LS-CATEGORY OF MOMENT-ANGLE MANIFOLDS AND HIGHER ORDER MASSEY PRODUCTS

PIOTR BEBEN AND JELENA GRBIĆ

ABSTRACT. Using the combinatorics of the underlying simplicial complex K, we give various upper and lower bounds for the Lusternik-Schnirelmann (LS) category of moment-angle complexes $\mathcal{Z}_K$. We describe families of simplicial complexes and combinatorial operations which allow for a systematic description of the LS category. In particular, we characterise the LS category of moment-angle complexes $\mathcal{Z}_K$ over triangulated d-manifolds K for $d \leq 2$, as well as higher dimension spheres built up via connected sum, join, and vertex doubling operations. We show that the LS category closely relates to vanishing of Massey products in $H^*(\mathcal{Z}_K)$ and therefore to the Golod property of the simplicial complex K. Through this connection we describe first structural properties of Massey products in moment-angle manifolds. Some of further applications include calculations of the LS category and the description of conditions for vanishing of Massey products for moment-angle manifolds over fullerenes, Pogorelov polytopes and k-neighbourly complexes, which double as important examples of hyperbolic manifolds.

1. INTRODUCTION

A covering of a topological space X is said to be *categorical* if every set in the covering is open and contractible in X, that is, the inclusion map of each set into X is nullhomotopic. The *Lusternik-Schnirelmann category* (or simply *category*) $\mathrm{cat}(X)$ of X is the smallest integer k such that X admits a categorical covering by $k+1$ open sets $\{U_0, \ldots, U_k\}$.

In general it is not easy to compute these invariants. Lusternik and Schnirelmann [47, 46] introduced it initially in connection to variational problems. Poincaré studying dynamical systems suggested that the existence and form of solutions of differential equations, that is, the complexity of flows should be related to the topological complexity of the underlying manifold. In this context, a first step was to estimate the number of invariant points for the particular case of gradient flows, equivalently, to estimate the minimal number of critical points of functions on the manifold. The Lusternik-Schnirelmann invariant, nowadays known as the LS category, gives a lower bound on the number of critical points for any smooth function on the manifold. While this was analytical in nature, there has been a host of useful applications in geometry and algebraic topology. For example, G. Whitehead [54, page 464] showed that for a topological space X and a group-like space G, a lower bound of the category of X is given in terms of nilpotency of the group of homotopy classes of maps from X to G, that is, $\mathrm{cat}(X) \geq \mathrm{nilp}([X, G])$. Here the nilpotency $\mathrm{nilp}(A)$ of a group A is the smallest integer k such that all length k product are trivial.

In this paper, motivated by the study of the Lusternik-Schnirelmann category and related invariants for polyhedral products of the form $(X, *)^K$ for certain nice spaces X in [21, 29], we are focusing on the LS category of moment-angle complexes $\mathcal{Z}_K$ which are a particular example of polyhedral product $(X, A)^K$ for $X = D^2$ and $A = S^1$.

A large portion of the attention that moment-angle complexes have received has been due to their relevance to algebraic and complex geometry, combinatorics and algebra, in particular homology of local rings. The relation between cohomology of moment-angle complexes and homology of local

2010 *Mathematics Subject Classification.* Primary 55M30, 55U10, Secondary 32Q55, 05E40.

Key words and phrases. polyhedral product, moment-angle complex, toric topology, Massey products, Lusternik-Schnirelmann category, Golod rings, simplicial complexes, simple polytopes.

rings can be seen through the notion of Golod rings. To a simplicial complex K on n vertices, the Stanley-Reisner algebra $R[K] = R[v_1, \ldots, v_n]/I_K$, where R is a commutative ring with unit, can be associated and there is a ring isomorphism $H^*(\mathcal{Z}_K; R) \cong \mathrm{Tor}_{R[v_1,\ldots,v_n]}(R[K], R)$. In particular, the case of $\mathcal{Z}_K$ being a co-H-space, that is, $\mathrm{cat}(\mathcal{Z}_K) = 1$, is closely related to the Stanley-Reisner ring $R[K]$ being a Golod ring.

Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let M be a finitely generated module over R. The Poincaré series $P^R_M(t)$ of M over R is a formal power series in $\mathbb{Z}[|t|]$ defined as

$$P^R_M(t) = \sum_{i \geq 0} \beta^R_i(M) t^i \in \mathbb{Z}[|t|]$$

where $\beta^R_i(M) = \dim_k \mathrm{Tor}^i_R(M, k)$ denotes the i-th Betti number of M. In the 1950s Serre and Kaplansky asked whether the Poincaré series $P^R_k(t)$ is a rational function. This question ties in closely with an analogous question in algebraic topology attributed to Serre on the rationality of the Poincaré series of loop spaces of finite simply connected CW complexes (see [51]). Several authors made an attempt to find an affirmative answer. However in 1982 Anick found a counterexample answering the question in negative [1]. Research on this topic intensified since the appearance of Anick's example. The question of the Stanley-Reisner ring $R[K]$ being Golod translates to cup products and all higher Massey products vanishing in $H^*(\mathcal{Z}_K)$. In fact, there is a fairly large literature that is focused on determining those K for which the Stanley-Reisner ring $R[K]$ is Golod (see [5, 6, 32, 34, 33, 38, 7]).

Away from algebra, moment-angle complexes have found intimate connections and applications to complex and symplectic geometry. For example, Bosio and Meersseman [9] studied a class of complete intersections of real quadrics in $\mathbb{C}^n$ called links and showed that all links, after taking products with circles in odd dimensional cases, can be given a complex-analytic structure. Some of them are non-Kähler complex manifolds which generalise the class of Hopf and Calabi-Ekmann manifolds. Crucially, Bosio and Meersseman established a connection between complex geometry and toric topology by showing that links coincide with moment-angle manifolds $\mathcal{Z}_P$ over a simple polytope P. Deligne, Griffiths, Morgan and Sullivan [17] proved that compact Kähler manifolds are formal. Therefore they have trivial Massey products with field coefficients. In this way, moment-angle manifolds admitting non-trivial Massey products describe some of the few known explicit families of complex manifolds beyond Kähler manifolds. This is one of the main motivations to study Massey products and more generally topology of moment-angle manifolds.

The above two questions; Golodness of Stanley-Reisner rings and Massey products in complex manifolds, provide a motivation to study the LS category and its relations to Massey products of the large class of moment-angle complexes known as *moment-angle manifolds*. Moment-angle complexes $\mathcal{Z}_K$ take the form of topological manifolds when K is a simplicial sphere or equivalently when moment-angle complexes are considered over simple polytopes. Their topology and cohomology is very intricate, with many questions remaining open even for low dimensional K (see for example [13, 18, 27, 16, 48, 9, 44, 45]).

In Section 3 we describe filtrations of simplicial complexes K which imply lower bounds on the LS-category of $\mathcal{Z}_K$. We continue by detecting combinatorial operations on simplicial complexes which in contrast to filtrations imply upper bounds on $\mathrm{cat}(\mathcal{Z}_K)$. As specific applications of these methods in Section 4, we characterise triangulations K of d-spheres where $d \leq 2$ for which $\mathcal{Z}_K$ has a given category, and we build up higher dimensional spheres K for which we can calculate $LS(\mathcal{Z}_K)$.

In Proposition 4.12 we further extend those results by calculating $\mathrm{cat}(\mathcal{Z}_K)$ for any triangulation of a closed oriented surface K. Up to now only $\mathrm{cat}(\mathcal{Z}_K) = 1$ has been considered.

Our further motivation is going back to the homology of rings and the study of a combinatorial and algebraic characterisation of Golod complexes K and *co-H*-space (category ≤ 1) moment-angle complexes $\mathcal{Z}_K$ given in [32] in the case of flag complexes K. The authors there showed that both of these concepts are equivalent, and moreover, that they both coincide with chordality of the 1-skeleton of K and the triviality of the multiplication on $H^*(\mathcal{Z}_K;R) \cong \mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$ for $R=\mathbb{Z}$ or R any field. An interesting consequence of this from the perspective of commutative algebra was that for K flag the trivial multiplication on $\mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$ implies that all higher Massey products are also trivial. This depended on the general fact that the cohomology ring of a space of category less than equal to 1 has trivial multiplication and Massey products vanish [26, 53]. It is natural to ask what the corresponding statement is for spaces with larger category, more so, if the characterisation for Golod flag complexes in [32] can be generalised in terms of Massey products. An answer to the first question was given by Rudyak in [52], which inspired us to give the following definition.

Definition 1.1. A simplicial complex K on vertex set $[n]$ is *m-annihilating* over R if

(1) $\mathrm{nill}(\mathrm{Tor}_{R[v_1,\dots,v_n]}(R[K],R)) \leq m+1$;
(2) Massey products $\langle v_1,\dots,v_k\rangle$ vanish in $\mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$ whenever $v_i = a_1\cdots a_{m_i}$ and $v_j = b_1\cdots b_{m_j}$, and $m_i+m_j > m$ for some odd i and even j and $a_s, b_t \in \mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$.

Proposition 1.2. *If $\mathrm{cat}(\mathcal{Z}_K) \leq m$, then K is m-annihilating.* □

Here the *nilpotency* $\mathrm{nill}\,A$ of a graded algebra A is the smallest integer k such that all length k products in the positive degree part A^+ vanish. Notice that K is $(m+1)$-annihilating whenever it is m-annihilating, and 1-annihilating of K coincides with K being Golod [28], namely, that all products and (higher) Massey products are trivial in $\mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$. All of this can be restated equivalently in terms of the cohomology of $\mathcal{Z}_K$ due to an isomorphism of graded commutative algebras $H^*(\mathcal{Z}_K;R) \cong \mathrm{Tor}_{R[v_1,\dots,v_n]}(R[K],R)$ when R is a field or $\mathbb{Z}$ [4]. We shall consider (co)homology with integer coefficients.

We note that the converse of Proposition 1.2 might not be true in general. For example, Iriye and Yano in [41] constructed an example of a Golod complex K such that $\mathcal{Z}_K$ is not a co-H-space. Inequality (1) can also be strict (using a construction of Katthän [43] in Example 5.8). However, we show the converse does hold (along with extension to larger LS-categories) for 1-spheres and 2-spheres, 2-dimensional closed oriented surfaces, as well as a natural class of higher dimensional spheres, and that they can be characterised combinatorially.

Proposition 1.3. *Let K be a triangulation of a closed oriented connected surface. Then*

$$1 \leq \mathrm{cup}(\mathcal{Z}_K) = \mathrm{cat}(\mathcal{Z}_K) \leq 3.$$

Moreover, letting $m := \mathrm{cat}(\mathcal{Z}_K)$, we have:
(i) $m=1$ iff K has no chordless cycles (equivalently $K=\partial\Delta^3$)
(ii) $m=2$ iff K has a chordless cycle, but none with more than 3 vertices
(iii) $m=3$ iff K has a chordless cycle with at least 4 vertices. □

Theorem 1.4. *If K is any triangulated d-sphere for $d \leq 2$, or built up as a connected sum of joins of one or more of such spheres (as long as spheres in the joins are not simplex boundaries, and connected sums are over disjoint faces), then the following are equivalent:*
(a) $\mathrm{cat}(\mathcal{Z}_K) \leq k$;
(b) K is k-annihilating;
(c) length $k+1$ cup products of positive degree elements in $H^*(\mathcal{Z}_K)$ vanish;
(d) there does not exist a spherical filtration *of full subcomplexes of K of length more than k. Moreover, $k \leq d+1$.* □

Dually, these operations on spheres can be stated in terms of simple polyotpes, that is, the join of simplicial complexes corresponds to the product of simple polytopes, while the connected sum of simplicial complexes corresponds to taking the vertex cut of corresponding polytopes and then gluing them along the new hyperplane which is subsequently removed.

As applications of the Theorem 1.4, we obtain the LS category of moment-angle manifolds over important classes of simple 3-polytopes which contain fullerenes, Löbel, Pogorelov polytopes. The moment-angle manifolds over these 3-polytopes have been of interest in various research areas, in particular there is a direct relation to hyperbolic geometry [14].

The question of determining higher Massey products in $H^*(\mathcal{Z}_K)$ is an important one but notoriously difficult and equally interesting for algebraist, topologists and geometers. Currently, a systematic answer is known in the case of moment-angle complexes associated to one dimensional simplicial complexes, and only for triple Massey products of three dimensional cohomological classes (see [18, 31, 30]). Using the Lusternik-Schnirelmann category of moment-angle complexes, we give the first structural results on higher Massey products in moment-angle complexes by considering n-Massey products of decomposable classes of arbitrary dimensions. For instance, k is precisely 3 when K is the boundary of the dual of a *fullerene* P.

Theorem 1.5. *For fullerenes P, $\mathrm{cat}(\mathcal{Z}_{\partial P^*}) = 3$ and ∂P^* is 3-annihilating. In particular, all Massey products of decomposable elements in $H^+(\mathcal{Z}_{\partial P^*})$ must vanish.* □

The last theorem can be generalised to any Pogorelov polytope.

Theorem 1.6. *For Pogorelov polytopes P, $\mathrm{cat}(\mathcal{Z}_{\partial P^*}) = 3$ and ∂P^* is 3-annihilating implying that all Massey products of decomposable elements in $H^+(\mathcal{Z}_{\partial P^*})$ vanish.*

Applying *vertex doubling operations*, the range of spheres in Theorem 1.4 can be extended.

Theorem 1.7. *If $K(J)$ is the simplicial wedge of K for some integer sequence $J = (j_1, \dots, j_n)$, then $\mathrm{cat}(\mathcal{Z}_{K(J)}) \leq \mathrm{cat}(\mathcal{Z}_K)$.* □

Our methods also apply to compute tight bounds on the LS-category for other important classes of simplicial complexes, for example, those that are graphs, and those that have the neighbourly property.

Theorem 1.8. *If G is a graph, $cat(\mathcal{Z}_G) \leq 2$. Thus, G is at least 2-annihilating, and all Massey products of decomposable elements in $\mathrm{Tor}^+_{\mathbb{Z}[v_1,\dots,v_n]}(\mathbb{Z}[G], \mathbb{Z}))$ vanish.*

Theorem 1.9. *If K is l-neighbourly, then $\mathrm{cat}(\mathcal{Z}_K) \leq \frac{1+\dim K}{l}$ and K is $\left(\frac{1+\dim K}{l}\right)$-annihilating.* □

We close the introduction by noting that many of the results in this paper extend to polyhedral products of the form $(Cone(X), X)^K$ in place of $\mathcal{Z}_K = (D^2, S^1)^K$.

2. Preliminary

Recall the following concepts from [42, 15, 22, 25]. The *geometric category* $\mathrm{gcat}(X)$ of a space X is the smallest integer k such that X admits a categorical covering $\{U_0, \dots, U_k\}$ of X with each U_i contractible (in itself), and the *category* $\mathrm{cat}(f)$ of a map $f\colon X \longrightarrow Y$ is the smallest k such that X admits an open covering $\{V_0, \dots, V_k\}$ such that f restricts to a nullhomotopic map on each V_i. It is easy to see that $\mathrm{cat}(X) = \mathrm{cat}(\mathbb{1}\colon X \longrightarrow X)$,

$$\mathrm{cat}(f\colon X \longrightarrow Y) \leq \min\{\mathrm{cat}(X), \mathrm{cat}(Y)\} \tag{1}$$

and $\mathrm{cat}(h \circ h') \leq \mathrm{cat}(h')$. For path-connected paracompact spaces,

$$\mathrm{cat}(f \times g) \leq \mathrm{cat}(f) + \mathrm{cat}(g)$$

which follows from the also well-known fact that $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$ together with the preceding inequalities. Unlike cat(), gcat() is not a homotopy invariant, though one can obtain a homotopy invariant from gcat() by defining the *strong category*

$$\mathcal{C}\mathrm{at}(X) = \min\{\mathrm{gcat}(Y) \mid Y \simeq X\}.$$

In fact, the strong category satisfies $\mathcal{C}\mathrm{at}(X) - 1 \leq \mathrm{cat}(X) \leq \mathcal{C}\mathrm{at}(X) \leq \mathrm{gcat}(X)$. We shall let $\mathrm{cup}(X) = \mathrm{nill}\, H^*(X) - 1$ denote the length of the longest non-zero cup product of positive degree elements in $H^*(X)$. The main use of this is the classical lower bound

$$\mathrm{cup}(X) \leq \mathrm{cat}(X).$$

2.1. **Some general bounds.** We begin by giving upper bounds for the Lusternik-Schnirelmann category of some general spaces.

