Common Bubble Detection in Large Dimensional Financial Systems *

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Abstract

Price bubbles in multiple assets are sometimes nearly coincident in occurrence. Such near-coincidence is strongly suggestive of co-movement in the associated asset prices and likely driven by certain factors that are latent in the financial or economic system with common effects across several markets. Can we detect the presence of such common factors at the early stages of their emergence? To answer this question, we build a factor model that includes I(1), mildly explosive, and stationary factors to capture normal, exuberant, and collapsing phases in such phenomena. The I(1) factor models the primary driving force of market fundamentals. The explosive and stationary factors model latent forces that underlie the formation and destruction of asset price bubbles, which typically exist only for subperiods of the sample. The paper provides an algorithm for testing the presence of and date-stamping the origination and termination of price bubbles determined by latent factors in a large-dimensional system embodying many markets. Asymptotics of the bubble test statistic are given under the null of no common bubbles and the alternative of a common bubble across these markets. We prove consistency of a factor bubble detection process for the origination and termination dates of the common bubble. Simulations show good finite sample performance of the testing algorithm in terms of its successful detection rates. Our methods are applied to real estate markets covering 89 major cities in China over the period January 2005 to December 2008. Results suggest the presence of a common bubble episode in what are known as China's Tier 1 and Tier 2 cities from June 2007 to February 2008. There is also a common bubble episode in Tier 3 cities but of shorter duration.

Keywords: Common Bubbles; Mildly Explosive Process; Factor Analysis; Date Stamping; Real Estate Markets.

JEL classification: C12, C13, C58

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1 Introduction

Financial bubbles are conventionally defined as explosive deviations of asset prices from market fundamentals followed by a subsequent collapse (Blanchard 1979, Diba & Grossman 1988, Evans 1991, Lee & Phillips 2016). There is now considerable accumulated empirical evidence of bubbles in historical records of financial asset prices, including equity, commodity, and real estate markets.[1] In a large-dimensional financial system, bubbles may arise concurrently in many of the variables in the system. For instance, using univariate bubble testing methods Pavlidis et al. (2016) found evidence of bubble presence in 22 international housing markets between 1975 and 2013, observing high synchronization in three of the bubble episodes. In a similar way using a univariate bubble detection technique, Narayan et al. (2013) discovered abundant evidence of bubbles in 589 firms listed on the NYSE over the period from 1998 to 2008. The detected bubble episodes were observed to appear in clusters according to financial sector. In a multi-country study using price-rent ratios, Engsted et al. (2016) uncovered substantial bubble synchronicity in the housing markets of 18 OECD countries in the early 2000s; and Greenaway-McGrevy et al. (2019) found evidence supporting the presence of a common explosive factor in house prices for 16 cities in two countries (Australia and New Zealand) over the period 1986-2015.

The focus of the current paper is the econometric detection of a common factor underlying the presence of bubbles that appear in a large-dimensional financial system. While evidence of a potential common bubble factor appeared in the empirical work of Greenaway-McGrevy et al. (2019) such phenomena have not been analyzed in the factor modeling literature. In consequence, there are no formal tests, dating schemes, or asymptotic theory available for use in estimation and inference concerning bubble factor detection. A common bubble factor refers to the circumstance that the dynamics of asset prices within a financial system are dominated by a pervasive common explosive factor, in the sense that the number of nonzero loadings for the common explosive factor passes to infinity as the number of assets $N \to \infty$. This formulation allows for a finite number (or small infinity) of assets in the system to have zero loading on the explosive factor, so these assets are unaffected by the common bubble. The concept of a common bubble factor is related to the idea of co-explosiveness in autoregressive models (with either distinct or common explosive roots) that has been studied in Magdalinos & Phillips (2009), Chen et al. (2017), Nielsen (2010), Phillips & Magdalinos (2013). But unlike the concept of a common bubble factor, the number of variables in co-explosive systems is finite and all variables in these systems display explosive dynamics. The goal of the present paper is to provide econometric methods to test for the presence of a common bubble factor that may be determining dominant time series behavior in a large-

[1]Amongst a large and growing literature, see Phillips et al. (2011), Phillips & Yu (2011), Gutierrez (2012), Kivedal (2013), Phillips & Yu (2013), Etienne et al. (2014*a*,*b*), Phillips et al. (2015*a*,*b*), Caspi et al. (2018), Adämmer & Bohl (2015), Figuerola-Ferretti et al. (2015, 2019), Harvey et al. (2015, 2016*a*), Pavlidis et al. (2016), Caspi (2016), Shi et al. (2016), Shi (2017), Greenaway-McGrevy & Phillips (2016), Hu & Oxley (2017*a*,*b*,*c*), Pavlidis & Peel (2017), Hu & Oxley (2018*a*,*b*), Phillips et al. (2018), Milunovich et al. (2019), Whitehouse (2019), Phillips & Shi (2020).

dimensional system and to date-stamp the origination of this common bubble.

The presence of asset price bubbles and potential commonality in bubble behavior across assets have important policy implications. Markets subject to common bubbles are extremely vulnerable to negative shocks and are exposed to the risk of system-wide failure, thereby entailing higher systemic risk (Brunnermeier & Oehmke 2013). In contrast, bubbles that occur independently in different markets without linkage or contamination seem likely to cause less system-wide damage. The procedures proposed in the present paper are intended to enable early identification of speculative behavior governed by a common latent factor that may expose financial markets to such system-wide risk. In addition, estimates of common explosive factors facilitate investigation of the underlying driving forces which produce this behavior and thereby offer potential guidance to governments and financial institution regulators in crafting policy to maintain economic and financial stability.

The identification of common bubble behavior also has important implications for the conduct of inference. Nielsen (2010) and Phillips & Magdalinos (2013) showed that maximum likelihood estimation of a vector autoregressive model is inconsistent when there are common explosive roots. Furthermore, the maximum likelihood estimator of co-explosive VAR models follows a mixed-normal limit distribution with Cauchy-type tail behavior rather than a normal distribution. To address the inconsistency, Phillips & Magdalinos (2013) propose an instrumental variable procedure for the consistent estimation of VAR models when the system contains co-explosive variables.

It is always possible to run univariate tests separately for bubble identification in each individual time series. But the presence of a common bubble characteristic across several time series, such as real estate prices in multiple regions or different metropolitan areas, is collective information of importance in understanding the phenomena and in assisting regulators to frame discretionary monetary policy. Cross section information from multiple time series is also necessary for identifying common bubbles. Furthermore, it is well known that the probability of making a false positive inference increases dramatically when univariate tests are applied repeatedly (in this case to a large number of assets), a phenomenon that is referred to as the multiplicity issue in the statistics literature.

The econometric procedure we propose here uses a factor model framework and involves two steps in the process of detecting a latent common bubble in the panel. In the first step we estimate the dominant common factor using a principal component (PC hereafter) approach. Factor estimation methods have been extensively used in applied economic research and asymptotic theory has been developed for stationary factor models in Bai & Ng (2002), Bai (2003), the I(1) factor model in Bai (2004), and most recently a mixed dynamic factor model with explosive, I(1), and stationary components in Chen et al. (2019). The latter work is most relevant for the present study.

The second step in our procedure applies the recursive explosive root testing algorithm of Phillips, Shi and Yu (2015a, 2015b, PSY hereafter) to the estimated dominant factor.

The PSY procedure is a commonly used bubble detection technique and has the capacity to consistently estimate bubble origination and termination dates (Phillips et al. 2015*b*).[2] The test statistic used here to detect a common factor bubble and provide date-stamping is referred to as a PSY-factor testing algorithm. Under the null hypothesis that there is no common bubble, asset prices are assumed to be driven by an I(1) factor and an idiosyncratic error term. The limit distribution of the PSY-factor test statistic under this null is shown to be the same as that of the original PSY statistic, although the derivation of this result is complicated by the additional step of factor estimation.

The alternative hypothesis allows for the presence of a common bubble factor in a subsample of the panel. In this formulation the initial trajectory is governed by an I(1) factor, representing a period of market normalcy. The middle trajectory is driven by an explosive factor and an I(1) factor. This phase represents a period of market abnormality in relation to fundamentals that is characterized by speculative behavior. The last part of the trajectory is governed by a stationary process, which represents the phase following the speculative bubble collapse. The estimated dominant first factor turns out to be a weighted average of the I(1), the explosive factor, the stationary factor, and idiosyncratic errors, with weightings that depend on the estimated factor loadings. Under certain regularity conditions, we show that the PSY-factor test statistic diverges to positive infinity for observations in the expansion phase. During the collapsing phase, the test statistic diverges to either positive or negative infinity (depending on the relative 'strength' of expansion and collapse) at a rate that is slower than that in the bubble expansion phase. So the presence of a common speculative component in the data that is embodied in the explosive factor is identified and the procedure is shown to consistently estimate the origination and termination dates of the common bubble.

Simulations are used to compare the asymptotic and finite sample distributions of the test statistic and investigate the successful detection rate (SDR),[3] and the estimation accuracy of the common bubble origination and collapse dates under various parameter settings. The results suggest satisfactory performance of the procedure in finite samples of the size typically used in empirical studies. As an empirical illustration of the methodology, we apply the common bubble detection procedure to real estate markets of 89 cities in China over the time period 2005 to 2008. One episode of common explosive behavior in real estate prices running from June 2007 to February 2008 is detected in 30 so-called Tier 1 and Tier 2 Chinese cities. A common bubble is evident also among the remaining 59 Tier 3 cities over the period from August 2007 to October 2008.

The present paper is related to work by Horie & Yamamoto (2016), who investigated whether the source of explosiveness lies in the common or idiosyncratic component. Our paper allows for explosiveness in the common component but not in the idiosyncratic com-

[2]The PSY approach has been used in many different empirical applications, including investigations to identify ballooning sovereign risks (Phillips & Shi 2019) and as a baseline technique for other bubble identification methods (Pavlidis et al. 2017, Shi & Phillips 2021).

[3]Successful detection occurs when the test indicates the presence of a common bubble in the data and the estimated origination date occurs on or after the true origination date.

ponent, reasoning that there may be a common driver to exuberant or speculative behavior that can be isolated empirically. Our framework allows for a structural break in the common component through a DGP that accommodates a three regime structure. The paper provides a test for explosive behavior in the common component, coupled with a date stamping algorithm to characterize the timing of explosiveness within these regimes. Our empirical application focuses attention on timing to detect when the emergence of market exuberance affects the system and leads to explosive behavior in asset prices initiating a pricing bubble. The approach by Horie & Yamamoto (2016) has a different focus, seeking to determine whether there is a nationwide bubble manifesting in the common component or whether there are local bubbles manifesting in the idiosyncratic components.

The rest of the paper is organized as follows. Section 2 describes the model specifications used for the null and alternative hypotheses. The econometric procedure for common bubble detection is introduced in Section 3. Section 4 provides the asymptotic properties of the test statistic under both the null and the alternative and shows the consistency of the estimated bubble origination and collapsing dates. Section 5 reports the results of the simulations investigating the finite sample performance of the procedure. The application to real estate markets in China is conducted in Section 6. Section 7 concludes. Proofs are collected in Appendices A-E. Appendix F contains tables and figures.

2 Model Specifications

A commonly used definition of bubble phenomena in financial markets is given by the present value identity

$$P_t = \sum_{s=0}^{\infty} \rho^s \mathbb{E}_t \left(R_{t+s} \right) + B_t, \tag{2.1}$$

where P_t is the price of the asset, R_t is the payoff received from the asset (i.e., rent for houses and dividends for stocks), and $\rho \in (0, 1)$ is the discount factor. The bubble component B_t satisfies the submartingale property[4]

$$\mathbb{E}_t \left(B_{t+1} \right) = \frac{1}{\rho} B_t, \tag{2.2}$$

which is a defining characteristic that assists in the development of econometric procedures of estimation and testing. Readers are referred to Phillips et al. (2011), Lee & Phillips (2016) and Shi & Phillips (2021) for more details on bubble process definitions, decision horizons, and generating mechanisms suited to empirical research.

In the absence of bubble phenomena, asset prices are governed by asset returns and

[4] Empirical experience shows that bubble processes typically collapse or stabilize periodically. Several models and data generating processes that satisfy the property of (2.2) are shown capable of generating periodically collapsing bubbles. See, for example, Blanchard & Watson (1982), Evans (1991) and Phillips (2016).

prevailing market conditions, often referred to as 'market fundamentals', and are commonly believed to be at most I(1). Conversely, in the presence of bubbles, B_t dominates the dynamics of asset prices and leads to explosive behavior of the data series P_t, consonant with the martigale property (2.2). Since the bubble component is itself unobserved, explosive root tests are conventionally applied to price-to-dividend or price-to-rent ratios for bubble identification. The dividend or rent series then serves as a proxy for market fundamentals. In our empirical application, rents are replaced by disposable incomes due to the unavailability of rent data.

We start the analysis with a simple model specification that differentiates normal and abnormal market behavior. In the absence of a common bubble factor (the null hypothesis), asset prices are assumed to be driven by an I(1) common factor and an idiosyncratic error, whereas in the presence of common speculative behavior (the alternative hypothesis), prices are determined by an I(1) factor, a mildly explosive factor, and an idiosyncratic term. The mildly explosive factor allows for mild deviations from unit root I(1) behavior in the explosive direction and have been found useful in analyzing potentially explosive processes. Autoregressive models with such mildly explosive roots have been extensively studied and utilized in empirical research following Phillips & Magdalinos (2007*a*).

2.1 Under the Null: No Common Bubble

In this case with no common bubble, dynamics for the asset price processes X_{it} are governed by market fundamentals so that

$$X_{it} = f_{0,t}\lambda_{0,i} + e_{it}, \tag{2.3}$$

where $f_{0,t}$ follows a unit root process

$$f_{0,t} = f_{0,t-1} + u_{0,t}. \tag{2.4}$$

The factor $f_{0,t}$ is assumed to capture the fundamental drivers of asset prices in normal market conditions subject to idiosyncratic errors e_{it}, which represent market variations.

In observation matrix form the model (2.3) can be rewritten as

$$X = F_0\Lambda_0' + E, \tag{2.5}$$

where $X = (\underline{X}_1 \ldots, \underline{X}_N)$ is an $T \times N$ matrix of the observed data with $\underline{X}_i = (X_{i1}, \ldots, X_{iT})'$, $F_0 = (f_{0,1} \ldots, f_{0,T})'$ is a $T \times 1$ vector, $\Lambda_0 = (\lambda_{0,1} \ldots, \lambda_{0,N})'$ is an $N \times 1$ vector of loading coefficients, and $E = (\underline{e}_1 \ldots, \underline{e}_N)$ is an $T \times N$ matrix of idiosyncratic errors with $\underline{e_i} = (e_{i1}, \ldots, e_{iT})'$. At time t

$$X_t = \Lambda_0 f_{0,t} + e_t, \tag{2.6}$$

where $X_t = (X_{1t}, \ldots, X_{Nt})'$ and $e_t = (e_{1t}, \ldots, e_{Nt})'$. For each i, we have

$$\underline{X_i} = F_0\lambda_{0,i} + \underline{e_i}.$$

2.2 Under the Alternative: Common Bubble Presence

Under the alternative hypothesis that there is a common bubble episode during the period of observation, asset prices are assumed to follow the factor dynamic mechanism

$$X_{it} = \begin{cases} f_{0,t}\lambda_{0,i} + e_{it} & \text{if } t \in A \\ f_{1,t}\lambda_{1,i} + f_{0,t}\lambda_{0,i} + e_{it} & \text{if } t \in B \\ f_{2,t}\lambda_{2,i} + e_{it} & \text{if } t \in C \end{cases}, \tag{2.7}$$

for $i = 1, \ldots, N$ and $t = 1, \ldots, T$, where $A = [1, T_e]$, $B = [T_e + 1, T_c]$, and $C = [T_c + 1, T]$ with $T_e = \lfloor \tau_e T \rfloor$ and $T_c = \lfloor \tau_c T \rfloor$. As in (2.5) we use $\Lambda_0 = (\lambda_{0,1} \ldots, \lambda_{0,N})'$ and define $\Lambda_1 = (\lambda_{1,1} \ldots, \lambda_{1,N})'$, $\Lambda_2 = (\lambda_{2,1} \ldots, \lambda_{2,N})'$.

The break points τ_e and τ_c are unknown in practice. The $\{f_{0,t}\}_1^T$ factor follows a unit root process as in (2.4), and the speculative-phase factor $\{f_{1,t}\}_{T_e+1}^{T_c}$ is assumed to follow an autoregressive process with a mildly explosive root (Phillips & Magdalinos 2007*a*) such that

$$f_{1,t} = \rho_T f_{1,t-1} + u_{1,t}, \tag{2.8}$$

where $\rho_T = 1 + \frac{d_1}{T^\alpha}$ with rate parameter $\alpha \in (0, 1)$ and localizing coefficient $d_1 > 0$. The larger the value α, the slower the rate of bubble expansion. The factor $\{f_{2,t}\}_{T_c}^T$ follows an autoregressive process with a mildly stationary root (Phillips & Magdalinos 2007*a*) such that

$$f_{2,t} = \phi_T f_{2,t-1} + u_{2,t} \tag{2.9}$$

where $\phi_T = 1 - \frac{d_2}{T^\beta}$ with $\beta \in (0, 1)$ and $d_2 > 0$. The smaller the value β, the faster the bubble collapses. When $\alpha > \beta$ (respectively $\alpha < \beta$), the rate of bubble expansion is slower (faster) than the rate of the bubble collapse.

The initial value $f_{0,0}$ is assumed to be $O_p(1)$. The bubble factor $f_{1,t}$ is assumed to emerge at some period $T_r = \lfloor r_0 T \rfloor$ with $r_0 \in [0, \tau_e]$ and represents emergent positive sentiment about the market that translates into market exuberance when this sentiment enters into the price determination system at $T_e + 1$. Similar assumptions on the initiation of second regimes are commonly made in structural break models (e.g. Perron & Zhu (2005)). This market exuberance impact on prices lasts until T_c, at which point negative market sentiment overtakes the price determination process, producing a bubble collapse regime that runs from $T_c + 1$ to the end of the sample period T.

The initial value f_{1,T_r} of the bubble factor is assumed to be $F_{1,r}T^{\alpha/2}$ for some $O_p(1)$ random variable $F_{1,r}$ so that $f_{1,T_r} = O_p\left(T^{\alpha/2}\right)$. It can easily be verified from the analysis in Phillips & Magdalinos (2007*a*) that the order of magnitude of the explosive factor at the break point f_{1,T_e} is then $O_p\left(T^{\alpha/2}\rho_T^{T_e - T_r}\right)$, which reduces to $O_p\left(T^{\alpha/2}\right)$ if the initial point coincides with the break date (i.e., $T_e = T_r$). This setting of the initial value is similar to, but slightly less restrictive than, that of Phillips & Magdalinos (2007*a*), where the order of the initial value of the mildly explosive process is assumed to be $o_p\left(T^{\alpha/2}\right)$. As in Phillips & Shi (2018), the initial value of the collapse factor f_{2,T_c} is set to be the same order of magnitude as the termination of the bubble factor $O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\right)$, i.e., $f_{2,T_c}=F_{2,c}T^{\alpha/2}\rho_T^{T_c-T_r}$ for some $O_p(1)$ random variable $F_{2,c}$.

Further, to avoid sudden jumps between regimes A and B we normalize $f_{1,t}$ to zero at the onset of regime B, i.e., $f_{1,\tau_e}=0$. There are two different ways to ensure a smooth transition from regime B to regime C. One is to impose the restrctions

$$c_0=\frac{\lambda_{0,i}}{\lambda_{2,i}},\ c_1=\frac{\lambda_{1,i}}{\lambda_{2,i}},\ \text{and}\ f_{2,T_c}=f_{0,T_c}c_0+f_{1,T_c}c_1, \tag{2.10}$$

where c_0 and c_1 are constants. Factor loadings are assumed to change in the same fashion for all assets when moving from one regime to another. In the special case of $c_0=c_1=1$, we have $\lambda_{0,i}=\lambda_{1,i}=\lambda_{2,i}$.[5] Another approach is to allow the loadings to change 'freely' and generate observations in regime C with a two-step procedure. First, generate

$$X_{it}^*=f_{2,t}\lambda_{2,i}+e_{it}^* \tag{2.11}$$

for $t=T_c,\cdots,T$, where e_{it}^* has the same properties as e_{it}. Next, normalize X_{it}^* to X_{iT_c} at period T_c and let $X_{it}=X_{it}^*-X_{iT_c}^*+X_{iT_c}$ for $t=T_c,\cdots,T$. The restriction is imposed on the aggregated data series X_{it}.

The idiosyncratic errors e_{it} in (2.7) may be serially correlated for each i. The factor specification error vector $u_t=(u_{0,t},u_{1,t},u_{2,t})'$ is taken to be $iid(0,\Sigma_u)$, in accordance with market efficiency in the first regime, followed by market exuberance and bubble collapse in the last two regimes. Further details on the error conditions are given in the assumptions in Section 4, where broader conditions are discussed.

It is convenient to represent the model (2.7) in matrix form as

$$X=G\Gamma'+E, \tag{2.12}$$

where $G=[g_1,g_2,\ldots,g_T]'$ is a $T\times 3$ matrix, with

$$g_t'=[g_{1t},g_{2t},g_{3t}]=\begin{cases}[0,0,f_{0,t}], & \text{if } t\in A\\ [0,f_{1,t},f_{0,t}], & \text{if } t\in B\\ [f_{2,t},0,0], & \text{if } t\in C\end{cases} \tag{2.13}$$

and $N\times 3$ matrix $\Gamma=[\gamma_1,\gamma_2,\ldots,\gamma_N]'$ with $\gamma_i=(\gamma_{i1},\gamma_{i2},\gamma_{i3})'$ for $i=1,\ldots,N$. The matrix G can be rewritten as $G=[G_1,G_2,G_3]$ with

$$G_1=(g_{11},\ldots,g_{1T})'=(0,\ldots,0,0,\ldots,0,f_{2,T_1+1},\ldots,f_{2,T})',$$

[5] Since $f_{0,T_c}=O_p(T^{1/2})$ and $f_{1,T_c}=O_p(T^{\alpha/2}\rho_T^{T_c-T_r})$ from Lemmas A.1 and A.3, the linear combination of f_{1,T_c} and f_{0,T_c} is $O_p(T^{\alpha/2}\rho_T^{T_c-T_r})$, which is consistent with the assumption on the initial value of $f_{2,t}$.

$$G_2 = (g_{21}, \ldots, g_{2T})' = (0, \ldots, 0, f_{1,T_0+1}, \ldots, f_{1,T_1}, 0, \ldots, 0)',$$
$$G_3 = (g_{31}, \ldots, g_{3T})' = (f_{0,1}, \ldots, f_{0,T_0}, f_{0,T_0+1}, \ldots, f_{0,T_1}, 0, \ldots 0)'.$$

The factor loading matrix in (2.12) is $\Gamma = [\Gamma_1, \Gamma_2, \Gamma_3]$ with $\Gamma_1 = \Lambda_2$, $\Gamma_2 = \Lambda_1$, and $\Gamma_3 = \Lambda_0$.

3 Econometrics of Common Bubble Identification

The proposed procedure consists of two steps. First, the leading common factor is estimated by principal components. With the above model specifications, the estimated first factor is at most I(1) under the null of no common bubbles (Lemma 4.1) but is explosive in the presence of the speculative bubbles (Lemma 4.4). As such, detecting common bubbles is equivalent to distinguishing a martingale first factor from observed explosive processes. In the second step we apply the PSY procedure to the estimated first factor to ascertain whether the leading factor manifests mildly explosive behavior.

3.1 Estimation of the First Common Factor

We estimate the first common factor using the following procedure. Assume the true number of factors is r for the data $\{X_{it}\}$ with $i = 1, \ldots, N$, and $t = 1, \ldots, T$. Recall that in the present framework $r = 1$ under the null model (2.3) and $r = 3$ under the alternative (2.7). Denote the common factors by the vector ξ_t $(r \times 1)$ and the corresponding factor loadings by l_i $(r \times 1)$. The objective function in the PC analysis is

$$\min_{\Xi, L} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(X_{it} - l_i' \xi_t\right)^2, \tag{3.1}$$

where $\Xi = (\xi_1, \ldots, \xi_T)'$ is a $T \times r$ matrix and $L = (l_1, \ldots, l_N)'$ is an $N \times r$ matrix. We impose a normalization condition on the loadings such that

$$\frac{1}{N} L'L = I_r. \tag{3.2}$$

The resulting solution for the factor loading, denoted by $\tilde{L}$, is $\sqrt{N}$ times the eigenvector matrix corresponding to the largest r eigenvalues (denoted by V_{NT}) of the $N \times N$ matrix $X'X$. The estimated r factors, denoted by $\tilde{\Xi} = \left(\tilde{\xi}_1, \ldots, \tilde{\xi}_T\right)'$, are

$$\tilde{\Xi} = X\tilde{L}\left(\tilde{L}'\tilde{L}\right)^{-1} = X\tilde{L}/N.$$

It is sufficient[6] to obtain the first common factor for the purpose of bubble identifica-

[6] Lemma 4.4 below demonstrates that the estimated first common factor is a linear combination of the common factors in the system. Consistency of the estimated bubble origination and collapse dates is established in Theorems 4.5-4.8.

tion. We denote the estimated first common factor by $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_T)'$, and let $\tilde{L}_1 = \left(\tilde{L}_{11}, \tilde{L}_{21}, \ldots, \tilde{L}_{N1}\right)'$ be the corresponding estimated factor loading, so that

$$\tilde{y} = X\tilde{L}_1/N.$$

3.2 The PSY Procedure

We apply the recursive evolving procedure of PSY to the estimated first common component $\tilde{y}_t$ to identify explosive behavior and characterize its nature, in particular to date-stamp the origination of any bubble that may be present. The regression model used for this purpose is

$$\Delta\tilde{y}_t = \delta + \gamma\tilde{y}_{t-1} + v_t, \tag{3.3}$$

where v_t is the equation residual. Under the null hypothesis of no common bubbles (2.3), the estimated first factor $\tilde{y}_t$ is a consistent estimator (up to a transformation) of the true factor (Lemma 4.1) and hence is at most I(1). In contrast, under the alternative model (2.7) the estimated first factor is a linear combination of the three factors (Lemma 4.4) and hence explosive in the presence of the bubble factor $f_{1,t}$. As such, the null and alternative hypotheses of the PSY-factor test can be translated into the hypotheses $\mathcal{H}_0 : \gamma = 0$ and $\mathcal{H}_1 : \gamma > 0$.

In describing the PSY mechanism it is conventional to use fractional notation to represent observations within the sample. Suppose the observation of interest is τ. To infer the presence of a common bubble characteristic at period τ, PSY suggest applying regression (3.3) recursively to a group of structured subsamples. Let $\tau_{\min}$ be the minimum sample proportion required to initiate the regression (3.3). The starting date of the subsample regressions τ_1 varies between 0 and $\tau - \tau_{\min}$, while the termination date τ_2 of all subsamples is fixed on the observation of interest (i.e., $\tau_2 = \tau$). The Dickey-Fuller (DF) statistics obtained from these subsample regressions are represented in the sequence $\{DF_{\tau_1,\tau_2}\}$ and defined as

$$DF_{\tau_1,\tau_2} = \hat{\gamma}_{\tau_2,\tau_2}\left[\frac{T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2}{\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_2,\tau_2}\tilde{y}_{t-1}\right)^2}\right]^{1/2}, \tag{3.4}$$

where $T_1 = \lfloor T\tau_1 \rfloor$, $T_2 = \lfloor T\tau_2 \rfloor$, $T_w = T_2 - T_1 = \lfloor T\tau_w \rfloor$ with the floor function $\lfloor . \rfloor$ returning the integer part of the argument, and $\hat{\delta}_{\tau_1,\tau_2}$ and $\hat{\gamma}_{\tau_2,\tau_2}$ are subsample estimates of δ and γ obtained by OLS regression. Inference concerning the presence of a common bubble is then based on the supremum of the DF statistic sequence, which is denoted $PSY_\tau(\tau_{min})$ and defined as

$$PSY_\tau(\tau_{min}) = \sup_{\tau_1\in[0,\tau-\tau_{\min}],\tau_2=\tau}\left\{DF_{\tau_1,\tau_2}\right\}.$$

Let β_T be the significance level and $cv_{\beta_T}(\tau_{min})$ be the $100(1-\beta_T)\%$ critical value of the test.

