gamma}_{0,\theta\theta}-\\
& 3\cot(\theta)\hat{\gamma}_{0,\theta}-1)+\frac{1}{2}\hat{U}_0 e^{2\hat{\gamma}_0-4\hat{\beta}_0}(4\hat{\beta}_{0,t}\hat{\gamma}_{0,\theta}+4\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}-2\cot(\theta)\hat{\beta}_{0,t}+\\
& 2\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}-4\hat{\gamma}_{0,t\theta}-3\cot(\theta)\hat{\gamma}_{0,t}+\hat{U}_{0,\theta}(2\hat{\beta}_{0,\theta}-3\hat{\gamma}_{0,\theta})-\hat{U}_{0,\theta\theta})+\\
& \frac{1}{2}e^{-4\hat{\beta}_0}(2e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2-e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}+4\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}-3\cot(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}-e^{4\hat{\beta}_0}+\\
& 4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2+2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}-2\cot(\theta)e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}+e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2-2e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt}+\\
& e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(2\hat{\beta}_{0,t}-\hat{\gamma}_{0,t})-e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2-\\
& 2e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t}-e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}+\cot(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t})
\end{aligned}
\tag{A.2.16c}
$$

$$
\begin{aligned}
g_{(2)\phi\phi} =& \frac{1}{2}\sin(\theta)(\hat{U}_0)^2 e^{-2(2\hat{\beta}_0+\hat{\gamma}_0)}(2\hat{\beta}_{0,\theta}(\cos(\theta)-2\sin(\theta)\hat{\gamma}_{0,\theta})+\sin(\theta)(\hat{\gamma}_{0,\theta})^2+\\
& 2\sin(\theta)\hat{\gamma}_{0,\theta\theta}+\cos(\theta)\hat{\gamma}_{0,\theta}+\sin(\theta))-\\
& \frac{1}{2}\sin(\theta)\hat{U}_0 e^{-2(2\hat{\beta}_0+\hat{\gamma}_0)}(4\sin(\theta)\hat{\beta}_{0,t}\hat{\gamma}_{0,\theta}+4\sin(\theta)\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}-2\cos(\theta)\hat{\beta}_{0,t}-\\
& 2\sin(\theta)\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}-4\sin(\theta)\hat{\gamma}_{0,t\theta}-\cos(\theta)\hat{\gamma}_{0,t}+\hat{U}_{0,\theta}(2\sin(\theta)\hat{\beta}_{0,\theta}-\\
& 5\sin(\theta)\hat{\gamma}_{0,\theta}+2\cos(\theta))-\sin(\theta)\hat{U}_{0,\theta\theta})+\\
& \frac{1}{2}\sin(\theta)e^{-4(\hat{\beta}_0+\hat{\gamma}_0)}(2\sin(\theta)e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2-\sin(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}-\\
& 4\sin(\theta)\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}-3\cos(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}-4\sin(\theta)e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-\sin(\theta)e^{4\hat{\beta}_0}-\\
& 2\sin(\theta)e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+2\cos(\theta)e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}+\sin(\theta)e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2+\\
& 2\sin(\theta)e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt}-\sin(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(2\hat{\beta}_{0,t}-3\hat{\gamma}_{0,t})+\sin(\theta)e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2+\\
& 2\sin(\theta)e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t}+\sin(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}-\cos(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t}).
\end{aligned}
\tag{A.2.16d}
$$

Given these $g_{(2)}$ coefficients, we can use formula (A.2.12) as a consistency check, using the non-zero coefficients of the Ricci tensor, $R_{(0)ab}$, and the Ricci scalar, $R_{(0)}$, of the boundary metric, given below.

Ricci Tensor

$$
\begin{aligned}
R_{(0)tt} = {}& e^{-2(2\hat{\beta}_0+\hat{\gamma}_0)}(e^{4\hat{\gamma}_0}(\hat{U}_0)^4(\hat{\gamma}_{0,\theta}(\cot(\theta)-2\hat{\beta}_{0,\theta})+\hat{\gamma}_{0,\theta\theta})+\\
& e^{4\hat{\gamma}_0}(\hat{U}_0)^3(-2\hat{\beta}_{0,t}\hat{\gamma}_{0,\theta}-2\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}+2\hat{\gamma}_{0,t\theta}+\cot(\theta)\hat{\gamma}_{0,t}+\hat{U}_{0,\theta}(-2\hat{\beta}_{0,\theta}+\\
& 2\hat{\gamma}_{0,\theta}+\cot(\theta))+\hat{U}_{0,\theta\theta})+e^{2\hat{\gamma}_0}(\hat{U}_0)^2(-e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}-2\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}-\\
& \cot(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}-2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}(\cot(\theta)-3\hat{\gamma}_{0,\theta})-4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+\\
& e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt}-e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(2\hat{\beta}_{0,t}-\hat{\gamma}_{0,t})+e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2+e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t}+e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta})-\\
& \hat{U}_0e^{4\hat{\beta}_0+2\hat{\gamma}_0}(2(\hat{\gamma}_{0,t}(\cot(\theta)-2\hat{\beta}_{0,\theta})-\cot(\theta)\hat{\beta}_{0,t}+\hat{\gamma}_{0,t\theta})+\\
& \hat{U}_{0,\theta}(-2\hat{\beta}_{0,\theta}+2\hat{\gamma}_{0,\theta}+\cot(\theta))+\hat{U}_{0,\theta\theta})-e^{4\hat{\beta}_0}(-2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}(\cot(\theta)-\\
& 2\hat{\gamma}_{0,\theta})-4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+2e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2-2e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(\hat{\beta}_{0,t}-\hat{\gamma}_{0,t})+\\
& e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2+e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}+\cot(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t}))
\end{aligned}
\tag{A.2.17a}
$$

$$
\begin{aligned}
R_{(0)t\theta} = R_{(0)\theta t} = {}& e^{-4\hat{\beta}_0}(e^{4\hat{\beta}_0}(2\hat{\gamma}_{0,t}(-\hat{\beta}_{0,\theta}-\hat{\gamma}_{0,\theta}+\cot(\theta))+\hat{\gamma}_{0,t\theta})-\\
& e^{2\hat{\gamma}_0}(\hat{U}_0)^3(\hat{\gamma}_{0,\theta}(\cot(\theta)-2\hat{\beta}_{0,\theta})+\hat{\gamma}_{0,\theta\theta})-\\
& e^{2\hat{\gamma}_0}(\hat{U}_0)^2(-2\hat{\beta}_{0,t}\hat{\gamma}_{0,\theta}-2\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}+2\hat{\gamma}_{0,t\theta}+\cot(\theta)\hat{\gamma}_{0,t}+\\
& \hat{U}_{0,\theta}(-2\hat{\beta}_{0,\theta}+2\hat{\gamma}_{0,\theta}+\cot(\theta))+\hat{U}_{0,\theta\theta})-\hat{U}_0(-2\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}-\\
& 2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}(\cot(\theta)-2\hat{\gamma}_{0,\theta})-4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+\\
& e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt}-e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(2\hat{\beta}_{0,t}-\hat{\gamma}_{0,t})+e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2+\\
& e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t}+e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}))
\end{aligned}
\tag{A.2.17b}
$$

$$
\begin{aligned}
R_{(0)\theta\theta} = {}& e^{-4\hat{\beta}_0}(-2e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2+2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta}+e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}-2\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}+\\
& 3\cot(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}+e^{4\hat{\beta}_0}-4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2-2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt}-\\
& e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(2\hat{\beta}_{0,t}-\hat{\gamma}_{0,t})+e^{2\hat{\gamma}_0}(\hat{U}_0)^2(\hat{\gamma}_{0,\theta}(\cot(\theta)-2\hat{\beta}_{0,\theta})+\hat{\gamma}_{0,\theta\theta})+\\
& e^{2\hat{\gamma}_0}\hat{U}_0(-2\hat{\beta}_{0,t}\hat{\gamma}_{0,\theta}-2\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}+2\hat{\gamma}_{0,t\theta}+\cot(\theta)\hat{\gamma}_{0,t}+\\
& \hat{U}_{0,\theta}(-2\hat{\beta}_{0,\theta}+2\hat{\gamma}_{0,\theta}+\cot(\theta))+\hat{U}_{0,\theta\theta})+\\
& e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2+e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t}+e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta})
\end{aligned}
\tag{A.2.17c}
$$