Lemma 2.1. *Let A be a subcomplex of X and S an open subset of A. Then S is a deformation retract of an open subset U of X such that $U \cap A = S$.*

Proof. Let $\mathcal{I}_j$ be an index set for the j-cells e^j_α of $X - A$, $\Phi_\alpha \colon D^j \longrightarrow X$ its characteristic map, and $\phi_\alpha \colon \partial D^j \hookrightarrow D^j \xrightarrow{\Phi_\alpha} X$ its attaching map. Given a subset $B \subseteq X$, let $V_{\alpha,B}$ be the image of $\phi_\alpha^{-1}(B) \times [0, \frac{1}{2}) \subseteq D^j \cong (\partial D^j \times [0,1])/(\partial D^j \times \{1\})$ under Φ_α. Notice that $V_{\alpha,B}$ deformation retracts onto a subspace of $\phi_\alpha(\partial D^j) \cap B$, and if $B \cap e^j_\alpha = \emptyset$, $B \cup V_{\alpha,B}$ deformation retracts onto B.

Construct $R_{i+1} \subseteq X$ such that $R_i \subseteq R_{i+1}$, R_i is a deformation retract of R_{i+1}, and $R_i \cap X^{\langle i \rangle}$ is open in the i-skeleton $X^{\langle i \rangle}$, by letting $R_0 = S$ and $R_{i+1} = R_i \cup \bigcup_{\alpha \in \mathcal{I}_{i+1}} V_{\alpha,S}$. Then $U = \bigcup_{i \geq 0} R_i$ is open in X, deformation retracts onto S, and $U \cap A = S$.

□

Lemma 2.2. *Given a filtration $X_0 \subseteq \cdots \subseteq X_m = X$ of subcomplexes of a CW-complex X, suppose $X_{i+1} - X_i$ is contractible in X for each i. Then $\mathrm{cat}(X) \leq \mathrm{cat}(X_0 \hookrightarrow X) + m \leq \mathrm{cat}(X_0) + m$.*

Proof. Let $k := \mathrm{cat}(X_0 \hookrightarrow X)$, and $\{U_0, \ldots, U_k\}$ be a categorical cover of the inclusion $X_0 \hookrightarrow X$. Note that $V_i = X_{i+1} - X_i$ is open in X_{i+1} since the subcomplex X_i is closed in X_{i+1}. Then iterating Lemma 2.1, we have open subsets $\bar{U}_i$ and $\bar{V}_i$ that deformation retract onto each U_i and V_i, respectively. Since the U_i's and V_i's cover X and are contractible in X, so do the $\bar{U}_i$'s and $\bar{V}_i$'s, thus they form a categorical cover of X.

□

For any spaces X and Y, and a fixed basepoint $* \in X$, we let $X \rtimes Y := (X \times Y)/(* \times Y)$ denote the *right half-smash* of X and Y, and $Y \ltimes X = (Y \times X)/(Y \times *)$ the *left half-smash*.

Lemma 2.3. *If X and Y are CW-complexes and X is path-connected, then* $\mathrm{cat}(X \rtimes Y) = \mathrm{cat}(X)$.

Proof. Let $\tilde{X}$ be given by attaching the interval $[0,1]$ to X by identifying $0 \in [0,1]$ with the basepoint $* \in X$, and fix $1 \in \tilde{X}$ to be the basepoint. Given $k = \mathrm{cat}(\tilde{X})$ and $\{U_0, \ldots, U_k\}$ a categorical cover of $\tilde{X}$, take the open cover $\{U_0 \rtimes Y, \ldots, U_k \rtimes Y\}$ of $\tilde{X} \rtimes Y = (\tilde{X} \times Y)/(1 \times Y)$ (here $U_i \rtimes Y = U_i \times Y$ if $1 \notin U_i$). Notice that the contractions of each U_i in $\tilde{X}$ can be taken so that 1 remains fixed if $1 \in U_i$. If U_i contracts to a point b_i in $\tilde{X}$, $U_i \rtimes Y$ deforms onto $\{b_i\} \rtimes Y$ in $\tilde{X} \rtimes Y$, which in turn contracts to the basepoint in $\tilde{X} \rtimes Y$ by homotoping the coordinate b_i to 1. Therefore $\mathrm{cat}(\tilde{X} \rtimes Y) \leq k$, and we have $\mathrm{cat}(X \rtimes Y) \leq k$ since $X \simeq \tilde{X}$ and $X \rtimes Y \simeq \tilde{X} \rtimes Y$. Moreover, $\mathrm{cat}(X \rtimes Y) \geq k$ since X is a retract of $X \rtimes Y$.

□

Let $\mathcal{S}$ be m copies of the interval $[0,1]$ glued together at the endpoints 1 in some order. Given a collection of maps $X \xrightarrow{f_i} Y_i$ for $i = 1, \ldots, m$, the *homotopy pushout* P of the maps f_i is the m-fold mapping cylinder

$$P := (Y_1 \sqcup \cdots \sqcup Y_m \sqcup (X \times \mathcal{S})) \,/ \sim$$

under the identification $(x,t) \sim f_i(x)$ whenever t is in the i^{th} copy of $[0,1]$ in $\mathcal{S}$ and $t = 0$.

Lemma 2.4. *Fix $m \geq 2$. For $i = 1, \ldots, m$, let A_i and C_i be basepointed CW-complexes, $B_i = \prod_{j \neq i} A_j$, and E be a contractible space. Suppose $A_i \times E \xrightarrow{f_i} C_i$ are nullhomotopic maps, and P is the homotopy pushout of the maps $A_i \times E \times B_i \xrightarrow{f_i \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \ldots, m$. Then*

$$\mathrm{cat}(P) \leq \max\{1, \mathrm{cat}(C_1), \ldots, \mathrm{cat}(C_m)\}.$$

Proof. We proceed by induction on m. Start with $m = 2$. By Lemma 7.1 in [34], there is a splitting $P \simeq (\Sigma A_1 \wedge A_2) \vee (C_1 \rtimes A_2) \vee (C_2 \rtimes A_1)$. Thus using Lemma 2.3,

$$\mathrm{cat}(P) = \max\{\mathrm{cat}(\Sigma A_1 \wedge A_2), \mathrm{cat}(C_1 \rtimes A_2), \mathrm{cat}(C_2 \rtimes A_1)\} = \max\{1, \mathrm{cat}(C_1), \mathrm{cat}(C_2)\}.$$

The statement holds when $m = 2$.

Take $\mathcal{B}_0 = *$, $\mathcal{B}_\ell = \prod_{j \leq \ell} A_j$, $B'_i = \prod_{j \neq i, j < m} A_j$, and B_i as basepointed subspaces of $\mathcal{B} = \prod_j A_j$. Let P' be the homotopy pushout of $f_i \times \mathbb{1}_{B'_i}$ for $i = 1, \ldots, m-1$ (these are all maps from $E \times B_m = E \times \mathcal{B}_{m-1}$). Suppose the lemma holds whenever $m < m'$ for some $m' > 2$. Let $m := m'$. Then $\mathrm{cat}(P') \leq \max\{1, \mathrm{cat}(C_1), \ldots, \mathrm{cat}(C_{m-1})\}$. Notice that P is the homotopy pushout of $f_m \times \mathbb{1}_{B_m}$ and the inclusion $A_m \times E \times B_m \xrightarrow{\mathbb{1}_{A_m} \times g} A_m \times P'$, where g is the inclusion $W_{m-1} \subset P'$, and $W_\ell = E \times \mathcal{B}_\ell \times \{1\}$. We can deform W_ℓ into $W_{\ell-1}$ in P' as follows. First deform W_ℓ onto $f_\ell(A_\ell \times E) \times \mathcal{B}_{\ell-1}$ by moving it down the mapping cylinder $M = ((E \times B_m \times [0,1]) \sqcup (C_\ell \times B_\ell))/\sim$ of P' and onto the base $C_\ell \times B_\ell$, then deform it onto $* \times \mathcal{B}_{\ell-1}$ in $C_\ell \times B_\ell$ using the nullhomotopy of f_ℓ. Finally, move $\mathcal{B}_{\ell-1}$ back up towards the top of the mapping cylinder M and into $W_{\ell-1}$. Composing these deformations for $\ell = m-1, \ldots, 1$ gives a contraction in P' of W_{m-1} to a point. Thus, g is nullhomotopic, as is f_m. Since the lemma holds for the base case $m = 2$, $\mathrm{cat}(P) = \max\{1, \mathrm{cat}(P'), \mathrm{cat}(C_m)\} \leq \max\{1, \mathrm{cat}(C_1), \ldots, \mathrm{cat}(C_m)\}$.

□

Lemma 2.5. *Fix $m \geq 2$, and for $i = 1, \ldots, m$, let A_i, C_i, E be basepointed CW-complexes, E is path-connected, and let $B_i := \prod_{j \neq i} A_j$. Suppose $A_i \times E \xrightarrow{f_i} C_i$ are maps such that the restriction $(f_i)_{|A_i \times *}$ of f_i to $A_i \times *$ is nullhomotopic, and P is the homotopy pushout of the maps $A_i \times E \times B_i \xrightarrow{f_i \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \ldots, m$.*

(i) Then

$$\mathrm{cat}(P) \leq \max\{1, \mathrm{cat}(C_1), \ldots, \mathrm{cat}(C_m)\} + \mathcal{C}\mathrm{at}(E).$$

(ii) Moreover, if each $C_i \times B_i$ is a subcomplex of some CW-complex X_i such that $X_i - C_i \times B_i$ is contractible in X_i, and P' is the homotopy pushout of the maps $A_i \times E \times B_i \xrightarrow{f_i \times \mathbb{1}_{B_i}} C_i \times B_i \hookrightarrow X_i$ for $i = 1, \ldots, m$, then also

$$\mathrm{cat}(P') \leq \max\{1, \mathrm{cat}(C_1), \ldots, \mathrm{cat}(C_m)\} + \mathcal{C}\mathrm{at}(E).$$

Proof. (i) Let $\mathcal{B} := A_1 \times \cdots \times A_m$, and $\mathcal{D} = \coprod_{i=1,\ldots,m}(C_i \times B_i)$, and let $\mathcal{S}_t$ for $t < 1$ be m copies of the interval $[t, 1]$ glued together at the endpoints 1, and $\mathcal{S}'_t$ be its interior, namely, m copies of $(t, 1]$ glued at 1.

Let $k := \mathcal{C}\mathrm{at}(E)$ and take $E' \simeq E$ to be such that $k = \mathrm{gcat}(E')$. Then P is homotopy equivalent to the homotopy pushout Q of the maps $A_i \times E' \times B_i \xrightarrow{f'_i \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \ldots, m$, where $A_i \times E' \xrightarrow{f'_i} C_i$ is the composite of f_i with the homotopy equivalence $A_i \times E' \xrightarrow{\mathbb{1}_{A_i} \times \simeq} A_i \times E$. Since E is path-connected and gcat() is unaffected by attaching an interval $[0, 1]$ to a space, we may assume that the homotopy equivalence $E' \xrightarrow{\simeq} E$ is basepointed for some $* \in E'$.

Let $U_0, \ldots, U_k$ be an open cover of E' with each U_i a contractible subspace. Take Q_j to be the homotopy pushout of $A_i \times U_j \times B_i \xrightarrow{g_{i,j} \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \ldots, m$, where $g_{i,j}$ is the restriction of f'_i to $A_i \times U_j$, and let $V_j = Q_j - \mathcal{D} \cong U_j \times \mathcal{B} \times \mathcal{S}'_0$. Since $g_{i,j} \times \mathbb{1}_{B_i}$ restricts $f'_i \times \mathbb{1}_{B_i}$, Q_j is a subspace of Q and V_j is open in Q. Moreover, we may contract V_j in Q to a point as follows. Let $\mathcal{B}_0 := *$ and $\mathcal{B}_\ell = \prod_{i \leq \ell} A_i \subseteq \mathcal{B}$, and take the subspace $W_\ell = * \times \mathcal{B}_\ell \times \{1\}$ of $E' \times \mathcal{B} \times \mathcal{S}'_0 \subset Q$. We can deform W_ℓ into $W_{\ell-1}$ in Q, first by deforming W_ℓ onto $f'_\ell(A_\ell \times *) \times \mathcal{B}_{\ell-1}$ by moving it down the mapping cylinder $M = ((E' \times \mathcal{B} \times [0,1]) \sqcup (C_\ell \times B_\ell))/\sim$ of Q and onto $C_\ell \times B_\ell$, then deforming it onto $* \times \mathcal{B}_{\ell-1}$ in $C_\ell \times B_\ell$ using the nullhomotopy of $(f'_\ell)_{|A_\ell \times *}$, and finally, moving $\mathcal{B}_{\ell-1}$ back up towards the top of the mapping cylinder M and into $E' \times \mathcal{B} \times \{1\}$. Composing these deformations of W_ℓ into $W_{\ell-1}$ in Q for $\ell = m, m-1, \dots, 1$, and deforming V_j onto W_m using contractibility of U_j and $\mathcal{S}'_0$ (onto 1), gives our contraction of V_j in Q to a point.

Assume $* \in U_0$. Since U_0 is contractible and $(g_{i,0})_{|A_i \times *} = (f'_i)_{|A_i \times *}$ is nullhomotopic, $g_{i,0}$ is also nullhomotopic. Lemma 2.4 then applies to Q_0, namely, we have

$$\mathrm{cat}(Q_0) \leq \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\}.$$

Let $\mathcal{R} := \mathcal{S}'_0 - \mathcal{S}_{\frac{1}{2}} \cong \coprod_{i=1,\dots,m}(0, \frac{1}{2})$ and $\bar{\mathcal{R}} = \mathcal{S}_0 - \mathcal{S}_{\frac{1}{2}} \cong \coprod_{i=1,\dots,m}[0, \frac{1}{2})$, and consider the open subspace $Q'_0 = Q_0 \cup (E' \times \mathcal{B} \times \mathcal{R})$ of Q. Notice Q'_0 deformation retracts in the weak sense onto Q_0 by deformation retracting the subspace of Q_0

$$\big((E' \times \mathcal{B} \times \bar{\mathcal{R}}) \sqcup \mathcal{D}\big)/\sim$$

onto $\mathcal{D}$, this being done by contracting each copy of $[0, \frac{1}{2})$ in the factor $\bar{\mathcal{R}}$ to 0, at the same time expanding $(U_0 \times \mathcal{B}) \times \mathcal{S}_{\frac{1}{2}}$ in Q'_0 by expanding each copy of $[\frac{1}{2}, 1]$ in the factor $\mathcal{S}_{\frac{1}{2}}$ outwards to $[0,1]$. Then $\mathrm{cat}(Q'_0) = \mathrm{cat}(Q_0)$. So take $k' = \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\}$ and $\{U'_0, \dots, U'_{k'}\}$ to be a categorical cover for Q'_0. Notice that U'_i is open in Q since Q'_0 is, and $Q = \bigcup_{j=0}^n Q_j = Q'_0 \cup \bigcup_{j=1}^n V_j$. As each V_j is open and contractible in Q, then $\{U'_0, \dots, U'_{k'}, V_1, \dots, V_k\}$ is a categorical cover of Q. Therefore $\mathrm{cat}(P) = \mathrm{cat}(Q) \leq k' + k$.