For notational ease, we write the PSY test statistic and its critical value as PSY_τ and cv_{β_T} when there is no confusion. If a common bubble is detected, then its origination date, $\hat{\tau}_e$, is identified to be the first chronological observation where the test statistic sequence exceeds the critical value. That is,

$$\hat{\tau}_e = \inf_{\tau \in [\tau_{\min}, 1]} \left\{ \tau : PSY_\tau > cv_{\beta_T} \right\}.$$

The termination date, $\hat{\tau}_c$, is the first chronological observation after $\hat{\tau}_e$ that the test statistic falls below the critical value, i.e.,

$$\hat{\tau}_c = \inf_{\tau \in [\hat{\tau}_e, 1]} \left\{ \tau : PSY_\tau < cv_{\beta_T} \right\}.$$

4 Asymptotics

We start by stating assumptions on the common factors, loadings, and errors which assist in the development of the asymptotic theory. Throughout, the notation M is used to denote a positive constant whose value may change according to location, := and =: represent definitional equality, $\equiv$ denotes distributional equivalence, and $\Rightarrow$ signifies weak convergence on the relevant probability space. We assume that N and T pass to infinity at the same rate.

4.1 Model Assumptions

Assumption 4.1 ***(Common factors):*** *Define the filtration $\mathcal{F}_t = \sigma\{u_t, u_{t-1}, ...\}$ where $u_t = (u_{0,t}, u_{1,t}, u_{2,t})'$, and let $\{u_t, \mathcal{F}_t\}$ be a martingale difference sequence (mds) with $\mathbb{E}(u_t u_t' | \mathcal{F}_{t-1}) = \Sigma_u$,*

$$\Sigma_u = \begin{bmatrix} \sigma_{00} & \cdot & \cdot \\ \sigma_{10} & \sigma_{11} & \cdot \\ \sigma_{20} & \sigma_{21} & \sigma_{22} \end{bmatrix} > 0$$

and $\sup_t \mathbb{E} \|u_t\|^{2+\varsigma} \leqslant M$ for some $\varsigma > 0$ and for all $t \leqslant T$.

Assumption 4.2 ***(Factor loadings):***

(1) Under the null of no bubble factor, as in (2.3), deterministic loadings $\{\lambda_{0,i}\}$ are assumed to satisfy $|\lambda_{0,i}| \leqslant M$ and $\Lambda_0'\Lambda_0/N \to \Sigma_\Lambda$ as $N \to \infty$ where $\Sigma_\Lambda > 0$, while stochastic loadings are assumed to satisfy $\sup_i \mathbb{E} |\lambda_{0,i}|^4 \leqslant M$ with $\Lambda_0'\Lambda_0/N \to_p \Sigma_\Lambda > 0$ as $N \to \infty$.

(2) Under the alternative of a bubble factor, as assumed in model (2.12), deterministic loadings $\{\gamma_i\}$ are assumed to satisfy $|\gamma_{ii}| \leqslant M$, stochastic loadings to satisfy $\sup_i \mathbb{E} |\gamma_i|^4 \leqslant M$, and the loading moment matrix

$$\Gamma'\Gamma/N = \frac{1}{N} \begin{bmatrix} \Lambda_2'\Lambda_2 & \cdot & \cdot \\ \Lambda_1'\Lambda_2 & \Lambda_1'\Lambda_1 & \cdot \\ \Lambda_0'\Lambda_2 & \Lambda_0'\Lambda_1 & \Lambda_0'\Lambda_0 \end{bmatrix} \to_p \Pi := \begin{bmatrix} \pi_{22} & \cdot & \cdot \\ \pi_{12} & \pi_{11} & \cdot \\ \pi_{02} & \pi_{01} & \pi_{00} \end{bmatrix},$$

which is positive definite, with elements $\frac{1}{N}\Lambda_k'\Lambda_l \to \pi_{kl}$ *as* $N \to \infty$.

Assumption 4.3 ***(Time and cross section dependence and heteroskedasticity)****: For some number* $M < \infty$,

(1) $\mathbb{E}(e_{it}) = 0$, $\sup_{i,t} \mathbb{E}|e_{it}|^8 \leqslant M$;

(2) $\mathbb{E}\left(\underline{e}_i'\underline{e}_j/T\right) = \mathbb{E}\left(\frac{1}{T}\sum_{t=1}^{T} e_{it}e_{jt}\right) = \gamma_T(i,j)$ *and*

$$\sup_{N\geqslant 1} \frac{1}{N}\sum_{i,j=1}^{N} |\gamma_T(i,j)| \leqslant M;$$

(3) $\mathbb{E}(e_{it}e_{jt}) = \tau_{ij,t}$ *with* $|\tau_{ij,t}| \leq |\tau_{ij}|$ *for some* τ_{ij} *and for all* t, *and* $\frac{1}{N}\sum_{i,j=1}^{N} |\tau_{ij}| \leqslant M$;

(4) $\mathbb{E}(e_{it}e_{js}) = \tau_{ij,ts}$ *and* $\sup_{N\geqslant 1, T\geqslant 1} \frac{1}{NT}\sum_{i,j=1}^{N}\sum_{s,t=1}^{T} |\tau_{ij,ts}| \leqslant M$;

(5) For every (i,j), $\sup_{T\geqslant 1} \mathbb{E}\left|T^{-1/2}\sum_{t=1}^{T}\left[e_{it}e_{jt} - \mathbb{E}(e_{it}e_{jt})\right]\right|^4 \leqslant M$.

Assumption 4.4 *(1)* $\sup_{i\geqslant 1} |\frac{1}{T}\sum_{t=1}^{T} f_{0,t-1}e_{it}| = O_p(1)$ *as* $T \to \infty$;

(2) $\sup_{i\geqslant 1} |\frac{1}{T^{\alpha}\rho_T^T}\sum_{t=1}^{T} f_{1,t-1}e_{it}| = O_p(1)$ *as* $T \to \infty$.

(3) $\sup_{i\geqslant 1} |\frac{1}{T^{(\alpha+\beta)/2}\rho_T^T}\sum_{t=1}^{T} f_{2,t-1}e_{it}| = O_p(1)$ *as* $T \to \infty$.

Assumption 4.5 $\{\lambda_i\}, \{u_t\}$, *and* $\{e_{it}\}$ *are mutually independent.*

Assumption 4.1 concerns the common factor errors $u_t = \{u_{0,t}, u_{1,t}, u_{2,t}\}$ which are assumed to be *mds* with uniform $2+\zeta$ moments. This condition is convenient, treating the component errors $\{u_{0,t}, u_{1,t}, u_{2,t}\}$ in the three periods commonly. It may be relaxed to allow (i) *mds* errors $\{u_{0,t}\}$ during the efficient market period, and (ii) more general weak dependence for $\{u_{1,t}\}$ during the explosive period and for $\{u_{2,t}\}$ during the collapse period, as in Phillips & Magdalinos (2007*b*) and Magdalinos & Phillips (2009). No distributional assumptions are needed and the uniform moment condition is weak, so the methods proposed can be applied widely in empirical work, including to financial market data.

Assumption 4.2 concerns the loading coefficients, whose moment matrices $\Lambda_0'\Lambda_0/N$, $\Gamma'\Gamma/N$ are assumed to converge to positive definite matrices as $N \to \infty$, a condition which helps to ensure identifiability of the factor structures. So, if a factor had only a finite number of nonzero loadings, it would not be treated as a common factor in our framework but would instead be absorbed within the idiosyncratic errors e_{it}.

Assumption 4.3 allows for time and cross section dependence and conditional heteroskedasticity, as in Bai (2004). Assumption 4.4 requires the uniform boundness over i of the time series sample covariances between $f_{0,t-1}$ and e_{it} and is stronger than simply requiring weak convergence of such sample covariances for all i as in Bai (2004). In addition, we assume uniform boundness over i of the time series sample covariances between $f_{1,t-1}$ and e_{it}, and between $f_{2,t-1}$ and e_{it}. The independence between u_t and e_{it} in Assumption 4.5 eliminates endogeneity in our framework, just as in the cointegrated factor model of Bai (2004). Under the null hypothesis, the situation is analogous to that of Bai (2004) with an integrated factor. In such cases, the model can be rewritten as a dynamic factor model by projection of e_{it} on u_t and suitably augmenting the regression equation, leading to a dynamic factor model as discussed in Bai (2004).[7] However, in our case under the alternative, the presence of a mildly explosive factor accommodates dependence between u_t, and e_{it} as shown in the cointegrating regression analysis of Magdalinos & Phillips (2009) with mildly explosive regressors. We therefore expect that the procedures for identifying and estimating the explosive factor in our framework retain validity under endogeneity, although formal analysis of this extension is not pursued in the present paper and left for subsequent work.

Additional assumptions used in the general setting of Bai (2004) are not required in the present paper. This is because in the model structure employed here there is no need to estimate the number of factors or to show uniform consistency of the estimated first factor.

4.2 Asymptotics Under the Null Hypothesis

The following Lemma shows consistency of the estimated first factor. This result is useful in developing an asymptotic theory of inference for quantities that relate to this estimated factor $\tilde{y}_t$. In particular, the theory is employed in deriving asymptotics for the bubble identification procedure.

Lemma 4.1 *Under the data generating process* (2.3) *and Assumptions 4.1-4.5, we have*

$$\delta_{NT}^2 \left(\frac{1}{T} \sum_{t=1}^{T} \left| \tilde{y}_t - H_{NT}^0 f_{0,t} \right|^2 \right) = O_p(1) \tag{4.1}$$

where $\delta_{NT} = \min\left(\sqrt{N}, T\right)$ *and* $H_{NT}^0 = \lambda_{NT} \left(\frac{F_0' F_0}{T^2} \right)^{-1} \left(\frac{\Lambda_0' \tilde{L}_1}{N} \right)^{-1}$ *with* λ_{NT} *being the largest eigenvalue of* $\frac{1}{NT^2} X'X$*.*

Lemma 4.1 reveals that the first factor can be identified up to a transformation given by H_{NT}^0. The proof of 4.1 follows directly as in Bai (2004) and Chen et al. (2019) and is given for convenience in the Online Supplement (Chen et al. 2021). While Bai (2004) shows consistency of factor estimates in the presence of $I(1)$ factors (and uniform consistency under stronger moment conditions) subject to a normalization condition for the factors of the form $\Xi' \Xi / T^2 = I_r$, Chen et al. (2019) provide consistency results under a factor model specification that includes an explosive factor as well as I(1) and stationary factors.

[7] In other work that does not involve explosive or nonstationary processes, Pesaran (2006) allows for endogeneity between the factor and the residuals by using cross section averages in a multifactor regression model.

Next, we develop asymptotics for a standard unit root test constructed from the first estimated factor, $\tilde{y}_t$, under the null (2.3).

Theorem 4.2 *Under the null specification (2.3) and 4.1-4.5, as $N, T \to \infty$,*

$$T\hat{\gamma}_{\tau_1,\tau_2} \Rightarrow \frac{\tau_w \int_{\tau_1}^{\tau_2} W(r)\, dW(r) - [W(\tau_2) - W(\tau_1)] \int_{\tau_1}^{\tau_2} W(r)\, dr}{\tau_w \int_{\tau_1}^{\tau_2} W(r)^2\, dr - \left[\int_{\tau_1}^{\tau_2} W(r)\, dr\right]^2}, \tag{4.2}$$

and

$$DF_{\tau_1,\tau_2} \Rightarrow \frac{\tau_w \int_{\tau_1}^{\tau_2} W(r)\, dW(r) - [W(\tau_2) - W(\tau_1)] \int_{\tau_1}^{\tau_2} W(r)\, dr}{\tau_w^{1/2}\left[\tau_w \int_{\tau_1}^{\tau_2} W(r)^2\, dr - \left[\int_{\tau_1}^{\tau_2} W(r)\, dr\right]^2\right]^{1/2}}, \tag{4.3}$$

where $\tau_w = \tau_2 - \tau_1$ and $W(\cdot)$ is standard Brownian motion.

Derivation of the asymptotic behavior of $T\hat{\gamma}_{\tau_1,\tau_2}$ and the DF statistic (3.4) follows standard lines. Although complicated by the fact that the test relies on the estimated factor, the derivation proceeds as usual because the fast convergence of $\tilde{y}_t - H_{NT}^0 f_{0,t}$ ensures that the limit distribution is unaffected by factor estimation and is identical to those of $T\hat{\gamma}_{\tau_1,\tau_2}$ and the DF statistic computed from the original data, as in Phillips et al. (2015*a*). An outline of the derivations is provided in Appendix B. The estimated coefficient $\hat{\gamma}_{\tau_1,\tau_2}$ converges to zero as $N, T \to \infty$. With these results in hand, the limit behavior of the PSY test applied to the fitted factor also follows in the standard manner (Phillips et al. 2015*a*,*b*).

Theorem 4.3 *Under the null specification of model (2.3) and 4.1-4.5, as $N, T \to \infty$,*

$$PSY_\tau(\tau_{min}) \Rightarrow \sup_{\tau_1 \in [0, \tau - \tau_{\min}], \tau_2 = \tau} \left\{ \frac{\tau_w \int_{\tau_1}^{\tau_2} W(r)\, dW(r) - [W(\tau_2) - W(\tau_1)] \int_{\tau_1}^{\tau_2} W(r)\, dr}{\tau_w^{1/2}\left[\tau_w \int_{\tau_1}^{\tau_2} W(r)^2\, dr - \left[\int_{\tau_1}^{\tau_2} W(r)\, dr\right]^2\right]^{1/2}} \right\}$$
$$=: \Upsilon_\tau(\tau_{min}) \tag{4.4}$$

The proof applies functional limit theory of the component elements of the statistic under the null and a version of continuous mapping applied to certain indexed functionals of these elements, just as in Theorem 1 of Phillips et al. (2015*a*). The limit result (4.4) for the PSY-factor test statistic is then identical to that of the original PSY statistic (i.e., $F_{r_2}(W, r_0)$ in Phillips et al. (2015*a*)). The details of the proof are provided in the Online Supplement (Chen et al. 2021).

4.3 Asymptotics Under the Alternative

We start with a useful representation of the first common factor under the alternative.

Lemma 4.4 *Under the alternative (2.7) and Assumption 4.2 and 4.3, the estimated first common factor has the form*

$$\tilde{y}_t = a_{N,T} g_{1t} + b_{N,T} g_{2t} + c_{N,T} g_{3t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} \tag{4.5}$$

$$= \begin{cases} c_{N,T} f_{0,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} & \text{if } t \in A \\ b_{N,T} f_{1,t} + c_{N,T} f_{0,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} & \text{if } t \in B \\ a_{N,T} f_{2,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} & \text{if } t \in C \end{cases}, \tag{4.6}$$

where $a_{N,T} = \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \lambda_{2,i}$*,* $b_{N,T} = \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \lambda_{1,i}$*, and* $c_{N,T} = \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \lambda_{0,i}$ *with* $\tilde{L}_{i1}$ *being the estimated loading of the first factor for asset* i*. Further,* $\{a_{N,T}, b_{N,T}, c_{N,T}\} \to_p \{a, b, c\}$ *with explicit expressions for the limit quantities* $\{a, b, c\}$ *given in* (C.9)-(C.11) *of Lemma C.4.*

Under the alternative (2.7), the estimated first factor is a weighted average of the component factors ($f_{0,t}$ for regime A, $f_{1,t}$ and $f_{0,t}$ for regime B, and $f_{2,t}$ for regime C) and the idiosyncratic errors. The weights depend on the estimated factor loadings $\tilde{L}_{i1}$ and the true loading coefficients $\lambda_{0,i}$, $\lambda_{1,i}$, and $\lambda_{2,i}$ and have explicit limits as $N, T \to \infty$ determined by primitive conditions on the factor loadings, as shown in Lemma C.4 of Appendix C.

Next, we derive the asymptotic properties of the unit root statistic DF_{τ_1,τ_2} under three regime settings: (1) $\tau_1 \in A$ and $\tau_2 \in B$; (2) $\tau_1 \in A$ and $\tau_2 \in C$; (3) $\tau_1, \tau_2 \in C$. The subsample in Case (1) starts from the normal regime and ends in the bubble expansion regime. In Case (2), the sample spans across all three regimes and includes two structural break points (τ_e and τ_c). In Case (3), the subsample falls completely in the collapse regime. The derivation of the limit properties is based on results in Lemmas C.4 - C.7.

Theorem 4.5 *Under the alternative (2.7) and Assumption 4.1-4.5, the following asymptotics hold as* $N, T \to \infty$*: when* $\tau_1 \in A$ *and* $\tau_2 \in B$*,*

$$\hat{\gamma}_{\tau_1,\tau_2} = \frac{d_1}{T^\alpha} + O_p\left(T^{-1}\right),$$

$$DF_{\tau_1,\tau_2} = T^{3/2-\alpha} \frac{d_1 r_w^{3/2}}{2} \left[1 + o_p(1)\right] = O_p\left(T^{3/2-\alpha}\right) \to +\infty. \tag{4.7}$$

According to Theorem 4.5, although there is a structural break within the subsample in Case (1), the bubble regime B dominates the normal regime and $\hat{\gamma}_{\tau_1,\tau_2}$ can be regarded as consistent for the deviation $\rho_T - 1$ in (2.8). The order of magnitude of the DF statistic depends asymptotically on the power parameter $\alpha \in (0, 1)$ that defines the magnitude of this local alternative and thereby the explosive strength of the factor transmitted through the autoregressive coefficient $\rho_T = 1 + d_1 T^{-\alpha}$, with explosive strength rising as α decreases towards zero. Correspondingly, the DF statistic diverges to positive infinity at the rate $O_p\left(T^{3/2-\alpha}\right)$ which increases according to explosive strength, measured by α.

Theorem 4.6 *Under the alternative (2.7) and Assumption 4.1-4.5, the following asymptotics hold when $\tau_1 \in A$ and $\tau_2 \in C$ as $N, T \to \infty$,*

$$\hat{\gamma}_{\tau_1,\tau_2} \sim \begin{cases} \frac{d_1}{T^{\alpha}}\left[1 - \frac{a^2}{b^2}\frac{F_{2,c}^2}{\left(F_{1,r}+N_{d_1}\right)^2}\right] = O_p\left(T^{-\alpha}\right) & \text{if } \alpha > \beta \\ -\frac{d_2}{T^{\beta}}\left[1 - \frac{b^2}{a^2}\frac{\left(F_{1,r}+N_{d_1}\right)^2}{F_{2,c}^2}\right] = O_p\left(T^{-\beta}\right) & \text{if } \alpha < \beta \end{cases},$$

where $N_{d_1} \sim \mathcal{N}\left(0, \frac{\sigma_{11}}{2d_1}\right)$, and the order magnitudes of the DF statistic are

$$\begin{aligned} DF_{\tau_1,\tau_2} &\sim \begin{cases} T^{(1-\alpha+\beta)/2} r_w^{1/2} \left(\frac{d_1}{d_2}\right)^{1/2}\left[\frac{b\left(F_{1,r}+N_{d_1}\right)}{aF_{2,c}} - \frac{aF_{2,c}}{b\left(F_{1,r}+N_{d_1}\right)}\right] & \text{if } \alpha > \beta \\ T^{(1+\alpha-\beta)/2} r_w^{1/2} \left(\frac{d_2}{d_1}\right)^{1/2}\left[\frac{b\left(F_{1,r}+N_{d_1}\right)}{aF_{2,c}} - \frac{aF_{2,c}}{b\left(F_{1,r}+N_{d_1}\right)}\right] & \text{if } \alpha < \beta \end{cases} && (4.8) \\ &= O_p\left(T^{(1-|\alpha-\beta|)/2}\right). && (4.9) \end{aligned}$$

When the sample period includes all three regimes, the limit properties including the signs of $\hat{\gamma}_{\tau_1,\tau_2}$ and DF_{τ_1,τ_2} depend on the relative rates of bubble expansion α and bubble collapse β. The estimated coefficient $\hat{\gamma}_{\tau_1,\tau_2}$ in the DF regression equation has order of magnitude $O_p\left(T^{-\alpha}\right)$ when $\alpha > \beta$ and $O_p\left(T^{-\beta}\right)$ when $\alpha < \beta$. Moreover, when the condition

$$a^2 F_{2,c}^2 > b^2 \left(F_{1,r} + N_c\right)^2 \tag{4.10}$$

is satisfied, the DF_{τ_1,τ_2} statistic diverges to negative infinity at rate $O_p\left(T^{(1-|\alpha-\beta|)/2}\right)$; otherwise it diverges to positive infinity at rate $O_p\left(T^{(1-|\alpha-\beta|)/2}\right)$. The condition in (4.10) matches the intuition that the collapsing regime plays a more prominent role in determining asymptotic behavior of the bubble test when the weight of the (squared) collapse factor a^2, and the (squared) initial value of the collapse regime, $F_{2,c}^2$, are large relative to the corresponding parameters of the expansion regime. In the opposite case where $a^2 F_{2,c}^2 < b^2 \left(F_{1,r} + N_c\right)^2$ the bubble test statistic DF_{τ_1,τ_2} diverges to positive infinity and the expansion regime dominates asymptotic behavior. In effect, the asymptotic outcome of the test depends on the strength of the collapse period parameters relative to those of the explosive period measured by the balancing of these parametric strengths via the inequality (4.10).

Theorem 4.7 *Under the alternative (2.7) and 4.1-4.5, the following asymptotics hold when $\tau_1, \tau_2 \in C$ as $N, T \to \infty$,*

$$\begin{aligned} \hat{\gamma}_{\tau_1,\tau_2} &= -\frac{d_2}{T^{\beta}} + O_p\left(T^{-1}\right), && (4.11) \\ DF_{\tau_1,\tau_2} &\sim -T^{3/2-\beta}\frac{d_2 r_w^{3/2}}{2}\left[1 + o_p(1)\right] = O_p\left(T^{3/2-\beta}\right) \to -\infty. && (4.12) \end{aligned}$$

When the subsample falls within the collapse regime, the DF statistic diverges to negative infinity. Under the alternative of model (2.7), the sample period has three regimes: A, B and

C, as defined in (2.7). Apart from Cases (1)-(3), there are potentially three other types of subsample regressions for the PSY procedure: (4) $\tau_1, \tau_2 \in A$; (5) $\tau_1 \in B$ and $\tau_2 \in C$; and (6) $\tau_1, \tau_2 \in B$. From Theorem 4.2, the order of magnitude of DF_{τ_1,τ_2} for Case (4) is $O_p(1)$. For Case (5), the estimated $\hat{\gamma}_{\tau_1,\tau_2}$ is a weighted average of $\frac{d_1}{T^\alpha}$ and $-\frac{d_2}{T^\beta}$, with slight changes in the weights. The orders of magnitude of DF_{τ_1,τ_2} under these scenarios are identical to those in (4.8). The order of magnitude of the DF statistic under Case (6) is identical to that of (4.7). The proofs for Case (5) and Case (6) follow directly from those of Theorem 4.6 and Theorem 4.5, and are omitted. Consequently, we have the following asymptotic behavior of the PSY statistic

$$PSY_\tau = \begin{cases} O_p(1) & \text{if } \tau \in A \\ O_p\left(T^{3/2-\alpha}\right) & \text{if } \tau \in B \\ O_p\left(T^{(1-|\alpha-\beta|)/2}\right) & \text{if } \tau \in C \end{cases}. \tag{4.13}$$

We are now able to deduce the limit behavior of the bubble origination date estimator $\hat{\tau}_e$ and the collapse date estimator $\hat{\tau}_c$ in the factor model (2.7).

Theorem 4.8 *Under the alternative (2.7), $\hat{\tau}_e \to \tau_e$ and $\hat{\tau}_c \to \tau_c$ if the divergence rate of the PSY critical value $cv_{\beta_T} \to \infty$ satisfies the following conditions:*

$$\frac{T^{(1-|\alpha-\beta|)/2}}{cv_{\beta_T}} + \frac{cv_{\beta_T}}{T^{3/2-\alpha}} \to 0.$$

Theorem 4.8 provides rate conditions on the localizing power cofficient α under which the bubble origination and collapse dates may be consistently estimated. The proof follows directly from Phillips et al. (2015*b*) and is given in the Online Supplement (Chen et al. 2021) for completeness. We draw attention to the fact that the orders of magnitude of the DF statistic under the various cases and conditions for the consistency of $\hat{\tau}_e$ and $\hat{\tau}_c$ differ slightly from those given in Phillips et al. (2015*b*). These differences arise from the distinct assumptions regarding the initialization of the explosive regime/factor. In the present work, the emergence of explosive sentiment is allowed to pre-date its impact on market prices, and for the reasons explained earlier, it is here assumed that $f_{1,T_r} = O_p(T^{\alpha/2})$, whereas the explosive regime of Phillips et al. (2015*b*) is assumed (implicitly) to start from a value of $O_p(T^{1/2})$. These differences lead to the results given in Theorem 4.8.

5 Simulations

We first compare the asymptotic and finite sample distributions of the test statistic PSY_τ. The asymptotic distribution is simulated from Υ_τ in (4.4) with 2,000 replications and standard Brownian motion is approximated using independent increments over $2,000$ steps. To obtain finite sample distributions, we generate data from (2.3)-(2.4). The factor loadings $\lambda_{0,i}$ are drawn randomly from a uniform distribution over the interval $[0, 1]$. The standard deviation of the idiosyncratic error σ_e is set to 0.1. These parameter settings are compatible with our later empirical application to Chinese housing markets.[8] Note that under the null hypothesis the parameter settings of $f_{0,0}$, σ_e, and σ_{00} do not affect the distribution of the ADF statistic, consistent with the results of Theorem 4.2. The common bubble detection procedure is applied to the simulated data with the initial sample size setting $\tau_{\min} = 0.01 + 1.8/\sqrt{T}$, as in Phillips et al. (2015a). The process is repeated for 2,000 replications.

Figure 1: Asymptotic and finite sample distributions of PSY_τ under the null hypothesis with $\tau = 1$ and $T \in \{60, 100, 140\}$.

(a) $T = 60$ (b) $T = 100$ (c) $T = 140$

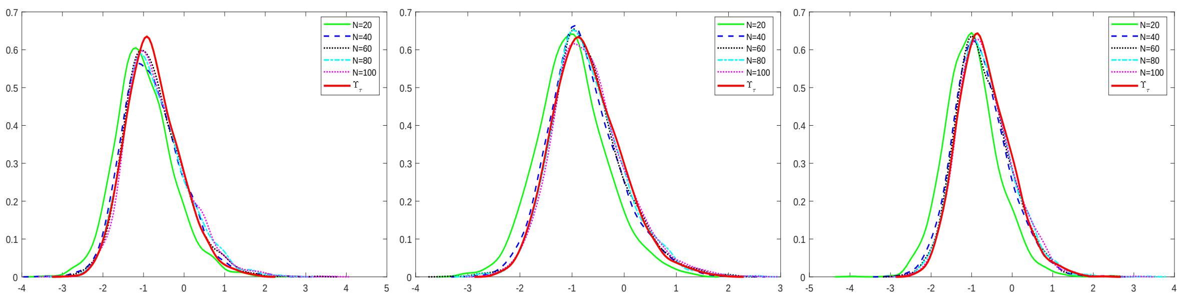


Figure 1 graphs the asymptotic and finite sample distributions (from kernel density estimates) of PSY_τ for $T = 60, 100, 140$ and with N varying from 20 to 100. The parameter τ is set to unity in all graphs and similar patterns were observed with other choices of τ. There is a small but visible gap between the finite sample distribution for $N = 20$ and the asymptotic distribution. Also, the finite sample distribution lies to the left of the asymptotic, which implies slight undersizing if asymptotic critical values are employed in bubble testing. The finite sample distribution evidently converges rapidly to Υ_1 as the number of cross sectional units N increases and the sample period T rises. We use finite sample critical values at the 5% significance level (i.e., the 95^{th} percentile of the finite sample distributions) for investigating the performance of the common bubble testing procedure.

The data generating process (DGP) is (2.7) - (2.9) under the alternative. As in the DGP under the null, the factor loadings $\lambda_{0,i}$, $\lambda_{1,i}$, and $\lambda_{2,i}$ are drawn randomly from $U[0, 2]$ and $\sigma_e = 0.1$. In addition, we set $\sigma_{00} = 0.01$, $\sigma_{11} = 0.1$, and $\sigma_{22} = 0.1$. The standard errors are calibrated to our Chinese housing market application. Specifically, we calibrate the $f_{0,t}$ process to the normal period in the estimated first factor from January 2005 to May 2007, the $f_{1,t}$ process to the fast expansion period (from June 2007 to February 2008), and the $f_{2,t}$ process to the collapse period from March 2008 to December 2008. The selections of the sample periods for $f_{0,t}$, $f_{1,t}$, and $f_{2,t}$ are guided by the empirical results. We estimate (2.8) and (2.9) by the indirect inference approach to reduce autoregressive biases as in Phillips et al. (2011). We fix the bubble origination date $\tau_e = 0.4$ and the bubble collapse date $\tau_c = 0.7$.