$$
\begin{aligned}
R_{(0)\phi\phi} = {} & \sin(\theta)e^{-4(\hat{\beta}_0+\hat{\gamma}_0)}(-2\sin(\theta)e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2 + \sin(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}+ \\
& 2\sin(\theta)\hat{\beta}_{0,t}e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t} + 3\cos(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta} + 2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}(\sin(\theta)\hat{\gamma}_{0,\theta} - \cos(\theta))+ \\
& \sin(\theta)e^{4\hat{\beta}_0} - \sin(\theta)e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt} + e^{2\hat{\gamma}_0}(\hat{U}_0)^2(2\hat{\beta}_{0,\theta}(\sin(\theta)\hat{\gamma}_{0,\theta} - \cos(\theta))- \\
& \sin(\theta)(\hat{\gamma}_{0,\theta\theta} + 1) - \cos(\theta)\hat{\gamma}_{0,\theta}) - e^{2\hat{\gamma}_0}\hat{U}_0(-2\sin(\theta)\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}+ \\
& 2\hat{\beta}_{0,t}(\cos(\theta) - \sin(\theta)\hat{\gamma}_{0,\theta}) + 2\sin(\theta)\hat{\gamma}_{0,t\theta} + \cos(\theta)\hat{\gamma}_{0,t}+ \\
& 2\hat{U}_{0,\theta}(\sin(\theta)\hat{\gamma}_{0,\theta} - \cos(\theta))) - \sin(\theta)e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta}\hat{U}_{0,t}- \\
& \sin(\theta)e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}\hat{U}_{0,\theta} + \cos(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t})
\end{aligned}
\tag{A.2.17d}
$$

Ricci Scalar

$$
\begin{aligned}
R_{(0)} = {} & 2e^{-2(2\hat{\beta}_0+\hat{\gamma}_0)}(-2e^{4\hat{\beta}_0}(\hat{\gamma}_{0,\theta})^2 + 4e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta} + e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}+ \\
& 3\cot(\theta)e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta} + e^{4\hat{\beta}_0} - 4e^{4\hat{\beta}_0}(\hat{\beta}_{0,\theta})^2 - 2e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta} - 2\cot(\theta)e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}+ \\
& e^{2\hat{\gamma}_0}(\hat{\gamma}_{0,t})^2 - e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}(2\hat{\beta}_{0,t} - \hat{\gamma}_{0,t}) - e^{2\hat{\gamma}_0}(\hat{U}_0)^2(2\cot(\theta)\hat{\beta}_{0,\theta} - (\hat{\gamma}_{0,\theta})^2+ \\
& \cot(\theta)\hat{\gamma}_{0,\theta} + 1) + e^{2\hat{\gamma}_0}\hat{U}_0(-2\cot(\theta)\hat{\beta}_{0,t} + 2\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t} - \cot(\theta)\hat{\gamma}_{0,t}+ \\
& \hat{U}_{0,\theta}(-2\hat{\beta}_{0,\theta} + \hat{\gamma}_{0,\theta} + 2\cot(\theta)) + \hat{U}_{0,\theta\theta})+ \\
& e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta})^2 + e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta} + \cot(\theta)e^{2\hat{\gamma}_0}\hat{U}_{0,t}).
\end{aligned}
\tag{A.2.18}
$$

A.2.3 Explicit expressions for $g_{(3)}$

Finally, we want to obtain $g_{(3)}$. To do this we extend our transformation in the coordinates to $\mathcal{O}(\rho^4)$

$$
\begin{aligned}
& r_* \to \rho + b_1(t,\theta)\rho^2 + c_1(t,\theta)\rho^3 + d_1(t,\theta)\rho^4 \\
& t \to t + \alpha_1(t,\theta)\rho + b_2(t,\theta)\rho^2 + c_2(t,\theta)\rho^3 + d_2(t,\theta)\rho^4 \\
& \theta \to \theta + \alpha_2(t,\theta)\rho + b_3(t,\theta)\rho^2 + c_3(t,\theta)\rho^3 + d_3(t,\theta)\rho^4,
\end{aligned}
\tag{A.2.19}
$$

where α_i, b_i, c_i are the functions already obtained from previous orders. As in the previous orders, we obtain $d_{1,2,3}$ by forcing the vanishing of the $d\rho$ terms, now at $\mathcal{O}(1/\rho)$. The expressions for d_i are too long to be reported here but they can be found in Mathematica file included in the arXiv submission of [52]. Using this transformation we can finally extract $g_{(3)ab}$, which may be manipulated to the form (3.4.38) with $\mathcal{U}_3, \mathcal{W}_3, \mathcal{G}_3$ given by

$$
\begin{aligned}
\mathcal{U}_3 =&\frac{1}{12}e^{-2(\hat{\beta}_0+\hat{\gamma}_0)}[-12\cot^3(\theta)+15\csc^2(\theta)\cot(\theta)+72\hat{\gamma}_{0,\theta}^2\cot(\theta)-\\
&8\hat{\beta}_{0,\theta\theta}\cot(\theta)-30\hat{\gamma}_{0,\theta\theta}\cot(\theta)-24\cot(\theta)-32\hat{\gamma}_{0,\theta}^3+\\
&32\hat{\beta}_{0,\theta}^2(\cot(\theta)-2\hat{\gamma}_{0,\theta})+\hat{\gamma}_{0,\theta}(-2(13\cos(2\theta)+2)\csc^2(\theta)+16\hat{\beta}_{0,\theta\theta}+\\
&36\hat{\gamma}_{0,\theta\theta})+\hat{\beta}_{0,\theta}(-7\cot^2(\theta)+92\hat{\gamma}_{0,\theta}\cot(\theta)-60\hat{\gamma}_{0,\theta}^2+48\hat{\gamma}_{0,\theta\theta}+24)-\\
&8\hat{\gamma}_{0,\theta\theta\theta}]\hat{U}_0^2+\frac{1}{12}e^{-2(\hat{\beta}_0+\hat{\gamma}_0)}[-16\hat{\beta}_{0,t}\cot^2(\theta)-16\hat{\gamma}_{0,t}\cot^2(\theta)-9\hat{U}_{0,\theta\theta}\cot(\theta)+\\
&32\hat{\beta}_{0,\theta}\hat{\beta}_{0,t}\cot(\theta)+48\hat{\gamma}_{0,\theta}\hat{\beta}_{0,t}\cot(\theta)+76\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t}\cot(\theta)+104\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}\cot(\theta)-\\
&8\hat{\beta}_{0,t\theta}\cot(\theta)-46\hat{\gamma}_{0,t\theta}\cot(\theta)+24\hat{\beta}_{0,\theta}\hat{U}_{0,\theta\theta}+6\hat{\gamma}_{0,\theta}\hat{U}_{0,\theta\theta}-\\
&\hat{U}_{0,\theta}(\cot^2(\theta)+12\hat{\gamma}_{0,\theta}\cot(\theta)+5\csc^2(\theta)+32\hat{\beta}_{0,\theta}^2-12\hat{\gamma}_{0,\theta}^2+\\
&2\hat{\beta}_{0,\theta}(\cot(\theta)-18\hat{\gamma}_{0,\theta})-8\hat{\beta}_{0,\theta\theta}+18\hat{\gamma}_{0,\theta\theta}+2)-4\hat{U}_{0,\theta\theta\theta}+8\csc^2(\theta)\hat{\beta}_{0,t}-\\
&32\hat{\gamma}_{0,\theta}^2\hat{\beta}_{0,t}-64\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta}\hat{\beta}_{0,t}+16\hat{\gamma}_{0,\theta\theta}\hat{\beta}_{0,t}-10\csc^2(\theta)\hat{\gamma}_{0,t}-\\
&64\hat{\beta}_{0,\theta}^2\hat{\gamma}_{0,t}-64\hat{\gamma}_{0,\theta}^2\hat{\gamma}_{0,t}-88\hat{\beta}_{0,\theta}\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}+16\hat{\beta}_{0,\theta\theta}\hat{\gamma}_{0,t}+20\hat{\gamma}_{0,\theta\theta}\hat{\gamma}_{0,t}+24\hat{\gamma}_{0,t}+\\
&16\hat{\gamma}_{0,\theta}\hat{\beta}_{0,t\theta}+80\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t\theta}+52\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t\theta}-16\hat{\gamma}_{0,t\theta\theta}]\hat{U}_0+\\
&\frac{1}{12}e^{-2(\hat{\beta}_0+2\hat{\gamma}_0)}[3e^{2\hat{\gamma}_0}(\cot(\theta)+3\hat{\beta}_{0,\theta}-2\hat{\gamma}_{0,\theta})\hat{U}_{0,\theta}^2-\\
&e^{2\hat{\gamma}_0}(3\hat{U}_{0,\theta\theta}-2(4(\cot(\theta)-4\hat{\beta}_{0,\theta})\hat{\beta}_{0,t}+(5\cot(\theta)+2\hat{\beta}_{0,\theta}-2\hat{\gamma}_{0,\theta})\hat{\gamma}_{0,t}+\\
&4\hat{\beta}_{0,t\theta}-5\hat{\gamma}_{0,t\theta}))\hat{U}_{0,\theta}+2(8e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}^3+\\
&(8e^{2\hat{\gamma}_0}\hat{U}_{0,t}-12e^{4\hat{\beta}_0}\cot(\theta))\hat{\gamma}_{0,\theta}^2-2(3e^{4\hat{\beta}_0}\csc^2(\theta)+2e^{4\hat{\beta}_0}+8e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}^2+\\
&8e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+6e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta}+6e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t}+8e^{2\hat{\gamma}_0}\hat{\beta}_{0,t}\hat{\gamma}_{0,t}-4e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt})\hat{\gamma}_{0,\theta}+\\
&16e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,t}^2+16e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}^2(\cot(\theta)-2\hat{\gamma}_{0,\theta})+4e^{4\hat{\beta}_0}\cot(\theta)\hat{\beta}_{0,\theta\theta}+\\
&6e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta\theta}+4e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta\theta}+2e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta\theta}+4e^{2\hat{\gamma}_0}\cot^2(\theta)\hat{U}_{0,t}-\\
&2e^{2\hat{\gamma}_0}\csc^2(\theta)\hat{U}_{0,t}-4e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta\theta}\hat{U}_{0,t}+4e^{2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}\hat{\beta}_{0,t}+3e^{2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}\hat{\gamma}_{0,t}+\\
&16e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,t}\hat{\gamma}_{0,t}-2e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t\theta}+8e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}\hat{\beta}_{0,t\theta}+8e^{2\hat{\gamma}_0}\hat{\beta}_{0,t}\hat{\gamma}_{0,t\theta}+\\
&10e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}\hat{\gamma}_{0,t\theta}-2e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta\theta}-8e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,tt}-2\hat{\beta}_{0,\theta}(2e^{4\hat{\beta}_0}\csc^2(\theta)-4e^{4\hat{\beta}_0}+\\
&7e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}^2-8e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}+4e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t}-8\hat{\gamma}_{0,\theta}(e^{4\hat{\beta}_0}\cot(\theta)+e^{2\hat{\gamma}_0}\hat{U}_{0,t})+\\
&16e^{2\hat{\gamma}_0}\hat{\beta}_{0,t}\hat{\gamma}_{0,t}-4e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta}-8e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt})-4e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt\theta})]
\end{aligned}
\tag{A.2.20}
$$

