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General Enhancements of Atom Interferometers

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by

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Atomic physics, a branch of physics whose modern formulation is about a century old has claimed a large number of Nobel Prizes over the last few decades. Despite rapid advancements in research, the development of practical devices using quantum effects has only recently taken off with incentives such as the UK Quantum Technology Programme. This programme has spent £120M from 2014 to 2019 into four different ‘hubs’: Sensors & Timing, Computing & Simulations, Imaging and Communication. Each hub comprising of a large collaboration of research groups and industrial partners. This marks a strong technological push towards practical and commercial quantum-base devices.

The work discussed in this thesis is partially funded by the Sensors & Timing hub with other contributions from EPSRC and DSTL. All three parties are interested in the advancements of quantum technologies to commercial levels using atomic physics, a challenge currently limited by the engineering complexity and the sensitivity quantum systems have to their surroundings. It are these two challenges this thesis aims to provide solutions for.

Our investigation into the different parts of the interferometer have identified twenty-six enhancements of atom interferometers of different scale. We report some on their effectiveness including the testing of composite pulse techniques adapted from nMRI and proposed characterisation of laser systems using speckle-based spectrometers. In particular, we discuss the finalisation of a mode-locked laser characterisation for providing a frequency reference to our experiments along with proposed modifications to typical set-ups. Progress towards a new quantum rotation sensor is also reported on along with a review of systematic errors and their origin in atom interferometers.
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Declaration of Authorship

I, David E. Elcock Jr., declare that this thesis titled, *General Enhancements of Atom Interferometers* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University.
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as: [120, 158].

Signed:

Date:

________________________________________________________________________

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Acknowledgements

I repeatedly experienced a lack of support from the University’s Graduate School. Both prior and during the Covid-19 pandemic, apart from examiners Prof M. Keller & Dr P. Horak, no thesis advisor has advised or made any recommendations to this thesis. To avoid further harm from discrimination I could not rely on my previous supervisors leaving me with no thesis advisor during the writing of this thesis. My frequent and recurring requests for a new supervisor were repeatedly denied which has undoubtedly negatively affected the quality of writing and presentation of material in this thesis.

The research project experienced delays of essential equipment in excess of 4+ years, obstructed access to the two laboratories, logistical problems and a lack of supervision as confirmed by an internal investigation into the project’s supervisory team. These supervisory problems led to excessive transfers to different, and sometimes unrelated projects. This thesis covers the major projects I worked on from 2016 to 2018 that fit the title’s narrative.

Despite these unfortunate circumstances, I would like to thank M. Carey and Dr. J. Woods for sharing their understanding on their experiments, J. Saywell for the various useful discussions on theoretical, atomic and laser physics and Dr. M. Belal for teaching me several tricks and techniques of experimental laser physics.

Throughout my MPhil I frequently visited the Telecom Lab of the ORC collective research group who generously provided me with equipment and tools I needed for my experiments. In particular, I like to thank Dr. Kyle Bottrill with whom I’ve had lengthy discussions on physics and none-physics related topics and whom always provided me with useful insight in technical difficulties I was facing.

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From the Integrated Atom Chip Group I like to thank my co-supervisor Dr. Matt Himsworth on his guitar training and how he often intrigued me with recent technological developments and techniques being used in atomic physics. I’d also like to thank Dr. M. Aldous for his brief introduction training with lasers and subsequent discussions and Dr. T. Freegarde for inviting me to join his research group back in 2016.
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David Emanuel Elcock Jr.

’Hot-cold, light-dark, emotion-intellect, the universe runs on the collision of such forces, as we shall see. Gather around.’ said Dr. Alexander Hartdegen.

’What is it?’ a student asks.

’A windmill?’ another student asks.

’Not hardly, this device which is called a reverse radiometer sits motionless, sealed and protected from the wind and yet when exposed to the light of the sun...’

’Could we maybe watch the universal collision of forces from inside?’ Mr. Lowmen asks.

’Patience Mr. Lowmen, time is your friend... so long as you have enough of it.’

– Deleted Scene from The Time Machine (2002), Simon Wells
Research Funding Description

The original research project description proposed by Dr. T. Freegarde:

Atom interferometers exploit the principle of quantum superposition to compare the evolution speeds of superposed wavefunctions by beating the oscillations of these inbuilt atomic clocks. Since atoms and their constituents have mass, charge and magnetization, atom interferometers can measure inertial motion and electric and magnetic fields, by recording their differential effect upon the interfering matterwaves.

We have developed an atom interferometer that can distinguish and control the velocities of ultracold atoms, and for enhanced fidelity we have applied nMR composite-pulse techniques. We shall use these techniques and apparatus to form a sensitive atom interferometric gyroscope that exploits the sensitivity of atomic phase to Coriolis acceleration. Ballistic atom cloud expansion maps atomic velocity to position, causing the Coriolis-dependent phase to be manifest spatially as a rotation-dependent fringe pattern, which can be imaged and the rotation rate deduced. Whereas conventional mechanical and optical gyroscopes are prone to toppling and mirror 'lock-in', the freely-moving atoms are unperturbed by the apparatus.

Beyond demonstrating and characterizing the atomic interferometer gyroscope, we shall demonstrate its miniaturization with a sealed-off atom/vacuum chamber, develop the composite-pulse sequences, and explore a quantum-enhanced read-out method borrowing from Raman quantum memory.
Convention of Figures

Unless stated otherwise, all data plots in this thesis are made using *Matlab* version 2018a. A particular style has been selected as convention for these figures as described here (to avoid unnecessary repetition).

For measured data we use a scatter plot of the individual data-points. Error estimates in the measured variable (y-axis) are represented using the *patch* function. Error estimates in the control variable (x-axis) are represented as conventional error-bars or given in the figure captions if constant. In that case they are depicted in decimal form as \(x.y(z) = x.y \pm 0.z\).

For double axes the measured variable appears on the left y-axis and the control variable on the bottom x-axis. The error estimates will be in units of the measured and control variable unless stated otherwise.

All error analysis associated with data figures in this thesis will use the equations and methodology as per [1] unless stated otherwise.

With regards to figures showing schematics of sketches, these were made using the *Inkscape* vector-graphics editor. Apart from figures in courtesy of external references (appropriately cited in its caption), no imported components were used in the making of the figures.
List of Abbreviations

General

EPSRC  (the) Engineering (and) Physical Sciences Research Council
DSTL  (the) Defence Science (and) Technology Laboratory
IOP  (the) Institute Of Physics
ORC  (the) Optoelectronics Research Centre (University of Southampton)
ESA  (the) European Space Agency

FT  Fourier Transform

nMR  nuclear Magnetic Resonance
nMRI  nMR Imaging

PP  Point-(to)-Point (composite pulse)
GR  General Rotator (composite pulse)

GRAPE  Gradient Ascent Pulse Engineering (or Parameter Estimation)

TBP  Time-Bandwidth Product
RMS  Root Mean Squared

FWHM  (the) Full Width (at) Half Maximum
HWHM  (the) Half Width (at) Half Maximum

Cov  Covariance

TDSE  Time-Dependent Schrödinger Equation
TISE  Time-Independent Schrödinger Equation

PCA  Principle Component Analysis
ANOVA  ANalysis Of VAriance

s.t.  such that
w.l.o.g.  without loss of generality
N/A  Not Applicable
Latin

i.e.  
*e.g.  
*c.f.  
*N.B.  
*q.o.d.  
*et. al.  
*etc.  

ad hoc  
a priori

Methods & Techniques

CW  Continuous Wave (operation/lasing)
OFC  Optical Frequency Comb
CPML  Colliding Pulse Mode-Locking
PCML  Pulse-Clustered Mode-Locking
FML  Fundamental Mode-Locking
ASP  Ancillary Stretched Pulse
SPIDER  Spectral Phase Interferometry (for) Direct Electric Field Reconstruction
FROG  Frequency-Resolved Optical Gating
OIL  Optical Injection-Locking
SHG  Second Harmonic Generation
SFG  Sum-Frequency Generation
MOT  Magneto-Optical Trap
DAVELL  Dichroic-Atomic-Vapor Laser Lock
STIRAP  Stimulated-Raman Adiabatic Passage
LMT  Large-Momentum Transfer Scheme
Optical Terminology & Components

UV  Ultra-Violet (light)
AR  Anti-Reflection (coating)
TEM Transverse Electro-Magnetic (mode)
ASE Amplified Spontaneous Emission
FSR Free-Spectral Range (frequency)
CEO Carrier Envelope Offset (frequency)
CO  Common Offset (frequency)
SA  Saturable Absorber
BD  (laser) Beam Dump
BSO (laser) Beam-Shaping Optics
PPLN Periodically-Poled Lithium Niobate ($LiNbO_3$)
SMF Single-Mode Fiber
PMF Polarisation-Maintaining Fiber
PBSC Polarising Beam Splitting Cube
BSC (non-polarising) Beam-Splitting Cube
PBS Pellicle Beam Splitter
DM  Dichroid Mirror
SPDM Short-Pass Dichroic Mirror
MML Mode-Matching Lens
MS  Microscope Slide
IRF Infra-Red Filter

Optical Instrumentation & Devices

FP  Fabry-Perot (interferometer)
MZI Mach-Zehnder Interferometer
PSI Point-Source Interferometer
Electro-Optical Terminology & Components

MLL  Mode-Locked Laser
RWL  Ridge Waveguide Laser
DBR  Distributed Bragg Reflector (laser)
DFB  Distributed Feed-Back (laser)
ECDL External Cavity Diode Laser
VECSEL Vertical External Cavity Surface-Emitting Laser
VCSEL Vertical Cavity Surface-Emitting Laser
BAL  Broad-Area Laser
LD   Laser Diode

PD   Photo-Diode
PMT  Photo-Multiplying Tube

TA   Tapered Amplifier
BoosTA TA (specific model made by Toptica)
EDFA Erbium-Doped Fiber Amplifier

FOI  Faraday Optical Isolator

SSBM Single Side-Band Modulator
AOM  Acousto-Optical Modulator
EOM  Electro-Optical Modulator

Electro-Optical Instrumentation & Devices

OSA  Optical Spectrum Analyser
M-OSA Monochromatic OSA
FP-OSA FP OSA

CCD Charge-Coupled Device (camera sensor)
CMOS Complementary Metal-Oxide-Semiconductor (camera sensor)
CID  Charge Injected Device (camera sensor)

OPLL Optical Phase-Locking Loop
**Electronic Terminology & Components**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td>RF</td>
<td>Radio Frequency</td>
</tr>
<tr>
<td>DC</td>
<td>Direct Current (constant)</td>
</tr>
<tr>
<td>AC</td>
<td>Alternating Current (non-constant)</td>
</tr>
<tr>
<td>AM</td>
<td>Amplitude Modulation (or)</td>
</tr>
<tr>
<td>FM</td>
<td>Frequency Modulation (or)</td>
</tr>
<tr>
<td>PM</td>
<td>Phase Modulation (or)</td>
</tr>
<tr>
<td>NTC</td>
<td>Negative Thermal Coefficient (thermistor)</td>
</tr>
<tr>
<td>LDA</td>
<td>Laser Diode Anode (pin)</td>
</tr>
<tr>
<td>TEC</td>
<td>Thermo-Electric Cooler</td>
</tr>
<tr>
<td>BT</td>
<td>Bias Tee</td>
</tr>
<tr>
<td>PS</td>
<td>Power Splitter</td>
</tr>
<tr>
<td>PCB</td>
<td>Printed Circuit Board</td>
</tr>
<tr>
<td>AMP</td>
<td>Amplifier</td>
</tr>
<tr>
<td>DAC</td>
<td>Digital-to-Analogue Converter</td>
</tr>
<tr>
<td>ADC</td>
<td>Analogue-to-Digital-Converter</td>
</tr>
<tr>
<td>LPF</td>
<td>Low-Pass Filter</td>
</tr>
<tr>
<td>HPF</td>
<td>High-Pass Filter</td>
</tr>
<tr>
<td>I&amp;Q</td>
<td>(RF) I&amp;Q Modulator</td>
</tr>
<tr>
<td>VCO</td>
<td>Voltage Controlled Oscillator</td>
</tr>
</tbody>
</table>

**Electronic Instrumentation & Devices**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC</td>
<td>Personal Computer</td>
</tr>
<tr>
<td>RFSA</td>
<td>RF Spectrum Analyser</td>
</tr>
<tr>
<td>PID</td>
<td>Proportional, Integral, Derivative (servo)</td>
</tr>
<tr>
<td>ICE</td>
<td>Integrated Control Electronics (servos)</td>
</tr>
<tr>
<td>CC</td>
<td>Current Controller (servo)</td>
</tr>
<tr>
<td>TC</td>
<td>Temperature Controller (servo)</td>
</tr>
<tr>
<td>TTL</td>
<td>Transistor-Transistor Logic (signal generator)</td>
</tr>
<tr>
<td>RF-S</td>
<td>RF Signal (generator)</td>
</tr>
<tr>
<td>AWG</td>
<td>Arbitrary Waveform Generator</td>
</tr>
</tbody>
</table>
Dedicated to my honourable, loving and hard-working parents:
David Emanuel Elcock Sr. and Jetske Hager.
Chapter 1

Introduction
CHAPTER 1. INTRODUCTION

1.1 Motivations and Aims

One of the longest lasting assumptions of classical physics is the 'atomic hypothesis', namely the idea that all matter is made from indivisible components called 'atoms'. This concept first introduced in the 5th and 4th centuries B.C. by Greek philosophers [2] coarsely persists up to today, excluding sub-atomic particles and field theory descriptions\(^1\).

The atomic hypothesis historically describing atoms as particles is put under question when considering atom interferometry experiments such as by O.Carnal et. al. in 1991 [24]. These experiments amongst others put into question the particle nature of atoms and are nowadays commonly treated by invoking a matterwave description first proposed by de Broglie 1924.

Whilst more is understood about atom interferometry and quantum duality over the years [25], its only recent that the practical application of these matterwaves are under development. One of these applications is using atomic matterwaves to make an inertial guidance system [26]. The end goal of this thesis is towards the development of a prototype atom interferometer for inertial guidance using ultra-cold Rubidium atoms.

We discuss the preliminary construction of an atomic interferometer that uses light-pulses to exercise control over the density matrix of an ensemble of atoms (distribution in wavefunctions of the atoms). A pulse of the appropriate amplitude (coherence) and duration (\textit{Rabi oscillations}) can place an atom in a coherent superposition, analogous to matter partitioning light into multiple paths. Each atom’s wavefunction is made to form a \textit{Mach-Zehnder Interferometer} whose output is sensitive to changes in the interrogating laser beam axis (which is paralell to the quantisation axis).

Due to the scale of the engineering involved: the complex interaction of the various technologies comprising the interferometer, we took interest in studying the limiting factors: \textit{noise, perturbations, imperfections, etc.}. Atomic systems are known to be sensitive to their environment, hence parallel to building a new interferometer we aim to identify, implement and present enhancement schemes of interest across the board of atomic physics experiments. As the new interferometer is in early stages of development, this thesis will focus on our findings regarding different levels of enhancement.

\(^1\)Key historical events were reviewed using the following references (in chronological order): [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].
1.2 Thesis Structure

Atom interferometers make use of a wide range of technologies and systems described by different fields of physics. This thesis makes suggestions and provides examples of enhancements in the different parts that comprise atomic interferometers. For this reason this thesis is divided into four sub-theses with a final sub-thesis dedicated to consolidating our findings. Each sub-thesis (part) focuses on various levels and types of modifications on a particular part of the atomic interferometer as illustrated in figure 1.1.

Figure 1.1: Following this introductory chapter, the thesis is comprised of five sub-theses. In order of appearance they focus on laser enhancements (Lasers), novel laser characterisation by speckles (Speckles), enhancements of laser-atom interactions (Laser-atom) and hardware enhancements (Hardware), all which are reviewed, summarised and classified towards the end of the thesis (Summary of Enhancements). The classification is based on the level of enhancement methods provide along with their implementation difficulty. Appendices can be found post the sub-theses.

For the most part, each sub-thesis is effectively independent of the others. They contain their own introductory to conclusion chapters and can thus honestly be viewed as actual mini-theses. One exception is the cross-references between sub-thesis III and IV. The second exception is sub-thesis V which depends on the previous four sub-theses. Regarding reading order we advised to read through sub-thesis III before reading sub-thesis IV and reading sub-thesis V at the end.

Beyond sub-thesis V, several insightful appendices can be found including a dedicated list of the conferences and schools the author attended during the MPhil along with the posters, titles and comments on publications. Note however that several projects have not found their way into this thesis as they did not fit under the umbrella of general enhancements. Examples include the supervision of a 350 MHz acousto-optical modulator quadrupole pass Raman laser system, etc. which are not discussed in this work.
1.2.1 Diode Laser Enhancements

The main technology used to prepare, interrogate and detect the ultra-cold atoms are lasers. Playing a multitude of roles of the atom interferometer, we dedicate sub-thesis I on diode lasers, the laser class used in our atom interferometers. In particular, we append to the doctoral work of Dr. J. R. C. Woods (2016) titled 'A Mode-Locked diode laser frequency comb for ultracold atomic physics experiments' [27], a laser system that we hope can provide a frequency reference to the multitude of atom interferometer lasers.

1.2.2 Characterisation using Speckles

The characterisation of the laser system developed by Dr. J. Woods (and by extension other laser systems) play an important role in optimising atom interferometers indirectly. Namely, being able to characterise a laser system to a greater depth and resolution enables higher interferometer signal optimisation. With recent development in speckle-based spectrometers, we therefore investigate their application onto our laser systems in sub-thesis II.

1.2.3 Laser-atom Enhancements

Sub-thesis III discusses work on an existing atom interferometer used for testing enhancement schemes. In particular, we discuss the experimental work on the application of composite mirror pulses. We report on preliminary results and categorise systematic errors of atom interferometers into two classes: off-resonance and pulse-length errors. This formal classification will keep track of the various errors and their effect on atom interferometers.

1.2.4 Hardware Enhancements

In sub-thesis (IV) we aim to realise the 1993 US7317184B2 patent by M. A. Kasevich & B. Dubetsky: 'Kinematic sensors employing atom interferometer phases'. Early development is discussed along with an outline of future work.

1.2.5 Summary of Enhancements

In sub-thesis V we evaluate and characterise the level of enhancement taken from and inspired by the previous sub-theses. The sources of experimental perturbations are highlighted with different mitigation methods identified and discussed. The results are then coarsely ordered into three classes of enhancement: minor, modest and major offering a coarse scale on implementation difficulty and level of enhancement on proposed techniques.
Sub-thesis I

Enhancements of Diode Lasers
Chapter 2

Introduction

Enhancement of an atom interferometer involves understanding the experimental and theoretical limitations in all aspect of the experiment. In this sub-thesis, we focus on the diode laser, the main tool used to prepare, interrogate and measure the state of our atoms.

In particular, we characterise a recently purchased distributed feedback diode laser (DFB) aimed to be used in atomic physics experiments. We also complement the mode-locking diode laser characterisation by former PhD student Dr. J. Woods. His PhD project aimed at generating a mini optical frequency comb (OFC) around the D2 line of Rubidium could namely provide several enhancements to our proposed atom interferometer along with freeing up space on the optical bench.

By characterising the laser’s behaviour in continuous wave operation (CW), we can anticipate knock-on effects on the laser-atom interactions or on mode-locking. Likewise we aimed to characterise the frequency and phase stability of the frequency-ruler (OFC) with the aim of controlling the single frequency detuning of the Raman interrogating laser for our atomic physics experiments. Please note that the propagation of the CW effects on laser-atom interactions however we forward to sub-thesis III.
2.1 Sub-thesis Structure

This sub-thesis is divided into two parts: continuous wave (CW) diode lasers (chapters 3 & 4) and mode-locking (chapters 5, 6 & 7). Each part has their own literature review and experimental characterisation chapter(s).

As the housing of diode lasers are not designed to be opened, our CW review aims to reveal the inaccessible parts, response and purpose in operating diode laser. Likewise, our mode-locking review aims to provide a synopsis from which we can hypothesise and infer traits/features from observations.

Towards the end of this sub-thesis in chapter 8 we discuss our primary findings and list several hypotheses on our observations. This is summarised in our conclusion chapter 9 at the end of the sub-thesis.

2.2 Disclaimer

This sub-thesis started as an attempt to finish off the characterisation of the Mode-Locked Laser (MLL) by Dr. J. Woods in his thesis [27]. Initially set as a short introduction project to the PhD, the project was extended after an unexplained performance improvement of the MLL was measured. Unfortunately, after the departure of a fellow PhD student (who fell ill and cancelled their candidature) the project was later terminated due to an urgent transfer to another project. Due to compulsory involvement in several other projects, this sudden departure limits the number of significant findings.

The design, construction and preliminary characterisation of the MLL documented in [27] are fully credited to Dr. J. Woods. Our experimental work discussed in this sub-thesis however regarding the MLL involved supervision and collaboration with Dr. J. Woods and continues from the findings in his doctoral thesis. About half of the work on the MLL discussed in this work are thus co-credited to Dr. J. Woods and append to his previous thesis.

The *Eagleyard* diode laser characterisation in this thesis was supervised by Dr. M. Belal with a helpful laboratory introduction given by Dr. M. Aldous. This acted as a separate PhD introduction project with the aim of confirming the laser’s specifications and finding the control variable values at which the recently purchased laser mode-hops. Whilst more results are available on this characterisation, this project met the same fate of terminating due to a mandatory and sudden transfer to another project.
Chapter 3

The Diode Laser

3.1 Principle of Operation

3.1.1 Population Inversion

As per Fermi-Dirac statistics, the occupation probability of a(n) hole (electron) in the valence (conduction) band is given respectively by

\[ p_v(E_1) = \frac{1}{1 + e^{\frac{E_1 - E_{f,v}}{kT}}} \quad p_c(E_2) = \frac{1}{1 + e^{\frac{E_2 - E_{f,c}}{kT}}} \]

where \( E_1 \in \{ E_v \} \) (valence band), \( E_2 \in \{ E_c \} \) (conduction band) and parameters \( E_{f,v} \) and \( E_{f,c} \) represent the quasi-Fermi levels of the valence and conduction band respectively. The quasi-Fermi levels define the upper boundary of energy-levels occupied by holes (electrons) in the valence (conduction) band. At higher charge carrier densities, \( E_{f,c} \) increases whilst \( E_{f,v} \) decreases.

As both electrons in the conduction band and holes in the valence band contribute to population inversion, per figure 3.1, lasing requires \( p(E_2) > p(E_1) \) which happens when \( h\omega > E_g \) i.e. the resultant photon’s energy \( h\omega \) exceeds the energy band-gap. This is equivalent to the charge carrier density \( N \geq N_{th} \) where the difference in quasi-Fermi levels at the threshold charge carrier density’s \( N_{th} \) is given by

\[ E_{f,c}(N_{th}) - E_{f,v}(N_{th}) = E_g \]

Note that this form of the Fermi-Dirac distribution specifies the occupation probability inside a particular band. Sometimes the Fermi-Dirac distribution takes a slightly different form by setting quasi-Fermi level to intrinsic/extrinsic Fermi-energy to specify the occupation probability in any band.
CHAPTER 3. THE DIODE LASER

Figure 3.1: The formation of energy bands (shaded regions) from discrete atomic energy levels by decreasing the spacing $\Delta x$ between adjacent atoms in a semiconductor. The discrete ground ($E_1$) and excited ($E_2$) energy-levels of atoms are replaced by a valence ($\{E_v\}$) and conduction ($\{E_c\}$) band respectively. The bands comprise of multiple discrete states populated which charge carriers upto their respective quasi-Fermi level $E_{f,c}, E_{f,v}$ which can undergo transitions of energy $\hbar \omega$ to states in other bands separated by a gap $E_g = E_c - E_v$.

For general $N$ the lasing criterion can be described by the 1961 Bernard-Duraffourg condition [28]

$$E_{f,c}(N) - E_{f,v}(N) > \hbar \omega > E_g$$

In diode lasers, the carrier density $N$ can be increased by forward-biasing the pn-junction as shown in figure 3.2. Forward biasing injects electrons into the n-side and holes into the p-side (together a larger $N$) resulting in a different Fermi level per junction side, a lower barrier voltage and a more narrow depletion zone, all proportional to the biasing potential $V_o$. This increases the rate of diffusion of charge carriers into the depletion zone which enhances the rate of electron-hole recombinations that generate light.

Placing the pn-junction inside a cavity we find that at sufficiently high $N$, the light generated from stimulated emission recombinations eventually starts to compete with the round-trip losses of the cavity. It is at this point that the pn-junction transitions from being a light source (e.g. light emitting diode) and becomes a diode laser. More details on pn-junctions and the GaAs derivative we use can be found in [29, 30] & [28, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] respectively.
3.1. PRINCIPLE OF OPERATION

Figure 3.2: pn-homojunction, (a) forward biasing by $V_0$; (b) reverse biasing; (c) p-doped side (left) with acceptor localised states $E_A$ right after contact with the n-doped side (right) hosting donor states $E_D$; (d) formation of the depletion zone potential barrier $V_b$ from charges $\pm Q$ and flow rates $i_1, i_2$ of electrons; (e) energy-band diagram at forward bias following the distortion of the valence (conduction) band edges $E_v (E_c)$ and the intrinsic (extrinsic) Fermi-energy $E_i (E_f)$. Finally, $E_{f,v}$ and $E_{f,c}$ represent the quasi-Fermi levels of the valence and conduction band respectively.

Figure 3.3: The double heterostructure when (a) doped and forward biased. The light intensity profiles (solid blue) and refractive index profile (dashed black) are shown in (b), (c) and (d) for the homo-, single hetero- and double heterostructure respectively.

In practice, a pn-junction where each sides share a common bandgap energy (homojunction) has a high carrier density threshold $N_{th}$ making lasing at room temperature impossible. Commercial diode lasers thus utilise a double heterostructure where multiple p- and n-doped layers are appropriately stacked to optimise spatial confinement of charge carriers within an active layer. Figure 3.3 shows a typical structure of such a configuration. Along with index guiding of light, this design lowers $N_{th}$ to allow for room temperature lasing and provides control over the gain-region. More details on diode lasers can be found here: [31, 41, 42, 43, 44, 45].
3.1.2 Optical Resonator

Typically the cleaved facets (ends) of the semiconductor pn-junction form a cavity from the Fresnel reflections at those interfaces [46]. An optical resonator is formed whose longitudinal modes are generated by the constructive interference of the forward and backward travelling waves inside the resonator/cavity. By definition these modes have a well-defined phase-relationship i.e. are phase coherent.

Qualitatively the coherent property$^2$ of modes originates from the type/level of interference (constructive/destructive) between the forward and backward travelling waves that comprise the mode. Per mode, opposite travelling waves of different frequencies interfere more destructively making same frequency (phase coherent) waves favourable. We can derive this result using boundary conditions and including (amplified) spontaneous emission (ASE).

The Dirichlet and Neumann boundary conditions$^3$ at the facets of the diode laser of length $L$ and uniform refractive index $n$ restrict the solutions$^4$ of the allowed cavity modes to waves whose wavelength $\lambda_q$ fit inside the cavity as per equation 3.1. Hence a laser acquires a discrete spectrum of supported cavity modes even though the laser-gain medium has a smooth spectrum. As such we say the cavity supports standing-wave modes whose wavelength $\lambda_q$ fits inside the cavity as illustrated in figure 3.5 b).

$$q\lambda_q = 2nL, q \in \mathbb{N}$$

Whilst supported cavity modes experience the smallest loss from optimising round-trip constructive interference, introducing an infinitesimal shift in wavelength ($\lambda_q \rightarrow \lambda_q \pm \delta \lambda$) still allows light with higher losses to persist inside the cavity albeit at lower amplitudes. Amplitudes are reduced by the level of round-trip destructive interference (e.g. increasing $\delta \lambda$) giving the cavity a fundamental longitudinal linewidth about the supported modes $\lambda_q$.

Note that the phase requirement for total constructive interference inside the cavity aids the stimulated emission phase property of a laser. As such, both spatial and temporal coherence of a laser depend in part on the filtering of supported cavity modes.

---

$^2$N.B. we are not stating that there is a natural requirement for coherence between different modes. But a coherence requirement does exist naturally for the backwards and forwards travelling waves that make up a mode.

$^3 f(\alpha) = \beta$ and $f'(\alpha) = \beta$ respectively. Also called first-type and second-type boundary conditions respectively.

$^4$When solving the wave-equation for the electro-magnetic fields inside the cavity.
3.1.3 Mode Competition

A cavity supports infinitely many longitudinal modes as per equation 3.1. Exciting multiple modes simultaneously results in multi-mode operation/lasing of that laser. This can either be stable (e.g. bistable for two modes at the same time) or unstable (e.g. bimodal for jumping randomly between two modes; metastable between two modes). In general, spontaneous emission encourages multimode operation as this type of laser-atom interaction operates over the entire gain-medium spectrum [47]. In practice however, single-mode operation in semiconductor diode lasers is possible through a process known as mode competition.

There are several mechanisms that induce or affect competition between modes, all having the same effect: to provide preference to a single or a few modes. Three of interest are discussed briefly here.

1. **Gain saturation.** A mode $\omega_q$ that lases ($N(\omega_q) > N_{th}(\omega_q)$) lowers the gain profile of the laser due to gain saturation. This results in fewer charge carriers $N$ being available to the other modes making them less competitive to $\omega_q$. Some modes experience a gain below the lasing threshold and become non-preferred modes which do not lase. As such, gain saturation reduces the number of modes that lase to a finite set of modes with frequencies near the peak of the gain spectrum [48].

2. **Laser geometry.** If the diode laser contains other elements inside its cavity such that the active gain medium does not fully fill the space between the facets (ends) of the laser (cavity), the spectrum is affected by another form of mode competition. This mode competition is governed by spatial hole burning of the gain profile [48]. Depending on the position and length of the gain medium inside the cavity, the mode competition can be enhanced or reduced. The position of intra-cavity elements can have a similar effect [46].

3. **Optical losses.** The bandwidth of the reflectivity of the mirrors (ends) of an optical resonator tend to peak around at a frequency. Along with other optical losses, the frequency dependence of such losses introduce a preference to those modes with lowest losses [49].
CHAPTER 3. THE DIODE LASER

Gain Saturation

The gain $g$ of a diode laser can according to [28] be approximated as

$$g(\omega) \propto (p_c(E_2) - p_v(E_1))\sqrt{\hbar \omega - E_g}$$  \hspace{1cm} (3.2)

The factor in brackets is positive only for $N > N_{th}$ demonstrating the need for the probability of an electron in the conduction band to exceed the probability for a hole to occupy the valence band. The remaining factor stems depends on the density of states of the laser’s gain medium.

Due to optical losses, lasing requires $g > g_{th}$ for some threshold gain value $g_{th}$. As such, only modes with sufficiently large $g$ end up building enough energy to lase. This is illustrated in figure 3.4.

Assuming the dominating broadening mechanism is natural broadening, our diode lasers acquire Lorentzian lineshapes. As such, we find the gain saturation lowers the gain profile globally as the charge carriers are shared amongst all modes. In the presence of inhomogeneous broadening mechanisms, the resulting lineshape would deform into a Voigt profile from the additional Gaussian lineshape contribution. Also, the gain saturation phenomena would occur more locally around each cavity mode depending on the level of inhomogeneous broadening [48, 50].

Figure 3.4: Diagram illustrating mode competition from gain saturation. (a) An gain and laser spectrum is drawn right after turning the laser on. After a short time, for an (b) homogeneous lineshape the gain profile lowers and a few modes lase. In the case of an inhomogeneous lineshape (c), the gain profile is only affected locally. Due to spontaneous emission, all modes have a minimum 'gain'/energy-density.
3.1. **PRINCIPLE OF OPERATION**

**Laser Geometry**

The case of geometry-based mode competition can be understood by the following simple analysis. Let the spatial phase of the standing wave mode $q$ of wavelength $\lambda_q$ along the spatial/optical axis $z$ be defined as

$$\phi_q = \frac{2\pi}{\lambda_q}z$$

We consider two longitudinal modes of indices $p$ and $q$. The spatial dependence on the difference in phase between these two modes is thus given by

$$\Delta \phi = \phi_p - \phi_q = \frac{2\pi}{\lambda_1} \Delta z \ , \ \Delta z := z(p - q) \quad (3.3)$$

where $\lambda_1 = 2L$ is the fundamental longitudinal mode. As $\Delta \phi$ is a phase, we can wind it to construct a map $\Delta \phi \mapsto \Delta \phi \in [0, \frac{\pi}{2}]$. From equation 3.3 we can infer how a similar winding-map can be made for $\Delta z$. The outcome is shown in figure 3.5 (a) and shows how the spacing between nodes of a pair of modes varies inside the optical resonator.

**Figure 3.5:** Diagram illustrating (a) the mode spacing $\Delta z$ between nodes of a pair of modes along the cavity’s optical axis $z$ of length $L$. (b) a time-snap of the electric field $E$ of the three lowest order modes. We make use of the wavelength mode spacing $\lambda_1 = 2L$ and difference in mode index $\Delta q$.

Suppose we place thin slab of negligible thickness at the centre of the cavity. This interface would introduce partial reflections forming two intra-cavities, each with one of the mirrors of the optical resonator and the slab. Due to the reflections at the slab being partial we find the spectra of this modified cavity depends on the spectra of the intra-cavities and of the unmodified cavity.
In other words, by placing the slab we modify the boundary conditions for intra-cavity modes. From subsection 3.1.2 we can thus expect a change in the modified laser’s spectrum.

From figure 3.5 we can infer that the modified cavity would be left preferring the even modes of the unmodified cavity. This follows since the fundamental mode (odd) of the unmodified cavity has an anti-node (maximised $\Delta z$) at the slab and $\Delta q = \text{odd}$ unmodified modes share a node (minimised $\Delta z$) at the centre of the cavity. That is to say, all unmodified cavity odd modes do not satisfy the new boundary conditions (node at the cavity ends) at the slab hence are not supported by the intra-cavity.

Alternatively we can say that the modes of the intra-cavity have twice the mode-spacing such that only even unmodified modes are supported. The consequence of this is fewer modes competing against a common laser gain. The mode competition is thus reduced from the partial reflections as the odd modes are supported only by the unmodified cavity compared to the even modes which become preferred in the modified cavity. This enhances the risk for multi-mode lasing by lowering the number of competing modes.

Similar to inter-cavity elements, if the active gain media does not fill the entire cavity’s length, its location and length inside the cavity can also lower the mode competition. For instance, if the gain medium of length $L_g$ is sufficiently small ($L_g << L$), placing the gain medium near the centre will lower the mode competition as the even modes experience a node here making their amplification difficult [46]. Placing the gain medium near a laser end-mirror however avoids this as there is little difference between the node spacing between pairs of modes as shown in figure 3.5 a).

In general, we thus find that the length $L_g$ of the gain medium, its position and other optical elements inside the cavity can lower the competition of modes. The laser geometry can reduce the amplitudes of certain modes making them less competitive against other modes. We can understand this by realising the geometry of the laser changes the boundary conditions of certain modes thus generating a preference for certain modes over others.

Likewise, we can look at figure 3.5 (b) and infer that the location of the gain medium along with its distribution will affect different modes differently. Essentially, geometric based mode-competition considers the spatial distribution of the gain medium and the resulting effect on different modes.
3.1. PRINCIPLE OF OPERATION

Optical losses

Suppose a diode laser of cavity length $L$ has output facet field-amplitude reflectivities $r_1$ and $r_2$. Per [51] the mirror loss $\alpha_m$ is then given by

$$\alpha_m = \frac{1}{2L} \ln \left( \frac{1}{r_1 r_2} \right)$$  \hspace{1cm} (3.4)

Typical power reflectivity values for semiconductor diode lasers are $R_i \lesssim 0.5$ with $i \in \{1, 2\}$ i.e. on the order of tens of percent [46, 49].

The threshold lasing condition can be recast in terms of the cavity’s confinement factor $\Gamma$. This factor defines the fraction of the squared electric field that is confined to the active gain region of the laser. For a typical diode laser $\Gamma \sim 0.2 - 0.5$ indicating up to 50% of the optical field operates outside the gain medium [49, 52]. Using this confinement factor we find the threshold gain $g_{th}$ to satisfy

$$\Gamma g_{th} = \alpha_a + \alpha_m$$  \hspace{1cm} (3.5)

where $\alpha_a$ of typical value 45 $cm^{-1}$ represents the absorption coefficient of the gain medium. Given that equation 3.4 has a wavelength dependence (through $r_1$ and $r_2$) with a trough, we find the lasing threshold is lowered for modes with lower mirror (cavity-ends) losses. This results in those modes near the trough becoming more competitive thus dominant.

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5 Not to be confused by the power reflectivities $R_i = |r_i|^2$, $i \in \{1, 2\}$

6 The mirror loss $\alpha_m$ is defined for a full round-trip inside the cavity of length $L$. This introduces spatial dimensions through the $\frac{1}{2L}$ factor, simplifying the appearance of equation 3.5 by both loss terms ($\alpha_a, \alpha_m$) acquiring the same units.

Figure 3.6: A sketch of a typical mirror loss $\alpha_m$ spectra in wavelength $\lambda$ of the lasing mode (depicted as $\circ$). The mode with the lowest loss (coloured in red) has the competitive advantage hence lases.
3.2 Internal Cavity Optics

Depending on the intended application, the double heterostructure of diode lasers have different shapes and dimensions. These include

1. Ridge Waveguide Laser (RWL)
   - Uses a ridge-shaped structure ontop of a substrate and is also called a ‘Fabry-Perot Laser’.
   - Good for lower power laser applications needing a good beam quality. Whilst available in single-mode, also available in multi-mode operation.
   - We use this laser for our mode-locked laser experiment.

2. Broad Area Laser (BAL)
   - Similar to a RWL but with a wider active region ridge.
   - Good for high power laser applications, but provides lower laser beam quality than RWL.

3. Tapered Amplifier (TA)  
   - The active region has a tapered shape for amplification purposes.
   - We use TA in our experiments to obtain higher optical powers.

4. Distributed Bragg Reflector laser (DBR)
   - Contains a bragg grating region (and sometimes a phase control region) for optical feedback. Very good for single-mode lasing.
   - Our cooling and repump laser beams when preparing and reading-out the state of our cold atoms use a DFB yet is mislabelled by the manufacturer as DBR.

5. Distributed Feed-Back laser (DFB)
   - Unlike the DBR, the bragg grating region spans the cavity’s full length for optical feedback, also very good for single-mode lasing.
   - Used as our interrogation laser beams when applying interferometric sequences onto our atoms.
3.2. INTERNAL CAVITY OPTICS

The RWL can be considered the modern base laser, with the other 4 categories simply having a modification from the RWL structure. In our atomic physics experiments we make use of the DFB diode laser for atom-laser interactions. We thus include a description of the dispersive element used by the DFB. For completeness we provide a quick review on the DBR as they are strongly related.

So far, we have only discussed so called 'Edge-emmitting lasers'. There is another class of diode lasers called 'Surface-emitting lasers' which can be further classed in two types: VCSEL and VECSEL\(^7\). These lasers emits light perpendicular to the wafer’s plane unlike Edge-emmiting lasers who emit light parallel. The VCSEL uses doped bragg reflector layers sandwiching a gain medium consisting of quantum wells \([44]\). The VECSEL very similar uses an external pump laser directly shining on the quantum wells at the surface of the laser, having an external output coupler to form a cavity \([45]\).

Surface-edge emitting diode laser provides several advantages including a smaller cavity, better beam profile\(^8\) that does not need beamshaping correction and a lower lasing-gain threshold. Whilst VCSELs are frequently used in sensing, medical and printing equipment, they are not readily available at our frequency specifications. Likewise, the VECSELs are relatively new hence have not penetrated the market during the time of purchase \([44, 45]\). They therefore were not used in our experiments.

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\(^7\)Standing for 'Vertical Cavity Surface-Emitting Laser' and 'Vertical External Cavity Surface-Emitting Laser' respectively.