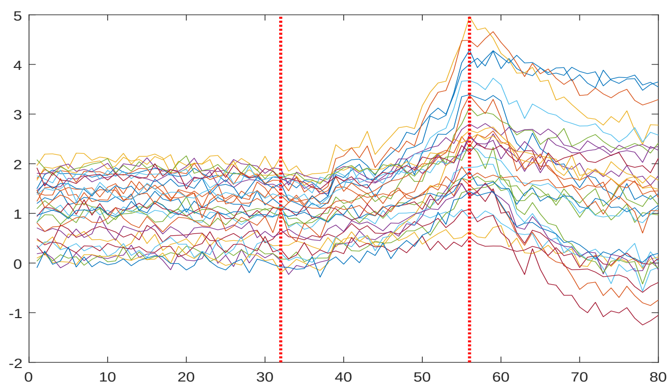
[8]The estimated loadings range between 0.3 and 1.7, while the estimated standard deviation of the idiosyncratic error term is around 0.1 for both Group I (Tier 1 and 2 cities) and Group II (Tier 3 cities).

The initial values of the I(1) and explosive factors are set to unity (i.e., $f_{0,0} = 1$ and $f_{1,T_r} = 1$). To avoid sudden dramatic jumps at the break point $T_e + 1$, we subtract the simulated $f_{1,t}$ for $t \in [T_r, T_c]$ by the value of f_{1,T_e} so that the explosive factor takes the value zero at period T_e. Together with the simulated loadings and idiosyncratic noise, the data X_t is generated from equation (2.7) for the period running from 1 to T_c. The initial value of the collapse factor f_{2,T_c} is set to be f_{1,T_c} and a data sequence $\{X_t^*\}_{t=T_c}^{T}$ is generated using (2.11). The remainder of the sequence X_{it} for $t = T_c + 1, \cdots, T$, is generated as

$$X_{it} = X_{it}^* - X_{iT_c}^* + X_{iT_c},$$

so that there is no discontinuity in the sequence. Figure 2 displays a typical realization of the data generating process under the specified alternatives with $\alpha = 0.8$, $\beta = 0.7$, $N = 30$, and $T = 80$.

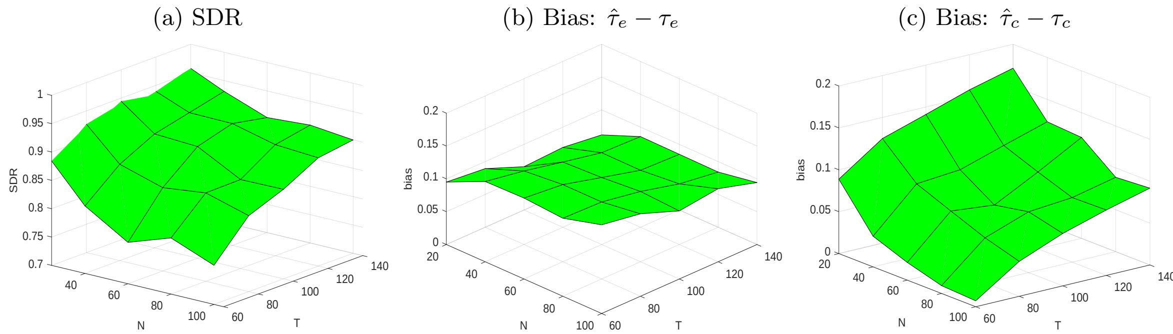
Figure 2: One typical realization of the data generating process under the alternative. Parameter settings are: $f_{0,0} = f_{1,T_r} = 1, \sigma_e = 0.1, \sigma_0 = 0.01, \sigma_1 = \sigma_2 = 0.1, \tau_e = 0.4, \tau_c = 0.7, \alpha = 0.8, \beta = 0.7, r_0 = \tau_e - 0.05, N = 30$ and $T = 80$. The vertical lines indicate the start and collapse dates of the common bubble episode.



We report the successful detection rates (i.e., the proportions of replications where the estimated bubble origination date falls between the true start and end dates), and the average bias of the estimated bubble origination (i.e., $\frac{1}{2000}\sum_{s=1}^{2000} \hat{\tau}_e^s - \tau_e$) and collapse dates (i.e., $\frac{1}{2000}\sum_{s=1}^{2000} \hat{\tau}_c^s - \tau_c$) under the alternative. The number of replications is 2,000 in all simulations. In Figure 3, we allow the time period T and the number of assets N to take various values. Specifically, we have $T = \{60, 80, 100, 120, 140\}$ and $N = \{20, 40, 60, 80, 100\}$. The bubble expansion α and collapsing rates β are fixed and set to be 0.8 and 0.7, respectively.

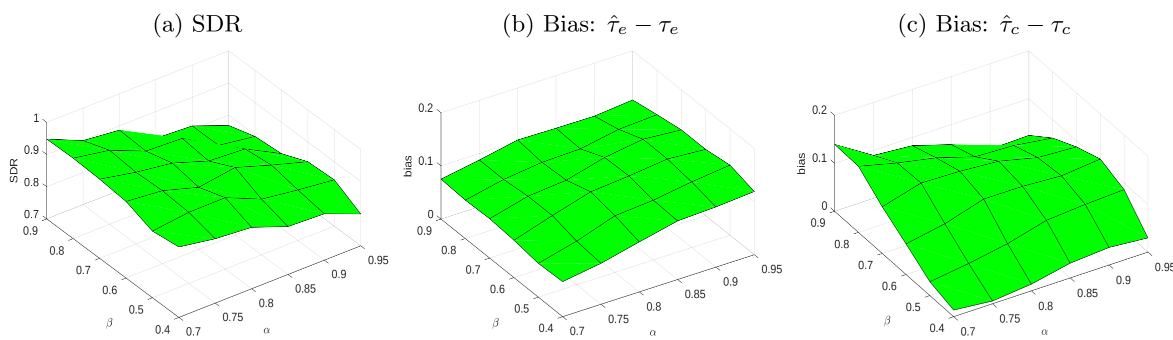
The following comments are in order. First, as the time span T lengthens, the SDR of the PSY-factor procedure increases and the bias of $\hat{\tau}_e$ reduces substantially. Additional time dimension information therefore lends considerable assistance in identifying explosive dynamics. Second, the SDR declines and the bias of $\hat{\tau}_e$ becomes more significant as N increases. The more cross-sectional units, the noisier the data and hence the harder for the PSY-factor procedure to identify the origination date of the common bubble. Third, the bias of the termination date is more considerable when there are fewer assets and the time span is longer. As an example, the bias of the estimated collapse date is 0.7% of the sample period when $N = 100$ and $T = 60$, while it is 17% when $N = 20$ and $T = 140$.

Figure 3: The successful detection rates and bias of the estimated bubble origination and termination points. Parameter settings: $\tau_e = 0.4$, $r_0 = \tau_e - 0.05$, $\tau_c = 0.7$, $\alpha = 0.8$, and $\beta = 0.7$.



In Figure 4, we fix T and N but allow the explosive rate α and the collapse rate β to take various values. The rate α changes from 0.7 (i.e., $\rho_T = 1.040$) to 0.95 (i.e., $\rho_T = 1.013$) with increments of 0.05, while β takes value between 0.4 (i.e., $\gamma_T = 0.842$) and 0.9 (i.e., $\gamma_T = 0.984$) with increments of 0.1. The rates of bubble expansion and collapse increase as α and β decrease, respectively. As expected, it is much easier to detect episodes that expand at a greater rate (i.e., when α is further below unity). From panel (a) and (b), we see that as α becomes smaller, the SDR rises rapidly and the bias of the estimated origination date reduces. The collapse rate β does not seem to have any obvious impact on SDR and the estimation accuracy of bubble origination. The bias of the origination date ranges between 0.06 and 0.13. Interestingly, we see a nonlinear pattern for the bias of the estimated termination date $\hat{\tau}_c$, which varies between 0.005 and 0.14. The most accurate estimate of termination is obtained when the bubble expands quickly and collapses rapidly (i.e., $\alpha = 0$ and $\beta = 0.4$).

Figure 4: The successful detection rates and bias of the estimated bubble origination and termination. Parameter settings: $\tau_e = 0.4$, $r_0 = \tau_e - 0.05$, $\tau_c = 0.7$, $N = 30$, and $T = 80$.



Additional simulations are reported in the Online Supplement (Chen et al. 2021). First, we consider a quasi-real-time implementation of the PSY-factor procedure. Specifically, instead of estimating the first factor from the entire sample, for each observation of interest τ we compute the factor from a sample starting with the first available observation and ending with the current observation at τ using only historical information up to this point in time. The SDRs and estimation accuracy of the bubble origination and collapse dates by the recursive procedure are presented in Figure 1 of the supplement. No major differences between the finite sample performance of the PSY-factor procedure and this real-time implementation are observed.

Second, we consider a setting where there are no common bubbles but some idiosyncratic bubbles in a fraction of assets. Specifically, the idiosyncratic errors of those assets switch from being white noise to mildly explosive. The simulations show that idiosyncratic bubbles do not have a visible impact on the performance of the common bubble test at least when the proportion of idiosyncratic bubbles is less than 30%. Third, we examine the performance of the PSY-factor procedure in identifying multiple bubbles. The data generating process includes two common bubbles, which are assumed to have the same expansion rate. We find that the testing procedure performs well in identifying both bubble episodes. In particular, the SDRs are high for both bubbles, and the bias magnitudes for the bubble origination and termination dates are similar to those under a single bubble DGP. Since the focus of this paper is not on multiple bubble detection, more detailed analysis of multiples bubbles is left to future work.

6 Empirical Application to China Real Estate Markets

6.1 Data Description

We study housing markets in 89 major Chinese cities. The sample includes 4 Tier 1, 26 Tier 2, and 59 Tier 3 cities. A list of these cities is given in Table 1. Monthly house prices are compiled by Fang et al. (2016), based on sequential sales of new homes within the same housing development. The sample period is chosen to lead up to and around the 2008 global financial crisis, running from January 2005 to December 2008 (48 observations). Underlying market fundamentals are proxied by urban disposable income per capita, which measures per capita income received by urban residents within each of the cities.[9] The data are obtained from the China City Yearbook.

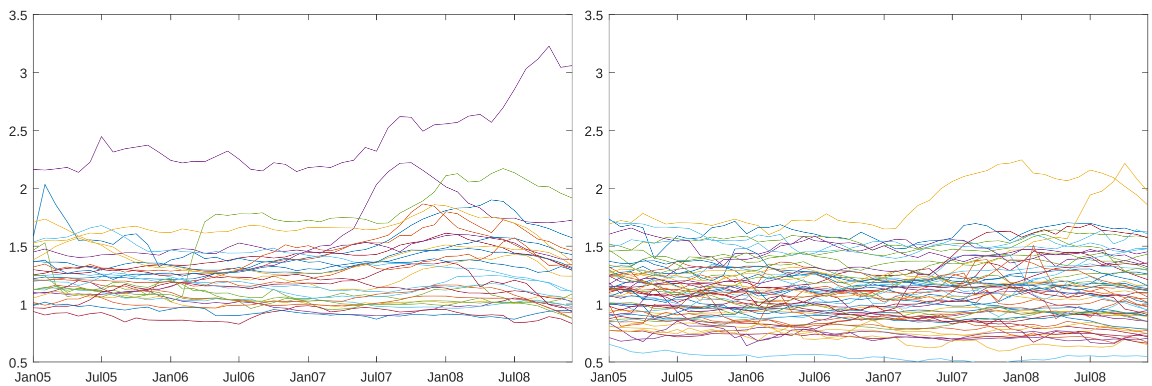
The sample is split into two groups. The first group includes all Tier 1 and 2 cities (group I), while the second group contains Tier 3 cities (group II). Figure 5 presents the housing price-to-income ratios (PIR) of group I (left panel) and group II (right panel). The variation within group I is larger than group II. We observe a dramatic increase of the price-to-income ratio in group I around 2007-2008, led by cities Wenzhen, Tianjin, Shenzhen and Ningbo.

[9] The number of cities included in this empirical analysis is mainly constrained by the availability of disposable income data.

Figure 9 displays the average PIR over the sample period for each city. Similar to what is observed in Figure 5, the average PIR of Wenzhou is the highest and is well above the national average.

Figure 5: The price-to-income ratios of 89 cities in China.

(a) Group I: Tier 1 and 2 (b) Group II: Tier 3



6.2 Common Bubbles

We apply the PSY-factor procedure to the price-to-income ratios in each group. To implement the PSY test, we set the minimum window size to 12 observations, based on the suggested rule in Phillips et al. (2015*a*) (i.e., $\tau_{\text{min}} = 0.01 + 1.8/\sqrt{T}$), so the evolving test recursion begins in December 2005. The DF regression model in (3.3) is augmented with lags and lag order is selected by BIC with a maximum lag order setting of 4 for each subsample.[10] The 95th percentiles of the finite sample distributions are used as critical values.

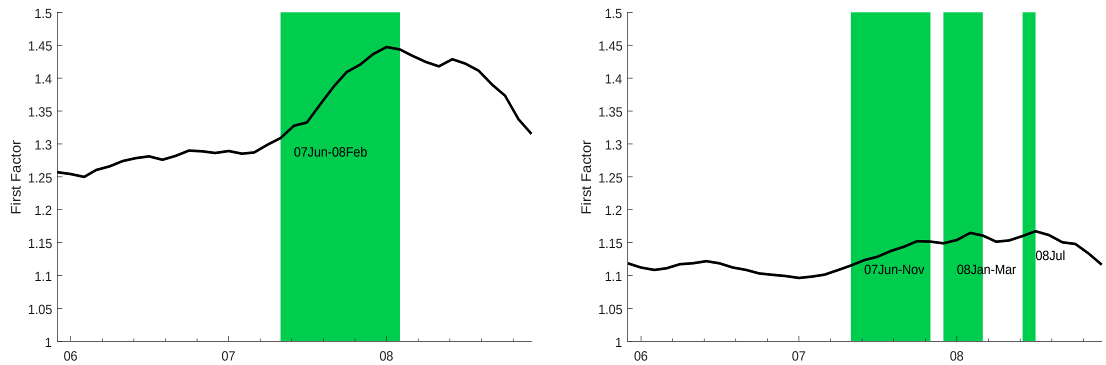
Figure 6 presents the estimated first common components (black lines) and the identified bubble periods (green shaded areas). Overall, the two estimated first common components show similar dynamics but the fluctuations in group I are far more dramatic. There is a period of rapid expansion in the first factor of both groups around 2007-2008. We apply the PSY procedure to the estimated first factors and find evidence of common bubbles in both groups. The explosive episode in Group I runs from June 2007 to February 2008. By comparision, the explosive episode in Group II starts at the same time but terminates one month later (with a one month break in December 2007). The procedure also detects common explosive behaviour of Group II in July 2008.

Interestingly, the origination date of the bubble episode is three months before the "927 Housing Mortgage Policy" (**?**) implemented by the People's Bank of China (September 2007). This policy requires that the down payment for first home buyers is no lower than 20% for units less than 90 square meters and no lower than 30% for units above 90 square meters.

[10]See Phillips et al. (2015*a*) for a detailed discussion on the finite sample performance of the PSY test under various lag order selection schemes.

Figure 6: The identified bubble periods. The solid (black) lines are the estimated first factors from respective groups. The shaded (green) areas, with dates, show the periods when the PSY-factor test rejects the null hypothesis of a unit root against the explosive alternative for the first common factor.

(a) Tier 1 and 2 (b) Tier 3



For those who apply for a second loan, the down payment should not be lower than 40% and the interest rate for such a loan should not be lower than 1.1 times the benchmark interest rate.

6.3 Robustness Check

We investigate the sensitivity of the empirical results with respect to a real-time implementation of the procedure and a bootstrapping method for critical values.

Real-time implementation

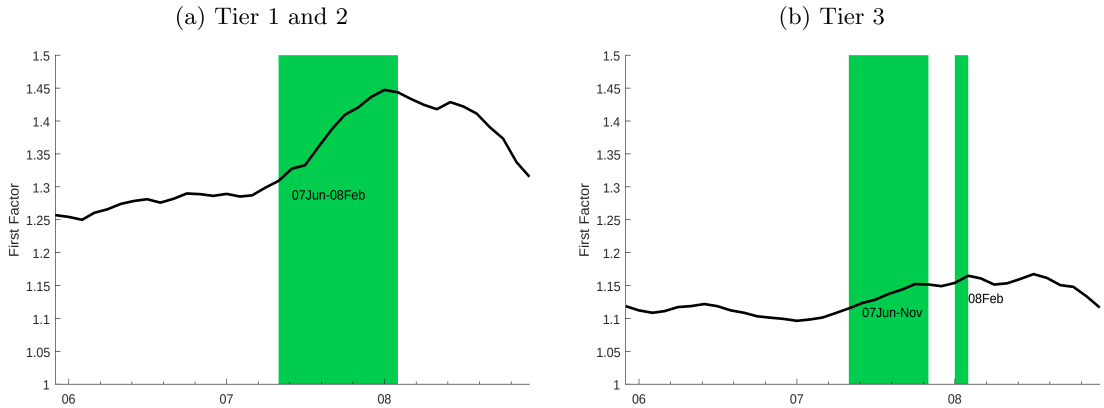
We undertake a pseudo-real-time implementation of the PSY-factor procedure on the real estate markets, i.e., using only information up to the observation of interest for estimation of the primary common factor in the first step. Critical values remain the same. The identified bubble episodes are displayed in Figure 7. For Group I, the identified episodes are identical to those in Figure 6(a). For Group II, the starting date of the common bubble episode is exactly the same but the termination date is in one month earlier. There is a two-month break (December 2007 and January 2008) in between the start and end dates. Additionally, unlike Figure 6(b), the explosive signal is not switched on in July 2008.

Bootstrap critical values

To account for potential heteroskedastic and serially correlated innovations in the monthly price-to-income ratios and the multiplicity issue induced by recursive testing,[11] we use a

[11] The probability of making a Type I error rises with the number of hypotheses in a recursive test sequence, a phenomenon that is called the multiplicity issue in testing. This tendency towards oversizing may be controlled

Figure 7: Pseudo-real-time identification of common bubbles. The black lines are the estimated first factors using the whole sample. The shaded (green) areas are the identified explosive periods.



composite bootstrap procedure for calculating critical values. The procedure combines the ideas of the sieve boostrap for serially correlated innovations (Park 2003, Chang & Park 2003, Palm et al. 2008, Pedersen & Schütte 2020), wild bootstrapping for non i.i.d errors (Liu et al. 1988, Mammen 1993, Harvey et al. 2016*b*), and bootstrap methods for dealing with multiplicity issues (White 2000, Phillips & Shi 2020, Shi et al. 2020).

Suppose T_b is the number of observations in the window where we aim to control the multiplicity issue. The probability of making at least one false-positive rejection over the window with T_b observations is 5%. The choice of the window is subjective. A larger window leads to more conservative bubble detection results. Let $T_{min} = \lfloor T\tau_{min} \rfloor$. The procedure is detailed in full below.

Step 1: Estimate the augmented Dickery Fuller regression model using $\tilde{y}_t$ (the first common factor estimated from the PIRs) such that

$$\Delta\tilde{y}_t = \delta + \gamma\tilde{y}_{t-1} + \sum_{j=1}^{k} \psi_j \Delta\tilde{y}_{t-j} + v_t, \tag{6.1}$$

where k is selected by BIC with maximum lag order 4. We obtain the estimated coefficients $\hat{\psi}_j$ and residuals $\{e_t\}_{t=k+1}^{T}$.

Step 2: Generate residuals recursively from

$$v_t^b = \sum_{j=1}^{k} \hat{\psi}_j v_{t-j}^b + e_t^b \text{ with } e_t^b = w_t e_l,$$

by using a familywise critical value. See PSY for discussion and for the development of a bootstrap procedure which assists in controlling size in such cases.

where w_t follows a standard normal distribution and e_l is bootstrapped from the residuals obtained in Step 1 (with replacement).

Step 3: Obtain a bootstrap sample with $T_{min} + T_b - 1$ observations using the formula

$$\tilde{y}_t^b = \tilde{y}_{t-1}^b + v_t^b. \tag{6.2}$$

Step 3: Compute from the bootstrap sample

$$\mathcal{M}_t^b = \max_{t \in [T_0, T_0 + T_b - 1]} \left(PSY_t^b \right).$$

Step 4: Repeat Steps 2-3 for $B = 1,999$ times.

Step 5: The 95% percentiles of the $\left\{ \mathcal{M}_t^b \right\}_{b=1}^B$ sequence serves as the critical value of the PSY-factor procedure.

In the Online Supplement we also consider a data generating process with no common bubble but serially correlated innovations for the I(1) factor. The empirical size is controlled over the entire sample period, i.e., $T_b = T$. We show that this procedure is conservative (under-sized) when there is no or weak serial correlations. But empirical size increases when the correlation becomes stronger. A comprehensive study of the bootstrap procedure implemented here is left for furture work. In practice, one could use a smaller control window to enable a less conservative test. For our empirical application, we allow the possibility of making at least one false-positive conclusion over a one-year window to be 5%, i.e., $T_b = 12$.

Figure 8: The PSY-factor procedure with boostrap critical values. The black lines are the estimated first factors using the whole sample. The shaded (green) areas are the indentified periods of explosiveness.

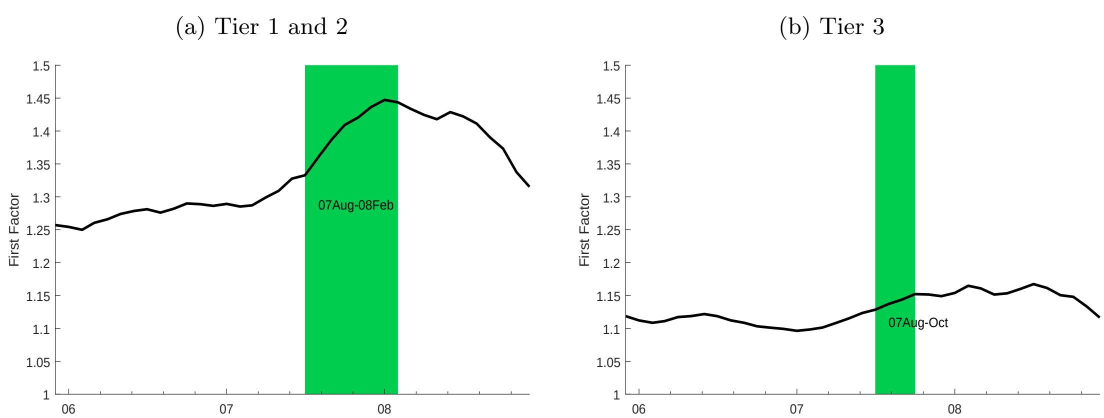


Figure 8 shows testing results from the PSY-factor procedure with bootstrap critical values for the Chinese housing markets. The identified common bubble period in Group I is almost identical to that in Figure 6 (with simulated finite sample critical values), except that the signal was switched off temporarily in July 2007. For Group II, the identified bubble period is much shorter, starting in August 2007 and terminating three months later. This result is expected as the bootstrap procedure allows for a lower chance of making false-positive conclusions and accounts for potential heteroskedasticity and serially correlated innovations.

7 Conclusion

Price bubbles in the financial system and asset markets such as those in real estate pose a significant threat to economic and financial stability. Such disturbances from normal market behavior have led to the introduction in many countries of macroprudential and microprudential policy regulations that are designed to moderate market behavior. In many cases, emergent speculative elements in financial and real estate asset markets are influenced by driving factors of the behavioral kind that are not directly observed. It is therefore particularly useful to have econometric methods that enable the detection of such behavior via the estimation and testing of the unobserved factors that may be driving speculative activity. Based on earlier methods in Phillips et al. (2015*a*,*b*) that were designed for observed data, this paper provides tools that enable such identification and empirical detection of an unobserved common explosive factor influencing market behavior coupled with a real-time mechanism for their dating and identification.

The factor methods developed here may be applied to large dimensional financial data sets and simulation results show good performance in the detection of unobserved common bubble factors in terms of successful detection rates, and the estimation accuracy of bubble origination and termination dates. The empirical application to real estate markets in major Chinese cities reveals strong evidence of a common driving factor affecting markets in all cities, starting from June 2007 and terminating in early 2008. Results that match well against government regulatory policies that were introduced as cooling measures to mitigate housing price bubble activity in the real estate market. Unobserved factor methods of the typed developed here therefore seem to offer some promise as a potential guide to regulatory authorities faced with emergent speculative behavior.

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A Preliminary Lemmas

Lemma A.1 *Under Assumption 4.1 and 4.3, as $T \to \infty$, we have the following:*

(1) $\sum_{t=T_1}^{T_2} e_{it-1} = O_p\left(T^{1/2}\right)$ *and* $\sum_{t=T_1}^{T_2} \left(e_{it} - e_{it-1}\right) = O_p\left(1\right)$;

(2) $\sum_{t=T_1}^{T_2} e_{it}e_{it-1} = O_p\left(T\right)$ *and* $\sum_{t=T_1}^{T_2} e_{it-1}^2 = O_p\left(T\right)$;

(3) $\sum_{t=T_1}^{T_2} u_{0,t} = O_p\left(T^{1/2}\right); \sum_{t=T_1}^{T_2} u_{0,t}^2 = O_p\left(T\right)$;

(4) $\sum_{t=T_1}^{T_2} u_{1,t} = O_p\left(T^{1/2}\right); \sum_{t=T_1}^{T_2} u_{1,t}^2 = O_p\left(T\right)$;

(5) $\sum_{t=T_1}^{T_2} u_{2,t} = O_p\left(T^{1/2}\right); \sum_{t=T_1}^{T_2} u_{2,t}^2 = O_p\left(T\right)$;

(6) $\sum_{t=T_e}^{T_c} u_{1,t}e_{it-1} = O_p\left(T\right); \sum_{t=T_c+1}^{T} u_{2,t}e_{it-1} = O_p\left(T\right)$.

Proof. The results follow directly from Assumptions 4.1 and 4.3 by application of suitable laws of large numbers and central limit theory, as in Bai (2004). ■

Lemma A.2 *Under Assumption 4.1 and 4.3, as $T \to \infty$ we have*

$$\frac{1}{T^2}\sum_{t=T_1}^{T_2} f_{0,t-1}^2 \Rightarrow \int_{\tau_1}^{\tau_2} B\left(r\right)^2 dr;$$

$$\frac{1}{T^{3/2}}\sum_{t=T_1}^{T_2} f_{0,t-1} \Rightarrow \int_{\tau_1}^{\tau_2} B\left(r\right) dr;$$

$$\frac{1}{T}\sum_{t=T_1}^{T_2} f_{0,t-1}u_{0,t} \Rightarrow \int_{\tau_1}^{\tau_2} B\left(r\right) dB\left(r\right);$$

where $B\left(r\right)$ is Brownian motion with variance σ_{00}.

Proof. The proofs follow standard methods (Phillips 1987). ■

Lemma A.3 *Under Assumption 4.1, 4.4, and 4.3, as $N, T \to \infty$, for any $T_s = \lfloor Ts \rfloor \in (T_e, T_c]$,*

(1) $f_{1,t} = O_p\left(T^{\alpha/2}\rho_T^{t-T_r}\right)$

(2) $\sum_{t=T_e+1}^{T_s} f_{1,t-1}u_{0,t} = O_p\left(T^{\alpha}\rho_T^{T_s-T_r}\right)$, $\sum_{t=T_e+1}^{T_s} f_{1,t-1}u_{1,t} = O_p\left(T^{\alpha}\rho_T^{T_s-T_r}\right)$, $\sum_{t=T_e+1}^{T_s} f_{1,t-1}e_{i,t} = O_p\left(T^{\alpha}\rho_T^{T_s-T_r}\right)$;

(3) $\sum_{t=T_e+1}^{T_s} f_{1,t-1} = O_p\left(T^{3\alpha/2}\rho_T^{T_s-T_r}\right)$;

(4) $\sum_{t=T_e+1}^{T_s} f_{1,t-1}^2 = O_p\left(T^{2\alpha}\rho_T^{2(T_s-T_r)}\right)$;

(5) $\sum_{t=T_e+1}^{T_s} f_{1,t-1}f_{0,t-1} = O_p\left(T^{(1+3\alpha)/2}\rho_T^{T_s-T_r}\right)$;

(6) $\sum_{t=T_e+1}^{T_s} f_{1,t-1}\left(e_{it} - e_{it-1}\right) = O_p\left(T^{\alpha}\rho_T^{T_s-T_r}\right)$.