$$
\begin{aligned}
\mathcal{W}_3 = & -\frac{1}{8}e^{-4\hat{\beta}_0}(\cot(\theta) - 2\hat{\gamma}_{0,\theta})[-\cot^2(\theta) + 4\hat{\gamma}_{0,\theta}\cot(\theta) + 3\csc^2(\theta) + \\
& 8\hat{\beta}_{0,\theta}(\cot(\theta) - 2\hat{\gamma}_{0,\theta}) + 8\hat{\gamma}_{0,\theta\theta} + 1]\hat{U}_0^3 + \\
& \frac{1}{4}e^{-4\hat{\beta}_0}[\hat{U}_{0,\theta}(2\cot^2(\theta) - 8\hat{\gamma}_{0,\theta}\cot(\theta) + \csc^2(\theta) + 12\hat{\gamma}_{0,\theta}^2 + \\
& 8\hat{\beta}_{0,\theta}(\cot(\theta) - 2\hat{\gamma}_{0,\theta}) + 4\hat{\gamma}_{0,\theta\theta} + 1) + 2\{2(-2\hat{\gamma}_{0,t\theta}(\cot(\theta) - 2\hat{\gamma}_{0,\theta}) - \\
& (\cot(\theta) - 2\hat{\gamma}_{0,\theta})^2\hat{\beta}_{0,t} + (2\cot(\theta)\hat{\gamma}_{0,\theta} + 4\hat{\beta}_{0,\theta}(\cot(\theta) - 2\hat{\gamma}_{0,\theta}) + \\
& 2\hat{\gamma}_{0,\theta\theta} + 1)\hat{\gamma}_{0,t}) - (\cot(\theta) - 2\hat{\gamma}_{0,\theta})\hat{U}_{0,\theta\theta}\}]\hat{U}_0^2 + \\
& \frac{1}{4}e^{-2(2\hat{\beta}_0 + \hat{\gamma}_0)}[-e^{2\hat{\gamma}_0}(3\cot(\theta) + 4\hat{\beta}_{0,\theta} - 8\hat{\gamma}_{0,\theta})\hat{U}_{0,\theta}^2 + \\
& 2e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta\theta} + 4(\cot(\theta) - 2\hat{\gamma}_{0,\theta})\hat{\beta}_{0,t} - 2\cot(\theta)\hat{\gamma}_{0,t} - 8\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t} + 8\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t} + \\
& 4\hat{\gamma}_{0,t\theta})\hat{U}_{0,\theta} + 2\{8e^{4\hat{\beta}_0}(\cot(\theta) - 2\hat{\gamma}_{0,\theta})\hat{\beta}_{0,\theta}^2 - \\
& 2(e^{4\hat{\beta}_0}\cot^2(\theta) - 12e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}\cot(\theta) - 3e^{4\hat{\beta}_0} + e^{4\hat{\beta}_0}\csc^2(\theta) + 8e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}^2 + \\
& 4e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}^2 - 4e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta})\hat{\beta}_{0,\theta} + e^{2\hat{\gamma}_0}(\hat{U}_{0,t}(\cot(\theta) - 2\hat{\gamma}_{0,\theta})^2 + 2\cot(\theta)\hat{\gamma}_{0,t}^2 + \\
& 2\hat{U}_{0,\theta\theta}\hat{\gamma}_{0,t} + 8\cot(\theta)\hat{\beta}_{0,t}\hat{\gamma}_{0,t} - 16\hat{\gamma}_{0,\theta}\hat{\beta}_{0,t}\hat{\gamma}_{0,t} - \cot(\theta)\hat{U}_{0,t\theta} + 2\hat{\gamma}_{0,\theta}\hat{U}_{0,t\theta} + \\
& 8\hat{\gamma}_{0,t}\hat{\gamma}_{0,t\theta} - 2\cot(\theta)\hat{\gamma}_{0,tt} + 4\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,tt})\}]\hat{U}_0 + \frac{1}{4}e^{-2(2\hat{\beta}_0 + \hat{\gamma}_0)}[e^{2\hat{\gamma}_0}\hat{U}_{0,\theta}^3 - \\
& 4e^{2\hat{\gamma}_0}(\hat{\beta}_{0,t} - \hat{\gamma}_{0,t})\hat{U}_{0,\theta}^2 + 2(-8e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}^2 + 4e^{4\hat{\beta}_0}\cot(\theta)\hat{\beta}_{0,\theta} + e^{2\hat{\gamma}_0}(2\hat{\gamma}_{0,t}^2 - \\
& 8\hat{\beta}_{0,t}\hat{\gamma}_{0,t} - (\cot(\theta) - 2\hat{\gamma}_{0,\theta})\hat{U}_{0,t} + \hat{U}_{0,t\theta} + 2\hat{\gamma}_{0,tt}))\hat{U}_{0,\theta} - \\
& 4\{8e^{4\hat{\beta}_0}\hat{\gamma}_{0,t}\hat{\beta}_{0,\theta}^2 - 2e^{4\hat{\beta}_0}(\hat{U}_{0,\theta\theta} + 2(2(\cot(\theta) - \hat{\gamma}_{0,\theta})\hat{\gamma}_{0,t} + \hat{\gamma}_{0,t\theta}))\hat{\beta}_{0,\theta} + \\
& e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}((\cot(\theta) - 2\hat{\gamma}_{0,\theta})\hat{U}_{0,t} + 4\hat{\beta}_{0,t}\hat{\gamma}_{0,t} - \hat{U}_{0,t\theta} - 2\hat{\gamma}_{0,tt})\}]
\end{aligned}
\tag{A.2.21}
$$