\(^8\)With this term we refer to the beam profile spatial distribution, level of astigmatism, proportion of higher order transverse electromagnetic (TEM) modes etc.
The Distributed Bragg Reflector Laser

The DBR makes use of an corrugated waveguide layer in its structure. This is an optical dispersive element which allows for controlled tunability and aids single-mode lasing. The corrugation in the waveguide typically spans a section of the laser called the 'Bragg wavelength control region'. The active layer spans the remaining part of the laser called the 'active region'. Sometimes an additional part called the 'phase control region' is used which assists laser tunability with a different response. The different configurations and structures are show in figure 3.7.

![Figure 3.7: The different configurations of DBR diode lasers. (a) dual control & (b) only Bragg wavelength control. If the Bragg control wavelength electrical terminal is not used in (a), we get phase control on its own. The optical axis $z$ is shown also with the example of light travelling a distance $y$ from the origin 0 to the point $z'$.](image)

Depending on the presence or usage of a layer, the DBR comes in different operating modes:

1. Bragg wavelength control
2. Phase control
3. Dual control

It are the boundary conditions between the different regions of the lasers that determine at which mode lasing takes place, the different configurations mainly changing the tuning response of the laser as discussed in 3.4.3.
The corrugated waveguide layer in a DBR forms a ‘Bragg reflector’, a frequency selective optical system consisting of cyclic high and low refractive index. The field-amplitude reflectivity $r$ of a Bragg reflector is given as

$$
r = -i\kappa \frac{\sinh (\gamma L)}{\cosh (\gamma L) + (\alpha_i + i\Delta\beta) \sinh (\gamma L)}, \quad \gamma^2 = \kappa^2 + (\alpha_i + i\Delta\beta)^2 \quad (3.6)$$

with $\kappa$ as corrugation coupling coefficient, $\alpha_i$ as optical loss and $L$ the length of the laser region [49]. $\Delta\beta$ is defined as

$$\Delta\beta = \frac{2\pi}{\lambda} n_B - \frac{\pi}{\Lambda}$$

where $\Lambda$ is the corrugation period, $n_B$ the equivalent refractive index of the Bragg reflector and $\lambda$ the optical wavelength for propagation constant $\beta$. Note that the different regions have their own propagation constant:

$$\beta_{a,p} = \frac{2\pi}{\lambda} n_{a,p}, \quad \begin{cases} a \text{- active region} \\
p \text{- phase control region} \end{cases}$$

Hence $n_{a,p}$ is the refractive index of the $a,p$ region of the laser with corresponding propagation constant $\beta_{a,p}$. Likewise, we can index the length of the active and phase control regions respectively as $L_a$ and $L_p$.

Depending on these constants and the propagation length of the optical field, as illustrated in 3.7 the accumulated phase $\phi_1$ it accrues at point $z'$ during a length of travel $y$ along the optical axis $z$ is given by

$$\phi_1(z', y) = \int_{z'-y}^{z'} \beta \, dz$$

Hence in general at the Phase—Bragg wavelength control interface by $\phi_1 := \phi_1(L_a + L_p, L_a + L_p)$ give as

$$\phi_1 = \beta_a L_a + \beta_p L_p$$

The Bragg reflector induces a phase change $\phi_2$ which we can infer from $r$ as

$$r = |r| e^{i\phi_2}$$

As we know the phase-shift per laser region, we can use a Dirichlet boundary condition at the Phase—Bragg wavelength control interface to describe the lasing modes. Lasing modes are then found [51] to satisfy the following phase-matching condition

$$\phi_1 = \phi_2 + 2q\pi, \quad q \in \mathbb{Z} \quad (3.7)$$

As such, a corrugated waveguide layer imposes the phase-matching condition in equation 3.7.
The Distributed Feedback Laser

The Distributed FeedBack laser (DFB) differs from the DBR mainly in length of the corrugated waveguide layer. For a DFB namely the grating stretches along the entire length of the laser cavity \((L \sim 100-1000 \mu m)\) making the active and Bragg wavelength control regions the same. Sometimes the DFB also has an additional phase tuning region similar to the DBR.

![Figure 3.8: An example of an fabricated DFB diode laser in [53]. (a) vertical cross section along the optical axis (b) perpendicular to the lasing direction & (c) bottom view of the laser. Miller indices \([hkl]\) used to show the crystal orientation. Note that \([0\bar{1}\bar{1}]\) is the optical axis.

So, the active (pump) region in a DBR is separate from the phase or bragg wavelength control (passive) regions whilst a DFB only might have a phase control passive region. A detailed structure diagram for the pump region can be found in figure 3.8.

Whilst they vary from manufacturer, estimates of typical thicknesses of layers can be found in [54]. The layer’s are typically \(\sim 10 - 1000 \, nm\) in cross-section depending on the layer’s application and the laser’s specific structure.
3.2. INTERNAL CAVITY OPTICS

Notice how in figure 3.8 the laser is cleaved at an angle to reduce optical feedback. These angled facets induce the longitudinal confinement, with the barrier layers\footnote{The n & p layers adjacent to the active layer of typical thickness \( \sim 100 \, \text{nm} \).} and buried confining the charge carriers transversely. The buried layer also help ensuring lasing is encouraged in the centre active layer and not the sides.

In the DFB the corrugated waveguide layer acquires a simpler mathematical form like

\[ n_B(z) = n_o + n_1 \sin \left( \frac{2q\pi z}{\Lambda} + \varphi \right), \quad q \in \mathbb{Z} \]

where \( \varphi \) determines starting point at the facets of the laser and \( n_o \) and \( n_1 \) represent the undoped and maxim doped refractive indices of the waveguide layer respectively. Note that various manufacturers use different waveforms to engineer their grating like triangle waveforms instead of a sinusoidal one.

Whilst the phase-matching condition of the DFB is a more complex version on the discussion of the DBR, they can be understood from to arise from the same Bragg condition \cite{40} given by

\[ \Lambda = \frac{\lambda}{2n_o} \]

I.e. the pitch of the corrugation (\( \Lambda \)) must fit the pitch of the optical field (\( \frac{\lambda}{2} \)) in the layer (\( \frac{1}{n_o} \)). Engineering the grating pitch \( \Lambda \) and utilising the current dependence on \( n_0 \) allows for designing the tuning characteristics of the DFB.
CHAPTER 3. THE DIODE LASER

3.3 External Cavity Optics

The laser housing typically hosts additional components that lie outside the laser’s cavity. These external elements lie between the output facet of the laser cavity and the optical exit terminal of the laser housing. These include

1. Cylindrical Lens
   - To counter the effect of diffraction and help create circular beam from an otherwise typically elliptical beam. As the active layer typically has a rectangular shape, diffraction induces astigmatism from the different beam divergences $\theta_{\parallel,\perp} = \frac{2\lambda}{\pi d_{\parallel,\perp}}$ where $d_{\parallel,\perp}$ is the parallel/perpendicular dimension of the active layer.
   - Given $d_{\parallel,\perp} \sim 1 \mu m$, a typical beam divergence has $\theta_{\parallel,\perp} \sim 0.5^\circ$.

2. Faraday Optical Isolator
   - To prevent optical feedback from reflections further in the optical system.
   - Not included for applications involving optical feedback like laser injection locking.

We have now described the contents inside a typical $\sim 1"$ diode laser housing along with their purpose and principle of operation. What remains is a brief comment on the ECDL, a popular laser configuration used in this work and still popular in modern atomic physics experiments.

3.3.1 The External Cavity Diode Laser

Some lasers have external optics outside their housing that play functional roles. One type such laser is called the External Cavity Diode Laser (ECDL) which makes use of an external cavity for fine and coarse wavelength tuning.

To reduce mode-competition from intra-cavity modes\(^{10}\), the output facet of the laser is typically coated with an anti-reflection coating (AR). The external cavity is formed by placing an output coupler in-front of the AR coated facet for optical feedback. Together the internal cavity (facet $\leftrightarrow$ AR-facet) and external (AR-facet $\leftrightarrow$ coupler) form a composite cavity.

\(^{10}\)The modes generated by the laser cavity.
The advantage of implementing the composite cavity is that with a good AR coating, it primarily controls the laser’s properties such as linewidth and longitudinal/transverse mode spectrum. The external cavity can thus be used to tune a regular RWL and improve its properties.

For tunability, a reflective diffraction grating acts as the output coupler. The grating imposes a wavelength dependence on the directionality of the laser beam. Hence the optical feedback of a diffraction order becomes wavelength dependent through geometry.

There are two typical configurations of the ECDL: the Littrow and Littman-Metcalf configuration. Their typical layout is sketched in figure 3.9. Their main difference is the angle of the grating (Littrow $\sim 30^\circ$ vs Littman-Mercal $\sim 80^\circ$) and the use of a mirror for double pass of the first order in the Litman-Metcalf configuration.

![Figure 3.9: (a) An tunable ECDL in the (a) Littrow (b) Littman-Metcalf configuration. The inset show the diffraction order indices. The light from the laser first reaches the reflective diffraction grating before (a) reflecting off a mirror (b) undergoing a double pass at the -1 order. The diffraction order convention we use is the conventional sign convention as defined in [55].](image)

Higher diffraction orders have a lower diffraction efficiency and a double grating pass cascades optical losses. Hence the Littman-Metcalf configuration with the feedback mirror at the first (typically -1) diffraction order experiences higher optical losses. The Littrow configuration (also typically using -1 diffraction order) is therefore preferred for higher power demands [56, 57].

Given optical feedback is essential to the ECDL, there are no dispersive elements typically inside the laser cavity. To reduce mode competition for single-mode lasing, sometimes an etalon is inserted between the grating and the AR coated facet. The etalon (slab of fused silica) acts as an optical filter whose internal longitudinal modes can be shifted by tilting the slab.
3.4 Diode Laser Tuning

3.4.1 Frequency-shifts from Thermal Drifts

Temperature drifts can affect the frequency of a laser as the optical path of the light inside the cavity is altered. In general, this follows from either a variation of the cavity length $L$ or the refractive index $n$ inside the cavity. A change in optical path affects the bandgap $E_g$ of the semiconductor hence the position of the peak of the gain medium as per equation 3.2. Given a DFB laser has its active media fill the entire length of the cavity, a temperature fluctuation $\Delta T$ gives rise to the following frequency shift $\Delta \omega$ around $\omega$

$$\frac{\Delta \omega}{\omega} = -\frac{\Delta L}{L} - \frac{\Delta n}{n}, \quad \frac{\Delta L}{L} = \alpha \Delta T$$

where $\alpha$ is the thermal expansion coefficient of the active medium [48]. In the case of GaAs the thermal expansion coefficient takes the value$^{12}$ of $5.7 \cdot 10^{-6} / K$ [32] (or $6.03 \cdot 10^{-6} / K$ in [35]) at $300 \, K$. Likewise, the thermal refractive index gradient $\frac{dn}{dT}$ takes the value of $15 \cdot 10^{-5} / K$ at a wavelength of $870 \, nm$ [35] (and $14.7 \cdot 10^{-5} / K$ at $10 \, \mu m$ in [32]).

The thermal frequency shift would at sufficiently high (optical-path) strains of the cavity also result in a mode-hop to the upper adjacent mode as the spectra of the cavity shifts up under a thermal increase. This can be seen from the cavity longitudinal mode spacing (FSR - free spectral range):

$$\Delta \omega_{FSR} = \frac{\pi c}{nL}$$

Under thermal heating the cavity’s optical path $nL$ increases inducing a positive shift of the supported longitudinal cavity modes. We thus find that at small thermal drifts the gain medium peak and longitudinal mode spectra shifts causing a small monotonic shift in frequency. But at larger thermal shifts we end up encountering a mode-hop as the mode competition would prefer the adjacent mode $\omega_{p+1}$. As such, proper thermal stabilisation is crucial to a frequency sensitive measurements over long periods of time.

Note that this tuning is linear as $\frac{dL}{dT}$ and $\frac{dn}{dT}$ are constants at fixed wavelengths. Hence linearly tuning the temperature of the laser results in linear shifts in the wavelength up to a mode-hop. Further tuning repeats this haviour at lower temperatures.

$^{11}$A temperature change in bandgap energy $\frac{dE_g}{dT}$ at $300 \, K$ for GaAs is $-0.45 \, meV / K$ [35].

$^{12}$Defined as strain per unit temperature change.
3.4. DIODE LASER TUNING

3.4.2 Current-induced Frequency-shifts

Tuning the laser’s injection/driver current provokes thermal variations in the laser resulting in laser frequency-shifts described in 3.4.1. Ignoring these, we find that current variations result in two similar types of laser frequency shifts. The first type of shift is in the peak of the gain medium. This happens because the laser’s current $I$ affects the refractive index $n$ of the laser hence the optical path $i.e. \frac{dn}{dI} \neq 0$ [58]. From the energy-momentum dispersion relation we can infer that the bandgap is altered by this hence the laser gain as per equation 3.2.

At larger current variations we can encounter the second type of frequency shifts: mode-hopping. This follows from the change in optical path as seen in 3.4.1. We thus see that the current-induced frequency shifts have a common mechanism to thermal-induced frequency-shifts. It is therefore generally expected for the spectral response to be similar $i.e.$ bar the sign of curves and possible scale difference in tuning, the overall shape of the shifts (linear, quadratic) should be common.

3.4.3 Current Tuning with Phase Matching

In the case of the DBR/DFB laser, tuning is controlled by adding a corrugated waveguide layer. Since injecting current changes the refractive index, the phase-matching condition is altered. The injection of current into the different regions give different tuning responses hence great control over the lasing mode. A complete review regarding the dynamics of tuning a DBR or DFB can be found in [40, 51]. The DFB case requires numerical solving hence we only highlight the main results here.

A current controller injects current in the different terminals of the DBR. For 

Bragg wavelength control, apart from a constant current in the active region for lasing, a current is injected into the bragg grating region inducing a shift in its phase $\phi_2$. Assuming no current is injected in the phase control region, the phase $\phi_1$ across the active and phase control regions remain constant. The mirror loss $\alpha_m$ however also shifts in frequency as the Bragg reflectivity is altered. The net result is a frequency shift in both phase-matching conditions and mirror loss inducing long-range tunability with mode-hopping.

---

13The momenta $\hbar k$ has dependence on refractive index $n$.

14Essentially, the optical path is altered causing the laser gain peak and the supported longitudinal modes to shift inducing gradual frequency shifts and mode-hopping under small and large current/thermal changes respectively.
If alternatively current is injected into the phase control region for tuning, the bragg phase and mirror loss curve stays fixed\textsuperscript{15} with the phase $\phi_1$ curves shifting in frequency. With this tuning called Phase control, only the phase-matching conditions change resulting in cyclic tuning over a single mode-hop. Both tuning methods are sketched in figure 3.10.

In general DBR operation, all laser regions are used providing a more complex change in phase-matching (modes) tuning. This is especially true for a DFB where the bragg wavelength and active regions are not separated. Given the great variety of laser structures and parameters, this is taken to be beyond the scope of this thesis.

\textsuperscript{15}The reflectivity of the waveguide layer is affected less by a current increase in the active region as the charge carriers recombine in the active layer.
3.4. DIODE LASER TUNING

3.4.4 Tunable ECDL

In the Littrow configuration, the laser beam undergoes a translational shift parallel to the optical bench during tuning\(^{16}\). It uses an optical feedback that utilises the -1 diffraction order of the grating, set at the Littrow-angle such that the grating condition\(^{17}\) for mode \(m\) and diffraction order \(q\) becomes

\[ q\lambda = 2\Lambda \sin (\theta_{m,q}) , q \in \mathbb{Z}, m \in \mathbb{N} \] (3.9)

Figure 3.11 (a) shows step-like mode-hopping upon grating rotations. The cavity length remains fixed during rotations so that the only wavelength changes are mode-hops (dashed lines) when the adjacent diffraction order goes into the intra-cavity.

Figure 3.11: The effect of tuning the current of an ECDL in the Littrow configuration by (a) rotating the grating along the optical axis (b) translationally displacing the grating along the optical axis \(z\). The respective configurations of (a) and (b) are sketched above appropriate tuning method graph.

In the (b) case, the translational grating displacement along the optical axis induces a ‘stretch’ of the longitudinal modes. According to equation 3.9 this alters the diffraction angle \(\theta_{m,q}\). Even at a common diffraction order \(q\), different longitudinal modes \(m\) have different diffraction angles \(\theta_{m,q}\). Hence at sufficiently high translational displacements, \(\theta_{m,-1} = \theta_{m+1,-1}\). The optical feedback receives competition from the \(m+1\) mode which now has the same optical loss \(\alpha_{m+1} = \alpha_{m}\) resulting in a mode-hop. Further translational displacement gives cyclical repetition of over a fixed wavelength range corresponding to a single mode-hop analogous to the phase control current tuning in a DBR as per figure 3.10.

\(^{16}\)The Littman-Metcalf configuration does not suffer from this problem as the grating is fixed during tuning. The feedback mirror is tilted to change the wavelength.

\(^{17}\)Otherwise, the grating condition becomes \(\lambda = \Lambda(\sin(\theta_{\text{incident}}) + \sin(\theta_{m,q}))\). The Littrow-angles is met when \(\theta_{\text{incident}} = \theta_{m,q}\) for some \(m\) and \(q\).
Like the DBR/DFB dual current tuning, we can combine the translational displacement with rotation of the grating\footnote{When using an etalon, it is important that it rotates appropriately with the grating/feedback-mirror to avoid mode-hopping from varying cavity modes.}. A large mode-hop free single-mode tuning can be achieved with a linear tuning response. In the case of the Littrow ECDL, this can be achieved by pivoting the grating around the intersection point of the plane normal to the optical axis (set at the AR-coated intra-cavity facet) and the plane of the diffraction grating\cite{59}.
Chapter 4

CW Characterisations of Diode Lasers

Since atom interferometers involve frequency-sensitive and power sensitive atom-laser interactions, the diode laser used heavily throughout our interferometer(s) require characterisation. This way, we can identify at what experimental settings the laser becomes unstable and what optimal parameters the associated laser operates. When identifying problems later in our experiments, we can thus infer from these measurements how the performance of the laser might have impacted our measurements.
4.1 Eagleyard DFB

The atomic experiments use two laser systems: the laser used for coherent manipulation of the $^{85}\text{Rb}$ atoms (‘Raman’ laser) along with the lasers for laser cooling a cloud of atoms in a Magneto-Optical Trap (MOT) configuration. For the composite pulse experiments an Eagleyard EYP-DFB-0780-00080-1500-BFW01-0005 semiconductor diode laser was used. As explained in further detail in section 16.2, the Raman laser was left free-running giving rise to the interest in its characterisation\(^1\).

Due to the strict requirements on frequency (MOT) and power (Raman) stability in our experiments, one of the three purchased lasers was thus characterised in continuous wave (CW) mode operation. This laser had the serial number: GF–00121, and is discussed in this thesis\(^2\). A transfer to another project and time constrains prevented the other laser’s from being characterised or further characterisation of GF–00121.

4.1.1 Tools of Measurements

Optical power measurements we made using an Thorlabs PM100D power meter with an Thorlabs S120C reflective coated power sensor. The sensor was placed after at least one Faraday Optical Isolator (FOI) unless stated otherwise to avoid optical feedback.

The wavelength was measured by coupling the light into an Advantest TQ8325 wavemeter (1 pm resolution). Frequency measurements were made using an CVI Spectral Products 150 finesse, 2 GHz SAC-2 scanning Fabry-Perot optical spectrum analyser (OSA). The spectrum was viewed on a Tektronix MDO-4104-6 Mixed Domain Oscilloscope.

\(^1\)To the author’s understanding, this laser’s characterisation was not as extensive as the characterisation presented here. Thus the interest was to characterise a laser of the same model without disrupting the ongoing atomic physics experiments involving the laser.

\(^2\)The ‘Raman laser’ used for atomic physics experiment was of different serial number. Its characterisation was not done by the author nor included in this thesis. The third purchased DFB had not yet been characterised at all.
4.1. Driving the Eagleyard laser

The *Eagleyard* diode laser resides in a *Thorlabs* LM14S2 butterfly mount and is held in place by a zero force insertion lock. The laser\(^3\) is driven using an *Thorlabs* ITC502 Combi controllers to control both the injection current and the temperature of the diode laser. In our case, the temperature control measures the resistance across the negative thermal coefficient thermistor (NTC) built in to the laser housing. Feedback is provided by an PID\(^4\) circuit that uses the built-in Thermo-Electric Cooler (TEC) inside the butterfly mount.

Following the datasheet of the laser, the ITC502 controller [60] and engineering notes from *Spectrum Sensors & Controls* [61], the relationship between temperature \(T\) and resistance \(R\) of the thermistor is given by

\[
T(R) = \frac{BT_0}{T_0 \ln R/R_0 + B}
\]  \hspace{1cm} (4.1)

where \(B\) is a constant and \(R_0\) is the thermistor nominal resistance at nominal (laboratory) temperature \(T_0\) of \(\sim 25.0(1) ^\circ C = 298.2(1) K\). This equation was used to convert between the measured thermistor resistance and temperature values. \(R_0\) and \(B\) are both specific to the thermistor hence the laser. The datasheet of the diode laser gave \(R_0 = 10 k\Omega\) and \(B = 3976 K\).

The ITC502 can operate lasers in two modes: *constant current* or *constant power*. The latter requires the laser to host a built-in photodiode that monitors the power to appropriately alter the injection current. For a more stable frequency however we instead maintain the set injection current.

A setting resolution of \(1 mK\) is specified when using a \(10 k\Omega\) thermistor in PID feedback [60]. Apart from a non-optimised PID loop and aged electronics, we thus anticipate temperature stability of the thermistor’s resistance (temperature) up to a few \(mK\). Due to the conversion by equation 4.1 and the uncertainty in the variables used, the actual quoted errors become larger\(^5\).

To reduce optical feedback, two Faraday Optical Isolators (FOI) (*Leyson* 5/57 followed by an *Caldwell* IO-5-780-HP) were placed in front of the diode laser. The second follows as the laser housing didn’t include an internal FOI.

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\(^3\)The actual diode laser is housed inside a package (‘housing’) which is mounted onto the LM14S2.

\(^4\)A Proportional Integral Derivative Servo.

\(^5\)The temperature errors include the systematic error from converting to temperature from \(R, R_0\) and \(B\). The temperature’s stability is actually more like \(\sim mK\).
4.1.3 P-I Relation

Since atom interferometers make use of Rabi-flopping whose frequency is dependent on the power of the laser beam, we set out to measure the power $P$ of the laser as a function of injection current $I$ and compare those with the lasers specifications (taken at 38.0 °C). The measurements were repeated at three temperatures and are shown in figure 4.1 (a), (b) and (c).

![Graphs showing power vs. injection current for three different temperatures: 35.0(1) °C, 38.0(1) °C, and 41.4(1) °C.

Figure 4.1: The 'brutto' optical output power $P$ against injection current $I$ \{(0.01 mA error\) of the Eagleyard laser at (a) 35.0(1) °C, (b) 38.0(1) °C and (c) 41.4(1) °C. The optical power is compared to the laser datasheet \{38.0(1) °C\} which was retrieved using the Webplotdigitizer software. (d) 'Netto' optical power output by measuring post the two FOI at 38.0(1) °C.

From figure 4.1 (a), (b) and (c) we estimate the current threshold $I_{th}$ to \{36.0(5), 37.0(5), 39.0(5)\} mA at \{35.0(1), 38.0(1) and 41.4(1)\} °C respectively. This suggests a shift in $I_{th}$ of about $\sim (0.3 + 0.6)/2 = 0.45$ mA/K.
4.1. EAGLEYARD DFB

The differential quantum efficiency $\eta$ describes the level of conversion of electrical (input) pump to optical power (output) energy. The external efficiency $\eta_{\text{ext.}}$ compares the input pump to the output optical power outside the laser cavity. If $e$ is the elementary charge of an electron and $\lambda$ the laser’s wavelength, then according to [62] $\eta_{\text{ext.}}$ is given as

$$\eta_{\text{ext.}} = \frac{e\lambda}{\pi\hbar c} \frac{dP}{dI}$$

Linear regression of the datasheet measurements returns a $\chi^2 > 99\%$ and estimates $\eta_{\text{ext.}} = 0.576 \pm 0.004$. Performing a linear regression over our own measurements proved difficult since we observe non-linearities in the $P-I$ data near mode-hop regions as measured in figure 4.3. This is particularly visible in figure 4.1 (b) in which our measurements yield slightly higher powers at low currents to the datasheet specifications, and lower powers at currents post the mode-hop region (around $\sim 60 \text{ mA}$).

Comparing figures 4.1 (a) & (c) with figure 4.3 we can infer the mode-hop regions do not overlap with all non-linear regions suggesting mode-hopping is not responsible: For the 41.4(1) °C data the power curves over 10 mA too early whilst for the 35.0(1) °C case a similar shift to lower currents would place it below the current lasing threshold. The non-linearity might however be caused by a systematic error stemming from the thermal drifts seen in subsection 4.1.6. By measurements having varied duration, or by taking the power measurement post a different active running time of the laser, the power might have shifted non-uniformly for different datapoints.

The measurements in figure 4.1 (a), (b) and (c) measure the total output power of the laser by placing the power sensor directly in-front of the laser. However, the laser’s specifications suggest the use of two FOI’s for stable single-mode lasing and tunability. In figure 4.1 (d) we therefore measured the power post two isolators of respective\(^6\) transmission $\sim 74$ and 95% to estimate the practical/netto output power.

The effect of the isolators are to drop the $P-I$ slope to $\sim 0.43(2) \text{ W/A}$ from 0.916(7) W/A to give a gross efficiency $\eta_{\text{ext.}}^{\text{gross}} \sim 0.27(1)$. This is just short of half the ‘brutto’ quantum efficiency $\eta_{\text{ext.}}$ that can be used for our experiments.

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\(^6\)The second isolator has a higher transmission as the polarisation has been filtered by the first isolator. Also, lower transmission is found in the first isolator due to its optical alignment as discussed in subsection 4.1.5.
4.1.4 Shifts (Hops) of the Mode-hop Region

We wish to determine the parameter values at which the laser hops between two modes so we can smoothly tune the laser and lock its frequency. The experimental control variables are temperature and current, both which can induce mode-hopping. So to avoid mode-hopping, we measure the wavelength response at different control variable settings.

Temperature-induced Hopping

As discussed in 3.4.1, varying the temperature of a laser results in a shift in output wavelength $\lambda_q$. At appropriate tuning lengths, the laser is forced to hop to an adjacent cavity mode $\lambda_r$ or $\lambda_p$ depending on the tuning direction of the variable (temperature). In figure 4.2 (a) we plot the measured wavelength-temperature response (at two injection currents) revealing mode-hopping induced by temperature variations. The measurement is repeated for two injection currents, demonstrates the shift in mode-hopping parameters (injection current) as depicted in 4.2 (b).

![Figure 4.2: (a) The measured mode-hopping of the Eagleyard diode laser wavelength $\lambda$ upon variation in temperature $T$ (dual axis with thermistor resistance $R$ of error 0.001 k$\Omega$). Data was collected for two currents 80.00(1) and 45.02(1) mA. (b) Schematic representation of the mode-hopping between modes $\lambda_p$, $\lambda_q$ & $\lambda_r$ for different injection currents. We indicate mode-hops arrows in both measurement (a) and schematic (b).](image-url)
By converting the x-axis error bars in figure 4.2 (a) to temperature, we see that our thermal feedback system (ITC502) struggles to lock the set temperature of the laser when the laser is driven closer the nominal temperature of the laboratory. Intuitively this is expected as the magnitude level of the error signal of the PID feedback scales with the difference between the nominal and thermistor temperature. Operating the laser closer to nominal temperature thus gives a smaller error signal making the temperature lock less stable.

Figure 4.2 (a) gives the perception that the tuning rate is non-linear. However note that we plot the wavelength versus a linear thermistor resistance $R$ scale\(^7\). As per equation 4.1 this implies a non-linear temperature scale. We find by appropriate linear regression outside the mode-hop region that the data provides a linear tuning rate (gradient) with a good fit of $\chi^2 > 99\%$ in both datasets. Hence our datasets actually resemble figure 4.2 (b) when plotting using a linear $T$ scale.

We fit a line to the higher temperature datapoints (which has the smallest control variable error) upto the mode-hop region for the 45.02(1) mA dataset dataset. For the 80.00(1) mA dataset dataset the limited number of datapoints means we needed to use the lower temperature datapoints.

Linear regression gave a tuning rate (gradient) of\(^8\) 0.06(1) nm/K for both 40.02(1) mA and 80.00(1) mA datasets. This is in agreement with the supplementary datasheet of the laser which quotes the tuning rate as:

$$\frac{d\lambda}{dT} \sim 0.06 \text{ nm/K}$$

Finally, the shift of the mode-hop region when comparing the datasets we can estimate to be $\sim 0.15$ K/mA. Note however that this estimate is calculated from the difference between two available datasets hence is not reliable.

\(^7\)We do this as $R$ is our laboratory control variable measured and set by the ITC502.

\(^8\)In frequency this respectively corresponds to 29.6 ± 0.5 GHz/K using equation 4.2.
Current-induced Hopping

Mode-hopping can also be found at appropriate injection currents of the laser at constant temperatures. Our measurements in figure 4.3 (a) of the wavelength demonstrate the existence of either bistable or bimodal modes of lasing (c.f. appendix B). As depicted in figure 4.3 (b), the laser when tuned sufficiently (in current) hops to the adjacent mode. From figure 4.3 (a) we infer the wavelengths of our datasets are separated by $\sim 0.1$ nm at common laser injection current $I$. This is by temperature of the laser whose effect is to shift the current at which the bistable/modal region is located.

![Figure 4.3](image_url)

**Figure 4.3:** (a) The measured mode-hopping of the *Eagleyard* diode laser wavelength $\lambda$ upon variation in injection current $I$ (0.01 mA error). Data was collected for three temperatures 35.0(1), 38.0(1) and 41.4(1) $^\circ$C. (b) Schematic representation of the mode-hopping between modes $\lambda_p \leftrightarrow \lambda_{p+1}$, $\lambda_q \leftrightarrow \lambda_{q-1}$, $\lambda_r \leftrightarrow \lambda_{r-1}$. The shaded regions depict bimodal or bistable modes of lasing.

Using linear regression to estimate the gradient, we estimate the tuning parameter to be $0.0030(4)$ $nm/\text{mA}$ ($1.5 \pm 0.2 \text{ GHz/mA}$) with a $\chi^2 = 91\%$. This estimate was taken from the 41.4(1)$^\circ$C dataset and found to be consistent with the other datasets. Our estimate is also in good agreement with the laser’s datasheet which specifies a typical tuning parameter $\frac{d\lambda}{dI} \sim 0.003 \text{ nm/mA}$.

We observe that compared to figure 4.2 (a), the mode-hopping regions occupy a larger control variable range. The mode-hop region also hops by $\sim 6 \text{mA/K}$. This agrees with our previous estimate of $0.15 \text{ K/} \text{mA} \sim (6 \text{ mA/K})^{-1}$ for the hopping of the mode-hop region.
4.1. EAGLEYARD DFB

By comparing the y-intercept of the fitted line before and after the mode-hop of the 38.0(1) $^0C$ dataset, we estimate the mode-hop to cause a wavelength jump of $\sim 0.15$ nm at $\lambda \sim 780$ nm. This wavelength modespacing $\Delta \lambda_{\text{FSR}}$ is related to the physical length of the laser’s cavity which we can estimate by taking the derivative of the wavelength–frequency relation

$$\nu = \frac{c}{\lambda} \to |\Delta \nu| = \left| \frac{c}{\lambda^2} \Delta \lambda \right| \quad (4.2)$$

Given the longitudinal free spectral range $\Delta \nu_{\text{FSR}}$ of the cavity is given by

$$\Delta \nu_{\text{FSR}} = \frac{c}{2nL}$$

we can infer the following relation

$$\Delta \lambda_{\text{FSR}} = \frac{\lambda^2}{2nL}$$

Hence we infer the laser to have a length $L = 0.55(5)$ mm (approximating $n = 3.6$ at 780 nm (valid for 870 nm [35]) and a Free Spectral Range (FSR)\(^9\) of 75(5) GHz. This is consistent with the typical cavity length of DFB lasers.

To get better estimates we viewed the mode-hops on the optical spectrum analyser (OSA). It became apparent however that outside the mode-hop regions, whilst the shot-to-shot Full Width Half Maximum (FWHM) of modes was around 10 MHz\(^10\), all modes suffered from frequency-jitter\(^11\) with a Half Width Half Maximum (HWHM) of $\sim 13$ MHz. Given the laser’s datasheet specifies a linewidth of $\sim 0.6$ MHz (not measured in this work), and as this jitter was not seen in other laser systems, it suggested the laser was suffering from significant levels of noise giving a time-average linewidth broadening to 26 MHz, above the OSA’s resolution.

During a mode-hop, the OSA showed rapid oscillations between two modes as sketched in figure 4.4 (a). The noise thus affected the mode-hopping and possibly the size of the mode-hopping region we observed in figure 4.3. Our mode-hop regions do seem stretched as our litterature review suggests an ideal mode-hop would take place at a single value of the control variable. Hence we set out to review possibles sources of noise that could be disrupting the laser.

\(^9\)More detail on the FSR of a cavity can be found in subsection 3.4.1.

\(^10\)Consistent with a finesse of $\sim 150$ as per equation 6.2, the estimated finesse according to the OSA’s datasheet. The linewidth of the modes are thus smaller than the $\sim 10$ MHz resolution of the OSA.

\(^11\)With jitter we imply the range of shift in frequency of the lineshape of the modes.
4.1.5 Noise from Grounding Loop

The LM14S2 butterfly mount is accompanied by a 'Type 1' or 'Type 2' configuration card that specifies how the electrical connections from the diode laser are connected to the pins of the driver input at the back of the butterfly mount. Our *Eagleyard* laser characterisation data used the 'Type 1' card as it is typically recommended for this type of laser.

After analysing our data revealed significant noise problems, we conducted an electronics review that revealed the need for using a custom configuration card. In this subsection we like to comment on this analysis and our findings on improving the laser’s performance. Note however that due to time constrains following a subsequent transfer to another project, limited data is available on the characteristics of the laser using this configuration card.

According to the laser’s and LM14S2 datasheet [63], neither of the standard configuration card provided the proper electrical connections with the laser. Hence for any scientific use of the laser we made used of a custom configuration card. By making use of the pin layout diagram of the diode lasers datasheet, we solder the appropriate connections onto the custom configuration card using single core electrical bridges.

We found that a common grounding problem in driving lasers are so called 'grounding loops', electrical bridges between multiple grounds in an electrical system. This issue most commonly seen when grounding the anode of the laser stems from currents flowing between the signal and earth grounds of the circuit. We thus attempted to isolate the grounds from each other. Using the custom card, the laser’s housing pin was left unconnected so that the housing is no longer connected to the laser diode anode (LDA) pin like with the 'Type 1' card. This limits the grounding loop shown in figure 4.4 (b) through the earth-grounded optical bench by breaking the Signal–LDA–Housing–LM14S2–Optical Bench–Earth electrical bridge.

The OSA confirms that configuring the custom card reduces the frequency-jitter’s HFHM from $\sim 13 \text{ MHz}$ to $\sim 7 \text{ MHz}$. Hence significant jitter remains over twenty-three times larger than the laser’s linewidth ($\sim 0.6 \text{ MHz}$). The time-averaged $14 \text{ MHz}$ linewidth also exceeds the $1 \text{ MHz}$ stability requirement suggesting the laser cannot be used in our atomic experiments.

\footnote{The advantage of correctly earth-grounding the anode include a reduced electrical noise i.e. improved laser performance [64].}

\footnote{Half the maximum frequency shift in peak of lineshape}
Given the LM14S2 can only be set to either ground the laser’s anode or cathode, without access to the mount’s inner circuitry it remains possible for a loaded grounding loop to remain after instalment of the custom configuration card. This is because even if we electrically isolate the LM14S2 mount from the grounded optical bench, a loaded electrical connection might remain through the cables\textsuperscript{14} connecting the mount to the ITC502 driver, which itself is earth grounded. Thus it may be of interest to make use of a different laser mount and driver that does not risk having a grounding loop. Regardless, using the custom configuration card we saw the mode-hop region width half from $\sim 20 \ mA$ (Type 1) in figure 4.3 to about $\sim 10 \ mA$ (custom).

Another hypothesis we considered was optical feedback from the the Leysop isolator closest to the laser. We found that tilting this isolator reduced the mode-hop region width was further from $\sim 10 \ mA$ to $\sim 5 \ mA$ in figure 4.3. This suggests the laser is very sensitive to optical feedback, even from the polarising cube facet of the first isolator.

We support our findings on improvement of the laser’s stability by reviewing the laser on the 2 GHz CVI SAC-2 OSA and comparing its spectrum of the ‘Type 1’ to the custom configuration card. We observed a ‘smoother’ mode-hop using the custom card compared as illustrated in figure 4.4 (a).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4_4.png}
\caption{(a) Sketch of the observed effect of grounding loops when viewing the lasers output using an Optical Spectrum Analyser (OSA) at different control variables ($I$, $T$). We viewed the OSA on an oscilloscope hence voltage $V$ at time $t$. The dashed lines was only seen using the ‘Type 1’ card. (b) circuit diagram of laser diode and current driver demonstrating the two identified grounding loops through the optical bench and laser driver.}
\end{figure}

\textsuperscript{14}I.e. a LDA–Housing–LM14S2–Cables–ITC502–Earth electrical bridge shown in figure 4.4 (b).
4.1.6 Impulse Response

When the laser was turned on, a particular response was observed with regards to its output power. The typical shape of this time-dependent response is shown in figure 4.5 (a) taken at 80.00(1) mA of injection current and 35.0(1) $^0C$. Our analysis of this behaviour divides the response into three time-scales: short ($< \tau_1 = 1 \min$), intermediate ($[\tau_1, \tau_2] = 1-10 \min$) and long ($> \tau_2 = 10 \min$) time-scales.

Figure 4.5: (a) The impulse response of the optical output power of an Eagle-yard diode laser operating when turned on at 80.00(1) mA, set to 35.0(1) $^0C$. (b) The typical PID feedback response from the proportional ($P$), integral ($I$) and derivative ($D$) amplifier-based feedback elements from a step-function impulse [48]. A 0.02 mW background light was observed.

The output power of the laser demonstrates largest fluctuation (error) at shorter time-scales. The largest and fastest drift in power (of smaller fluctuations) is observed at intermediate time-scales. This is followed by smaller and slower drifts (of comparable fluctuations) at long-term timescales.

Given the laser is temperature stabilised using a PID control system, this behaviour may follow from an impulse response when turning the laser driver on. The sudden spike in current gives rise to an impulse in temperature whose stabilisation is coordinated by the PID temperature control circuit (Thorlabs ITC502). The 3 different time-scales thus could indicate the output response of an step-function input when turning the laser on.
The elements of a PID system have a well-understood characteristic response to impulses such as a step-function. From electronic control theory the shapes of the responses can be shown to agree with those shown in figure 4.5 (b). Thus comparing shapes (a) and (b) in figure 4.5 suggests that at the appropriate values of the gain coefficients the time-dependence of the output power might be described well from considering the thermal impulse from turning on the laser.

Let the gain coefficients of the proportional ($P$), integral ($I$) and differential ($D$) amplifier be given as $K_P$, $K_I$ & $K_D$ respectively. The corresponding transfer function $f(s)$ of a PID controller is then given by

$$f(s) = K_P + \frac{K_I}{s} + sK_D, \ s = i\omega$$ \hspace{1cm} (4.3)

We see from equation 4.3 that $I$ responds better to smaller frequencies $\omega$, $D$ to higher frequencies and $P$ in-between. This can be shown to be in agreement with figure 4.5 (b). Within the paradigm of our inference, the intermediate (long) time-scale response thus follows from the dominance of $D$ ($I$ & $P$) for a step-function impulse. The short time-scale we speculate to stem from an impedance mismatch causing reflections of the driver signal. The measured time dependence of the output power thus could stem from the feedback of the PID controller and possible back reflections from an impedance match.