Proof. (1) By definition,

$$f_{1,t} = f_{1,T_r}\rho_T^{t-T_r} + \sum_{j=T_r+1}^{t} \rho_T^{t-j}u_{1,j}.$$

Since $f_{1,T_r} = O_p\left(T^{\alpha/2}\right)$ and $T^{-\alpha/2}\sum_{j=T_r+1}^{t} \rho_T^{T_r-j}u_{1,j} \Rightarrow N_{d_1} = \mathcal{N}\left(0, \frac{\sigma_{11}}{2d_1}\right)$ from Lemma 4.2 of Phillips & Magdalinos (2007*a*),

$$\frac{f_{1,T_s}}{T^{\alpha/2}\rho_T^{t-T_r}} = \frac{f_{1,T_r}}{T^{\alpha/2}} + \frac{1}{T^{\alpha/2}}\sum_{j=T_r+1}^{t} \rho_T^{T_r-j}u_{1,j} \Rightarrow F_{1,r} + N_{d_1}.$$

Thus, $f_{1,t} = O_p\left(T^{\alpha/2}\rho_T^{t-T_r}\right)$.

(2) This follows directly from Phillips & Magdalinos (2007*a*).

(3) Since $f_{1,t} = \rho_T f_{1,t-1} + u_{1,t}$, we have

$$\sum_{t=T_e+1}^{T_s} f_{1,t} = \rho_T \sum_{t=T_e+1}^{T_s} f_{1,t-1} + \sum_{t=T_e+1}^{T_s} u_{1,t}.$$

It follows that

$$(1-\rho_T)\sum_{t=T_e+1}^{T_s} f_{1,t-1} = -f_{1,T_s} + f_{1,T_e} + \sum_{t=T_e+1}^{T_s} u_{1,t} = -f_{1,T_s}\left[1 + o_p(1)\right]$$

since $f_{1,T_e} = O_p\left(T^{\alpha/2}\rho_T^{T_e-T_r}\right)$ and $f_{1,T_s} = O_p\left(T^{\alpha/2}\rho_T^{T_s-T_r}\right)$ from Lemma A.3 (1) and

$\sum_{t=T_e+1}^{T_s} u_{1,t} = O_p\left(T^{1/2}\right)$ from Lemma A.1. Therefore,

$$\sum_{t=T_e+1}^{T_s} f_{1,t-1} = \frac{T^\alpha}{d_1} f_{1,T_s}\left[1+o_p(1)\right] = O_p\left(T^{3\alpha/2}\rho_T^{T_s-T_r}\right).$$

(4) By squaring equation $f_{1,t} = \rho_T f_{1,t-1} + u_{1,t}$, substracting $f_{1,t-1}^2$ from both sides, summing from T_e+1 to T_s, reorganizing the equation, and dividing by $\rho_T^2 - 1$, we have

$$\begin{aligned}\sum_{t=T_e+1}^{T_s} f_{1,t-1}^2 &= \frac{1}{\rho_T^2-1}\left[f_{1,T_s}^2 - f_{1,T_e}^2 - \sum_{t=T_e+1}^{T_s} u_{1,t}^2 - 2\rho_T \sum_{t=T_e+1}^{T_s} f_{1,t-1}u_{1,t}\right] \\ &= \frac{1}{\rho_T^2-1} f_{1,T_s}^2\left[1+o_p(1)\right] = O_p(1)\end{aligned}$$

since $f_{1,T_s}^2 = O_p\left(T^\alpha \rho_T^{2(T_s-T_r)}\right)$, $f_{1,T_e}^2 = O_p\left(T^\alpha \rho_T^{2(T_e-T_r)}\right)$, and $\sum_{t=T_e+1}^{T_s} f_{1,t-1}u_{1,t} = O_p\left(T^\alpha \rho_T^{T_s-T_r}\right)$ from Lemma A.3 (1)-(2), $\sum_{t=T_e+1}^{T_s} u_{1,t}^2 = O_p(T)$ from Lemma A.1, and $\frac{1}{T^\alpha}\frac{1}{\rho_T^2-1} = O_p(1)$. Therefore,

$$\sum_{t=T_e+1}^{T_s} f_{1,t-1}^2 = O_p\left(T^{2\alpha}\rho_T^{2(T_s-T_r)}\right).$$

(5) The sum of the cross product between $f_{1,t-1}$ and $f_{0,t-1}$ over $[T_e+1, T_s]$ is

$$\begin{aligned}\sum_{t=T_e+1}^{T_s} f_{1,t-1}f_{0,t-1} &= T^{(1+\alpha)/2}\sum_{t=T_e+1}^{T_s} \frac{f_{1,t-1}}{T^{\alpha/2}\rho_T^{t-T_r}}\frac{f_{0,t-1}}{T^{1/2}}\rho_T^{t-T_r} \\ &\leq T^{(1+\alpha)/2}\max_{t\in[T_e+1,T_s]}\left\{\frac{f_{1,t-1}}{T^{\alpha/2}\rho_T^{t-T_r}}\right\}\max_{t\in[T_e+1,T_s]}\left\{\frac{f_{0,t-1}}{T^{1/2}}\right\}\sum_{t=T_e+1}^{T_s}\rho_T^{t-T_r} \\ &= O_p\left(T^{(1+3\alpha)/2}\rho_T^{T_s-T_r}\right).\end{aligned}$$

(6) From Assumption 4.3, $\mathbb{E}\left(e_{it}-e_{it-1}\right) = 0$ and $Var\left(e_{it}-e_{it-1}\right) < \infty$, so that we have $\sum_{t=T_e+1}^{T_s} f_{1,t-1}\left(e_{it}-e_{it-1}\right) = O_p\left(T^\alpha \rho_T^{T_s-T_r}\right)$, which follows directly from Phillips & Magdalinos (2007*a*). ■

Lemma A.4 *Under Assumption 4.1, as $N, T \to \infty$, for any $T_2 \in (T_c, T]$, we have*

(1) $f_{2,t} = \begin{cases} O_p\left(T^{\beta/2}\right) & \text{if } \alpha > \beta \\ O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\phi_T^{t-T_c}\right) & \text{if } \alpha < \beta \end{cases}$ *for* $t > T_c$;

(2) $\sum_{t=T_{c+1}}^{T_2} f_{2,t-1}e_{i,t} = O_p\left(T^{(\alpha+\beta)/2}\rho_T^{T_c-T_r}\right)$ *and* $\sum_{t=T_{c+1}}^{T_2} f_{2,t-1}u_{2,t} = O_p\left(T^{(\alpha+\beta)/2}\rho_T^{T_c-T_r}\right)$;

(3) $\sum_{t=T_c+1}^{T_2} f_{2,t-1} = O_p\left(T^{\alpha/2+\beta}\rho_T^{T_c-T_r}\right)$;

(4) $\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 = O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right)$;

(5) $\sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} = \begin{cases} O_p\left(T^{\alpha/2+\beta}\rho_T^{2(T_c-T_r)}\right) & \textit{if } \alpha > \beta \\ O_p\left(T^{3\alpha/2}\rho_T^{(T_c-T_r)+(T_2-T_r)}\phi_T^{T_2-T_c}\right) & \textit{if } \alpha < \beta \end{cases}$;

Proof. (1) By definition,

$$f_{2,t} = \phi_T^{t-T_c} f_{2,T_c} + \sum_{j=T_c+1}^{t} \phi_T^{t-j} u_{2,j}.$$

By assumption, $f_{2,T_c} = O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\right)$ and $\phi_T^{t-T_c} f_{2,T_c} = O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\phi_T^{t-T_c}\right)$. Since

$$\mathbb{E}\left[\left(\frac{1}{T^{\beta/2}}\sum_{j=T_c+1}^{t}\phi_T^{t-j}u_{2,j}\right)^2\right] = \frac{1}{T^{\beta}}\sum_{j=T_c+1}^{t}\phi_T^{2(t-j)}\mathbb{E}\left(u_{2,j}^2\right) \to \frac{\sigma_{22}}{2d_2},$$

from Lemma B.1(3) in Phillips & Shi (2018),

$$\frac{1}{T^{\beta/2}}\sum_{j=T_c+1}^{t}\phi_T^{t-j}u_{2,j} \Rightarrow X_{d_2} = \mathcal{N}\left(0, \frac{\sigma_{22}}{2d_2}\right).$$

That is, $\sum_{j=T_c+1}^{t}\phi_T^{t-j}u_{2,j} = O_p\left(T^{\beta/2}\right)$. Therefore,

$$f_{2,t} = \begin{cases} \sum_{j=T_c+1}^{t}\phi_T^{t-j}u_{2,j}\left[1+o_p(1)\right] = O_p\left(T^{\beta/2}\right) & \text{if } \alpha > \beta \\ \phi_T^{t-T_c} f_{2,T_c}\left[1+o_p(1)\right] = O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\phi_T^{t-T_c}\right) & \text{if } \alpha < \beta \end{cases}.$$

(2) We have

$$\begin{aligned}
\sum_{t=T_{c+1}}^{T_2} f_{2,t-1}e_{i,t} &= \sum_{t=T_{c+1}}^{T_2}\left(\phi_T^{t-1-T_c}f_{2,T_c} + \sum_{j=T_c+1}^{t-1}\phi_T^{t-j}u_{2,j}\right)e_{i,t} \\
&= T^{\beta/2}f_{2,T_c}\left(T^{-\beta/2}\sum_{t=T_{c+1}}^{T_2}e_{i,t}\phi_T^{t-1-T_c}\right) + T^{\beta/2}\sum_{t=T_{c+1}}^{T_2}e_{i,t}\left(\frac{1}{T^{\beta/2}}\sum_{j=T_c+1}^{t-1}\phi_T^{t-j}u_{2,j}\right) \\
&= T^{\beta/2}f_{2,T_c}\left(T^{-\beta/2}\sum_{t=T_{c+1}}^{T_2}e_{i,t}\phi_T^{t-1-T_c}\right) + O_p\left(T^{(1+\beta)/2}\right) \\
&= O_p\left(T^{(\alpha+\beta)/2}\rho_T^{T_c-T_r}\right)
\end{aligned}$$

since $f_{2,T_c} = O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\right)$, $\frac{1}{T^{\beta/2}}\sum_{j=T_c+1}^{t-1}\phi_T^{t-1-j}u_{2,j} \Rightarrow X_{d_2}$, and from Phillips & Shi

(2018),

$$T^{-\beta/2}\sum_{t=T_c+1}^{T_2}\phi_T^{t-1-T_c}e_{i,t}=T^{-\beta/2}\sum_{j=0}^{T_2-T_c-1}\phi_T^{j}e_{i,j+T_c+1}\Rightarrow N\left(0,\sigma_e^2/2d_2\right).$$

Similarly, we can show that

$$\sum_{t=T_{c+1}}^{T_2}f_{2,t-1}u_{2,t}=O_p\left(T^{(\alpha+\beta)/2}\rho_T^{T_c-T_r}\right).$$

(3) Since $f_{2,t}=\phi_T f_{2,t-1}+u_{2,t}$, we have

$$\sum_{t=T_c+1}^{T_2}f_{2,t}=\phi_T\sum_{t=T_c+1}^{T_2}f_{2,t-1}+\sum_{t=T_c+1}^{T_2}u_{2,t}.$$

It follows that

$$(1-\phi_T)\sum_{t=T_c+1}^{T_2}f_{2,t-1}=-f_{2,T_2}+f_{2,T_c}+\sum_{t=T_c+1}^{T_2}u_{2,t}.$$

Since $f_{2,T_2}=O_p\left(T^{\beta/2}\right)$ if $\alpha>\beta$ and $f_{2,T_2}=O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\phi_T^{T_2-T_c}\right)$ if $\alpha<\beta$ from Lemma A.4 (1), $f_{2,T_c}=O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\right)$, and $\sum_{t=T_c+1}^{T_2}u_{2,t}=O_p\left(T^{1/2}\right)$ from Lemma A.1, we have

$$\sum_{t=T_c+1}^{T_2}f_{2,t-1}=\frac{T^{\beta}}{d_2}f_{2,T_c}\left[1+o_p(1)\right]=O_p\left(T^{\alpha/2+\beta}\rho_T^{T_c-T_r}\right).$$

(4) By squaring equation $f_{2,t}=\phi_T f_{2,t-1}+u_{1,t}$, subtracting $f_{2,t-1}^2$ from both sides, summing from T_c+1 to T_2, reorganizing the equation, and dividing by ϕ_T^2-1, we have

$$\begin{aligned}\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2 &= \frac{1}{\phi_T^2-1}\left[f_{2,T_2}^2-f_{2,T_c}^2-\sum_{t=T_c+1}^{T_2}u_{2,t}^2-2\phi_T\sum_{t=T_c+1}^{T_2}f_{2,t-1}u_{2,t}\right]\\ &= \frac{1}{1-\phi_T^2}f_{2,T_c}^2\left[1+o_p(1)\right]=O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right),\end{aligned}$$

since $f_{2,T_c}^2=O_p\left(T^{\alpha}\rho_T^{2(T_c-T_r)}\right)$, $\sum_{t=T_c+1}^{T_2}u_{2,t}^2=O_p(T)$, $1-\phi_T^2=2d_2T^{-\beta}\left[1+o_p(1)\right]$, $\sum_{t=T_{c+1}}^{T_2}f_{2,t-1}u_{i,t}=O_p\left(T^{(\alpha+\beta)/2}\rho_T^{T_c-T_r}\right)$, and

$$f_{2,T_2}^2=\begin{cases}O_p\left(T^{\beta}\right) & \text{if } \alpha>\beta\\ O_p\left(T^{\alpha}\rho_T^{2(T_c-T_r)}\phi_T^{2(T_2-T_c)}\right) & \text{if } \alpha<\beta\end{cases}.$$

(5) Since $f_{2,t} = \phi_T f_{2,t-1} + u_{2,t}$, we have

$$\sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t} = \phi_T \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} + \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} u_{2,t}.$$

Re-organizing the equation and adding $\sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1}$ to both sides, we have

$$\begin{aligned}
& (1-\phi_T) \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} \\
= & \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} - \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t} + \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} u_{2,t} \\
= & \rho_T^{T_c-T_r} f_{2,T_c} - \rho_T^{T_2-1-T_r} f_{2,T_2} + \sum_{t=T_c+2}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} - \sum_{t=T_c+1}^{T_2-1} \rho_T^{t-1-T_r} f_{2,t} + \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} u_{2,t} \\
= & \rho_T^{T_c-T_r} f_{2,T_c} - \rho_T^{T_2-1-T_r} f_{2,T_2} + \frac{\rho_T - 1}{\rho_T} \sum_{t=T_c+2}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} + \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} u_{2,t} \\
= & \rho_T^{T_c-T_r} f_{2,T_c} - \rho_T^{T_2-1-T_r} f_{2,T_2} - \frac{\rho_T - 1}{\rho_T} \rho_T^{T_c-T_r} f_{2,T_c} + \frac{\rho_T - 1}{\rho_T} \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} + \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} u_{2,t}.
\end{aligned}$$

It follows that

$$\left(1 - \phi_T - \frac{\rho_T - 1}{\rho_T}\right) \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} = \rho_T^{T_c-T_r} f_{2,T_c} - \rho_T^{T_2-1-T_r} f_{2,T_2} - \frac{\rho_T - 1}{\rho_T} \rho_T^{T_c-T_r} f_{2,T_c} + \sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} u_{2,t}.$$

We have $\rho_T^{T_c-T_r} f_{2,T_c} = O_p\left(T^{\alpha/2} \rho_T^{2(T_c-T_r)}\right)$, $\frac{\rho_T-1}{\rho_T} \rho_T^{T_c-T_r} f_{2,T_c} = O_p\left(T^{-\alpha/2} \rho_T^{2(T_c-T_r)}\right)$,

$$\begin{aligned}
& \rho_T^{T_2-1-T_r} f_{2,T_2} = \begin{cases} O_p\left(T^{\beta/2} \rho_T^{T_2-T_r}\right) & \text{if } \alpha > \beta \\ O_p\left(T^{\alpha/2} \rho_T^{(T_c-T_r)+(T_2-T_r)} \phi_T^{T_2-T_c}\right) & \text{if } \alpha < \beta \end{cases}, \\
& T^{\alpha/2} \rho_T^{T_2-1-T_r} \left(T^{-\alpha/2} \sum_{t=T_c+1}^{T_2} \rho_T^{t-T_2} u_{2,t}\right) = O_p\left(T^{\alpha/2} \rho_T^{T_2-T_r}\right), \\
& \frac{T^{\alpha/2} \rho_T^{(T_c-T_r)+(T_2-T_r)} \phi_T^{T_2-T_c}}{T^{\alpha/2} \rho_T^{2(T_c-T_r)}} = (\rho_T \phi_T)^{T_2-T_c} \to \infty,
\end{aligned}$$

and

$$\frac{1}{1 - \phi_T - \frac{\rho_T-1}{\rho_T}} = \begin{cases} O_p\left(T^{\beta}\right) & \text{if } \alpha > \beta \\ O_p\left(T^{\alpha}\right) & \text{if } \alpha < \beta \end{cases}.$$

Therefore,

$$\sum_{t=T_c+1}^{T_2} \rho_T^{t-1-T_r} f_{2,t-1} = \begin{cases} O_p\left(T^{\alpha/2+\beta}\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha > \beta \\ O_p\left(T^{3\alpha/2}\rho_T^{(T_c-T_r)+(T_2-T_r)}\phi_T^{T_2-T_c}\right) & \text{if } \alpha < \beta \end{cases}.$$

■

B Proofs Under the Null Hypothesis

Lemma B.1 *Under the null specification of model (2.3) and Assumption 4.1, 4.2(1), 4.3, 4.4, and 4.5, we have:*

$$(1) \quad \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = \sum_{t=T_1}^{T_2} H_{NT}^0 f_{0,t-1}\left[1+o_p(1)\right] = O_p\left(T^{3/2}\right);$$

$$(2) \quad \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = \left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} f_{0,t-1}^2\left[1+o_p(1)\right] = O_p\left(T^2\right);$$

$$(3) \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1} = \left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} f_{0,t-1}u_{0,t}\left[1+o_p(1)\right] = O_p(T);$$

$$(4) \quad \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t^2 = \left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} u_{0,t}^2\left[1+o_p(1)\right] = O_p(T);$$

$$(5) \quad \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t = H_{NT}^0 \sum_{t=T_1}^{T_2} u_{0,t}\left[1+o_p(1)\right] = O_p\left(T^{1/2}\right).$$

Proof. (1) The quantity

$$\frac{1}{T^{3/2}}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1} = \frac{1}{T^{3/2}}\sum_{t=T_1}^{T_2}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right) + H_{NT}^0\frac{1}{T^{3/2}}\sum_{t=T_1}^{T_2} f_{0,t-1}.$$

By the Cauchy-Schwarz inequality and Lemma 4.1, we have

$$\left[\sum_{t=T_1}^{T_2}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t}\right)\right]^2 \leqslant \sum_{t=T_1}^{T_2}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t}\right)^2 \leq \sum_{t=1}^{T}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t}\right)^2 = O_p\left(T\delta_{NT}^{-2}\right)$$

and hence

$$\frac{1}{T^{3/2}}\sum_{t=T_1}^{T_2}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right) = O_p\left(T^{-1/2}\delta_{NT}^{-1}\right).$$

From Lemma A.2, $T^{-3/2}\sum_{t=T_1}^{T_2} f_{0,t-1} = O_p(1)$ and $H_{NT}^0 = O_p(1)$, from Lemma S.1 in the

Online Supplement. Therefore,

$$\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = H_{NT}^0 \sum_{t=T_1}^{T_2} f_{0,t-1} \left[1 + o_p(1)\right] = O_p\left(T^{3/2}\right).$$

(2) Similarly,

$$\frac{1}{T^2}\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = \frac{1}{T^2}\sum_{t=T_1}^{T_2} \left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1} + H_{NT}^0 f_{0,t-1}\right)^2$$
$$= \frac{1}{T^2}\sum_{t=T_1}^{T_2} \left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right)^2 + \frac{2}{T^2} H_{NT}^0 \sum_{t=T_1}^{T_2} \left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right) f_{0,t-1} + \frac{1}{T^2}\left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} f_{0,t-1}^2. \tag{B.1}$$

By Lemma 4.1, the first term of (B.1) is $\frac{1}{T^2}\sum_{t=T_1}^{T_2}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right)^2 = O_p\left(T^{-1}\delta_{NT}^{-2}\right)$. The second term is

$$\frac{1}{T^2}\left|H_{NT}^0 \sum_{t=T_1}^{T_2}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right) f_{0,t-1}\right| \leqslant \frac{1}{T^{1/2}}\left|H_{NT}^0\right| \left(\frac{1}{T}\sum_{t=T_1}^{T_2}\left|\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right|^2\right)^{1/2} \left(\frac{1}{T^2}\sum_{t=T_1}^{T_2} f_{0,t-1}^2\right)^{1/2}$$
$$= O_p\left(T^{-1/2}\delta_{NT}^{-1}\right).$$

The last term is $O_p(1)$ from Lemma A.2 and Lemma S.1 in the Online Supplement. Combining the above we have

$$\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = \left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} f_{0,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^2\right).$$

(3) The quantity

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} = \sum_{t=T_1}^{T_2} \tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2$$
$$= \sum_{t=T_1}^{T_2}\left(\tilde{y}_t - H_{NT}^0 f_{0,t}\right)\tilde{y}_{t-1} + H_{NT}^0 \sum_{t=T_1}^{T_2} f_{0,t}\tilde{y}_{t-1} - \left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} f_{0,t-1}^2 \left[1 + o_p(1)\right] \tag{B.2}$$
$$= \sum_{t=T_1}^{T_2}\left(\tilde{y}_t - H_{NT}^0 f_{0,t}\right)\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right) + H_{NT}^0 \sum_{t=T_1}^{T_2}\left(\tilde{y}_t - H_{NT}^0 f_{0,t}\right) f_{0,t-1} \tag{B.3}$$
$$+ H_{NT}^0 \sum_{t=T_1}^{T_2} f_{0,t}\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right) + \left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} f_{0,t} f_{0,t-1} - \left(H_{NT}^0\right)^2 \sum_{t=T_1}^{T_2} f_{0,t-1}^2.$$

The first component of (B.3) is

$$\left|\sum_{t=T_1}^{T_2}\left(\tilde{y}_t - H_{NT}^0 f_{0,t}\right)\left(\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right)\right| \leq T\left(\frac{1}{T}\sum_{t=T_1}^{T_2}\left|\tilde{y}_t - H_{NT}^0 f_{0,t}\right|^2\right)^{1/2}\left(\frac{1}{T}\sum_{t=T_1}^{T_2}\left|\tilde{y}_{t-1} - H_{NT}^0 f_{0,t-1}\right|^2\right)^{1/2}$$
$$= O_p\left(T\delta_{NT}^{-2}\right),$$

using (4.1). Similarly, the second component of (B.3) is

$$\left|H_{NT}^0\sum_{t=T_1}^{T_2}\left(\tilde{y}_t - H_{NT}^0 f_{0,t}\right)f_{0,t-1}\right| \leq T^{\frac{3}{2}}\left|H_{NT}^0\right|\left(\frac{1}{T}\sum_{t=T_1}^{T_2}\left|\tilde{y}_t - H_{NT}^0 f_{0,t}\right|^2\right)^{1/2}\left(\frac{1}{T^2}\sum_{t=T_1}^{T_2} f_{0,t-1}^2\right)^{1/2} = O_p\left(T^{\frac{3}{2}}\delta_{NT}^{-1}\right)$$

By the same argument, the third component of equation (B.3) is at most $O_p\left(T^{\frac{3}{2}}\delta_{NT}^{-1}\right)$. The fourth component is of order $O_p\left(T^2\right)$ since $\left|H_{NT}^0\right| = O_p(1)$ and

$$\sum_{t=T_1}^{T_2} f_{0,t}f_{0,t-1} = \sum_{t=T_1}^{T_2} f_{0,t-1}^2 + \sum_{t=T_1}^{T_2} f_{0,t-1}u_{0t} = \sum_{t=T_1}^{T_2} f_{0,t-1}^2\left[1 + o_p(1)\right] = O_p\left(T^2\right).$$

The fifth component is $O_p\left(T^2\right)$. Therefore, we have

$$\begin{aligned}\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1} &= \left[\left(H_{NT}^0\right)^2\sum_{t=T_1}^{T_2} f_{0,t}f_{0,t-1} - \left(H_{NT}^0\right)^2\sum_{t=T_1}^{T_2} f_{0,t-1}^2\right]\left[1 + o_p(1)\right] \\ &= \left(H_{NT}^0\right)^2\sum_{t=T_1}^{T_2} f_{0,t-1}u_{0,t}\left[1 + o_p(1)\right] = O_p(T).\end{aligned}$$

(4) The quantity

$$\begin{aligned}&\frac{1}{T}\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t^2 = \frac{1}{T}\sum_{t=T_1}^{T_2}\tilde{y}_t^2 - \frac{2}{T}\sum_{t=T_1}^{T_2}\tilde{y}_t\tilde{y}_{t-1} + \frac{1}{T}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 \\ &= \frac{1}{T}\left(H_{NT}^0\right)^2\sum_{t=T_1}^{T_2}\left(f_{0,t}^2 - 2f_{0,t}f_{0,t-1} + f_{0,t-1}^2\right)\left[1 + o_p(1)\right] \\ &= \frac{1}{T}\left(H_{NT}^0\right)^2\sum_{t=T_1}^{T_2}\left(f_{0,t} - f_{0,t-1}\right)^2\left[1 + o_p(1)\right] = \frac{1}{T}\left(H_{NT}^0\right)^2\sum_{t=T_1}^{T_2} u_{0,t}^2\left[1 + o_p(1)\right] = O_p(1).\end{aligned}$$

using Lemma A.1. (5) The quantity

$$\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t = \sum_{t=T_1}^{T_2} H_{NT}^0\left(f_{0,t} - f_{0,t-1}\right)\left[1 + o_p(1)\right] = H_{NT}^0\sum_{t=T_1}^{T_2} u_{0,t}\left[1 + o_p(1)\right] = O_p\left(T^{1/2}\right)$$

using Lemma A.1. ■

Proof of Theorem 4.2

Proof. We first derive the limiting distribution of $T\hat{\gamma}_{\tau_1,\tau_2}$. Let $T_w = T_2 - T_1 + 1 = [T\tau_w]$. The OLS estimator $\hat{\gamma}_{\tau_1,\tau_2}$ is

$$\hat{\gamma}_{\tau_1,\tau_2} = \frac{T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}}{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2}. \tag{B.4}$$

The denominator of (B.4) is

$$T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2 = \left(H_{NT}^0\right)^2 \left[T_w \sum_{t=T_1}^{T_2} f_{0,t-1}^2 - \left(\sum_{t=T_1}^{T_2} f_{0,t-1}\right)^2\right] [1 + o_p(1)]$$

using Lemma B.1. The numerator is

$$T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = \left(H_{NT}^0\right)^2 \left[T_w \sum_{t=T_1}^{T_2} f_{0,t-1} u_{0,t} - \sum_{t=T_1}^{T_2} u_{0,t} \sum_{t=T_1}^{T_2} f_{0,t-1}\right] [1 + o_p(1)].$$

Thus,

$$\begin{aligned} T\hat{\gamma}_{\tau_1,\tau_2} &= T \frac{T_w \sum_{t=T_1}^{T_2} f_{0,t-1} u_{0,t} - \sum_{t=T_1}^{T_2} u_{0,t} \sum_{t=T_1}^{T_2} f_{0,t-1}}{T_w \sum_{t=T_1}^{T_2} f_{0,t-1}^2 - \left(\sum_{t=T_1}^{T_2} f_{0,t-1}\right)^2} [1 + o_p(1)] \\ &\Rightarrow \frac{\tau_w \int_{\tau_1}^{\tau_2} B(r)\, dB(r) - [B(\tau_2) - B(\tau_1)] \int_{\tau_1}^{\tau_2} B(r)\, dr}{\tau_w \int_{\tau_1}^{\tau_2} B(r)^2\, dr - \left[\int_{\tau_1}^{\tau_2} B(r)\, dr\right]^2}, \qquad \text{(B.5)} \\ &= \frac{\tau_w \int_{\tau_1}^{\tau_2} W(r)\, dW(r) - [W(\tau_2) - W(\tau_1)] \int_{\tau_1}^{\tau_2} W(r)\, dr}{\tau_w \int_{\tau_1}^{\tau_2} W(r)^2\, dr - \left[\int_{\tau_1}^{\tau_2} W(r)\, dr\right]^2}, \qquad \text{(B.6)} \end{aligned}$$

where $B(\cdot)$ is Brownian motion with variance σ_{00} and $W(\cdot)$ is standard Brownian motion. Next we find the limit distribution of the least squares estimate

$$\hat{\delta}_{\tau_1,\tau_2} = \frac{\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)}{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2}. \tag{B.7}$$