$$
\begin{aligned}
\mathcal{G}_3 =& \frac{1}{48} e^{-6\hat{\beta}_0}[-19\cot^3(\theta) + 18\csc^2(\theta)\cot(\theta) + 36\hat{\gamma}_{0,\theta}^2\cot(\theta) + 16\hat{\beta}_{0,\theta\theta}\cot(\theta)+ \\
& 24\hat{\gamma}_{0,\theta\theta}\cot(\theta) - 18\cot(\theta) - 24\hat{\gamma}_{0,\theta}^3 - 64\hat{\beta}_{0,\theta}^2(\cot(\theta) - 2\hat{\gamma}_{0,\theta})- \\
& 2\hat{\gamma}_{0,\theta}(7\cot^2(\theta) + 2\csc^2(\theta) + 16\hat{\beta}_{0,\theta\theta} + 2) - 8\hat{\beta}_{0,\theta}(\cot^2(\theta) + 8\hat{\gamma}_{0,\theta}\cot(\theta)+ \\
& \csc^2(\theta) + 12\hat{\gamma}_{0,\theta\theta} + 5) + 16\hat{\gamma}_{0,\theta\theta\theta}]\hat{U}_0^3+ \\
& \frac{1}{48} e^{-6\hat{\beta}_0}[\hat{U}_{0,\theta}(-7\cot^2(\theta) + 84\hat{\gamma}_{0,\theta}\cot(\theta) + 10\csc^2(\theta)+ \\
& 64\hat{\beta}_{0,\theta}^2 - 36\hat{\gamma}_{0,\theta}^2 + 64\hat{\beta}_{0,\theta}(\cot(\theta) - 3\hat{\gamma}_{0,\theta}) - 16\hat{\beta}_{0,\theta\theta} + 72\hat{\gamma}_{0,\theta\theta} + 22)+ \\
& 2(-4\hat{\beta}_{0,t}\cot^2(\theta) - 7\hat{\gamma}_{0,t}\cot^2(\theta) + 16\hat{\beta}_{0,t\theta}\cot(\theta) + 24\hat{\gamma}_{0,t\theta}\cot(\theta) + 4\hat{U}_{0,\theta\theta\theta}- \\
& 4\csc^2(\theta)\hat{\beta}_{0,t} - 48\hat{\gamma}_{0,\theta\theta}\hat{\beta}_{0,t} - 20\hat{\beta}_{0,t} - 2\csc^2(\theta)\hat{\gamma}_{0,t} + 64\hat{\beta}_{0,\theta}^2\hat{\gamma}_{0,t}- \\
& 36\hat{\gamma}_{0,\theta}^2\hat{\gamma}_{0,t} - 16\hat{\beta}_{0,\theta\theta}\hat{\gamma}_{0,t} - 2\hat{\gamma}_{0,t} + 4\hat{\gamma}_{0,\theta}(3\hat{U}_{0,\theta\theta} - 8\cot(\theta)\hat{\beta}_{0,t} + 9\cot(\theta)\hat{\gamma}_{0,t}- \\
& 8\hat{\beta}_{0,t\theta}) - 8\hat{\beta}_{0,\theta}(3\hat{U}_{0,\theta\theta} + 4(2(\cot(\theta) - 2\hat{\gamma}_{0,\theta})\hat{\beta}_{0,t} + \cot(\theta)\hat{\gamma}_{0,t} + 3\hat{\gamma}_{0,t\theta}))+ \\
& 24\hat{\gamma}_{0,t\theta\theta})]\hat{U}_0^2+ \\
& \frac{1}{48} e^{-2(3\hat{\beta}_0+\hat{\gamma}_0)}[-3e^{2\hat{\gamma}_0}(\cot(\theta) + 16\hat{\beta}_{0,\theta} - 10\hat{\gamma}_{0,\theta})\hat{U}_{0,\theta}^2 + 4e^{2\hat{\gamma}_0}(6\hat{U}_{0,\theta\theta}+ \\
& 4(3\cot(\theta) + 8\hat{\beta}_{0,\theta} - 10\hat{\gamma}_{0,\theta})\hat{\beta}_{0,t} + 15\cot(\theta)\hat{\gamma}_{0,t} - 24\hat{\beta}_{0,\theta}\hat{\gamma}_{0,t} - 18\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t}- \\
& 8\hat{\beta}_{0,t\theta} + 24\hat{\gamma}_{0,t\theta})\hat{U}_{0,\theta} - 4(8e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta}^3 - 12e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta}^2+ \\
& 2(2e^{4\hat{\beta}_0}\cot^2(\theta) - 4e^{2\hat{\gamma}_0}\hat{U}_{0,t}\cot(\theta) - 5e^{4\hat{\beta}_0}\csc^2(\theta) - 16e^{2\hat{\gamma}_0}\hat{\beta}_{0,t}^2+ \\
& 9e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}^2 - 8e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta} - 6e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta} - 5e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta} + 4e^{2\hat{\gamma}_0}\hat{\beta}_{0,tt})\hat{\gamma}_{0,\theta}+ \\
& 16e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,t}^2 - 9e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,t}^2 + 16e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta}^2(\cot(\theta) - 2\hat{\gamma}_{0,\theta})+ \\
& 4e^{4\hat{\beta}_0}\cot(\theta)\hat{\beta}_{0,\theta\theta} + 6e^{4\hat{\beta}_0}\cot(\theta)\hat{\gamma}_{0,\theta\theta} + 4e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta\theta} + 2e^{4\hat{\beta}_0}\hat{\gamma}_{0,\theta\theta\theta} - 5e^{2\hat{\gamma}_0}\hat{U}_{0,t}- \\
& e^{2\hat{\gamma}_0}\cot^2(\theta)\hat{U}_{0,t} - e^{2\hat{\gamma}_0}\csc^2(\theta)\hat{U}_{0,t} - 12e^{2\hat{\gamma}_0}\hat{\gamma}_{0,\theta\theta}\hat{U}_{0,t} + 12e^{2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}\hat{\beta}_{0,t}- \\
& 2e^{2\hat{\gamma}_0}\hat{U}_{0,\theta\theta}\hat{\gamma}_{0,t} + 16e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,t}\hat{\gamma}_{0,t} + 2e^{2\hat{\gamma}_0}\cot(\theta)\hat{U}_{0,t\theta} + 16e^{2\hat{\gamma}_0}\hat{\gamma}_{0,t}\hat{\beta}_{0,t\theta}+ \\
& 48e^{2\hat{\gamma}_0}\hat{\beta}_{0,t}\hat{\gamma}_{0,t\theta} - 4e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta\theta} - 4e^{2\hat{\gamma}_0}\cot(\theta)\hat{\beta}_{0,tt} - 6e^{2\hat{\gamma}_0}\cot(\theta)\hat{\gamma}_{0,tt}- \\
& 2\hat{\beta}_{0,\theta}(7e^{4\hat{\beta}_0}\cot^2(\theta) + 8e^{2\hat{\gamma}_0}\hat{U}_{0,t}\cot(\theta) + 3e^{4\hat{\beta}_0} - 5e^{4\hat{\beta}_0}\csc^2(\theta) - 8e^{4\hat{\beta}_0}\hat{\beta}_{0,\theta\theta}- \\
& 8\hat{\gamma}_{0,\theta}(e^{4\hat{\beta}_0}\cot(\theta) + 2e^{2\hat{\gamma}_0}\hat{U}_{0,t}) + 32e^{2\hat{\gamma}_0}\hat{\beta}_{0,t}\hat{\gamma}_{0,t} - 6e^{2\hat{\gamma}_0}\hat{U}_{0,t\theta} - 12e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt})- \\
& 12e^{2\hat{\gamma}_0}\hat{\gamma}_{0,tt\theta})]\hat{U}_0+ \\
& \frac{1}{48} e^{-2(3\hat{\beta}_0+\hat{\gamma}_0)}[e^{2\hat{\gamma}_0}(\hat{U}_{0,\theta}^3 - 2(16\hat{\beta}_{0,t} + 5\hat{\gamma}_{0,t})\hat{U}_{0,\theta}^2 - 4(-16\hat{\beta}_{0,t}^2+ \\
& 16\hat{\gamma}_{0,t}\hat{\beta}_{0,t} + 9\hat{\gamma}_{0,t}^2 + 2(2\cot(\theta) + 2\hat{\beta}_{0,\theta} - 5\hat{\gamma}_{0,\theta})\hat{U}_{0,t} - 4\hat{U}_{0,t\theta} + 4\hat{\beta}_{0,tt}- \\
& 6\hat{\gamma}_{0,tt})\hat{U}_{0,\theta} + 8(-3\hat{\gamma}_{0,t}^3 + 16\hat{\beta}_{0,t}^2\hat{\gamma}_{0,t} + \hat{U}_{0,t\theta}\hat{\gamma}_{0,t} - 4\hat{\beta}_{0,tt}\hat{\gamma}_{0,t}+ \\
& \hat{U}_{0,\theta\theta}\hat{U}_{0,t} - 6\hat{\beta}_{0,t}\hat{U}_{0,t\theta} + \hat{U}_{0,t}(6(\cot(\theta) - 2\hat{\gamma}_{0,\theta})\hat{\beta}_{0,t}+ \\
& (\cot(\theta) - 4\hat{\beta}_{0,\theta})\hat{\gamma}_{0,t} + 6\hat{\gamma}_{0,t\theta}) - \cot(\theta)\hat{U}_{0,tt} + 2\hat{\gamma}_{0,\theta}\hat{U}_{0,tt} - 12\hat{\beta}_{0,t}\hat{\gamma}_{0,tt} + \hat{U}_{0,tt\theta}+ \\
& 2\hat{\gamma}_{0,ttt})) - 4e^{4\hat{\beta}_0}\{-2(4(\csc^2(\theta) + 4\hat{\beta}_{0,\theta}^2 - \hat{\gamma}_{0,\theta}^2 - 2\cot(\theta)\hat{\beta}_{0,\theta}+ \\
& \cot(\theta)\hat{\gamma}_{0,\theta} + 2\hat{\beta}_{0,\theta\theta} + \hat{\gamma}_{0,\theta\theta})\hat{\gamma}_{0,t} + 2(\cot(\theta) - 4\hat{\beta}_{0,\theta})\hat{\beta}_{0,t\theta}- \\
& \cot(\theta)\hat{\gamma}_{0,t\theta} + 2\hat{\gamma}_{0,\theta}\hat{\gamma}_{0,t\theta} - 2\hat{\beta}_{0,t\theta\theta} - \hat{\gamma}_{0,t\theta\theta})\}]
\end{aligned}
\tag{A.2.22}
$$