To confirm the impulse response hypothesis, the measurement of figure 4.5 could be repeated at different values of $P$, $I$ and $D$ to confirm dependence on $\tau_2$ and locate the optimum PID feedback. Defining the optimum feedback for when $\tau_2$ is minimised, the study of the impulse response of the laser may help stabilise the lasers power. As for the impedance mismatch hypothesis, measuring the impedance of the laser ports and laser driver could verify the speculation on impedance mismatch. Also, taking more datapoints during this stage ($t \in [0, \tau_I]$) also helps confirm if the fluctuations are caused by an impedance mismatch. Due to time constrains and the transfer to another project, these further analyses of the impulse response were left unfinished.

In summary, we suspect a poorly-tuned PID feedback (ITC502) to cause a slow temperature stabilisation of our laser on the order of hours. This would have affected power measurements as in figure 4.1 taken before we were aware of this problem. Aging of the driver might also be responsible for the enhanced voltage ringing (power variations) when the laser is turned on. Finally, this also explains the high sensitivity of the experiments to changes in the laboratory such as opening doors and walking past the optical bench.


4.2 Diode Laser Power Limit

Characterisation of the mode-locked laser made use of two identical ECDLs; each ECDL was made up of a Thorlabs L785P090 edge-emitting Fabry-Perot diode laser as source in the Littrow configuration with a Thorlabs G13-18V diffraction grating for optical feedback. During operation we used a Thorlabs TED200C temperature controller and a Thorlabs LDC202 current controller.

As much of the characterisation was done by Dr. J. Woods in his thesis (c.f. [27] for ECDL details), and due to time constraints and a transfer to other projects, we focused our characterisation on the power output at higher currents. This is as we were struggling with powers in our experiments and wanted to determine the highest injection current the laser provided.

Figure 4.6: The power-current characteristic curve of one Littrow-ECDL. The power is measured in the same fiber used to measure its wavelength. Due to optical losses the power is thus not absolute, including the y-axis error which is scaled by 100. The diffraction grating orientation was fixed during measurements.

Our measurements are shown in figure 4.6 and suggest thermal roll-over starts around 90 mA of injection current. At these levels of injection current, the active layer has heated typically by $\leq 100$ K from the non-photonic recombination events of charge carriers. The threshold at which thermal roll-over becomes significant is higher in pulsed laser systems [54], we only measured the CW thermal roll-over threshold here.

Thermal roll-over (decreasing quantum efficiency with increasing injection current) is a complicated process caused by more then heating. Other mechanisms include: Auger re-combination events, carrier leakage, carrier delocalization, electron overflow, poor hole injection, etc. [65]. Lateral effects such as current spreading, filamentation and thermal lensing along with spectral effects typically are insignificant in comparison [54]. Many effects listed however degrade the device over time, especially heating. The laser’s operational lifetime is thus reduced by operating the laser at higher operational currents.
4.3 Mode–locked Eagleyard RWL

For mode-locking an Eagleyard EYP-RWE-0790-04000-0750-SOT01-000 semiconductor diode laser is used with its front emission facet Anti-Reflection (AR) coated at $\sim 0.03\%$. The active medium of this laser uses a GaAs active layer and a ridge waveguide structure contained in a Fabry-Perot resonator cavity (i.e. a RWL). Collimation of the laser output is achieved using a Thorlabs TME240-B 0.5 NA 8 mm focal length aspheric lens.

The laser is mounted in a custom diode mount featuring an Layertec GMBH (model 11009) 50(3)% output coupler mirror to form an external cavity. A 20 $\mu$m uncoated fused silica etalon is placed before the coupler which itself is tilted from the lasing optical axis as an attempt to reduce optical feedback into the laser gain medium. Housed in a custom nitrogen-flushed acrylic chamber, the laser mount is temperature locked to a set-point using a Thorlabs LDC202 laser diode controller. For driving the laser, a Mini Circuits ZX85-12G+ Bias tee is used to combine the DC laser driver current (Thorlabs LDC202 laser diode controller) with the RF modulation (Rhode & Schwarz SMC100A-3.2 GHz) in series with an QuBig RF amplifier. During certain experiments the laser has been injection-locked using one of the ECDLs from section 4.2. The light from the ECDL would pass through the second port of the second FOI in-front of the laser into the RWL. In this work however we didn’t injection lock the laser.

Previous CW laser characterisation by Dr. J. Woods conclude a threshold current $I_{th} = 8.96(0.04)\ mA$ and a highly unstable laser output frequency at driver currents $I \in \{10, 20\} \ mA$. No instability in the laser output power was reported for both a present and absent output coupler (i.e. external cavity). Due to time constrains and the transfer to another project these measurements were not repeated post Dr. J. Woods thesis submission.

As the CW operation is disrupted, we might be able to explain future anomalous features of mode-locking this laser. Our literature review suggests:

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15 To prevent water condensation at lower operating temperatures.
16 The LDC202 hosts a built-in PID controller whose input is connected to an AD590 integrated circuit monitoring the laser mount temperature. The PID output is connected to an Thermo-Electric Cooler (TEC) attached to the laser mount also.
CHAPTER 4. CW CHARACTERISATIONS OF DIODE LASERS

1. Partial optical feedback despite the tilted output coupler.
   - Laser cavities are very sensitive to optical feedback as observed in subsection 4.1.5 with an Eagleyard DFB laser. Thus even the small percentage of transmitted light indifferent to the coupler’s orientation might induce lasing instability.
   - Similar to injection locking, the mode competition by gain saturation would favour modes of the seeding optical feedback. This might lower the competition from optical losses which prefer a different mode(s).

2. Imperfect AR coating of laser front emission facet.
   - This could support CW multimode operation as (geometrical) mode competition would no longer be maximised from the visibility of the laser’s (intra) cavity to its external cavity.
   - The introduction of the etalon inside the external cavity was found to improve the laser’s CW output spectrum, reducing the spectral bandwidth hence supporing this hypothesis.
   - Multimode operation from a misaligned cavity has on occasion been observed during CW operation on a scanning Fabry-Perot Optical Spectrum Analyser (1.5 GHz, Thorlabs SA200-6A).

3. Lack of dispersive optical element inside the laser cavity.
   - Apart from the etalon (only in CW), the laser does not host any other dispersive elements in its composite cavity. It could be that the composite cavity suffers from low mode competition.
   - The former popular $C^3$ diode laser (Fabry-Perot cavity derivative) also was without an dispersive element inside its cavity. Whilst that laser could also operate in single mode with a tunable frequency, it was notorious for its instability. The analogy is discussed in further detail in Appendix B.

   - We sometimes observed 'fake' lasing: two non-coherent faint beams were emitted in different directions with each beam having low intensity and a very wide spectrum on the OSA. Realigning the output coupler solved this problem.

Any or several of these sources might account for the laser’s instability in CW and anomalies during mode-locking.
Chapter 5

Birth of a Mode-locked Laser

The next three chapters in this sub-thesis will focus on the characterisation of the Mode-Locked Laser (MLL) designed and discussed in Dr. J. Woods thesis [27]. This MLL was aimed at providing a frequency ruler to several CW lasers in atomic physics experiments. In this chapter we review some basic theory behind MLL’s with the subsequent two chapters focusing on different characterisations performed on the MLL.
5.1 Mode-locking

Mode-locking of a laser is a method to generated pulsed lasing. By modulating the laser’s cavity gain medium, refractive index, polarisation, length or optical losses, multiple cavity modes can be made to lase simultaneously. This results in the formation of narrow optical pulses consisting of a wide spectrum of cavity modes about a centre mode $\nu_c$ (multi-mode lasing). These pulses can be used for several applications such as molecular fingerprinting, frequency counters for atomic clocks, arbitrary waveform measurements and more [66].

5.1.1 Time-domain Pulse Train

We consider typical multi-mode lasing, a cavity in which $N \in \mathbb{N}/\{1\}$ cavity modes are lasing simultaneously. On this precedent, the pair-wise relative phase between the $\{\ldots, \nu_{c-1}, \nu_c, \nu_{c+1}, \ldots\}$ modes stems from spontaneous emission i.e. is random. Apart from the initial randomness and subsequent decoherence events, these relative phases remain otherwise static. This results in a cyclical temporal output intensity profile as shown in figure 5.1 (a).

Figure 5.1: (a) The intensity profile over time from $N$ random phase-related cavity modes. (b) A theoretical MLL output from a definite phase relationship between multiple modes. A train of pulses of length $\Delta t$ separated $\Delta T_R$ apart are output by the laser. Every next pulse experiences an increase in the carrier envelope offset (CEO) by $\Delta t_{CEO}$.
5.1. MODE-LOCKING

In a MLL, multiple modes are also made to lase simultaneously, but instead of a random static phase relation between the \( N \) modes, the phase relationships are *definite* resulting in an electric field profile as shown in figure 5.1 (b). This generates a train of wavepackets (optical pulses), each pulse having an electric field that can be described as

\[
E(t) = A(t)e^{2\pi i\nu_c t} + \text{c.c.} = \sum_{q=q_j}^{q_j+N} A_q e^{2\pi i\nu_q t} + \text{c.c.}
\]  

(5.1)

where \( A(t) \) is the periodic pulse-envelope, \( \nu_c \) the carrier frequency, \( \nu_q \) are the lasing mode frequencies with amplitude \( A_q \) and \( \text{c.c} \) stands for the complex conjugated term.

The superposition between the definite phase-related \( N \) modes generates pulses of duration \( \Delta t \). These pulses are separated by the *cavity Round-trip time* of a single pulse \( \Delta T_R \). Two adjacent pulses are not identical however as the pulse’s envelope moves at a different velocity (group-\( v_g \)) compared to the carrier (phase-\( \nu_p \)). Every next pulse therefore undergoes a temporal shift of \( \Delta t_{CEO} \) between the carrier and the envelope.

Assuming a single pulse traverses inside the cavity of length \( L \), the relationship between the group velocity and the cavity round-trip time can be shown to equal

\[
\Delta T_R = \frac{2L}{v_g}
\]
5.1.2 Frequency–domain Comb

Let \( \nu_m \) be the frequency at which the laser’s cavity is being modulated. Mode-locking requires this modulation to take place near the FSR of the cavity i.e. \( \nu_m \simeq \Delta \nu_{\text{FSR}} \). For a single optical pulse oscillating inside the cavity, \( \nu_m = \Delta T_R^{-1} \). Thus \( \nu_m \) also defines the laser-pulse exit/repetition rate.

The (longitudinal\(^1\)) modes \( \nu_p \) of a cavity are given by

\[
\nu_p = p\Delta \nu_{\text{FSR}} \quad , \quad p \in \mathbb{N}
\]

(5.2)

Due to non-linear processes during the dynamic pulse formation stage, even if the cavity is modulated exactly at the FSR, there will be some Common Offset frequency \( \nu_{\text{CO}} = \Delta t_{\text{CEO}}^{-1} \). This results in the formation of a ’comb-like structure’ in the frequency domain that follows the comb-equation

\[
\nu_q = q\nu_m + \nu_{\text{CO}} \quad , \quad q \in \mathbb{N}_0
\]

(5.3)

For the convenience of clarity, the modes \( \nu_q \) which make up the optical pulses shall henceforth be referred to as ’comb-lines’. Due to \( \nu_{\text{CO}} \) they typically differ from the cavity’s longitudinal (and transverse) modes \( \nu_p \). The cavity modes & comb-line structure from mode-locking are illustrated in figure 5.2.

\[\text{FT}[E(t)](\nu)\]

\[\nu_p \]

\[\Delta \nu_{\text{FSR}}\]

\[\nu_{\text{CO}}\]

\[\nu_{\text{CO}}\]

\[\nu_q\]

\[\nu_c\]

\[\Delta \nu_g\]

\[\Delta \nu\]

Figure 5.2: This diagram illustrates the spectrum (dark-red) of an infinite train of pulses from a MLL. The comb-lines \( \nu_q \) are excited around a centre mode \( \nu_c \) with decreasing amplitude away from the centre. A comb-like structure (lightly-shaded red) appears with mode spacing \( \nu_m \simeq \Delta \nu_{\text{FSR}} \) shifted by \( \nu_{\text{CO}} \) from the cavity modes \( \nu_p \). The MLL spectrum has a linewidth of \( \Delta \nu \) bound by the gain bandwidth \( \Delta \nu_g \).

\(^1\)For most cavity designs, when including the effects for transverse modes the laser’s cavity spectrum becomes a tad more complicated as discussed in 6.2.
To see how this comb-structure comes forth we take the Fourier Transform (FT) of equation 5.1. Using \( \sim \) to denote the FT we can simplify it as

\[
\text{FT}[E(t)](\nu) = \tilde{A}(\nu - \nu_c)
\]

When we extend our FT to an infinite train of optical pulses of a MLL we find

\[
\text{FT}\left[ \sum_{j=-\infty}^{\infty} A(t - j\Delta T_R)e^{2\pi i\nu_c(t - j\Delta T_R) + ji\Delta \phi_o}(\nu) \right] = \tilde{A}(\nu - \nu_c) \sum_{q=0}^{\infty} \delta(\nu - q\nu_m - \nu_{CO})
\]

(5.4)

where \( \Delta \phi_o = 2\pi \nu_c \Delta t_{CEO} = 2\pi \frac{\nu_{CO}}{\nu_c} \) represents the pulse-to-pulse phase slip and \( \delta(\nu - q\nu_m - \nu_{CO}) \) the Dirac delta function centred by the comb-equation. This shows that the spectrum of infinite pulses of a MLL has a spectrum defined by the comb-equation and the FT of individual pulses [67].

It is interesting to note that equation 5.4 shows that the MLL’s spectrum is constructed from the interference of multiple pulses despite them being separated in time. We see that the pulse-to-pulse phase slip and pulse separation bring forth the comb-equation as each pulse acquires a different instantaneous complex phase (despite having the same carrier frequency). This non-intuitive behaviour is of importance to characterising MLL’s in techniques such as SPIDER discussed in subsection 5.4.3.

### 5.1.3 Frequency-Phase Relation

The coherence of the carrier-envelope pulse phase-slip \( \phi_o \) is partially representative of the coherence in the harmonic structure of the comb-lines. This can be seen from its relationship to \( \nu_{CO} \) as seen in

\[
\nu_{CO} = \frac{1}{2\pi} \frac{d\phi_o}{dt}
\]

(5.5)

A decoherence event can thus induce a chirp or shift in the collective comb-like structure of the MLL. Decoherence can also manifest through decoherence in the laser-pulse exit rate \( \nu_m \) giving the MLL two free variables: \( \nu_{CO} \) and \( \nu_m \). Proper mode-locking requires these frequencies to be coherent.
5.1.4 Limit to Mode–Locking

The spectral width $\Delta \nu$ of the MLL is limited first-most by the width of the Gain medium $\Delta \nu_g$. By the time-bandwidth theorem this also sets a lower bound to the duration of a pulse $\Delta t$. Due to the size of most gain bandwidths however this limit is treated as lower order. A higher order limit is imposed directly by the time-bandwidth theorem onto the product between the temporal and spectral bandwidths of the laser. In terms of the Root Mean Squared (RMS) bandwidth, for a chirp-free Gaussian intensity pulse-profile the Time-Bandwidth Product (TBP) is

$$\text{TBP}_{\text{RMS}} = \Delta t_{\text{RMS}} \Delta \nu_{\text{RMS}} \geq \frac{1}{2}$$  \hspace{1cm} (5.6)

Typically however the Full Width Half Maximum (FWHM) bandwidths of the intensity profile are used indicating the $-3 \text{ dB}$ cut-off line. These are easier to measure in practice but give different TBP boundaries. Apart from bandwidth definition the TBP is also pulse-shape dependent. For example, for a chirp-free Gaussian Pulse we find $\text{TBP}_{\text{FWHM}} \geq \frac{2 \ln(2)}{\pi} \approx 0.44$. A chirp-free secant ($\text{sech}^2$) pulse however has a $\text{TBP}_{\text{FWHM}} \geq \left( \frac{2 \ln(1+\sqrt{2})}{\pi} \right)^2 \approx 0.31$ with a Lorentzian pulse going smaller to $\text{TBP}_{\text{FWHM}} \geq \frac{\ln 2 \sqrt{2-1}}{\pi} \approx 0.14$.

Pulses that operate at the minimum TBP are known as 'Transform-Limited Pulses'. Depending on the broadening mechanism the pulse experiences however, the definition of the TBP might also change. Homogeneous broadening for example has a different formula [41] for calculating the TBP:

$$\text{TBP}_{\text{h-broad}} = \Delta t \sqrt{\nu_m \Delta \nu_g}$$

In addition to the broadening mechanism, the effect of a laser chirp also affect what constitutes a transform-limited pulse. For a Gaussian pulse for example, a transform limited pulse has a TBP$= \frac{2 \ln(2)}{\pi} \sqrt{1 + \left( \frac{a}{b} \right)^2}$ where $a$ is the pulse’s envelope broadening parameter and $b$ is the frequency chirp parameter [28].

Hence consideration have to be taken when determining how near transform-limited the pulses of a MLL are. Likewise, when using the TBP as a measure of pulse quality, the bandwidth definition, pulse-shape, broadening and chirp must be considered. Under those reflections the TBP can prove to be an useful dimensionless quantity to refer to.
5.2 Typical Characteristics

The modulation of the laser’s cavity can be classified in three classes:

1. Active
   - This involves modulating the cavity using an external oscillator.
   - Modulation is set near or at the FSR of the laser’s cavity.

2. Passive
   - This involves the modulation of cavity losses using an ‘saturable absorber’ (SA).
   - The modulation frequency is set a priori.

3. Hybrid
   - This involves the dual use of a SA and an external oscillator.
   - In some laser systems, the SA can be externally modulated.

Both active and passive mode-locking have successfully achieved near transform-limited pulses in ECDL’s with pulse-repetition rates easily upto $20 \, GHz$ and pulse durations $\sim 1 \, ps$. Indeed, the first demonstrations of active and passive mode-locking in laser diodes operated around these bandwidths. Those systems could easily have spectral bandwidths upto $1 \, THz$ depending on $\nu_c$.

Pulse-repetition rates up to $100 \, GHz$ have also been demonstrated in laser diodes using an electro-absorption modulator. Interestingly, if left unbiased these modulators act as SA allowing for passive or hybrid mode-locking.

Also remarkable is the effects on the dynamics of pulse formation on placing the SA. So far the aforementioned passive mode-locked examples had a single SA situated at the end of the cavity. Placing the SA elsewhere however can result in multiple pulses coexisting inside the cavity leading to so called ‘Colliding Pulse Mode-Locking’ (CPML).

\footnote{A medium in which higher intensities of light lower the absorption levels.}

\footnote{Through the \textit{quantum-confined Stark effect}, the application of an AC signal onto the device allows for modulating the light passing through it.}
If we let $M$ denote the number of pulses coexisting in the cavity we find

$$\nu_m \simeq M \frac{v_g}{2L}$$

A cavity with a single SA in the centre will have $M = 2$ by symmetry. This CPML laser will feature a second harmonic comb mode-spacing and exhibit shorter and more stable pulses. The results are virtually transform-limited pulses such as the $(M = 2)$ InGaAsP passively mode-locked diode laser which was found to have $\sim 1$ ps secant pulses from $80 \text{ GHz}$ up to several hundreds of $\text{GHz}$ [70]. Even a hybrid CPML (e.g. the cited GaInAs in [71]) has achieved pulsewidths from $6 - 8$ ps, slightly longer due to a frequency chirp.

The last characteristic can be seen when extending the three-mode frequency-domain model to $N$ comb-lines. As per [72] it can be shown that

$$\Delta t = 4\sqrt{\ln(2)} \frac{L}{Nv_g}$$

5.2.1 Clustered Mode–locking

So far the discussion has been restricted to so called ‘fundamental mode-locking’ (FML). But like the $C^3$ laser (c.f. appendix B) we again find the need to remark on the dynamical solutions of mode-locking. The unstable solutions at higher pump powers in $v$-shaped passively locked VESCSEL suggest the formation of clusters of pulses. Compared to the FML, depending on the Hopf bifurcation $N_H$ the main pulse is bifurcated into $M_p$ equidistant sub-pulses of separation $\Delta T_{\text{spacing}}$ given by

$$\Delta T_{\text{spacing}} \simeq \frac{\Delta t}{2N_H M_p}$$

These dynamical regions of pulse-clustered mode-locking (PCML-stable) are separated by regions of unstable side-pulse mode-locking. They can also feature in higher order mode-locking (CPML) [73]. Hence our diode laser might also feature injection-current dependent clusters of pulses.

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4Higher $M$ are achieved by placing multiple SA equidistant inside the laser’s cavity. Every additional SA increases $M$ by 1.

5Due to the coherent (interference of pulses) and incoherent [bleaching of the SA] colliding pulse effects when multiple pulses are incident on (each other) [the same SA] at the same time.

6Within a few percent from perfectly transform-limited pulses when calculating as $(\text{TBP} - \text{TBP}_{\text{min}})/\text{TBP}_{\text{min}}$. 
5.3 Optical Frequency–Comb

Recall that one degree of freedom of a typical MLL that is subject to vary from fluctuations in the laboratory is $\nu_{\text{CO}}$. As such, the comb-line structure of a MLL can experience side-way frequency shifts. An absolute frequency reference can be made however by fixing this offset frequency. This leads to the formation of an **Optical Frequency Comb (OFC)**.

An OFC can be defined as a *phase-stabilised* MLL. By stabilising $\phi_o$, not only is $\nu_{\text{CO}}$ statically defined but through a stable $\nu_m$ and the comb equation also the rest of the comb-lines $\nu_q$. For a proper frequency ruler for phase-sensitive atom-laser interactions this is of interest.

The original aim in [27] was to make a mini-frequency-comb for $Rb$ based atomic physics experiments. Our characterisation are aimed to quantify the optical pulses and phase-stability of the MLL along with demonstrating the locking of multiple CW laser’s onto the MLL. Therefore the following two section will summarise the standard techniques used to characterise an OFC along with some novel alternatives we had to implement.

### 5.3.1 Phase-Stabilisation

**Self-referencing**

If the spectrum of a MLL is octave spanning, a *self-referencing scheme* is typically used for phase-stabilisation. These schemes work by creating a beat-note that is equal to or a multiple of $\nu_{\text{CO}}$. Once such scheme does this by doubling the frequency of a comb-line $\nu_q$ on the red (lower) end of the spectrum through a non-linear process. By beating the red comb-line against the $\nu_{2q}$ blue (upper) comb-line the beat-note is found to be

$$2\nu_q - \nu_{2q} = \nu_{\text{CO}}$$

By frequency-locking the measured beat-note we can infer from the comb-equation that the comb-structure is fixed [74]. In addition, by equation 5.5 we also find that the carrier-envelope offset and therefore the phase relationship between modes is now defined. Note however that the phase relationships in CPML and PCML are typically more complicated\(^7\) then FML.

\(^7\)In PCML for example, the main pulse bifurcation might destroy the coherence amongst the sub-pulses. Therefore locking the beat-note might only phase-stabilise the sub-pulses that contain $\nu_q$ and $\nu_{2q}$.
 Whilst self-referencing helps anchoring the frequency-comb comb-lines offset from DC, there is no common technique known to the author that is used to counter the 'breathing' of OFC’s. Whilst mathematically it can be shown that ideal amplitude, phase and frequency modulation of a laser’s cavity set’s up a definite phase relationship, in practice some comb-lines can experience some 'liberty' around their centre value. The concern is that coherence might be established over separate groups of comb-lines as opposed to the entire OFC. This would result in an accordion-like compression/expansion ('breathing') of the spectrum between these groups of comb-lines. Whilst this ties into PCML, further details are beyond the scope of this sub-thesis.

5.3.2 Optical Injection-Locking

When a MLL is not octave spanning, an alternative to broadening its spectrum / self-stabilisation is to use a master-slave laser set-up. A frequency-locked CW "master" laser injects seed-light into the cavity of a "slave’s" (MLL) cavity. This synchronises the "slave’s" phase (and by extension frequency) to the "master’s" phase (and frequency). This technique called 'Optical Injection Locking' (OIL) does not require a wide spectral bandwidth.

OIL comes with several benefits including linewidth reduction (e.g. from 1.2 MHz to \(\simeq 5\) kHz slave using a 5 kHz master) and chirp reduction approaching the "master's" chirp in CW. Upon modulation of a semiconductor "slave", chirp reduction is found to be governed and limited by\(^8\) the frequency chirp in the modulation \(\delta \nu\) which is given by

\[
\delta \nu(t) = \frac{\alpha}{4\pi} \left[ g(N - N_{tr}) - \gamma_p \right]
\]  

(5.7)

where \(g\) is the laser gain coefficient, \(\alpha\) the linewidth enhancement factor, \(N\) the charge carriers number, \(N_{tr}\) the transparency charge carrier number and \(\gamma_p\) the photon delay time\(^\text{[75]}\).

To stabilising the MLL, OIL synchronises the seeding mode of the master laser \(\nu_M\) with the nearest comb-line \(\nu_S\). With a coherent spacing between comb-lines, OIL frequency-locks not only \(\nu_S\) but the surrounding modes. This we can understand by re-shuffling the comb-equation: \(\nu_{CO} = \nu_q - q\nu_m\), \(q \in \mathbb{N}_0\). Setting \(q = S\) and assuming we modulation \(\nu_m\) is coherent, we see we immediately restrict \(\nu_{CO}\), defining the two free variables of a MLL and enhancing its coherence whilst also achieving phase-stabilisation.

\(^8\)The \(\alpha\)-factor in equation 5.7 shows a narrow linewidth "master" laser also helps reduce frequency chirp of the slave laser along with other factors.
5.4 Characterising an OFC

5.4.1 Autocorrelators

With current commercial photodiodes limited to bandwidths up to tens of GHz\(^9\), pulses of duration \(\tau_p \lesssim 1\) ns cannot be studied using photodiodes. Pulses of duration \(\tau_p \gtrsim 100\) fs can be studied using a commercial streak camera\(^10\), but shorter duration pulses require a higher resolution measurement such as provided by autocorrelators [28]. Here we discuss the principle behind the autocorrelator as we use it as a characterisation tool.

To determine the pulse duration for ultra-short pulses (\(\tau_p \leq ps\)) an autocorrelator probes the pulse with a delayed copy of itself. By utilising the scanning Michelson-Morley interferometer structure as in figure 5.3, the probe pulse is convolved with the pulse at a relative delay of \(\tau_{rel}\). By slowly scanning this delay at actuator speeds, conventional detectors can be used to resolve \(ps\) and even \(fs\) pulses. The timescale-resolution is limited mainly by the pulse perturbations induced by the optics\(^11\). In essence, the fidelity of the probe pulse (and pulse perturbations of the non-probe pulse) fundamentally limits the temporal resolution of the autocorrelator.

Figure 5.3: Diagram of a general autocorrelator setup. An incident pulse equally partitioned using a 50 : 50 pellicle beamsplitter generating a probe. The probe travels an scanning optical path providing a varying delay \(\tau_2\). The remaining pulse experiences a fixed delay \(\tau_1\). Both pulses are incident on a detector providing a signal \(S_{auto}(\tau_{rel})\) that depends on the relative delay \(\tau_{rel} = \tau_2 - \tau_1\) between the two pulses. The set-up makes use of a PhotoDiode (PD), Second Harmonic Generation crystal (SHG), Polarising Beam Splitting Cube (PBSC), Beam Dump (BD) and half [quarter] wave plates (\(\lambda /2\) [\(\lambda /4\)]). N.B. other non-linear electro-optics\(^a\) might be used.

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\(^9\)I.e. rise/fall-times at least tens of picoseconds.

\(^10\)Though developments in strong field ionisation physics has led to the development of a streak camera that can resolve up to attoseconds in resolution [76].

\(^11\)Such as their bandwidth or chromatic dispersion.

\(^a\)For example, avalanche photodiode can frequently be found in commercial autocorrelators.
Depending on their resolution, autocorrelators operate in two regimes: interferometric (higher) and intensity (lower resolution). Both types produce symmetric signals about $\tau_{rel} = 0$ and provide a trace $S_{auto}(\tau_{rel})$ that is proportional to the overlap of the probe and non-probe pulse. In general (interferometric), this proportionality\(^\dagger\) can be described by

$$S_{auto}(\tau_{rel}) \propto \int_{-\infty}^{\infty} (|E(t) + E_{probe}(t + \tau_{rel})|^2)^2 dt$$

where $\tau_1$ and $\tau_2$ represent the path time-delay per interferometer arm and $\tau_{rel}$ the relative time-delay between the arms generated from scanning the probe arm ($\tau_2$) of the interferometer \([28]\).

For an intensity autocorrelator of lower temporal resolution, the interference terms vanish resulting in an intensity autocorrelation trace $S_{auto}^i(\tau_{rel})$ to follow the following proportionality

$$S_{auto}^i(\tau_{rel}) \propto \int_{-\infty}^{\infty} I^2(t)dt + \int_{-\infty}^{\infty} I_{probe}^2(t + \tau_{rel})dt + 4(I \ast I_{probe})(\tau_{rel})$$

with $I \propto |E|^2$ and the linear convolution term defined as

$$(I \ast I_{probe})(\tau_{rel}) = \int_{-\infty}^{\infty} I(t)I_{probe}(t + \tau_{rel})dt$$

Since the linear convolution operation distorts the shape of the pulse, an autocorrelator is unable to predict the pulse duration of a pulse of an unknown shape. To use an autocorrelator thus requires a priori knowledge of the pulse-shape to correct for this distortion using deconvolution factors. For reference, a Gaussian pulse profile gives a deconvolution factor of $\sqrt{2}$ so that $\Delta t_{measured} = \sqrt{2}\Delta t$.

In practice, the measured $S_{auto}$ will be more complex due to coherence spikes in PCML \([77]\), accompanying satellite (pre/post) pulses from optical reflections and (or) non-linear effects \([78]\) and the pedestal of the main pulse arising from amplified spontaneous emission (ASE). Optical pulses can however be formed without a pedestal in some laser systems e.g. by laser-probing cold atomic clouds. Abrupt phase changes to the laser-probe are able to generate non-pedestal pulses through destructive interference \([79]\).

\(^\dagger\)The constant of proportionality can be determined from the specific optical-to-electrical gain factor of the detector.
5.4. CHARACTERISING AN OFC

5.4.2 FROG

A useful characterisation tool can be made from adapting the autocorrelator. Essentially, by replacing the intensity detector with a spectrometer, a pulse-delay ($\tau_{\text{rel}}$) dependent spectrum ($\omega$) can be measured. By using an appropriate algorithm both the phase and intensity profile can be retrieved simultaneously. This technique is called Frequency-Resolved Optical Gating or FROG for short [80] and has a trace that is proportional to

$$S_{\text{FROG}}(\omega, \tau_{\text{rel}}) \propto |\text{FT}[E(t)E(t + \tau_{\text{rel}})]|^2.$$}

5.4.3 SPIDER

So far the previous techniques characterise the properties of the optical pulse in the time-domain. For a full characterisation of an OFC it is conventional to complement a time-domain data-set with a frequency-domain measurement. A popular technique that operates in the frequency-domain is Spectral Phase Interferometry for Direct Electric-field Reconstruction or SPIDER for short.

Whilst many adaptations of SPIDER exist, the classical configuration is shown in figure 5.4. Light from the OFC is split into two parts that are further on recombined onto a sum frequency generation crystal (SFG). The lower arm has its pulses ancillary stretched (ASP) using a pair of diffraction gratings. The upper arm generates a copy of the pulse delayed by $\tau_{\text{rel}}$, each copy experiencing a different spectral shear from the SFG with the ASP. The two pulses are then made to interfere\(^\text{13}\) on the spectrometer [81].

If we let $\phi$ be the complex phase of a pulse and $\Omega$ the shearing parameter\(^\text{14}\) then the trace is given by

$$S_{\text{SPIDER}}(\omega, \tau_{\text{rel}}) \propto I(\omega) + I(\omega + \Omega) + \sqrt{I(\omega)I(\omega + \Omega)} \cos \left[\phi(\omega) - \phi(\omega + \Omega) + \omega \tau_{\text{rel}}\right]$$

SPIDER can thus reconstruct the intensity and phase profile of optical pulses from measurements in the frequency-domain.

\(^{13}\)Recall that we showed with equation 5.4 that temporally separated pulses can interfere with each other.

\(^{14}\)The frequency-shift between the two pulse copies
CHAPTER 5. BIRTH OF A MODE-LOCKED LASER

5.4.4 RF-Spectrum Measurement

Another measurement in the frequency domain involves a frequency-locked CW laser beating with the OFC on a fast photodiode and monitoring the spectra on a vector network analyser. By monitoring the frequency and phase of the different beat-notes from different comb-lines, the coherence of the phase relationship can be measured over a few comb-lines directly. Then, by tuning the CW wavelength to another part of the OFC’s spectrum, other comb-lines can be inspected.

5.4.5 OFC Synopsis

Several measurements are typically taken to reaffirm and quantify the phase-stabilisation of the MLL. The measurement of the TBP using a monochromator and an autocorrelator, an intensity and phase reconstruction using FROG and SPIIDER, and an RF-spectrum measurement against a stabilised CW laser together provides a useful diagnostic tool-kit to the OFC.
Chapter 6

Mode-Locking Characterisation

Apart from an autocorrelator and a monochromator, our research group did not have the parts and components needed to perform a FROG or SPIDER measurement. It was therefore impossible to perform the standard frequency-comb characterisation methods discussed in the previous chapter. Instead, to finalise the data in [27] into a research paper, we opted for a novel technique that would provide a coarse characterisation of the MLL. In this chapter we discuss this technique and our findings with regards to the frequency-jitter of the MLL.

Please note that this chapter includes work that involved collaboration with Dr. J. Woods. The computational modelling used in this chapter along with one of two datasets of measurements are credited to Dr. J. Woods. Since the rest of the work in this chapter is by the author and as the analysis on the MLL will make use these measurements, the work specific to Dr. J. Woods is included (and highlighted) for completeness.
6.1 Estimating Frequency-Jitter

A novel method we can use to estimate the frequency coherence of the various frequency components of the frequency comb involves an Fabry-Perot OSA. We inject the various comb-lines of the mode-locked laser (MLL) into the OSA cavity whose FSR matches that of the MLL. In a single scan of the OSA this will produce a single spectral line-shaped signal stemming from the overlap of multiple comb-lines with OSA cavity modes as shown in figure 6.5. It is this spectral width that will be of interest as it depends on

1. **The linewidth of the Fabry-Perot OSA.**
   The transmission $T$ through the OSA can be shown to be \([82]\)
   \[ T_\phi(\phi) = \frac{1}{1 + \frac{4F^2}{\pi^2} \sin (\phi/2)} , \quad \phi = \frac{4\pi n L}{\lambda} \quad (6.1) \]
   where \(F\) called 'Finesse' represents the quality factor of the cavity and is given by
   \[ F = \frac{\Delta\nu_{FSR}}{\Delta\nu_{FWHM}} \quad (6.2) \]
   Typically \(F \sim 100-1000, \Delta\nu_{FSR} \sim GHz\) and \(\Delta\nu_{FWHM} \sim 1-10 MHz\).

2. **The linewidth of the frequency components of the frequency comb.**
   The MLL had a sub-MHz linewidth in both CW operation and during mode-locking (per comb-line) \([27]\).

3. **The maximum frequency-jitter of the frequency components of the comb relative to the Fabry-Perot.**
   Dr. J. Wood’s thesis tentatively confirmed the laser was cluster mode-locking. Under this assumption, the jitter estimate is between the clusters as we expect a stronger phase-coherence within each cluster.

The advantage of this method is implementation, the measurement can be done using few modifications to the experimental set-up and without the need of purchasing expensive equipment or parts. It is the latter advantage that limited our characterisation of the MLL by general techniques. The measurement is however limited heavily by the linewidth of the OSA and provides no information on the distribution of noise amongst the comb-lines.
6.2 Mode Matching

The only available OSA at the time in our inventory was the Thorlabs SA200-6A Fabry-Perot OSA which had half the required FSR (3 GHz). But since this OSA makes use of a confocal geometry, we could mode-match the MLL light into the cavity to match their FSR.

So far our literature review in subsection 3.1.2 focused on longitudinal modes. However, a deeper analysis finds the cross-sectional spatial confinement introduces another boundary condition that leads to so called 'transverse modes'. These transverse modes TEM_{nm}, n \wedge m \in \mathbb{N} are the solution to the paraxial wave equation which are Hermite-Gaussian polynomials in Cartesian coordinates or Laguerre-Gaussian polynomials in Cylindrical coordinates. Figure 6.1 shows the spatial intensity distributions of these transverse modes in both coordinate systems.

Figure 6.1: Spatial Intensity Distribution of the transverse modes TEM_{nm}. The indices in the figure represent the mode indices n, m. [Left] Hermite-Gaussian using Cartesian coordinates. [Right] Laguerre-Gaussian using Cylindrical coordinates. Both figures originate from [83].

As is evident from figure 6.1 the indices n, m represent the number of nodes for each Cartesian axis (2D). Higher order modes also occupy a larger space compared to the 'ground state' TEM_{00}. Interestingly enough these modes can be derived from calculating 'self-replicating shapes' over a full cavity round-trip. Similar to finding standing waves when calculating the longitudinal resonance cavity modes, by demanding the transverse electric field E(x, y) at (x, y) to have the same shape\(^1\) for any number of round trips in the cavity when returning to (x, y) we find the transverse modes TEM_{nm}. So these transverse modes can be thought of as being the only transverse maintaining shapes in the cavity just like standing wave modes are the only longitudinal maintaining shapes the cavity supports.

\(^1\)I.e. only possibly a smaller amplitude or phase shift
By including both transverse and longitudinal modes the cavity supports the resonance frequencies become [84]

$$\omega_{qm} = \Delta\omega_{FSR}[q + (n + m + 1)\frac{\cos^{-1} \pm \sqrt{g_1g_2}}{\pi}]$$ (6.3)

where the resonator parameters $g_i = 1 - L/R_i$ are functions of the radius of curvature $R_i$ of the $i^{th}$ mirror with $i \in \{1, 2\}$. The $\pm$ in the inverse cosine argument reflects where on the stability curve (figure 6.2) i.e. positive or negative quadrant the resonator is located at. N.B. we continue to define $\Delta\omega_{FSR}$ as the longitudinal FSR of the cavity which evidently can now differ from the observed frequency mode spacing.

For a stable resonator it can be shown that

$$0 \leq g_1g_2 \leq 1$$

This is often represented using a so called 'Resonator Stability Diagram' as shown in figure 6.2. Essentially this diagram shows that stable resonators require at least one mirror to be sufficiently concave.

![Figure 6.2: The Resonator Stability Diagram demonstrating the regions (shaded regions in upper-left inset) with stable resonance using the resonator $g$ parameters. Three common cavity configurations are shown with their respective position on the stability diagram. We label the radius of curvature $R$ per mirror with $R \to \infty$ for planar mirrors.](image-url)
As is evident from equation 6.3 the transverse modes occupy different frequencies which now become supported cavity modes. Hence scanning the cavity means multiple peaks show up representing the different allowed higher order TEM modes. There are a few\(^2\) cavity configurations in which the frequencies become degenerate which includes the confocal cavity. Here \(\frac{\cos^{-1}(\pm \sqrt{\frac{g_1}{g_2}})}{\pi} = \frac{1}{2}\) leading to half-longitudinal modes as shown in figure 6.3.

These half-longitudinal modes are at frequencies between the typical longitudinal modes \(\omega_q\) i.e. \(\omega_{q\pm1/2}\). As a result the 'effective' FSR (mode frequency separation) becomes half of the usual definition which is why the 1.5 GHz is quoted by the manufacturer Thorlabs for the SA200-6A OSA.

For the confocal cavity, if \(m + n\) is odd (i.e. 'odd transverse modes') than they have an resonant frequency at \(\omega_q\). Similarly, if \(m + n\) is even (i.e. an 'even transverse mode') than they only occupy half-longitudinal modes with resonant frequencies \(\omega_{q\pm1/2}\). So if light is coupled into the OSA cavity such that only the 'ground state' TEM\(_{00}\) gets 'excited' than the cavity will 'skip' every \(\omega_q\) resonant frequency\(^3\) such that the free spectral range doubles. This is done through a process called Mode Matching and will give us the 3 GHz free spectral range needed for the frequency-jitter estimation.