The denominator of (B.7) is identical to that of (B.4). The numerator is

$$\begin{aligned} &\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right) \\ &= \left(H_{NT}^0\right)^3 \left[\sum_{t=T_1}^{T_2} f_{0,t-1}^2 \sum_{t=T_1}^{T_2} u_{0,t} - \sum_{t=T_1}^{T_2} f_{0,t-1} u_{0,t} \sum_{t=T_1}^{T_2} f_{0,t-1}\right] [1 + o_p(1)], \qquad \text{(B.8)} \end{aligned}$$

using Lemma B.1. Therefore, we have

$$T^{1/2}\hat{\delta}_{\tau_1,\tau_2} = T^{1/2}H^0_{NT}\frac{\sum_{t=T_1}^{T_2} f_{0,t-1}^2 \sum_{t=T_1}^{T_2} u_{0,t} - \sum_{t=T_1}^{T_2} f_{0,t-1}u_{0,t}\sum_{t=T_1}^{T_2} f_{0,t-1}}{T_w\sum_{t=T_1}^{T_2} f_{0,t-1}^2 - \left(\sum_{t=T_1}^{T_2} f_{0,t-1}\right)^2}\left[1+o_p(1)\right]$$
$$\Rightarrow H^0_{NT}\frac{\int_{\tau_1}^{\tau_2} B(r)^2 dr\left[B(\tau_2)-B(\tau_1)\right] - \int_{\tau_1}^{\tau_2} B(r)\,dB(r)\int_{\tau_1}^{\tau_2} B(r)\,dr}{\tau_w\int_{\tau_1}^{\tau_2} B(r)^2 dr - \left[\int_{\tau_1}^{\tau_2} B(r)\,dr\right]^2},$$

using Lemma A.2. Since $H^0_{NT} = O_p(1)$, we have

$$T^{1/2}\hat{\delta}_{\tau_1,\tau_2} = O_p(1). \tag{B.9}$$

The sum of squared errors $\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2$ can be written as

$$\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = \sum_{t=T_1}^{T_2}\Delta\tilde{y}_t^2 + T_w\hat{\delta}^2_{\tau_1,\tau_2} + \hat{\gamma}^2_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - 2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t$$
$$- 2\hat{\gamma}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\Delta\tilde{y}_t + 2\hat{\delta}_{\tau_1,\tau_2}\hat{\gamma}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}$$
$$= \sum_{t=T_1}^{T_2}\Delta\tilde{y}_t^2\left[1+o_p(1)\right] = \left(H^0_{NT}\right)^2\sum_{t=T_1}^{T_2}u_{0,t}^2\left[1+o_p(1)\right] = O_p(T).$$

The first term dominates the other terms since $\hat{\delta}_{\tau_1,\tau_2} = O_p\left(T^{-1/2}\right)$, $\hat{\gamma}_{\tau_1,\tau_2} = O_p\left(T^{-1}\right)$, and

$$\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t^2 = O_p(T);$$
$$\hat{\gamma}^2_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 = O_p\left(T^{-2}\right)O_p\left(T^2\right) = O_p(1);$$
$$2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t = O_p\left(T^{-1/2}\right)O_p\left(T^{1/2}\right) = O_p(1);$$
$$2\widehat{\gamma}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\Delta\tilde{y}_t = O_p\left(T^{-1}\right)O_p(T) = O_p(1);$$
$$2\hat{\delta}_{\tau_1,\tau_2}\widehat{\gamma}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\tilde{y}_{t-1} = O_p\left(T^{-1/2}\right)O_p\left(T^{-1}\right)O_p\left(T^{3/2}\right) = O_p(1).$$

The limit form of the DF test statistic is then obtained as follows

$$DF_{\tau_1,\tau_2} = \hat{\gamma}_{\tau_1,\tau_2}\left[\frac{T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2}{\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2}\right]^{1/2}$$

$$
\begin{aligned}
&= T\hat{\gamma}_{\tau_1,\tau_2}\left[\frac{1}{T^2}\frac{T_w\sum_{t=T_1}^{T_2} f_{0,t-1}^2 - \left(\sum_{t=T_1}^{T_2} f_{0,t-1}\right)^2}{\sum_{t=T_1}^{T_2} u_{0,t}^2}\right]^{1/2} \\
&\Rightarrow \frac{\tau_w\int_{\tau_1}^{\tau_2} W(r)\,dW(r) - [W(\tau_2)-W(\tau_1)]\int_{\tau_1}^{\tau_2} W(r)\,dr}{\tau_w^{1/2}\left[\tau_w\int_{\tau_1}^{\tau_2} W(r)^2\,dr - \left[\int_{\tau_1}^{\tau_2} W(r)\,dr\right]^2\right]^{1/2}},
\end{aligned}
$$

where $W(\cdot)$ is standard Brownian motion. ■

C Proofs Under the Alternative

Proof of Lemma 4.4

Proof. Since $\tilde{\xi}_t = \tilde{L}'X_t/N$ and $X_t = \Gamma g_t + e_t$, we have $\tilde{\xi}_t = \tilde{L}'(\Gamma g_t + e_t)/N$. Recall that $\Gamma_1 = \Lambda_2$, $\Gamma_2 = \Lambda_1$, $\Gamma_3 = \Lambda_0$ and

$$
g_t' = [g_{1t}, g_{2t}, g_{3t}] = \begin{cases} [0,0,f_{0,t}], & \text{if } t \in A \\ [0, f_{1,t}, f_{0,t}], & \text{if } t \in B \\ [f_{2,t}, 0, 0], & \text{if } t \in C \end{cases}.
$$

The estimated first common factor

$$
\begin{aligned}
\tilde{y}_t &= \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\gamma_{i1}g_{1t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\gamma_{i2}g_{2t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\gamma_{i3}g_{3t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it} \\
&= \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\lambda_{2,i}g_{1t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\lambda_{1,i}g_{2t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\lambda_{0,i}g_{3t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it} \qquad \text{(C.1)} \\
&= a_{N,T}g_{1t} + b_{N,T}g_{2t} + c_{N,T}g_{3t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it} \qquad \text{(C.2)} \\
&= \begin{cases} c_{N,T}f_{0,t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it} & \text{if } t \in A \\ b_{N,T}f_{1,t} + c_{N,T}f_{0,t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it} & \text{if } t \in B \\ a_{N,T}f_{2,t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it} & \text{if } t \in C \end{cases}
\end{aligned}
$$

where $a_{N,T} := \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\lambda_{2,i}$, $b_{N,T} := \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\lambda_{1,i}$, and $c_{N,T} := \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\lambda_{0,i}$. By Cauchy-Schwarz

$$
\left|\frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}\lambda_{2,i}\right| \leqslant \frac{1}{N}\sum_{i=1}^N \left|\tilde{L}_{i1}\lambda_{2,i}\right| \leqslant \left(\frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^N \lambda_{2,i}^2\right)^{1/2} = O_p(1),
$$

since $\frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}^2 = O_p(1)$ from the normalization constraint, and $\frac{1}{N}\sum_{i=1}^N \lambda_{2,i}^2 = O_p(1)$ by Assumption 4.2. Thus, $a_{N,T} = O_p(1)$. Using the same argument, we have $b_{N,T} = O_p(1)$,

$c_{N,T} = O_p(1)$, and $\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it} = O_p(1)$. Explicit limit expressions for $a_{N,T}$, $b_{N,T}$ and $c_{N,T}$ are derived in Lemma C.4 and given in (C.9) - (C.11) below. ■

Lemma C.1 *Let $V_{NT,1}$ denote the first eigenvalue of $X'X$. Under the alternative (2.7) and Assumption 4.1-4.3, we have*

$$\begin{array}{ll} \frac{V_{NT,1}}{NT^{2\alpha}\rho_{1T}^{2(T_c-T_r)}} \Rightarrow \pi_{22}\frac{(F_{1,r}+N_c)^2}{2d_1} & \text{if } \alpha>\beta \\ \frac{V_{NT,1}}{NT^{\alpha+\beta}\rho_{1T}^{2(T_c-T_r)}} \Rightarrow \pi_{11}\frac{1}{2d_2}F_{2,c}^2 & \text{if } \alpha<\beta \end{array}.$$

Proof. This proof follows arguments similar to those in Chen et al. (2019) and is outlined here for completeness. Under the alternative, $X = G\Gamma' + E$. The quantity $X'X$ can be rewritten as

$$X'X = \Gamma G'G\Gamma' + \Gamma G'E + E'G\Gamma' + E'E. \tag{C.3}$$

From Assumption 4.2, $\|\Gamma'\| = \sqrt{N}\left(\sum_{j=1}^{3}\frac{1}{N}\sum_{i=1}^{N}\gamma_{ij}^2\right)^{1/2} = O_p\left(\sqrt{N}\right)$. Further,

$$\|G'G\| = \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\left(G_i'G_j\right)^2\right)^{1/2} = \begin{cases} \left(\sum_{t=T_e}^{T_c} f_{1,t}^2\left[1+o_p(1)\right]\right) = O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha>\beta \\ \left(\sum_{t=T_c}^{T} f_{2,t}^2\left[1+o_p(1)\right]\right) = O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha<\beta \end{cases}$$

since, by construction, $G_1'G_2 = 0$, $G_1'G_3 = 0$, $G_3'G_2 = \sum_{t=T_e}^{T_c} f_{0,t}f_{1,t}$, $G_1'G_1 = \sum_{t=T_c}^{T} f_{2,t}^2$, $G_2'G_2 = \sum_{t=T_e}^{T_c} f_{1,t}^2$, and $G_3'G_3 = \sum_{t=1}^{T_c} f_{0,t}^2$. The orders of magnitude are from Lemmas A.3 and A.4. The quantity

$$\|\Gamma'E'G\| \leqslant \|\Gamma'\|\,\|E'G\| = \begin{cases} O_p\left(N^{1/2}T^{\alpha}\rho_T^{T-T_c}\right) & \text{if } \alpha>\beta \\ O_p\left(N^{1/2}T^{(\alpha+\beta)/2}\rho_T^{T-T_c}\right) & \text{if } \alpha<\beta \end{cases},$$

since $\|\Gamma'\| = O_p\left(\sqrt{N}\right)$,

$$\sum_{j=1}^{3}\left(\underline{e}_i'G_j\right)^2 = \begin{cases} \left(\underline{e}_i'G_2\right)^2\left[1+o_p(1)\right] = O_p\left(T^{2\alpha}\rho_T^{2(T-T_c)}\right) & \text{if } \alpha>\beta \\ \left(\underline{e}_i'G_1\right)^2\left[1+o_p(1)\right] = O_p\left(T^{\alpha+\beta}\rho_T^{2(T-T_c)}\right) & \text{if } \alpha<\beta \end{cases},$$

from Assumption 4.4, and

$$\|E'G\| = \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{3}\left(\underline{e}_i'G_j\right)^2\right)^{1/2} = \begin{cases} O_p\left(\sqrt{N}T^{\alpha}\rho_T^{T-T_c}\right) & \text{if } \alpha>\beta \\ O_p\left(\sqrt{N}T^{(\alpha+\beta)/2}\rho_T^{T-T_c}\right) & \text{if } \alpha<\beta \end{cases}.$$

The quantity

$$\|E'E\| = NT\left(\frac{1}{N^2}\sum_{i,j}\left(\frac{1}{T}\sum_{t=1}^{T}e_{ti}e_{tj}\right)^2\right)^{1/2} \leq NT\left(\frac{1}{N^2}\sum_{i,j}M^2\right)^{1/2} = O_p(NT),$$

from Assumption 4.3(2). Therefore, the first term in (C.3) dominates the other terms and

$$X'X = \Gamma G'G\Gamma' \left[1 + o_p\left(1\right)\right]. \tag{C.4}$$

Next, by the fundamental property of the eigenvalue $V_{NT,1}$

$$\left|X'X - V_{NT,1} I_N\right| = 0 \tag{C.5}$$

with I_N the $N \times N$ identity matrix. Multiplying the equation by $|\Gamma'|$ from the left and $|\Gamma|$ from right and scaling by $1/N$, we obtain

$$\begin{aligned} & \left|\frac{1}{N}\Gamma' X' X \Gamma - V_{NT,1}\frac{1}{N}\Gamma'\Gamma\right| = 0 \\ \Longrightarrow \quad & \left|\frac{1}{N}\Gamma' X' X \Gamma \left(\frac{1}{N}\Gamma'\Gamma\right)^{-1} - V_{NT,1} I_3\right| = 0 \\ \Longrightarrow \quad & \left|\frac{1}{N}\Gamma'\Gamma\left(NG'G\right)\left[1 + o_p\left(1\right)\right] - V_{NT,1} I_3\right| = 0 \end{aligned} \tag{C.6}$$

using result (C.4). If $\alpha > \beta$,

$$\frac{\frac{1}{N}\Gamma'\Gamma\left(NG'G\right)}{NT^{2\alpha}\rho_T^{2(T_c - T_r)}} \Rightarrow \begin{bmatrix} \pi_{11} & \pi_{21} & \pi_{31} \\ \pi_{21} & \pi_{22} & \pi_{32} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(F_{1,r}+N_c)^2}{2d_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \pi_{21}\frac{(F_{1,r}+N_c)^2}{2d_1} & 0 \\ 0 & \pi_{22}\frac{(F_{1,r}+N_c)^2}{2d_1} & 0 \\ 0 & \pi_{32}\frac{(F_{1,r}+N_c)^2}{2d_1} & 0 \end{bmatrix} \tag{C.7}$$

since $\frac{1}{NT^{2\alpha}\rho_T^{2(T_c-T_r)}} NG_2'G_2 = \frac{1}{T^{2\alpha}\rho_T^{2(T_c-T_r)}} \sum_{t=T_e}^{T_c} f_{1,t}^2 \sim \frac{1}{2d_1}\left(F_{1,r} + N_{d_1}\right)^2$ from Lemma A.3. It follows from (C.6) and (C.7) that

$$\frac{1}{NT^{2\alpha}\rho_T^{2(T_c-T_r)}} V_{NT,1} \Rightarrow \pi_{22}\frac{\left(F_{1,r}+N_c\right)^2}{2d_1}.$$

Similarly, if $\alpha < \beta$, from Lemma A.4, $\frac{1}{NT^{\alpha+\beta}\rho_T^{2(T_c-T_r)}} NG_1'G_1 = \frac{1}{T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}} \sum_{t=T_c}^{T} f_{2,t}^2 \sim \frac{1}{2d_2}F_{2,c}^2$ and

$$\frac{1}{NT^{\alpha+\beta}\rho_T^{2(T_c-T_r)}} \frac{1}{N}\Gamma'\Gamma\left(NG'G\right) \Rightarrow \begin{bmatrix} \pi_{11}\frac{1}{2d_2}F_{2,c}^2 & 0 & 0 \\ \pi_{21}\frac{1}{2d_2}F_{2,c}^2 & 0 & 0 \\ \pi_{31}\frac{1}{2d_2}F_{2,c}^2 & 0 & 0 \end{bmatrix}.$$

It follows that

$$\frac{1}{NT^{\alpha+\beta}\rho_T^{2(T_c-T_r)}} V_{NT,1} \Rightarrow \pi_{11}\frac{1}{2d_2}F_{2,c}^2.$$

■

Lemma C.2 *Let $\tilde{L}_1$ be the estimated first factor loading and*

$$H_{1,NT} = \begin{cases} G_2' G_2 \Gamma_2' \tilde{L}_1 V_{NT,1}^{-1} & \text{if } \alpha > \beta \\ G_1' G_1 \Gamma_1' \tilde{L}_1 V_{NT,1}^{-1} & \text{if } \alpha < \beta \end{cases}.$$

Under the alternative (2.7) and Assumptions 4.1-4.4, we have

$$\begin{Vmatrix} \tilde{L}_1 - \Gamma_2 H_{1,NT} \\ \tilde{L}_1 - \Gamma_1 H_{1,NT} \end{Vmatrix} \begin{matrix} = O_p\left(N^{1/2} T^{\beta-\alpha}\right) & \text{if } \alpha > \beta \\ = O_p\left(N^{1/2} T^{\alpha-\beta}\right) & \text{if } \alpha < \beta \end{matrix}.$$

Proof. This proof follows arguments similar to those in Chen et al. (2019) and is outlined here for completeness. By definition $\tilde{L}_1$ equals $\sqrt{N}$ times the eigenvector corresponding to the first eigenvalue of $X'X$. We have the usual eigenvector matrix equality $X'X\tilde{L}_1 = \tilde{L}_1 V_{NT,1}$. It follows that

$$\begin{aligned} \tilde{L}_1 &= X'X\tilde{L}_1 V_{NT,1}^{-1} = \Gamma G' G \Gamma' \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right] \\ &= \left[\Gamma_1 G_1' G_1 \Gamma_1' + \Gamma_2 G_2' G_2 \Gamma_2' + \Gamma_2 G_2' G_3 \Gamma_3' + \Gamma_3 G_3' G_2 \Gamma_2' + \Gamma_3 G_3' G_3 \Gamma_3'\right] \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right], \end{aligned}$$

using equation (C.4) and by construction, $G_1' G_2 = 0$ and $G_1' G_3 = 0$. By definition, if $\alpha > \beta$,

$$\tilde{L}_1 - \Gamma_2 H_{1,NT} = \left[\Gamma_1 G_1' G_1 \Gamma_1' + \Gamma_2 G_2' G_3 \Gamma_3' + \Gamma_3 G_3' G_2 \Gamma_2' + \Gamma_3 G_3' G_3 \Gamma_3'\right] \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right];$$

and if $\alpha < \beta$,

$$\tilde{L}_1 - \Gamma_1 H_{1,NT} = \left[\Gamma_2 G_2' G_2 \Gamma_2' + \Gamma_2 G_2' G_3 \Gamma_3' + \Gamma_3 G_3' G_2 \Gamma_2' + \Gamma_3 G_3' G_3 \Gamma_3'\right] \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right].$$

From Lemma A.2, A.3, and A.4, $G_1' G_1 = \sum_{t=T_c}^{T} f_{2,t}^2 = O_p\left(T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right)$, $G_3' G_2 = \sum_{t=T_e}^{T_c} f_{0,t} f_{1,t} = O_p\left(T^{(3\alpha+1)/2} \rho_T^{T_c - T_r}\right)$, $G_2' G_2 = \sum_{t=T_e}^{T_c} f_{1,t}^2 = O_p\left(T^{2\alpha} \rho_T^{2(T_c - T_r)}\right)$, and $G_3' G_3 = \sum_{t=1}^{T_c} f_{0,t}^2 = O_p\left(T^2\right)$. From Assumption 4.2(2), $\|\Gamma_i\| = O_p\left(\sqrt{N}\right)$ with $i = 1, 2, 3$. It follows that

$$\begin{aligned} \left\|\Gamma_1 G_1' G_1 \Gamma_1'\right\| &\leq \|\Gamma_1\| \left\|G_1' G_1\right\| \left\|\Gamma_1'\right\| = O_p\left(N T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right) \\ \left\|\Gamma_2 G_2' G_2 \Gamma_2'\right\| &\leq \|\Gamma_2\| \left\|G_2' G_2\right\| \left\|\Gamma_2'\right\| = O_p\left(N T^{2\alpha} \rho_T^{2(T_c - T_r)}\right), \\ \left\|\Gamma_2 G_2' G_3 \Gamma_3'\right\| &\leq \|\Gamma_2\| \left\|G_2' G_3\right\| \left\|\Gamma_3'\right\| = O_p\left(N T^{(3\alpha+1)/2} \rho_T^{T_c - T_r}\right), \\ \left\|\Gamma_3 G_3' G_3 \Gamma_3'\right\| &\leq \|\Gamma_3\| \left\|G_3' G_3\right\| \left\|\Gamma_3'\right\| = O_p\left(N T^2\right). \end{aligned}$$

Therefore, if $\alpha > \beta$

$$\tilde{L}_1 - \Gamma_2 H_{1,NT} = \Gamma_1 G_1' G_1 \Gamma_1' \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right]; \tag{C.8}$$

and if $\alpha < \beta$

$$\tilde{L}_1 - \Gamma_1 H_{1,NT} = \Gamma_2 G_2' G_2 \Gamma_2' \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right].$$

From the normalization condition, we have $\tilde{L}_1' \tilde{L}_1 / N = 1$ and hence $\left\| \tilde{L}_1 \right\| = O_p\left(\sqrt{N}\right)$. Then

$$\begin{aligned}
& \left\| \tilde{L}_1 - \Gamma_2 H_{1,NT} \right\| \\
= & \left\| \Gamma_1 G_1' G_1 \Gamma_1' \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right] \right\| \\
\leq & \left\| \Gamma_1 \right\| \left\| G_1' G_1 \right\| \left\| \Gamma_1' \right\| \left\| \tilde{L}_1 \right\| \left\| V_{NT,1}^{-1} \right\| \\
= & O_p\left(\sqrt{N}\right) O_p\left(T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right) O_p\left(\sqrt{N}\right) O_p\left(\sqrt{N}\right) O_p\left(N^{-1} T^{-2\alpha} \rho_{1T}^{-2(T_c - T_r)}\right) \\
= & O_p\left(N^{1/2} T^{\beta - \alpha}\right)
\end{aligned}$$

if $\alpha > \beta$; and

$$\begin{aligned}
& \left\| \tilde{L}_1 - \Gamma_1 H_{1,NT} \right\| \\
= & \left\| \Gamma_2 G_2' G_2 \Gamma_2' \tilde{L}_1 V_{NT,1}^{-1} \left[1 + o_p(1)\right] \right\| \\
\leq & \left\| \Gamma_2 \right\| \left\| G_2' G_2 \right\| \left\| \Gamma_2' \right\| \left\| \tilde{L}_1 \right\| \left\| V_{NT,1}^{-1} \right\| \\
= & O_p\left(\sqrt{N}\right) O_p\left(T^{2\alpha} \rho_T^{2(T_c - T_r)}\right) O_p\left(\sqrt{N}\right) O_p\left(\sqrt{N}\right) O_p\left(N^{-1} T^{-(\alpha+\beta)} \rho_{1T}^{-2(T_c - T_r)}\right) \\
= & O_p\left(N^{1/2} T^{\alpha - \beta}\right)
\end{aligned}$$

if $\alpha < \beta$, using results in Lemma C.1. ■

Lemma C.3 *Under the alternative (2.7) and Assumption 4.1-4.5, we have the following limits as* $N, T \to \infty$,

$$H_{1,NT} \to_p H_1 := \begin{cases} \pi_{22}^{-1/2} & \text{if } \alpha > \beta \\ \pi_{11}^{-1/2} & \text{if } \alpha < \beta \end{cases}.$$

where $H_{1,NT}$ *is a scalar defined as*

$$H_{1,NT} = \begin{cases} G_2' G_2 \Gamma_2' \tilde{L}_1 V_{NT,1}^{-1} & \text{if } \alpha > \beta \\ G_1' G_1 \Gamma_1' \tilde{L}_1 V_{NT,1}^{-1} & \text{if } \alpha < \beta \end{cases}.$$

Proof. From Lemmas A.3, A.4, and C.1, Assumption 4.2, and the normalization condition $\tilde{L}_1' \tilde{L}_1 / N = 1$, we have

$$\begin{aligned}
& \left| H_{1,NT} \right| = \left\| G_2' G_2 \Gamma_2' \tilde{L}_1 V_{NT,1}^{-1} \right\| \leq \left\| G_2' G_2 \right\| \left\| \Gamma_2' \right\| \left\| \tilde{L}_1 \right\| \left\| V_{NT,1}^{-1} \right\| \\
& = O_p\left(T^{2\alpha} \rho_T^{2(T_c - T_r)}\right) O_p\left(N^{1/2}\right) O_p\left(N^{1/2}\right) O_p\left(\frac{1}{N T^{2\alpha} \rho_{1T}^{2(T_c - T_r)}}\right) = O_p(1)
\end{aligned}$$

if $\alpha > \beta$, and

$$\begin{aligned}
|H_{1,NT}| &= \left\| G_1' G_1 \Gamma_1' \tilde{L}_1 V_{NT,1}^{-1} \right\| \leq \|G_1' G_1\| \|\Gamma_1'\| \left\|\tilde{L}_1\right\| \left\| V_{NT,1}^{-1}\right\| \\
&= O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right) O_p\left(N^{1/2}\right) O_p\left(N^{1/2}\right) O_p\left(\frac{1}{NT^{\alpha+\beta}\rho_{1T}^{2(T_c-T_r)}}\right) = O_p(1)
\end{aligned}$$

if $\alpha < \beta$.

Let $D_1 = \frac{\tilde{L}_1' \tilde{L}_1}{N} = 1$. Consider the case $\alpha > \beta$ and let $D_0 = H_{1,NT}^2 \frac{\Gamma_2' \Gamma_2}{N}$. We have

$$\begin{aligned}
|D_1 - D_0| &= \left| \frac{\tilde{L}_1' \tilde{L}_1}{N} - H_{1,NT}^2 \frac{\Gamma_2' \Gamma_2}{N} \right| \\
&= \frac{1}{N} \left| \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} + \Gamma_2 H_{1,NT} \right)' \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} + \Gamma_2 H_{1,NT} \right) - H_{1,NT}^2 \Gamma_2' \Gamma_2 \right| \\
&\leqslant \frac{1}{N} \left[\left| \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} \right)' \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} \right) \right| + 2 \left| H_{1,NT} \Gamma_2' \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} \right) \right| \right] \\
&= 2 \frac{1}{N} \left| H_{1,NT} \Gamma_2' \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} \right) \right| [1 + o_p(1)] = O_p\left(T^{\beta-\alpha}\right),
\end{aligned}$$

since

$$\begin{aligned}
\left| \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} \right)' \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} \right) \right| &\leq \left\| \tilde{L}_1' - \Gamma_2' H_{1,NT} \right\| \left\| \tilde{L}_1 - \Gamma_2 H_{1,NT} \right\| = O_p\left(NT^{2(\beta-\alpha)}\right), \\
2 \left| H_{1,NT} \Gamma_2' \left(\tilde{L}_1 - \Gamma_2 H_{1,NT} \right) \right| &\leq 2 |H_{1,NT}| \|\Gamma_2'\| \left\| \tilde{L}_1 - \Gamma_2 H_{1,NT} \right\| = O_p\left(NT^{\beta-\alpha}\right).
\end{aligned}$$

The result $|D_1 - D_0| = O_p\left(T^{\beta-\alpha}\right)$ implies that $\left|1 - H_{1,NT}^2 \frac{\Gamma_2'\Gamma_2}{N}\right| = o_p(1)$. Therefore, we have

$$H_{1,NT}^2 \frac{\Gamma_2' \Gamma_2}{N} \to_p 1 \Rightarrow H_{1,NT} \to_p \pi_{22}^{-1/2}.$$

Similarly, when $\alpha < \beta$, we let $D_0 = H_{1,NT}^2 \frac{\Gamma_1' \Gamma_1}{N}$. It follows that

$$\begin{aligned}
|D_1 - D_0| &= \left| \frac{\tilde{L}_1' \tilde{L}_1}{N} - H_{1,NT}^2 \frac{\Gamma_1' \Gamma_1}{N} \right| \\
&= \frac{1}{N} \left| \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} + \Gamma_1 H_{1,NT} \right)' \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} + \Gamma_1 H_{1,NT} \right) - H_{1,NT}^2 \Gamma_1' \Gamma_1 \right| \\
&\leqslant \frac{1}{N} \left[\left| \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} \right)' \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} \right) \right| + 2 \left| H_{1,NT} \Gamma_1' \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} \right) \right| \right] \\
&= 2 \frac{1}{N} \left| H_{1,NT} \Gamma_1' \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} \right) \right| [1 + o_p(1)] = O_p\left(T^{\alpha-\beta}\right),
\end{aligned}$$

since

$$\left| \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} \right)' \left(\tilde{L}_1 - \Gamma_1 H_{1,NT} \right) \right| \leq \left\| \tilde{L}_1' - \Gamma_1' H_{1,NT} \right\| \left\| \tilde{L}_1 - \Gamma_1 H_{1,NT} \right\| = O_p\left(NT^{2(\alpha-\beta)}\right),$$

$$2\left|H_{1,NT}\Gamma_1'\left(\tilde{L}_1-\Gamma_1 H_{1,NT}\right)\right| \leq 2\left|H_{1,NT}\right|\left\|\Gamma_1'\right\|\left\|\tilde{L}_1-\Gamma_1 H_{1,NT}\right\| = O_p\left(NT^{\alpha-\beta}\right).$$

The result $|D_1 - D_0| = O_p\left(T^{\alpha-\beta}\right)$ implies that $\left|1 - H_{1,NT}^2\frac{\Gamma_1'\Gamma_1}{N}\right| = o_p(1)$. Therefore, we have

$$H_{1,NT}^2\frac{\Gamma_1'\Gamma_1}{N} \to_p 1 \Rightarrow H_{1,NT} \to_p \pi_{11}^{-1/2}.$$