where all of the metric coefficients are functions of (t, θ) (as indicated by the hats over the functions).

A.3 Logarithmic terms in the presence of matter

In this appendix we explore how matter can affect the asymptotic expansions, inducing logarithmic terms that are related to conformal anomalies. The latter is well-understood within the context of holography (see [60]). When logarithmic terms in the asymptotic solutions appear then the on-shell gravitational action also has closely logarithmic divergences[1]. The presence of such divergences implies that the theory depends not only on the conformal class fixed at the conformal boundary but also on the specific representative picked: there is a conformal anomaly. Via AdS/CFT this anomaly should match a corresponding quantum anomaly in the dual QFT (and it does [58, 60]). In the context of Bondi gauge analysis for $\Lambda \neq 0$, it was noted in [48] that the metric functions acquire logarithmic contributions given specific fall-off conditions on the bulk stress energy tensor, and we now explain how such terms emerge.

Using the Fefferman-Graham gauge (5.1.1), the fall-off conditions on the bulk stress energy tensor that lead to logarithmic terms in the metric expansions in [48] are

$$\mathcal{T}_{\rho\rho} \sim \rho; \qquad \mathcal{T}_{ab} \sim \rho. \tag{A.3.1}$$

This can be understood easily from the Einstein equations in this gauge. The $(\rho\rho)$ equation is

$$-\frac{\rho}{4}\mathrm{Tr}(g^{-1}g_{,\rho})^2 + \frac{\rho}{2}\mathrm{Tr}(g^{-1}g_{,\rho\rho}) - \frac{1}{2}\mathrm{Tr}(g^{-1}g_{,\rho}) = \rho\bar{\mathcal{T}}_{\rho\rho} \tag{A.3.2}$$

where $\bar{\mathcal{T}}_{\mu\nu}$ is the trace adjusted bulk stress tensor and the subscript denotes a derivative; the trace is over the indices (a, b). The (ab) equations are

$$\begin{aligned} &-\frac{1}{2}\mathrm{Tr}(g^{-1}g_{,\rho})g_{ab} - (g_{ab})_{,\rho} \\ &+\rho\left(\frac{1}{2}(g_{ab})_{,\rho\rho} - \frac{1}{2}(g_{,\rho}g^{-1}g_{,\rho})_{ab} - \hat{R}_{ab} + -\frac{1}{4}\mathrm{Tr}(g^{-1}g_{,\rho})(g_{ab})_{,\rho}\right) = \rho\bar{\mathcal{T}}_{ab}, \end{aligned} \tag{A.3.3}$$

where $\hat{R}_{ab}$ is the Ricci curvature of g_{ab}. We do not give the (ρa) equations as we will not need them below.

[1]The logarithmic terms both in the on-shell action and the asymptotic solution are local functions of the fields specifying the boundary conditions for gravity coupled to matter. The logarithmic term in the asymptotic solution of a given field is given by the functional derivative of the on-shell logarithmic term w.r.t. the corresponding boundary condition [60].

In the absence of a bulk stress tensor these equations admit asymptotic solutions with

$$g_{ab} = g_{(0)ab} + g_{(2)ab}\rho^2 + g_{(3)ab}\rho^3 + \cdots \tag{A.3.4}$$

where $g_{(2)}$ is determined by the curvature of $g_{(0)}$ and $g_{(3)}$ is traceless and divergenceless. (The tracelessness of $g_{(3)}$ follows from differentiating (A.3.2) and (A.3.3) with respect to ρ and then setting $\rho \to 0$.) If we now impose the falloff conditions above:

$$\bar{\mathcal{T}}_{\rho\rho} = \bar{\mathcal{T}}_{(1)\rho\rho}\rho + \cdots \qquad \bar{\mathcal{T}}_{ab} = \bar{\mathcal{T}}_{(1)ab}\rho + \cdots \tag{A.3.5}$$

then the asymptotic expansion is modified to

$$g_{ab} = g_{(0)ab} + g_{(2)ab}\rho^2 + (g_{(3)ab} + h_{(3)ab}\log\rho)\rho^3 + \cdots \tag{A.3.6}$$

with

$$\mathrm{Tr}(g_{(0)}^{-1}h_{(3)}) = 0; \qquad \mathrm{Tr}(g_{(0)}^{-1}g_{(3)}) = \frac{2}{3}\bar{\mathcal{T}}_{(1)\rho\rho} \tag{A.3.7}$$

and

$$h_{(3)ab} = \frac{2}{3}\bar{\mathcal{T}}_{(1)\rho\rho}g_{(0)ab} + \frac{2}{3}\bar{\mathcal{T}}_{(1)ab}. \tag{A.3.8}$$

Note that self consistency requires that

$$\bar{\mathcal{T}}_{(1)\rho\rho} + \frac{1}{3}g_{(0)}^{ab}\bar{\mathcal{T}}_{(1)ab} = 0. \tag{A.3.9}$$

The (ρa) equations determine the divergence of $g_{(3)}$ and $h_{(3)}$; apart from the trace and divergence constraints, $g_{(3)}$ remains undetermined by the field equations and describes the energy momentum tensor of the dual theory.

Thus the falloff conditions (A.3.5) imposed on the bulk stress tensor induce logarithmic terms in the asymptotic expansion, along with non-zero trace and divergence of $g_{(3)}$. Such effects are associated with conformal anomalies.

An explicit example of bulk matter that induces such a conformal anomaly is the following. Consider a bulk scalar field ϕ of mass $m^2 = -2$, corresponding to a scalar operator of dimension two in the conformal field theory, and let the field have a cubic interaction i.e. the field equation is

$$(\Box + 2)\phi = \lambda\phi^2 \tag{A.3.10}$$

where λ is the cubic coupling. The asymptotic expansion of the field ϕ is of the form

$$\phi = \phi_{(1)}\rho + \cdots \tag{A.3.11}$$

where $\phi_{(1)}(x)$ is the source for the dual operator in the field theory. The cubic interaction induces terms of the form (A.3.5) in the bulk stress tensor, and hence logarithmic terms

$h_{3)}$ and non-zero trace and divergence of $g_{(3)}$. These are associated with a conformal anomaly in the dual stress energy tensor of the form

$$g^{ab}_{(0)}\langle T_{ab}\rangle \sim \lambda\phi^3_{(1)}. \tag{A.3.12}$$

It follows that there is a conformal anomaly associated with the 3-point function of the operator of dimension 2, in agreement with the QFT analysis in [229].