![Figure 6.3: The angular resonant frequencies \(\omega_{qmn}\) supported by a near confocal cavity. \(x\) represents the sum of the transverse mode indices \(n, m\) with \(q\) the longitudinal mode index. The cavity has an angular free spectral range of \(\Delta \omega_{FSR}\). The insets show the practical scenario of the higher order modes not being perfectly degenerate for two resonant frequencies for two different cases. For a shift on the stability curve towards concentric the frequencies are shifted down whilst a shift towards plano-plano as a up frequency shift. Note how the odd and even modes are separated.](image)

\(^2\)E.g. when at least one of the \(g\) parameters vanish.

\(^3\)Hence all odd transverse modes.
A computer simulation model by Dr. J. Woods found a 98% overlap of the laser beam to the OSA’s cavity TEM\textsubscript{00} mode using a plano-convex lens of focal length $f = 100$ mm ($f = 125$ mm gave 95%). As such a $f = 100$ mm Thorlabs LA1509-A lens was placed with its focal point at the centre of the OSA along its optical axis. The lens was placed on a linear translation stage for fine-tuning its distance from the OSA as shown in figure 7.1.

Due to the OSA’s saturation at high photodiode gain of even modes, this setting cannot be used to estimate the level of mode-matching against odd modes. Once the signal from the odd modes reduced to the level of visible sensitivity against tuning/touching the steering mirrors\footnote{We ended up needing to use Newport fine-adjustment kinematic mounts instead of the Thorlabs KM100T mounts for the rest of the optics. The KM100T namely demonstrated higher sensitivity to drift and touch, lowering the level of achievable mode-matching.} we switched the OSA photodiode to low gain. At this gain we couldn’t see the odd modes any more as shown in the right of figure 6.4 demonstrating mode-matching.

![Figure 6.4: Partial mode-matching into the OSA using the high photodiode gain setting of 100 kV/A. False spikes and a base signal step are caused by the OSA’s trigger are marked. Right] Full mode-matching of the OSA is demonstrated using the low photodiode gains setting of 10 kV/A. The suppressed mode location is marked once with the unsuppressed modes no longer experiencing saturation (hight is now affected by the base step). N.B. the y-axis in the right is scaled by $10^{-3}$ as indicated.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure64.png}
\caption{[Left] Partial mode-matching into the OSA using the high photodiode gain setting of 100 kV/A. False spikes and a base signal step are caused by the OSA’s trigger are marked. [Right] Full mode-matching of the OSA is demonstrated using the low photodiode gains setting of 10 kV/A. The suppressed mode location is marked once with the unsuppressed modes no longer experiencing saturation (hight is now affected by the base step). N.B. the y-axis in the right is scaled by $10^{-3}$ as indicated.}
\end{figure}
6.3 Measurements

By weakly\(^5\) modulating the MLL at 0.2 GHz we could obtain a more accurate estimate (\(\leq 0.001\) GHz uncertainty) of the OSA’s FSR (\(\nu_{\text{FSR-OSA}}\)). Using the measured OSA carrier spacing \(a\) and sideband spacing \(b\) we can calculate \(\nu_{\text{FSR-OSA}}\) as

\[
\frac{a}{b} \cdot 0.2\,\text{GHz} = \nu_{\text{FSR-OSA}}
\]

This estimate found \(\nu_{\text{FSR-OSA}} \sim 2.995\,\text{GHz}\) which Dr. J. Woods then set as the reference modulation frequency of the MLL. By viewing the MLL output on the autocorrelator he\(^6\) could optimise the length of the cavity of the MLL to make its FSR (\(\nu_{\text{FSR-MLL}}\)) match that of the OSA. The autocorrelation pulse is namely narrower when the modulation frequency \(\nu_m\) driving the MLL is closer to \(\nu_{\text{FSR-MLL}}\). Hence \(\nu_{\text{FSR-MLL}}\) can be matched to \(\nu_{\text{FSR-OSA}}\) by minimising the autocorrelation pulse at \(\nu_m \sim 2.995\,\text{GHz}\).

With the MLL now properly mode-matched into the OSA the estimation of the frequency-jitter could commence. The MLL was modulated at +35 dBm around 2.995 GHz ± 1.5 MHz in steps of 0.1 MHz. Per \(\nu_m\) value the transmission through the mode-matched OSA is recorded. This constructs a 2D transmission plot against the OSA cavity scan and MLL driving frequency. This transmission measurements can then be compared to a simple transmission model to retrieve for interpretation.

A transmission model by Dr. J. Woods was used for preliminary interpretation. It modelled 35 comb-lines as Gaussians lineshapes with a FWHM of 1 MHz. Each comb-line is separated by \(\nu_{\text{FSR-MLL}}\) with an random added frequency-jitter bounded to \([-500, 500]\) kHz. These comb-lines are then convolved with a sinusoidal function oscillating at \(\nu_{\text{FSR-MLL}}\) from 0.5 to 1.0 to include the effect of the 3 dB spectral modulation effect of the imperfect AR coating on the laser’s intra-cavity\(^27\). The scanning of the OSA is modelled by convolving the comb-lines with a moving Gaussian window of FWHM at 4.8 MHz, the same as the measured FSR of the OSA. This simulation and the accompanying data taken by Dr. J. Woods are shown in figure 6.5.

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\(^5\)We want to operate in the weakly perturbed regime as it behaves similar to the CW lasing regime.

\(^6\)As Dr. J. Woods had completed his PhD on this experimental set-up, it was judged as more practical to have him tune the length of the flimsy MLL cavity.
CHAPTER 6. MODE–LOCKING CHARACTERISATION

Figure 6.5: Normalised measurement (left) and simulation (right) of the Fabry-Perot (FP) OSA mode-matched with the MLL around a modulation frequency of $\nu_m = 2.995 \text{ GHz} \pm 1.5 \text{ MHz}$ in steps of 0.1 MHz at +35 dBm. The convolved OSA signal is plotted at various $\nu_m$ (MLL Mode Spacing Detuning) across a full OSA scan (FP Scan). The simulation shown used 35 comb-lines of 1 MHz FWHM with $\pm 500 \text{ kHz}$ frequency-jitter. In both cases, zero detuning represents 2.995 GHz. The simulation, measurements and figures are fully credited to Dr. J. Woods.

Dr. J. Woods model does not give a full representation of the mode-matched MLL spectrum as measured by the Fabry-Perot (FP). Therefore an attempt was made to improve the simulation using equation 6.1. Due to time constraints and a transfer to another project this simulation was not completed hence not implemented. The theoretical background however can be found in appendix C where we describe how we can account for PCML more accurately. Analysis on frequency-jitter of the MLL thus remains limited to the estimation by Dr. J. Woods model.

What we can infer from figure 6.5 is that the spectral resolution of each of the 35 comb-lines has an upper-bound of 1 MHz: the selected frequency-jitter upper-bound in the simulation. This 1 MHz jitter-bound is out of 102 GHz as the spectral span of 35 comb-lines is $34 \cdot 2.995 \text{ GHz} \simeq 102 \text{ GHz}$. This shows the frequency-jitter falls within the $\leq 1 \text{ MHz}$ linewidth and $\leq 1 \text{ MHz}$/Experiment drift requirements for typical atomic experiments.
Chapter 7

Laser Frequency Ruler

The shortest optical pulses measured on the autocorrelator were 20 \( ps \) long with a corresponding spectral FWHM of 0.16 \( nm \) (77.6 \( GHz \)) about 786 \( nm \) on a Yokogawa AndoAQ-6315A (0.05 \( nm \)) monochrometer OSA. This gives an time-bandwidth product (TBP) of \( \sim 1.55 \), a factor of 3.5 larger than the time-bandwidth limit for a Gaussian pulse (0.44). This also marks a change from the measured TBP in Dr. J. Woods thesis [27] which was found to be around 13 without injection locking and 1.44 with side-mode suppression from injection locking. Given we were not injection-locking the MLL laser in the most recent characterisations, this sudden improvement sparked the interest to repeat some of the characterisations in [27], in particular the beat signal analysis experiment therein. This chapter reports on the progress towards this experiment made by the author.

As some of the electrical & optical components had been re-purposed in other experiments, parts of the mode-locking laser experimental set-up had to be re-purchased and re-built from bare components. The original components were also sometimes replaced with other parts as they were still in use in other experiments or lost. These changes to parts might play a role in the improved performance of the MLL and are therefore documented in this chapter.
7.1 Phase-Locking to a Mode-Locked Laser

As the MLL was intended to function as a frequency ruler for multiple CW lasers, a triple beat-signal experiment was set-up to measure the beat-signals between two ECDL’s and against the MLL. The beat-signal between an ECDL and the MLL would be converted into an error signal for phase-locking the ECDL to the nearest comb-line of the MLL. By having each ECDL phase-lock to a comb-line on opposite end of the MLL spectrum, the measurement of the beat-signal between the two ECDL’s would provide insight on the stability and drift of the MLL and the attainable phase-lock of a laser onto it. Hence in summary, the beat-signal experiment aims to quantify the stability of the MLL as a frequency-ruler and establish the level of frequency-jitter between two MLL-referenced CW lasers.

7.1.1 Experimental Set-Up

The two ECDL’s discussed in [27] and briefly in section 4.2 are driven by a dual diode laser driver and set to operate in CW. By measuring their wavelength on a wavemeter their grating and temperature are coarsely tuned near the edges of the MLL spectrum. The injection current is initially set near the maximum value as discussed in section 4.2 to get the strongest possible beat-signal. Driving the MLL is briefly summarised in section 4.3. N.B. the amplifier used for the MLL is a different model than used for the two beat-signals on photodiodes 1 & 2.

Several beam-blockers can be toggled on/off to select which laser is viewed on the wavemeter. They also control which laser is viewed on the Fabry-Perot OSA (FP-OSA). Note that the two ECDL’s have a lower finesse on the FP-OSA as they are incident at a different angle on the FP-OSA steering mirrors. One beam-blocker can be used to switch on/off the injection-locking the MLL by ECDL-1. The injection-locking is achieved by using the second port of the FOI in-front of the MLL. Injection-locking would be of interest at a later stage to fix the carrier-envelope-offset of the MLL.

Post optical isolation, both ECDL’s are incident on a polarising beam splitting cube (PBSC) to filter their polarisation. Using a half-wave plate ECDL-1 is given s-polarisation whilst ECDL-2 left at p-polarisation. Their respective beams are combined onto a non-polarising beam splitting cube (BSC) with one port travelling to the FP-OSA. The other port takes both polarisations to the optical circuit seen and described in [27].
7.1. PHASE-LOCKING TO A MODE-LOCKED LASER

The MLL beam reaches the upper-left BSC at a diagonal \( \left( \pm \sqrt{2} \right) \) polarisation. Both the MLL and the two ECDL’s are thus incident on the PBSC that optically separates PD1 and PD2 from each other. By the principle of superposition, the MLL polarisation contains equal parts of \( s \) and \( p \)-polarisation leading to PD1 measuring a MLL ↔ ECDL-1 beat-signal and PD2 a MLL ↔ ECDL-2 beat-signal. Both photodiodes would use a 50 mm plano-convex lens to get a better signal. PD3 would measure the ECDL-1 ↔ ECDL-2 beat-signal by rotating the \( s \) and \( p \)-polarisations 45° and passing it through a PBSC. To measure a \( \gtrsim 40 \) GHz beat-signal, a very fast photodiode is needed which whilst not secured we narrowed down to three potential models. All three fast photodiode contenders would require a 75 mm plano-convex lens placed in-front of PD3 which was already installed pre-emptively.

After some filtering of the beat-signals on PD1 and PD2, they would pass onto a custom Optical Phase-Locking Loop (OPLL) built and modified by Dr. M. Himsworth based on the Dr. J. Appel design [85]. These OPLL’s provide up to 7 GHz of bandwidth for phase-locking a CW laser to the MLL comb-line meaning our ECDL’s could be phase-locked at any frequency in-between two adjacent (3 GHz apart) comb-lines of the MLL’s spectrum. It is the utilisation of these large bandwidth OPLL’s with a MLL that would provide atomic physics experiments with unprecedented resolution and bandwidth on the laser’s frequency.

The custom OPLL had an internal 30 MHz reference oscillator \( \nu_{LO} \) along with an external reference oscillator port for locking the ECDL’s at any other frequency-offset from the nearest comb-line. With a frequency reference divider of \( R = 2 \) and feedback divider of \( N = 64 \), using the internal oscillator would provide phase-lock at \( \nu_{lock} = \frac{N}{R} \nu_{LO} = 960 \) MHz as beat-signal. The high and low-pass filters were therefore chosen to filter around this frequency.

In practice, these OPLL’s generate two error signals separated that differ in their frequency. The lower frequency ‘piezo’ error signal \( \epsilon_P \) is fed to the piezo-port on the ECDL to drive the grating whilst the higher frequency ‘current’ error signal \( \epsilon_L \) is fed to the current port. Due to the observation in [27] of enhanced electronic noise using the error-signal port on the ECDL laser-drivers, a custom design was considered where the error-signal would by-pass the driver and be electrically isolated using a diode D. This proposal along with the beat-signal experimental set-up is shown in figure 7.1. Further documentation on these OPLL’s can be found in the doctoral thesis [86]. The instrumentation used in this set-up can be found in table 7.1.
Figure 7.1: Experimental set-up for the proposed repetition of the beat-signal experiment from [27]. Two continuous wave External Cavity Diode Lasers (ECDL-i) are made to beat against each other on Photodiode PDD3 and against the Mode-Locked Laser (MLL) on Photodiodes PDi, i ∈ {1, 2}. The latter beat-signals are amplified (AMP), filtered by a Low-Pass Filter (LPF) and a high-pass filter (HPF) before converted into two error signals using an optical phase-locking loop (OPLL). The higher frequency current Error signal $\epsilon_L$ is fed into the Laser Driver’s (L-Driver) Current Controller (CC) as shown in the lower right corner schematic. The electrical Diode D provides electrical isolation of the error signal to the Laser Diode (LD), with a safety Resistor R to limit the error-current. The lower frequency piezo error signal $\epsilon_P$ is forwarded to the piezo-driver P-Driver controlling the grating of the ECDL. The higher beat-signal on PD3 is viewed directly on a Radio-Frequency Spectrum Analyser (RFSA). Coarse tuning of the ECDL’s use a wavemeter whilst tuning of the MLL uses a RF signal generator (RF-S) which combines with a CC onto a bias tee (BT). A separate Temperature Controller (TC) provides a thermal lock on the MLL. The triple beat-signal generation uses Faraday Optical Isolators (FOI), [polarising] beam splitting cubes ([P]BSC), half-wave plates ($\lambda/2$) and beam-dumps (BD). Removing beamdump * takes ECDL-1 light to an polarisation spectroscopy set-up. Measuring the time-bandwidth product of the MLL is done using an Autocorrelator (A) a monochrome-ter optical spectrum analyser (M-OSA). A Fabry-Perot Optical Spectrum Analyser (FP-OSA) is also used for estimating the frequency-jitter using a Mode-Matching Lens (MML) on a linear translation stage.
### 7.1. PHASE-LOCKING TO A MODE-LOCKED LASER

<table>
<thead>
<tr>
<th>Brand &amp; Model</th>
<th>Instrument</th>
<th>Comments</th>
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<tr>
<td><strong>Thorlabs</strong></td>
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<tr>
<td>SA200-6A</td>
<td>Optical Spectrum Analyser</td>
<td>3 GHz Confocal Fabry-Perot</td>
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<td>SA201</td>
<td>SA200-6A Driver</td>
<td>'Driver'</td>
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<td>LDC 202C</td>
<td>Current Controller</td>
<td>'CC'</td>
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<td>200C</td>
<td>Temperature Controller</td>
<td>'TC'</td>
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<td>ITC502</td>
<td>Dual Diode Laser Driver</td>
<td>'L-Driver'</td>
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<tr>
<td>ZRL-1150LN+</td>
<td>Low Noise RF Amplifier</td>
<td>35 dB gain</td>
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<tr>
<td>LPF-VLF-1000+</td>
<td>Low-Pass Filter</td>
<td>DC to 1 GHz</td>
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<tr>
<td>HPF-SHP-800+</td>
<td>High-Pass Filter</td>
<td>750 MHz to 3 GHz</td>
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<tr>
<td>ZX85-12G+</td>
<td>Bias Tee</td>
<td>0.2 to 12 000 MHz</td>
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<td><strong>Tektronix</strong></td>
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<tr>
<td>TDS 1002C-EDU</td>
<td>Oscilloscope</td>
<td>60 MHz, 2 channel</td>
</tr>
<tr>
<td>DPO4104</td>
<td>Mixed Domain Oscilloscope (RF Spectrum Analyser)</td>
<td>1 GHz Oscilloscope (DC to 6 GHz)</td>
</tr>
<tr>
<td><strong>N/A</strong></td>
<td>Optical Phase Locking Loop</td>
<td>Custom design by Dr. Himsworth [86]</td>
</tr>
<tr>
<td><strong>Rohde &amp; Schwarz</strong></td>
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<tr>
<td>SMC100A-B103-B1</td>
<td>RF Signal Generator</td>
<td>9 kHz to 3.2 GHz, thermally locked</td>
</tr>
<tr>
<td><strong>Hamamatsu</strong></td>
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<td>G4176-03</td>
<td>Ultrafast Photodetectors</td>
<td>450-870 nm with 30 ps rise/fall time</td>
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<td><strong>Yokogawa</strong></td>
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<td>AndoAQ-6315A</td>
<td>Optical Spectrum Analyser</td>
<td>0.05 nm resolution</td>
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<td>TQ8325</td>
<td>Wavemeter</td>
<td>1 pm resolution</td>
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<td>QDG8</td>
<td>RF Amplifier</td>
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<td><strong>Femtochrome</strong></td>
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<tr>
<td>FR-103XL</td>
<td>Intensity Autocorrelator</td>
<td>0.1 ps resolution</td>
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</tbody>
</table>

Table 7.1: List of instruments used for charactering the MLL. The labels in single quotes refer to the abbreviations used in figure 7.1, the experimental set-up used to characterised the MLL.
7.2 Quantitative Expectations

As stated in this chapter’s introduction, the MLL was found to have a spectral bandwidth FWHM of $\sim 0.16$ nm ($77.6\ GHz$) and optical pulses lasting $\sim 20$ ps in duration. This is a reduction by about eight\(^1\) from the TBP reported in Dr. J. Woods’s thesis ([27]—the same MLL) where pulses of $\sim 21$ ps in duration were observed with a spectral bandwidth of 1 nm. The improvement is thus mainly seen in the narrowing of the spectral bandwidth i.e. better ‘utilisation’ of the comb-lines in the optical pulse.

To clarify, the change in spectrum and pulse duration suggest that comb-lines between 0.08 nm to 0.5 nm apart from the center of the MLL’s spectrum made a negligible collective contribution to the MLL’s pulses during Dr. J. Woods analysis. Energy was being dumped in these comb-lines, but a low coherence was established compared to the inner 26 comb-lines\(^2\) (about the MLL’s spectrum’s centre). In our measurements however, modulating the MLL had reduced the number of comb-lines above the $-3$ dB line closer to the fourier transform limit, increasing the power of these 26 comb-lines but not substantially altering the pulses duration.

When we compare the spectral FWHM in this analysis to the previously observed $\sim 1$ nm ($485.3\ GHz$) spectral FWHM, we see a reduction in the maximum comb-line separation from 162. Given the candidates for PD3 however, for the beat-signal experiment we were aiming around 40 GHz (13 comb-lines) of separation between the ECDL’s. The beat-signal would then pass through a frequency divider\(^3\) to place it within the 6 GHz bandwidth of the RF spectrum analyser (RFSA).

With regards to the MLL $\leftrightarrow$ ECDL$-i$, $i \in \{1, 2\}$ beat-signal, previously in [27] a FWHM ($-3$ dB) of 250 $\pm$ 100 kHz was observed when taking single-shot measurements. Dr. J. Wood postulated in that thesis that frequency-jitter caused the broadening of the beat-note up to 65 MHz when increasing the sweeping/measurement time on the RFSA. Unexplained peaks were also reported in those RF spectrums with varying responses to the detuning of the MLL modulation frequency from the MLL’s FSR. We refer the reader to the details in Dr. J. Woods thesis on his defense of his postulate.

\(^1\)Without injection-locking of the MLL.
\(^2\)A 77.6 GHz spectral FWHM MLL about 786 nm consists primarily of 26 comb-lines.
\(^3\)The specific model intended to use is not included in table 7.1 as we had not selected one for purchasing.
These anomaly peaks were previously situated 10-100 $MHz$ from the beat-note in [27] and had smaller amplitudes. Likewise, a few $MHz$ linewidth beat-note is expected on PD3 with the narrowest linewidth observed when both piezo and current error-signals are implemented. It is possible these features could also be seen in our measurements.

Given the lower TBP, we were also anticipating a similar/narrower beat-note. In particular, as the ECDL’s error-signals would bypass the laser’s driver, we were anticipating an improvement of the ECDL-1 ↔ ECDL-2 beat-note to $\sim 100 \ kHz$. The broadening of the MLL ↔ ECDL$_i$, $i \in \{1, 2\}$ beat-note might also drop to $1 \ MHz$. If there’s no narrowing, [27] still suggests frequency-locks on the order of tens of minutes, providing a stable frequency-lock and sub-$MHz$ frequency-jitter for our atomic physics experiments.

Summarising the improved performance: from Dr. J. Woods frequency-jitter postulate we can suggest that the changes in spectrum and pulse duration might be as a consequence of a reduced frequency jitter in the comb-lines. Given the primary change between the two theses are instrumentation, the set-up used in our measurements introduce less frequency jitter then in Dr. J. Woods measurements in his thesis. The changes are summarised below.

### 7.3 Changes to Set-Up

The modulation of the MLL in [27] used a $+3 \ dBm$ Voltage Controlled Oscillator (VCO, Mini Circuits ZX95-3600+) as modulation source followed by a bandpass filter (Mini Circuits VBF-2900+) and about $+27 \ dBm$ amplification (Mini Circuits TVA-11-422). Selecting the modulation frequency used a custom-made voltage source. As the research group acquired new equipment and had lost sight of the originally used components, a different modulation source and amplifier were used in the work of this thesis. A change in performance of the MLL is thus likely to stem from the change in instrumentation, as supported by our discussion in the previous section 7.2.

It is important to note the RF Signal Generator is temperature stabilised internally whilst the modulation source in [27] was not. From the ZX95-3600+ datasheet we know there is also a temperature sensitivity on the VCO’s output frequency and power. Likewise the bandpass filter and amplifier also demonstrate a temperature sensitivity which must be considered when comparing the performance of the MLL to its performance in [27]. The presence of a grounding loop in the previous set-up should also be considered.
The optical schematic shown in figure 7.1 also differs from [27] with the presence of a permanent wavemeter and mode-matched FP–OSA. The principle of using different polarisations to generate three beat-signals however remains the same.

Future changes to the set-up include a repetition of the MLL characterisation at 780 nm from the current 786 nm to make it relevant for our Rubidium-based atomic physics experiments. This would require cooling the MLL below condensation temperatures meaning the MLL would operated in a nitrogen-flushed chamber as discussed in [27]. The mode-matching data would also need to be repeated to match the 26 or new number of comb-lines stemming from different lasing conditions. It could then be of benefit to make use of the improved model detailed in appendix C to analyse this data and possibly learn more about the frequency-jitter of the comb-lines. As the project was terminated\(^4\) however, the model has not been extended to a numerical computational format.

\(^4\)A fellow PhD student fell ill leaving their project underdeveloped and abandoned. With a subsequent upcoming deadline imposed by the research funders, the supervisory team called for an urgent transfer of the author to that project leading to this project’s termination.
Chapter 8
Discussion

8.1 Continuous Wave

Before discussing the scraps of data and observations with regards to our MLL, we will first comment on our findings regarding the CW characterisations. Some of the unexpected and undesired behaviours seen at CW might correlate namely with the low level of performance seen in the MLL. Whilst a large chunk of our upcoming analysis is speculative, it will nevertheless set the current level of understanding of these specific laser systems and generate hypotheses that might be of interest for further pursuit.

To recall and summarise, the anomalous/undesired features observed in the Eagleyard DFB CW laser are

1. Jumps in the measured Power-Current ($P-I$) relation.
2. Broadened mode-hop regions from current variations.
3. Electronic driver grounding problems.
4. Slow temperature impulse response.

The remaining unwelcome observations in CW relate to the Eagleyard RWL. These observations described in [27] along with our own subsequent observations can be summarised as

5. Unstable lasing at particular driving currents.
6. An notably delicate laser cavity.
Having investigated the DFB and noticing correlations between the responses of the two lasers, some of the features could stem from a common source. If so, the same proposed solutions might mitigate the driving issues seen with the MLL and improve its performance.

1) – The most puzzling observation by far are the jumps in the $P-I$ datasets. Our literature review from chapter 3 suggests there should be a linear relationship\(^1\) between the laser’s output power $P$ against injection/driver current $I$ above some threshold $I_{th}$. However, our $P-I$ datasets at different temperatures consistently found jumps with both sides of the line appearing to be parallel to the laser’s datasheet. Given the external differential quantum efficiency $\eta_{ext}$ is defined by the gradient of the $P-I$ dataset, this suggests $\eta_{ext}(I)$ is constant apart from dipping at these shifts.

The natural intuitive suggestion is to blame mode-hopping between two longitudinal modes for these non-linear $P-I$ regions. However, when we compare figure 4.1 with figures 4.2 and 4.3 we see the mode-hop regions do not match for all datasets as we remarked on in subsection 4.1.3. Additionally, this does not explain why the power does not return back to the pre-mode-hop line. We can therefore reject longitudinal mode-hops for causing the non-linearities $P-I$ datasets.

We note that the constant offset post the dataset jump suggests a few mW of optical output power is permanently leaking out of the cavity. This is not characteristic of a standard longitudinal mode-hop where the $P-I$ relation on either side of the mode-hop region are co-linear. One hypothesis we therefore consider are mode-hops between two transverse modes. As different transverse modes have different amounts of their electrical field propagate outside the gain region (different confinement factor), a mode-hop to a different transverse mode could explain why the power loss remains. This is neither in contradiction with our findings in figures 4.2 and 4.3 as for a Fabry-Perot cavity the transverse modes overlap with the longitudinal modes in wavelength. Therefore no wavelength change would be visible.

One possible cause for transverse mode-hops could arise from thermal lensing. This transverse effect is known to affect the intensity distribution of light and could explain why the modes change at certain injection currents (temperatures of the gain medium). If so, the $P-I$ datasets should share a common current instances of non-linearity between successive datasets.

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\(^1\)Limited by the thermal roll-over effect as discussed in section 4.2.
Another possible cause for the non-linearities are the drifts in output power as demonstrated during out impulse response tests in sub-section 4.1.6. If we left the laser to stabilise for different times prior to each individual measurement, and/or if each measurement takes different times, we could have introduced a systematic error. The drift-induced systematic error would however change the location of the non-linearities between successive datasets. This hypothesis can be tested against the transverse mode-hopping hypothesis. Unfortunately, due to time constrains this test couldn’t be done as we couldn’t retake $P-I$ measurements.

2 & 3) With regards to the broadened mode-hop regions, our main suspect are the grounding loops identified and discussed in subsection 4.1.5. Whilst we managed to eliminate the grounding loop through the optical bench, one possible signal ground to earth connection remained through the laser’s driver. Therefore a different driver was used for the ‘Raman laser’ in the atomic physic experiments as discussed in section 16.2. A coarse characterisation of that laser by a fellow PhD student confirmed the frequency-jitter had reduced to levels within the linewidth of the Thorlabs OSA.

4) The fourth listed feature demonstrates a thermal stabilisation problem of the laser. Ideally, within a few seconds the laser’s temperature should be stabilised. Figure 4.5 however suggests the PID settings were not calibrated properly as the latency in stabilisation exceeded 30 min. Tuning the gain coefficients could have helped confirm and possibly thereby optimise the stabilisation. Such as optimisation although not part of this work is generally of interest in new experiments.

The initial power fluctuations in figure 4.5 from the thermal impulse we hypothesise to possible impedance matching causing a ‘ringing’ of the driver voltage when switching on the laser driver. More prevalent in older generation and aged electronics, a transfer to another (new) driver in hensight might also remove this feature and save time. Indeed, the ITC502 driver we used for characterising the DFB broke down a few months later, suspected from old age by the manufacturers.

The static atomic physics experiments initially used an ITC502 driver without problems, but eventually we started using a different driver (c.f. section 16.2) for more precise measurements. It continued to be required however for the laser to be left on the same settings for a good half hour or longer suggesting temperature stabilisation issues might have persisted.
Apart from these four observations, for the DFB we also found that the results from section 4.2 tie into the drop in output power drops when the laser is driven at higher temperatures. In general, driving a diode laser at lower temperatures lowers the threshold $I_{th}$ and provides higher optical powers. The thermal roll-over is also pushed to higher currents as the active layer overheating is reduced. Note however that the temperature stabilisation is less robust closer to room/nominal temperatures (e.g. figure 4.2) as the error-signal is reduced. So running the diode laser at colder temperature (taking consideration for possible condensation and staying clear from nominal laboratory temperature) is proposed.

The tuning parameter for temperature was also confirmed to be larger then for current demonstrating why the injection current tends to be modulated for frequency-stabilising/locking the laser. Temperature tends to be used for coarse wavelength control with current for finer control. We set the current several modulation depths away from $I_{th}$ to allow for frequency-stabilisation at the highest possible output powers as discussed in 4.2.

8.1.1 CW Effects on the MLL

Moving onto the RWL used for mode-locking, recall we hypothesised four suggestions regarding the laser’s instability in CW back in section 4.3. Using our findings from characterising the MLL outside CW along with the DFB in CW we can start considering these and new suggestions.

The partial optical feedback hypothesis: Even the FOI surface can provide a few percent reflection of light. These back-reflections in CW can be significant enough to cause instability from the beat-note between the incident and reflected beam. A racing-condition would form between the two beams leading to uncontrolled (unstable) injection locking.

This hypothesis is related to the second suggestion: the imperfect AR coating of the output’s facet of the RWL. Apart from lowering the geometrical mode competition, this would also affect the spectrum of the MLL. The reflections from the AR coating would namely result in at least two pulses oscillating inside the cavity at a time. These pulses might not start off at the same intensity, but could over successive cavity round-trips equilibrate in power\(^2\).

\(^2\)Especially as diode lasers have high gain active regions.
To clarify, one pulse would represent the light transmitted through the AR coating with the other satellite-pulses being generated from reflection(s). Over time, some of these satellite-pulses could acquire the same intensity as the transmitted pulse hence become indistinguishable from the ‘original’ pulse. The term ‘satellite-pulse’ is therefore reserved for theoretical analysis.

Satellite-pulses of sufficiently high intensity could give the MLL characteristics akin to a CPML laser. Most likely however the spacing between the pulses against the satellite-pulses and copies of themselves wouldn’t be uniform giving the MLL some characteristics of a PCML laser. In this case the time-domain would manifest itself closer to the intensity profile seen in 5.1 (a), i.e. closer to a multi-mode laser with a pseudo-random phase-relationship between modes. There is namely no mechanism that sets up a definite phase between the pulses and sub-pulses. The cavity modulation (AM, FM or PM\(^3\)) only sets up a definite phase-relationship through the side-bands within a single pulse. Hence each (sub-)pulse comb-lines contributing to a different part of the MLL’s full spectrum would qualitatively be self-coherent but not to adjacent comb-lines from the other (sub-)pulses.

The etalon has also been put in question during mode-locking. In CW the etalon was found to reduce the instability. However, the MLL performed best without the etalon. Hence contrary to our third hypothesis on *intra-cavity dispersive elements*, the lack of a dispersive element might actually have improved the performance of the MLL, and not only from eliminating the reflections from the etalon’s surface.

To explain this, we recall that in subsection 3.4.4 when discussing the tuning of an ECDL, we commented on the requirement for the etalon to rotate appropriately with changes to the grating. The equivalent requirement for keeping the mode number constant in a DFR we showed in subsection 3.4.3 to lie with the injection current. Hence tuning a RWL requires the etalon to be rotated appropriately to avoid mode-hopping. But to keep the FSR of the MLL constant, the etalon if left inside the cavity was not altered.

During mode-locking, the injection current is driven between two levels to generate sidebands with a definite phase relationship\(^4\). Given the high driving power of the MLL, the varying current together with a static etalon could induce mode-hops of the carrier frequency of a pulse.

\(^3\)Standing for *Amplitude*, *Frequency* and *Phase Modulation* respectively.

\(^4\)Who in turn also generate their own sidebands cascading the generation of the comb.
The energy leaked out of the pulse from the mode-hop\(^5\) could amplify easily from the high gain factor of the diode laser over a few round-trips leading to the formation of a second pulse of the same intensity. Similar to the satellite-pulses from reflections, these satellite-pulse would also not be guaranteed to be coherent to the other pulses (or any present sub-pulses within themselves) thus lowering the performance of the MLL. Whilst not tentatively confirming, the observation of a coherence peak in [27] suggests elements of PCML.

The fourth hypothesis regarding the RWL cavity alignment/instability has manifested itself in another way. The cavity of the MLL became very sensitive to vibrations and was prone to drifting even in the temperature stabilised laboratory and nitrogen floating optical bench. Our mode-matching into the OSA as discussed in section 6.2 demonstrated heightened sensitivity to drift and vibrations to the point where the Thorlabs mirror mounts had to be swapped out for higher class Newport mounts. The cavity of the MLL however still uses Thorlabs mounts hence continues to be delicate. Our previous observation of 'fake' lasing therefore does not come as a surprise.

With regards to the optical pulses, these hypotheses can introduce 'breathing' of the \(\nu_p\) cavity modes as the cavity’s FSR and/or the common offset frequency \(\nu_{CO}\) is fluctuating. Given the comb-lines \(\nu_q\) are generated from the cascading side-band generation of the cavity modes, the mode-spacing \(\nu_m\) of the comb becomes prone to fluctuations. Hence the cavity could experience some 'breathing' \textit{i.e.} variations in \(\nu_m\) and 'shifts' from variations in \(\nu_{CO}\). This is detrimental for the MLL’s coherence.

Analogous to the cavity instability, we suspect that 'breathing' was introduced from electronic noise in the original set-up of the MLL. In section 7.3 we compare the previous and substituted driving system with the latter which gave an improved performance of the MLL. Not only can we identify a grounding loop through the optical bench, but the datasheets of the VCO, filter and amplifier confirmed a temperature sensitivity. In the previous work [27] these components were not temperature stabilised like the substituted driving system was. The electronic noise from a grounding loop we demonstrated in subsection 4.1.5 introduces frequency-jitter over 20 mA tuning range in CW, the same range as Dr. J. Woods observed in his characterisation of the RWL in CW [27]. During mode-locking this would directly affect \(\nu_m\) and \(\nu_{CO}\) leading to lower coherence.

\(^5\)E.g. a transverse mode-hop as hypothesised in 8.1.
8.2 Mode-Locking

From our MLL’s review in section 5.2 we know our MLL is performing poorly. Instead of $\geq THz$ level bandwidths, our latest measurement demonstrated under 100 $GHz$ spectral bandwidth. Likewise, our pulses were longer in duration than the first demonstrations of mode-locking in ECDLs. Hence only 26 out of typically thousands of comb-lines are above the $-3 dB$ line.

Apart from possible PCML from CW effects, there are also some suggestions that a chirp increased the TBP of the MLL. In subsection 5.3.2 we namely discussed how a chirp is introduced by modulating a laser. This chirp limits the minimum experimentally attainable TBP, reviewed further in subsection 5.1.4. One such observation supporting thus chirp postulate is the reduction of the TBP after the substitution for new and temperature stabilised driving electronics (previously not stabilised). The grounding/driving noise reduction$^6$ from newer and higher specification instrumentation would reduce the chirp in the modulation of the MLL thus enhancing its coherence.

The chirp postulate is in agreement with our review from section 5.2 which suggests the number of coherent comb-lines might not have changed. According to the extended three-mode frequency-domain model namely, the width of a pulse is inversely proportional to the number of comb-lines $N$ of the MLL’s spectrum. Hence we suspect the spectrum in [27] falsely appeared wider with the majority (162-26=136) of the comb-lines above the $-3 dB$ line not actually contributing to the optical pulse. This is supported by the observation that the TBP reduction primarily came from the decrease in spectral bandwidth. The reduction in the TBP during optical injection locking (OIL) also supports our postulate as OIL can introduce significant chirp and/or linewidth reduction$^7$.

Whilst our frequency-jitter estimate confirmed these comb-lines collectively operate within the 1 $MHz$ linewidth and drift requirements for atomic physics experiments, they have not ruled out linewidth broadening below 1 $MHz$. The persistence of the coherence spike after the change in instrumentation suggests the TBP problems might not only be limited to the driving electronics but lie with the optical cavity itself. A coherence spike is featured in PCML which the CW effects on the MLL do enable.

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$^6$Evidence of linewidth broadening appeared in several measurements in Dr. J. Woods thesis: [27].

$^7$C.f. sub-section 5.3.2 for details on OIL
Finally, depending on the application of the laser, the MLL might not be suitable as a frequency ruler. Typical applications of cooling, repump or depump lasers are not phase sensitive hence can be frequency stabilised onto the MLL. But for phase-sensitive atom-laser interactions such composite pulses by Raman lasers, the unconfirmed and unlikely phase coherence of the MLL makes it unsuitable as a reliable frequency ruler.
Chapter 9

Conclusion

Characterisation of an *Eagleyard* EYP-DFB-0780-00080-1500-BFW01-0005 diode laser revealed \( \sim 70\% \) of the optical power can be utilised, and that at least two optical isolators (FOI) are needed for stable operation. The limit stems from the laser’s polarisation purity along with the insertion-loss of the FOI’s. The brutto output power is in agreement with the datasheet, giving an external quantum efficiency of \( 0.576 \pm 0.004 \). This is reduced to a netto output power efficiency of \( \sim 0.27(1) \) when using two FOI’s.

Higher powers can be achieved at lower laser temperatures by current-shifting thermal roll-over. We therefore recommend to drive diode lasers up to thermal roll-over at the lowest possible temperature for the desired wavelength. Prepare for possible condensation and consider avoiding the nominal temperature of the laboratory as it reduces the error signal in the servo feedback.

Tuning parameters were measured to be \( 0.06(1) \, nm/K \) (29.6 \( \pm \) 0.5 \( GHz/K \)) and \( 0.0030(4) \, nm/mA \) (1.5 \( \pm \) 0.2 \( GHz/mA \)). This is in good agreement with the datasheet specifications. The hopping of the mode-hop is estimated at \( \sim 0.15 \, K/mA \) (temperature-induced) and \( \sim 6 \, mA/K \) (current induced). These are coarsely in agreement with each other. A cavity length of \( 0.55(5) \, mm \) is estimated with a mode-separation (free spectral range) of \( 75(5) \, GHz \). The laser threshold current \( I_{th} \) was found to shift by \( \sim 0.45 \, mA/K \).

Power-current measurements featured non-linearities that originate from either thermal lensing induced transverse mode-hops or power drifts induced from an inadequately tuned thermal-stabilisation feedback. The former would introduce non-linearities at specific current values whilst the latter would randomise the systematic error. Future successive measurements could help identify the right hypothesis. Identified electronic grounding loops and an impulse response test support the latter hypothesis, further supported by the observation that the laser diver breaking down a few months later.
Continuing the doctoral work of Dr. J. Woods on his custom-assembled actively mode-locked laser (MLL), we characterised his *Eagleyard* EYP-RWE-0790-04000-0750-SOT01-000 diode laser placed in an custom external cavity. Due to lost instrumentation after an inactive period in the project, the electronic driving system was altered in the set-up. We suspect the instrumentation change plays a large factor in improving the time-bandwidth product (TBP) we measured in this work under the supervision of Dr. J. Woods compared to his doctoral work measurements of the same laser. Under Dr. J. Woods’s supervision, we measured a TBP of \(\sim 1.55\) with 20 ps pulses and a spectrum that is 77.6 GHz wide (26 comb-lines), measured when modulating the MLL at 2.995 GHz at +35 dBm. We confirm the frequency-jitter of the comb-lines to be sub-MHz level using a mode-matching set-up we implemented and a quantative simulation computed by Dr. J. Woods. This confirms the MLL can be used for frequency-dependent laser-atom applications such as laser cooling. We have not established if the MLL is suitable for phase-sensitive atom-laser interactions like composite pulses.