(ii) Since $C_i \times B_i$ is a subcomplex of X_i, P is a subspace of P' with

$$P' - P = \coprod_{i=1,\dots,m} (X_i - C_i \times B_i),$$

so $P' - P$ is open and contractible in P'. Notice each V_j is an open (and contractible) subset of P', while $S_j = U'_j \cap (\coprod_{i=1,\dots,m} X_i)$ is an open subset of $\mathcal{D}$. By Lemma 2.1, there exists an open subset R_j of $\coprod_{i=1,\dots,m} X_i$ that deformation retracts onto S_j such that $R_j \cap \mathcal{D} = S_j$. Then $R'_j = R_j \sqcup (U'_j - S_j)$ is an open subset of P' that deformation retracts onto U'_j, thus is contractible in P'. Since $P' - P$ and V_j are both open in P', and $(P' - P) \cap V_j = \emptyset$, then the subspace $(P' - P) \sqcup V_j$ is contractible in P'. We can therefore take $\{R'_0, \dots, R'_{k'}, (V_1 \sqcup (P' - P)), V_2, \dots, V_k\}$ as a categorical cover for P', so $\mathrm{cat}(P') \leq k' + k$. □

3. Moment-Angle Complexes

Given a simplicial complex K on vertex set $[n]$ and a sequence of pairs of spaces

$$\mathcal{S} := ((X_1, A_1), \dots, (X_n, A_n)),$$

$A_i \subseteq X_i$, the *polyhedral product* $\mathcal{S}^K$ is the subspace of $X^{\times n}$ defined by

$$\mathcal{S}^K := \bigcup_{\sigma \in K} Y_1^\sigma \times \cdots \times Y_n^\sigma,$$

where $Y_i^\sigma = X_i$ if $i \in \sigma$, or $Y_i^\sigma = A_i$ if $i \notin \sigma$. If the pairs (X_i, A_i) are all equal to the same pair (X, A), we usually write $\mathcal{S}^K$ as $(X, A)^K$. The *moment-angle complex* $\mathcal{Z}_K$ is defined as the polyhedral product $(D^2, \partial D^2)^K$, and the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ is the polyhedral product $(D^1, \partial D^1)^K$.

The *join* of two simplicial complexes K and L is the simplicial complex $K * L = \{\sigma \sqcup \tau \mid \sigma \in K,\ \tau \in L\}$, and one has $|K * L| \cong |K| * |L| \simeq \Sigma |K| \wedge |L|$ and $\mathcal{Z}_{K*L} \cong \mathcal{Z}_K \times \mathcal{Z}_L$. If $I \subseteq [n]$, $K_I = \{\sigma \in K \mid \sigma \subseteq I\}$ denotes the *full subcomplex* of K on vertex set I, in which case $\mathcal{Z}_{K_I}$ is a retract of $\mathcal{Z}_K$. Notice that if K_I and L_J are full subcomplexes of K and L, then $K_I * L_J$ is the full subcomplex $(K * L)_{I \sqcup J}$ of $K * L$. As a convention, we let $\mathcal{Z}_\emptyset := *$ when $\emptyset$ is on empty vertex set.

We let S^0 denote both the 0-sphere and the simplicial complex $\partial\Delta^1$ consisting of only two vertices. Generally, we assume our simplicial complexes (except $\emptyset$) are non-empty and have no ghost vertices, unless stated otherwise.

3.1. **The Hochster theorem.** When R is a field or $\mathbb{Z}$, it was shown in [12, 23, 4, 37] that there are isomorphisms of graded commutative algebras

$$H^*(\mathcal{Z}_K;R) \cong \mathrm{Tor}_{R[v_1,\dots,v_n]}(R[K],R) \cong \bigoplus_{I\subseteq[n]} \tilde{H}^*(\Sigma^{|I|+1}|K_I|;R). \tag{2}$$

The isomorphism on the left is induced by a quasi-isomorphism of DGAs between the Koszul complex of the Stanley-Reisner ring $R[K]$ and the cellular cochain complex of $\mathcal{Z}_K$ with coefficients in R. The multiplication on the right is given by maps $H^*(K_I)\otimes H^*(K_J) \longrightarrow H^{*+1}(K_{I\cup J})$ that are zero when $I\cap J \neq \emptyset$, otherwise they are induced by maps $\iota_{I,J}\colon\ |K_{I\cup J}| \longrightarrow |K_I * K_J| \cong |K_I|*|K_J| \simeq \Sigma|K_I|\wedge|K_J|$ geometrically realising the canonical inclusions $K_{I\cup J} \hookrightarrow K_I * K_J$. One can iterate so that any length ℓ product $\bigotimes_{i=1}^{\ell} H^*(K_{I_i}) \longrightarrow H^{*+\ell-1}(K_{I_1\cup\dots\cup I_\ell})$ is induced by the inclusion

$$\iota_{I_1,\dots,I_\ell}\colon\ |K_{I_1\cup\dots\cup I_\ell}| \hookrightarrow |K_{I_1} * \dots * K_{I_\ell}|$$

where the I_i's are mutually disjoint.

3.2. **A necessary condition.** The Hochster theorem lets us make statements about general bounds on the category $\mathrm{cat}(\mathcal{Z}_K)$ in terms of combinatorics and topology of K and its full subcomplexes. Suppose $\mathrm{cat}(\mathcal{Z}_K) \leq \ell-1$, so cup products of length l vanish in $H^+(\mathcal{Z}_K)$. Then in light of the Hochster theorem, the inclusions $\iota_{I_1,\dots,I_\ell}$ must induce trivial maps on cohomology. In fact, their suspensions must be nullhomotopic.

Proposition 3.1. *If* $\mathrm{cat}(\mathcal{Z}_K) \leq \ell-1$, *then*

$$\Sigma^{m+1}\iota_{I_1,\dots,I_\ell}\colon\ |K_{I_1\cup\dots\cup I_\ell}| \hookrightarrow |K_{I_1} * \dots * K_{I_\ell}|$$

is nullhomotopic for all mutually disjoint $I_1,\dots,I_\ell \subseteq [n]$.

Proof. Let $\widehat{\mathcal{Z}}_K := \mathcal{Z}_K/\{(x_1,\dots,x_n)\in\mathcal{Z}_K \mid \text{at least one } x_i = *\}$. Fix $m = |I_1\cup\dots\cup I_\ell|$, $Y = \mathcal{Z}_{K_{I_1\cup\dots\cup I_\ell}}$, and $\hat{Y} = \widehat{\mathcal{Z}}_{K_{I_1\cup\dots\cup I_\ell}}$. Since Y is a retract of $\mathcal{Z}_K$, $\mathrm{cat}(Y) \leq \ell-1$. Recall from [42] that a path-connected basepointed CW-complex such as Y satisfies $\mathrm{cat}(Y) \leq \ell-1$ if and only if there is a map $Y \xrightarrow{\psi} FW_\ell(Y)$ such that the diagonal map $Y \xrightarrow{\Delta} Y^{\times\ell}$ factors up to homotopy as $Y \xrightarrow{\psi} FW_\ell(Y) \xrightarrow{include} Y^{\times\ell}$. Here $FW_\ell(Y) = \big\{(y_1,\dots,y_\ell)\in Y^{\times\ell} \bigm| \text{at least one } y_i = *\big\}$ is the fat wedge. This implies the reduced diagonal map $\bar{\Delta}\colon Y \xrightarrow{\Delta} Y^{\times\ell} \longrightarrow Y^{\times\ell}/FW_\ell(Y) \xrightarrow{\cong} Y^{\wedge\ell}$ is nullhomotopic. Then so is $\zeta\colon Y \xrightarrow{\bar{\Delta}} Y^{\wedge\ell} \longrightarrow \bigwedge_j \mathcal{Z}_{K_{I_j}} \longrightarrow \bigwedge_j \widehat{\mathcal{Z}}_{K_{I_j}}$, where the second last map is the smash of the coordinate-wise projection maps onto each $\mathcal{Z}_{K_{I_j}}$, and the last map is the smash of quotient maps. This last nullhomotopic map ζ coincides with $Y \xrightarrow{q} \hat{Y} \xrightarrow{\hat{\iota}} \bigwedge_j \widehat{\mathcal{Z}}_{K_{I_j}}$, where q is the quotient map and $\hat{\iota}$ is the inclusion given simply by rearranging coordinates. Moreover, $\hat{\iota}$ is homeomorphic to $\Sigma^{m+1}\iota_{I_1,\dots,I_\ell}$ and Σq has a right homotopy inverse (c.f. [2], and also the proof of Proposition 2.5 and pg. 23 in [5]). It follows that $\Sigma^{m+1}\iota_{I_1,\dots,I_\ell}$ is nullhomotopic. $\square$

3.3. **Filtrations and lower bounds.** To give combinatorial lower bounds for $\mathrm{cat}(\mathcal{Z}_K)$ we construct cup products in $\mathrm{cat}(\mathcal{Z}_K)$ using information coming from the combinatorics of K. We state this in terms of being able to construct certain filtrations of K of a bounded length. Let

$$K\backslash L := \{\tau\in K \mid \sigma \not\subseteq \tau \text{ for any } \sigma\in L\}.$$

denote the *deletion* of subcomplex L from K. Notice $K\backslash L$ and $K\backslash(K\backslash L)$ are full subcomplexes of K on complementary vertex sets: if K is on vertex set $[n]$ and $K\backslash L$ is on vertex set $I \subset [n]$, then

$K\backslash(K\backslash L)$ is on the same vertex set $[n]-I$ as L, and we can take the canonical inclusion

$$K \hookrightarrow K_I * K_{[n]-I} \;=\; (K\backslash L) * K\backslash(K\backslash L).$$

If L itself is a full subcomplex, then $L = K\backslash(K\backslash L)$.

Lemma 3.2. *If K is a triangulated d-sphere on vertex set $[n]$ and L is a full subcomplex that is a triangulated $(d-1)$-sphere, then $|K\backslash L| \simeq |K|\backslash|L| \simeq |E_1| \sqcup |E_2|$ such that E_1 and E_2 are disjoint contractible subcomplexes of K.*

More generally, if K is a triangulated closed d-manifold and L is a full subcomplex that is a triangulated $(d-1)$-sphere which bounds a triangulated d-disk in K, then at least one of E_1 or E_2 is contractible.

Proof. For the case $|K| \cong S^d$, since L and K are finite, the embedding of $|L|$ in $|K|$ can be thickened on either side to an embedding of a thickened sphere. Thus by the generalised Schoenflies theorem $|L|$ bounds two distinct d-disks (hemispheres) $|D_1|$ and $|D_2|$ of the sphere $|K|$, and $|K_{[n]-I}| \cong |E_1| \sqcup |E_2|$ for full subcomplexes $E_i \subsetneq D_i$ obtained by deleting vertices from D_i that are in L. Notice that $|D_i|$ deformation retracts onto $|E_i|$ as follows. Take an open collar neighbourhood $C := [0,1) \times S^{d-1}$ of $|L|$ in $|D_i|$ small enough such that the collar does not intersect $|E_i|$ (i.e. is contained only in the d-faces that have vertices on the boundary $|L| = \partial|D_i|$), and deformation retract $|D_i|$ onto $|D_i|\backslash|C|$ by contracting the collar onto S^{d-1}. Take a d-face σ in D_i with vertices on L, and write $\sigma := \{v_1, \ldots, v_k, w_1, \ldots, w_{d-k}\}$ so that the v_i's are in L and w_i's are in E_i. There must be at least one of each type since L is a full subcomplex with no d-faces. For any point $x \in |\sigma|$ that is not on the $(d-1)$-face $\{v_1, \ldots, v_k\}$, x lies on the line interpolating barycentric coordinates

$$l(t) \;:=\; (1-t)(d_1, \ldots, d_k, e_1, \ldots, e_{d-k}) + t\,\frac{(0, \ldots, 0, e_1, \ldots, e_{d-k})}{e_1 + \ldots + e_{d-k}}$$

in $|\sigma|$, where $(d_1, \ldots, d_k, e_1, \ldots, e_{d-k})$ are the barycentric coordinates of x in $|\sigma|$. When $x \in D_i\backslash C$, these lines end at $t = 1$ on a point in E_i since at least one of $e_i > 0$. Also, they vary continuously as x varies, and they agree on shared boundaries of d-faces. Thus following these lines from $t = 0$ to $t = 1$ defines a deformation retractions of $|D_i|\backslash|C|$ onto $|E_i|$, so $|E_i| \simeq |D_i|\backslash|C| \simeq |D_i| \simeq |*|$ and we are done. The argument when $|K|$ is a d-manifold is similar. □

Lemma 3.3. *If K is a triangulated d-sphere on vertex set $[n]$ and L is a full subcomplex that is a triangulated $(d-1)$-sphere, then the inclusion*

$$\iota:\; |K| \hookrightarrow |(K\backslash L) * L| \;\cong\; |K\backslash L| * |L|$$

is a homotopy equivalence $S^d \longrightarrow S^d$.

More generally, if K is a triangulated closed d-manifold and L is a full subcomplex that is a triangulated $(d-1)$-sphere which bounds a triangulated d-disk in K, then ι induces an isomorphism on degree d cohomology mapping a generator to the fundamental class of K.

Proof. For the case $|K| \cong S^d$, by Lemma 3.2, $|K\backslash L| \cong |E_1| \sqcup |E_2|$ for some disjoint contractible subcomplexes $|E_i|$ of $|K|$. Let

$$h:\; |K\backslash L| * |L| \longrightarrow S^0 * |L| \;\cong\; S^d$$

be the join of the identity map on the right factor with the map collapsing $|E_1|$ and $|E_2|$ to -1 and 1 in $S^0 = \{-1, 1\}$ on the right factor. Then h is a homotopy equivalence since each $|E_i|$ is contractible. Likewise, $h \circ \iota$ is the quotient map that collapses each $|E_i|$ to a distinct point, so it is a homotopy equivalence since the E_i's are disjoint contractible subcomplexes of K. Therefore, ι is a homotopy equivalence.

To prove the general case, assume by Lemma 3.2 that E_2 is contractible. Composing h with the map q that collapses the bottom hemisphere of S^d to -1, we see that $q \circ h \circ \iota$ is the map that collapses everything outside the interior of d-disk D in K that bounds L to the point -1, and collapses the subcomplex E_2 in the interior of D to the point 1. But since E_2 is contractible, $q \circ h \circ \iota$ is homotopy equivalent to the map that simply collapses the everything outside the interior of D, which is the quotient map of the top d-cell of the d-manifold K to a d-sphere. Since this induces a map on cohomology that sends the fundamental class of S^d to the fundamental class of K, we are done. □

Definition 3.4. Given K is a triangulated closed connected d-manifold on vertex set $[n]$, suppose there is a sequence $I_\ell \subsetneq \cdots \subsetneq I_1 = [n]$ such that the filtration of full subcomplexes

$$K_{I_\ell} \subsetneq \cdots \subsetneq K_{I_1} := K$$

satisfies

(1) K_{I_i} is a triangulation of a $(d+1-i)$-sphere when $i \geq 2$;
(2) K_{I_2} bounds the triangulation of a d-disk in $K = K_{I_1}$ (i.e. there exists $I_2 \subsetneq J \subsetneq [n]$ such that $|K_J| \cong D^d$).

Then we say that this is a *spherical filtration* of K of length ℓ.

Remark: Condition (2) is redundant when K is a triangulated d-sphere by the generalised Schoenflies theorem.

Definition 3.5. For any triangulated closed manifold K, define its *full filtration length* $\operatorname{filt}(K)$ to be the largest integer ℓ such that K admits a spherical filtration of length ℓ.