A.4 Equivalence of Bondi and Abbott-Deser masses in asymptotically AdS spacetimes

In this appendix we will show that our candidate for the Bondi mass (3.4.56) agrees with the well-known Abbott-Deser mass [230] in asymptotically AdS spacetime. We recall that the Abbott-Deser mass is defined relative to a reference background spacetimes which for asymptotically AdS spacetimes is taken to be pure AdS. Specifically, we write the spacetime metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{A.4.1}$$

where $\bar{g}_{\mu\nu}$ is the metric of pure AdS_4 and $h_{\mu\nu}$ is a perturbation chosen such that $g_{\mu\nu}$ solves (2.1.4) and $h_{\mu\nu}$ vanishes at $\mathscr{I}$. Note that the vanishing condition at $\mathscr{I}$ ensures that $g_{\mu\nu}$ is asymptotically AdS as it has the same conformal structure induced at $\mathscr{I}$ as $\bar{g}_{\mu\nu}$, the metric for pure AdS_4. In this appendix we will restrict our attention to $\Lambda < 0$ and $T_{\mu\nu} = 0$. We will use the normalisation of $l = 1$ ($\Lambda = -3$) which can of course be reintroduced via dimensional analysis.

We first recall the definition of the Abbott-Deser energy-momentum for asymptotically AdS spacetimes as given in [230] (using units where $G = 1$):

$$E[\bar{\xi}] = \frac{1}{8\pi}\lim_{S_a\to\mathscr{I}_a}\oint dS_a\sqrt{-\bar{g}}[\bar{D}_\beta K^{ta\nu\beta} - K^{tb\nu a}\bar{D}_b]\bar{\xi}_\nu \tag{A.4.2}$$

where the integral is taken over a spacelike 2-surface at the conformal boundary $\mathscr{I}$. $\bar{\xi}$ is a Killing vector associated with the background metric $\bar{g}_{\mu\nu}$ (which is also used to raise and lower indices) and $\bar{D}_\mu$ its associated covariant derivative operator. In the equation above we continue to use the convention that Greek indices β, ν run over all spacetime values and Roman indices a, b over spatial values (the index t is of course the time coordinate). The rank four tensor K is known as the *superpotential* and is given by

$$K^{\mu\alpha\nu\beta} = \frac{1}{2}[\bar{g}^{\nu\beta}H^{\nu\alpha} + \bar{g}^{\nu\alpha}H^{\mu\beta} - \bar{g}^{\mu\nu}H^{\alpha\beta} - \bar{g}^{\alpha\beta}H^{\mu\nu}] \tag{A.4.3}$$

where

$$H^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}g^{\mu\nu}h^{\alpha}{}_{\alpha}. \tag{A.4.4}$$

In order to compute the Abbott-Deser mass we follow the prescription of [230] and evaluate (A.4.2) when $\bar{\xi}$ is a timelike Killing vector, namely

$$\bar{\xi}^{\mu} = -\left(\frac{\partial}{\partial t}\right)^{\mu} = (-1, 0). \tag{A.4.5}$$

To evaluate this integrand (and to make connection with our earlier discussion of the Bondi mass) we will work in the Fefferman-Graham gauge. We note that we have

$$\bar{g}_{\mu\nu}dx^{\mu}dx^{\nu} = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}\left(g_{(0)ab} + \rho^2 g_{(2)ab} + \rho^4 g_{(4)ab}\right)dx^a dx^b \tag{A.4.6}$$

where the terms in the expansion on the RHS have the line-elements

$$\begin{aligned} ds^2_{(0)} &= -dt^2 + d\Omega^2 \\ ds^2_{(2)} &= \frac{1}{2}(-dt^2 - d\Omega^2) \\ ds^2_{(4)} &= \frac{1}{16}(-dt^2 + d\Omega^2) \end{aligned} \tag{A.4.7}$$

($g_{(0)}$ and $g_{(2)}$ were already given in equations (3.4.17) and (3.4.18) respectively). Enforcing the requirements that $g_{\mu\nu}$ solves the field equations and $h_{\mu\nu}$ vanishes at $\mathscr{I}$, the most general form for $h_{\mu\nu}$ is

$$h_{ab}dx^a dx^b = \rho g_{(3)ab}dx^a dx^b + \mathcal{O}(\rho^2) \tag{A.4.8}$$

where $g_{(3)}$ is given by (3.4.42) and we note that the Fefferman-Graham gauge forces $h_{\rho\mu} = 0$. The higher order terms do not contribute to the Abbott-Deser mass (they vanish in the limit to $\mathscr{I}$), so we focus on the $g_{(3)}$ term.

With the coordinates, timelike Killing vector and perturbation specified, we are ready to compute the Abbott-Deser mass. In Fefferman-Graham coordinates the limit in (A.4.2) simply becomes $\rho \to 0$ (recall $\mathscr{I} = \{\rho = 0\}$) and we can apply formulae (A.4.6)-(A.4.7) for the background metric and (A.4.8) for the perturbation in order to write the superpotential (A.4.3) and thus the Abbott-Deser mass. Explicitly the Abbott-Deser mass, $\mathcal{M}_{AD}$, is given by

$$\mathcal{M}_{AD} = \frac{1}{8\pi}\lim_{S_a \to \mathscr{I}_a}\oint dS_a m^a \tag{A.4.9}$$

with

$$m^a = \sqrt{-\bar{g}}[\bar{D}_{\beta}K^{ta\nu\beta} - K^{tb\nu a}\bar{D}_b]\bar{\xi}_{\nu}. \tag{A.4.10}$$

Given that we are working in the Fefferman-Graham gauge, the only component which we will need in order to compute $\mathcal{M}_{AD}$ is m^ρ:

$$m^\rho = \frac{(\rho^4 + 48)\sin\theta \hat{W}_3(t,\theta)}{12\,(\rho^2 - 4)}\,(1 + \mathcal{O}(\rho)) \tag{A.4.11}$$

Taking the limit to $\mathscr{I}$ gives

$$\begin{aligned} \mathcal{M}_{AD} &= \frac{1}{8\pi}\lim_{\rho\to 0}\oint dS_\rho m^\rho \\ &= -\frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \hat{W}_3(t,\theta)\sin\theta \\ &= \frac{1}{4\pi}\int_{S^2} m_B(t,\theta) = \mathcal{M}_B \end{aligned} \tag{A.4.12}$$

where in going from the second to the third line we have used the relationship (3.4.43) to rewrite the integral in terms of the Bondi mass aspect. Thus we have shown that the Bondi and Abbott-Deser masses are the same for asymptotically AdS spacetimes.

APPENDIX B

Appendices for chapter 5

B.1 Fefferman-Graham terms in de-Sitter

In this appendix we present the long expressions which result in the Fefferman-Graham expansion of an asymptotically locally dS Bondi-Sachs metric. For the duration of this section we have used the normalisation of $l_{dS} = 1$.