We suspect the enhanced TBP of the MLL to stem from three factors: (1) partial reflections of the FOI combined with an imperfect anti-reflection (AR) coating of the MLL’s intra-cavity, (2) the flimsy nature of the external cavity and (3) the chirp introduced by modulating the laser. These factors introduce variations in the comb-line spacing and common offset frequency of the MLL which reduces coherence. Factor (1) destabilises the laser from racing conditions between the outwards and retro-reflected light from the FOI’s surface, which we need to rotate to stabilise the laser’s frequency. The AR coating issue could also led to either colliding pulse mode locking and/or pulse clustered mode locking (PCML) with the observation of a coherence spike during autocorrelation measurements suggesting PCML. Regarding (2): we found the MLL’s external cavity was found to be highly sensitive to minor vibrations on the optical bench and demonstrated instability in Continuous Wave (CW) including frequent ‘false’ lasing during misalignments. When we attempted to mode-matching the MLL with a Fabry-Perot cavity we found the Thorlabs mounts used in the MLL’s set-up limited the mode suppression. The external cavity using a few Thorlabs mounts was thus prone to additional sensitivities to the laboratory environments. For (3) we note the previous instrumentation that Dr. J. Woods used in his work was not temperature stabilised and appears to have suffered from a grounding loop when comparing his instability data to our other diode laser characterisation. This increases the chirp of the MLL when modulated, explaining why a change in instrumentation or optical injection locking lowered the TBP.
Sub-thesis II

Speckle-enhanced Laser Characterisation
Chapter 10

Introduction

Monochromatic and coherent light propagating through or reflecting from disordered media give rise to complex matter-light interaction that produce light speckles. These speckles are the local extrema in a random granular pattern (called a speckle pattern) whose size and position depend on the specifics of the matter-light interactions. Small changes in these interactions produce substantial changes in speckle patterns making them useful in practical applications such as atom interferometers. One such application we investigate in this sub-thesis is high bandwidth and resolution laser characterisation.

Most of the work in thus sub-thesis will focus however on a different related question: investigating the reliability of speckles-wavelength reconstruction, a problem about orthogonality and uniqueness we approach from an ideal and simple mathematical model. By constructing a theoretical basis for solving related speckle problems we aim to discover statistical properties and pave a roadmap for future dynamical and more realistic studies of speckle patterns.

Before we discuss the physics of speckles, we first introduce the origins of speckles studies (section 10.1), describing our research motivational question (section 10.2). The novel speckle-based spectrum analyser under consideration (section 10.3) for future experimentation provides a strong characterisation tool namely, not only of interest to the previous sub-thesis I but also to characterising the various laser systems part of atom interferometers.
CHAPTER 10. INTRODUCTION

10.1 Classical Speckles

The speckle phenomena has been investigated since the time of Sir Isaac Newton on scintillation observations for stars and the lack of this twinkling for planets. Speckle patterns based on interference however really became an active field a few centuries later. Newton and Fraunhofer diffraction rings were well understood phenomenon before 1877 and at least since 1877 the finer speckle pattern associated with these phenomenon started to become of interest. The Quêtelets fringes i.e. radial\(^1\) granular speckle patterns from small particles on an smoother surface\(^2\) was namely sketched by K. Exner in 1877 [87] and later photographed by M. von Laue in 1914 as in figure 10.1.

![Figure 10.1: This figure shows one of the earliest work on modern Speckle analysis. (a) An sketch by K. Exner made in 1877 of the radial granular structure from an glass plate he breathed on using an candle as light source. (b) A photograph taken by M. von Laue from 1914 showing an diffraction pattern of an lycopodium powder covered glass plate using an carbon arc lamp as light source through a prism. The 420-430 nm wavelength region of the prism was used. Both figures were taken from reference [87].](image)

From this point onwards the nature of the speckle pattern was studied more detailed by looking at the statistical properties. In this work we make use of some of these derived results by using these statistical properties and sometimes adding more statistical assumptions ourselves. This avoids having to ”re-invent the wheel”, helps us remain in agreement with the literature and ensures we use the common notation of this research field.

\(^1\) N.B. Not all Quêtelets fringes are radial.

\(^2\) E.g. condensation from breathing on glass or using lycopodium powder on glass. An glass plate is smooth compared to the distribution of the smaller particles hence why we use the term "smoother".
10.2 Research Motivation

A common claim regarding speckle phenomena is that each wavelength will produce a unique speckle pattern allowing for the construction of ideal speckle based wavemeters and spectrometers with perfect wavelength/spectral resolution. But just like the claim of the uniqueness of the human fingerprint [88] or the commonly accepted uniqueness of the snowflake, the uniqueness of the speckle pattern against different wavelengths appears to be unfounded. We therefore investigate this claim’s bounds and limits as understanding this claim helps finding the practical limits in resolution and reconstruction reliability of wavelength/spectrum by speckle pattern reconstruction methods (from a wavelength pre-calibrated system).

A limit on the uniqueness would be detrimental to the reliability of practical applications in two ways: 1) lowering the attainable practical resolution and 2) enabling the mix-up of reconstructed variables (e.g. wavelength). Whilst we briefly look into limit 1) in this sub-thesis, we focus mainly on limit 2) as a mix-up of measurements renders the reconstruction totally unreliable compared to limit 1) which only effects the precision of measurements.

The problem is simple: Consider two optical fields of wavelengths $\lambda_1$ and $\lambda_2$ where $\lambda_1 \neq \lambda_2$. Let the fields be incident on an optically rough surface and made visible some distance away from the rough surface. Can we be certain that the generated speckle pattern $I_{\lambda_1+\lambda_2}$ does not match the speckle pattern $I_{\lambda_3}$ of a third optical field of wavelength $\lambda_3$? In essence, can the overlap of speckle patterns of wavelengths $\lambda_1$ and $\lambda_2$ reduce the reliability of reconstruction by matching a speckle pattern of a different wavelength $\lambda_3$?

If the difference in wavelength $\delta \lambda = |\lambda_2 - \lambda_1|$ is small we expect a correlation to manifest between speckle patterns which could be used to define the resolution of the speckle spectrometer and thereby study limit 1). The resolution would depend on the reconstruction algorithm, the speckle imaging resolution, $\delta \lambda$ and the medium’s correlation length $\xi$: the measure of constraint between variations (e.g. height) of neighbouring points of the media [89].

For large $\delta \lambda$ the overlapping speckle patterns could in principle still match that of a third wavelength $\lambda_3$ introducing a systematic reconstruction error (limit 2). The problem becomes even more difficult if we measure the spectrum of $N$ wavelengths: $\{\lambda_1, \lambda_2, \ldots, \lambda_{N-1}, \lambda_N\} = \Lambda_N$. Can the overlapping speckle-pattern by $\Lambda_N$ match the speckle pattern of a single wavelength $\lambda$?
Speckle spectrometers provide no guarantee the measured wavelength was not mixed up with other wavelengths. Can one wavelength’s speckle pattern match that of another one (at large $\delta \lambda$)? Or could the spectrum $\Lambda$ return a false measurement of a different set (cardinality $> 2$) of wavelengths? In this sub-thesis we investigate wavelength orthogonality of speckle patterns in relation to limits 1) and 2) and any other queries raised such as the role of roughness like the correlation length of the surface, possible filtering effects from image plane to interference region separation and phase correlations.

10.2.1 Orthogonality in Practical Devices

Both 1) and 2) are problems of orthogonality. If the speckle patterns of different wavelengths form an orthogonal set then it becomes impossible for $I_{\lambda_3} = I_{\lambda_1 + \lambda_2}$ even if $\lambda_3 = \lambda_1 + \lambda_2$. That would allow for two optical fields of wavelengths $\lambda_1$ and $\lambda_2$ to be distinguishable to the speckle spectrometer from any other wavelength $\lambda_3$. The resolution is then set by the degree of orthogonality (level of correlation) between speckle patterns by different wavelengths.

Consider a spectrometer (for multiple wavelengths detection) making use of the reconstruction of speckle patterns for determining the wavelength. If the speckle patterns of different wavelengths are shown to be statistically independent of each other, than we can think of them as forming a linearly independent spanning set. By normalising the set, any new speckle pattern could then be rewritten in terms of this basis as a linear combination.

If we index each speckle pattern $|I_{\lambda_i}\rangle$ by the corresponding wavelength $\lambda_i$, this vector space $V_I$ would be equivalent to a vector space $V_\Lambda$ of wavelengths constructed by taking the basis of $\mathbb{R}^N$, $N \in \mathbb{N}_0$ and scaling each basis vector with the wavelength of interest. Thus every vector $|\lambda_i\rangle \in V_\Lambda$ could than be identified with a vector $|I_{\lambda_i}\rangle \in V_I$. A speckle pattern reconstruction method then becomes a map between $V_I$ and $V_\Lambda$.

Apart from limitations imposed by calibration and resolution (limit 1), our main concern lies in a lack of statistical independence between speckle patterns (limit 2) for then this map between $V_I$ and $V_\Lambda$ is no longer unique risking jumbling during the reconstruction method. Limit 2) could also affect single wavelength measurements as the wavelength reconstruction from the speckle pattern may interpret a single wavelength $\lambda$ speckle pattern as that of multiple wavelengths $\Lambda_N$. Reliability of speckle spectrometers is therefore in question for single and multiple wavelength spectrum of light.
10.3 Speckle-based Spectro/wave-meters

10.3.1 Transmission - Multimode Fiber Medium

Speckle-based spectrometers have been made using a wide range of random dispersive systems, ranging from disordered photonic crystal lattices to Bragg-fibre arrays with modest resolution. One recent novel adaptation provides a competitive resolution speckle spectrometer with low-insertion loss. The set-up uses a multimode fibre as dispersive medium to provide a wavelength dependent spatial-to-spectral map through interference between speckles by different guided modes. This gives a spectral resolution $\delta\lambda$ ($8\, \text{pm}$ at $L = 20\, \text{m}$) that scale inversely with the length $L$ of the fiber [90, 91].

A laser is coupled into a polarisation maintaining single-mode fibre in series with the multi-mode fiber. The output of the fibre is then imaged on a camera to record the speckle pattern. For stability purposes, the multi-mode fiber is typically temperature stabilised and isolated from other sources of noise using a partial vacuum chamber or other infrastructure.

To map between $V_\lambda$ and $V_I$, the speckle spectrometer of bandwidth $\Delta\lambda$ records the speckle patterns $|I_{\lambda_i}\rangle$ for $N = \left\lfloor \frac{\Delta\lambda}{\delta\lambda} \right\rfloor$ known wavelengths $|\lambda_i\rangle$ where $i \in [1, \ldots, N]$. The $N \times N$ speckle patterns $|I_{\lambda_i}\rangle$ are then placed in the successive columns of a $N \times N \times N$ transmission matrix $T = (|I_{\lambda_1}\rangle \ldots |I_{\lambda_N}\rangle)$. Any future spectrum $|\Lambda\rangle$ of wavelengths within the spectrometers bandwidth $\Delta\lambda$ ($|\lambda_i\rangle \in V_\lambda$) can then be reconstructed from a measured speckle pattern $|I_{\Lambda}\rangle$ by inverting $T$ using the following equation: $|\Lambda\rangle = T^{-1} |I_{\Lambda}\rangle$.

As $T$ is typically asymmetrical, a Moore-Penrose (left generalised) inverse matrix is used in calculating $T^{-1}$. To reduce corruption in the reconstruction from the propagation of experimental noise, reconstruction algorithms typically modify $T^{-1}$ to the pseudo inverse of the truncated singular value decomposed matrix $T_{\text{trunc}}^{-1}$ [91]. Multi-variable data can then be further decorrelated using principle component analysis (PCA) [92, 93].

Not all pixels of speckle patterns are always sampled: if the average speckle size i.e. the spatial correlation length $\xi_I$ of a speckle pattern exceeds the size of a pixel, fewer pixels are needed in $T$ for an optimum spatial-to-spectral map. Also, $\Delta\lambda$ of fiber speckle spectrometers is limited to a $400\text{-}2400\, \text{nm}$ transmission window of the fiber’s silica medium which caps $\Delta\lambda$ [94].

---

3 Measured by an external spectrometer, limits precision of reconstruction.
10.3.2 Reflection - Ulbricht Sphere Medium

Reliable reflection speckle patterns can be made using lasers and integrating (Ulbricht) spheres\(^4\). The generation of speckle patterns using Ulbricht spheres has been demonstrated since at least 1989 \([95]\). In the scheme, a laser beam of wavelength \(\lambda\) is injected into the Ulbricht sphere whose inner lining is diffusively coated with a reflective paint. The optical path from the input and output ports\(^5\) is a complex spatial-to-spatial map involving multiple bounces inside the sphere. The output port is then imaged on a camera\(^6\).

If a square aperture of size \(l\) is placed a distance \(L\) from the camera, a spatial frequency limit\(^7\) \(\nu_s^+\) of the laser speckle pattern is imposed of

\[
\nu_s^+ = \frac{l}{\lambda L}
\]

The camera itself has a limit: the Nyquist (spatial) frequency which can be calculated from the spacing \(\delta x\) between individual sensors that make up the camera’s detector. The Nyquist (spatial) frequency is given by

\[
\nu_N = \frac{1}{2\delta x}
\]

To avoid the aliasing effect, the upper spatial frequency \(\nu_s^+\) is typically set to the the spatial Nyquist frequency \(\nu_N\) using an aperture in-front of the Ulbricht sphere’s output port. This is an important part as the question of orthogonality is affected by the placing of the aperture.

A similar calibration technique is used as to the multimode fiber speckle spectrometer, the calculation of the transmission matrix combined with a sophisticated computer algorithm to retrieve wavelengths in-between adjacent input calibration spectra \([93]\).

The theoretical formulation of the scattering inside the Ulbricht sphere has experimentally been confirmed to be very reliable \([96]\). The pursuit of the level and conditions on the orthogonality of speckles on an analytical level is therefore reasonable given the strength of the existing literature.

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\(^4\)SphereOptics provide a good introduction/review on integrating spheres \([97]\).

\(^5\)The combined ports make up under 1% of the sphere’s surface area.

\(^6\)A CMOS (Complementary Metal Oxide Semiconductor), CCD (Charge Coupled Device), CID (Charge Injection Device) array or other electro-optical based camera.

\(^7\)This result can be derived from the spatial limit imposed on the wave vectors \(k\) by the aperture. Two wave vectors \(k_1, k_2\) must originate from sufficiently adjacent points on the speckle pattern to pass through the aperture.
10.3. Resolution of Speckle Spectro/wave-meters

We can estimate the resolution of speckle spectro/wave-meters as the wavelength shift $\delta \lambda$ needed to change the speckle pattern’s intensity distribution distinctly. This estimate can be approximated by constraining the maximum phase difference $\phi_{\text{max}}$ between variations in optical paths to a $\pi$ \cite{91}.

Transmission-based

For transmission-based speckle spectro/wave-meters of length $L$ with refractive index $n$ and $M$ guided modes indexed by propagation constant $\beta_i$, $i \in \{1, \ldots, M\}$, this phase constraining condition is determined by

$$\left| \frac{d\phi_{\text{max}}(\lambda)}{d\lambda} \right| \delta \lambda = \pi, \ \phi_{\text{max}} = (\beta_1 - \beta_M) L$$

We can approximate the first and $M^{th}$ propagation constants as $\beta_1 = \frac{2 \pi n}{\lambda}$ and $\beta_M = \frac{2 \pi n}{\lambda} \cos (NA)$ respectively where NA is the numerical aperture of the multimode fiber. This sets the resolution $\delta \lambda$ for transmission speckle devices to

$$\delta \lambda = \frac{\lambda^2}{2nL}(1 - \cos (NA))^{-1} \approx \frac{\lambda^2}{nL(NA)^2}$$

(10.1)

A multimode fiber of length $L = 20$ m and NA = 0.22 at 1550 nm thus gives a resolution of 1-2 pm depending on the refractive index $n$ of the fiber’s core (step index\(^8\)). This is comparable to the 8 pm resolution quoted in \cite{91} which was subject to experimental errors.

In deriving equation 10.1 we assume different guided modes don’t mix \textit{i.e.} there is no evanescent coupling between modes from stress (\textit{e.g.} bends and twists) in the fiber and the mode coupling length $L_c$ is smaller than the length $L$ of the fiber. In this limit the resolution is inversely proportional to the length of the fiber \textit{i.e.} $\delta \lambda \propto L^{-1}$. Introducing mode mixing changes the constant of proportionality and the dependence on $L$ \textit{s.t.} $\delta \lambda \propto L^{-1/2}$ \cite{98}.

Due to the added complexity that mode mixing introduces the resolution $\delta \lambda$ is usually calculated using computational simulations. We also note that with mode mixing the constant of proportionality would include a $L_c$ dependent factor/term thereby be dependent on the ’strength’\(^9\) of mode mixing.

---

\(^8\)For a graded-index multimode fiber with small NA the average refractive index of the core can be used.

\(^9\)We can think of the effect of mode mixing being more pronounced or stronger if it becomes significant over shorter distances \textit{i.e.} if $L_c$ is small instead of large.
CHAPTER 10. INTRODUCTION

Reflection-based

For reflection-based speckle spectro/wave-meters whose surface has a correlation length $\xi$, the phase constraining condition can be modelled from a $\phi_{\text{max}}$ defined as the difference between phase between the rough $(k(L + \xi))$ and an ideal smooth surface $(kL)$ for light with wave-number $k$ propagating in a medium with index of refraction $n$. This approximates the phase constraining condition for a single reflection to

$$\left| \frac{d\phi_{\text{max}}(\lambda)}{d\lambda} \right| \delta\lambda = \pi, \quad \phi_{\text{max}} = k([L - L] + \xi) = \frac{2\pi n}{\lambda} \xi$$

For $M$ bounces inside the Ulbricht sphere we use a $\xi \rightarrow M\xi$ substitution. This gives a resolution $\delta\lambda$ for reflection speckle devices with $M$ bounces of

$$\delta\lambda = \frac{\lambda^2}{2nM\xi} \quad (10.2)$$

Whilst different integrating spheres use different coatings with different thicknesses and materials, we use surface measurements of typical tin oxide coatings using an atomic force microscope to estimate the correlation length as $\xi = 50 - 100 \text{ nm}$ for 50-1000 nm coatings [99]. For a resolution of $0.3 \text{ fm}$ at $\lambda = 780 \text{ nm}$ ([93]), an Ulbricht sphere filled with air at room temperature and pressure ($n \approx 1$) returns per equation $10.2$ $M = 1 \cdot 10^4 - 2 \cdot 10^{10}$ bounces of light inside the sphere. For a typical commercial Ulbricht sphere of diameter $D_{\text{sphere}} = 1-200 \text{ cm}$, this number of bounces corresponds to a total optical path of $nD_{\text{sphere}}(M + 1) = 1 \cdot 10^2 - 2 \cdot 10^{10} \text{ m}$ inside the sphere leading to a duration of $0.3 \mu\text{s}$ to $33 \text{ s}$ for light to exit the sphere.

The calculate typical number of bounces $M$ seems at least a hundred times larger than typical number of reflections quoted as tens or hundreds by different manufacturers. This error we appears to comes from an over-simplification in deriving equation $10.2$. Because reflection in Ulbricht spheres is diffusive, light doesn’t take a single path making the substitution $\xi \rightarrow M\xi$ imprecise.

In practice each path taken inside the Ulbricht sphere generates a speckle pattern with the interference between speckle patterns from different paths not accounted for in our calculation. Thus unlike the transmission-based speckle device where mode mixing can be ignored in certain limits, in reflection-based speckle devices the 'equivalent' to mode-mixing\footnote{Superimposing the speckle patterns generated by different paths to produce the final speckle pattern.} needs to always be included.
10.3. SPECKLE-BASED SPECTRO/WAVE-METERS

Our calculation of $M$ thus shows the equivalent number $M_{\text{equiv.}}$ of bounces the light has if the final speckle pattern is generated by only one optical path (unphysical). Calculating the resolution of a reflection-based speckle device therefore needs to account for the bifurcation number of optical paths or some other variable quantifying the spread of light distribution inside the sphere per reflection. Unfortunately due to the complexity of this problem and the novelty (recency) of reflection-based speckle devices there is no further work in the literature on the partitioning and recombination of light taking different paths inside Ulbricht spheres. In this sub-thesis we therefore attempt to lay the foundations in analysing this complex ensemble of light interferometers\(^\text{11}\) taken inside the Ulbricht sphere with possible varying number of bounces per interferometer arm.

Bounded Resolution

If $\delta\lambda$ is set too large we risk losing correlation of speckles within one spectral channel of width $d\lambda = \delta\lambda/2$. For example, for large $\delta\lambda$ the speckle patterns at wavelengths $\lambda \pm \frac{d\lambda}{2}$ decorrelate even if $\lambda$ is placed at the center of spectral channels. This would introduce variations in speckle patterns within a single spectral channel thus prevent reliable wavelength or spectrum reconstruction using speckle devices. The maximum value for resolution $\delta\lambda$ is thus capped at the value the reconstruction becomes unreliable from decorrelation within a spectral channel. Similarly the minimum attainable resolution is limited by physical properties of the medium-light interactions such as fiber length $L$ or correlation length $\xi$. If we decide to use more spectral channels $N$ in our transmission matrix $T$ we introduce errors in our reconstruction as different spectral channels become correlated i.e. start to ‘overlap’. The resolution $\delta\lambda$ of speckle devices is therefore bounded above and below.

10.3.4 Bandwidth of Speckle Spectro/wave-meters

Apart from the properties of the propagation medium, speckle devices are limited further by two additional reconstruction errors: in resolution $\delta\lambda$ by the algorithmic error introduced in calculating $T_{\text{trunc}}^{-1}$ and in bandwidth $\Delta\lambda$ by the difference between the number of spectral $(N = \left\lfloor \frac{\Delta\lambda}{\delta\lambda} \right\rfloor)$ and spatial $(N = \left\lfloor \frac{\Delta r}{(\delta r)^2} \right\rfloor)$ channels. Here $\Delta r$ represents the total number of pixels in the image and $\delta r$ the spatial channel spacing (units of px) in both horizontal and vertical axes of the image that we sample over.

---

\(^\text{11}\)The path bifurcation of light per diffusive reflection along with the unification at the output port of light inside the Ulbricht sphere is equivalent to a light interferometer.
If $N < \mathcal{N}$ i.e. the number of spectral channels exceeds the number of spatial channels, the speckle device oversamples the imaged speckle patterns making the reconstruction unreliable. For $\mathcal{N} < N$ however the undersampling spatial-to-spectral map involves $N - \mathcal{N}$ redundant equations whose effect is to bring the attainable resolution closer to its lower bound value [91].

Whilst the number of spectral channels $\mathcal{N}$ is limited by the practical bounds of resolution $\delta\lambda$, the number of spatial channels $N$ is bounded below by $\delta r \geq \delta x$ (1 px) or the correlation length $\xi_I$ of the imaged speckles to avoid aliasing effects. The upper bound of $N$ can be set from when $\delta r$ equals the total number of pixels of the image i.e. we sample the whole image as one spatial channel. This limits the difference in magnitude between $\mathcal{N}$ and $N$ for practical applications\(^\text{12}\).

A consequence of limits on $N - \mathcal{N}$ is a constraint on the dimensions of transmission matrix $T$ which affects the bandwidth of speckle devices. This follows from the spatial correlation width $\delta r$ of speckle patterns which sets the number of spatial channels $N$. To avoid oversampling, a value of $N$ caps the number of spectral channels $\mathcal{N}$. Since the resolution $\delta\lambda$ has an upper bound, this limits the maximum bandwidth. Speckle spectrometers are thus affected by limit 2): If the size of $V_\lambda$ is smaller than that of $V_I$ the redundant equations make the spatial-to-spectral map a many-to-one map which introduces limit 2). Larger number of redundant equations (lower resolution) thus decreases the bandwidth. Even if $N - \mathcal{N}$ is minimised we could find that superimposed speckle patterns of multiple wavelengths of light correlate with other speckle pattern making the spectrometer unreliable. The resemblance between speckle patterns comprised of different spectrum is complex and we attempt in this sub-thesis to study it by analysing speckle statistics.

### 10.3.5 Performance of Speckle Spectro/wave-meters

A high performing spectral characterisation device has a large bandwidth $\Delta\lambda$ and can resolve small differences in wavelengths (resolution $\delta\lambda$). To compare the performance of different devices we thus define their performance as the ratio $\frac{\Delta\lambda}{\delta\lambda}$. In table 10.1 we compare the resolution, bandwidth and performance of various commercial and research competing devices. When available we include the interferometer/technique used per device.

\(^{12}\)The size of the vector space of imaged speckle patterns $V_I$ restricts the size of reconstructable wavelengths in vector space $V_\lambda$ under any spatial-to-spectral map. Hence spatial imaging resolution and size of speckle patterns map to spectral resolution and bandwidth.
10.3. SPECKLE-BASED SPECTRO/WAVE-METERS

<table>
<thead>
<tr>
<th>Device</th>
<th>Resolution $\delta \lambda$</th>
<th>Bandwidth $\Delta \lambda$</th>
<th>Performance $\frac{\Delta \lambda}{\delta \lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ulbricht sphere speckle [w]</td>
<td>$0.3 \text{ fm}$ [at 780 nm]</td>
<td>488-1064 nm</td>
<td>$1.9 \cdot 10^9$ [uniform $\delta \lambda$]⁵</td>
</tr>
<tr>
<td>HighFinesse WS8-2 [fw]</td>
<td>$4.0 \text{ fm}$ [at 375-1180 nm]</td>
<td>330-1180 nm</td>
<td>$0.2 \cdot 10^9$ [for 375-1180 nm]⁶</td>
</tr>
<tr>
<td>Bristol Instrum. 671A [mw]</td>
<td>$22.3 \text{ fm}$ [at 780 nm] (or 520-1700 nm)</td>
<td>375-1100 nm</td>
<td>$32.5 \cdot 10^6$ (52.9-10⁶) [uniform $\delta \lambda$]⁷</td>
</tr>
<tr>
<td>HighFinesse WS8-2 [fw]</td>
<td>$2.0-20.0 \text{ fm}$</td>
<td>750-795 nm</td>
<td>$2.3-22.5 \cdot 10^5$</td>
</tr>
<tr>
<td>R. S. S. - Merck LW-10 [mw]</td>
<td>$40.0 \text{ fm}$ [at 700-1000 nm]</td>
<td>630-1100 nm</td>
<td>$10.0 \cdot 10^6$ (for 700-1100 nm)⁸</td>
</tr>
<tr>
<td>APEX Tech. AP-2687A [u]</td>
<td>$40.0 \text{ fm}$</td>
<td>1520-1630 nm (or 1265-1345 nm)</td>
<td>$2.8 \cdot 10^6$ (2.0-10⁶) [spectral width]⁹</td>
</tr>
<tr>
<td>Advantest TQ-8325 [mw]</td>
<td>$1.0 \text{ pm}$</td>
<td>480-1650 nm</td>
<td>$1.2 \cdot 10^6$ [used in sub-thesis I]¹⁰</td>
</tr>
<tr>
<td>HORIBA 1250M [gs]</td>
<td>$6.0 \text{ pm}$</td>
<td>0-1500 nm</td>
<td>$0.3 \cdot 10^6$</td>
</tr>
<tr>
<td>Tapered fiber speckle [s]</td>
<td>$40.0, 10.0 \text{ pm}$ [at 635, 1550 nm]</td>
<td>500-1600 nm</td>
<td>$27.5 \cdot 10^4$, $0.1 \cdot 10^6$ [uniform $\delta \lambda$]¹¹</td>
</tr>
<tr>
<td>Yokogawa AQ-6370D [gs]</td>
<td>$10.0-40.0 \text{ pm}$ [100 pm, full $\Delta \lambda$]</td>
<td>600-1700 nm</td>
<td>$27.5 \cdot 10^4$-0.1-10⁶ [10.0-10⁶]</td>
</tr>
<tr>
<td>Fabry-Perot [s]</td>
<td>$2.0-20.0 \text{ fm}$</td>
<td>1-10 GHz for $\lambda = 300-2000 \text{ nm}$</td>
<td>$15.0$ to $65.0 \cdot 10^3$ [for $\lambda$ dependent]</td>
</tr>
<tr>
<td>Ocean Insight Optics [gs]</td>
<td>$20.0 \text{ pm}$</td>
<td>200-1100 nm</td>
<td>$45.0 \cdot 10^3$</td>
</tr>
<tr>
<td>Yokogawa AQ-6315A [gs]</td>
<td>$50.0 \text{ pm}$</td>
<td>350-1750 nm</td>
<td>$28.0 \cdot 10^4$ [used in sub-thesis I]</td>
</tr>
<tr>
<td>Stabilised fiber speckle [s]</td>
<td>$2.0 \text{ pm}$</td>
<td>1530-1537 nm</td>
<td>$3.5 \cdot 10^7$ [for $\lambda$ dependent limit 2]¹²</td>
</tr>
<tr>
<td>Multimode fiber broadband speckle [s]</td>
<td>$1.0. \text{ nm}$ (or 1.0 pm)</td>
<td>400-750 nm (1500.0-1500.1 nm)</td>
<td>$350$ (100) [limit 2]¹²</td>
</tr>
<tr>
<td>Photonic chip speckle [s]</td>
<td>$75.0 \text{ pm}$</td>
<td>1500-1525 nm</td>
<td>$33.3$ [limit 2]¹²</td>
</tr>
<tr>
<td>Non-stabilised speckle [s]</td>
<td>$400.0 \text{ pm}$ (8.0 pm)</td>
<td>1450-1550 nm (1500-1501 nm)</td>
<td>$250.0$ (125.0) [limit 2]¹²</td>
</tr>
<tr>
<td>Spiral fiber speckle [s]</td>
<td>$10.0 \text{ pm}$</td>
<td>1520-1522 nm</td>
<td>$200.0$ [limit 2]¹²</td>
</tr>
</tbody>
</table>

Table 10.1: Comparing the resolution $\delta \lambda$, bandwidth $\Delta \lambda$ in order of performance $\frac{\Delta \lambda}{\delta \lambda}$ of different devices. [f] = Fizeau interferometer-based, [m] = Michelson interferometer-based, [u] = Unspecified, [g] = Grating-based, [w] = wavemeter & [s] = spectrometer. This table was corrected, updated and extended from a supplementary table courtesy of [93]. N.B. a, b, c & d are additional comments shown in the following four paragraphs respectively.
Comments to Table 10.1

(a) The Ulbricht sphere speckle wavemeter’s resolution has only been tested at 780 nm whilst it depends on lights wavelength as per equation 10.2. As 780 nm lies near the centre of the bandwidth, we expect our performance estimate in table 10.1 (calculated assuming uniform $\delta \lambda$) to lie near the integral $P = \int_{\Delta \lambda} \frac{\delta \lambda (\lambda')}{X} d\lambda$. Integral $P$ represents our idea of performance more precisely by including variations in resolution whilst including the full bandwidth of the device.

(b) From the specifications and brochure of the Bristol Instruments 671A wavemeter, the variations in $\delta \lambda$ are small and restricted to the lower part of the bandwidth. Our estimate in table 10.1 is therefore close to the integral $P$.

(c) The estimate by integral $P$ for the tapered fiber speckle spectrometer will lie between the two extremes set by the two different resolution values (10 & 40 pm) over the same bandwidth $\Delta \lambda$. We therefore calculate the performance using the extreme resolution values over the full bandwidth providing us the upper and lower performance bounds of the device (i.e. uniform $\delta \lambda$). A similar resolution-bandwidth behaviour is reported by the Yokogawa AQ6370D grating spectrometer which however quotes the minimum resolution when utilising the full bandwidth in single-shot measurements. For the tapered speckle device in comparison the resolution remains within 10-40 pm even in full-bandwidth single-shot measurements.

(d) The performance of several speckle spectrometers is limited by the relation between resolution and bandwidth discussed in subsection 10.3.4. As per limit 2, to avoid correlations between speckle patterns at larger $\delta \lambda$ the bandwidth of simple speckle spectrometers is severely restricted within the total transmission bandwidth of the optical medium that the light propagates through. Depending on the resolution, this restriction can reduce the bandwidth (width) from 400-2400 nm (2000 nm) to 1500.0-1500.1 nm (0.1 nm) i.e. by a factor of 20,000 for transmission-based speckle devices. The restriction scales with resolution thus could be higher in higher performing speckle devices. For wavemeter applications the bandwidth is less restricted as the correlation between speckle patterns from smaller spectrum decrease [114]. Utilising alternative designs such as tapering fibers in transmission-based spectrometers also reduces the restriction in bandwidth [94].

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13In subsections 10.3.1 (and 10.3.2) we describe simple transmission-based (and reflection-based) speckle devices.
10.3. SPECKLE-BASED SPECTRO/WAVE-METERS

Comments on Performance

(e) As expected, the wavemeters in table 10.1 have the highest performance $\frac{\Delta \lambda}{\delta \lambda}$ and best resolution $\delta \lambda$. We also infer that changes in bandwidth $\Delta \lambda$ of speckle devices scale more rapidly with changes in resolution resulting in a reduced performance per speckle device when we lower $\delta \lambda$. We see this for example with the Non-stabilised speckle spectrometer ([91]) when $\delta \lambda$ reduces from 400 to 8 $pm$ resulting in the performance being halved.

(f) As per subsection 10.3.3, the resolution of transmission-based speckle devices depends on several variables including the length $L$ of the fiber. For ‘short’ fibers whose length is shorter than the mode coupling length ($L < L_c$) the minimal attainable\(^{14}\) resolution is given by equation 10.1. This equation applies to the 8 $pm$ resolution from a 20 $m$ long fiber [91]. For $L > L_c$ (‘long’ fibers) the constant of proportionality changes and the minimum resolution becomes proportional to $L^{-1/2}$. This relation is used when calculating the 1 $pm$ resolution of the 100 $m$ long fiber used in the multimode fiber broadband speckle spectrometer [113]. The values in table 10.1 should therefore be considered together with the relevant parameters of the experimental setup if compared.

(g) For the Ulbricht sphere speckle wavemeter ([93]), following comment d the bandwidth is expected to be restricted if used as a spectrometer. The performance of an Ulbricht sphere spectrometer is thus expected to be lowered by the bandwidth restriction factor. In this sub-thesis we aim to increase understanding of and deriving the relation between speckles and their correlation for single and multiple wavelengths of light including smooth spectra. It is namely not clear what relations determine the (de)correlation between speckle patterns by Ulbricht spheres which sets the bounds of resolution, bandwidth and hence performance (c.f. subsection 10.3.4). This was the original motivation behind this sub-thesis.

(h) Atomic physics experiments require lasers that are stable and reliable to 1 $MHz$ over the time-scale of the experiment. A speckle device therefore needs to have a resolution of 1 $MHz$ which for $\lambda = 300$-$1550$ $nm$ corresponds to $\delta \lambda = 0.3$-$8.0$ $fm$. Speckle devices can therefore be used in direct and simultaneous measurements of the stability and reliability of all the lasers used in atomic physics experiments.

\(^{14}\)Equation 10.1 doesn't take include the experimental errors in reconstructing wavelength from imaged speckle patterns hence offers the ideal minimal attainable resolution.
In comparison as per table 10.1 we find that most competitive alternatives do not have the necessary resolution thus require indirect measurements or laser specific devices (narrow bandwidth) such as the Fabry-Perot interferometer.

(i) A clever solution to increase the bandwidth of speckle spectrometers is demonstrated in [94] where Snell-Decartes law combined with a tapered fiber separates out speckle patterns of different wavelengths along the fiber’s axis (tapered region). By measuring the leaked light exiting the fiber’s tapered region the bandwidth limit of speckle devices can be exceeded by a factor of 550, possibly higher. A 10 pm resolution tapered fiber speckle spectrometer has namely 550 times larger bandwidth then the 10 pm resolution spiral fiber speckle spectrometer in table 10.1. This increases the performance by 550 using the tapered design over a spiral design highlighting the importance of geometry. Likewise we see how stabilisation (temperature and pressure) also improves the performance by a factor of around 10 when comparing the stabilised against the non-stabilised speckle spectrometer in table 10.1.

The tapered design introduced a spatial dependence in propagation constants $\beta_i$ along with a spatial dependency in the propagation losses per wavelength$^{15}$ of light. Speckle patterns by different wavelengths therefore separate spatially along the tapered region providing a coarse spatial-to-spectral map whilst each speckle pattern comprises from interference by a different number of guided modes providing a fine spatial-to-spectral map. This reduces the risk of mixing speckle patterns by different spectra of light: limit 2).

(j) We suspect that the Ulbricht sphere’s curvature combined with refraction during propagation between different interiors of the sphere might have a similar effect as tapering the fiber: Different wavelengths of light per Snell-Decartes law should take different trajectories as they refract at different angles to different interior points inside the sphere. At sufficiently large exit port diameters of the sphere this wavelength-trajectory dependence might result in different exit angles of speckle patterns of different wavelengths which introduces a coarse spatial-to-spectral map. This effect is enhanced if we submerge the Ulbricht inside a medium with high refractive index $n$ and high dispersion of light. From equations 10.1 and 10.2 we know larger values of $n$ provide better resolutions resulting in an increase in bandwidth and resolution for submerged reflection-based speckle spectrometers.

$^{15}$The wavelength dependence on the refractive index of the core and cladding different leads to different exit angles (Snell-Decartes law) in the tapered region when shifting the wavelengths across the transmission window of the fiber. The leaking from the tapered region therefore becomes analogous to refraction of light by a prism.
10.3. SPECKLE-BASED SPECTRO/WAVE-METERS

(k) Transmission-based speckle spectrometer can and are used to simultaneously monitor hundreds/thousands of comb-lines of an optical frequency comb (OFC) in single acquisition measurements [112]. This tool is useful in characterising the laser systems like the mode-locked laser discussed in the previous sub-thesis I. Given comment h we could also characterise the other laser systems used in atomic physics experiments together with an OFC.

Summary of Performance

- We can re-phrase our motivational question as follows: Where does the resolution and bandwidth limit lie when comparing small differences in wavelengths (limit 1, wavemeter) and comparing large differences in wavelength (limit 2, spectro/wave-meter)?

- Our review shows speckle wavemeters have high performance (bandwidth/resolution) exceeding alternative technologies whilst speckle spectrometers are limited in bandwidth by limit 2) giving them a lower performance.

- The restriction of bandwidth (easily factors of 20 000) for speckle spectrometers can be reduced by spatially separating out speckle patterns generated by different wavelengths of light. For transmission-based system a tapered design offers factors fo 550 times less bandwidth restriction and we hypothesise the reflection-based system might also benefit from enhancing this separation through submerging the Ulbricht sphere in a highly dispersive medium with a high refractive index $n$.

- The performance of speckle devices can be enhanced by factors of ten by stabilising the experimental set-up in temperature and pressure (using a vacuum arrangement).

- A speckle wavemeter’s resolution will drop if the spectral bandwidth of a laser is as wide/similar to the Advantest wavemeter resolution listed in table 10.1. Therefore we find that limit 2) correlates with limit 1).
10.4 Sub-thesis Structure

The sub-thesis is divided into two core chapters providing alternative methods to determine the orthogonality of speckle patterns. The first chapter aims to derive the analytical statistical distribution functions of speckles for different wavelengths. With these functions, the statistical independence can be tested directly and the orthogonality conundrum solved.

The second chapter aims to apply the null hypothesis, disproving orthogonality by discussing the mathematically simpler correlations. If speckles can be shown to be correlated namely, an orthogonal basis cannot be formed. This would limit the retrieval and reliability of spectral information from laser systems such as a MLL using the speckle spectrometer.