Proposition 3.6. *If some full subcomplex $K_I \subseteq K$ is a triangulated closed manifold with a spherical filtration of length ℓ, then $\ell \leq \operatorname{cup}(\mathcal{Z}_K)$.*

Proof. Let $K_{I_\ell} \subsetneq \cdots \subsetneq K_{I_1} := K_I$ be a spherical filtration of K_I length ℓ. Let $J_{i+1} = I_i - I_{i+1}$ for $i < \ell$. By Lemma 3.3, each inclusion

$$\iota_i : |K_{I_i}| \hookrightarrow |K_{I_{i+1}} * K_{J_{i+1}}| \cong |K_{I_{i+1}}| * |K_{J_{i+1}}|$$

is a homotopy equivalence when $i \geq 2$. Take the composite of inclusions

$$|K_{I_1}| \xhookrightarrow{\iota_1} |K_{I_2} * K_{J_2}| \xhookrightarrow{\iota_2'} |K_{I_3} * K_{J_3} * K_{J_2}| \xhookrightarrow{\iota_3'} \cdots \xhookrightarrow{\iota_{\ell-1}'} |K_{I_\ell} * K_{J_\ell} * \cdots * K_{J_2}| \tag{3}$$

where the i^{th} map $(i \geq 2)$ in this composite

$$\iota_i' : |K_{I_i} * K_{J_i} * \cdots * K_{J_2}| \hookrightarrow |K_{I_{i+1}} * K_{J_{i+1}} * K_{J_i} * \cdots * K_{J_2}|$$

is the join of the homotopy equivalence ι_i and the identity $|K_{J_i} * \cdots * K_{J_2}| \xrightarrow{\mathbb{1}_i} |K_{J_i} * \cdots * K_{J_2}|$. Since each $|K_{J_j}| \simeq S^0$ for $j \geq 3$, then $\mathbb{1}_i$ is homotopy equivalent to the identity $S^{i-2} \xrightarrow{\mathbb{1}} S^{i-2}$. Each ι_i' (and ι_i) is therefore a homotopy equivalence when $i \geq 2$, and then so is the composite in (3) of all the maps except possibly the first. The first map ι_1 on the other hand induces a non-trivial map to the fundamental class of K_I by Lemma 3.3. Therefore (3) induces a non-trivial map on cohomology mapping the fundamental class of S^d to the fundamental class of K_I, so the Hochster theorem implies there is a non-trivial length ℓ cup product in $H^+(\mathcal{Z}_{K_{I_1}})$. □

Corollary 3.7. *If K is a triangulated closed connected manifold, then*

$$\operatorname{filt}(K) \leq \operatorname{cup}(\mathcal{Z}_K) \leq \operatorname{cat}(\mathcal{Z}_K).$$

More generally for any K, since each $\mathcal{Z}_{K_I}$ is a retract of $\mathcal{Z}_K$, then

$$\operatorname{filt}(K_I) \leq \operatorname{cup}(\mathcal{Z}_K) \leq \operatorname{cat}(\mathcal{Z}_K)$$

whenever K_I is a triangulated closed connected manifold. □

With some additional effort the statement in Lemma 3.3 can be generalized so that it does away with the triangulated sphere needing to be a full subcomplex. In particular, Proposition 3.6 still holds when we modify the definition of spherical filtration so that the second full subcomplex K_{I_2} in the filtration is replaced with any triangulated $(d-1)$-sphere L that is not necessarily a full subcomplex, as long as it satisfies the hypothesis in Lemma 3.8 below. The proof of Proposition 3.6 then follows as before, this time using Lemma 3.8 in place of Lemma 3.3 for the map ι_1.

Lemma 3.8. *Suppose K is a triangulated closed connected d-manifold, $L \subsetneq K$ is a triangulation of a $(d-1)$-sphere that bounds a triangulation of a d-disk D in K such that $D\backslash L$ is non-empty and connected, and $K\backslash L$ consists of two non-empty disconnected components A_0 and A_1. Then the inclusion*

$$\iota :\ |K| \hookrightarrow |(K\backslash L) * K\backslash(K\backslash L)|$$

induces an isomorphism on degree d cohomology, mapping the fundamental class of S^d to the fundamental class of K.

Proof. Denote $A := A_0 \sqcup A_1 := K\backslash L$ and $B := K\backslash A$, and let $D \subsetneq K$ be a choice of triangulated d-disk that is bounded by the triangulated $(d-1)$-sphere L. Notice that D cannot be a subcomplex of B and all vertices of D not in L must all be in exactly one of A_0 or A_1 since $D\backslash L$ is non-empty and connected.

Consider the inclusion $g\colon L \longrightarrow B$ restricting to the identity on vertices. Topologically, the inclusion $g\colon |L| \longrightarrow |B|$ has a left homotopy inverse

$$r :\ |B| \longrightarrow |L| \cong S^{d-1}$$

by the following argument. Note that B has no faces above dimension d. Also, no d-faces can form a cycle in the simplicial chain group $C_d(B)$. If they did, then this would at the same time be a cycle in $C_d(K)$ that is distinct from the cycle ξ corresponding to the fundamental class of $H^*(K)$ consisting of all d-faces of K (since $K\backslash L$ has two non-empty disconnected components), and they could not be homologous to themselves or zero since there are no $(d+1)$-faces, meaning $H_d(K)$ has two $\mathbb{Z}$ generators, which contradicts Poincaré duality. Then $H_*(B)$ is trivial in degrees d and above, implying B is homotopy equivalent to at most a $(d-1)$-dimensional CW-complex, and so we can quotient the $(d-2)$-skeleton of B to obtain a map $|B| \longrightarrow \bigvee_\alpha S^{d-1}$ into a possibly empty wedge of $(d-1)$-spheres that induces a surjection on H_{d-1}. Thus, all that is left to show is that g induces a non-trivial map on H_{d-1}. In this direction, take the cycle in $C_{d-1}(L)$ consisting of all $(d-1)$-faces of L yielding its fundamental class, which is a cycle γ in $C_{d-1}(B)$ under the inclusion g. We are done if we can show γ is not a boundary in $C_{d-1}(B)$. To see this, suppose conversely that γ is a boundary of an element $a \in C_d(B) \subsetneq C_d(K)$. Since the d-disk D is bounded by L, γ is also boundary of the element $b \in C_d(K)$ consisting of the d-faces of D, and $a \neq b$ since D is not a subcomplex of B. Then $a - b$ is a cycle in $C_d(K)$ distinct from the cycle ξ (since it does not contain those d-faces with a vertex in one of A_0 or A_1), which contradicts Poincaré duality by the same argument as before.

Now consider the composite

$$f\colon |K| \xrightarrow{\iota} |A * B| \cong |A| * |B| \xrightarrow{\mathbb{1}*r} |A| * S^{d-1} \xrightarrow{q*\mathbb{1}} S^0 * S^{d-1} \cong S^d$$

where the last map $q * \mathbb{1}$ collapses the components A_0 and A_1 of A to either point of S^0. Thus f maps $|A_0|$ and $|A_1|$ to distinct hemispheres of $S^d \cong S^0 * S^{d-1}$ sharing the equator $S^{d-1} \subsetneq S^d$. In turn, f maps the pair $(|D|, |L|)$ to one of the hemispheres such that f restricts on $|L|$ to $r \circ g$, mapping $|L|$ to the equator. In other words, f restricts to a map of pairs $(|D|, |L|) \xrightarrow{f'} (D^d, S^{d-1})$, which represents an element of $\pi_d(D^d, S^{d-1})$ since $(|D|, |L|) \cong (D^d, S^{d-1})$. Since the boundary map $\pi(D^d, S^{d-1}) \xrightarrow{\partial} \pi_{d-1}(S^{d-1})$ is an isomorphism $\mathbb{Z} \mapsto \mathbb{Z}$ in the homotopy long exact sequence of the pair (D^d, S^{d-1}), and since $\partial([f']) = [r \circ g]$ and $r \circ g$ is homotopic to the identity, then f' represents the identity, meaning f' is homotopic to a homeomorphism $(|D|, |L|) \longrightarrow (D^2, S^{d-1})$. Then using the homotopy extension property, f is homotopic to a map that restricts to such a homeomorphism, while still mapping everything outside $|D|$ to the opposing hemisphere. Now collapsing the opposing hemisphere to a point, f is homotopic to a map that collapses everything outside a d-disk in the d-manifold K to a point. Namely, this is the map quotienting the d-cell of $|K|$ to a sphere that induces an isomorphism on degree d cohomology between their fundamental classes. Since f factors through ι, we are done.

□

3.4. **Skeleta and suspension on coordinates.** Let K be a simplicial complex on vertex set $[n]$ and let $K^{(i)}$ denote the i-skeleton of K, and $K^{(-1)} := \emptyset$. An inclusion of simplicial complexes $L \hookrightarrow K$ induces a canonical inclusion of CW-complexes $\mathcal{Z}_L \hookrightarrow \mathcal{Z}_K$. This gives $\mathcal{Z}_{K^{(i)}}$ and $\mathcal{Z}_{K^{(-1)}} = (\partial D^2)^{\times n} = (S^1)^{\times n}$ as CW-subcomplexes of $\mathcal{Z}_K$.

Lemma 3.9 (Corollary 3.3 in [34])**.** *If K is on vertex set $[n]$ with no ghost vertices, then $\mathcal{Z}_{K^{(-1)}} = (\partial D^2)^{\times n}$ is contractible in $\mathcal{Z}_K$.* □

Lemma 3.10. *If $0 \leq l \leq \dim K$, then $\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}}$ is contractible in $\mathcal{Z}_K$.*

Proof. We have a decomposition

$$\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}} = \coprod_{\sigma\in K,\, |\sigma|=\ell+1} \tilde{Y}_1^\sigma \times \cdots \times \tilde{Y}_n^\sigma$$

where $\tilde{Y}_i^\sigma = D^2 - \partial D^2$ if $i \in \sigma$ and $\tilde{Y}_i^\sigma = \partial D^2$ if $i \notin \sigma$. This being a disjoint union of open subspaces of $\mathcal{Z}_{K^{(l)}}$, each of which can be deformed into $\mathcal{Z}_{K^{(-1)}}$ in $\mathcal{Z}_K$ by contracting $\tilde{Y}_i^\sigma$ to a point in ∂D^2 whenever $i \in \sigma$. Thus $\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}}$ can also be deformed into $\mathcal{Z}_{K^{(-1)}}$. Then $\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}}$ is contractible in $\mathcal{Z}_K$ by Lemma 3.9.

□

Lemma 3.11. *If $-1 \leq j \leq \dim K$, then*

$$\operatorname{cat}(\mathcal{Z}_K) \leq \operatorname{cat}(\mathcal{Z}_{K^{(j)}} \hookrightarrow \mathcal{Z}_K) + \dim K - j.$$

In particular,

$$\operatorname{cat}(\mathcal{Z}_K) \leq \dim K + 1.$$

and

$$\operatorname{cat}(\mathcal{Z}_K) \leq \operatorname{cat}(\mathcal{Z}_{K^{(j)}}) + \dim K - j.$$

Remark: Lemma 3.10 and 3.11 can be generalised to any filtration $L_j \subseteq \cdots \subseteq L_k = K$ satisfying $\partial\sigma \subseteq L_i$ whenever $\sigma \in L_{i+1}$ in place of the skeletal filtration.

Proof. The skeletal filtration $K^{(j)} \subseteq \cdots \subseteq K^{(\dim K)} = K$ induces a filtration of subcomplexes $\mathcal{Z}_{K^{(j)}} \subseteq \cdots \subseteq \mathcal{Z}_K$, and for $0 \leq j \leq \dim K$, $\mathcal{Z}_{K^{(j)}} - \mathcal{Z}_{K^{(j-1)}}$ is contractible in $\mathcal{Z}_K$ by Lemma 3.10. The result then follows using Lemma 2.2. In particular, when $j = -1$, we get $\operatorname{cat}(\mathcal{Z}_K) \leq \dim K + 1$ since $\operatorname{cat}(\mathcal{Z}_{K^{(-1)}} \hookrightarrow \mathcal{Z}_K) = 0$ by Lemma 3.9. The last bound follows from equation 1. □

Proposition 3.12. *Consider the sequences of pairs of spaces $\mathcal{S} := ((X_1, A_1), \ldots, (X_n, A_n))$ and $\mathcal{T} := ((\Sigma^{m_1} X_1, \Sigma^{m_1} A_1), \ldots, (\Sigma^{m_n} X_n, \Sigma^{m_n} A_n))$ for some integers m_i and connected basepointed X_i. Then for any K with no ghost vertices,*

$$\operatorname{cat}(\mathcal{T}^K) \leq \operatorname{cat}(\mathcal{S}^K).$$

Proof. Let K be on vertex set $[n]$, $k := \operatorname{cat}(\mathcal{S}^K)$, and take a categorical cover $\{U_0, \ldots, U_k\}$ of $\mathcal{S}^K$. For any open subset V of $\mathcal{S}^K$, define the following open subset V^1 of $\mathcal{T}^K$

$$V^1 := \left\{ ((t_1, x_1), \ldots, (t_n, x_n)) \in \prod_{i=1}^n \Sigma^{m_i} X_i \;\middle|\; (x_1, \ldots, x_n) \in V,\, t_i \in D^{m_i} \right\}.$$

In particular, $\mathcal{T}^K = (\mathcal{S}^K)^1$. Then $\{U_0^1, \ldots, U_k^1\}$ is an open cover of $\mathcal{T}^K$. Since K has no ghost vertices, $A_i \subseteq X_i$, and each X_i is path-connected, then $\mathcal{S}^K$ is path-connected. Since $\Sigma^{m_i} X_i$ is the reduced suspension of the basepointed space X_i, we have identifications $(t, *) \sim * \in \Sigma^{m_i} X_i$. Then we can define a contraction of U_i^1 in $\mathcal{T}^K$ by contracting U_i in $\mathcal{S}^K$ to a point p and homotoping p to the basepoint $(*, \ldots, *) \in \mathcal{S}^K$. Therefore, $\{U_0^1, \ldots, U_k^1\}$ is a categorical cover of $\mathcal{T}^K$.

□

Notice that the $(i+1)$-skeleton $(\mathbb{R}\mathcal{Z}_K)^{(i+1)}$ of $\mathbb{R}\mathcal{Z}_K$ is equal to $\mathbb{R}\mathcal{Z}_{K^{(i)}}$ although this is not true for the complex moment-angle complex $\mathcal{Z}_K$.

Corollary 3.13. *For any K with no ghost vertices,*

$$\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathbb{R}\mathcal{Z}_K) \tag{4}$$

and if $\mathbb{R}\mathcal{Z}_K$ is not contractible and $i \geq 0$, then

$$\mathrm{cat}(\mathcal{Z}_{K^{(i)}}) \leq \mathrm{cat}(\mathbb{R}\mathcal{Z}_{K^{(i)}}) \leq \mathrm{cat}(\mathbb{R}\mathcal{Z}_K). \tag{5}$$

Proof. Inequality (4) and the first inequality in (5) follow from Proposition 3.12. By the main corollary of Theorem 1 in [20], the i-skeleton $X^{(i)}$ of any connected non-contractible CW-complex X satisfies that $\mathrm{cat}(X^{(i)}) \leq \mathrm{cat}(X)$. Since $(\mathbb{R}\mathcal{Z}_K)^{(i+1)} = \mathbb{R}\mathcal{Z}_{K^{(i)}}$ holds for real moment-angle complexes, the last inequality follows.

□

It is plausible that the second bound can be strengthened to $\mathrm{cat}(\mathcal{Z}_{K^{(i)}}) \leq \mathrm{cat}(\mathcal{Z}_K)$. In any case, even if it is true, we will sometimes need a sharper bound.

Let X and Y be path-connected paracompact spaces, and $\mathcal{U} := \{U_0, \ldots, U_k\}$ and $\mathcal{V} := \{V_0, \ldots, V_\ell\}$ be categorical covers of X and Y, respectively. We recall James' construction of a categorical cover $\mathcal{W} := \{W_0, \ldots, W_{k+\ell}\}$ of $X \times Y$ from the covers $\mathcal{U}$ and $\mathcal{V}$ (see [42], page 333).