B.1.1 $\tilde{g}_{(2)ab}$

Here we present the $g_{(2)}$ term in the expansion component by component

$$
\begin{aligned}
g_{(2)\tilde{t}\tilde{t}} =& -\frac{1}{2}e^{2\gamma_0-4\beta_0}U_0^4+\frac{1}{2}e^{2\gamma_0-4\beta_0}\gamma_{0,\theta}^2U_0^4-e^{2\gamma_0-4\beta_0}\cot(\theta)\beta_{0,\theta}U_0^4- \\
&\frac{3}{2}e^{2\gamma_0-4\beta_0}\cot(\theta)\gamma_{0,\theta}U_0^4+2e^{2\gamma_0-4\beta_0}\beta_{0,\theta}\gamma_{0,\theta}U_0^4-e^{2\gamma_0-4\beta_0}\gamma_{0,\theta\theta}U_0^4+ \\
&e^{2\gamma_0-4\beta_0}U_{0,\theta}\beta_{0,\theta}U_0^3-\frac{3}{2}e^{2\gamma_0-4\beta_0}U_{0,\theta}\gamma_{0,\theta}U_0^3-\frac{1}{2}e^{2\gamma_0-4\beta_0}U_{0,\theta\theta}U_0^3+ \\
&abe^{2\gamma_0-4\beta_0}\cot(\theta)\beta_{0,\tilde{t}}U_0^3-2abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta}\beta_{0,\tilde{t}}U_0^3+ \\
&\frac{3}{2}abe^{2\gamma_0-4\beta_0}\cot(\theta)\gamma_{0,\tilde{t}}U_0^3-2abe^{2\gamma_0-4\beta_0}\beta_{0,\theta}\gamma_{0,\tilde{t}}U_0^3-abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta}\gamma_{0,\tilde{t}}U_0^3+ \\
&2abe^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}\theta}U_0^3+\cot^2(\theta)U_0^2-\csc^2(\theta)U_0^2-\frac{1}{2}e^{2\gamma_0-4\beta_0}U_{0,\theta}^2U_0^2-2\beta_{0,\theta}^2U_0^2+ \\
&\frac{3}{2}\gamma_{0,\theta}^2U_0^2+\frac{1}{2}e^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}}^2U_0^2-2\cot(\theta)\beta_{0,\theta}U_0^2-3\cot(\theta)\gamma_{0,\theta}U_0^2+4\beta_{0,\theta}\gamma_{0,\theta}U_0^2- \\
&\beta_{0,\theta\theta}U_0^2-\frac{3}{2}\gamma_{0,\theta\theta}U_0^2-\frac{1}{2}abe^{2\gamma_0-4\beta_0}\cot(\theta)U_{0,\tilde{t}}U_0^2+abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta}U_{0,\tilde{t}}U_0^2- \\
&abe^{2\gamma_0-4\beta_0}U_{0,\theta}\beta_{0,\tilde{t}}U_0^2+\frac{1}{2}abe^{2\gamma_0-4\beta_0}U_{0,\theta}\gamma_{0,\tilde{t}}U_0^2+2e^{2\gamma_0-4\beta_0}\beta_{0,\tilde{t}}\gamma_{0,\tilde{t}}U_0^2+ \\
&\frac{1}{2}abe^{2\gamma_0-4\beta_0}U_{0,\tilde{t}\theta}U_0^2-e^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}\tilde{t}}U_0^2+U_{0,\theta}\beta_{0,\theta}U_0-\frac{3}{2}U_{0,\theta}\gamma_{0,\theta}U_0- \\
&\frac{1}{2}U_{0,\theta\theta}U_0-ab\cot(\theta)\beta_{0,\tilde{t}}U_0+8abe^{-2\beta_0+2(\beta_0-\gamma_0)+2\gamma_0}\beta_{0,\theta}\beta_{0,\tilde{t}}U_0- \\
&8ab\beta_{0,\theta}\beta_{0,\tilde{t}}U_0+\frac{5}{2}ab\cot(\theta)\gamma_{0,\tilde{t}}U_0-4ab\beta_{0,\theta}\gamma_{0,\tilde{t}}U_0-ab\gamma_{0,\theta}\gamma_{0,\tilde{t}}U_0+ \\
&2ab\gamma_{0,\tilde{t}\theta}U_0-\frac{1}{2}e^{4\beta_0-2\gamma_0}-\frac{1}{2}U_{0,\theta}^2-2e^{4\beta_0-2\gamma_0}\beta_{0,\theta}^2+e^{4\beta_0-2\gamma_0}\gamma_{0,\theta}^2-\frac{3}{2}\gamma_{0,\tilde{t}}^2- \\
&e^{4\beta_0-2\gamma_0}\cot(\theta)\beta_{0,\theta}-\frac{3}{2}e^{4\beta_0-2\gamma_0}\cot(\theta)\gamma_{0,\theta}+2e^{4\beta_0-2\gamma_0}\beta_{0,\theta}\gamma_{0,\theta}-e^{4\beta_0-2\gamma_0}\beta_{0,\theta\theta}- \\
&\frac{1}{2}e^{4\beta_0-2\gamma_0}\gamma_{0,\theta\theta}+\frac{1}{2}ab\cot(\theta)U_{0,\tilde{t}}-4abe^{-2\beta_0+2(\beta_0-\gamma_0)+2\gamma_0}\beta_{0,\theta}U_{0,\tilde{t}}+ \\
&4ab\beta_{0,\theta}U_{0,\tilde{t}}-abU_{0,\theta}\beta_{0,\tilde{t}}+\frac{3}{2}abU_{0,\theta}\gamma_{0,\tilde{t}}+\frac{1}{2}abU_{0,\tilde{t}\theta}
\end{aligned}
\tag{B.1.1a}
$$

$$
\begin{aligned}
2g_{(2)\tilde{t}\theta} =& - abe^{2\gamma_0-4\beta_0}U_0^3 + abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta}^2 U_0^3 - 2abe^{2\gamma_0-4\beta_0}\cot(\theta)\beta_{0,\theta}U_0^3 - \\
& 3abe^{2\gamma_0-4\beta_0}\cot(\theta)\gamma_{0,\theta}U_0^3 + 4abe^{2\gamma_0-4\beta_0}\beta_{0,\theta}\gamma_{0,\theta}U_0^3 - 2abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta\theta}U_0^3 + \\
& 2abe^{2\gamma_0-4\beta_0}U_{0,\theta}\beta_{0,\theta}U_0^2 - 3abe^{2\gamma_0-4\beta_0}U_{0,\theta}\gamma_{0,\theta}U_0^2 - abe^{2\gamma_0-4\beta_0}U_{0,\theta\theta}U_0^2 + \\
& 2e^{2\gamma_0-4\beta_0}\cot(\theta)\beta_{0,\tilde{t}}U_0^2 - 4e^{2\gamma_0-4\beta_0}\gamma_{0,\theta}\beta_{0,\tilde{t}}U_0^2 + 3e^{2\gamma_0-4\beta_0}\cot(\theta)\gamma_{0,\tilde{t}}U_0^2 - \\
& 4e^{2\gamma_0-4\beta_0}\beta_{0,\theta}\gamma_{0,\tilde{t}}U_0^2 - 2e^{2\gamma_0-4\beta_0}\gamma_{0,\theta}\gamma_{0,\tilde{t}}U_0^2 + 4e^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}\theta}U_0^2 + \\
& 2ab\cot^2(\theta)U_0 - 2ab\csc^2(\theta)U_0 - abe^{2\gamma_0-4\beta_0}U_{0,\theta}^2 U_0 + \\
& 8abe^{-2\beta_0+2(\beta_0-\gamma_0)+2\gamma_0}\beta_{0,\theta}^2 U_0 - 12ab\beta_{0,\theta}^2 U_0 + 2ab\gamma_{0,\theta}^2 U_0 + abe^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}}^2 U_0 + \\
& abU_0 - 2ab\cot(\theta)\beta_{0,\theta}U_0 - 3ab\cot(\theta)\gamma_{0,\theta}U_0 + 4ab\beta_{0,\theta}\gamma_{0,\theta}U_0 - 2ab\beta_{0,\theta\theta}U_0 - \\
& ab\gamma_{0,\theta\theta}U_0 - e^{2\gamma_0-4\beta_0}\cot(\theta)U_{0,\tilde{t}}U_0 + 2e^{2\gamma_0-4\beta_0}\gamma_{0,\theta}U_{0,\tilde{t}}U_0 - \\
& 2e^{2\gamma_0-4\beta_0}U_{0,\theta}\beta_{0,\tilde{t}}U_0 + e^{2\gamma_0-4\beta_0}U_{0,\theta}\gamma_{0,\tilde{t}}U_0 + 4abe^{2\gamma_0-4\beta_0}\beta_{0,\tilde{t}}\gamma_{0,\tilde{t}}U_0 + \\
& e^{2\gamma_0-4\beta_0}U_{0,\tilde{t}\theta}U_0 - 2abe^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}\tilde{t}}U_0 + 4abe^{-2\beta_0+2(\beta_0-\gamma_0)+2\gamma_0}U_{0,\theta}\beta_{0,\theta} - \\
& 4abU_{0,\theta}\beta_{0,\theta} - 8e^{-2\beta_0+2(\beta_0-\gamma_0)+2\gamma_0}\beta_{0,\theta}\beta_{0,\tilde{t}} + 8\beta_{0,\theta}\beta_{0,\tilde{t}} + \\
& 4\cot(\theta)\gamma_{0,\tilde{t}} - 4\beta_{0,\theta}\gamma_{0,\tilde{t}} - 4\gamma_{0,\theta}\gamma_{0,\tilde{t}} + 2\gamma_{0,\tilde{t}\theta}
\end{aligned}
\tag{B.1.1b}
$$