■

Lemma C.4 *Under the alternative (2.7) and Assumptions 4.1-4.5, we have the following explicit limits as* $N, T \to \infty$,

$$a_{N,T} \to_p a \ :=\begin{cases} \pi_{22}^{-1/2}\pi_{21} & \text{if } \alpha>\beta \\ \pi_{11}^{1/2} & \text{if } \alpha<\beta \end{cases}, \tag{C.9}$$

$$b_{N,T} \to_p b \ :=\begin{cases} \pi_{22}^{1/2} & \text{if } \alpha>\beta \\ \pi_{22}^{-1/2}\pi_{12} & \text{if } \alpha<\beta \end{cases}, \tag{C.10}$$

$$c_{N,T} \to_p c \ :=\begin{cases} \pi_{22}^{-1/2}\pi_{23} & \text{if } \alpha>\beta \\ \pi_{11}^{-1/2}\pi_{13} & \text{if } \alpha<\beta \end{cases}. \tag{C.11}$$

Proof. (1) Consider first the case $\alpha > \beta$. Rewrite $a_{N,T}$ as

$$\begin{aligned} a_{N,T} &= \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\lambda_{2,i} = \frac{1}{N}\tilde{L}_1'\Gamma_1 = \frac{1}{N}\left(\tilde{L}_1-\Gamma_2 H_{1,NT}\right)'\Gamma_1 + \frac{1}{N}H_{1,NT}\Gamma_2'\Gamma_1 \\ &= \frac{1}{N}H_{1,NT}'\Gamma_2'\Gamma_1 + O_p\left(T^{\beta-\alpha}\right) \to_p a := \pi_{22}^{-1/2}\pi_{21}, \end{aligned}$$

since

$$\begin{aligned} \left|\frac{1}{N}\left(\tilde{L}_1-\Gamma_2 H_{1,NT}\right)'\Gamma_1\right| &= \frac{1}{N}\left\|\left(\tilde{L}_1-\Gamma_2 H_{1,NT}\right)'\right\|\left\|\Gamma_1\right\| \\ &= \frac{1}{N}O_p\left(N^{1/2}T^{\beta-\alpha}\right)O_p\left(N^{1/2}\right) = O_p\left(T^{\beta-\alpha}\right), \end{aligned}$$

and $\frac{1}{N}H_{1,NT}\Gamma_2'\Gamma_1 \to_p H_1\pi_{21}$ from Lemma C.3 and Assumption 4.2. Similarly, we obtain

$$\begin{aligned} b_{N,T} &= \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\lambda_{1,i} = \frac{1}{N}\tilde{L}_1'\Gamma_2 = \frac{1}{N}\left(\tilde{L}_1-\Gamma_2 H_{1,NT}\right)'\Gamma_2 + \frac{1}{N}H_{1,NT}\Gamma_2'\Gamma_2 \\ &= \frac{1}{N}H_{1,NT}\Gamma_2'\Gamma_2 + O_p\left(T^{\beta-\alpha}\right) \to_p b := \pi_{22}^{1/2}, \end{aligned}$$

and

$$c_{N,T} = \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\lambda_{0,i} = \frac{1}{N}\tilde{L}_1'\Gamma_3 = \frac{1}{N}\left(\tilde{L}_1-\Gamma_2 H_{1,NT}\right)'\Gamma_3 + \frac{1}{N}H_{1,NT}\Gamma_2'\Gamma_3$$

$$= \frac{1}{N} H_{1,NT} \Gamma_2' \Gamma_3 + O_p\left(T^{\beta-\alpha}\right) \to_p c := \pi_{22}^{-1/2} \pi_{23}.$$

Next, consider the case $\alpha < \beta$. Rewrite $a_{N,T}$ as

$$\begin{aligned} a_{N,T} &= \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \lambda_{2,i} = \frac{1}{N} \tilde{L}_1' \Gamma_1 = \frac{1}{N} \left(\tilde{L}_1 - \Gamma_1 H_{1,NT}\right)' \Gamma_1 + \frac{1}{N} H_{1,NT} \Gamma_1' \Gamma_1 \\ &= \frac{1}{N} H_{1,NT}' \Gamma_1' \Gamma_1 \left[1 + o_p(1)\right] \to_p a := \pi_{11}^{1/2}, \end{aligned}$$

since

$$\begin{aligned} \left| \frac{1}{N} \left(\tilde{L}_1 - \Gamma_1 H_{1,NT}\right)' \Gamma_1 \right| &= \frac{1}{N} \left\| \left(\tilde{L}_1 - \Gamma_1 H_{1,NT}\right)' \right\| \|\Gamma_1\| \\ = \frac{1}{N} O_p\left(N^{1/2} T^{\alpha-\beta}\right) O_p\left(N^{1/2}\right) &= O_p\left(T^{\alpha-\beta}\right), \end{aligned}$$

and $\frac{1}{N} H_{1,NT} \Gamma_1' \Gamma_1 \to_p H_1 \pi_{11}$ from Lemma C.3 and Assumption 4.2. Similarly,

$$\begin{aligned} b_{N,T} &= \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \lambda_{1,i} = \frac{1}{N} \tilde{L}_1' \Gamma_2 = \frac{1}{N} \left(\tilde{L}_1 - \Gamma_1 H_{1,NT}\right)' \Gamma_2 + \frac{1}{N} H_{1,NT} \Gamma_1' \Gamma_2 \\ &= \frac{1}{N} H_{1,NT} \Gamma_1' \Gamma_2 + O_p\left(T^{\beta-\alpha}\right) \to_p b := \pi_{22}^{-1/2} \pi_{12}, \end{aligned}$$

and

$$\begin{aligned} c_{N,T} &= \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \lambda_{0,i} = \frac{1}{N} \tilde{L}_1' \Gamma_3 = \frac{1}{N} \left(\tilde{L}_1 - \Gamma_1 H_{1,NT}\right)' \Gamma_3 + \frac{1}{N} H_{1,NT} \Gamma_1' \Gamma_3 \\ &= \frac{1}{N} H_{1,NT} \Gamma_1' \Gamma_3 + O_p\left(T^{\beta-\alpha}\right) \to_p c := \pi_{11}^{-1/2} \pi_{13}, \end{aligned}$$

giving the required results. ■

Lemma C.5 *Under the alternative (2.7) and Assumptions 4.1, 4.2(2), 4.3, 4.4, and 4.5, when $\tau_1 \in [0, \tau_e]$ and $\tau_2 \in (\tau_e, \tau_c]$, we have*

$$(a) \quad \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = b \sum_{t=T_e+1}^{T_2} f_{1,t-1} \left[1 + o_p(1)\right] = O_p\left(T^{3\alpha/2} \rho_T^{T_2 - T_r}\right);$$

$$(b) \quad \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = b^2 \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{2\alpha} \rho_T^{2(T_2 - T_r)}\right);$$

$$(c) \quad \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} = b^2 \frac{d_1}{T^\alpha} \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha} \rho_T^{2(T_2 - T_r)}\right);$$

$$(d) \quad \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t^2 = b^2 \frac{d_1^2}{T^{2\alpha}} \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(\rho_T^{2(T_2 - T_r)}\right);$$

$$(e)\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t = b\frac{d_1}{T^{\alpha}}\sum_{t=T_e+1}^{T_2} f_{1,t-1}\left[1+o_p\left(1\right)\right] = O_p\left(T^{\alpha/2}\rho_T^{T_2-T_r}\right).$$

Proof. (a) From Lemma 4.4, we can rewrite $\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}$ as

$$\sum_{t=T_1}^{T_2}\tilde{y}_{t-1} = b_{N,T}\sum_{t=T_e+1}^{T_2} f_{1,t-1} + c_{N,T}\sum_{t=T_1}^{T_2} f_{0,t-1} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\left(\sum_{t=T_1}^{T_2} e_{it-1}\right).$$

By Cauchy-Schwarz

$$\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\left(\sum_{t=T_1}^{T_2} e_{it-1}\right) \leqslant T^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}^2\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T^{1/2}}\sum_{t=T_1}^{T_2} e_{it-1}\right)^2\right)^{1/2} = O_p\left(T^{1/2}\right).$$

Using Lemma 4.4, A.3, and Lemma C.4 we know that

$$\sum_{t=T_1}^{T_2}\tilde{y}_{t-1} = b\sum_{t=T_e+1}^{T_2} f_{1,t-1} + O_p\left(T^{3/2}\right) = O_p\left(T^{3\alpha/2}\rho_T^{T_2-T_r}\right). \tag{C.12}$$

(b)

$$\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 = \sum_{t=T_1}^{T_e}\tilde{y}_{t-1}^2 + \sum_{t=T_e+1}^{T_2}\tilde{y}_{t-1}^2.$$

The first term

$$\begin{aligned}
\sum_{t=T_1}^{T_e}\tilde{y}_{t-1}^2 &= \sum_{t=T_1}^{T_e}\left(c_{N,T}f_{0,t-1} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right)^2 \\
&= c_{N,T}^2\left(\sum_{t=T_1}^{T_2} f_{0,t-1}^2\right) + \left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right)^2 + 2c_{N,T}\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\left(\sum_{t=T_1}^{T_2} f_{0,t-1}e_{it-1}\right) \\
&= c^2\left(\sum_{t=T_1}^{T_2} f_{0,t-1}^2\right)\left[1+o_p\left(1\right)\right] = O_p\left(T^2\right).
\end{aligned}$$

The second term

$$\begin{aligned}
&\sum_{t=T_e+1}^{T_2}\tilde{y}_{t-1}^2 = \sum_{t=T_e+1}^{T_2}\left(b_{N,T}f_{1,t-1} + c_{N,T}f_{0,t-1} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right)^2 \\
&= b_{N,T}^2\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2\right) + c_{N,T}^2\left(\sum_{t=T_e+1}^{T_2} f_{0,t-1}^2\right) + \sum_{t=T_e+1}^{T_2}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right)^2 \\
&+ 2b_{N,T}c_{N,T}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}f_{0,t-1}\right) + 2b_{N,T}\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}e_{it-1}\right) + 2c_{N,T}\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\left(\sum_{t=T_e+1}^{T_2} f_{0,t-1}e_{it-1}\right.
\end{aligned}$$

$$= b^2 \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + O_p\left(T^{(1+3\alpha)/2}\rho_T^{T-T_r}\right) = O_p\left(T^{2\alpha}\rho_T^{2(T_2-T_r)}\right).$$

using Lemma 4.4 and A.3. Therefore, the second term dominates and

$$\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = b^2 \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{2\alpha}\rho_T^{2(T_2-T_r)}\right). \tag{C.13}$$

(c) Similarly,

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1} = \sum_{t=T_1}^{T_e} \Delta\tilde{y}_t\tilde{y}_{t-1} + \sum_{t=T_e+1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1}.$$

The first term

$$\begin{aligned}
\sum_{t=T_1}^{T_e} \Delta\tilde{y}_t\tilde{y}_{t-1} &= \sum_{t=T_1}^{T_e}\left(c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it}\right)\left(c_{N,T}f_{0,t-1} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right) \\
&= c_{N,T}^2 \sum_{t=T_1}^{T_e} u_{0,t}f_{0,t-1} + c_{N,T}\sum_{t=T_1}^{T_e} u_{0,t}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right) \\
&\quad + c_{N,T}\sum_{t=T_1}^{T_e} f_{0,t-1}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it}\right) + \sum_{t=T_1}^{T_e}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it}\right)\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right) \\
&= O_p(T)
\end{aligned}$$

since $\sum_{t=T_1}^{T_e} u_{0,t}f_{0,t-1} = O_p(T)$, $\sum_{t=T_1}^{T_e} u_{0,t} = O_p\left(T^{1/2}\right)$, $\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1} = O_p(1)$, and

$$\sum_{t=T_1}^{T_e} f_{0,t-1}\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it-1} \leqslant T\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\right)^{1/2}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=T_1}^{T_e} f_{0,t-1}\Delta e_{it-1}\right)^2\right)^{1/2} = O_p(T).$$

The second term

$$\begin{aligned}
&\sum_{t=T_e+1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1} \\
&= \sum_{t=T_e+1}^{T_2}\left(b_{N,T}(\rho_T - 1) f_{1,t-1} + b_{N,T}u_{1,t} + c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it}\right) \\
&\quad \left(b_{N,T}f_{1,t-1} + c_{N,T}f_{0,t-1} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right) \\
&= b_{N,T}^2(\rho_T - 1)\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + b_{N,T}c_{N,T}(\rho_T - 1)\sum_{t=T_e+1}^{T_2} f_{1,t-1}f_{0,t-1} \\
&\quad + b_{N,T}(\rho_T - 1)\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}e_{it-1}\right) + b_{N,T}^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}u_{1,t}
\end{aligned}$$

$$+ b_{N,T}c_{N,T}\sum_{t=T_e+1}^{T_2} f_{0,t-1}u_{1,t} + b_{N,T}\sum_{t=T_e+1}^{T_2} u_{1,t}\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}$$
$$+ b_{N,T}c_{N,T}\sum_{t=T_e+1}^{T_2} u_{0,t}f_{1,t-1} + c_{N,T}^2\sum_{t=T_e+1}^{T_2} u_{0,t}f_{0,t-1} + c_{N,T}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\right)\left(\sum_{t=T_e+1}^{T_2} u_{0,t}e_{it-1}\right)$$
$$+ b_{N,T}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\right)\left[\sum_{t=T_e+1}^{T_2} f_{1,t-1}\Delta e_{it}\right] + c_{N,T}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\right)\left[\sum_{t=T_e+1}^{T_2} f_{0,t-1}\Delta e_{it}\right]$$
$$+ \sum_{t=T_e+1}^{T_2}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it}\right)\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right) \tag{C.14}$$
$$= b^2(\rho_T - 1)\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + O_p\left(T^{(1+\alpha)/2}\rho_T^{T_2-T_r}\right) = O_p\left(T^{\alpha}\rho_T^{2(T_2-T_r)}\right), \tag{C.15}$$

using Lemma 4.4 and A.3. Therefore, we have

$$\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1} = b^2\frac{d_1}{T^{\alpha}}\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2\left[1 + o_p(1)\right] = O_p\left(T^{\alpha}\rho_T^{2(T_2-T_r)}\right). \tag{C.16}$$

(d) The quantity

$$\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t^2 = \sum_{t=T_1}^{T_e}\Delta\tilde{y}_t^2 + \sum_{t=T_e+1}^{T_2}\Delta\tilde{y}_t^2.$$

The first term

$$\begin{aligned}\sum_{t=T_1}^{T_e}\Delta\tilde{y}_t^2 &= \sum_{t=T_1}^{T_e}\left[c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}(e_{it} - e_{it-1})\right]^2 \\ &= c_{N,T}^2\sum_{t=T_1}^{T_e}u_{0,t}^2 + \sum_{t=T_1}^{T_e}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}(e_{it} - e_{it-1})\right)^2 + 2c_{N,T}\sum_{t=T_1}^{T_e}u_{0,t}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}(e_{it} - e_{it-1})\right) \\ &= O_p(T).\end{aligned}$$

The second term

$$\sum_{t=T_e+1}^{T_2}\Delta\tilde{y}_t^2 = \sum_{t=T_1}^{T_2}\left[b_{N,T}(f_{1,t} - f_{1,t-1}) + c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}(e_{it} - e_{it-1})\right]^2$$
$$= \sum_{t=T_1}^{T_2}\left[b_{N,T}\frac{d_1}{T^{\alpha}}f_{1,t-1} + b_{N,T}u_{1,t} + c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}(e_{it} - e_{it-1})\right]^2$$
$$= b_{N,T}^2\frac{d_1^2}{T^{2\alpha}}\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + \sum_{t=T_1}^{T_2}\left[b_{N,T}u_{1,t} + c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}(e_{it} - e_{it-1})\right]^2$$
$$+ 2b_{N,T}\frac{d_1}{T^{\alpha}}\sum_{t=T_e+1}^{T_2} f_{1,t-1}\left[b_{N,T}u_{1,t} + c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}(e_{it} - e_{it-1})\right]$$

$$= b^2 \frac{d_1^2}{T^{2\alpha}} \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + O_p\left(T^{\alpha/2}\rho_T^{(T_2-T_r)}\right) = O_p\left(\rho_T^{2(T_2-T_r)}\right),$$

using Lemmas 4.4 and A.3. Therefore, the second term dominates and

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t^2 = b^2 \frac{d_1^2}{T^{2\alpha}} \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(\rho_T^{2(T_2-T_r)}\right).$$

(e) The quantity

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t = \sum_{t=T_1}^{T_e} \Delta\tilde{y}_t + \sum_{t=T_e+1}^{T_2} \Delta\tilde{y}_t.$$

The first term

$$\begin{aligned} \sum_{t=T_1}^{T_e} \Delta\tilde{y}_t &= \sum_{t=T_1}^{T_e} \left[c_{N,T}\left(f_{0,t} - f_{0,t-1}\right) + \frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1}\left(e_{it} - e_{it-1}\right) \right] \\ &= c_{N,T} \sum_{t=T_1}^{T_2} u_{0,t} + \frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1} \sum_{t=T_1}^{T_e} \left(e_{it} - e_{it-1}\right) = O_p\left(T^{1/2}\right). \end{aligned}$$

The second term

$$\begin{aligned} \sum_{t=T_e+1}^{T_2} \Delta\tilde{y}_t &= \sum_{t=T_e+1}^{T_2} \left[b_{N,T}\left(f_{1,t} - f_{1,t-1}\right) + c_{N,T}\left(f_{0,t} - f_{0,t-1}\right) + \frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1} \sum_{t=T_1}^{T_2} \left(e_{it} - e_{it-1}\right) \right], \\ &= b_{N,T} \frac{d_1}{T^{\alpha}} \sum_{t=T_e+1}^{T_2} f_{1,t-1} + b_{N,T} \sum_{t=T_e+1}^{T_2} u_{1,t} + c_{N,T} \sum_{t=T_e+1}^{T_2} u_{0,t} + \frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1} \sum_{t=T_e+1}^{T_2} \left(e_{it} - e_{it-1}\right) \\ &= b \frac{d_1}{T^{\alpha}} \sum_{t=T_e+1}^{T_2} f_{1,t-1} + O_p\left(T^{1/2}\right) = O_p\left(T^{\alpha/2}\rho_T^{T_2-T_r}\right), \end{aligned}$$

from Lemma A.3. Therefore, the second term dominates, giving

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t = b \frac{d_1}{T^{\alpha}} \sum_{t=T_e+1}^{T_2} f_{1,t-1} \left[1 + o_p(1)\right] = O_p\left(T^{\alpha/2}\rho_T^{T_2-T_r}\right).$$

■

Lemma C.6 *Under the alternative (2.7) with Assumptions 4.1, 4.2 (2), 4.3, 4.4, and 4.5, when $\tau_1 \in [0, \tau_e]$ and $\tau_2 \in (\tau_c, T]$ we have*

$$(a) \quad \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = \begin{cases} b \sum_{t=T_e+1}^{T_c} f_{1,t-1}\left[1 + o_p(1)\right] = O_p\left(T^{3\alpha/2}\rho_T^{T_c-T_r}\right) & \text{if } \alpha > \beta \\ a \sum_{t=T_c+1}^{T_2} f_{2,t-1}\left[1 + o_p(1)\right] = O_p\left(T^{\alpha/2+\beta}\rho_T^{T_c-T_r}\right) & \text{if } \alpha < \beta \end{cases};$$

$$(b)\quad \sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2=\begin{cases} b^2\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2\left[1+o_p(1)\right]=O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha>\beta \\ a^2\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\left[1+o_p(1)\right]=O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha<\beta \end{cases};$$

$$(c)\quad \sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1}=\left[b^2\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2-a^2\frac{d_2}{T^\beta}\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\right]\left[1+o_p(1)\right]=O_p\left(T^\alpha\rho_T^{2(T_c-T_r)}\right);$$

$$(d)\quad \sum_{t=1}^{T}\Delta\tilde{y}_t^2=\begin{cases} a^2\frac{d_2^2}{T^{2\beta}}\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\left[1+o_p(1)\right]=O_p\left(T^{\alpha-\beta}\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha>\beta \\ b^2\frac{d_1^2}{T^{2\alpha}}\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2\left[1+o_p(1)\right]=O_p\left(\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha<\beta \end{cases};$$

$$(e)\sum_{t=T_1}^{T_c}\Delta\tilde{y}_t=\left[b\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T_2}f_{1,t-1}-a\frac{d_2}{T^\beta}\sum_{t=T_c+1}^{T_2}f_{2,t-1}\right]\left[1+o_p(1)\right]=O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\right).$$

Proof. (a) We can rewrite $\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}$ as

$$\begin{aligned}
&\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}=\sum_{t=T_1}^{T_2}\left(a_{N,T}g_{1t}+b_{N,T}g_{2t}+c_{N,T}g_{3t}+\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it}\right)\\
&=a_{N,T}\sum_{t=T_c+1}^{T_2}f_{2,t-1}+\left[b_{N,T}\sum_{t=T_e+1}^{T_c}f_{1,t-1}+c_{N,T}\sum_{t=T_1}^{T_c}f_{0,t-1}+\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\left(\sum_{t=T_1}^{T_2}e_{it-1}\right)\right]\\
&=O_p\left(T^{\alpha/2+\beta}\rho_T^{T_c-T_r}\right)+O_p\left(T^{3\alpha/2}\rho_T^{T_c-T_r}\right)\\
&=\begin{cases} b\sum_{t=T_e+1}^{T_c}f_{1,t-1}\left[1+o_p(1)\right]=O_p\left(T^{3\alpha/2}\rho_T^{T_c-T_r}\right) & \text{if } \alpha>\beta \\ a\sum_{t=T_c+1}^{T_2}f_{2,t-1}\left[1+o_p(1)\right]=O_p\left(T^{\alpha/2+\beta}\rho_T^{T_c-T_r}\right) & \text{if } \alpha<\beta \end{cases},
\end{aligned}$$

from Lemma C.5(a) and Lemma A.4.

(b) The quantity

$$\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2=\sum_{t=T_1}^{T_e}\tilde{y}_{t-1}^2+\sum_{t=T_e+1}^{T_c}\tilde{y}_{t-1}^2+\sum_{t=T_c+1}^{T_2}\tilde{y}_{t-1}^2.$$

From Lemma C.5, the first and second terms are

$$\sum_{t=T_1}^{T_e}\tilde{y}_{t-1}^2+\sum_{t=T_e+1}^{T_c}\tilde{y}_{t-1}^2=b^2\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2\left[1+o_p(1)\right]=O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right).$$

The third term

$$\begin{aligned}
\sum_{t=T_c+1}^{T_2}\tilde{y}_{t-1}^2 &= \sum_{t=T_c+1}^{T_2}\left(a_{N,T}f_{2,t-1}+\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it}\right)^2\\
&= a_{N,T}^2\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2+\sum_{t=T_c+1}^{T_2}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it}\right)^2+2a_{N,T}\sum_{t=T_c+1}^{T_2}f_{2,t-1}\left(\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it}\right)
\end{aligned}$$

$$= \quad a^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right).$$

Therefore,

$$\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = \begin{cases} b^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{2\alpha} \rho_T^{2(T_c - T_r)}\right) & \text{if } \alpha > \beta \\ a^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right) & \text{if } \alpha < \beta \end{cases}.$$

(c) We have

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} = \sum_{t=T_1}^{T_e} \Delta\tilde{y}_t \tilde{y}_{t-1} + \sum_{t=T_e+1}^{T_c} \Delta\tilde{y}_t \tilde{y}_{t-1} + \sum_{t=T_c+1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1}.$$

From Lemma C.5(c),

$$\sum_{t=T_1}^{T_e} \Delta\tilde{y}_t \tilde{y}_{t-1} + \sum_{t=T_e+1}^{T_c} \Delta\tilde{y}_t \tilde{y}_{t-1} = b^2 (\rho_T - 1) \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha} \rho_T^{2(T_c - T_r)}\right).$$

The third term

$$\begin{aligned}
\sum_{t=T_c+1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} &= \sum_{t=T_c+1}^{T_2} \left[a_{N,T} (\phi_T - 1) f_{2,t-1} + a_{N,T} u_{2,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it}\right] \left(a_{N,T} f_{2,t-1} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) \\
&= -a_{N,T}^2 d_2 T^{-\beta} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 + a_{N,T}^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1} u_{2,t} + a_{N,T} \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \sum_{t=T_c+1}^{T_2} f_{2,t-1} \Delta e_{it} \\
&\quad - a_{N,T} d_2 T^{-\beta} \sum_{t=T_c+1}^{T_2} f_{2,t-1} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) + a_{N,T} \sum_{t=T_c+1}^{T_2} u_{2,t} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) \\
&\quad + \sum_{t=T_c+1}^{T_2} \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) \\
&= -a^2 d_2 T^{-\beta} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha} \rho_T^{2(T_c - T_r)}\right),
\end{aligned}$$

using results from Lemmas 4.4 and A.4. Therefore,

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} = \left[b^2 d_1 T^{-\alpha} \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2 d_2 T^{-\beta} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right] \left[1 + o_p(1)\right] = O_p\left(T^{\alpha} \rho_T^{2(T_c - T_r)}\right).$$

(d) The quantity

$$\sum_{t=1}^{T} \Delta\tilde{y}_t^2 = \sum_{t=T_1}^{T_e} \Delta\tilde{y}_t^2 + \sum_{t=T_e+1}^{T_c} \Delta\tilde{y}_t^2 + \sum_{t=T_c+1}^{T_2} \Delta\tilde{y}_t^2.$$

From Lemma C.5(d), we have

$$\sum_{t=T_1}^{T_e} \Delta\tilde{y}_t^2 + \sum_{t=T_e+1}^{T_c} \Delta\tilde{y}_t^2 = b^2 \frac{d_1^2}{T^{2\alpha}} \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(\rho_T^{2(T_c-T_r)}\right).$$

The third term

$$\sum_{t=T_c+1}^{T_2} \Delta\tilde{y}_t^2 = \sum_{t=T_c+1}^{T_2} \left[a_{N,T}(\phi_T - 1) f_{2,t-1} + a_{N,T} u_{2,t} + \frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1}\Delta e_{it}\right]^2$$

$$= a_{N,T}^2 (\phi_T - 1)^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 + \sum_{t=T_c+1}^{T_2} \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1}\Delta e_{it}\right)^2 + 2a_{N,T}^2 (\phi_T - 1) \sum_{t=T_c+1}^{T_2} f_{2,t-1} u_{2,t}$$

$$+ a_{N,T}^2 \sum_{t=T_c+1}^{T_2} u_{2,t}^2 + 2a_{N,T}(\phi_T - 1) \sum_{t=T_c+1}^{T_2} f_{2,t-1} \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1}\Delta e_{it}\right) + 2a_{N,T} \sum_{t=T_c+1}^{T_2} u_{2,t} \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1}\Delta e_{it}\right)$$

$$= a^2 (\phi_T - 1)^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha-\beta}\rho_T^{2(T_c-T_r)}\right),$$

using Lemmas 4.4 and A.4. Therefore,

$$\sum_{t=1}^{T} \Delta\tilde{y}_t^2 = \begin{cases} a^2 \frac{d_2^2}{T^{2\beta}} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha-\beta}\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha > \beta \\ b^2 \frac{d_1^2}{T^{2\alpha}} \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(\rho_T^{2(T_c-T_r)}\right) & \text{if } \alpha < \beta \end{cases}.$$

(e) The quantity

$$\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t = \sum_{t=T_1}^{T_e} \Delta\tilde{y}_t + \sum_{t=T_e+1}^{T_c} \Delta\tilde{y}_t + \sum_{t=T_c+1}^{T_2} \Delta\tilde{y}_t.$$

From Lemma C.5(e),

$$\sum_{t=T_1}^{T_e} \Delta\tilde{y}_t + \sum_{t=T_e+1}^{T_c} \Delta\tilde{y}_t = b(\rho_T - 1) \sum_{t=T_e+1}^{T_c} f_{1,t-1} \left[1 + o_p(1)\right] = O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\right).$$

The third term

$$\sum_{t=T_c+1}^{T_2} \Delta\tilde{y}_t = \sum_{t=T_c+1}^{T_2} \left[a_{N,T}(\phi_T - 1) f_{2,t-1} + a_{N,T} u_{2,t} + \frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1}\Delta e_{it}\right]$$

$$= a_{N,T}(\phi_T - 1) \sum_{t=T_c+1}^{T_2} f_{2,t-1} + a_{N,T} \sum_{t=T_c+1}^{T_2} u_{2,t} + \frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1} \sum_{t=T_c+1}^{T_2} \Delta e_{it}$$

$$= a(\phi_T - 1) \sum_{t=T_c+1}^{T_2} f_{2,t-1} \left[1 + o_p(1)\right] = O_p\left(T^{\alpha/2}\rho_T^{T_c-T_r}\right),$$

from Lemma A.4. Therefore,

$$\sum_{t=T_1}^{T_2} \Delta \tilde{y}_t = \left[b\left(\rho_T - 1\right) \sum_{t=T_e+1}^{T_2} f_{1,t-1} + a\left(\phi_T - 1\right) \sum_{t=T_c+1}^{T_2} f_{2,t-1} \right] \left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha/2} \rho_T^{T_c - T_r}\right).$$

■

Lemma C.7 *Under the alternative (2.7) and with Assumptions 4.1, 4.2 (2), 4.3, 4.4, and 4.5, when* $\tau_1, \tau_2 \in (\tau_c, T]$ *we have*

$$(a) \quad \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = a \sum_{t=T_1}^{T_2} f_{2,t-1} \left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha/2+\beta} \rho_T^{T_c - T_r}\right);$$

$$(b) \quad \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 \left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right);$$

$$(c) \quad \sum_{t=T_1}^{T_2} \Delta \tilde{y}_t \tilde{y}_{t-1} = -a^2 \frac{d_2}{T^\beta} \sum_{t=T_1}^{T_2} f_{2,t-1}^2 \left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha} \rho_T^{2(T_c - T_r)}\right);$$

$$(d) \quad \sum_{t=1}^{T} \Delta \tilde{y}_t^2 = a^2 \frac{d_2^2}{T^{2\beta}} \sum_{t=T_1}^{T_2} f_{2,t-1}^2 \left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha-\beta} \rho_T^{2(T_c - T_r)}\right);$$

$$(e) \sum_{t=T_1}^{T_c} \Delta \tilde{y}_t = -a \frac{d_2}{T^\beta} \sum_{t=T_1}^{T_2} f_{2,t-1} \left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha/2} \rho_T^{T_c - T_r}\right).$$

Proof. (a) We can rewrite $\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}$ as

$$\begin{aligned}
\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} &= \sum_{t=T_1}^{T_2} \left(a_{N,T} g_{1t} + b_{N,T} g_{2t} + c_{N,T} g_{3t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} \right) \\
&= a_{N,T} \sum_{t=T_1}^{T_2} f_{2,t-1} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \left(\sum_{t=T_1}^{T_2} e_{it-1} \right) \\
&= a \sum_{t=T_1}^{T_2} f_{2,t-1} \left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha/2+\beta} \rho_T^{T_c - T_r}\right),
\end{aligned}$$

from Lemma A.4. (b) The quantity

$$\begin{aligned}
\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 &= \sum_{t=T_1}^{T_2} \left(a_{N,T} f_{2,t-1} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} \right)^2 \\
&= a_{N,T}^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 + \sum_{t=T_1}^{T_2} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} \right)^2 + 2 a_{N,T} \sum_{t=T_1}^{T_2} f_{2,t-1} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} \right)
\end{aligned}$$

$$= \quad a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right).$$

Therefore,

$$\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha+\beta} \rho_T^{2(T_c - T_r)}\right).$$

(c) We have

$$\begin{aligned}
\sum_{t=T_1}^{T_2} \Delta \tilde{y}_t \tilde{y}_{t-1} &= \sum_{t=T_1}^{T_2} \left[a_{N,T}(\phi_T - 1) f_{2,t-1} + a_{N,T} u_{2,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it}\right] \left(a_{N,T} f_{2,t-1} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) \\
&= -a_{N,T}^2 d_2 T^{-\beta} \sum_{t=T_1}^{T_2} f_{2,t-1}^2 + a_{N,T}^2 \sum_{t=T_1}^{T_2} f_{2,t-1} u_{2,t} + a_{N,T} \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \sum_{t=T_1}^{T_2} f_{2,t-1} \Delta e_{it} \\
&- a_{N,T} d_2 T^{-\beta} \sum_{t=T_1}^{T_2} f_{2,t-1} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) + a_{N,T} \sum_{t=T_1}^{T_2} u_{2,t} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) \\
&+ \sum_{t=T_1}^{T_2} \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it-1}\right) \\
&= -a^2 d_2 T^{-\beta} \sum_{t=T_1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha} \rho_T^{2(T_c - T_r)}\right),
\end{aligned}$$

using results from Lemmas 4.4 and A.4.