Let $\{\pi_j\}_{j \in \{0,\ldots,k\}}$ be a partition of unity subordinate to the cover $\mathcal{U}$. For any subset $S \subseteq \{0, \ldots, k\}$, define

$$W_{\mathcal{U}}(S) := \{x \in X \mid \pi_j(x) > \pi_i(x) \text{ for any } j \in S \text{ and } i \notin S\},$$

and for any point $p \in X$, let

$$S_{\mathcal{U}}(p) := \{j \in \{0, \ldots, k\} \mid \pi_j(p) > 0\}.$$

Since the context is clear, let $W(S) := W_{\mathcal{U}}(S)$ and $S(p) = S_{\mathcal{U}}(p)$. Then $W(S)$ is an open subset of X. Given $x \in X$, $x \in W(S)$ where $S = \{i \mid \pi_i(x) = \max\{\pi_1(x), \ldots, \pi_k(x)\}\}$), we have $X = \bigcup_{S \subseteq \{0,\ldots,k\}} W(S)$. Moreover, $W(S') \cap W(S) = \emptyset$ when $S \nsubseteq S'$ and $S' \nsubseteq S$, in particular, when $|S| = |S'|$ and $S \neq S'$, and $W(S) \subseteq U_j$ whenever $j \in S$. Therefore $W(S)$ is contractible in X. Then so is the disjoint union of open sets

$$U_i' := \coprod_{\substack{S = S(p) \text{ for some } p \in X \\ |S| = i+1}} W(S). \tag{6}$$

Since $W(S) = \emptyset$ when $S \neq S(p)$ for every $p \in X$, the set $\{U_0', \ldots, U_k'\}$ forms a categorical cover of X. We obtain a categorical cover $\{V_0', \ldots, V_\ell'\}$ of Y from $\mathcal{V}$ by an analogous construction.

Now let $\bar{U}_i := U_{k-i}' \cup \cdots \cup U_k'$ and $\bar{V}_j := V_{\ell-j}' \cup \cdots \cup V_\ell'$, and for $-1 \leq s \leq k + \ell$, let $C_{-1} := \emptyset$ and

$$C_s := \bigcup_{\substack{i+j=s \\ i \leq k,\, j \leq \ell}} \bar{U}_i \times \bar{V}_j.$$

Take $W_s := C_s - C_{s-1}$. Notice that

$$W_s = \coprod_{\substack{i+j=s \\ i\leq k,\, j\leq \ell}} U'_i \times V'_j. \tag{7}$$

This defines a categorical cover $\mathcal{W}$ of $X \times Y$.

Given subcomplexes $B \subseteq Y$ and $A \subseteq X$, consider the polyhedral product

$$\mathcal{X}^{S^0} := X \times B \cup_{A\times B} A \times Y$$

over the sequence $\mathcal{X} := ((X, A), (Y, B))$, where S^0 is considered as a simplicial complex of two disjoint points.

Lemma 3.14. *If $X - A$ is contractible in X and $Y - B$ is contractible in Y, then*

$$\mathrm{cat}(\mathcal{X}^{S^0}) \leq \mathrm{cat}(A) + \mathrm{cat}(B) + 1.$$

Proof. Suppose we have categorical covers $\{R_1, \ldots, R_k\}$ and $\{S_1, \ldots, S_\ell\}$ of A, and B. By Lemma 2.1, we have open subsets $U_i \subseteq X$ and $V_i \subseteq Y$ such that U_i and V_i deformation retract onto R_i and S_i respectively, and $U_i \cap A = R_i$ and $V_i \cap B = S_i$ for $i \geq 1$. Then taking $U_0 := X - A$ and $V_0 := Y - B$, $\mathcal{U} := \{U_0, \ldots, U_k\}$ and $\mathcal{V} := \{V_0, \ldots, V_\ell\}$ are categorical covers of X and Y.

Notice that

$$X \times Y - U_0 \times V_0 = \mathcal{X}^{S^0},$$

and since $R_i = U_i \cap A = U_i - U_0$ and $S_j = V_j \cap B = V_j - V_0$ for $i, j \geq 1$,

$$D_{i,j} := U_i \times V_j - U_0 \times V_0 = (R_i \times V_j) \cup_{R_i\times S_j} (U_i \times S_j). \tag{8}$$

Notice that $D_{i,j}$ is contractible in $\mathcal{X}^{S^0}$ by deformation retracting the factor U_i onto R_i and V_j onto S_j, then contracting $R_i \times S_j$ in $A \times B$.

Take the categorical cover $\mathcal{W} := \{W_0, \ldots, W_{k+\ell}\}$ of $X \times Y$ constructed from $\mathcal{U}$ and $\mathcal{V}$ as above. By (7),

$$W_s - U_0 \times V_0 = \coprod_{\substack{i+j=s \\ i\leq k,\, j\leq \ell}} (U'_i \times V'_j - U_0 \times V_0),$$

and by (6),

$$U'_i \times V'_j - U_0 \times V_0 = \coprod_{\substack{S=S_{\mathcal{U}}(p) \text{ for some } p\in X \\ T=S_{\mathcal{V}}(q) \text{ for some } q\in Y \\ |S|=i+1,\, |T|=j+1}} (W_{\mathcal{U}}(S) \times W_{\mathcal{V}}(T) - U_0 \times V_0).$$

These are disjoint unions of open subsets of $\mathcal{X}^{S^0}$. Since $W_{\mathcal{U}}(S)$ is contained in some $U_{i'}$ and $W_{\mathcal{V}}(T)$ is contained in some $V_{j'}$, it follows that $(W_{\mathcal{U}}(S) \times W_{\mathcal{V}}(T) - U_0 \times V_0)$ is contained in $D_{i',j'}$, so it is contractible in $\mathcal{X}^{S^0}$. Therefore, so are the disjoint unions $U'_i \times V'_j - U_0 \times V_0$ and $W_s - U_0 \times V_0$. Moreover, since $W_{k+\ell} = U'_k \times V'_\ell$, and $U'_k = W_{\mathcal{U}}(\{0, \ldots, k\})$ and $V'_\ell = W_{\mathcal{V}}(\{0, \ldots, \ell\})$ are contained in $U_{i'}$ and $V_{j'}$ respectively for each $i' \in \{0, \ldots, k\}$ and $j' \in \{0, \ldots, \ell\}$, $W_{k+\ell} - U_0 \times V_0 = \emptyset$. Then

$$\{(W_0 - U_0 \times V_0), \ldots, (W_{k+\ell-1} - U_0 \times V_0)\}$$

is a categorical cover of $\mathcal{X}^{S^0}$.

□

Corollary 3.15. *Let K and L be simplicial complexes with $d := \dim K$ and $d' := \dim L$. Then*

$$\mathrm{cat}(\mathcal{Z}_{(K*L)^{(d+d')}}) \leq \mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) + \mathrm{cat}(\mathcal{Z}_{L^{(d'-1)}}) + 1.$$

Proof. Recall that $\dim K * L = d + d' + 1$. Notice that

$$(K * L)^{(d+d')} = (K * L^{(d'-1)}) \cup_{(K^{(d-1)} * L^{(d'-1)})} (K^{(d-1)} * L),$$

$$\mathcal{Z}_{K*L} = \mathcal{Z}_K \times \mathcal{Z}_L, \text{ so}$$

$$\begin{aligned}\mathcal{Z}_{(K*L)^{(d+d')}} &= (\mathcal{Z}_{K*L^{(d'-1)}}) \cup_{\mathcal{Z}_{(K^{(d-1)}*L^{(d'-1)})}} (\mathcal{Z}_{K^{(d-1)}*L}) \\ &= (\mathcal{Z}_K \times \mathcal{Z}_{L^{(d'-1)}}) \cup_{\mathcal{Z}_{K^{(d-1)}} \times \mathcal{Z}_{L^{(d'-1)}}} (\mathcal{Z}_{K^{(d-1)}} \times \mathcal{Z}_L),\end{aligned}$$

and $\mathcal{Z}_K - \mathcal{Z}_{K^{(d-1)}}$ and $\mathcal{Z}_L - \mathcal{Z}_{L^{(d'-1)}}$ are contractible in $\mathcal{Z}_K$ and $\mathcal{Z}_L$ by Lemma 3.10. The result follows by Lemma 3.14. □

3.5. **Missing face complexes.** Take K on vertex set $[n]$. We fix the basepoint in the unreduced suspension $\Sigma|K| = (|K| \times [0,1])/\sim$ to be the tip of the double cone corresponding to 1 under the identifications $(x,0) \sim 0$ and $(x,1) \sim 1$. Let $MF(K) := \{\sigma \subseteq [n] \mid \sigma \notin K, \partial\sigma \subseteq K\}$ be the collection of minimal missing faces of K. With this we can define a large class of category 1 moment-angle complexes that include those over chordal graphs (this is a somewhat more flexible alternative to the *directed missing face complexes* defined in [36]). By filtering through skeleta as in the previous section, tight upper bounds can sometimes be obtained when the category of the moment angle complex over the 1-skeleton is known to be small.

Definition 3.16. A simplicial complex K on vertex set $[n]$ is called a *homology missing face complex* (or *HMF-complex*) if for each non-empty $I \subseteq [n]$, K_I is a simplex or there exists a subcollection $\mathcal{C}_I \subseteq MF(K_I)$ such that the wedge sum of suspended inclusions

$$\gamma_I : \bigvee_{\sigma \in \mathcal{C}_I} \Sigma|\partial\sigma| \longrightarrow \Sigma|K_I|$$

induces an isomorphism on homology. Consequently γ_I is a homotopy equivalence since it is a map between suspensions.

Remark 3.17. Given $H_*(K_I)$ is torsion-free, since each $\Sigma|\partial\sigma|$ is a sphere, one needs only to find γ_I that induces surjection on homology in order for K to be an HMF-complex.

Proposition 3.18. *If K is an HMF-complex, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres or is contractible. Therefore* $\operatorname{cat}(\mathcal{Z}_K) \le 1$ *and* $\mathcal{C}\operatorname{at}(\mathcal{Z}_K) \le 1$.

Proof. For each $I \subseteq [n]$, either K_I is a simplex, boundary of a simplex, or else for each $\sigma \in \mathcal{C}_I$, we can pick an $i_\sigma \in I$ such that $\partial\sigma \subseteq K_{I-\{i_\sigma\}}$, so each inclusion $|\partial\sigma| \longrightarrow |K_I|$ factors through inclusions $|\partial\sigma| \longrightarrow |K_{I-\{i_\sigma\}}| \longrightarrow |K_I|$. Take the composite

$$f : \Sigma|K_I| \xrightarrow{\gamma_I^{-1}} \bigvee_{\sigma \in \mathcal{C}_I} \Sigma|\partial\sigma| \longrightarrow \bigvee_{i \in I} \Sigma|K_{I-\{i_\sigma\}}| \longrightarrow \Sigma|K_I|$$

where γ_I^{-1} is a homotopy inverse of γ_I, the second last map includes the summand $\Sigma|\partial\sigma|$ into the summand $\Sigma|K_{I-\{i_\sigma\}}|$, and the last map is the standard inclusion on each summand. Since the composite of the last two maps is γ_I, f is a homotopy equivalence. Then K is an *extractible complex* as defined in [38]. Therefore $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres or contractible by Corollary 3.3 therein. □

3.6. **Gluings and connected sums.** Now we look at the effect on category of moment angle complexes when two simplicial complexes are glued along a full subcomplex, or along a simplex with interior then deleted (i.e. a connected sum). If L and K are simplicial complexes and C is a full subcomplex common to both L and K, then we obtain a new simplicial complex $L \cup_C K$ by gluing L and K along C. One can always glue along simplices since they are always full subcomplexes. When $C = \emptyset$, $L \cup_C K$ is just the disjoint union $L \sqcup K$.

Given $\sigma \in K$, define the *deletion* of the face σ from K to be the simplicial complex given by

$$K \backslash \sigma := \{\tau \in K \mid \sigma \not\subseteq \tau\}.$$

If σ is a common face of L and K, define the *connected sum* $L \#_\sigma K$ to be the simplicial complex $(L\backslash\sigma) \cup_{\partial\sigma} (K\backslash\sigma)$. In other words, $L\#_\sigma K$ is obtained by deleting σ from L and K and gluing along the boundary $\partial\sigma$. As a convention, we let $\mathcal{Z}_\emptyset := *$ when $\emptyset$ is on empty vertex set.

Proposition 3.19. *If C is a (possibly empty) full subcomplex common to $K_1, \dots, K_m$, then*

$$\mathrm{cat}(\mathcal{Z}_{K_1 \cup_C \cdots \cup_C K_m}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_{K_1}), \dots, \mathrm{cat}(\mathcal{Z}_{K_m})\} + \mathcal{C}\mathrm{at}(\mathcal{Z}_C).$$

Moreover, if each K_i is the $(d_i - 1)$-skeleton of some d_i dimensional simplicial complex $\bar{K}_i$, and C is also a full subcomplex of each $\bar{K}_i$, then

$$\mathrm{cat}(\mathcal{Z}_{\bar{K}_1 \cup_C \cdots \cup_C \bar{K}_m}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_{K_1}), \dots, \mathrm{cat}(\mathcal{Z}_{K_m})\} + \mathcal{C}\mathrm{at}(\mathcal{Z}_C).$$

Proof. Let K_i be on vertex set $[n_i]$, and C has ℓ vertices. If C is on vertex set $[n_i]$, possibly with ghost vertices, the inclusion $C \hookrightarrow K_i$ induces a coordinate-wise inclusion $(\partial D^2)^{\times n_i - \ell} \times \mathcal{Z}_C \xhookrightarrow{f_i} \mathcal{Z}_{K_i}$. By Lemma 3.9, f_i is nullhomotopic when restricted to $(\partial D^2)^{\times n_i - \ell} \times *$. Let $N_i := \Sigma_{j \neq i} n_j$. Note $\mathcal{Z}_{K_1 \cup_C \cdots \cup_C K_m}$ is the pushout of $(\partial D^2)^{\times n_i - \ell} \times \mathcal{Z}_C \times (\partial D^2)^{\times N_i - \ell} \xhookrightarrow{f_i \times \mathbb{1}} \mathcal{Z}_{K_i} \times (\partial D^2)^{\times N_i - \ell}$ for $i = 1, \dots, m$. Since each of these maps are inclusions of subcomplexes, $\mathcal{Z}_{K_1 \cup_C \cdots \cup_C K_m}$ is homotopy equivalent to the homotopy pushout P of these maps. The first inequality therefore follows from the first part of Lemma 2.5.

By Lemma 3.10, $\mathcal{Z}_{\bar{K}_i} - \mathcal{Z}_{K_i}$ is contractible in $\mathcal{Z}_{\bar{K}_i}$, so the second equality follows from the second part of Lemma 2.5.

□

Example 3.20. In particular, when C is a simplex $\mathrm{cat}(\mathcal{Z}_{L \cup_C K}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_L), \mathrm{cat}(\mathcal{Z}_K)\}$ and $\mathrm{cat}(\mathcal{Z}_{\bar{L} \cup_C \bar{K}}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_L), \mathrm{cat}(\mathcal{Z}_K)\}$ since $\mathcal{Z}_C$ is contractible. These also hold when C is the empty simplex and $\mathcal{Z}_C = *$ in which case $\bar{L} \cup_C \bar{K} = \bar{L} \sqcup \bar{K}$ and $L \cup_C K = L \sqcup K$. When C is the boundary of a simplex, $\mathrm{cat}(\mathcal{Z}_{L \cup_C K}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_L), \mathrm{cat}(\mathcal{Z}_K)\} + 1$ since $\mathcal{Z}_C$ here is a sphere.

The bound in Proposition 3.19 is not always optimal, sometimes far from it. If K and Δ^{n-1} are on vertex set $[n]$ and L is formed by gluing Δ^{n-1} and $\{n+1\} * K$ along K, then $\mathcal{Z}_L$ is a *co-H*-space by [38] so $\mathrm{cat}(\mathcal{Z}_L) = 1$. In fact, it is not difficult to directly show that $\mathcal{Z}_L \simeq \Sigma^2 \mathcal{Z}_K$. On the other hand, Proposition 3.19 gives $\mathrm{cat}(\mathcal{Z}_L) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_K)\} + \mathcal{C}\mathrm{at}(\mathcal{Z}_K)$ since $\mathcal{Z}_{\{n+1\}*K} \cong D^2 \times \mathcal{Z}_K \simeq \mathcal{Z}_K$ and $\mathcal{Z}_{\Delta^n}$ is contractible.