As the author was transferred to another project, only limited results can be shown per core chapter. Nonetheless, some of the identified routes along with preliminary results are summarised in the conclusion at the end.

10.5 Disclaimer

For completeness, parts of this sub-thesis adapts several theorems, calculations, derivations, etc. Apart from locally referencing the relevant sources, we wish to highlight in particular the two sources we made use of in chapter 11: [87, 115].

To complete the characterisation of the mode-locked laser (MLL) discussed in the previous sub-thesis I, we aimed to test our MLL onto the experimental set-up discussed in [93]. This sub-thesis started with the aim to derive a theory that describes the speckle pattern of multiple wavelengths being injected into an integrating (Ulbricht) sphere to append to speckle characterisation of the MLL. The proposed experiment involving the MLL in sub-thesis I would serve as a preliminary experimental test prior to testing the novel speckle-based spectrometer. Unfortunately, as the mode-locked laser project was terminated by the supervisory team, progress on this sub-thesis followed.

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16This would have provided a very detailed analysis of the MLL. In particular, we could view all the comb-lines simultaneously in single-shot measurements from which we could infer the level of ‘breathing’ (inter comb-line spacing variations) and any global frequency offset changes (variations in the common offset frequency).
Chapter 11

Detailed Analysis of Speckles

In this chapter we aim to derive the principal probability density of speckles as a function/variable of wavelength. The principle we hope to exploit is analytically calculating the covariance of speckles of different wavelengths. Using an existing template involving the central limit theorem and random walks, we highlight a possible path to solve the problem of speckle orthogonality and demonstrate our progress on it.

It should be noted that literature review sections 11.1, 11.2 and 11.3 are adapted from [87, 115]. We apply their results in sections 11.4 where we demonstrate our progress towards solving the orthogonality progress along with an outline of future tasks in section 11.5.

11.1 Introduction to Speckle Statistics

In this section we will cover some common mathematical tools developed to model speckle generation through reflection of optical fields from optical rough surfaces: surfaces with a correlation length $\xi$ on the order of the optical wavelength $\lambda$ of light used to generated the speckle pattern. Starting with some standard wave mechanics we will introduce the analogy of the speckle effect to an random walk and finish with some standard and relevant propositions for speckle statistics that will be used in later calculations.
11.1.1 Basic Wave Mechanics

We start by intuitively modelling the light field as the real part of an ideal monochromatic wave of linear frequency $\nu$ of standard form

$$u(x, y, z, t) = A(x, y, z)e^{i2\pi\nu t} \quad (11.1)$$

where the spatial dependent total amplitude of the field $A(x, y, z)$ in Cartesian coordinates $\{x, y, z\}$ is given by

$$A(x, y, z) = |A(x, y, z)|e^{i\theta(x, y, z)} \quad (11.2)$$

and $\theta(x, y, z)$ is the total phase of the field [87].

Using equation 11.1 and equation 11.2 we can show that at the point $(x, y, z)$, the intensity $I(x, y, z)$ solely depends on the field amplitude $A(x, y, z)$ at that point. This relation for the intensity is calculated from its definition as the average power in a time $T$ as shown below

$$I(x, y, z) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |u(x, y, z, t)|^2 dt = |A(x, y, z)|^2 \quad (11.3)$$

From equations 11.2 and 11.3 we can see that the intensity thus takes the form

$$I = [A^{(r)}]^2 + [A^{(i)}]^2 \quad (11.4)$$

where $A^{(r)}$ and $A^{(i)}$ are the real and imaginary parts respectively of the complex field amplitude $A(x, y, z)$ i.e

$$A^{(r)} = Re\{A\}$$
$$A^{(i)} = Im\{A\} \quad (11.5)$$

Later when calculating the Jacobi determinant we will make use of two more equations shown below which directly follow from equations 11.4 and 11.5.

$$A^{(r)} = \sqrt{I} \cos \theta$$
$$A^{(i)} = \sqrt{I} \sin \theta \quad (11.6)$$

$I$ has units of $W/m^2$ and $u, A, A^{(r)} & A^{(i)}$ units of $\sqrt{W}/m$.

\[\text{1} The power per unit area i.e. energy per unit time per unit area. The term 'Irradiance' is also sometimes used instead of intensity and shares the same meaning in most contexts, differentiating in the emphasis on directionality of the energy flux (irradiance) over total energy flux (intensity). For speckles the terms are typically synonymous.
11.1. INTRODUCTION TO SPECKLE STATISTICS

11.1.2 Speckle Random Walk on the Argand Plane

For speckle analysis we can use Huygens construction\(^2\) to calculate the field amplitude \(A(x, y, z)\) ([116]). Huygens construction is a theorem that can be stated as

**Theorem 1** (Huygens construction). *Each element of a wave-front may be regarded as the centre of a secondary disturbance which gives rise to spherical wavelets. The position of the wave-front at any later time is the envelope of all such wavelets.*

This allows us to express the field amplitude as

\[
A(x, y, z) = \sum_{k=1}^{N} \frac{1}{\sqrt{N}} a_k(x, y, z) = \sum_{k=1}^{N} \frac{1}{\sqrt{N}} |a_k| e^{i\phi_k} \tag{11.7}
\]

Here \(a_k(x, y, z)\) are the individual phasor contributions from the \(N\) local sites on the optically rough surface with respective phase \(\phi_k\).

The scattering of light from the optically rough surface is illustrated in figure 11.1. There we show how Huygens construction of spherical wavelet generation at different sites of the rough surface generates an speckle pattern through path length difference resulting in phase delays between different phasors whose additive sum/superposition will thus vary randomly in the imaging plane.

Equation (11.7) is analogous to an random walk in the complex Argand space by taking each step to be of length \(|a_k|\) with direction given by \(\phi_k\) i.e. each step given as a phasor \(a_k e^{i\theta_k}, \ k \in \{1, 2, ..., N\} \subset \mathbb{N}\). In this analogy, \(\theta(x, y, z)\) becomes the net phase i.e. direction following all the individual phasors/steps in the random walk and \(|A(x, y, z)|\ the total amplitude i.e. displacement from the starting point (origin) of the random walk. Thus the field amplitude defined in equation 11.2 gives the final position from the origin on the Argand plane following an random walk. This analogy is illustrated in figure 11.2.

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\(^2\)Sometimes also referred to as the *Huygens-Fresnel principle*.\)
CHAPTER 11. DETAILED ANALYSIS OF SPECKLES

Incident wavefronts

Phasor trajectories

Imaging plane

$O(x, y, z)$

Figure 11.1: This figure illustrates the light scattering from an optically rough surface from a side view angle. The (orange) incident planar wavefronts from an coherent light source come in contact with the (red) optically rough surface producing (green) spherical wavelets by the Huygens construction. As standard, the wavevector orthogonal to the wavefronts than define the direction of phasors for different observation points $O(x, y, z)$. The figure inset shows this in more detail for the 7th phasor. A speckle pattern will thus be visible at the imaging plane. The equivalent dashed line in the optically rough surface represents the average thickness.
11.1. INTRODUCTION TO SPECKLE STATISTICS

11.1.3 Propositions of Speckle Statistics

Our aim here is to derive the principal probability density of the intensity and phase distribution of the speckle field post reflection from the rough surface. We start with some propositions and in the next section apply them to our random walk picture introduced in section 11.1.2.

Axioms of Speckle Statistics

Two axioms\(^3\) commonly taken in speckle statistics are

**Axiom 1.** Consider an optical field operating at wavelength $\lambda$. Let $k$ be an index representing positions/local sites on the optically rough surface illuminated to generate a speckle pattern.

\(^3\)We have identified a method that can possibly verify the validity of these axioms or at least determine the 'strength' of these axioms. That way we should be able to qualitatively understand if these axioms make for an good model or not. The method is currently studied and follows from [117] when we use the random walk picture in 11.2.
a) The amplitudes of the complex phasors $|a_k|$ and $|a_l|$ are statistically independent of each other for $k \neq l$.
b) The phases of the complex phasors $\phi_k$ and $\phi_l$ are statistically independent of each other for $k \neq l$.
c) The amplitudes and phases of the phasors are statistically independent of each other\(^4\) even if $k = l$.

which can be expressed more concisely as

a) $|a_k| \perp \perp |a_l| , \forall k, l \in [1, 2, ..., N] \setminus \{k = l\}$
b) $\phi_k \perp \perp \phi_l , \forall k, l \in [1, 2, ..., N] \setminus \{k = l\}$
c) $|a_k| \perp \perp \phi_l , \forall k, l \in [1, 2, ..., N]$ including $k = l$.

**Axiom 2.** Consider an optical field operating at wavelength $\lambda$. Let $k$ be an index representing positions/local sites on the optically rough surface illuminated to generate an speckle pattern.

The phases $\phi_k$ of the complex phasors are uniformly distributed over a whole number of unit cycle revolution(s) (e.g. $[-\pi, \pi]$, $[2\pi, 4\pi]$, etc.) over an ensemble of phasors $k \in [1, 2, ..., N]$ which is isomorphic\(^5\) to a single full unit cycle revolution.

These two axioms describe the nature of height variations between different positions/local sites on the optically rough surface: random. In practice adjacent points won’t have uncorrelated height variations. Axioms 1 & 2 do apply in practice however as imaging systems have a limited resolution which introduces a minimum distance between points that can be resolved in an image: the spatial channel spacing $\delta r$. At sufficiently long $\delta r$ the correlation between heights of adjacent spatial channels vanishes making the axioms valid. The measure of how realistic the two axioms are thus scales with the level of decorrelation between spatial channels which decreases for larger $\delta r$.

These two axioms have proven to be reliable with quantitative agreement found between experiment and theory on statistical properties of speckle [96]. We therefore expect speckle statistics can assist us in finding the intensity correlation function as a function of wavelength of light (and geometry/dimensions/type of speckle generation) to test for orthogonality.

\(^4\)To clarify, axiom 1 part c states that the amplitudes are statistically independent of the phases of the phasors, even at the same location.

\(^5\)A bijective homomorphism as described by Category Theory.
Axioms 1 & 2 will prove to be great tools of simplification when calculating expectation values. To understand why we will start with a few propositions and show an example in which we apply the propositions and the axioms for calculating the expectation value of the field amplitude $A$.

**Expectation Values on Sets**

**Lemma 1.** Let $f$ and $g$ be smooth functions and let $\alpha_i$ and $\beta_j$ be statistically independent variables $\forall i, j \in [1, 2, \ldots, N] \subseteq \mathbb{N}$ for some $N \in \mathbb{N}$. We then have

$$\langle f(\alpha_i)g(\beta_j) \rangle = \langle f(\alpha_i) \rangle \langle g(\beta_j) \rangle$$

which is to say that the covariance $\text{Cov}[f(\alpha_i), g(\beta_j)]$ vanishes

$$\text{Cov}[f(\alpha_i), g(\beta_j)] = 0$$

Here $\langle \ldots \rangle$ represents the average over an ensemble i.e. the optically rough surface. Furthermore if the statistical independence only holds for $i \neq j$ than these results can only be inferred for $i \neq j$ when the covariance vanishes.

Lemma 1 can be easily derived from the definition of the expectation value for each term. It follows directly form the vanishing covariance of the variables $\alpha_i$ and $\beta_j$ which is an result from the statistical independence of $\alpha_i$ and $\beta_j$.

**Proof.** Let $\rho_{ij}^{\alpha\beta}$ be the joint probability density factor of $\alpha_i$ and $\beta_j$. This implies that the expectation value $\langle f(\alpha_i)g(\beta_j) \rangle$ is given by

$$\langle f(\alpha_i)g(\beta_j) \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} f(\alpha_i)g(\beta_j)\rho_{ij}^{\alpha\beta}$$

From statistical independence between $\alpha_i \& \beta_j$ ($\alpha_i \perp \beta_j$), we can infer that $\rho_{ij}^{\alpha\beta} = \rho_i^{\alpha} \rho_j^{\beta}$ where $\rho_i^{\alpha}$ is the probability factor for $\alpha_i$ and $\rho_j^{\beta}$ the probability factor for $\beta_j$. This allows us to rewrite the expectation as

$$\langle f(\alpha_i)g(\beta_j) \rangle = \sum_{i=1}^{N} f(\alpha_i)\rho_i^{\alpha} \sum_{j=1}^{N} g(\beta_j)\rho_j^{\beta}$$

$$= \langle f(\alpha_i) \rangle \langle g(\beta_j) \rangle$$

where in the last step we used the definition of the expectation value again.

In the case of statistical dependence at $i = j$, the factor $\rho_{ii}^{\alpha\beta}$ no longer factors into $\rho_i^{\alpha}$ and $\rho_j^{\beta}$ making the factorisation of expectations impossible. $\square$

Lemma 1 may be applied for the case of self-statistical independence e.g. $|a_i|$ and $|a_j|$. Using Lemma 1 we can than infer an orthogonality results for the phases of our phasors that is given by the upcoming Lemma 2.
Lemma 2. Let \( \{ \phi_m \}_{m \in \mathbb{N}} \) be the set of statistically independent phases from axiom 2 that also obey axiom 1. If we take \( < \cdots > \) to indicate an average over our set \( \{ \phi_m \}_{m \in \mathbb{N}} \) than the following orthogonality rules hold

\[
< \cos \phi_m \cos \phi_n > = < \sin \phi_m \sin \phi_n > = \frac{1}{2} \delta_{mn}
\]

\[
< \cos \phi_m \sin \phi_n > = < \sin \phi_m \cos \phi_n > = 0
\]

Proof. For the \( m \neq n \) case we can apply Lemma 1 and use the property of the sinusoidal functions that their average of an set of variables that spans a whole number of cycles on the unit circle (i.e. obeys axiom 2) vanishes (\( < \cos \phi_m > = < \sin \phi_m > = 0 \)).

For the \( m = n \) case a different property is used namely that the square of an sinusoidal function acting on an set of variables that spans over a whole number of cycles on the unit circle returns \( \frac{1}{2} \) when taking the average. Direct calculation details are left as an exercise to the reader. \( \square \)

11.1.4 Field Amplitude Expectation Value

Lemma 1 and 2 form great mathematical tools when combined with the random walk picture as we will demonstrate below:

Using the random walk picture we can re-write equation 11.5 as

\[
A^{(r)} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |a_k| \cos \phi_k
\]

\[
A^{(i)} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |a_k| \sin \phi_k
\]

Taking the average of equations 11.8 we get

\[
< A^{(r)} > = \frac{1}{\sqrt{N}} < \sum_{k=1}^{N} |a_k| \cos \phi_k > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} < |a_k| \cos \phi_k >
\]

\[
< A^{(i)} > = \frac{1}{\sqrt{N}} < \sum_{k=1}^{N} |a_k| \sin \phi_k > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} < |a_k| \sin \phi_k >
\]

If we set \( f(\alpha_i) = \alpha_i \) and \( g(\beta_i) = \begin{cases} \cos \beta_i, & A^{(r)} > \\ \sin \beta_i, & A^{(i)} > \end{cases} \), we can apply Lemma 1 by setting \( \alpha_i = |a_k| \) and \( \beta_i = \phi_k \). This allows us to rewrite the expectation values as
11.1. INTRODUCTION TO SPECKLE STATISTICS

\[
< A^{(r)} > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} < |a_k| > < \cos \phi_k >
\]

\[
< A^{(i)} > = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} < |a_k| > < \sin \phi_k >
\]

Given \(< \cos \phi_k > = < \sin \phi_k > = 0\) we can infer that

\[
< A^{(r)} > = 0
\]
\[
< A^{(i)} > = 0
\]

(11.9)

Following equation 11.5 i.e. \( A = A^{(r)} + i A^{(i)} \) we thus conclude that \(< A > = 0\). Hence the field amplitude of a Speckle Field has an vanishing average.

Of course in practice the intensity is measured and for understanding correlations between intensities we will need to calculate the expectation values of the intensity. Luckily enough the techniques employed here can be used later to calculate \(< I >\) and will be of use for deriving the statistical distribution of the intensity and phase.

We proceed with calculating \(< [A^{(r)}]^2 >, < [A^{(i)}]^2 > \) and \(< [A^{(r)} A^{(i)}] >\).

\[
< [A^{(r)}]^2 > = \frac{1}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} < |a_m| |a_n| \cos \phi_m \cos \phi_n >
\]

Infer Lemma 1

\[
= \frac{1}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} < |a_m| |a_n| > < \cos \phi_m \cos \phi_n >
\]

Infer Lemma 2

\[
= \frac{1}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} < |a_m| |a_n| > \left( < \cos \phi_m \cos \phi_n > \right) = \frac{1}{2} \delta_{mn}
\]

\[
= \frac{1}{N} \sum_{m=1}^{N} < |a_m|^2 >
\]

\[
= \frac{1}{2}
\]
Similarly, we find that $< [A^{(i)}]^2 > = \frac{1}{N} \sum_{m=1}^{N} \frac{\langle |a_m|^2 \rangle}{2}$. That leaves one more expectation value to be calculated, namely $< [A^{(r)} A^{(i)}] >$. We proceed as before

$$< [A^{(r)} A^{(i)}] > = \frac{1}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} < |a_m| |a_n| > < \cos \phi_m \sin \phi_n >$$

Infer Lemma 2

$$= 0$$

It should be clear that this results holds under the replacement $i \leftrightarrow r$ i.e. $< [A^{(r)} A^{(i)}] > \big|_{i \leftrightarrow r} = < [A^{(i)} A^{(r)}] > = 0$. We thus conclude

$$< [A^{(a)}][A^{(b)}] > = \frac{1}{N} \sum_{m=1}^{N} \frac{\langle |a_m|^2 \rangle}{2} \delta_{ab} , a \land b \in \{ r, i \}$$

and recall that from equation (11.9) we can infer

$$< A > = 0$$

Equations 11.10 and 11.11 are expected to be of significance in the future when we properly link speckles with the random walk picture we introduced in section 11.1.2 by making use of the Central Limit Theorem 2. For now, they demonstrate the application of axioms 1, 2 and provide insight into some statistical properties of the field amplitude.

11.2 Simple 2D Random Walks

A powerful theorem we will be using for describing speckles is Liapounoff’s formulation of the Central Limit Theorem. This theorem links the speckle statistics to a random walk in the complex Argand space and provides us with an asymptotic limit of the probability density of the random walk. It is this probability density we intend to use to test for orthogonality thus making this method essential. The particular formulation of the theorem can be expressed as:
11.2. SIMPLE 2D RANDOM WALKS

**Theorem 2** (Liapounoff formulation of the Central Limit Theorem[115]). Let \( \{x^{(1)}_k, x^{(2)}_k, \ldots, x^{(m)}_k, \ldots, x^{(M)}_k\} \) be \( M \) sets of \( n \) be independent\(^6\) random variables each with \( k \in \{1, 2, \ldots, n\} \), \( m \in \{1, 2, \ldots, M\} \) and finite expectation value \( < x^{(m)}_k > < \infty, \forall k, m \). Also let \( w^{(m)}_k(x^{(m)}_k) \) be the probability density for \( x^{(m)}_k \) and our \( M \) sets have finite variance \( \sigma^2_m = \int_{-\infty}^{\infty} < (x^{(m)}_k - < x^{(m)}_k >)^2 > w^{(m)}_k(x^{(m)}_k)dx^{(m)}_k < \infty \)

We define\(^7\) \( C^{(m)}_n := \sum_{k=1}^{n} < (x^{(m)}_k - < x^{(m)}_k >)^2 > \) and \( D^{(m)}_n := \sum_{k=1}^{n} \beta^{m}_{k,2+\delta} \) for some \( \delta > 0 \) where the absolute moment \( \beta^{m}_{k,2+\delta} \) is defined as

\[
\beta^{m}_{k,2+\delta} = < |x^{(m)}_k - < x^{(m)}_k > |^{2+\delta} >
\]

If for some \( \delta > 0 \), \( \exists \beta^{m}_{k,2+\delta} \) and the limit

\[
\lim_{n \to \infty} \frac{D^{(m)}_n}{(C^{(m)}_n)^{1+\frac{\delta}{2}}} \to 0, \forall m
\]

than the probability density of the \( M \) standardised random variables

\[
u_m = \sum_{k=1}^{n} (x^{(m)}_k - < x^{(m)}_k >)/(C^{(m)}_n)^{\frac{1}{2}}
\]

asymptotically approaches an normal distribution with \( <\nu_m> = 0 \) and \( <\nu^2_m> = 1, \forall m \).

Defining the \( M \times M \) covariance matrix as

\[
k = < \nu_i \nu_j > , i \land j \in \{1, 2, \ldots, M\}
\]

the joint probability density of the set of random variables \( \{\nu_m\}_m \) asymptotically and uniformly approaches a distribution given by

\[
\lim_{n \to \infty} W_n(\nu_1, \ldots, \nu_m) \to (2\pi)^{-M/2} |k|^{-\frac{1}{2}} e^{-(2\nu^T k \nu)^{-1}}
\]

Here \( |k| \) is the determinant of the covariance matrix \( k \) and \( w^T k^j \) is the contraction of co-vector (row) \( u_i \) and contra-variant (column) vector \( u^j \) where we make use of index summation notation of Ricci calculus. This can alternatively be expressed as \( u^T k u \) with \( u^T \) being the transpose of \( u \).

---

\(^6\)This condition can be relaxed in certain circumstances [117] and it is of interest to see if speckle generation falls into that category of acceptable relaxations. That way we can mathematically confirm that Liapounoff’s formulation of the Central Limit Theorem can/cannot be applied to the field amplitudes of the optical fields and thus if the random walk approach is a good model (or not).

\(^7\)\(C^{(m)}_n = D^{(m)}_n|_{\delta=0} \) given all the \( M \cdot n \) random variables take real values only.
Theorem 2 can be applied on the simple case of a random walk with steps of equal length as will be demonstrated shortly. In our case, regarding speckle orthogonality we want to include the possibility of steps having uneven lengths. As such, we aim to adapt the formulation of this calculation in the future to our interest. We therefore have interest to display the existing calculation here which we have adapted from [115] regarding a simple 2D random walk. We start with some useful definitions:

**Definition 1** (Probability Density and Mean of Random Variables $x$). Let $\omega(x)$ be the **probability density function** of an random variable $x$. Let $\bar{x} = < x >$ represent the **mean (average)** of an distribution from an ensemble of random variables $x$ each following a probability distribution $\omega(x)$.

**Definition 2** (Characteristic Function). Let $\xi$ be an auxiliary variable. We define the **1-Dimensional Characteristic Function** $F_x(\xi)$ of an distribution of random variables $x$ as the Fourier Transform of the probability density $\omega(x)$ using auxiliary variable $\xi$

$$F_x(\xi) = < e^{i\xi x} > = \int_{-\infty}^{\infty} \omega(x) e^{i\xi x} dx$$

**Definition 3** (Generalized Characteristic Function). Let $g$ be an $m$-multi-valued variable obtained from an set of transformation of $n$-multivalued variables $\{y_1, ..., y_n\}$ i.e $g_k = g_k(y_1, ..., y_k)$. Here we implicitly assumed that $m \leq n$.

The **Generalized Characteristic Function** of multi-variable transformed variables is given as

$$F_g(\xi) = \int_{-\infty}^{\infty} \cdots \int W_n(y_1, ..., y_n) e^{i\xi^T dy} dy$$

*Here $\xi^T$ represents the transpose of the vector $\xi$ and $W_n(y_1, ..., y_n)$ the joint probability density function of the $n$-variables $\{y_1, ..., y_n\}$. Finally, $dy = dy_1 ... dy_n$.*

---

8Sometimes in the literature, authors can refer to this function as 'distribution density', or 'frequency function'. It may also be referred to as 'derivative of distribution function'. We will stick with the conventional 'probability density'.
Lemma 3. If the set of $n-$variables are statistically independent of each other and $\xi^T \cdot g = \xi \sum_{k=1}^{n} f(y_k)$ for some smooth function $f$ than we can simplify the generalized characteristic function $F_g(\xi)$ to

$$F_g(\xi) = \prod_{k=1}^{n} \int_{-\infty}^{\infty} \omega_k(y_k) e^{i\xi f(y_k)} dy_k = \prod_{k=1}^{n} F_f(y_k)(\xi)$$

Proof. The set \{\(y_1, \ldots, y_n\)\} is mutually independent iff $\omega_i(y_i) || \omega_j(y_j)$, $\forall i, j \in \{1, \ldots, n\} \subseteq \mathbb{N}$. This implies that $W_n(y_1, \ldots, y_n) = \prod_{k=1}^{n} \omega_k(y_k)$. With this knowledge we start

$$F_g(\xi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W_n(y_1, \ldots, y_n) e^{i\xi^T \cdot 2 dy}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\prod_{k=1}^{n} \omega_k(y_k)) e^{i\xi^T \cdot 2 dy}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\prod_{k=1}^{n} \omega_k(y_k)) e^{i\xi f(y_k)} dy$$

$$= \prod_{k=1}^{n} \int_{-\infty}^{\infty} \omega_k(y_k) e^{i\xi f(y_k)} dy_k$$

as required. Here we set up the $n-$multi variable auxiliary vector as $\xi = [\xi \ \xi \ldots \xi]^T$ ($\xi$ is a column vector with all elements taking a common value of $\xi = |\xi|$).

Our random walk will consists of discrete steps with random direction $\phi_k \in [0, 2\pi]$ and common length $\rho_k = \rho$, $\forall k \in \{1, \ldots, n\} \subseteq \mathbb{N}$. Together, the set \{\(\rho_k, \phi_k\)\} form a Polar coordinate system which is linked with an Cartesian coordinate system through the usual transformation equations

$$x_k = \rho_k \cos \phi_k$$
$$y_k = \rho_k \sin \phi_k$$

(11.12)

With equation 11.12 we can define a ‘random step vector’ $\mathbf{r}_k$ that fully describes each random step in the complex Argand space. This vector we defined the on the next page as definition 4.
Definition 4 (Random Step Vector). Let \( \hat{j}_x \) be the unit vector along the real \( x \)-axis and \( \hat{j}_y \) a unit vector along the imaginary \( y \)-axis in the usual complex Argand space. We define the Random Step Vector \( \mathbf{r}_k \) for each discrete random step of the random walk as

\[
\mathbf{r}_k = \hat{j}_x x_k + \hat{j}_y y_k
\]

If we start at the origin \( 0 \) and consider \( n \) steps we end up with a final displacement vector \( \mathbf{R} = \sum_{k=1}^{n} \mathbf{r}_k \). Every random walk will generate a different displacement vector \( \mathbf{R} \) and it is the probability distribution of the generation of these \( \mathbf{R} \) vectors that we are looking for implicitly.

For deriving the probability density it helps to transform the random step vector as

Definition 5 (Transformed Random Step Vector). Let \( \gamma > 0 \) and let \( \hat{j} = \hat{j}_x \cos \theta + \hat{j}_y \sin \theta \) be a unit vector pointing along \( \mathbf{R} \) s.t. \( \mathbf{R} = R(\hat{j}_x \cos \theta + \hat{j}_y \sin \theta) \) with \( R = |\mathbf{R}| \). We define the Transformed Random Step Vector as

\[
\mathbf{v} = \hat{j} \sum_{k=1}^{n} |\mathbf{r}_k - \mathbf{r}_k|\gamma
\]

Given that \( \mathbf{r}_k = 0 \) (given \( <\cos \phi_k> = <\sin \phi_k> = 0 \)) and since \( \rho_k = \rho > 0, \forall k \in \{1, \ldots, n\} \subseteq \mathbb{N} \), we can simplify the transformed random step vector to

\[
\mathbf{v} = \hat{j} \sum_{k=1}^{n} \rho_k \gamma
\]

(11.13)

We’re almost able to calculate the generalized characteristic function of the transformed random step vector \( F_{\mathbf{v}}(j\xi) \). The only missing ingredient are the input probability densities \( \omega_k(\mathbf{r}_k) = \omega_k(\rho_k, \phi_k) \) for each step \( \mathbf{r}_k \) where \( \omega_A(\rho_k) \) and \( \omega_P(\phi_k) \) respectively are the phasors amplitude and phase probability densities. Given all steps have the same magnitude \( (\rho_k = \rho, \forall k \in \{1, \ldots, n\} \subseteq \mathbb{N} \) and the direction \( \phi_k \) is completely random in the interval \([0, 2\pi]\) we have

\[
\omega_A(\rho_k) = \delta(\rho_k - \rho), \quad \omega_P(\phi_k) = \begin{cases} 
\frac{1}{2\pi}, & \text{for } \phi_k \in [0, 2\pi] \\
0, & \text{for } \phi_k \notin [0, 2\pi]
\end{cases}
\]

giving input probability densities \( \omega_k(\mathbf{r}_k) \) as

\[
\omega_k(\mathbf{r}_k) = \omega_k(\rho_k, \phi_k) = \omega_A(\rho_k)\omega_P(\phi_k) = \begin{cases} 
\frac{\delta(\rho_k - \rho)}{2\pi}, & \text{for } \phi_k \in [0, 2\pi] \\
0, & \text{for } \phi_k \notin [0, 2\pi]
\end{cases}
\]

(11.14)
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N.B. In equation 11.14 we invoked axiom 2 to describe the direction $\phi_k$ of each step when the phase lies outside the $[0, 2\pi]$ interval ($\phi_k < 0$ or $\phi_k > 2\pi$). Recall that axiom 2 allows us to make use of the existing isomorphism of angles to a full revolution on the unit cycle. This effectively maps phases outside the closed interval $[0, 2\pi]$ to within the closed interval $[0, 2\pi]$. The probability density $\omega_P(\phi_k)$ is therefore equivalent to restricting the phases $\phi_k$ to the closed interval $[0, 2\pi]$, the form used in equation 11.14.

Using equations 11.13 and 11.14 and lemma\textsuperscript{9} 3 we can now calculate the generalized characteristic function of the transformed random step vector $F_{\nu}(j\xi)$ as

$$F_{\nu}(j\xi) = \prod_{k=1}^{n} \int_{-\infty}^{\infty} \omega_k(\nu_k) e^{i\xi \nu_k} d\nu_k$$

$$= \prod_{k=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega_k(\rho_k, \phi_k) e^{i\xi \nu_k} d\rho_k d\phi_k$$

$$= \prod_{k=1}^{n} \int_{0}^{2\pi} \int_{-\infty}^{\infty} \delta(\rho_k - \rho) \frac{e^{i\xi \nu_k}}{2\pi} d\rho_k d\phi_k$$

$$= \prod_{k=1}^{n} e^{i\xi \rho_k}$$

$$= e^{i\xi \sum_{k=1}^{n} \rho_k}$$

To continue using this characteristic function we will make of the useful identity Lemma 4:

**Lemma 4.** The linear sum over index $k$ of the absolute moment $\beta_{k, \gamma}$ can be calculated as

$$\sum_{k=1}^{n} \beta_{k, \gamma} = \langle \sum_{k=1}^{n} |\nu_k - \xi_k| \rangle = -i \frac{d}{d\xi} F_{\nu}(j\xi) |_{\xi=0}$$

**Proof.** Recall that

$$F_{\nu}(j\xi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W_n(y_1, \ldots, y_n) e^{i\xi y^T \cdot \nu} d\nu$$

where we define $d\nu = d\nu_1 \cdots d\nu_n$. The derivative is then given by

$$\frac{d}{d\xi} F_{\nu}(j\xi) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (i\xi y^T \cdot \nu) W_n(y_1, \ldots, y_n) e^{i\xi y^T \cdot \nu} d\nu$$

$$= i \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \sum_{k=1}^{n} |\nu_k - \xi_k| \right) W_n(y_1, \ldots, y_n) e^{i\xi y^T \cdot \nu} d\nu$$

\textsuperscript{9}Since $\xi y^T \cdot \nu = \xi y^T \cdot \sum_{k=1}^{n} \rho_k = \xi \sum_{k=1}^{n} \rho_k$ using the fact that $\sum_{k=1}^{n} \nu_k = 1$. 

where we used definition 5 (without simplification) in the last step. If we now set \( \xi = 0 \) than we get
\[
\frac{d}{d\xi} F_v(j\xi)|_{\xi=0} = i \int_{-\infty}^{\infty} \left( \sum_{k=1}^{n} |r_k - \xi|^\gamma \right) W_n(y_1, \ldots, y_n) \, dr
\]
\[
= i < \sum_{k=1}^{n} |r_k - \xi|^\gamma >
\]
Scaling this with \(-i\) and we get the desired result
\[
< \sum_{k=1}^{n} |r_k - \xi|^\gamma > = -i \frac{d}{d\xi} F_v(j\xi)|_{\xi=0}
\]

The derivative mentioned in lemma 4 can be calculated as
\[
-i \frac{d}{d\xi} F_v(j\xi)|_{\xi=0} = -i \frac{d}{d\xi} \left( e^{i\xi \sum_{k=1}^{n} \rho^\gamma} \right)|_{\xi=0}
\]
\[
= \sum_{k=1}^{n} \rho^\gamma e^{i\xi \sum_{k=1}^{n} \rho^\gamma}|_{\xi=0}
\]
\[
= \sum_{k=1}^{n} \rho^\gamma
\]

Using lemma 4 and equation 11.15 we have thus shown that
\[
\sum_{k=1}^{n} \beta_{k,\gamma} = \sum_{k=1}^{n} \rho^\gamma
\]

Hence by sum index comparison that
\[
\beta_{k,\gamma} = \rho^\gamma
\]

Note that \( M = 1 \) in our 2D-random walk example with reference to theorem 2 hence the only values of \( m \) are \( m = 1 \). We can see from the theorem 2 and equation 11.16 that we have already calculated one of the necessary functions \( D_n^{(m=1)} \) of theorem 2
\[
D_n^{(1)} := \sum_{k=1}^{n} \beta_{1,2+\delta}^{(1)}
\]

We only have to perform the substitution \( \gamma \rightarrow 2 + \delta \) in equation 11.17 to induce that
\[
D_n^{(1)} = \sum_{k=1}^{n} \rho^{2+\delta}
\]

Here we have used the fact that only \( m = 1 \) is available such that \( \beta_{k,\alpha} = \beta_{k,\alpha}^{(1)}, \forall \alpha \).
Similarly it can be noted that we can calculate the other function \( C_n^{(1)} \) using a similar substitution as in equation 11.19 namely \( \gamma \rightarrow 2 \ i.e. \ \delta = 0 \).

\[
C_n^{(1)} := \sum_{k=1}^{n} \beta_{k,2}^1 \\
= \sum_{k=1}^{n} \rho^2
\]

This leaves the limit of the ratio to be calculated

\[
\lim_{n \to \infty} \frac{D_n^{(1)}}{(C_n^{(1)})^{1+\frac{\delta}{2}}} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \rho^{2+\delta}}{\left(\sum_{k=1}^{n} \rho^2\right)^{1+\frac{\delta}{2}}} = \lim_{n \to \infty} \frac{n \rho^{2+\delta}}{(n \rho^2)^{1+\frac{\delta}{2}}} \\
= \lim_{n \to \infty} \frac{n}{n^{1+\frac{\delta}{2}} \rho^{2+\delta}} = \lim_{n \to \infty} n^{-\frac{\delta}{2}}
\]

\( \delta > 0 \)

We have thus satisfied all requirements of theorem 2 and can now start implementing it to acquire the probability distribution of random walks of constant step length. With regards to speckle analysis, part of this implementation has already been demonstrated as discussed in section 11.3, needing only adaptation for wavelength dependence.
11.3 Principal Probability Densities

To discuss the statistics of speckles, we have to obtain the probability density function. We will start with the dual intensity and phase density function $\rho_{I,\theta}(I, \theta)$. We can derive this function analytically by allowing the number of elementary phasors $N$ to diverge towards $\infty$ and applying the Central Limit Theorem 2 (c.f. section 11.2).

The joint probability density function of the real and imaginary parts of the field can according to [115] be shown to be

$$\rho_{r,i}(A^{(r)}, A^{(i)}) = \frac{1}{2\pi \sigma^2} e^{-\frac{[|A^{(r)}|^2 + |A^{(i)}|^2]/(2\sigma^2)}}$$

This density function is called the Circular Gaussian Density Function.

We can use equation 11.20 to derive $\rho_{I,\theta}(I, \theta)$ using a change of variables. In particular, from equations 11.3 we can infer

$$\theta = \tan^{-1}\left(\frac{A^{(i)}}{A^{(r)}}\right)$$

which together with equation 11.4 gives a Jacobian determinant $\|J\|$ of

$$\|J\| = \| \begin{pmatrix} \frac{\partial I}{\partial A^{(r)}} & \frac{\partial A^{(r)}}{\partial \theta} \\ \frac{\partial I}{\partial A^{(i)}} & \frac{\partial A^{(i)}}{\partial \theta} \end{pmatrix} \| = \frac{1}{2}$$

resulting in a dual density $\rho_{I,\theta}(I, \theta)$ given by

$$\rho_{r,i}(\sqrt{I}\cos(\theta), \sqrt{I}\sin(\theta))\|J\| = \left\{ \begin{array}{ll} \frac{1}{2\pi} e^{-\frac{1}{2\sigma^2}}, & I \geq 0 \land \theta \in [-\pi, \pi] \\ 0, & \text{otherwise} \end{array} \right\}$$

We can separate the variables $I$ and $\theta$ by appropriate integrals to obtain the intensity density function $\rho_I(I)$ and phase density function $\rho_\theta(\theta)$:

$$\rho_I(I) = \int_{-\pi}^{\pi} \rho_{I,\theta}(I, \theta) d\theta = \left\{ \begin{array}{ll} \frac{1}{2\pi^2} e^{-\frac{1}{2\sigma^2}}, & I \geq 0 \\ 0, & \text{otherwise} \end{array} \right\}$$

$$\rho_\theta(\theta) = \int_{0}^{\infty} \rho_{I,\theta}(I, \theta) dI = \left\{ \begin{array}{ll} \frac{1}{2\pi}, & \theta \in [-\pi, \pi] \\ 0, & \text{otherwise} \end{array} \right\}$$

We can infer from equation 11.21 and equations D.1.2 that the intensity and phase of speckles are statistically independent as

$$\rho_{I,\theta}(I, \theta) = \rho_I(I)\rho_\theta(\theta)$$
11.4  Iterative Field Partitioning

For multiple bounces, does the formulation described in section 11.1 still hold? That formulation considered a single reflection of a single wavelength from an optically rough surface producing a reflection speckle pattern. With regards to the Ulbricht sphere, the light bounces multiple times before exiting producing higher-order reflection speckle patterns. The iterated partitioning of the optical field produces a more complicated speckle pattern whose properties we investigate in this section by modelling such a random walk.

11.4.1  First Order Reflective Speckles

Before considering higher order reflective speckle generation, it helps breaking-down/re-formalising the generation of the first order reflective speckle pattern in more detail. We can re-formalise a reflective speckle as follows: Let an incident optical field \( u(x, y, z) \) of amplitude \( \mathcal{A}(x, y, z) \) and phase \( \theta^A(x, y, z) \) be incident on an optically rough surface. According Huygens construction we get an ensemble (index \( m \)) of complex phasors of amplitude \( a_m(x, y, z) \) and phases \( \phi_m(x, y, z) \). We can express Huygens construction using \( \mapsto \) as

\[
u(x, y, z) = |\mathcal{A}(x, y, z)|e^{i\theta^A(x, y, z)} \mapsto \{ |a_m(x, y, z)|e^{i\phi_m(x, y, z)} \}_m\]

The phases \( \phi_m(x, y, z) \) can be calculated from the wave vector of the optical field \( \mathbf{k} \) and source-to-surface-site displacement vector \( \mathbf{r}_m \) as

\[
\phi_m(x, y, z) = \theta^A(x, y, z) + \mathbf{k} \cdot \mathbf{r}_m + \pi
\]

where the third term (\( \pi \)) stems from the phase-reversal upon reflection\(^{10}\). The second term stems from free-space travelling from the laser source to the rough surface with the first term the incident phase which we set to zero.