(d) The quantity

$$\begin{aligned}
\sum_{t=T_1}^{T_2} \Delta \tilde{y}_t^2 &= \sum_{t=T_1}^{T_2} \left[a_{N,T}(\phi_T - 1) f_{2,t-1} + a_{N,T} u_{2,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it}\right]^2 \\
&= a_{N,T}^2 (\phi_T - 1)^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 + \sum_{t=T_1}^{T_2} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it}\right)^2 + 2a_{N,T}^2 (\phi_T - 1) \sum_{t=T_1}^{T_2} f_{2,t-1} u_{2,t} \\
&+ a_{N,T}^2 \sum_{t=T_1}^{T_2} u_{2,t}^2 + 2a_{N,T}(\phi_T - 1) \sum_{t=T_1}^{T_2} f_{2,t-1} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it}\right) + 2a_{N,T} \sum_{t=T_1}^{T_2} u_{2,t} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it}\right) \\
&= a^2 (\phi_T - 1)^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 \left[1 + o_p(1)\right] = O_p\left(T^{\alpha-\beta} \rho_T^{2(T_c - T_r)}\right),
\end{aligned}$$

using Lemmas4.4 and A.4.

(e) The quantity

$$\sum_{t=T_1}^{T_2} \Delta \tilde{y}_t = \sum_{t=T_1}^{T_2} \left[a_{N,T}(\phi_T - 1) f_{2,t-1} + a_{N,T} u_{2,t} + \frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} \Delta e_{it}\right]$$

$$= a_{N,T}\left(\phi_T - 1\right)\sum_{t=T_1}^{T_2} f_{2,t-1} + a_{N,T}\sum_{t=T_1}^{T_2} u_{2,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\sum_{t=T_1}^{T_2}\Delta e_{it}$$

$$= a\left(\phi_T - 1\right)\sum_{t=T_1}^{T_2} f_{2,t-1}\left[1 + o_p\left(1\right)\right] = O_p\left(T^{\alpha/2}\rho_T^{T_c - T_r}\right),$$

from Lemma A.4. ■

Proof of Theorem 4.5

Proof. The OLS estimator

$$\widehat{\gamma}_{\tau_1,\tau_2} = \frac{T_w\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1} - \sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}}{T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2}.$$

The denominator is

$$T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2 = T_w b^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + O_p\left(T^{3\alpha}\rho_T^{2(T_2-T_r)}\right) = O_p\left(T^{1+2\alpha}\rho_T^{2(T_2-T_r)}\right). \tag{C.17}$$

since $\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 = O_p\left(T^{2\alpha}\rho_T^{2(T_2-T_r)}\right)$ and $\sum_{t=T_1}^{T_2}\tilde{y}_{t-1} = O_p\left(T^{3\alpha/2}\rho_T^{T_2-T_r}\right)$ from Lemma C.5. The numerator is

$$T_w\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1} - \sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\sum_{t=T_1}^{T_2}\tilde{y}_{t-1} = \left[b^2\frac{d_1}{T^\alpha}T_w\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 - b^2\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2\right]\left[1 + o_p\left(1\right)\right], \tag{C.18}$$

since

$$\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1} = b^2\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2\left[1 + o_p\left(1\right)\right] = O_p\left(T^\alpha\rho_T^{2(T_2-T_r)}\right),$$

and

$$\sum_{t=1}^{T_2}\Delta\tilde{y}_t\sum_{t=1}^{T_2}\tilde{y}_{t-1} = b^2\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2\left[1 + o_p\left(1\right)\right] = O_p\left(T^{2\alpha}\rho_T^{2(T_2-T_r)}\right),$$

from Lemma C.5. Therefore,

$$\begin{aligned}
\widehat{\gamma}_{\tau_1,\tau_2} &= \frac{b^2\frac{d_1}{T^\alpha}T_w\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 - b^2\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2}{T_w b^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2}\left[1 + o_p\left(1\right)\right] \\
&= \frac{d_1}{T^\alpha} - \frac{\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2}{T_w\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2}\left[1 + o_p\left(1\right)\right] \qquad \text{(C.19)} \\
&= \frac{d_1}{T^\alpha} + O_p\left(T^{-1}\right).
\end{aligned}$$

Next, we derive the order of magnitude of $\hat{\delta}_{\tau_1,\tau_2}$. By definition we have

$$\hat{\delta}_{\tau_1,\tau_2} = \frac{\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)}{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2}.$$

The numerator is

$$\begin{aligned}
&\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right) \\
&= \left[b^2\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2\right) + 2bc\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}f_{0,t-1}\right) + O_p\left(T^\alpha \rho_T^{(T_2-T_r)}\right)\right]\left[b\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T_2} f_{1,t-1} + O_p\left(T^{1/2}\right)\right] \\
&- \left[b^2\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T} f_{1,t-1}^2 + bc\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T_2} f_{1,t-1}f_{0,t-1} + O_p\left(T^\alpha \rho_T^{(T_2-T_r)}\right)\right]\left[b\sum_{t=T_e+1}^{T_2} f_{1,t-1} + O_p\left(T^{3/2}\right)\right] \\
&= b^2 c\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}f_{0,t-1}\right)\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right) + O_p\left(T^{\alpha+1/2}\rho_T^{T-T_r}\right) \\
&= O_p\left(T^{2\alpha+\frac{1}{2}}\rho_T^{2(T_2-T_r)}\right),
\end{aligned}$$

using Lemma 4.5. Therefore,

$$\hat{\delta}_{\tau_1,\tau_2} = \frac{b^2 c\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}f_{0,t-1}\right)\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right) + O_p\left(T^{\alpha+1/2}\rho_T^{T-T_r}\right)}{T_w b^2 \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + O_p\left(T^{3\alpha}\rho_T^{2(T_2-T_r)}\right)} = O_p\left(T^{-1/2}\right).$$

Next, we obtain the order of $\sum_{t=1}^{T}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)$. The quantity can be written as

$$\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = \sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 + T_w\hat{\delta}_{\tau_1,\tau_2}^2 - 2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right).$$

Consider the term

$$\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = \sum_{t=T_1}^{T_e}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 + \sum_{t=T_e+1}^{T_2}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2. \qquad \text{(C.20)}$$

Let $\xi_{0t} = c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it} - \widehat{\gamma}_{\tau_1,\tau_2}\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}$. The first term in (C.20) is

$$\begin{aligned}
\sum_{t=T_1}^{T_e}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 &= \sum_{t=T_1}^{T_e}\left(-c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2}f_{0,t-1} + c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}\Delta e_{it} - \widehat{\gamma}_{\tau_1,\tau_2}\frac{1}{N}\sum_{i=1}^{N}\tilde{L}_{i1}e_{it-1}\right)^2 \\
&= \sum_{t=T_1}^{T_e}\left(-c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2}f_{0,t-1} + \xi_{0t}\right)^2 = \sum_{t=T_1}^{T_e}\left(c_{N,T}^2\widehat{\gamma}_{\tau_1,\tau_2}^2 f_{0,t-1}^2 + \xi_{0t}^2 - 2c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2}f_{0,t-1}\xi_{0t}\right)
\end{aligned}$$

$$= c_{N,T}^2\widehat{\gamma}_{\tau_1,\tau_2}^2 \sum_{t=1}^{T_e} f_{0,t-1}^2 + \sum_{t=1}^{T_e} \xi_{0t}^2 - 2c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2} \sum_{t=T_1}^{T_e} f_{0,t-1}\xi_{0t} = O_p\left(\max\left\{T^{2(1-\alpha)}, T\right\}\right).$$

Let $\xi_{1t} = b_{N,T}u_{1,t} + c_{N,T}u_{0,t} + \frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it} - (1+\widehat{\gamma}_{\tau_1,\tau_2})\frac{1}{N}\sum_{i=1}^N \tilde{L}_{i1}e_{it-1}$. The second term in (C.20) is

$$\sum_{t=T_e+1}^{T_2} (\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1})^2 = \sum_{t=T_e+1}^{T_2} \left[b_{N,T}(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2}) f_{1,t-1} - \widehat{\gamma}_{\tau_1,\tau_2}c_{N,T}f_{0,t-1} + \xi_{1t}\right]^2$$
$$= b_{N,T}^2(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2})^2 \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + c_{N,T}^2\widehat{\gamma}_{\tau_1,\tau_2}^2 \sum_{t=T_e+1}^{T_2} f_{0,t-1}^2 + \sum_{t=T_e+1}^{T_2} \xi_{1t}^2$$
$$- 2b_{N,T}c_{N,T}(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2})\widehat{\gamma}_{\tau_1,\tau_2} \sum_{t=T_e+1}^{T_2} f_{1,t-1}f_{0,t-1}$$
$$+ 2b_{N,T}(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2}) \sum_{t=T_e+1}^{T_2} f_{1,t-1}\xi_{1t} - 2c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2} \sum_{t=T_e+1}^{T_2} f_{0,t-1}\xi_{1t}$$
$$= b^2(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2})^2 \sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 + O_p(\rho_T^{T_2-T_r})$$
$$= b^2\frac{d_1^2}{T^{2\alpha}} \frac{\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^4}{T_w^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2} + O_p(\rho_T^{T_2-T_r})$$
$$= O_p\left(T^{2\alpha-2}\rho_T^{2(T_2-T_r)}\right).$$

since $\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2} = \frac{\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2}{T_w\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2}[1+o_p(1)] = O_p\left(T^{-1}\right)$ using results in (C.19). Therefore,

$$\begin{aligned}\sum_{t=T_1}^{T_2} (\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1})^2 &= \sum_{t=T_e+1}^{T_2} (\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1})^2[1+o_p(1)] \\ &= b^2\frac{d_1^2}{T^{2\alpha}} \frac{\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^4}{T_w^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2} + O_p(\rho_T^{T_2-T_r}) \qquad \text{(C.21)} \\ &= O_p\left(T^{2\alpha-2}\rho_T^{2(T_2-T_r)}\right).\end{aligned}$$

Moreover,

$$2\hat{\delta}_{\tau_1,\tau_2} \sum_{t=T_1}^{T_2} (\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1})$$
$$= 2\hat{\delta}_{\tau_1,\tau_2}\left[\sum_{t=T_1}^{T_e} (\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}) + \sum_{t=T_e+1}^{T_2} (\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1})\right]$$

$$
\begin{aligned}
&= 2\hat{\delta}_{\tau_1,\tau_2}\left[-c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_e} f_{0,t-1} + \sum_{t=T_1}^{T_e}\xi_{0t} + b_{N,T}\left(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2}\right)\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right. \\
&\qquad \left. -\widehat{\gamma}_{\tau_1,\tau_2}c_{N,T}\sum_{t=T_e+1}^{T_2} f_{0,t-1} + \sum_{t=T_e+1}^{T_2}\xi_{1t}\right] \\
&= 2\hat{\delta}_{\tau_1,\tau_2}b_{N,T}\left(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2}\right)\sum_{t=T_e+1}^{T_2} f_{1,t-1}\left[1 + o_p(1)\right] = O_p\left(T^{3(\alpha-1)/2}\rho_T^{T_2-T_r}\right).
\end{aligned}
$$

Since $T_w\hat{\delta}^2_{\tau_1,\tau_2} = O_p(1)$,

$$
\begin{aligned}
\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 &= \sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2\left[1 + o_p(1)\right] \\
&= b^2\frac{d_1^2}{T^{2\alpha}}\frac{\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^4}{T_w^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2}\left[1 + o_p(1)\right] \qquad \text{(C.22)} \\
&= O_p\left(T^{2\alpha-2}\rho_T^{2(T_2-T_r)}\right).
\end{aligned}
$$

Using results from (C.17), (C.18), and (C.22), the DF statistic is

$$
\begin{aligned}
DF_{\tau_1,\tau_2} &= \widehat{\gamma}_{\tau_1,\tau_2}\left[\frac{T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2}{\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2}\right]^{1/2} \\
&= \frac{T_w\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1} - \sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}}{\left[T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2\right]^{1/2}\left[\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2\right]^{1/2}} \\
&= \frac{b^2\frac{d_1}{T^\alpha}T_w\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 - b^2\frac{d_1}{T^\alpha}\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2}{\left[T_w b^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2\right]^{1/2}\left[b^2\frac{d_1^2}{T^{2\alpha}}\frac{\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^4}{T_w^2\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2}\right]^{1/2}}\left[1 + o_p(1)\right] \\
&= T_w^{1/2}\frac{T_w\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 - \left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2}{\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2}\left[1 + o_p(1)\right] \\
&= T_w^{3/2}\frac{\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2}{\left(\sum_{t=T_e+1}^{T_2} f_{1,t-1}\right)^2}\left[1 + o_p(1)\right].
\end{aligned}
$$

From the proof of Lemma A.3 (3) and (4), we have $\sum_{t=T_e+1}^{T_2} f_{1,t-1} = \frac{T^\alpha}{d_1}f_{1,T_2}\left[1 + o_p(1)\right]$ and $\sum_{t=T_e+1}^{T_2} f_{1,t-1}^2 = \frac{1}{\rho_T^2 - 1}f_{1,T_2}^2\left[1 + o_p(1)\right]$. It follows that

$$
DF_{\tau_1,\tau_2} = T^{3/2-\alpha}\frac{1}{2}d_1 r_w^{3/2}\left[1 + o_p(1)\right] = O_p\left(T^{3/2-\alpha}\right).
$$

■

C.1 Proof of Theorem 4.6

Proof. (1) Consider the case $\alpha > \beta$. The OLS estimator

$$\widehat{\gamma}_{\tau_1,\tau_2} = \frac{T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}}{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2}. \tag{C.23}$$

The denominator

$$T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2 = T_w b^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 + O_p\left(T^{3\alpha}\rho_T^{2(T_c-T_r)}\right) = O_p\left(T^{1+2\alpha}\rho_T^{2(T_c-T_r)}\right),$$

since $\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right)$ and $\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = O_p\left(T^{3\alpha/2}\rho_T^{T_c-T_r}\right)$ from Lemma C.6. The numerator is

$$\begin{aligned} & T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} + O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right) \\ = & \left[b^2 \frac{d_1}{T^\alpha} T_w \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2 \frac{d_2}{T^\beta} T_w \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right] + O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right), \end{aligned}$$

since $T_w \sum_{t=1}^{T} \Delta\tilde{y}_t \tilde{y}_{t-1} = O_p\left(T^{1+\alpha}\rho_T^{2(T_c-T_r)}\right)$ and $\sum_{t=1}^{T} \Delta\tilde{y}_t \sum_{t=1}^{T} \tilde{y}_{t-1} = O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right)$ from Lemma C.6. Therefore,

$$\begin{aligned} \widehat{\gamma}_{\tau_1,\tau_2} &= \frac{\left[b^2 \frac{d_1}{T^\alpha} T_w \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2 \frac{d_2}{T^\beta} T_w \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right] + O_p\left(T^{2\alpha}\rho_T^{2(T_c-T_r)}\right)}{T_w b^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 + O_p\left(T^{3\alpha}\rho_T^{2(T_c-T_r)}\right)} \\ &= \frac{d_1}{T^\alpha} - \frac{a^2}{b^2}\frac{d_2}{T^\beta}\frac{\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2}{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2} + O_p\left(T^{-1}\right). \end{aligned}$$

Moreover, from the proof of Lemma A.3(4) and A.4(4), we have

$$\frac{\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2}{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2} = \frac{\frac{1}{1-\phi_T^2} f_{2,T_c}^2 \left[1 + o_p(1)\right]}{\frac{1}{\rho_T^2-1} f_{1,T_c}^2 \left[1 + o_p(1)\right]} \sim \frac{\frac{1}{1-\phi_T^2} F_{2,c}^2 T^\alpha \rho_T^{2(T_c-T_r)}}{\frac{1}{\rho_T^2-1} T^\alpha \rho_T^{2(T_c-T_r)} \left(F_{1,r} + N_c\right)^2} = T^{\beta-\alpha}\frac{d_1}{d_2}\frac{F_{2,c}^2}{\left(F_{1,r} + N_c\right)^2},$$

with $N_c \sim N\left(0, \frac{\sigma_{11}}{2c}\right)$, since $f_{2,T_c} = F_{2,c} T^{\alpha/2} \rho_T^{T_c-T_r}$ by virtue of the initial conditions and

$$\frac{f_{1,T_c}}{T^{\alpha/2}\rho_T^{T_c-T_r}} \Rightarrow F_{1,r} + N_c$$

from the proof of Lemma A.3(1). Thus,

$$\widehat{\gamma}_{\tau_1,\tau_2} = \left[\frac{d_1}{T^\alpha} - \frac{a^2}{b^2} \frac{d_2}{T^\beta} \frac{\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2}{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2} \right] [1 + o_p(1)] \sim \frac{d_1}{T^\alpha} \left[1 - \frac{a^2}{b^2} \frac{F_{2,c}^2}{(F_{1,r} + N_c)^2} \right] = O_p\left(T^{-\alpha}\right).$$

Next, we derive the order of magnitude of $\hat{\delta}_{\tau_1,\tau_2}$. By definition, we have

$$\hat{\delta}_{\tau_1,\tau_2} = \frac{\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)}{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2}.$$

The numerator is

$$\begin{aligned}
& \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right) \\
= & \left(b^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2\right)\left[b\frac{d_1}{T^\alpha} \sum_{t=T_e+1}^{T_c} f_{1,t-1} - a\frac{d_2}{T^\beta} \sum_{t=T_c+1}^{T_2} f_{2,t-1}\right][1 + o_p(1)] \\
& - \left[b^2 \frac{d_1}{T^\alpha} \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2 \frac{d_2}{T^\beta} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right]\left(b \sum_{t=T_e+1}^{T_c} f_{1,t-1}\right)[1 + o_p(1)] \\
= & ab\frac{d_2}{T^\beta}\left[-b \sum_{t=T_c+1}^{T_2} f_{2,t-1} \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 + a \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}\right][1 + o_p(1)] \\
= & O_p\left(T^{5\alpha/2} \rho_T^{3(T_c - T_r)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\delta}_{\tau_1,\tau_2} = & \frac{ab\frac{d_2}{T^\beta}\left[-b\sum_{t=T_c+1}^{T_2} f_{2,t-1} \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 + a\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}\right]}{T_w b^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2}[1 + o_p(1)] \\
= & a\frac{d_2}{T^\beta}\frac{1}{T_w}\left[-\sum_{t=T_c+1}^{T_2} f_{2,t-1} + \frac{a}{b}\frac{\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}}{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2}\right][1 + o_p(1)] \\
= & O_p\left(T^{\alpha/2-1} \rho_T^{T_c - T_r}\right).
\end{aligned}$$

The sum of squared errors $\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2$ can be rewritten as

$$\sum_{t=T_1}^{T_2}\left(\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = \sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 + T_w\hat{\delta}_{\tau_1,\tau_2}^2 - 2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right).$$

The quantity

$$\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t-\hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2=\sum_{t=T_1}^{T_e}\left(\Delta\tilde{y}_t-\hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2+\sum_{t=T_e+1}^{T_c}\left(\Delta\tilde{y}_t-\hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2+\sum_{t=T_c+1}^{T_2}\left(\Delta\tilde{y}_t-\hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2. \tag{C.24}$$

When $\alpha>\beta$, $\phi_T-1-\widehat{\gamma}_{\tau_1,\tau_2}=O_p\left(T^{-\beta}\right)$ and $\rho_T-1-\widehat{\gamma}_{\tau_1,\tau_2}=O_p\left(T^{-\alpha}\right)$. The first term in (C.24) is

$$\sum_{t=T_1}^{T_e}\left(\Delta\tilde{y}_t-\widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2=O_p\left(\max\left\{T^{2(1-\alpha)},T\right\}\right),$$

as in the proof of Theorem 4.5. The second term in (C.24) is

$$\begin{aligned}
&\sum_{t=T_e+1}^{T_c}\left(\Delta\tilde{y}_t-\widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2=\sum_{t=T_e+1}^{T_c}\left[b_{N,T}\left(\rho_T-1-\widehat{\gamma}_{\tau_1,\tau_2}\right)f_{1,t-1}-\widehat{\gamma}_{\tau_1,\tau_2}c_{N,T}f_{0,t-1}+\xi_{1t}\right]^2\\
=\;&b_{N,T}^2\left(\rho_T-1-\widehat{\gamma}_{\tau_1,\tau_2}\right)^2\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2+c_{N,T}^2\widehat{\gamma}_{\tau_1,\tau_2}^2\sum_{t=T_e+1}^{T_c}f_{0,t-1}^2+\sum_{t=T_e+1}^{T_c}\xi_{1t}^2\\
&-2b_{N,T}c_{N,T}\left(\rho_T-1-\widehat{\gamma}_{\tau_1,\tau_2}\right)\widehat{\gamma}_{\tau_1,\tau_2}\sum_{t=T_e+1}^{T_c}f_{1,t-1}f_{0,t-1}\\
&+2b_{N,T}\left(\rho_T-1-\widehat{\gamma}_{\tau_1,\tau_2}\right)\sum_{t=T_e+1}^{T_c}f_{1,t-1}\xi_{1t}-2c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2}\sum_{t=T_e+1}^{T_c}f_{0,t-1}\xi_{1t}\\
=\;&b^2\left(\rho_T-1-\widehat{\gamma}_{\tau_1,\tau_2}\right)^2\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2+O_p(T^{(1-\alpha)/2}\rho_T^{T_2-T_r})\\
=\;&\frac{a^4}{b^2}\frac{d_2^2}{T^{2\beta}}\frac{\left(\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\right)^2}{\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2}\left[1+o_p\left(1\right)\right]\\
=\;&O_p\left(\rho_T^{2(T_c-T_r)}\right).
\end{aligned}$$

Let $\xi_{2t}=a_{N,T}u_{2,t}+\frac{1}{N}\sum_{i=1}^N\tilde{L}_{i1}e_{it}-(1+\hat{\gamma})\frac{1}{N}\sum_{i=1}^N\tilde{L}_{i1}e_{it-1}$. The third term in (C.24) is

$$\begin{aligned}
&\sum_{t=T_c+1}^{T_2}\left(\Delta\tilde{y}_t-\hat{\gamma}\tilde{y}_{t-1}\right)^2=\sum_{t=T_c+1}^{T_2}\left[a_{N,T}\left(\phi_T-1-\hat{\gamma}\right)f_{2,t-1}+\xi_{2t}\right]^2\\
=\;&a_{N,T}^2\left(\phi_T-1-\hat{\gamma}\right)^2\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2+\sum_{t=T_c+1}^{T_2}\xi_{2t}^2+2a_{N,T}\left(\phi_T-1-\hat{\gamma}\right)\sum_{t=T_c+1}^{T_2}f_{2,t-1}\xi_{2t}\\
=\;&a^2\left(\phi_T-1-\hat{\gamma}\right)^2\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\left[1+o_p\left(1\right)\right]\\
=\;&a^2\frac{d_2^2}{T^{2\beta}}\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\left[1+o_p\left(1\right)\right]=O_p\left(T^{\alpha-\beta}\rho_T^{2(T_c-T_r)}\right).
\end{aligned}$$

since $\phi_T - 1 - \hat{\gamma} = (\phi_T - 1)[1 + o_p(1)]$ when $\alpha > \beta$. Therefore, the third term dominates and hence

$$\sum_{t=T_1}^{T_2} \left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = a^2 \frac{d_2^2}{T^{2\beta}} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 [1 + o_p(1)] = O_p\left(T^{\alpha-\beta}\rho_T^{2(T_c-T_r)}\right).$$