Corollary 3.21. *Suppose*

$$K = L_1 \#_{\sigma_1} L_2 \#_{\sigma_2} \cdots \#_{\sigma_{k-1}} L_k$$

where $\dim L_i = d$, *σ_i is a d-face common to L_i and L_{i+1}, and $\sigma_i \cap \sigma_j = \emptyset$ when $i \neq j$. Then*

$$\mathrm{cat}(\mathcal{Z}_K) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_{{L_1}^{(d-1)}}), \dots, \mathrm{cat}(\mathcal{Z}_{{L_k}^{(d-1)}})\} + 1.$$

Proof. Take the disjoint unions

$$C := \partial\sigma_1 \sqcup \cdots \sqcup \partial\sigma_{k-1}$$

$$K_1 := \bigsqcup_{1 \leq 2i+1 \leq k-1} L_{2i+1}$$

$$K_2 := \bigsqcup_{2\le 2i\le k-1} L_{2i}$$

and take the iterated face deletions $K_1' := K_1\backslash(\sigma_1 \sqcup \cdots \sqcup \sigma_{k-1})$ and $K_2' := K_2\backslash(\sigma_1 \sqcup \cdots \sqcup \sigma_{k-1})$. Then C is a full subcomplex common to both K_1' and K_2', and to both ${K_1'}^{(d-1)}$ and ${K_2'}^{(d-1)}$. Moreover, ${K_1'}^{(d-1)} = {K_1}^{(d-1)}$ and ${K_2'}^{(d-1)} = {K_2}^{(d-1)}$, and $K = K_1' \cup_C K_2'$, so by the second part of Proposition 3.19,

$$\mathrm{cat}(\mathcal{Z}_K) \le \max\{1, \mathrm{cat}(\mathcal{Z}_{{K_1}^{(d-1)}}), \mathrm{cat}(\mathcal{Z}_{{K_2}^{(d-1)}})\} + \mathcal{C}\mathrm{at}(\mathcal{Z}_C).$$

It is clear that C is an HMF-complex, so $\mathcal{Z}_C$ is homotopy equivalent to a wedge of spheres and $\mathcal{C}\mathrm{at}(\mathcal{Z}_C) = 1$. Alternatively, this follows from Theorem 10.1 in [33]. Moreover, we can think of $\mathcal{Z}_{{K_1}^{(d-1)}}$ as being built up iteratively by gluing $\bigsqcup_{1\le 1\le j} \mathcal{Z}_{{L_{2i+1}}^{(d-1)}}$ and $\mathcal{Z}_{{L_{2j+3}}^{(d-1)}}$ along the empty simplex, so iterating the first inequality in Example 3.20,

$$\mathrm{cat}(\mathcal{Z}_{{K_1}^{(d-1)}}) \le \max\{\, 1,\, \mathrm{cat}(\mathcal{Z}_{{L_1}^{(d-1)}}),\, \mathrm{cat}(\mathcal{Z}_{{L_3}^{(d-1)}}),\, \dots\}.$$

Likewise, $\mathrm{cat}(\mathcal{Z}_{{K_2}^{(d-1)}}) \le \max\{\, 1,\, \mathrm{cat}(\mathcal{Z}_{{L_2}^{(d-1)}}),\, \mathrm{cat}(\mathcal{Z}_{{L_4}^{(d-1)}}),\, \dots\}$. The inequality in the lemma follows. □

4. Some Specific Applications: Triangulated Surfaces and Spheres

We now apply the general tools developed in the previous sections to compute exact values for the LS-category of moment angle complexes over triangulated closed oriented surfaces and certain classes of triangulated higher dimensional spheres that are dual to boundaries of polytopes built up iteratively via vertex cuts and products.

4.1. **Triangulated spheres.** Let $\mathcal{C}_0 := \{S^0\}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ consist of all triangulated 0,1, and 2-spheres, and for $d \ge 3$, let $\mathcal{C}_d$ be the class of triangulated d-spheres defined by $K \in \mathcal{C}_d$ if

(1) $K = L_1 * \cdots * L_k$ for some $L_i \in \mathcal{C}_{d_i}$, $d_i \le 2$, and $d_1 + \cdots + d_k = d - k + 1$;
(2) $K = K_1\#_{\sigma_1} \cdots \#_{\sigma_{\ell-1}} K_\ell$ where σ_i is a d-face common to K_i and K_{i+1} with $\sigma_i \cap \sigma_j = \emptyset$ when $i \ne j$, and each $K_i = L_{1,i} * \cdots * L_{k_i,i}$ is of the form (1) such that each $L_{j,i}$ is not the boundary of a simplex.

Recall that the join $L * L'$ is the simplicial complex $\{\sigma \sqcup \sigma' \mid \sigma \in L,\, \sigma' \in L'\}$, and the connected sum $K\#_\sigma K'$ is given topologically by gluing triangulations K and K' of S^d along a common d-face σ, and deleting its interior.

Remark 4.1. If L and L' are boundaries ∂P^* and $\partial P'^*$ of the duals of simple polytopes P and P', then $L * L'$ is the boundary of $(P \times P')^*$, while $L\#_\sigma L'$ is the boundary of dual Q^*, where Q is obtained by taking the vertex cut at the vertices of P and P' that are dual to σ, gluing along the new hyperplane and removing it after gluing.

Our goal in the next few subsections will be to show the following.

Theorem 4.2. *If K on vertex set $[n] = \{1, \dots, n\}$ is any triangulated d-sphere for $d = 0, 1, 2$, or $K \in \mathcal{C}_d$ for $d \ge 3$, then the following are equivalent.*

(1) *K is m-annihilating over $\mathbb{Z}$;*
(2) $\mathrm{nill}(\mathrm{Tor}_{\mathbb{Z}[v_1,\dots,v_n]}(\mathbb{Z}[K], \mathbb{Z})) \le m + 1$ *(equivalently* $\mathrm{cup}(\mathcal{Z}_K) \le m$*);*
(3) *for any filtration of full subcomplexes*

$$\partial\Delta^{d+2-\ell} = K_{I_\ell} \subsetneq K_{I_{\ell-1}} \subsetneq \cdots \subsetneq K_{I_1} = K$$

such that $|K_{I_i}| \cong S^{d+1-i}$, we have $\ell \le m$;
(4) $\mathrm{cat}(\mathcal{Z}_K) \le m$.

Moreover, $1 \le m \le d + 1$; that is, K satisfies any of the above for some m which cannot be greater than $d + 1$. □

4.2. **Filtration length on spheres.** The simplest example is $\mathrm{filt}(\partial\Delta^{d+1}) = 1$. Generally, there are the following bounds with respect to joins, connected sums, and cup product length.

Lemma 4.3. *If K and L are both triangulations of S^d, and σ is a d-face common to K and L, then*

$$\mathrm{filt}(K\#_\sigma L) \geq \max\{2, \mathrm{filt}(K), \mathrm{filt}(L)\}.$$

Proof. A full subcomplex $N_{\mathcal{I}}$ of $K\#_\sigma L$ satisfying $|N_{\mathcal{I}}| \cong S^k$ for some $k < d$ must either be a full subcomplex of exactly one of K or L, or else $N_{\mathcal{I}} = \partial\sigma$, otherwise $N_{\mathcal{I}}$ would have a $(k-1)$-face contained in three k-faces. Moreover, $K\#_\sigma L$ always has the length 2 spherical filtration $\partial\sigma \subsetneq K\#_\sigma L$. The lemma follows immediately. □

Lemma 4.4. *If K and L are any triangulated spheres, then*

$$\mathrm{filt}(K * L) \geq \mathrm{filt}(K) + \mathrm{filt}(L).$$

Proof. Let K and L be on vertex sets $[n]$ and $[m]$. Let $d := \dim K$, $d' := \dim L'$, $\ell := \mathrm{filt}(K)$, $\ell' := \mathrm{filt}(L)$, and take $K_{I_\ell} \subsetneq \cdots \subsetneq K_{I_1} = K$ and $L_{J_{\ell'}} \subsetneq \cdots \subsetneq L_{J_1} = L$ to be spherical filtrations of K and L.

Since $|K_{I_i} * L_{J_j}| \cong |K_{I_i}| * |L_{J_j}| \cong S^{d+1-i} * S^{d'+1-j} \cong S^{d+d'-i-j+3}$, and $|K_{I_i} * L_{J_j}|$ is a full subcomplex of $|K_{I_{i'}} * L_{J_{j'}}|$ when $i \leq i'$ and $j \leq j'$, we have a length $\ell + \ell'$ spherical filtration of $K * L$

$$(K_{I_\ell} * L_{J_{\ell'}}) \subsetneq \cdots \subsetneq (K_{I_\ell} * L_{J_2}) \subsetneq (K_{I_\ell} * L_{J_1}) \subsetneq (K_{I_{\ell-1}} * L_{J_1}) \subsetneq \cdots \subsetneq (K_{I_1} * L_{J_1}) = K * L.$$

Therefore $\mathrm{filt}(K * L) \geq \ell + \ell'$. □

4.3. **Triangulated d-spheres for $d = 0, 1, 2$.** The only triangulated 0-sphere is S^0, and the only triangulated 1-sphere with $n \geq 3$ vertices is the n-gon, both of which have an LS-category that is easy to characterize. For higher dimensional spheres the following definition will be useful.

Definition 4.5. We will say that C is a *chordless cycle* in K with $m \geq 3$ vertices if C is a full subcomplex K_I of K, and C is an m-gon for some $m \geq 3$. When K is a graph, this is the same as C being an induced cycle of K. A simplicial complex K is said to be *chordal* if it contains no chordless cycles with 4 or more vertices.

Lemma 4.6. *If K is a triangulation of S^1 on vertex set $[n]$, then*

(i) $\mathrm{filt}(K) \geq 2$ *whenever K has at least 4 vertices.*

If K is a triangulation of S^2, then

(ii) $\mathrm{filt}(K) \geq 2$ *whenever K has a chordless cycle;*

(iii) $\mathrm{filt}(K) \geq 3$ *whenever K has a chordless cycle with at least 4 vertices.*

Proof. Let $|K| \cong S^1$. If K has at least $n \geq 4$ vertices, then K being an n-gon means we can take $I' \subset [n]$, $|I'| = 2$, such that $|K_{I'}| = S^0$. Then $S^0 \subsetneq K$ is a length 2 spherical filtration of K.

Let $|K| \cong S^2$, and $C := K_I$ be a chordless cycle for some $I \subset [n]$. We have $|K_I| \cong S^1$ and $K_I = \partial\Delta^2$ when K_I has 3 vertices, in which case $K_I \subsetneq K$ is a length 2 spherical filtration. Otherwise, when K_I has at least 4 vertices, $\mathrm{filt}(K) \geq 3$ since there is a spherical filtration $S^0 \subsetneq K_I \subsetneq K$ by part (i). □

Proposition 4.7. *Let K be a triangulated d-sphere, $d = 0, 1, 2$. Then*

$$1 \leq \mathrm{filt}(K) = \mathrm{cup}(\mathcal{Z}_K) = \mathrm{cat}(\mathcal{Z}_K) \leq d + 1.$$

In particular, letting $m := \mathrm{cat}(\mathcal{Z}_K)$, when $d = 1$ we have:
(i) *$m = 1$ iff $K = \partial\Delta^2$*
(ii) *$m = 2$ iff K has at least 4 vertices.*

When $d = 2$ we have:
(i) *$m = 1$ iff K has no chordless cycles of any length (equivalently $K = \partial\Delta^3$)*

(ii) $m = 2$ *iff* K *has a chordless cycle, but none with more than* 3 *vertices*
(iii) $m = 3$ *iff* K *has a chordless cycle with at least* 4 *vertices.*

Proof. The case $d = 0$ is immediate since S^0 is the only triangulated 0-sphere, and $\mathcal{Z}_{S^0} \cong S^3$. By Lemma 3.11 and Corollary 3.7

$$1 \leq \mathrm{filt}(K) \leq \mathrm{cup}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_K) \leq d + 1,$$

so it remains to show that $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{filt}(K)$.

Fix $d = 1$. Using Lemma 4.6, if K has at least 4 vertices, then $\mathrm{filt}(K) \geq 2$, so $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{filt}(K)$ since $\mathrm{cat}(\mathcal{Z}_K) \leq d + 1 = 2$. Otherwise, if K has 3-vertices, then $K = \partial\Delta^2$ and $\mathcal{Z}_K \cong S^5$, so $\mathrm{filt}(K) = \mathrm{cat}(\mathcal{Z}_K) = 1$.

Now fix $d = 2$. We break the argument up into three cases: K has no chordless cycles; K has only chordless cycles of length no more than 3; and K has at least one chordless cycle of length 4 or more. To start, by Lemma 4.6, $\mathrm{filt}(K) \geq 3$ whenever K has a chordless cycle with at least 4 vertices, so $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{filt}(K)$ since $\mathrm{cat}(\mathcal{Z}_K) \leq d + 1 = 3$. On the other hand, suppose K has chordless cycles, but none with more than 3 vertices. Then $\mathrm{filt}(K) = 2$ by Lemma 4.6, and the 1-skeleton $K^{(1)}$ is a chordal graph. The chordal property is closed under vertex deletion (taking full subcomplexes). Moreover, recall from [24] that chordal graphs have a *total elimination ordering*, that is, they can be built up one vertex v at a time in some order such that at each step the neighbours of v form a clique. Inducting on this ordering, one sees that chordal graphs are HMF-complexes, therefore $\mathrm{cat}(\mathcal{Z}_{K^{(1)}}) \leq 1$ by Proposition 3.18 (this also follows the main result in [39]). Using Lemma 3.11, we have $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_{K^{(1)}}) + 1 \leq 2 = \mathrm{filt}(K)$. Otherwise, $K = \partial\Delta^3$ when K has no chordless cycles at all, so we have $\mathcal{Z}_K \cong S^7$ and $\mathrm{filt}(K) = \mathrm{cat}(\mathcal{Z}_K) = 1$.

□

Lemma 4.8. *Let* K *be a triangulated* d*-sphere,* $d = 1, 2$*. Then*

$$\mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_K) - 1\} = \max\{1, \mathrm{filt}(K) - 1\}.$$

Proof. The last equality $\mathrm{filt}(K) = \mathrm{cat}(\mathcal{Z}_K)$ is from the previous proposition. Let $m := \mathrm{cat}(\mathcal{Z}_K)$. Using Proposition 4.7, the statement simplifies to $\mathrm{cat}(\mathcal{Z}_{K^{(0)}}) \leq 1$ when $d = 1$, or $d = 2$ and $m = 1$. This is true since $\mathcal{Z}_{K^{(0)}}$ has the homotopy type of a wedge of spheres by [35], or by Proposition 3.18 as $K^{(0)}$ is a collection of disjoint points. Fix $d := 2$. As in the proof of Proposition 4.7, $\mathrm{cat}(\mathcal{Z}_{K^{(1)}}) \leq 1 < m$ when $m = 2$, since $\mathrm{filt}(K) = 2$. This last inequality also holds when $m = 3$ since $\mathrm{cat}(\mathcal{Z}_{K^{(1)}}) \leq \dim K^{(1)} + 1 = 2$ is always true by Lemma 3.11. □

4.4. **Triangulated d-spheres for $d \geq 3$.**

Proposition 4.9. *Suppose* $K \in \mathcal{C}_d$*,* $d \geq 0$*. Then*

$$1 \leq \mathrm{filt}(K) = \mathrm{cup}(\mathcal{Z}_K) = \mathrm{cat}(\mathcal{Z}_K) \leq d + 1.$$

Proof. By Lemmas 3.11 and Corollary 3.7,

$$1 \leq \mathrm{filt}(K) \leq \mathrm{cup}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_K) \leq d + 1.$$

It remains to show that $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{filt}(K)$. The $d = 0, 1, 2$ case is Proposition 4.7.