We can simplify the second term further by defining the roughness of the surface by an offset vector \( \delta_m \) (with \( |\delta_m| \leq \xi \)) from an ideal smooth surface (e.f. figure 11.3 (b) for an illustration). If we take the vector from the laser source to the image plane by bouncing from the ideal smooth surface to be \( \tilde{\mathbf{r}} \), then we find the vector \( \mathbf{r}_m \) from the laser source to the image plane by bouncing from the optically rough surface can be given as

\[
\mathbf{r}_m = \tilde{\mathbf{r}} + \delta_m \quad \text{(11.23)}
\]

\(^{10}\)This is the main difference between reflection and transmission generated speckle patterns.
CHAPTER 11. DETAILED ANALYSIS OF SPECKLES

Since we are interested in the speckle pattern at some imaging plane, we need to account for the additional path from the rough surface to the imaging plane. Depending on the orientation of $\delta_m$, as shown in figure 11.3 (a) the phasor phases at the imaging plane can be recast as

$$
\phi_m(x, y, z) = \begin{cases} 
\pi + k \cdot [\tilde{r}^a + \left(1 + \cos \vartheta_m - \sin \vartheta_m \sin \vartheta_m \right) \delta_m^a], & \text{co} \\
\pi + k \cdot [\tilde{r}^a + \delta_m^a], & \text{counter}
\end{cases}
$$

Here $\vartheta_m$ models the local reflection (at site $m$) and equals the angle from the light ray to the normal of local tangents whose orientation align with the local derivative at site $m$. These angles adhere to the law of reflection relative to their local tangents.

Figure 11.3: We define the offset vector $\delta_m$ for the local site $m$ on an optically rough surface. The lower dashed line represents the average of the surface \textit{i.e.} ideal smooth surface giving a travel path of $\tilde{r}^a$ for light. Depending on the surface being raised (co) or lowered (counter) from the average, the travel path for light is altered by a vector shown at the bottom in (a). For co-propagating vectors $\tilde{r}^a$ (incident part) and $\delta_m^a$, the extra travel path is dependent on the local application of the law of reflection ($\vartheta$). For reference, we illustrate the definitions of $\delta_m^a$ and $\tilde{r}^a$ in (b).

Note that the displacement vector $\tilde{r}^a$ is defined from the source to the imaging plane following a reflection at the ideal smooth surface. Therefore moving the surface up or down with respect to the smooth average does not produce the same phase shift in different directions.
11.4. ITERATIVE FIELD PARTITIONING

The beauty of this arbitrary and general approach is that $\tilde{r}^a$ can describe a multitude of geometries including curved geometries of a Ulbricht sphere. We also note that if the wavelength $\lambda$ of the laser is assumed to remain unaltered through the reflections, we have $|k| = 2\pi/\lambda$ i.e. wavelength dependence.

With regards to the phasor amplitudes $|a_m(x, y, z)|$, we can deduce a simple relation using a scaling factor $\chi_m^A \in [0, 1]$ such that

$$|a_m(x, y, z)| = \chi_m^A |A(x, y, z)|$$

The exact distribution of $\chi_m^A$ depends on the type of random walk we wish to model. If $N$ steps are of constant length for example, we can infer that $\chi_m^A = 1/\sqrt{N}$.

If we combine the local complex phasors to form an envelope, using Huygen’s construction we can infer the global optical field of the speckle pattern to the the form

$$|A(x, y, z)|e^{i\theta A(x, y, z)} = \sum_{m=1}^{N} \frac{1}{\sqrt{N}} |a_m(x, y, z)|e^{i\phi_m(x, y, z)}$$

Note that we changed the superscript $A \mapsto A$ and factored out the $1/\sqrt{N}$ factor into the $N$ phasor amplitudes for the first order speckle pattern.

**Summary**

We have built on the discussion in section 11.1 where the complex phasors were shown to represent steps in a random walk. Each step can be modelled as the step from an smooth surface (described by the law of reflection) with an offset vector $\delta_m$ and phase-reversal generating speckle patterns.

The speckle pattern can be formulated in a local description comprising of the ensemble of $N$ complex phasors. Alternatively, as per Huygen’s construction we can use a global representation of the optical field.

The probability distribution of a random walk can be calculated using the central limit theorem as shown in sections 11.2 and 11.3. Adapting those demonstrated calculations to include wavelength, the orthogonality can be tested for directly. This includes taking the continuous limit $N \to \infty$ to describe the rough surface.
11.4.2 Higher Order Reflective Speckles

By Huygen’s construction, the global incident first order speckle pattern gives a local second order speckle pattern \( i.e. \)

\[
|A(x, y, z)|e^{i\theta^A(x,y,z)} \mapsto \{|b_n(x, y, z)|e^{i\phi^b_n(x,y,z)}\}_n
\]

From subsection 11.4.1 we can infer the phasors to have amplitudes \( |b_n(x, y, z)| \) that can be modelled as

\[
|b_n(x, y, z)| = \frac{\chi_n^A|A(x, y, z)|}{\sqrt{N}}
\]

and phases \( \phi^b_n(x, y, z) \) given by

\[
\phi^b_n(x, y, z) = \begin{cases} 
\theta^A(x, y, z) + \pi + \vec{k} \cdot \vec{r}_b + \left( 1 + \cos \vartheta^b_n \right) \delta^b_n & (\uparrow) \\
\theta^A(x, y, z) + \pi + \vec{k} \cdot \vec{r}_b + \delta^b_n & (\downarrow) 
\end{cases}
\]

where (\( \uparrow \), \( \downarrow \)) respectively represent (counter, co) vector orientation of \( \delta^b_n \) relative to the incident half of \( \vec{r}_b \). Note that we cannot set the incident phase \( \theta^A(x, y, z) \) to zero this time as we already defined the global phase by the previous zeroing of \( \theta^A(x, y, z) \) in the first order speckle pattern.

The global envelope of the second order speckle pattern is therefore given by

\[
|B(x, y, z)|e^{i\theta^B(x,y,z)} = \sum_{n=1}^{N} \frac{1}{\sqrt{N}}|b_n(x, y, z)|e^{i\phi^b_n(x,y,z)} \tag{11.24}
\]

The speckle pattern given by equation 11.24 can describe the output optical field following a second bounce of light inside the Ulbricht sphere. Higher order reflective speckle patterns (or bounces inside a Ulbricht sphere) can be derived iteratively as the second order. If we use the Roman alphabet to label each order with the upper (lower) case for global (local) speckle representation, an \( M \) order reflective speckle pattern can be modelled analytically. The details of this scheme are shown below in figure 11.4.

![Figure 11.4: Repeated application of Huygen’s construction leading to an \( M^{th} \) order speckle pattern. The Roman upper (lower) case/row represent global (local) variables with \( m \) and \( n \) replaced with general Greek alphabet in the general scheme.](image-url)
11.4. Iterative Field Partitioning

11.4.3 Second Order Speckle Statistics

We can apply the random walk of a second order reflective speckle pattern to provide analogous equations to 11.8 regarding the real and imaginary parts of $B(x, y, z)$. We find

\[
    B^{(r)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |b_n| \cos \phi_n
\]

\[
    B^{(i)} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |b_n| \sin \phi_n
\]

(11.25)

We can take the average of equations 11.25 to get

\[
    \langle B^{(r)} \rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \langle |b_n| \cos \phi_n \rangle
\]

\[
    \langle B^{(i)} \rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \langle |b_n| \sin \phi_n \rangle
\]

(11.26)

To simplify equations 11.26 we can modify the derivation of principal probability density functions shown in 11.3 to demonstrate that $|A| \parallel \theta^A$. This namely reduces complexity in the deducing $|b_n| \parallel \phi_n$. To avoid excessive iteration of calculations, we refer the reader to appendix D for the $|A| \parallel \theta^A$ demonstration.

It remains to show that $|A| \parallel \theta^A \Rightarrow |b_n| \parallel \phi_n$. If the latter part holds namely, we can simplify equations 11.26 as

\[
    \langle B^{(r)} \rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \langle |b_n| \rangle \langle \cos \phi_n \rangle
\]

\[
    \langle B^{(i)} \rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \langle |b_n| \rangle \langle \sin \phi_n \rangle
\]

(11.27)

It helps to consider the following simple result from Lemma 1:

**Corollary 1.** Let $h(x_1, x_2, \cdots, x_n)$ be a function depending on the variables $x_1, \cdots, x_n$. Let $\rho$ denote the probability density function. If the variables $x_1, \cdots, x_n$ are mutually statistically independent (i.e. $x_i \parallel x_j$, $\forall i, j \in \{1, \cdots, n\}$ or $\rho_{x_1, \cdots, x_n}(x_1, \cdots, x_n) = \prod_{i=1}^{n} \rho_{x_i}(x_i)$, then

\[
    \langle h(x_1, x_2, \cdots, x_n) \rangle = \prod_{i=1}^{n} \langle h_i(x_i) \rangle
\]

Proof. Given the mutual statistical independence of the $n$ variables, we can iteratively apply Lemma 1 using an auxiliary function $h'_n(x_1, \cdots, x_{m-1}) = f(\alpha_i)$ and $h(x_m) = g(\beta_j)$. Start with $m = 1$ and end at $m = n$. 

\[\square\]
Ignoring the limit variable \( N \), the phasor amplitudes \( |b_n| \) depends on the variables \( \{ A, \chi_n^A \} \) \( \text{i.e.} \ |b_n| = f(A, \chi_n^A) \). Likewise, for the phasor phases \( \phi_n^b \) we can write \( \phi_n^b = f(\theta^A, \vartheta_n^b, \tilde{r}_n^b, \delta_n^b) \) (for a constant wavelength). If we assume the site of the second bounce by the light is uncorrelated\(^{11}\) to the first light bounce site, the variables \( \{ \chi_n^A, \vartheta_n^b, \tilde{r}_n^b, \delta_n^b \} \) become mutually independent of each other. Given the incident field is independent of the rough surface, we find \( \{ |A|, \theta^A \} \| \{ \chi_n^A, \vartheta_n^b, \tilde{r}_n^b, \delta_n^b \} \). Applying corollary 1 onto this set we thus infer equations 11.27 to hold as \( |b_n| \| \phi_n^b \).

Starting with the phase factor ensemble average, we re-write it as

\[
< \cos [\phi_n^b] > =< \cos [\theta^A + \pi + k \cdot (\tilde{r}_n^b + \mathcal{M} \delta_n^b)] > =< \cos \varphi_1 > < \cos \varphi_2 > -< \sin \varphi_1 > < \sin \varphi_2 >
\]

where we introduced the auxiliary phases \( \varphi_1 = \theta^A + \pi + k \cdot \tilde{r}_n^b \), \( \varphi_2 = k \cdot \mathcal{M} \delta_n^b \) and the matrix \( \mathcal{M} \) describing the local reflection is given by

\[
\mathcal{M} = \left\{ \begin{array}{ccc}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1 \\
\end{array} \right\} + \left( \begin{array}{cc}
\cos \vartheta_n^b & -\sin \vartheta_n^b \\
\sin \vartheta_n^b & \cos \vartheta_n^b \\
\end{array} \right), \ \downarrow \ \text{co} \\
\left( \begin{array}{cc}
\cos \vartheta_n^b & -\sin \vartheta_n^b \\
\sin \vartheta_n^b & \cos \vartheta_n^b \\
\end{array} \right), \ \uparrow \ \text{counter}
\right\}
\]

If we apply lemma 2, we can infer \( < \cos [\phi_2] > = < \sin [\phi_2] > = 0 \). As such, repeating the double-angle formula for \( < \sin [\phi_n^b] > \) we find \( < B^{(i)} > = < B^{(i)} > = 0 \) such that

\[
< B > = 0
\]

If we compare this to equation 11.11, we find the second order speckle pattern global amplitude behaves similar to the first order speckle pattern. In fact, given we demonstrated (through corollary 1) that \( |b_n| \| \phi_n^b \), we can iterate the first-order speckle statistics calculations from subsection 11.1.4 to infer

\[
< [B^{(a)}][B^{(b)}] > = \frac{1}{N} \sum_{m=1}^{N} \frac{|b_m|^2}{2} \delta_{ab}, a \land b \in \{ r, i \} \quad (11.28)
\]

\(^{11}\)This does not hold in situations such as light reflecting at different angles/paths to the same site. As the number of bounces \( M \) increases, the risk of same-site reflections increases making the assumption invalid.
Equation 11.28 is the counter part to equation 11.10. We thus showed the second order reflective\textsuperscript{12} speckle pattern obeys the same statistics as the first order assuming the sites of reflections are uncorrelated. Extending to higher order reflections, we retrieve the same principal probability densities as found in section 11.3. The speckle patterns of higher orders thus maintain the same statistical properties to first order patterns but yield smaller resolution \( \delta \lambda \) (from smaller speckles) due to longer optical paths (c.f. subsection 10.3.3).

\section*{11.5 Theoretical Progression}

Whilst not demonstrated here in full, we anticipate that modifying the calculation of the probability density using the field amplitude \( A \) allows us to find the analytical form of the wavelength dependent probability densities \( \rho_{I,\lambda}(I,\lambda), \rho_{\theta,\lambda}(\theta,\lambda) \) & \( \rho_{I,\theta,\lambda}(A,\theta,\lambda) \). This would pave the way for testing statistical independence between speckles of different wavelengths \( \lambda_1, \lambda_2 \) as:

\[
\rho_{I_1,I_2,\theta_1,\theta_2,\lambda_1,\lambda_2} = \rho_{I_1,\theta_1,\lambda_1} \cdot \rho_{I_2,\theta_2,\lambda_2}
\]

If this condition holds, the speckles of different wavelengths form an orthogonal basis. The particular function of the covariance would inform us of the theoretical limit of a speckle-based spectrometer. Thus by extending the derivations to include wavelengths namely, the question regarding the limits of orthogonality of speckles could in principle be solved directly.

With the principal probability distributions known even for higher order speckle patterns, it remains to describe the covariance of speckle patterns as a function of wavelength. We note that the orthogonality question must be investigated under the continuous limit \( N \to \infty \) i.e. an divergent random walk ensemble of infinitesimal step size.

A typical Ulbricht sphere also experiences spatial filtering at its input and output. Together with the wavelength retrieval algorithm and spatial Nyquist frequency, the future work towards speckle orthogonality analysis requires consideration of experimental implementation of reflective speckle spectrometers. The laser for example has a finite linewidth, experiences frequency-jitter and drift. Any vibrations or coupling errors (in and out of the sphere) also introduce temporal variations in the observed speckle pattern, the effects thereof not covered in this sub-thesis.

\textsuperscript{12}Easily extendible to transmission patterns by removing the \( \pi \) terms in \( \phi_n^b \).
Thus, to establish if speckle patterns are orthogonal (or to what degree they are), such effects need to be included. A computer simulation from the principal probability distributions would be very useful here along with an experimental confirmation using the Southampton mode-locked laser from sub-thesis I and the novel St. Andrews spectrometer [93].

11.5.1 Future Considerations

If we list the anticipated steps for future considerations of extending this work, we can summarise them in a seven step plan.

1. Model the complex phasors (random walk steps).
2. Derive the relevant wavelength dependent probability distribution.
3. Test speckle orthogonality under the continuous limit.
4. Adapt calculations to include spatial low-pass filter.
5. Make a computer model simulation of the analysis.
6. Introduce linewidth, jitter and drift.
7. Test computer simulation with retrieval algorithm.

Whilst some preliminary progress was made towards step 1, we cannot report significant results demonstrable to this sub-thesis. We have however identified a useful mathematical description of speckle patterns generated by an Ulbricht sphere. Our demonstration in this chapter shows the formulation can be applied appropriately to analytically model the speckles from the Ulbricht sphere and establish orthogonality of speckles.
Chapter 12

Simple Analysis of Speckles

The attempt of this chapter is to use correlation of speckle patterns, a quantity that is simpler to calculate than covariance or the probability distribution function. Whilst less fundamental than the calculations in the previous chapter 11, the results will be of interest from an experimental perspective as in practice the correlation between speckle pattern and wavelength is used in reconstructing the spectrum of a light source.

As we have demonstrated in the previous chapter, the second (and higher) order speckle patterns obey the same statistics as the first order speckle pattern. We can thus simplify the notation to first order in this work to study properties of speckles, even those generated by multiple reflections inside integrating spheres.

12.1 Testing Intensities Correlation Directly

We are interested in determining the orthogonality of the produced Speckle Fields for different wavelengths. One naive approach is to look at the correlation between two intensities. Although this on its own does not provide enough material to prove orthogonality, it can still provide useful insight for us. We start with the approach adapted from [87].

12.1.1 Multi-Wavelength Speckle Intensities

We wish to demonstrate how speckle patterns by different wavelengths add so future work can address limit 2). The addition of speckle patterns can be understood by theorem 3 whose proof we include.
CHAPTER 12. SIMPLE ANALYSIS OF SPECKLES

Theorem 3 (Intensity-Sum Representation of Speckles). Consider $N'$ optical fields of different wavelength $\lambda_k$ generating each an speckle pattern $I_k$. The total intensity $I_{\text{tot}}$ of the resultant speckle pattern constructed from the $N'$ superimposed speckle patterns is given by

$$I_{\text{tot}} = \sum_{k=1}^{N'} I_k$$

Proof. Let every optical field of wavelength $\lambda_k$ be represented by the complex electric field $u_k(x, y, z, t) = A_k(x, y, z)e^{i2\pi\nu_k t}$. Here $u_k$, $A_k$ and $\nu_k$ respectively represent the $k^{th}$ total field, complex field amplitude and field linear frequency reminiscent of equation 11.1. The complex field amplitude has a magnitude of $A_k(x, y, z)$ and a phase of $\theta_k(x, y, z)$ as defined by equation 11.2. For $N'$ optical fields we thus have $k \in \{1, \ldots, N'\} \subseteq \mathbb{N}$ for some positive integer $N' \in \mathbb{N}$.

By the principle of superposition we can calculate the total field as

$$u_{\text{tot}}(x, y, z, t) = \sum_{k=1}^{N'} u_k(x, y, z, t)$$

Thus using equation 11.3 we can calculate the total intensity $I_{\text{tot}}$ as

$$I_{\text{tot}} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| \sum_{k=1}^{N'} u_k(x, y, z, t) \right|^2 dt = \left\langle \sum_{k=1}^{N'} u_k(x, y, z, t) \right\rangle_T >_T \quad (12.1)$$

Since the modulus of a complex number $z$ is equal to $zz^*$ (denoting complex conjugation), equation 12.1 can be expanded to

$$I_{\text{tot}} = \left\langle \sum_{n=1}^{N'} |A_n(x, y, z)| e^{i(\theta_n(x, y, z)+2\pi\nu_nt)} \sum_{m=1}^{N'} |A_m(x, y, z)| e^{-i(\theta_m(x, y, z)+2\pi\nu_mt)} \right\rangle_T$$

where we use different sum indices $n$ and $m$ to differentiate between terms. By calculating the product of $I_{\text{tot}}$ the expression becomes

$$I_{\text{tot}} = \left\langle \sum_{k=1}^{N'} |A_k(x, y, z)|^2 + J >_T \quad (12.2)$$

where $J$ are the cross-terms given by

$$J = \sum_{n=1}^{N'} |A_n(x, y, z)| \sum_{\substack{m=1 \atop \{m \neq n\}}}^{N'} |A_m(x, y, z)| e^{i(\theta_n-\theta_m+2\pi[\nu_n-\nu_m]t)} \quad (12.3)$$
12.1. TESTING INTENSITIES CORRELATION DIRECTLY

From equation 12.1 we can infer our average $< \cdots >_T$ is a linear operator which simplifies equation 12.2 to

$$I_{tot} = < \sum_{k=1}^{N'} |A_k(x, y, z)|^2 >_T + < J >_T$$  \hspace{1cm} (12.4)

Since $J$ is a complex number we infer that $< J >_T = 0$ s.t.

$$I_{tot} = < \sum_{k=1}^{N'} |A_k(x, y, z)|^2 >_T$$  \hspace{1cm} (12.5)

To eliminate the average we can utilise the following property of the absolute value: $|A_k(x, y, z)|e^{i(\theta_k(x, y, z) + 2\pi \nu_k t)} = |A_k(x, y, z)|$ (i.e. $|u_k(x, y, z, t)| = |A_k(x, y, z)|$). Applying the property to equation 12.5 we infer

$$I_{tot} = \sum_{k=1}^{N'} I_k$$

where $I_k = |A_k(x, y, z)|^2$ as per equation 11.3. We have thus shown that $I = \sum_{k=1}^{2} I_k$ holds.

12.1.2 Correlation Testing by Axiom Introduction

For a multi-wavelength source such as a frequency comb the total speckle field will consist of multiple sub-speckle intensity fields that overlap. This approach is thus very useful to us for understanding the correlation between the different speckle intensity fields when investigating orthogonality amongst speckle patterns/fields of different wavelength. We consider two fields $I_k$ and $I_l$ and calculate the covariance amongst them.

The correlation between speckle patterns $I_k$ and $I_l$ of respective wavelengths $\lambda_1$ and $\lambda_2$ is given by $C_{kl}$ which is defined as

$$C_{kl} = \frac{< I_k I_l > - < I_k > < I_l >}{\sqrt{< (I_k - < I_k >)^2 > < (I_l - < I_l >)^2 >}}$$  \hspace{1cm} (12.6)

$I_k$ & $I_l$ are uncorrelated $\iff C_{kl} = 0 \iff < I_k I_l > = < I_k > < I_l >$ which is equivalent to saying that the covariance Cov[$I_k, I_l$] vanishes. Statistical independence hence orthogonality of the speckle patterns implies a vanishing covariance thus the contrapositive can be used to disprove orthogonality. This demonstrates the advantage of calculating the covariance $C_{kl}$ between the two speckle patterns $I_k$ and $I_l$. 

\[\square\]
We can use some results from the previous section, but as we are looking at more than one speckle field we will have to use an additional index to differentiate between them. The extra index will prove to make the correlation calculation more difficult, however the calculation will also be familiar to work from the previous section.

Using results from the previous section we can infer

\[
< I_k > = \langle [A^{(r)}_k]^2 + [A^{(i)}_k]^2 \rangle \\
= \frac{1}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} a_m^k |a_n^k| \cos \phi_m^k \cos \phi_n^k + < |a_m^k| |a_n^k| \sin \phi_m^k \sin \phi_n^k >
\]

Infer axiom 1

\[
= \frac{1}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} |a_m^k| |a_n^k| > (\frac{\cos \phi_m^k \cos \phi_n^k}{N} + < \sin \phi_m^k \sin \phi_n^k >) = \frac{1}{2} \delta_{mn}
\]

Inferred axioms 1 and 2

\[
= \frac{1}{N} \sum_{m=1}^{N} |a_m^k|^2
\]

Similarly or by index replacement \( k \rightarrow l \) we get \( I_l > = \frac{1}{N} \sum_{m=1}^{N} |a_m^l|^2 > \).

It remains to calculate the other term in the covariance \( \text{Cov}[I_k I_l] \).

\[
I_k I_l = ([A^{(r)}_k]^2 + [A^{(i)}_k]^2) ([A^{(r)}_l]^2 + [A^{(i)}_l]^2)
\]

\[
\]

(12.7)

where

\[
[A^{(r)}_k]^2 = \frac{1}{N} \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} |a_{m_1}^k| |a_{m_2}^k| \cos \phi_{m_1}^k \cos \phi_{m_2}^k
\]

\[
[A^{(i)}_k]^2 = \frac{1}{N} \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} |a_{m_1}^k| |a_{m_2}^k| \sin \phi_{m_1}^k \sin \phi_{m_2}^k
\]

For \([A^{(r)}_l]^2 \) and \([A^{(i)}_l]^2 \) we get something similar using the following replacement: \( k \rightarrow l \). We can thus infer

\[
[A^{(r)}_k]^2 [A^{(r)}_l]^2 = \\
\frac{1}{N^2} \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} |a_{m_1}^k| |a_{m_2}^k| |a_{n_1}^l| |a_{n_2}^l| \cos \phi_{m_1}^k \cos \phi_{m_2}^k \cos \phi_{n_1}^l \cos \phi_{n_2}^l
\]
A similar result holds for the other 3 terms in equation 12.7 by \( r \leftrightarrow i \) and using \( \cos \phi \) and \( \sin \phi \) accordingly. The average of the derived quantity thus takes the following form:

\[
\frac{1}{N^2} \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} < |a_{m_1}^k| |a_{m_2}^k| |a_{n_1}^l| |a_{n_2}^l| \cos \phi_{m_1}^k \cos \phi_{m_2}^k \cos \phi_{n_1}^l \cos \phi_{n_2}^l >
\]

where by lemma 1 (holds since \( |a| \perp \phi \) ) we find

\[
< |a_{m_1}^k| |a_{m_2}^k| |a_{n_1}^l| |a_{n_2}^l| \cos \phi_{m_1}^k \cos \phi_{m_2}^k \cos \phi_{n_1}^l \cos \phi_{n_2}^l > =
\]

\[
< |a_{m_1}^k| |a_{m_2}^k| |a_{n_1}^l| |a_{n_2}^l| > < \cos \phi_{m_1}^k \cos \phi_{m_2}^k \cos \phi_{n_1}^l \cos \phi_{n_2}^l >
\]

Unfortunately, the different index \( k \) and \( l \) prevents us from applying Lemma 2 as we do not know if the statistical independence between the phases and amplitudes of the phasors hold for \( k \neq l \). Axiom 1 only holds for \( k = l \) (one wavelength \( \lambda_k \) ) and a naive progression would be to relax that constrain to multiple wavelengths. We thus introduce axiom 3:

**Axiom 3.** Let \( k \) and \( l \) be indices corresponding to wavelengths \( \lambda_1 \) and \( \lambda_2 \). Also let \( m \) and \( n \) be indices representing positions/local sites on the optically rough surface illuminated to generate an speckle pattern.

The amplitudes of the complex phasors \( |a_m^k| \) and \( |a_n^l| \) are statistically independent of each other for \( k \neq l \) even if \( m = n \). Similarly, the phases of the phasors \( \phi_m^k \) and \( \phi_n^l \) are statistically independent of each other. Finally, the amplitudes \( |a_m^k| \) and phases \( \phi_n^l \) are similarly statistically independent of each other for \( k \neq l \). The case for \( k = l \) i.e. same optical field and rough surface follows axiom 1. This can be expressed more concisely as

a) \( |a_m^k| \perp |a_n^l| \), \( \forall m, n \in [1, 2, ..., N] \) including \( m = n, \ k \neq l \)

b) \( \phi_m^k \perp \phi_n^l \), \( \forall m, n \in [1, 2, ..., N] \) including \( m = n, \ k \neq l \)

c) \( |a_m^k| \perp \phi_n^l \), \( \forall m, n \in [1, 2, ..., N] \) including \( m = n, \ k \neq l \)

d) c.f. axiom 1 for the \( k = l \) case

Axiom 3 is the extended version of axiom 1 by applying the statistical independence across different optical fields for the same optically rough surface.

---

1 Beware of the clash of notation. In this section, \( k \) and \( l \) are indices for different speckle fields whether by a different rough surface or an different wavelength of light or both. When we introduced axiom 1 we used the indices \( k \) and \( l \) to represent local sites on the rough surface.
We can restate axiom 3 by saying the random walks of the phasors in Argand space are statistically independent of each other for different wavelengths of light. Axiom 1 on the other hands states that the steps in a single random walk are statistically independent of each other in direction (phase) and length (amplitude).

Let \( \sum_{m_1, m_2, n_1, n_2 = 1}^{N} \) If we infer axiom 3 we can further simplify our expression for \(< [A^{(r)}_k]^2 [A^{(r)}_l]^2 >\) to

\[
\frac{1}{N^2} \sum_{m_1 = 1}^{N} \sum_{m_2 = 1}^{N} \sum_{n_1 = 1}^{N} \sum_{n_2 = 1}^{N} < |a^{m_1}_k||a^{m_2}_l| > < |a^{n_1}_k||a^{n_2}_l| > < \cos \phi^{m_1}_k \cos \phi^{m_2}_l > < \cos \phi^{n_1}_l \cos \phi^{n_2}_l > = \frac{1}{2} \delta^{m_1, m_2} = \frac{1}{2} \delta^{n_1, n_2}
\]

which simplifies to

\[
<A^{(r)}_k|^2[A^{(r)}_l]^2 > = \frac{1}{4N^2} \sum_{m_1 = 1}^{N} \sum_{n_1 = 1}^{N} < |a^{m_1}_k|^2 > < |a^{n_1}_l|^2 >
\]

where the index contraction is by the Kronecker-Delta function \((m_1 = m_2 := m \text{ and } n_1 = n_2 := n)\).

Given axiom 3 allows us to write \(< [A^{(a)}_k]^2 [A^{(b)}_l]^2 > < [A^{(a)}_k]^2 > < [A^{(b)}_l]^2 >, \forall a \land b \in \{r, i\}\) and since \(< \cos \phi_1 \cos \phi_2 > = < \sin \phi_1 \sin \phi_2 > = \frac{1}{2} \delta_{12}\) we can infer \(w.l.o.g\.) that all the terms in the average of equation 12.7 are the same.

Thus the average of equation 12.7 gives

\[
< I_k I_l > = \frac{1}{N^2} \sum_{m_1 = 1}^{N} \sum_{n_1 = 1}^{N} < |a^{m_1}_k|^2 > < |a^{n_1}_l|^2 >
\]

Our covariance \(\text{Cov}[I_k, I_l]\) is thus given by

\[
\text{Cov}[I_k, I_l] = \frac{1}{N^2} \sum_{m_1 = 1}^{N} \sum_{n_1 = 1}^{N} < |a^{m_1}_k|^2 > < |a^{n_1}_l|^2 > - \\
(\frac{1}{N} \sum_{m_1 = 1}^{N} < |a^{m_1}_k|^2 >) (\frac{1}{N} \sum_{n_1 = 1}^{N} < |a^{n_1}_l|^2 >)
\]

\(= 0\) (12.8)
Thus by equation 12.8 we infer that if axiom 3 holds then the speckle patterns of two different wavelengths are uncorrelated. The problem of orthogonality can thus be re-phrased as testing the conditions under which axiom 3 holds.

Axiom 3 is not part of typical speckle statistics and we know property b) regarding independence between phases is false: the difference in phase between two known wavelengths ($\lambda_1$ & $\lambda_2 > \lambda_1$) reflecting from the same site with optical path $L$ from the source to the imaging plane is $2\pi L \frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2}$ hence deterministic (i.e. not independent). As axiom 3 describes a random walk however we might be able to relax some of the assumptions it makes. In particular we consider the work by S. Bernstein [117] describing the requirement of the statistical independence between the variables involved with the random walk. Adapting this work to our problem should offer insight into redundant restrictions of random walks which might include our unrealistic oversimplifications in axiom 3. Further work into axiom 3 is unfortunately not available following the termination of the speckle analysis project.

12.2 Uniqueness through Phase Analysis

For axiom 3 we only made use of the vanishing correlation when calculating the correlation of the intensities $I_k$ and $I_l$. Thus whether or not the phases and amplitudes are actually independent or not does not matter as long as their covariance vanishes. Thus it suffices to show that for example the correlation of the phases does not vanish for the intensity correlation to be non vanishing also. Whether the phases are statistically dependent or not does not directly matter for the intensity correlation.

In more detail, axiom 3 requires the phases $\phi^k_m$ and $\phi^l_n$ to be uncorrelated for $k \neq l$ at all sites $m, n \in [1, 2, ..., N]$ of the optically rough surface. Here indices $k$ and $l$ again represent wavelengths $\lambda_1$ and $\lambda_2$ respectively. However, we will define $\lambda_2 = \lambda_1 + \delta \lambda$ where $\delta \lambda$ is the wavelength offset.

Thus we investigate the correlation of the phases of the complex phasors at generic sites of the optically rough surface for the two different fields. Through the contrapositive we may than be able to disprove the validity of axiom 3 or possibly determine under which conditions the axiom holds. In the process of trying to cheaply test axiom 3 we pave the way for testing the uniqueness of speckle patterns per wavelength. The results of this are shown at the end of this section.
12.2.1 Phasors Phases Correlation

Similarly to equation 12.6 we can define the correlation between the phases as

\[
c_{kl} = \frac{<\phi^k_m \phi^l_n> - <\phi^k_m> <\phi^l_n>}{[<\phi^k_m - <\phi^k_m>]^2 > (<\phi^l_n - <\phi^l_n>)^2]^{\frac{1}{2}}} \tag{12.9}
\]

As in the case of intensity correlations, the phases of different fields are uncorrelated \(\iff\) the numerator of equation 12.9 \(i.e.\) the covariance between two phasors vanishes \(\text{Cov}[\phi^k_m, \phi^l_n] = 0\).

The complex phase is given by

\[
\phi^l_n = k^l \cdot r_n
\]

where \(k^l\) is the wave-vector of the phasor and \(r_n\) is the vector from the imaging plane reflecting from the \(n^{th}\) site travelling to the observation point \(O(x, y, z)\) (same as equation 11.23). Recall that we already eliminated the \(\omega t\) (frequency, time respectively) phase term by separation of variables in expressing the monochromatic electric field in equation 11.1. Thus other than an constant phase offset \(\phi_{\text{offset}}\) which we can take to be zero, we can by the monochromatic sinusoidal wave approximation of the electric field assume there are no other phase terms.

Given \(k^l\) and \(r_n\) are co-propagating parallel vectors we can infer that \(k^l \cdot r_n = k^l r_n\). And given the wave-number \(k\) is defined as \(k = \frac{2\pi}{\lambda}\) we can infer that

\[
\phi^k_m = \frac{2\pi}{\lambda_1} r_m, \quad \phi^l_n = \frac{2\pi}{\lambda_2} r_n = \frac{2\pi}{\lambda_1 + \delta\lambda} r_n
\]

We can express the vector \(r_m\) as a linear sum of the vector reflecting from the average of the optical rough surface to the observation point \(O(x, y, z)\) and an vector \(\delta_m\) whose length and direction (sign) varies uniformly on the order of a few wavelengths with \(m\) \(c.f.\) equation 11.23. It is this vector \(\delta_m\) that stores the information of the optically rough surfaces.

So \(\delta_m = |\delta_m| \in [-m'\lambda_0, m'\lambda_0]\) where \(\lambda_0\) is an wavelength and \(m' \in \mathbb{R}^+\) the maximum order of wavelengths \(\lambda_0\) of variations the rough surfaces experiences (since the correlation length \(\xi = m'\lambda_0\)). We perform this representation of \(r_m\) to every phasor hence both fields \(k\) and \(l\) as \(r_m\) is the same regardless of the field. Indeed, \(r_m\) only depends on the rough surface hence why it only has one label \(m\) representing local site of rough surface.
12.2. UNIQUENESS THROUGH PHASE ANALYSIS

Of course, by construction we have that $\tilde{\delta}_m$ is co-propagating and parallel to $\mathbf{r}_m$ hence $|\mathbf{r}_m| = |\tilde{\mathbf{r}}_m| + |\tilde{\delta}_m|$ s.t. $\mathbf{r}_m = \tilde{\mathbf{r}}_m + \tilde{\delta}_m$. With all these parts we can calculate the ensemble averages $< \cdots >_m$ of the complex phasors as

$$< \phi_m > = \frac{2\pi}{\lambda_1} < \mathbf{r}_m > = \frac{2\pi}{\lambda_1} < \tilde{\mathbf{r}}_m > + < \tilde{\delta}_m > = \frac{2\pi}{\lambda_1} < \tilde{r}_m > + < \mathbf{b}_m >$$

(12.10)

where the vanishing of $< \delta_m >$ follows from the uniform distribution of $\delta_m$ on a continuous closed interval centred at zero.

Similarly, for the the optical field of wavelength $\lambda_2$ we get

$$< \phi_n > = \frac{2\pi}{\lambda_2} < \tilde{\mathbf{r}}_n > = \frac{2\pi}{\lambda_1 + \delta_\lambda} < \tilde{r}_n >$$

(12.11)

This leaves the average over an ensemble of their product to be calculated:

$$< \phi_m \phi_n > = \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} r_m r_n >$$

$$= \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} < \tilde{r}_m \tilde{r}_n > + < \tilde{r}_m \delta_m + \tilde{r}_n \delta_n >$$

$$= \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} (< \tilde{r}_m \tilde{r}_n > + < \tilde{r}_m \delta_n > + < \tilde{r}_n \delta_m > + < \delta_m \delta_n >)$$

If we infer statistical independence of $\tilde{r}_\alpha$ and $\mathbf{b}_\beta$, $\forall \alpha, \beta \in [1, 2, ..., N]$, this can be simplified as

$$= \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} (< \tilde{r}_m \tilde{r}_n > + < \tilde{r}_m \delta_n > + < \tilde{r}_n \delta_m > + < \delta_m \delta_n >)$$

$$= \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} (< \tilde{r}_m \tilde{r}_n > + < \delta_m \delta_n >)$$

$$= \begin{cases} \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} (< \tilde{r}_m \tilde{r}_n > + < \delta_m \delta_n >), & m \neq n \\ \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} (< \tilde{r}_m \tilde{r}_n > + < \delta^2 >), & m = n \end{cases}$$

Here $\rho_{mn}$ is the joint probability density of getting $\tilde{r}_m$ and $\tilde{r}_n$, $\rho_m$ the joint probability density of getting $\delta_m$ and $\delta_n$ and $\rho_m$ and $\rho_m$ the probability density of getting $\tilde{r}_m$ and $\delta_m$ respectively.

$$= \begin{cases} \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} (\sum_{n=1}^N \sum_{m=1}^N \tilde{r}_m \tilde{r}_n \rho_{mn} + \sum_{n=1}^N \sum_{m=1}^N \delta_m \delta_n \rho_{mn}^b), & m \neq n \\ \frac{4\pi^2}{\lambda_1(\lambda_1 + \delta_\lambda)} (\sum_{n=1}^N \sum_{m=1}^N \tilde{r}_m \tilde{r}_n \rho_{mn} + \sum_{n=1}^N \sum_{m=1}^N \delta^2 \rho_{mn}), & m = n \end{cases}$$

Here $\rho_{mn}$ is the joint probability density of getting $\tilde{r}_m$ and $\tilde{r}_n$, $\rho_m$ the joint probability density of getting $\delta_m$ and $\delta_n$ and $\rho_m$ and $\rho_m$ the probability density of getting $\tilde{r}_m$ and $\delta_m$ respectively.
Recall that the expectation $< x >$ really is asking a question: What do I get on average if I grab many $x$ from an ensemble set $X$ over which we take average? Similarly, $< xy >$ is the average of taking $x$ from $X$ and $y$ from $Y$ over an large ensemble (high cardinality $X \& Y$) and similar for averages of multiple components.

Given there is no constrain between $m$ and $n$ we can infer from the frequentist definition of probability that all probability densities are equal to the reciprocal cardinality product of the ensemble. Thus $\rho_{mn}^r = \rho_{mn}^b = \frac{1}{N^2}$ and $\rho_m^r = \rho_m^b = \frac{1}{N}$. This is because the lack of constrain makes all combinations of $m$ and $n$ equally probable s.t. $\rho_{ab} = \rho_a \rho_b \forall a, b$ and since both $m, n \in [1, 2, ..., N] \subseteq \mathbb{N}$ we get the $\frac{1}{N}$ factor.

We can thus infer the following

$$< \phi_m^k \phi_n^l > = \begin{cases} \frac{4 \pi^2}{\lambda_1 (\lambda_1 + \delta \lambda)} \left( \frac{1}{N} \sum_{m=1}^{N} \tilde{r}_m + \frac{1}{N} \sum_{n=1}^{N} \tilde{r}_n + \frac{1}{N} \sum_{m=1}^{N} \delta_m \frac{1}{N} \sum_{n=1}^{N} \delta_n \right) & , m \neq n \\ \frac{4 \pi^2}{\lambda_1 (\lambda_1 + \delta \lambda)} \left( \frac{1}{N} \sum_{m=1}^{N} \tilde{r}_m^2 + \frac{1}{N} \sum_{m=1}^{N} \delta_m^2 \right) & , m = n \end{cases}$$

Unlike the case for $m \neq n$, when $m = n$ there is only one sum so we cannot separate the term. We cannot infer from any know result that the variance of $\tilde{r}_m$ or $\delta_m^2$ is zero unlike the covariance $\tilde{r}_m\tilde{r}_n$ or $\delta_m\delta_n$ which we have indirectly shown to be zero. Thus, the result for $m = n$ cannot be broke down further. This was discussed previously in lemma 1.