Since $T_w \hat{\delta}_{\tau_1,\tau_2}^2 = O_p\left(T^{\alpha-1}\rho_T^{2(T_c-T_r)}\right)$ and

$$\begin{aligned}
& 2\hat{\delta}_{\tau_1,\tau_2} \sum_{t=T_1}^{T_2} \left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right) \\
= \; & 2\hat{\delta}_{\tau_1,\tau_2} \left[\sum_{t=T_1}^{T_e} \left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right) + \sum_{t=T_e+1}^{T_c} \left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right) + \sum_{t=T_c+1}^{T_2} \left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)\right] \\
= \; & 2\hat{\delta}_{\tau_1,\tau_2} \left[-c_{N,T}\widehat{\gamma}_{\tau_1,\tau_2} \sum_{t=T_1}^{T_e} f_{0,t-1} + \sum_{t=T_1}^{T_e} \xi_{0t} + b_{N,T}\left(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2}\right) \sum_{t=T_e+1}^{T_c} f_{1,t-1} - \widehat{\gamma}_{\tau_1,\tau_2} c_{N,T} \sum_{t=T_e+1}^{T_c} f_{0,t-1}\right. \\
& \left. + \sum_{t=T_e+1}^{T_c} \xi_{1t} + a_{N,T}\left(\phi_T - 1 - \hat{\gamma}\right) \sum_{t=T_c+1}^{T_2} f_{2,t-1} + \sum_{t=T_c+1}^{T_2} \xi_{2t}\right] \\
= \; & 2\hat{\delta}_{\tau_1,\tau_2} \left[b\left(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2}\right) \sum_{t=T_e+1}^{T_c} f_{1,t-1} + a\left(\phi_T - 1 - \hat{\gamma}\right) \sum_{t=T_c+1}^{T_2} f_{2,t-1}\right] [1 + o_p(1)] \\
= \; & O_p\left(T^{\alpha-1}\rho_T^{2(T_c-T_r)}\right),
\end{aligned}$$

we have

$$\begin{aligned}
\sum_{t=T_1}^{T_2} \left(\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 &= \sum_{t=T_1}^{T_2} \left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 [1 + o_p(1)] \\
&= a^2 \frac{d_2^2}{T^{2\beta}} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 [1 + o_p(1)] = O_p\left(T^{\alpha-\beta}\rho_T^{2(T_c-T_r)}\right).
\end{aligned}$$

Therefore, the DF test statistic has the following asymptotic behavior

$$\begin{aligned}
DF_{\tau_1,\tau_2} &= \widehat{\gamma}_{\tau_1,\tau_2} \left[\frac{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2}{\sum_{t=T_1}^{T_2} \left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2}\right]^{1/2} \\
&= \frac{T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}}{\left[T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2\right]^{1/2} \left[\sum_{t=T_1}^{T_2} \left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2\right]^{1/2}} \\
&= \frac{b^2 \frac{d_1}{T^\alpha} T_w \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2 \frac{d_2}{T^\beta} T_w \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2}{\left[T_w b^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2\right]^{1/2} \left[a^2 \frac{d_2^2}{T^{2\beta}} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right]^{1/2}} [1 + o_p(1)]
\end{aligned}$$

$$= T_w^{1/2}\left[T^{\beta-\alpha}\frac{bd_1}{ad_2}\frac{\left(\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2\right)^{1/2}}{\left(\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right)^{1/2}} - \frac{a}{b}\frac{\left(\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right)^{1/2}}{\left(\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2\right)^{1/2}}\right][1+o_p(1)].$$

From the proof of Lemma A.3(4) and Lemma A.4(4) , we have $\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 = \frac{1}{\rho_T^2-1} f_{1,T_c}^2 [1+o_p(1)]$ and $\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 = \frac{1}{1-\phi_T^2} f_{2,T_c}^2 [1+o_p(1)]$. It follows that

$$\begin{aligned}
DF_{\tau_1,\tau_2} &= T_w^{1/2}\left[T^{\beta-\alpha}\frac{bd_1}{ad_2}\left(\frac{1-\phi_T^2}{\rho_T^2-1}\right)^{1/2}\frac{f_{1,T_c}}{f_{2,T_c}} - \frac{a}{b}\left(\frac{\rho_T^2-1}{1-\phi_T^2}\right)^{1/2}\frac{f_{2,T_c}}{f_{1,T_c}}\right][1+o_p(1)] \\
&= T^{(1-\alpha+\beta)/2} r_w^{1/2}\left(\frac{d_1}{d_2}\right)^{1/2}\left(\frac{bf_{1,T_c}}{af_{2,T_c}} - \frac{af_{2,T_c}}{bf_{1,T_c}}\right)[1+o_p(1)] \\
&\sim T^{(1-\alpha+\beta)/2} r_w^{1/2}\left(\frac{d_1}{d_2}\right)^{1/2}\left[\frac{b(F_{1,r}+N_{d_1})}{aF_{2,c}} - \frac{aF_{2,c}}{b(F_{1,r}+N_{d_1})}\right] \\
&= O_p\left(T^{(1-\alpha+\beta)/2}\right)
\end{aligned}$$

since $1-\phi_T^2 = 2\frac{d_2}{T^\beta}[1+o(1)]$, $\rho_T^2 - 1 = 2\frac{d_1}{T^\alpha}[1+o(1)]$, $f_{1,T_c} \sim T^{\alpha/2}\rho_T^{T_c-T_r}(F_{1,r}+N_{d_1})$ from A.3(1), and $f_{2,T_c} \sim T^{\alpha/2}\rho_T^{T_c-T_r}F_{2,c}$ by assumption.

(2) Next, consider the case where $\alpha < \beta$. The denominator of (C.23) is

$$T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2 = T_w a^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 + O_p\left(T^{\alpha+2\beta}\rho_T^{2(T_c-T_r)}\right) = O_p\left(T^{1+\alpha+\beta}\rho_T^{2(T_c-T_r)}\right),$$

since $\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right)$ and $\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = O_p\left(T^{\alpha/2+\beta}\rho_T^{T_c-T_r}\right)$ from Lemma C.6. The numerator is

$$\begin{aligned}
&T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1} + O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right) \\
&= \left[b^2\frac{d_1}{T^\alpha}T_w \sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2\frac{d_2}{T^\beta}T_w \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right] + O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right),
\end{aligned}$$

since $T_w\sum_{t=1}^T \Delta\tilde{y}_t\tilde{y}_{t-1} = O_p\left(T^{1+\alpha}\rho_T^{2(T_c-T_r)}\right)$ and $\sum_{t=1}^T \Delta\tilde{y}_t \sum_{t=1}^T \tilde{y}_{t-1} = O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right)$ from Lemma C.6. Therefore,

$$\begin{aligned}
\widehat{\gamma}_{\tau_1,\tau_2} &= \frac{b^2\frac{d_1}{T^\alpha}T_w\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2\frac{d_2}{T^\beta}T_w\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 + O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right)}{T_w a^2\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2} \\
&= -\frac{d_2}{T^\beta} + \frac{b^2}{a^2}\frac{d_1}{T^\alpha}\frac{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2}{\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2} + O_p\left(T^{-1}\right).
\end{aligned}$$

As before, from the proof of Lemmas A.3(4) and A.4(4), we have the ratio

$$\frac{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2}{\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2} = \frac{\frac{1}{\rho_T^2-1} f_{1,T_c}^2 \left[1+o_p(1)\right]}{\frac{1}{1-\phi_T^2} f_{2,T_c}^2 \left[1+o_p(1)\right]} \sim T^{\alpha-\beta} \frac{d_2}{d_1 F_{2,c}^2} \left(F_{1,r}+N_{d_1}\right)^2$$

since $f_{2,T_c} = F_{2,c} T^{\alpha/2} \rho_T^{T_c-T_r}$ and $f_{1,T_c} \sim T^{\alpha/2} \rho_T^{T_c-T_r} \left(F_{1,r}+N_{d_1}\right)$. It follows that

$$\widehat{\gamma}_{\tau_1,\tau_2} = \left[-\frac{d_2}{T^\beta} + \frac{b^2}{a^2}\frac{d_1}{T^\alpha}\frac{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2}{\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2}\right]\left[1+o_p(1)\right] \sim -\frac{d_2}{T^\beta}\left[1-\frac{b^2}{a^2}\frac{\left(F_{1,r}+N_{d_1}\right)^2}{F_{2,c}^2}\right] = O_p\left(T^{-\beta}\right).$$

Next, we derive the limit behavior of $\hat{\delta}_{\tau_1,\tau_2}$. By definition, we have

$$\hat{\delta}_{\tau_1,\tau_2} = \frac{\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)}{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2}.$$

The numerator is

$$\begin{aligned}
&\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2\right)\left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t\tilde{y}_{t-1}\right)\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right) \\
=& \left(a^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right)\left[b\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T_c} f_{1,t-1} - a\frac{d_2}{T^\beta}\sum_{t=T_c+1}^{T_2} f_{2,t-1}\right]\left[1+o_p(1)\right] \\
&-\left[b^2\frac{d_1}{T^\alpha}\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 - a^2\frac{d_2}{T^\beta}\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2\right]\left(a\sum_{t=T_c+1}^{T_2} f_{2,t-1}\right)\left[1+o_p(1)\right] \\
=& ab\frac{d_1}{T^\alpha}\left[a\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1} - b\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}\right]\left[1+o_p(1)\right] \\
=& O_p\left(T^{3\alpha/2+\beta}\rho_T^{3(T_c-T_r)}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\delta}_{\tau_1,\tau_2} &= \frac{ab\frac{d_1}{T^\alpha}\left[a\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 \sum_{t=T_e+1}^{T_c} f_{1,t-1} - b\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}\right]}{T_w a^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2}\left[1+o_p(1)\right] \\
&= b\frac{d_1}{T^\alpha}\left[\frac{1}{T_w}\sum_{t=T_e+1}^{T_c} f_{1,t-1} - \frac{b}{a}\frac{\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2 \sum_{t=T_c+1}^{T_2} f_{2,t-1}}{T_w \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2}\right]\left[1+o_p(1)\right] \\
&= O_p\left(T^{\alpha/2-1}\rho_T^{T_c-T_r}\right).
\end{aligned}$$

The sum of squared errors $\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2$ can be rewritten as

$$\sum_{t=T_1}^{T_2}\left(\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = \sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 + T_w\hat{\delta}^2_{\tau_1,\tau_2} - 2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right).$$

The quantity

$$\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = \sum_{t=T_1}^{T_e}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 + \sum_{t=T_e+1}^{T_c}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 + \sum_{t=T_c+1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2. \tag{C.25}$$

When $\alpha < \beta$, $\phi_T - 1 - \hat{\gamma} = O_p\left(T^{-\beta}\right)$ and $\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2} = \frac{d_1}{T^\alpha}\left[1 + o_p(1)\right] = O_p\left(T^{-\alpha}\right)$. The first term in (C.25) is

$$\sum_{t=T_1}^{T_e}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = O_p\left(\max\left\{T^{2(1-\alpha)}, T\right\}\right),$$

as in the proof of Theorem 4.5. The second term in (C.25) is

$$\begin{aligned}\sum_{t=T_e+1}^{T_c}\left(\Delta\tilde{y}_t - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 &= b^2\left(\rho_T - 1 - \widehat{\gamma}_{\tau_1,\tau_2}\right)^2\sum_{t=T_e+1}^{T_c} f^2_{1,t-1}\left[1 + o_p(1)\right] \\ &= b^2\frac{d_1^2}{T^{2\alpha}}\sum_{t=T_e+1}^{T_c} f^2_{1,t-1}\left[1 + o_p(1)\right] = O_p\left(\rho_T^{2(T_c - T_r)}\right),\end{aligned}$$

and the third term is

$$\sum_{t=T_c+1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}\tilde{y}_{t-1}\right)^2 = a^2\left(\phi_T - 1 - \hat{\gamma}\right)^2\sum_{t=T_c+1}^{T_2} f^2_{2,t-1}\left[1 + o_p(1)\right] = O_p\left(T^{\alpha-\beta}\rho_T^{2(T_c - T_r)}\right).$$

Since $\alpha < \beta$, the second term dominates and hence

$$\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = b^2\frac{d_1^2}{T^{2\alpha}}\sum_{t=T_e+1}^{T_c} f^2_{1,t-1}\left[1 + o_p(1)\right] = O_p\left(\rho_T^{2(T_c - T_r)}\right).$$

Since $T_w\hat{\delta}^2_{\tau_1,\tau_2} = O_p\left(T^{\alpha-1}\rho_T^{2(T_c - T_r)}\right)$ and

$$2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}\tau_1, \tau_2\tilde{y}_{t-1}\right)$$

$$
\begin{aligned}
&= 2\hat{\delta}_{\tau_1,\tau_2}\left[b\left(\rho_T-1-\widehat{\gamma}_{\tau_1,\tau_2}\right)\sum_{t=T_e+1}^{T_c} f_{1,t-1}+a\left(\phi_T-1-\hat{\gamma}_{\tau_1,\tau_2}\right)\sum_{t=T_c+1}^{T_2} f_{2,t-1}\right]\left[1+o_p\left(1\right)\right]\\
&= O_p\left(T^{\alpha-1}\rho_T^{2(T_c-T_r)}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
\sum_{t=T_1}^{T_2}\left(\tilde{y}_t-\hat{\delta}_{\tau_1,\tau_2}-\hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 &= \sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t-\hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2\left[1+o_p\left(1\right)\right]\\
&= b^2\frac{d_1^2}{T^{2\alpha}}\sum_{t=T_e+1}^{T_c} f_{1,t-1}^2\left[1+o_p\left(1\right)\right]=O_p\left(\rho_T^{2(T_c-T_r)}\right).
\end{aligned}
$$

The DF statistic

$$
\begin{aligned}
DF_{\tau_1,\tau_2} &= \widehat{\gamma}_{\tau_1,\tau_2}\left[\frac{T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2-\left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2}{\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t-\hat{\delta}_{\tau_1,\tau_2}-\widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2}\right]^{1/2}\\
&= \frac{T_w\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\tilde{y}_{t-1}-\sum_{t=T_1}^{T_2}\Delta\tilde{y}_t\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}}{\left[T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2-\left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2\right]^{1/2}\left[\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t-\hat{\delta}_{\tau_1,\tau_2}-\widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2\right]^{1/2}}\\
&= \frac{\left[b^2\frac{d_1}{T^\alpha}T_w\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2-a^2\frac{d_2}{T^\beta}T_w\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\right]}{\left[T_wa^2\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\right]^{1/2}\left[b^2\frac{d_1^2}{T^{2\alpha}}\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2\right]^{1/2}}\left[1+o_p\left(1\right)\right]\\
&= T_w^{1/2}\left[\frac{b}{a}\frac{\left(\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2\right)^{1/2}}{\left(\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\right)^{1/2}}-\frac{ad_2}{bd_1}T^{\alpha-\beta}\frac{\left(\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2\right)^{1/2}}{\left(\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2\right)^{1/2}}\right]\left[1+o_p\left(1\right)\right].
\end{aligned}
$$

We know from the proof of Lemma A.3(4) and Lemma A.4(4) that $\sum_{t=T_e+1}^{T_c}f_{1,t-1}^2=\frac{1}{\rho_T^2-1}f_{1,T_c}^2\left[1+o_p\left(1\right)\right]$ and $\sum_{t=T_c+1}^{T_2}f_{2,t-1}^2=\frac{1}{1-\phi_T^2}f_{2,T_c}^2\left[1+o_p\left(1\right)\right]$. It follows that

$$
\begin{aligned}
DF_{\tau_1,\tau_2} &= T_w^{1/2}\left[\frac{b}{a}\left(\frac{1-\phi_T^2}{\rho_T^2-1}\right)^{1/2}\frac{f_{1,T_c}}{f_{2,T_c}}-\frac{ad_2}{bd_1}T^{\alpha-\beta}\left(\frac{\rho_T^2-1}{1-\phi_T^2}\right)^{1/2}\frac{f_{2,T_c}}{f_{1,T_c}}\right]\left[1+o_p\left(1\right)\right]\\
&= T_w^{1/2}T^{(\alpha-\beta)/2}\left(\frac{d_2}{d_1}\right)^{1/2}\left[\frac{b}{a}\frac{f_{1,T_c}}{f_{2,T_c}}-\frac{a}{b}\frac{f_{2,T_c}}{f_{1,T_c}}\right]\left[1+o_p\left(1\right)\right]\\
&\sim T^{(1+\alpha-\beta)/2}r_w^{1/2}\left(\frac{d_2}{d_1}\right)^{1/2}\left[\frac{b\left(F_{1,r}+N_{d_1}\right)}{aF_{2,c}}-\frac{aF_{2,c}}{b\left(F_{1,r}+N_{d_1}\right)}\right]\\
&= O_p\left(T^{(1+\alpha-\beta)/2}\right),
\end{aligned}
$$

since $1-\phi_T^2=2\frac{d_2}{T^\beta}\left[1+o\left(1\right)\right]$, $\rho_T^2-1=2\frac{d_1}{T^\alpha}\left[1+o\left(1\right)\right]$, $f_{1,T_c}\sim T^{\alpha/2}\rho_T^{T_c-T_r}\left(F_{1,r}+N_{d_1}\right)$ from A.3(1), and $f_{2,T_c}\sim T^{\alpha/2}\rho_T^{T_c-T_r}F_{2,c}$ by assumption. ■

C.2 Proof of Theorem 4.7

Proof. The denominator of (C.23) is

$$T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} \right)^2 = T_w a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 + O_p \left(T^{\alpha+2\beta} \rho_T^{2(T_c-T_r)} \right) = O_p \left(T^{1+\alpha+\beta} \rho_T^{2(T_c-T_r)} \right),$$

since $\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 = O_p \left(T^{\alpha+\beta} \rho_T^{2(T_c-T_r)} \right)$ and $\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} = O_p \left(T^{\alpha/2+\beta} \rho_T^{T_c-T_r} \right)$ from Lemma C.7. The numerator is

$$\begin{aligned} T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1} &= T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} + O_p \left(T^{\alpha+\beta} \rho_T^{2(T_c-T_r)} \right) \\ &= -a^2 \frac{d_2}{T^\beta} T_w \sum_{t=T_1}^{T_2} f_{2,t-1}^2 + O_p \left(T^{\alpha+\beta} \rho_T^{2(T_c-T_r)} \right), \end{aligned}$$

since $T_w \sum_{t=1}^{T} \Delta\tilde{y}_t \tilde{y}_{t-1} = O_p \left(T^{1+\alpha} \rho_T^{2(T_c-T_r)} \right)$ and $\sum_{t=1}^{T} \Delta\tilde{y}_t \sum_{t=1}^{T} \tilde{y}_{t-1} = O_p \left(T^{\alpha+\beta} \rho_T^{2(T_c-T_r)} \right)$ from Lemma C.7. Therefore,

$$\begin{aligned} \widehat{\gamma}_{\tau_1,\tau_2} &= \frac{-a^2 \frac{d_2}{T^\beta} T_w \sum_{t=T_1}^{T_2} f_{2,t-1}^2 - \left(-a \frac{d_2}{T^\beta} \sum_{t=T_1}^{T_2} f_{2,t-1} \right) \left(a \sum_{t=T_1}^{T_2} f_{2,t-1} \right)}{T_w a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2} \\ &= -\frac{d_2}{T^\beta} + \frac{d_2}{T^\beta} \frac{\left(\sum_{t=T_1}^{T_2} f_{2,t-1} \right)^2}{T_w \sum_{t=T_1}^{T_2} f_{2,t-1}^2} \\ &= -\frac{d_2}{T^\beta} + O_p \left(T^{-1} \right) = O_p \left(T^{-\beta} \right). \end{aligned}$$

Next, we derive the limiting properties of $\hat{\delta}_{\tau_1,\tau_2}$. By definition, we have

$$\hat{\delta}_{\tau_1,\tau_2} = \frac{\left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 \right) \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} \right) \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} \right)}{T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} \right)^2}.$$

The numerator is

$$\begin{aligned} & \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 \right) \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \right) - \left(\sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} \right) \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1} \right) \\ = & \left(a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 + 2a \sum_{t=T_1}^{T_2} f_{2,t-1} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} \right) \right) \left(-a \frac{d_2}{T^\beta} \sum_{t=T_1}^{T_2} f_{2,t-1} \right) [1 + o_p(1)] \\ & - \left[-a^2 \frac{d_2}{T^\beta} \sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 - a \frac{d_2}{T^\beta} \sum_{t=T_1}^{T_2} f_{2,t-1} \left(\frac{1}{N} \sum_{i=1}^{N} \tilde{L}_{i1} e_{it} \right) \right] \left(a \sum_{t=T_1}^{T_2} f_{2,t-1} \right) [1 + o_p(1)] \end{aligned}$$

$$= -a^2 \frac{d_2}{T^\beta}\left(\sum_{t=T_1}^{T_2} f_{2,t-1}\right)^2 \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1} e_{it}\right)[1+o_p(1)] = O_p\left(T^{\alpha+\beta}\rho_T^{2(T_c-T_r)}\right).$$

Therefore,

$$\hat{\delta}_{\tau_1,\tau_2} = \frac{-a^2 \frac{d_2}{T^\beta}\left(\sum_{t=T_1}^{T_2} f_{2,t-1}\right)^2 \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{L}_{i1} e_{it}\right)}{T_w a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2}[1+o_p(1)] = O_p\left(T^{-1}\right).$$

The sum of squared errors $\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2$ can be rewritten as

$$\sum_{t=T_1}^{T_2}\left(\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 = \sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 + T_w\hat{\delta}_{\tau_1,\tau_2}^2 - 2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right).$$

Since $\phi_T - 1 - \hat{\gamma}_{\tau_1,\tau_2} = -\frac{d_2}{T^\beta}\frac{\left(\sum_{t=T_1}^{T_2} f_{2,t-1}\right)^2}{T_w\sum_{t=T_1}^{T_2} f_{2,t-1}^2} = O_p\left(T^{-1}\right)$,

$$\begin{aligned}\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 &= a^2\left(\phi_T - 1 - \hat{\gamma}_{\tau_1,\tau_2}\right)^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2 [1+o_p(1)] \\ &= a^2 \frac{d_2^2}{T^{2\beta}}\frac{\left(\sum_{t=T_1}^{T_2} f_{2,t-1}\right)^4}{T_w^2\sum_{t=T_1}^{T_2} f_{2,t-1}^2}[1+o_p(1)] = O_p\left(T^{\alpha+\beta-2}\rho_T^{2(T_c-T_r)}\right).\end{aligned}$$

Since $T_w\hat{\delta}_{\tau_1,\tau_2}^2 = O_p\left(T^{-1}\right)$ and

$$2\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right) = 2a\left(\phi_T - 1 - \hat{\gamma}_{\tau_1,\tau_2}\right)\hat{\delta}_{\tau_1,\tau_2}\sum_{t=T_1}^{T_2} f_{2,t-1}[1+o_p(1)] = O_p\left(T^{\alpha/2+\beta-2}\rho_T^{T_c-T_r}\right),$$

we have

$$\begin{aligned}\sum_{t=T_1}^{T_2}\left(\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2 &= \sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2[1+o_p(1)] \\ &= a^2\frac{d_2^2}{T^{2\beta}}\frac{\left(\sum_{t=T_1}^{T_2} f_{2,t-1}\right)^4}{T_w^2\sum_{t=T_1}^{T_2} f_{2,t-1}^2}[1+o_p(1)] = O_p\left(T^{\alpha+\beta-2}\rho_T^{2(T_c-T_r)}\right).\end{aligned}$$

Using these results, we find that the DF test statistic has the following asymptotic order

$$DF_{\tau_1,\tau_2} = \widehat{\gamma}_{\tau_1,\tau_2}\left[\frac{T_w\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2}\tilde{y}_{t-1}\right)^2}{\sum_{t=T_1}^{T_2}\left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2}\right]^{1/2}$$

$$
\begin{aligned}
&= \frac{T_w \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \tilde{y}_{t-1} - \sum_{t=T_1}^{T_2} \Delta\tilde{y}_t \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}}{\left[T_w \sum_{t=T_1}^{T_2} \tilde{y}_{t-1}^2 - \left(\sum_{t=T_1}^{T_2} \tilde{y}_{t-1}\right)^2\right]^{1/2} \left[\sum_{t=T_1}^{T_2} \left(\Delta\tilde{y}_t - \hat{\delta}_{\tau_1,\tau_2} - \widehat{\gamma}_{\tau_1,\tau_2}\tilde{y}_{t-1}\right)^2\right]^{1/2}} \\
&= \frac{-a^2 \frac{d_2}{T^\beta} T_w \sum_{t=T_1}^{T_2} f_{2,t-1}^2}{\left[T_w a^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2\right]^{1/2} \left[a^2 \frac{d_2^2}{T^{2\beta}} \frac{\left(\sum_{t=T_1}^{T_2} f_{2,t-1}\right)^4}{T_w^2 \sum_{t=T_1}^{T_2} f_{2,t-1}^2}\right]^{1/2}} \left[1 + o_p(1)\right] \\
&= -T_w^{3/2} \frac{\sum_{t=T_1}^{T_2} f_{2,t-1}^2}{\left(\sum_{t=T_1}^{T_2} f_{2,t-1}\right)^2} \left[1 + o_p(1)\right]
\end{aligned}
$$

We know from Lemma A.4(3) and (4) that $\sum_{t=T_c+1}^{T_2} f_{2,t-1} = \frac{T^\beta}{d_2} f_{2,T_c} [1 + o_p(1)]$ and $\sum_{t=T_c+1}^{T_2} f_{2,t-1}^2 = \frac{1}{1-\phi_T^2} f_{2,T_c}^2 [1 + o_p(1)]$. It follows that

$$
\begin{aligned}
DF_{\tau_1,\tau_2} &= -T_w^{3/2} \frac{\frac{1}{1-\phi_T^2} f_{2,T_c}^2}{\frac{T^{2\beta}}{d_2^2} f_{2,T_c}^2} \left[1 + o_p(1)\right] = -T_w^{3/2} \frac{d_2}{2T^\beta} \left[1 + o_p(1)\right] \\
&\sim -T^{3/2-\beta} r_w^{3/2} \frac{d_2}{2} \left[1 + o_p(1)\right] = O_p\left(T^{3/2-\beta}\right).
\end{aligned}
$$

■

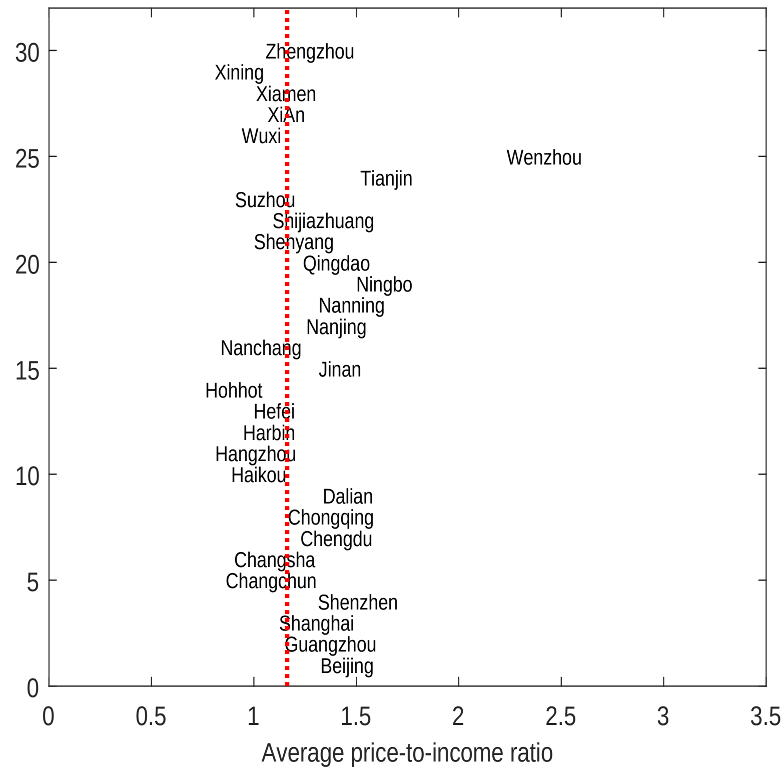
D Tables and Figures

Table 1: Tier 1, 2 and 3 cities

Tier 1	Beijing, Shanghai, Guangzhou, Shenzhen
Tier 2	Changchun, Changsha, Chengdu, Chongqin, Dalian, Haikou, Hangzhou, Harbin, Hefei, Hohhot, Jinan, Nanchang, Nanjing, Ningbo, Qingdao, Shenyang, Shijiazhuang, Suzhou, Tianjin, Wenzhou, Wuxi, Xi'an, Xiamen, Xining, Zhengzhou
Tier 3	Anqing, Anshan, Baoding, Baotou, Bengbu, Changde, Changzhou, Chuzhou, Dandong, Deyang, Dongguan, Huai'an, Huzhou, Jianyan, Jiaxing, Jieyang, Jiujiang, Kaifeng, Langfang, Leshan, Lianyungang, Luohe, Luoyang, Luzhou, Mianyang, Nanchong, Nantong, Nanyang, Ningde, Qinhuang, Quanzhou, Rizhao, Shangrao, Shantou, Shaoxing, Songyuan, Suqian, Taizhou, Tangshan, Wuhu, Wuludao, Xingtai, Xuancheng, Xuzhou, Yancheng, Yangzhou, Yichun, Yingkou, Zaozhuang, Zhangjiakou, Zhangzhou, Zhaoqing, Zhenjiang, Zhongshan, Zhumadian

Figure 9: The average price-to-income ratios of 89 cities in China. The vertical line indicates the national average price-to-income ratio over the sample period.

(a) Tiers 1 and 2



(b) Tier 3