Suppose $K = L_1 * \cdots * L_k \in \mathcal{C}_d$ for some $L_i \in \mathcal{C}_{d_i}$, $d_i \leq 2$, and $d_1 + \cdots + d_k = d - k - 1$. Then $\mathcal{Z}_K = \mathcal{Z}_{L_1} \times \cdots \times \mathcal{Z}_{L_k}$, and so using Proposition 4.7 and iterating Lemma 4.4,

$$\mathrm{cat}(\mathcal{Z}_K) \leq \sum_{i=1,\ldots,k} \mathrm{cat}(\mathcal{Z}_{L_i}) = \sum_{i=1,\ldots,k} \mathrm{filt}(L_i) \leq \mathrm{filt}(L_1 * \cdots * L_k) = \mathrm{filt}(K).$$

Moreover, iterating Corollary 3.15, and using Lemma 4.8, when each $L_i \neq S^0$

$$\mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) \leq \sum_{i=1,\ldots,k} \mathrm{cat}(\mathcal{Z}_{L_i{}^{(d_i-1)}}) + k - 1 \leq \sum_{i=1,\ldots,k} \max\{1, \mathrm{filt}(L_i) - 1\} + k - 1$$

and when each $L_i \neq \partial\Delta^{d_i+1}$, we have $\mathrm{filt}(L_i) > 1$, therefore

$$\mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) \leq \left(\sum_{i=1,\ldots,k} \mathrm{filt}(L_i)\right) - 1 \leq \mathrm{filt}(L_1 * \cdots * L_k) - 1 = \mathrm{filt}(K) - 1 \tag{9}$$

the second inequality by iterating Lemma 4.4.

Suppose $K = K_1\#_{\sigma_1} \cdots \#_{\sigma_{\ell-1}} K_\ell$ where each $K_i = L_{1,i} * \cdots * L_{k_i,i}$ is a join of the above form such that each $L_{j,i}$ is not the boundary of a simplex, and σ_i is a d-face common to K_i and K_{i+1} with $\sigma_i \cap \sigma_j = \emptyset$ when $i \neq j$. By Corollary 3.21 and inequality (9), we have

$$\begin{aligned}
\mathrm{cat}(\mathcal{Z}_K) &\leq \max\{1, \mathrm{cat}(\mathcal{Z}_{K_1^{(d-1)}}), \ldots, \mathrm{cat}(\mathcal{Z}_{K_\ell^{(d-1)}})\} + 1 \\
&\leq \max\{1, \mathrm{filt}(K_1) - 1, \ldots, \mathrm{filt}(K_\ell) - 1\} + 1 \\
&= \max\{2, \mathrm{filt}(K_1), \ldots, \mathrm{filt}(K_\ell)\} \\
&= \mathrm{filt}(K_1\#_{\sigma_1} \cdots \#_{\sigma_{\ell-1}} K_\ell) \\
&= \mathrm{filt}(K)
\end{aligned}$$

where the second last inequality follows from iterating Lemma 4.3.

□

4.5. **Proof of Theorem 4.2.** The following is an immediate consequence of Theorem 4.4 in [52] and the fact that *category weight*, *cwgt* as defined there is bounded below by 1, and linearly below with respect to cup products.

Theorem 4.10 (Rudyak [52])**.** *If* $\mathrm{cat}(X) \leq m$, *then*

(1) $\mathrm{cup}(X) \leq m$;
(2) *Massey products* $\langle v_1, \ldots, v_k\rangle$ *vanish in* $H^*(X)$ *whenever* $v_i = a_1 \cdots a_{m_i}$ *and* $v_j = b_1 \cdots b_{m_j}$, *and* $m_i + m_j > m$, *for some odd* i *and even* j *and* $a_s, b_t \in H^+(X)$. □

Then by our remarks in Section 3.1, and by definition $\mathrm{cup}(X) = \mathrm{nill}\, H^*(X) - 1$, we have a condition for K to be m-annihilating.

Proposition 4.11. *If* $\mathrm{cat}(\mathcal{Z}_K) \leq m$, *then* K *is* m-*annihilating.* □

Now Theorem 4.2 follows from Propositions 4.7 and 4.9, and the fact that, $\mathrm{cup}(\mathcal{Z}_K) \leq m$ when K is m-annihilating.

4.6. **Triangulated surfaces.** We extend the characterization of the LS-category of 2-spheres to closed connected 2-dimensional oriented surfaces. As before, the algebraic invariant that distinquishes between them is cup product length, though the combinatorial characerisation diverges. The main difference is that spherical filtration length does not play a part for surfaces as it did for spheres since the full subcomplex condition is a bit too strict here. Also, the different values for the LS-category do not pigeonhole nicely in terms of length of chordless cycles. Note the case of LS-category 1 has already been characterized in [40].

Proposition 4.12. *Let* K *be a triangulation of a closed oriented connected surface. Then*

$$1 \leq \mathrm{cup}(\mathcal{Z}_K) = \mathrm{cat}(\mathcal{Z}_K) \leq 3.$$

Moreover, letting $m := \mathrm{cat}(\mathcal{Z}_K)$, *we have:*
(i) $m = 1$ *iff* K *has no chordless cycles (equivalently* $K = \partial\Delta^3$*)*
(ii) $m = 2$ *iff* K *has a chordless cycle, but none with more than* 3 *vertices*
(iii) $m = 3$ *iff* K *has a chordless cycle with at least* 4 *vertices.*

Proof. For any vertex v, let D_v be the fan of 2-faces that contain v, and ∂D_v be its boundary, which consists of all vertices adjacent to v. To see that D_v is a triangulation of a 2-disk, notice that a small neighbourhood of v in $|K|$ is homeomorphic to a 2-disk since $|K|$ is a closed 2-manifold, so the interior of $|D_v|$ is an open 2-disk. At the same time the boundary ∂D_v has to be a triangulated circle, otherwise we would have a pair of distinct edges between the same pair of vertices, namely, vertex v and a vertex w in ∂D_v, which contradicts K being a triangulation.

Now we get to the heart of the proof. As in the proof of Proposition 4.7, we break the argument up into three cases: (1) K has no chordless cycles of any length; (2) K has chordless cycles of length no more than 3; and (3) K has at least one chordless cycle of length 4 or more.

The first difficulty extending to surfaces beyond 2-spheres is in case (3). Unlike the situation for 2-spheres, we cannot use a chordless cycle C of length ≥ 4 to obtain a length 3 spherical filtration since C might not always bound a 2-disk in K. We search for filtrations derived from C instead. Taking v to be a vertex on a chordless cycle C of length ≥ 4, since the vertices w_0 and w_1 that are adjacent to v in C are in ∂D_v, we have a length 3 filtration of K in terms of triangulated spheres

$$E := \{w_0, w_1\} \subsetneq \partial D_v \subsetneq K$$

where $|E| \cong S^0$, $|\partial D_v| \cong S^1$, and $|\partial D_v|$ bounds the 2-disk $|D_v|$. Also, E is a full subcomplex of K since there are no edges between w_0 and w_1 in C since they are both adjacent to v in C, and C is chordless with at least 4 vertices. So there are no edges between them in K as well since C is a full subcomplex of K by definition of chordless cycle. This then is almost a spherical filtration of length 3; the only thing missing is that ∂D_v might not be a full subcomplex of K. However, ∂D_v satisfies the hypothesis of Lemma 3.8. In particular, $K\backslash\partial D_v$ has two disconnected components: one containing v, and the other one containing another vertex w_3 in C besides v, w_0, or w_1 which must be disconnected from v in K since only w_1 and w_2 are adjacent to v in the full subcomplex C. Then by the remarks preceding Lemma 3.8, $3 \leq \mathrm{cup}(\mathcal{Z}_K)$. Namely we have a length 3 cup product induced by the composite of inclusions

$$K \longrightarrow \partial D_v * (K\backslash\partial D_v) \longrightarrow E * (\partial D_v\backslash E) * (K\backslash\partial D_v).$$

Since $\mathrm{cup}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_K) \leq 3$ by Lemma 3.11, we are done.

The difficulty in case (2) is similar. We have $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_{K^{(1)}}) + 1 \leq 2$ by the same argument as in Proposition 4.7, but again we cannot choose any length 3 chordless cycle to form a length 2 spherical filtration. Instead, as in case (3), we look at D_v for some vertex v in a chordless cycle C of length 3. Assume K is not a triangulated 2-sphere since that is addressed in Proposition 4.7. Note ∂D_v must have a chordless cycle. If it did not then ∂D_v would bound a triangulated 2-disk in D on the same vertex set as ∂D_v, and then $|K|$ would have to be a 2-sphere since $|D| \cup |D_v|$ is a 2-sphere. Also, since ∂D_v must have chordless cycles, then ∂D_v cannot have every vertex in K beside v. Otherwise, embedding $|K|$ in $\mathbb{R}^3$, we would have a path from v through the interior of $|K|$ and outside $|K|$ through this chordless cycle, contradicting that $|K|$ is a closed surface. Then ∂D_v satisfies the hypothesis of Lemma 3.8, and the filtration $\partial D_v \subsetneq K$ gives a length 2 cup product similarly as case (3). Thus $2 \leq \mathrm{cup}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_K)$ and we are done.

Case (1) is similar as for spheres. For any v, ∂D_v must have at least 3 vertices. In case (1) it cannot have more than 3 as follows. Suppose it had 4 or more vertices. Since we assume K has no chordless cycles, then the cycle ∂D_v must not be a full subcomplex. Namely, there must be an edge $\{a, b\}$ that is not in ∂D_v for two non-adjacent vertices $a, b \in \partial D_v$. But then (a, b, v) is a length 3 chordless cycle since $\sigma := \{a, b, v\}$ cannot be a 2-face of K. If it was a face in K then K would not be a surface since $\{b, v\}$ would be adjacent to three 2-faces: σ and the other two in the triangulated 2-disk D_v distinct from σ since $\sigma \notin D_v$. Now since ∂D_v has 3 vertices, and ∂D_v cannot be a length 3 chordless cycle, that is, cannot be a full subcomplex, it must be the boundary of a 2-face δ, so δ and D_v form the boundary of a 3-simplex in K, implying K itself must be this boundary. $\square$

5. Further Applications

5.1. **Fullerenes and Pogorelov polytopes.** A *fullerene* P is a simple 3-polytope all of whose 2-faces are pentagons and hexagons. These are mathematical idealisations of physical fullerenes - spherical molecules of carbon such that each carbon atom belongs to three carbon rings, and each carbon ring is either a pentagon or hexagon.

The authors in [19] have shown that the cohomology ring of moment-angle complexes is a complete combinatorial invariant of fullerenes, while Buchstaber and Erokhovets [11, 10] show that the finer details of their cohomology encode many interesting properties of fullerenes. For example, if P^* is the dual of P, then the bigraded Betti numbers of $\mathcal{Z}_{\partial P^*}$ count the number *k-belts* in P. Here, a k-belt of a simple polytope such as P is a sequence of 2-faces $(F_1,\dots,F_k)$ such that $F_k\cap F_1\neq\emptyset$, $F_i\cap F_{i+1}\neq\emptyset$ for $1\leq i\leq k-1$, and all other intersections are empty. Notice that the k-belts of P correspond to full subcomplexes of ∂P^* that are k-gons. But since fullerenes can have no 3-belts [11, 10], ∂P^* must only have n-gons as full subcomplexes for $n\geq 4$. Moreover, since ∂P^* is a triangulated 2-sphere that is not a boundary of the 2-simplex, it must have at least one such n-gon as a full subcomplex. Thus, $\mathrm{filt}(\partial P^*)=3$, and by Theorem 4.2, we can determine certain Massey products in $H^*(\mathcal{Z}_{\partial P^*})$.

Theorem 5.1. *For fullerenes P, $\mathrm{cat}(\mathcal{Z}_{\partial P^*})=3$ and ∂P^* is 3-annihilating. In particular, all Massey products consisting of decomposable elements in $H^+(\mathcal{Z}_{\partial P^*})$ must vanish.* □

Pogorelov polytopes [49] can be seen as a generalisation of fullerenes. A Pogorelov polytope is a combinatorial simple 3-polytope realisable in the Lobachevsky (hyperbolic) space as a bounded right-angled polytope. These polytopes are exactly simple 3-polytopes with cyclically 5-edge connected graphs. A Pogorelov polytope has no 3 and 4-gons and may have any prescribed numbers of k-gons, $k\geq 7$. Any simple polytope with only 5-, 6 and at most one 7-gon is Pogorelov.

Since Pogorelov polytopes P have no 3 and 4-belts of facets, ∂P^* must only have n-gons as full subcomplexes for $n\geq 5$. Now following the same proof as in the case of fullerenes, we have the following statement.

Theorem 5.2. *For Pogorelov polytopes P, $\mathrm{cat}(\mathcal{Z}_{\partial P^*})=3$ and ∂P^* is 3-annihilating. In particular, all Massey products consisting of decomposable elements in $H^+(\mathcal{Z}_{\partial P^*})$ must vanish.* □

5.2. **Neighbourly complexes.** For any finite simply-connected CW-complex X, let

$$\mathrm{hd}(X):=\max\left\{\max\left\{i\;\middle|\;\tilde{H}^i(X)\otimes\mathbb{Q}\neq 0\right\},\max\left\{i\;\middle|\;\mathrm{Torsion}(\tilde{H}^{i-1}(X))\neq 0\right\}\right\}$$

and

$$\mathrm{hc}(X):=\min\left\{i\;\middle|\;\tilde{H}^{i+1}(X)\neq 0\right\}.$$

These coincide with the dimension and connectivity of X up to homotopy equivalence. It is well known (c.f. [42]) that X satisfies

$$\mathrm{cat}(X)\leq\frac{\mathrm{hd}(X)}{\mathrm{hc}(X)+1}. \tag{10}$$

A version of the Hochster formula also holds for real moment-angle complexes, namely,

$$H^*(\mathbb{R}\mathcal{Z}_K)\cong\bigoplus_{I\subseteq[n]}\tilde{H}^*(\Sigma|K_I|). \tag{11}$$

Thus,

$$\mathrm{hd}(\mathbb{R}\mathcal{Z}_K)=1+\max\{\,\mathrm{hd}(|K_I|)\mid I\subseteq[n]\,\}\;\leq\;1+\dim K$$

and

$$\mathrm{hc}(\mathbb{R}\mathcal{Z}_K)=1+\min\{\,\mathrm{hc}(|K_I|)\mid I\subseteq[n]\,\},$$

and using the inequality $\mathrm{cat}(\mathcal{Z}_K)\leq\mathrm{cat}(\mathbb{R}\mathcal{Z}_K)$ from Corollary 3.13,

Proposition 5.3. *It holds that*

$$\mathrm{cat}(\mathcal{Z}_K) \leq \frac{\mathrm{hd}(\mathbb{R}\mathcal{Z}_K)}{\mathrm{hc}(\mathbb{R}\mathcal{Z}_K)}.$$

□

Comparing the Hochster formula for $H^*(\mathbb{R}\mathcal{Z}_K)$ to the Hochster formula for $H^*(\mathcal{Z}_K)$ in Section 3.1, one sees that the inequality $\frac{\mathrm{hd}(\mathbb{R}\mathcal{Z}_K)}{\mathrm{hc}(\mathbb{R}\mathcal{Z}_K)} \leq \frac{\mathrm{hd}(\mathcal{Z}_K)}{\mathrm{hc}(\mathcal{Z}_K)}$ usually holds, with the disparity between these two often being very large. In such case, the bound in Proposition 5.3 is an improvement over what one would get by applying (10) directly to $X = \mathcal{Z}_K$.

Consider, for instance, the case of k-neighbourly complexes. A simplicial complex K on vertex set $[n]$ is said to be k-*neighbourly* if every subset of k or less vertices in $[n]$ is a face of K. In this case $H_i(K_I) = 0$ for $i \leq k-2$ and each $I \subseteq [n]$, so $\mathrm{hc}(\mathbb{R}\mathcal{Z}_K) \geq k-1$. Therefore we have the following result.

Theorem 5.4. *If K is k-neighbourly, $\mathrm{cat}(\mathcal{Z}_K) \leq \frac{1+\dim K}{k}$. In particular, K is $\left(\frac{1+\dim K}{k}\right)$-annihilating.* □

Example 5.5. Suppose K is $\lceil \frac{n}{2} \rceil$-neighbourly. If $K \neq \Delta^{n-1}$, then $\dim K \leq n-2$. Thus, $\mathrm{cat}(\mathcal{Z}_K) \leq 1$ ($\mathcal{Z}_K$ is a *co-H*-space) and K is 1-annihilating.