$$
\begin{aligned}
g_{(2)\theta\theta} =& -\frac{1}{2}e^{2\gamma_0-4\beta_0}U_0^2 + \frac{1}{2}e^{2\gamma_0-4\beta_0}\gamma_{0,\theta}^2 U_0^2 - e^{2\gamma_0-4\beta_0}\cot(\theta)\beta_{0,\theta}U_0^2 - \\
& \frac{3}{2}e^{2\gamma_0-4\beta_0}\cot(\theta)\gamma_{0,\theta}U_0^2 + 2e^{2\gamma_0-4\beta_0}\beta_{0,\theta}\gamma_{0,\theta}U_0^2 - e^{2\gamma_0-4\beta_0}\gamma_{0,\theta\theta}U_0^2 + \\
& e^{2\gamma_0-4\beta_0}U_{0,\theta}\beta_{0,\theta}U_0 - \frac{3}{2}e^{2\gamma_0-4\beta_0}U_{0,\theta}\gamma_{0,\theta}U_0 - \frac{1}{2}e^{2\gamma_0-4\beta_0}U_{0,\theta\theta}U_0 + \\
& abe^{2\gamma_0-4\beta_0}\cot(\theta)\beta_{0,\tilde{t}}U_0 - 2abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta}\beta_{0,\tilde{t}}U_0 + \frac{3}{2}abe^{2\gamma_0-4\beta_0}\cot(\theta)\gamma_{0,\tilde{t}}U_0 - \\
& 2abe^{2\gamma_0-4\beta_0}\beta_{0,\theta}\gamma_{0,\tilde{t}}U_0 - abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta}\gamma_{0,\tilde{t}}U_0 + 2abe^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}\theta}U_0 - \\
& \frac{1}{2}e^{2\gamma_0-4\beta_0}U_{0,\theta}^2 - 8e^{-2\beta_0+2(\beta_0-\gamma_0)+2\gamma_0}\beta_{0,\theta}^2 + 6\beta_{0,\theta}^2 - \gamma_{0,\theta}^2 + \frac{1}{2}e^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}}^2 + \\
& \cot(\theta)\beta_{0,\theta} + \frac{3}{2}\cot(\theta)\gamma_{0,\theta} - \beta_{0,\theta\theta} + \frac{1}{2}\gamma_{0,\theta\theta} - \frac{1}{2}abe^{2\gamma_0-4\beta_0}\cot(\theta)U_{0,\tilde{t}} + \\
& abe^{2\gamma_0-4\beta_0}\gamma_{0,\theta}U_{0,\tilde{t}} - abe^{2\gamma_0-4\beta_0}U_{0,\theta}\beta_{0,\tilde{t}} + \frac{1}{2}abe^{2\gamma_0-4\beta_0}U_{0,\theta}\gamma_{0,\tilde{t}} + \\
& 2e^{2\gamma_0-4\beta_0}\beta_{0,\tilde{t}}\gamma_{0,\tilde{t}} + \frac{1}{2}abe^{2\gamma_0-4\beta_0}U_{0,\tilde{t}\theta} - e^{2\gamma_0-4\beta_0}\gamma_{0,\tilde{t}\tilde{t}} + \frac{1}{2}
\end{aligned}
\tag{B.1.1c}
$$

$$
\begin{aligned}
g_{(2)\phi\phi} =& \frac{1}{2}e^{-4\gamma_0}\sin^2(\theta) + \frac{1}{2}e^{-4\beta_0-2\gamma_0}U_0^2\sin^2(\theta) + \frac{1}{2}e^{-4\beta_0-2\gamma_0}U_{0,\theta}^2\sin^2(\theta) + \\
& 2e^{-4\gamma_0}\beta_{0,\theta}^2\sin^2(\theta) - e^{-4\gamma_0}\gamma_{0,\theta}^2\sin^2(\theta) + \frac{1}{2}e^{-4\beta_0-2\gamma_0}U_0^2\gamma_{0,\theta}^2\sin^2(\theta) + \\
& \frac{1}{2}e^{-4\beta_0-2\gamma_0}\gamma_{0,\tilde{t}}^2\sin^2(\theta) - e^{-4\beta_0-2\gamma_0}U_0U_{0,\theta}\beta_{0,\theta}\sin^2(\theta) + \\
& \frac{5}{2}e^{-4\beta_0-2\gamma_0}U_0U_{0,\theta}\gamma_{0,\theta}\sin^2(\theta) - 2e^{-4\beta_0-2\gamma_0}U_0^2\beta_{0,\theta}\gamma_{0,\theta}\sin^2(\theta) + \\
& \frac{1}{2}e^{-4\beta_0-2\gamma_0}U_0U_{0,\theta\theta}\sin^2(\theta) + e^{-4\gamma_0}\beta_{0,\theta\theta}\sin^2(\theta) + \frac{1}{2}e^{-4\gamma_0}\gamma_{0,\theta\theta}\sin^2(\theta) + \\
& e^{-4\beta_0-2\gamma_0}U_0^2\gamma_{0,\theta\theta}\sin^2(\theta) - abe^{-4\beta_0-2\gamma_0}\gamma_{0,\theta}U_{0,\tilde{t}}\sin^2(\theta) + \\
& abe^{-4\beta_0-2\gamma_0}U_{0,\theta}\beta_{0,\tilde{t}}\sin^2(\theta) + 2abe^{-4\beta_0-2\gamma_0}U_0\gamma_{0,\theta}\beta_{0,\tilde{t}}\sin^2(\theta) - \\
& \frac{3}{2}abe^{-4\beta_0-2\gamma_0}U_{0,\theta}\gamma_{0,\tilde{t}}\sin^2(\theta) + 2abe^{-4\beta_0-2\gamma_0}U_0\beta_{0,\theta}\gamma_{0,\tilde{t}}\sin^2(\theta) - \\
& abe^{-4\beta_0-2\gamma_0}U_0\gamma_{0,\theta}\gamma_{0,\tilde{t}}\sin^2(\theta) - 2e^{-4\beta_0-2\gamma_0}\beta_{0,\tilde{t}}\gamma_{0,\tilde{t}}\sin^2(\theta) - \\
& \frac{1}{2}abe^{-4\beta_0-2\gamma_0}U_{0,\tilde{t}\theta}\sin^2(\theta) - 2abe^{-4\beta_0-2\gamma_0}U_0\gamma_{0,\tilde{t}\theta}\sin^2(\theta) + \\
& e^{-4\beta_0-2\gamma_0}\gamma_{0,\tilde{t}\tilde{t}}\sin^2(\theta) - e^{-4\beta_0-2\gamma_0}\cos(\theta)U_0U_{0,\theta}\sin(\theta) + \\
& e^{-4\beta_0-2\gamma_0}\cos(\theta)U_0^2\beta_{0,\theta}\sin(\theta) - e^{-4\gamma_0}\cos(\theta)\beta_{0,\theta}\sin(\theta) + \\
& \frac{1}{2}e^{-4\beta_0-2\gamma_0}\cos(\theta)U_0^2\gamma_{0,\theta}\sin(\theta) + \frac{3}{2}e^{-4\gamma_0}\cos(\theta)\gamma_{0,\theta}\sin(\theta) + \\
& \frac{1}{2}abe^{-4\beta_0-2\gamma_0}\cos(\theta)U_{0,\tilde{t}}\sin(\theta) - abe^{-4\beta_0-2\gamma_0}\cos(\theta)U_0\beta_{0,\tilde{t}}\sin(\theta) - \\
& \frac{1}{2}abe^{-4\beta_0-2\gamma_0}\cos(\theta)U_0\gamma_{0,\tilde{t}}\sin(\theta)
\end{aligned} \tag{B.1.1d}
$$

with all other components zero.

B.1.2 $\tilde{g}_{(3)ab}$

The printout for the $\tilde{g}_{(3)}$ component is excessively long and the formulae span several pages of this document. Instead of printing them here we refer the reader to the supplementary MATHEMATICA file (‘`BS_AdS_dS_continuatiuon_FG.nb`’), which contains the full expressions for $\tilde{g}_{(3)}$

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