5.3. **Simplicial wedges.** We recall the *simplicial wedge* construction defined in [50, 3]. Let K be a simplicial complex on vertex set $\{v_1, ..., v_n\}$, and for any face $\sigma \in K$, define the *link* of σ in K the subcomplex of K given by

$$\mathrm{link}_K(\sigma) = \{\tau \in K \mid \tau \cap \sigma = \emptyset, \, \tau \cup \sigma \in K\}.$$

By *doubling* a vertex v_i in K, we obtain a new simplicial complex $K(v_i)$ on vertex set

$$\{v_1, \ldots, v_{i-1}, v_{i1}, v_{i2}, v_{i+1}, \ldots, v_n\}$$

defined by

$$K(v_i) = (v_{i1}, v_{i2}) * \mathrm{link}_K(v_i) \cup_{\{v_{i1},v_{i2}\}*\mathrm{link}_K(v_i)} \{v_{i1}, v_{i2}\} * K\backslash\{v_i\},$$

where (v_{i1}, v_{i2}) is the 1-simplex with vertices $\{v_{i1}, v_{i2}\}$. One can of course iterate this construction by reapplying the doubling operation to successive complexes, and the order of vertices on which this is done is irrelevant. To this end, taking any sequence $J = (j_1, \ldots, j_n)$ of non-negative integers, let u_j be the j^{th} vertex in the sequence $v_1, v_{12}, \ldots, v_{1j_1}, v_2, \ldots, v_n, v_{n2}, \ldots, v_{nj_n}$ and $N = j_1 + \cdots + j_n$, and define

$$K(J) = K_N$$

where $K_{j+1} = K_j(u_{j+1})$ and $K_0 = K$. In algebraic terms, the Stanley-Reisner ideal of $K(J)$ is obtained from the Stanley-Reisner ideal of K by replacing each vertex v_i by $v_{i1}, v_{i2}, \ldots, v_{ij_i}$ in each monomial. This construction arises in combinatorics (see [50]) and has the important property that if K is the boundary of the dual of d-polytope, then $K(J)$ is the boundary of the dual of a $(d+N)$-polytope.

Theorem 5.6. *For any J, $\mathrm{cat}(\mathcal{Z}_{K(J)}) \leq \mathrm{cat}(\mathcal{Z}_K)$.*

Proof. Let (D^J, S^J) be the sequence of pairs $((D^{2j_1+2}, S^{2j_1+1}), \ldots, (D^{2j_n+2}, S^{2j_n+1}))$. By [3, 34], there is a homeomorphism

$$\mathcal{Z}_{K(J)} = (D^2, S^1)^{K(J)} \cong (D^J, S^J)^K,$$

and by Proposition 3.12, $\mathrm{cat}((D^J, S^J)^K) \leq \mathrm{cat}((D^2, S^1)^K) = \mathrm{cat}(\mathcal{Z}_K)$. □

This result becomes algebraically useful when a good bound on $\mathrm{cat}(\mathcal{Z}_K)$ is known. For instance, there are many examples of complexes K for which $\mathrm{cat}(\mathcal{Z}_K) = 1$, duals of sequential Cohen Macaulay and shellable complexes, and chordal flag complexes to name a few [32, 38]. In each of these examples $\mathrm{cat}(\mathcal{Z}_{K(J)}) \leq 1$, so $K(J)$ is Golod. Generally, $K(J)$ is at least $(1 + \dim K)$-annihilating since $\mathrm{cat}(\mathcal{Z}_K) \leq 1 + \dim K$, even though $\dim K(J) - \dim K$ can be arbitrarily large.

Notice $K(J)$ is a triangulation of a $(d+N)$-sphere whenever K is a triangulation of a d-sphere. Combining Theorem 5.6 and Proposition 4.11, the range of spheres for which Theorem 4.2 holds generalises as follows.

Corollary 5.7. *Let K be be any triangulated d-sphere for $d = 0, 1, 2$, or $K \in \mathcal{C}_d$ when $d \geq 3$, and let $m := \mathrm{filt}(K)$ (equivalently $m = \mathrm{cup}(\mathcal{Z}_K)$). Then $\mathrm{cat}(\mathcal{Z}_{K(J)}) \leq m$ and $K(J)$ is m-annihilating.* □

5.4. **Examples of complexes for which the cup product length does not determine the category.** There are two examples of simplicial complexes that clearly stand out as they are counterexamples to a conjecture and a theorem on the homotopy theoretical structures of moment-angle complexes associated with Golod simplicial complexes. Using m-annihilating property, we shall prove that in both cases the cup product length is strictly less than the category of these moment-angle complexes.

Example 5.8. For some time it was thought that for any monomial ideal $\mathfrak{a}$ in a polynomial ring $S = \mathbb{k}[x_1, \ldots, x_n]$ over some field $\mathbb{k}$, the ring $R = S/\mathfrak{a}$ is Golod if and only if the product in the Koszul homology $H_*(K_R)$ of R is trivial [8]. Katthän [43] constructed a counterexample to this statement. Let $\mathbb{k}$ be a field, $S = \mathbb{k}[x_1, x_2, y_1, y_2, z]$ and let $\mathfrak{a} \subset S$ be the ideal with the following generators:

$$m_a := x_1x_2^2 \qquad m_{ab} := x_1x_2y_1y_2 \qquad m_{ab\#c} := x_1y_1z \tag{12}$$

$$m_b := y_1y_2^2 \qquad m_{bc} := y_2^2z^2 \qquad m_{bc\#a} := x_2^2y_2^2z \tag{13}$$

$$m_c := z^3 \qquad m_{ca} := x_2^2z^2. \tag{14}$$

Then the product in $H_*(K_{S/\mathfrak{a}}) = \mathrm{Tor}_*^S(S/\mathfrak{a}, \mathbb{k})$ is trivial, but $S/\mathfrak{a}$ is not Golod. More precisely, there is a nonzero ternary Massey product.

The polarisation of this ideal is the Stanley-Reisner ideal of some simplicial complex K of dimension 5. By taking its 4-skeleton, one obtains an example of a 4-dimensional simplicial complex L, such that $\mathbb{k}[L]$ is not Golod but has a trivial product in its Koszul homology. We shall show that

$$1 = \mathrm{cup}(\mathcal{Z}_L) < \mathrm{cat}(\mathcal{Z}_L).$$

As the cup product in $H^*(\mathcal{Z}_L)$ is trivial, $\mathrm{cup}(\mathcal{Z}_L) = 1$. If $\mathrm{cat}(\mathcal{Z}_L) = 1$, then by Proposition 4.11 the moment-angle complex $\mathcal{Z}_L$ would be Golod. Therefore, $\mathrm{cat}(\mathcal{Z}_L) > \mathrm{cup}(\mathcal{Z}_L)$.

Example 5.9. In [5] the question whether $\mathcal{Z}_K$ is a co-H-space for K Golod was studied and the authors showed that these two notions, one coming from algebra and another from topology, are very closely related. Recently, Iriye and Yano [41] described a Golod simplicial complex K for which $\mathcal{Z}_K$ is not a co-H-space. It is straightforward to see that for the simplicial complex K,

$$1 = \mathrm{cup}(\mathcal{Z}_K) < \mathrm{cat}(\mathcal{Z}_K)$$

as $\mathrm{cat}(\mathcal{Z}_K) \neq 1$ since K is not Golod.

References

1. David J. Anick, *A counterexample to a conjecture of Serre*, Ann. of Math. (2) **115** (1982), no. 1, 1–33. MR 644015
2. A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Adv. Math. **225** (2010), no. 3, 1634–1668. MR 2673742 (2012b:13053)
3. ———, *Operations on polyhedral products and a new topological construction of infinite families of toric manifolds*, Homology Homotopy Appl. **17** (2015), no. 2, 137–160. MR 3426378
4. I. V. Baskakov, V. M. Bukhshtaber, and T. E. Panov, *Algebras of cellular cochains, and torus actions*, Uspekhi Mat. Nauk **59** (2004), no. 3(357), 159–160. MR 2117435 (2006b:57045)
5. Piotr Beben and Jelena Grbić, *Configuration spaces and polyhedral products*, Adv. Math. **314** (2017), 378–425. MR 3658721

6. ———, $\frac{n}{3}$*-neighbourly moment-angle complexes and their unstable splittings*, Bol. Soc. Mat. Mex. (3) **23** (2017), no. 1, 141–152. MR 3633129
7. Alexander Berglund, *Homotopy invariants of Davis-Januszkiewicz spaces and moment-angle complexes.*
8. Alexander Berglund and Michael Jöllenbeck, *On the Golod property of Stanley-Reisner rings*, J. Algebra **315** (2007), no. 1, 249–273. MR 2344344 (2008f:13036)
9. Frédéric Bosio and Laurent Meersseman, *Real quadrics in* $\mathbf{C}^n$*, complex manifolds and convex polytopes*, Acta Math. **197** (2006), no. 1, 53–127. MR 2285318 (2007j:32037)
10. V. M. Buchstaber and N. Yu. Erokhovets, *Truncations of simple polytopes and applications*, Proc. Steklov Inst. Math. **289** (2015), no. 1, 104–133. MR 3486778
11. ———, *Constructions of families of three-dimensional polytopes, characteristic patches of fullerenes, and Pogorelov polytopes*, Izv. Ross. Akad. Nauk Ser. Mat. **81** (2017), no. 5, 15–91. MR 3706860
12. Victor M. Buchstaber and Taras E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002. MR 1897064 (2003e:57039)
13. ———, *Toric topology*, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015. MR 3363157
14. V. M. Bukhshtaber, N. Yu. Erokhovets, M. Masuda, T. E. Panov, and S. Pak, *Cohomological rigidity of manifolds defined by 3-dimensional polytopes*, Uspekhi Mat. Nauk **72** (2017), no. 2(434), 3–66. MR 3635437
15. Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, vol. 103, American Mathematical Society, Providence, RI, 2003. MR 1990857 (2004e:55001)
16. Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451. MR 1104531 (92i:52012)
17. Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274. MR 0382702
18. Graham Denham and Alexander I. Suciu, *Moment-angle complexes, monomial ideals and Massey products*, Pure Appl. Math. Q. **3** (2007), no. 1, Special Issue: In honor of Robert D. MacPherson. Part 3, 25–60. MR 2330154 (2008g:55028)
19. F. Fan, J. Ma, and X. Wang, *B-Rigidity of flag* 2*-spheres without* 4*-belt*, ArXiv e-prints (2015).
20. Yves Felix, Steve Halperin, and Jean-Claude Thomas, *Lusternik-Schnirelmann category of skeleta*, Topology Appl. **125** (2002), no. 2, 357–361. MR 1933583 (2004c:55025)
21. Yves Félix and Daniel Tanré, *Rational homotopy of the polyhedral product functor*, Proc. Amer. Math. Soc. **137** (2009), no. 3, 891–898. MR 2457428 (2009i:55011)
22. Ralph H. Fox, *On the Lusternik-Schnirelmann category*, Ann. of Math. (2) **42** (1941), 333–370. MR 0004108 (2,320f)
23. M. Franz, *The integral cohomology of toric manifolds*, Tr. Mat. Inst. Steklova **252** (2006), no. Geom. Topol., Diskret. Geom. i Teor. Mnozh., 61–70. MR 2255969 (2007f:14050)
24. D. R. Fulkerson and O. A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math. **15** (1965), 835–855. MR 0186421 (32 #3881)
25. T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois J. Math. **11** (1967), 417–427. MR 0229240 (37 #4814)
26. W. J. Gilbert, *Some examples for weak category and conilpotency*, Illinois J. Math. **12** (1968), 421–432. MR 0231375
27. Samuel Gitler and Santiago López de Medrano, *Intersections of quadrics, moment-angle manifolds and connected sums*, Geom. Topol. **17** (2013), no. 3, 1497–1534. MR 3073929
28. E. S. Golod, *Homologies of some local rings*, Dokl. Akad. Nauk SSSR **144** (1962), 479–482. MR 0138667
29. Jesús González, Bárbara Gutiérrez, and Sergey Yuzvinsky, *Higher topological complexity of subcomplexes of products of spheres and related polyhedral product spaces*, Topol. Methods Nonlinear Anal. **48** (2016), no. 2, 419–451. MR 3642766
30. J. Grbić and A. Linton, *Non-trivial higher massey products in moment-angle complexes*, arXiv:1911.07083, 1–51.
31. ———, *Lowest-degree triple Massey products in moment-angle complexes*, Uspekhi Mat. Nauk **75** (2020), no. 6(456), 175–176. MR 4181064
32. Jelena Grbić, Taras Panov, Stephen Theriault, and Jie Wu, *The homotopy types of moment-angle complexes for flag complexes*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6663–6682. MR 3461047
33. Jelena Grbić and Stephen Theriault, *The homotopy type of the complement of a coordinate subspace arrangement*, Topology **46** (2007), no. 4, 357–396. MR 2321037 (2008j:13051)
34. ———, *The homotopy type of the polyhedral product for shifted complexes*, Adv. Math. **245** (2013), 690–715. MR 3084441

35. E. Grbich and S. Terio, *Homotopy type of the complement of a configuration of coordinate subspaces of codimension two*, Uspekhi Mat. Nauk **59** (2004), no. 6(360), 203–204. MR 2138475 (2005k:55023)
36. ———, *Homotopy theory in toric topology*, Uspekhi Mat. Nauk **71** (2016), no. 2(428), 3–80. MR 3507473
37. Melvin Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975), Dekker, New York, 1977, pp. 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26. MR 0441987 (56 #376)
38. K. Iriye and D. Kishimoto, *Topology of polyhedral products and the Golod property of Stanley-Reisner rings*, ArXiv e-prints (2013).
39. ———, *Golodness and polyhedral products for two dimensional simplicial complexes*, ArXiv e-prints (2015).
40. Kouyemon Iriye and Daisuke Kishimoto, *Golodness and polyhedral products for two-dimensional simplicial complexes*, Forum Math. **30** (2018), no. 2, 527–532. MR 3770008
41. Kouyemon Iriye and Tatsuya Yano, *A Golod complex with non-suspension moment-angle complex*, Topology Appl. **225** (2017), 145–163. MR 3649878
42. I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), no. 4, 331–348. MR 516214 (80i:55001)
43. Lukas Katthän, *A non-Golod ring with a trivial product on its Koszul homology*, J. Algebra **479** (2017), 244–262. MR 3627285
44. I. Yu. Limonchenko, *Families of minimally non-Golod complexes and polyhedral products*, Dalnevost. Mat. Zh. **15** (2015), no. 2, 222–237. MR 3593998
45. ———, *Massey products in the cohomology of moment-angle manifolds of 2-truncated cubes*, Uspekhi Mat. Nauk **71** (2016), no. 2(428), 207–208. MR 3507477
46. L Lusternik and L. Schnirelmann, *Existence de trois lignés géodésiques fermées sur toute surface de genre* 0, Comptes Rend. **188** (1929), 534–536.
47. ———, *Sur un principe topologique en analyse*, Comptes Rend. **188** (1929), 295–297.
48. Dennis McGavran, *Adjacent connected sums and torus actions*, Trans. Amer. Math. Soc. **251** (1979), 235–254. MR 531977 (82c:57026)
49. A. V. Pogorelov, *Regular decomposition of the Lobačevskiĭ space*, Mat. Zametki **1** (1967), 3–8. MR 0208458
50. J. Scott Provan and Louis J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), no. 4, 576–594. MR 593648
51. Jan-Erik Roos, *Relations between Poincaré-Betti series of loop spaces and of local rings*, Séminaire d'Algèbre Paul Dubreil 31ème année (Paris, 1977–1978), Lecture Notes in Math., vol. 740, Springer, Berlin, 1979, pp. 285–322. MR 563510
52. Yuli B. Rudyak, *On category weight and its applications*, Topology **38** (1999), no. 1, 37–55. MR 1644063 (99f:55007)
53. Graham Hilton Toomer, *Lusternik-Schnirelmann category and the Moore spectral sequence*, Math. Z. **138** (1974), 123–143. MR 0356037
54. George W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978. MR 516508

School of Mathematics, University of Southampton,Southampton SO17 1BJ, United Kingdom
Email address: `P.D.Beben@soton.ac.uk`

School of Mathematics, University of Southampton,Southampton SO17 1BJ, United Kingdom
Email address: `J.Grbic@soton.ac.uk`