For $m \neq n$ we recall that the average over an ensemble of $\delta_n$ is zero. For the $\tilde{r}_m$ and $\tilde{r}_n$ averages we can see the averages are the same and differ only by index (which we average over so is contracted over). These arguments allow us to take the next steps below

$$< \phi_m^k \phi_n^l > = \begin{cases} \frac{4 \pi^2}{\lambda_1 (\lambda_1 + \delta \lambda)} \left( < \tilde{r}_m^2 > + < \delta_m^2 > \right) & , m \neq n \\ \frac{4 \pi^2}{\lambda_1 (\lambda_1 + \delta \lambda)} \left( < \tilde{r}_m^2 > \right) & , m = n \end{cases}$$

Using $< \phi_m^k >, < \phi_n^l > \& < \phi_m^k \phi_n^l >$ the covariance $\text{Cov}[\phi_m^k, \phi_n^l]$ can be calculated as shown in the following equation 12.12.
The covariance $\text{Cov}[\phi^k_m, \phi^l_n]$ is given by

$$\text{Cov}[\phi^k_m, \phi^l_n] = \begin{cases} 
4\pi^2 \frac{\lambda_1(\lambda_1 + \delta\lambda)}{\lambda_1(\lambda_1 + \delta\lambda)} \left( <\tilde{r}_m^2> + <\delta^2_m> - <\tilde{r}_m> <\tilde{r}_n> \right), & m \neq n \\
\frac{4\pi^2}{\lambda_1(\lambda_1 + \delta\lambda)} \left( <\tilde{r}_m>^2 - <\tilde{r}_m> <\tilde{r}_n> + <\delta^2_m> \right), & m = n \\
0, & m \neq n \\
\frac{4\pi^2}{\lambda_1(\lambda_1 + \delta\lambda)} \left( \sigma^2_{\tilde{r}_m} + <\delta^2_m> \right), & m = n 
\end{cases}$$

(12.12)

Equation 12.12 is what we expected namely that when $m = n$ there is a correlation due to the phasors being from the same location on the rough surface. Indeed, the phase can equal each other for a particular wavelength shift when the phases differ by a full cycle on the unit circle. From this we can understand that introducing a detuning of $\delta\lambda$ gives us a general correlation as the difference of the phases is cyclic in $\delta\lambda$ therefore describable by a function. We commented on this in part in section 11.4.3 regarding light bouncing at correlated sites.

When $m \neq n$ however the phasors come from different locations and the correlation is lost as the rough surface varies randomly and uniformly with location on the surface. So the correlation is lost due to the randomness of the optically rough surface for different locations. Thus the correlation $c_{kl}$ does not equal zero and axiom 3 does not hold. The main result of interest of axiom 3 namely the lack of correlation between different fields of the amplitudes and the phases and the phasors is also false because we demonstrated a direct correlation between the phases in this section.

### Length Dependence on Correlation

We place our Cartesian coordinate system $\{x, y, z\}$ so that the origin lies on the center of the average of the optically rough surface. We take the optically rough surface to be a symmetric object in the $xy$ plane to simplify the results here and place the imaging plane ($O(x, y, z)$ placed on the $z$-axis) a distance $L$ behind the source plane.
The ensemble average over \( \hat{r}_m \) will return the vector \([0, 0, NL]\) due to the symmetry of the rough surface, placement of planes and the positioning of the Cartesian axes. This allows us to rewrite the variance \( \sigma^2_{\hat{r}_m} \) of \( \hat{r}_m \) as

\[
\sigma^2_{\hat{r}_m} = <\hat{r}_m^2> - (NL)^2
\]

Similarly we can see that the \( <\hat{r}_m^2> \) term also equals \((NL)^2\) using the defined placement of the imaging and source planes. As such we infer that

\[
\sigma^2_{\hat{r}_m} = (NL)^2 - (NL)^2 = 0
\]

The exercise remains to formally confirm that \( \sigma^2_{\hat{r}_m} \) converges to zero in general under the \( \lim_{L \to \infty} \) limit.

We have thus shown that the covariance \( \text{Cov}[\phi^k_m, \phi^l_m] \) decreases with increasing \( L \). To be more precise we have shown that the covariance \( \text{Cov}[\phi^k_m, \phi^l_m] \) hence correlation \( c_{kl} \) depends on \( L^2 \), the square of the interference length. Hence the correlation has been demonstrated to decrease with increasing interference length \( L \). Of course the term \( <\delta_m^2> \) does not change with \( L \) and will act as an fundamental limit to the correlation of the phases.

### 12.2.2 Comparing Phases for Two Wavelengths

We now focus on the claimed uniqueness of the speckle patterns for each wavelength of light. Just because the speckle patterns are not orthogonal that does not mean that they cannot be unique, just like two distinct non orthogonal vectors being unique. To tackle this claim we start once again with the phases \( \phi^k_m \) and \( \phi^l_n \) where again \( k \) and \( l \) correspond to wavelengths \( \lambda_1 \) and \( \lambda_2 = \lambda_1 + \Delta \lambda \) respectively. Recall that

\[
\phi^k_m = \frac{2\pi}{\lambda_1} r_m^k
\]
\[
\phi^l_n = \frac{2\pi}{\lambda_1(\lambda_1 + \delta \lambda)} r_n^l
\]

If we fix the optically rough surface we fix the values for \( \delta_m \) hence of \( r_m \).

---

2. \( N \) parallel vectors from wavelets traveling a distance \( L \).
3. By taking the scalar product of \( \hat{r}_m \).
4. When the source subtends a triangle with the image at \( O(x,y,z) \), the increasing separation \( L \) between the imaging and source induces an ever increasing dominance on common terms in \( <\hat{r}_m^2> \) and \( <\hat{r}_m^2> \).
12.2. UNIQUENESS THROUGH PHASE ANALYSIS

For simplicity we consider the case of \( m = n \). Thus we expect that for some \( \Delta \lambda \) that the phases equal each other by 1 or more unit cycles of phase \( i.e. \)

\[
\phi^k_n = \phi^l_n + 2m\pi , \ m \in \mathbb{Z} \tag{12.13}
\]

Note that here \( m \) is a different index from \( n \), we recycled the index \( m \).

If we can show that these two phases are not simultaneously equal for every \( n \in [1, 2, ..., N] \) for two arbitrary wavelengths than the speckle patterns are unique. This is equivalent to saying that \( \exists n \in [1, 2, ..., N] \) s.t. \( \phi^k_n \neq \phi^l_n \) (phase matching wise \( i.e. \) including turning number of unit cycle phases do not match) which clearly indicates a different in speckle patterns.

Note that this does not account for the variation in the amplitude coefficients \( a_{k,l} \) which in principle may be able to undo the effect of \( \delta \lambda \). But given these amplitudes are only affected by the total reflection coefficient \( R = R(\lambda) \) we assume the dispersion with wavelength does not affect the amplitudes too much. This does become of interest however for larger \( \delta \lambda \) but unless \( R \) is co-cyclic in \( \lambda \) with the phase difference \( \phi^k_n - \phi^l_n \) our proof still holds for the effect of the amplitudes does not undo the effect of the phases.

Having that in mind we can start calculating the condition for which we get phase matching by algebraically manipulating equation 12.13 as below

\[
\frac{2\pi}{\lambda_1} r_m = \frac{2\pi}{\lambda_1(\lambda_1 + \delta \lambda)} r_m + 2m\pi , \ m \in \mathbb{Z}
\]

\[
2\pi \left[ \frac{\lambda_1 - (\lambda_1 + \delta \lambda)}{\lambda_1(\lambda_1 + \delta \lambda)} \right] = 2\pi m , \ m \in \mathbb{Z}
\]

\[
m = -\frac{\delta \lambda r_n}{\lambda_1(\lambda_1 + \lambda_2)}
\]

Recall that \( r_n = \tilde{r}_n + \delta_n \ s.t.

\[
m = m(n) = \frac{-\delta \lambda (\tilde{r}_n + \delta_n)}{\lambda_1(\lambda_1 + \lambda_2)} \tag{12.14}
\]

Given that both \( \tilde{r}_n \) and \( \delta_n \) are uniformly distributed over an closed interval with respect to local surface site index \( n \) we can infer that \( m \) is also uniformly distributed over \( n \). We can thus conclude that \( m(n) \notin \mathbb{Z}, \forall n \in [1, 2, ..., N] \) which tells us that the phases do not all match hence we get different speckle patterns.
CHAPTER 12. SIMPLE ANALYSIS OF SPECKLES

This proof shows that for the same phasor amplitudes $a_k$ the speckle patterns are unique for each wavelength. We also argued that it is very difficult for a material to have a reflection coefficient $R$ which ensures that $\forall n \in [1, 2, ..., N], |a_n^k|e^{i\phi_n^k} = |a_n^l|e^{i\phi_n^l}$ and the majority of the know materials lie in this category.

This proof cannot be easily extended for the $\lambda_1 + \lambda_2$ and $\lambda_3$ case because the complex field amplitudes $A_{\lambda_1+\lambda_2} = A_{\lambda_1} + A_{\lambda_2}$ would contain terms of the form $|a_n^{\lambda_1}|e^{i\phi_{n}^{\lambda_1}} + |a_n^{\lambda_2}|e^{i\phi_{n}^{\lambda_2}}$ which cannot simply be pairwise compared to $|a_n^{\lambda_3}|e^{i\phi_{n}^{\lambda_3}}$. Thus we cannot use this as an short cut to prove orthogonality.

Also care must be taken to ensure that the phases which differ for different wavelength do not add up in the sum and cancel each other out to produce the same speckle pattern. Some extra work is needed to confirm that.

\footnote{In practice this corresponds to $R(\lambda_1) \approx R(\lambda_2)$ which certainly is true for smaller $\delta \lambda$. Here $R$ is the reflection coefficient of the material.}
Chapter 13

Conclusion

Due to transfers to other higher priority projects, apart from sporadic instances the progress on speckle studies was unfortunately terminated. This subsequently introduced time-constraints which limited the amount of relevant and novel results regarding uniqueness and orthogonality in the map between speckle patterns and optical wavelength. Our findings however do help lay out a theoretical basis for finding results by navigating the experimentally supported theoretical description of speckles in the literature, and by demonstrating how this mathematics can be applied to approach the uniqueness question. Novel results such as the resolution of reflection-based speckle spectrometers or the iterative field partitioning calculation provide some progress towards solving the uniqueness and orthogonality questions and provide insight into properties of speckles that have not been discussed previously in the literature. We summarise these results in this chapter.

We took two paths in attempting to solve the orthogonality problem of speckles when varying the wavelength of the laser (light source). Our first approach attempts to derive the principal probability density functions with the aim of testing statistical independence directly. Adapting Huygen’s construction to include higher order speckle patterns from multiple internal reflections inside the integrating (Ulbricht) sphere, we demonstrated the higher order speckle statistics are the same as first order speckle patterns (single reflection), though with smaller speckles. This result holds assuming the reflection sites are uncorrelated and also holds for transmission speckle patterns under the same condition. We therefore identified a mathematical description that only needs to be modified in order to derive a wavelength dependent probability density of speckles from reflections inside integrating spheres. Continuing from this result we outlined a seven step plan to complete the speckle orthogonality analysis including a computer simulation and in-lab experiment.
Our second attempt involved investigating intensity and phase correlations with the notion that the measured quantity in experiments is correlation. Using this simpler analysis method we demonstrated the intensity speckle patterns of two different wavelength (on the same rough surface) have a vanishing covariance under the assumption of independent random walks (axiom 3). Whilst this assumption is unrealistic in its current form, we have identified a possible test for relaxing this assumption. This might allow us solve the problem of uniqueness and orthogonality by testing when the assumption holds and why. Findings of this analysis could then be compared to the formal principal probability density analysis as a confirmation test.

Investigation into the correlation of the phases of phasors of different (wavelength) random walks showed the covariance vanishes at different sites, but not at the same site. This is despite a demonstration of the phases being unique at the same site. The variance between the phases of two phasors however converges to zero under the divergent limit for interference length: $L \to \infty$. This is in agreement with the literature review of transmission-based speckle spectrometers and our model of reflection-based speckle spectrometers which demonstrate the resolution of speckle devices to scale with $L^x$ where $x < 0$. The value of $x$ depends on the specifics of the light-matter interaction (e.g. mode mixing in fibers).

Our literature review on speckle devices suggests the theory is in quantitative agreement with experiments. Existing speckle devices have demonstrated up to sub-femtometer resolution (0.5 MHz at 780 nm) and have resolved hundreds to thousands of comb-lines of optical frequency combs with competitive resolution. From this review we modelled the resolution of reflection-based speckle devices from which we propose that submerging the device should reduce their resolution limit by $\frac{1}{n}$ and enhance their bandwidth by a factor that scales with the level of dispersion.

We investigated the limits in the performance (bandwidth/resolution) of speckle devices and found the bandwidth of devices to be limited by factors of 20 000 depending on the resolution (typically a few pm). Introducing a coarse separation of speckle patterns from different parts of the light spectrum can reduce the bandwidth restriction by $\sim 550$. Temperature and pressure stabilisation found the performance increase by $\sim 10$.

Finally, we comment that an aperture can be used mitigate aliasing in images, a concern of interest regarding the imaging of fringes for the atom interferometer discussed in sub-thesis IV.
Sub-thesis III

Enhancing Laser-atom Interactions
Chapter 14

Introduction

Atom interferometers involve the coherent manipulation of the quantum state of atoms. This manipulation is achieved through the atom-laser interaction, a mechanism that can be enhanced by studying sources of errors: imperfections and perturbations. In this sub-thesis we review systematic errors along with preliminary results from testing a coarse mitigation method: novel composite pulses adapted for atomic physics experiments.

14.1 Sub-thesis Structure

This sub-thesis starts with a literature review chapter (15) on atomic physics covering the energy structure of the Rubidum atoms we use along with a brief introduction to atom interferometry. This is followed by an experimental chapter (16) covering the state-preparation, experimental-set-up and simple atom interferometry experiments on so called mirror optical pulses.

We dedicate the subsequent chapter (17) to introduce composite pulses and our preliminary findings of tested composite pulses with comments on further applications in research. The various systematic errors affecting the interferometer’s performance is then discussed in chapter 18 which includes a list of limits in our experiments. A summary of our findings can be found in our conclusion in chapter 19.
14.2 Disclaimer

The work discussed in this sub-thesis involves the contributions of several researchers. Regarding the composite pulse computational work discussed in chapter 17, this is fully credited to fellow PhD student J. C. Saywell under the co-supervision of Prof. I. Kuprov. The bulk of the experimental configuration described in chapter 16 was first built by former PhD students Dr. S. Patel ([118]) & Dr. A. J. Dunnings ([119]) followed by a revamp by fellow PhD student M. S. Carey and Postdoc Dr. M. Belal. Several subsequent enhancements were also made by the author during their membership in the research group\(^1\). These can be found in both this sub-thesis and in the next sub-thesis (sub-thesis IV).

The computational work involved discourse between J. Saywell, the author and M. Carey regarding suggestions on computing the shape of composite pulses. Suggestions by the author include techniques such as the method of least squares which was implemented for optimising composite pulses. However, we stress that the actual scripting and primary contribution to the composite pulse simulations are by J. Saywell. Likewise apart from the listed work by the author in this and the next sub-thesis, the design and assembling of the experimental set-up is credited to M. Carey. Finally, the data-set on composite pulses in chapter 17 includes joint collaboration between J. Saywell, the author and M. Carey with all three being involved in measurements.

For the purpose of completeness, this sub-thesis will include some work involving the contributions by the aforementioned researchers. Whenever this happens, we will label them appropriately and clarify the labour division. This includes related research papers\(^2\) such as [120] where the author secured a revised acknowledgements from IOP Publishing and the University of Southampton Research Integrity and Governance Team regarding their contributions following a dispute with the research group.

\(^1\)Disclaimer: My measurements related to this sub-thesis were not made available post my departure from the research group due to confidentiality clauses in the research’s funding agreement. Along with labaratory access and other obstacles, not all data could be transferred prior to my departure. This sub-thesis’s data is therefore incomplete, limiting the scope of this sub-thesis.

\(^2\)Disclaimer: The research paper [121] lists me as an author and involves research on atom interferometry with the experimental set-up used in the work of this sub-thesis during my stay in the research group. Since I didn’t contribute to that research in that paper in any form, I chose not to included the works of the paper into this sub-thesis. I also wish to clarify that I have made a corrigendum request to the journal and the University’s ePrints services which led to a correction in the listed authors.
15.1 Atomic Structure of Rubidium

15.1.1 Gross Structure

For our experiments we make use of Rubidium atoms of isotope 85 ($^{85}\text{Rb}$). Rubidium is an Alkali metal hence has one outer electron whose corresponding orbital is well described by the solutions of the hydrogen atom’s orbital. This is as the inner electrons of Rubidium (forming full orbitals) shield the charge of the nucleus from the outer electron. Thus the outer electron experiences a potential equivalent to the potential seen by the electron of the Hydrogen atom.

The first-order calculation of the energies $E_n$ of the orbitals of Hydrogen depend only on the principle quantum number $n$ and is called the Gross Structure. The Gross structure calculates the energy-spectrum of Hydrogen as

$$E_n = -\frac{m}{2}\left(\frac{\alpha c}{n}\right)^2 , n \in \mathbb{N}$$

where $m$ is the mass of an electron, $c$ the speed of light in vacuum, $\hbar$ the reduced Planck’s constant and $\alpha$ the Fine Structure constant defined by $e$ is the elementary charge and $\epsilon_0$ the permittivity of free space as

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$
15.1.2 Fine Structure

The Gross structure treats the proton-electron interaction using a classical Coulomb potential and solves the time-independent Schrödinger equation after assuming separation of variables. This process induces a degeneracy in the azimuthal quantum number \( L \) which represents the orbital angular momentum of the electron. As a result, the sub-levels indexed by \( m_l \) can have the same energy in the Gross structure.

This degeneracy is lifted when accounting for Spin-Orbit Coupling (and other effects) leading to the Fine Structure of Hydrogen. This introduces a new quantum number \( J \) describing the total angular momentum \( J = L + S \) of the electron by accounting for both its orbital \( L \) and spin angular momentum \( S \). This introduces an energy shift which according to \([122]\) is given by

\[
\Delta E_{n,J} = -\frac{\alpha^2}{n^2} \left( \frac{n}{J+1/2} - \frac{3}{4} \right), \ n \in \mathbb{N}, \ J \in \{0, 1, \cdots, n-1\} \quad (15.1)
\]

As evident from equation 15.1, the states with higher \( n \) and \( J \) acquire a smaller energy shift. Whilst these shifts lift the degeneracy in \( L \), the Fine structure maintains degeneracy through \( J \) (sub-levels of index \( m_j \)).

15.1.3 Hyperfine Structure

Typically the Hyperfine Structure is needed when considering the spectrum of hydrogen-like atoms for atomic physics experiments. This energy-spectrum follows from considering the coupling between the total angular momentum \( J \) of the outer-electron and the total nuclear spin angular momentum \( I \) of the nucleus. This introduces a new quantum number \( F \) representing the total angular momentum of the atom (electron’s): \( F = J + I \). Calculating the perturbation from this coupling according to \([123]\) gives

\[
\Delta E_{J,I,F} = B \frac{\frac{3}{2}K(K + 1) - 2I(I + 1)J(J + 1)}{4I(2I - 1)J(2J - 1)} - C \frac{5I(I + 1)J(J + 1)}{I(I - 1)(2I - 1)J(J - 1)(2J - 1)}
- C \frac{5K^2(K^2 + 1) + K[I(I + 1) + J(J + 1) - 3 - 3I(I + 1)J(J + 1)]}{I(I - 1)(2I - 1)J(J - 1)(2J - 1)} + \frac{1}{2} AK
\]

where \( K = -J(J+1) - I(I+1) + F(F+1) \), \( A \) the magnetic dipole constant, \( B \) the electric quadrupole constant and \( C \) the magnetic octupole constant.

\(^1\) Usually upper case quantum numbers are reserved for a system. So \( L \) and \( S \) represent angular momenta for the combined electron system of an atom when considering one atom. However, given Alkali atoms only have one outer electron this corresponds to the angular momenta corresponding to the outer electron. Hence \( L = l \) and \( S = s \)
Since these four constants are atomic-state/transition dependent, we do not quote their numerical values. The three different energy structures following the consecutive couplings are however qualitatively shown in figure 15.1 and quantitatively for $^{85}$Rb in figure 15.2. This provides a simple overview of how the Hyperfine structure has thus lifted the degeneracy in $J$ but has a degeneracy in $F$ through the sub-levels indexed by $m_f$.

Figure 15.1: The energy structure (not to scale) of $^{85}$Rb as more perturbations are included. The standard naming of the five lowest $L$ orbitals are found in the bottom-left inset table. Above the $P$ orbitals not all $F$ Hyperfine states are shown. The states $|1\rangle$ and $|2\rangle$ are the lower and upper levels ground-levels. The transitions of interest are typically between ground and $F'$. 

Figure 15.2: The Hyperfine energy structure of $^{85}$Rb about the D2-line as calculated by [86, 123]. For the ground-levels $|1\rangle$ & $|2\rangle$ the rounding error is shown to maintain the same number of significant figures.
15.1.4 Zeeman Sub-structure

Laser-atom interactions in atomic physics experiments more fundamentally operate on the sub-states of the Hyperfine structure indexed by $m_f$. Due to the 'Zeeman effect' these sub-states shift in energy under the influence of a magnetic field. Hence dubbed 'Zeeman sublevels', the presence of a magnetic field strength $B$ lifts the degeneracy of the Hyperfine structure. For a small $B$ this energy-shift can be shown to give

$$\Delta E_{F,m_f,B} = \mu_B g_f m_f B$$

where $\mu_B$ is the Bohr Magneton and $g_f$ the Landé $g$-factor.

The Zeeman sublevels indexed by $m_f$ are not shown in figure 15.1 or 15.2. They are often thought of as the z-component of the respective angular momentum vector $F$ such that

$$F_z = m_f \hbar$$

15.1.5 Quantum Numbers

Alkali metals only have one outer electron so $S = \frac{1}{2}$ and for $^{85}$Rb the nucleus has angular momentum\(^2\) $I = \frac{5}{2}$. As Rubidium is element $Z = 37$ its first unfilled orbital has $n = 5$ in the ground state electron configuration\(^3\). This results in the spectrum of Rubidium to be filled from the bottom in figure 15.1.

Apart from the sub-level quantum numbers, one more relevant remains to this review: Parity. Parity $\chi$ also called 'space inversion' is a quantum number that for spherical harmonic eigenstates such as the eigenstates of the Hydrogen atom is given by

$$\chi(L) = (-1)^L \eta$$

where $\eta$ the intrinsic parity of the eigenstate. For electrons, $\eta = 1$ hence $\chi(L) = \pm 1$ [82, 124].

In integer steps, the general possible values of all quantum numbers are

$$n \geq 0, \ 0 \leq L \leq n - 1, \ -L \leq m_L \leq L$$

$$|L - S| \leq J \leq L + S, \ -J, S \leq m_J, m_s \leq J, S$$

$$|J - I| \leq F \leq J + I, \ -F \leq m_F \leq F$$

---

\(^2\)In the ground-state electron configuration.

\(^3\)I.e. at low temperatures where the electrons fill the lowest energy-levels first.
15.1.6 Selection Rules

Optical transitions in atoms arise from the formation of a multipole of the atom’s charge distribution when interacting with electro-magnetic fields. For light, the largest multipoles in atoms are electric dipoles leading to a class of transitions called Electric Dipole Transitions. In all transition classes, some transition rates vanishes leading to so-called 'forbidden transitions’ between certain quantum states. The mathematics describing the possible and forbidden transitions are called Selection Rules.

For a transition from an initial \( |\psi_i\rangle \) and final \( |\psi_f\rangle \) state of density \( \rho_f \), the transition rate \( W_{|\psi_i\rangle \rightarrow |\psi_f\rangle} \) is given by Fermi’s Golden Rule. For a perturbation Hamiltonian \( \hat{H}' \) this rules states

\[
W_{|\psi_i\rangle \rightarrow |\psi_f\rangle} = \frac{2\pi}{\hbar} |\langle \psi_f | \hat{H}' | \psi_i \rangle |^2 \rho_f
\]  

(15.2)

Using an electric-dipole approximation as perturbation and applying the Wigner-Eckart Theorem (Laporte’s rule for parity \( \chi \)), the selection rules for the allowed electric dipole transitions can be calculated [125] to give

\[
\Delta S = 0, \ \Delta L = \pm 1, \ \Delta \chi = 0 \\
\Delta J = 0, \pm 1 \ (J = 0 \rightarrow J' = 0) \\
\Delta F = 0, \pm 1 \ (F = 0 \rightarrow F' = 0) \\
\Delta m_l = 0, \pm 1
\]  

(15.3)

If we consider the selection rules on the Zeeman sublevels we find a polarisation dependence emerges. We set \( z \) as our Cartesian propagation/quantisation axis and let \( \pi (\sigma) \) define linear (circular) polarisation of photons moving along this axis. Using the conventional definitions and notation as per [119], the polarisation dependencies can then be summarised as

\[
\Delta m_f = \begin{cases} 
\pm 1 & , \pi^+ = \frac{\sigma^+ + \sigma^-}{\sqrt{2}} \quad \text{Linear y-axis} \\
1 & , \sigma^+ \quad \text{Left Circular (Anti-clockwise)} \\
0 & , \pi^0 \quad \text{Linear z-axis} \\
-1 & , \sigma^- \quad \text{Right Circular (Clockwise)} \\
\pm 1 & , \pi^- = \frac{\sigma^+ - \sigma^-}{\sqrt{2}} \quad \text{Linear x-axis} 
\end{cases}
\]  

(15.4)

Note that transitions are driven when the wavector \( k \) of a linearly polarised photon is perpendicular to the quantisation axis \( z \) (\( \pi^0 \)) or when \( k \) of a circular polarised photon is parallel to the \( z \) (\( \sigma^\pm \)). Polarisations \( \pi^\pm \ (k \parallel z, y) \) drive superpositions. We sketch these allowed electric dipole interactions in figure 15.3 about the D2 line of \(^{85}\)Rb, including one Zeeman sublevel.
Figure 15.3: The Hyperfine energy structure of $^{85}\text{Rb}$ showing the allowed transitions (and $\pi^\pm$ driving superpositions) per electric-dipole transition selection rules. Dark arrows give the allowed Hyperfine structure transitions (equation 15.3) with the grey arrows examples of the polarisation dependent transitions on the Zeeman sublevels (equation 15.4). Finally, the $g_f$ factors are included in smaller font.

15.1.7 Experimental Techniques

There are several techniques which make use of the atomic structure of rubidium. Whilst together worthy of their own review research paper such as atomic clocks and atomic interferometers, here we highlight only two.

To set the initial state of atoms as a control variable of the experiment, we aim to prepare all atoms partaking in the interferometer into a common quantum state through a procedure called State Preparation. State preparation of atoms for experiments can be achieved using multiple techniques, one is called the Magneto-Optical Trap or MOT for short. First demonstrated in 1987, the technique combines laser cooling with spatial confinement through the Zeeman effect and a spatially-dependent magnetic field $B$ [126]. We discuss the MOT more extensively in subsequent chapters regarding its principle of operation, role and configuration in our experiments.

The highly frequency-selective transitions on the Hyperfine structure require specialised frequency-stabilisation techniques (locks). One such technique is called the Dichroic-Atomic-Vapor Laser Lock or DAVLL. Using $\sigma^-$ and $\sigma^+$ polarised light on a Doppler broadened absorption spectrum and creating an error signal by differential Zeeman shearing, this technique is able to stabilise the laser’s drift to $\leq MHz$ and also reduce its linewidth over a day period at a time [127]. They have also demonstrated $\leq 5 MHz$ stable frequency-lock’s of lasers with detunings up to several $GHz$ away from the D2 transition of $^{85}\text{Rb}$ [128]. This is on par to the specifications of our mode-locked laser characterised in sub-thesis I.
15.2 The Three ’R’s of Atom Interferometry

15.2.1 Rabi Flopping

Rabi flopping/oscillations is the term used to describe the strong field radiation interaction with a two-level atom. The resulting atomic response are oscillations between the probability amplitudes of the two levels at the Generalized Rabi frequency. By appropriate atomic-state preparation and frequency-stabilised laser systems, Rabi flopping can be used to conduct precise and controlled atomic physics experiments.

One Photon

We consider an two-level atom with lower level \(|1\rangle\) at energy \(\hbar \omega_1\) and upper level \(|2\rangle\) at energy \(\hbar \omega_2\) as illustrated in figure 15.4. Here \(\omega_j\) are angular frequencies corresponding to the state \(|a\rangle\), \(a \in \{1, 2\}\). The frequencies can be calculated using the Time-Independent Schrödinger equation (TISE).

![Figure 15.4: An atom modelled as a two level system. Two different energy levels \(|1\rangle\) (lower, blue) and \(|2\rangle\) (higher, red) operating at energies \(\hbar \omega_1\) and \(\hbar \omega_2\) respectively. An strong optical field incident on the atom of frequency \(\omega\) is detuned by \(\delta\) from the resonance frequency of the transition \(\omega_{21}\). Hence \(\omega = \omega_{21} + \delta\) with \(\omega_{21} = \omega_2 - \omega_1\).](image)

The Time Dependent Schrödinger Equation (TDSE) describes the time evolution for the wave function \(\Psi\) of a quantum system with Hamiltonian \(\hat{H}\) and is given by

\[
\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t} \tag{15.5}
\]

To describe the strong field, a near resonant optical field at angular frequency \(\omega \sim \omega_{21}\) is considered. This can be modelled as inducing a time-dependent perturbation term \(\hat{V}(t)\) in the Hamiltonian \(i.e.\)

\[
\hat{H} = \hat{H}_0(\vec{r}) + \hat{V}(t) \tag{15.6}
\]

where \(\vec{r}\) is the position vector of the electron relative to the atom’s reduced mass and \(\hat{H}_0(\vec{r})\) is the unperturbed time-independent Hamiltonian that satisfies the (TISE) given by

\[
\hat{H}_0(\vec{r})\psi_j(\vec{r}) = \hbar \omega_j \psi_j(\vec{r}) , j \in \{1, 2\} \tag{15.7}
\]
Here $\psi_1(r)$ and $\psi_2(r)$ are the two eigenstates of $\hat{H}_0(r)$ corresponding to the states $|1\rangle$ and $|2\rangle$ respectively. By separating variables and using the completeness theorem, solving the Schrödinger equation for a two level system gives the following general solution

$$\Psi(r, t) = c_1(t)\psi_1(r)e^{-i\omega_1 t/\hbar} + c_2(t)\psi_2(r)e^{-i\omega_2 t/\hbar} \quad (15.8)$$

where $|c_a|^2$, $a \in \{1, 2\}$ are the probability amplitudes of levels $|a\rangle$. Normalization of the probability requires that $|c_1|^2 + |c_2|^2 = 1$.

The work done on an electric dipole by displacing it from its neutral orientation gives a perturbation $\hat{V}(t)$ that is given by

$$\hat{V}(t) = d \cdot \mathbf{E}(t) \quad (15.9)$$

where $e$ is the elementary charge, $\mathbf{E}$ the electric field incident on the atom and the vector $d = -e\mathbf{r}$ describes the dipole charge separation and orientation [129]. The electric field can be taken as a sinusoidal wave of form

$$\mathbf{E}(t) = E_0 \cos(k \cdot r - \omega t)$$

where $k$ is the wave vector of the optical field. Given that the atom is very small compared to the wavelength of the optical field the spatial variations of $\mathbf{E}(t)$ can be ignored\(^4\): $|k \cdot r| \sim r/\lambda << 1$ for $r = |r|$ and $\lambda = 2\pi/|k|$. Thus the electric field is modelled as

$$\mathbf{E}(t) \simeq E_0 \cos (0 - \omega t) = E_0 \cos (\omega t)$$

using the even function property of cosine. Thus equation 15.9 becomes

$$\hat{V}(t) = \frac{e d \cdot E_0}{2}(e^{i\omega t} + e^{-i\omega t})$$

such that the matrix elements of $\hat{V}$ take the form

$$V_{ab}(t) = \frac{\langle a | d \cdot E_0 | b \rangle}{2}(e^{i\omega t} + e^{-i\omega t}) = -\frac{\mu_{ab} \cdot E_0}{2}(e^{i\omega t} + e^{-i\omega t})$$

where

$$\mu_{ab} = \langle a | d | b \rangle \quad (15.10)$$

is called the electric dipole moment/matrix element.

\(^4\)The term $k \cdot r$ is of first order correction and gives rise to magnetic dipole and electric quadrupole transitions. Higher powers of $k \cdot r$ give rise to higher pole magnetic and electric transitions corresponding to higher multi-pole moments [122].
As a transition is between two different states, we find $\mu_{aa} = \mu_{bb} = 0$ giving $V_{aa}(t) = V_{bb}(t) = 0$. From equation 15.10 we can also infer that $\mu_{ab} = \mu_{ba}^*$. Since $\hat{V}$ (and by extension $V_{ab}$) are measurable quantities, so must $\mu_{ab}$ take real values only giving $\mu_{ab} = \mu_{ba}$.

Using the TDSE with the prescribed semi-classical electric-dipole interaction, the probability amplitudes can be shown to take the following forms:

\[
\begin{align*}
\dot{c}_1(t) &= \frac{i\Omega R}{2} e^{-i\omega_{21}t} (e^{i\omega t} + e^{-i\omega t}) c_2(t) \\
\dot{c}_2(t) &= \frac{i\Omega R}{2} e^{i\omega_{21}t} (e^{i\omega t} + e^{-i\omega t}) c_1(t)
\end{align*}
\] (15.11)

where $\cdot$ represents a time-derivative and a new variable called the One-photon Rabi frequency has been defined as

\[
\Omega_R = \left| \frac{\mu_{12} \cdot E_0}{\hbar} \right|, \quad (\Omega_{ab} = \left| \frac{\mu_{ab} \cdot E_0}{\hbar} \right|)
\] (15.12)

According to the rotating-wave approximation, the fast oscillations $e^{i(\omega + \omega_{21})t}$ can be ignored which simplifies equation 15.11 to

\[
\begin{align*}
\dot{c}_1(t) &= \frac{i\Omega R}{2} e^{i\delta t} c_2(t) \\
\dot{c}_2(t) &= \frac{i\Omega R}{2} e^{-i\delta t} c_1(t)
\end{align*}
\] (15.13)

By taking another derivative of equation 15.13 a linear second order ordinary differential equation can be found taking the form

\[
\ddot{c}_2 + i\delta \dot{c}_2 + \frac{\Omega_R^2}{4} c_2 = 0
\] (15.14)

Solving this differential equation using the ansatz $c_2 = e^{-i\xi}$ we find

\[
c_2(t) = e^{-i\xi_{\pm} t}, \quad \xi_{\pm} = \frac{\delta \pm \sqrt{\Omega_R^2 + \delta^2}}{2}
\] (15.15)

Given that equation 15.14 is linear we infer from the completeness theorem that the general solution is given by a superposition between the two solutions in equation 15.15.

---

5Ignoring the physical interpretation of $a = b$, we can also infer these matrix element values form the selection rules: if $a = b$ then the parity and orbital angular momentum and sometimes $J,F$ selection rules are violating making the transition forbidden. This requires a vanishing electric dipole matrix element.
For a specific solution, the coefficients are determined by the initial conditions of the system. For a two-level system starting in $|1\rangle$ these are

$$c_1(t = 0) = 1 , c_2(t = 0) = 0$$
$$\dot{c}_1(t = 0) = 0 , \dot{c}_2(t = 0) = \frac{i\Omega_R}{2} e^{-i\delta t}$$

which returns the solution of the upper level probability as

$$|c_2(t)|^2 = \frac{\Omega_R^2}{\Omega^2} \sin^2 \left(\frac{\Omega t}{2}\right)$$

where the *Generalized One-photon Detuned Rabi frequency* is defined as

$$\Omega^2 = \Omega_R^2 + \delta^2 \quad (15.16)$$

By the probability normalization criterion the probability of finding the atom in the lower level is given by

$$|c_1(t)|^2 = 1 - |c_2(t)|^2 = 1 - \frac{\Omega_R^2}{\Omega^2} + \frac{\Omega_R^2}{\Omega^2} \cos^2 \left(\frac{\Omega t}{2}\right)$$

As evident form the sinusoidal functions appearing in the probabilities $|c_j|^2$ with $j \in \{1, 2\}$, there will be oscillations in time at the general Rabi frequency $\Omega$. As the cosine function is in anti-phase with its sine partner, the atom will exhibit flopping between occupying $|1\rangle$ and $|2\rangle$. For larger detunings $\delta$ it is evident from the pre-factor $\frac{\Omega_R^2}{\Omega^2}$ that a full population inversion is not possible. Hence the peak inversion fidelity is reduced by larger detunings $\delta$ from the transition resonant frequency $\omega_{21}$.

Please note that the calculation shown here does not include damping effects due to *population decay* and *dephasing processes*. A theoretical review on damped Rabi oscillations can be found in [130].
15.2.2 Raman Transitions

From the electric dipole selection rules we infer that a single laser cannot induce electric dipole transitions\(^6\) between \(|1\rangle\) and \(|2\rangle\). We can however induce such transitions using a two-photon laser-atom interaction called **Raman Scattering** which uses a *virtual level* that is detuned from a third state \(|3\rangle\). Note that the selection rules are not altered by using two optical fields as we demonstrate in appendix E.

Raman scattering was first observed in molecular systems and involves the exchange of energy of the atom with the photon. In *spontaneous* Raman scattering, the original photon of the laser is absorbed and a new photon is produced spontaneously\(^7\) [131]. Depending on which state the original photon acted on, either a photon of higher energy (*Anti-Stokes shifted*) is produced or of lower energy (*Stokes shifted*) as shown in figure 15.5. For stoke-shifted stimulated Raman scattering however as used in this work, the first photon of highest energy \(\hbar \omega_1\) is absorbed whilst simultaneously a photon of lower energy \(\hbar \omega_2\) is emitted from the atom due to stimulated emission by the second photon. The atom’s energy increases by the difference in energy \(\hbar \omega_{21}\) between \(|1\rangle\) and \(|2\rangle\). This dual laser arrangement we call ’Raman lasers’.

Whilst \{\(|1\rangle, |2\rangle, |3\rangle\}\) represent eigenstates, the virtual level detuned from \(|3\rangle\) is not. If we think of Raman scattering as a multi-step process, then virtual levels can be thought of as intermediate step where the atom has absorbed one photon but has not emitted the second one yet. Inelastic collisions between photons and atoms controls how large a detuning we can get whilst avoiding spontaneous emission following a direct excitation to \(|3\rangle\) [132, 133].

---

\(^6\)Magnetic dipole transitions can induce transitions between \(|1\rangle\) and \(|2\rangle\). Since the electric dipole dominates however, higher optical powers are needed to observe this.

\(^7\)The spontaneous Raman scattering process is typically quite rare thus requiring high-power lasers to be observed. This process can however be stimulated using a second laser at either the Stokes or the Anti-Stokes frequency.
Two-photon

When we use two Raman lasers at the Stokes and Anti-Stokes frequencies, the \( |1⟩ \leftrightarrow |2⟩ \) light-coupling is called Stimulated Raman Transitions. Here we can think of the two Raman scattering processes taking place simultaneously, each laser stimulating the other. Alternatively, we can understand stimulated Raman transitions as the beat-note between the two lasers driving the electric dipole forbidden transition with the \( ac \) stark-shift from each laser also affecting the other state. The later perspective we summarise here following an extensive review found in \[119, 134, 135, 136\].

Schematically, the two-photon Rabi flopping by stimulated Raman transitions involves the modelling of the atom as a three-level system as depicted in figure 15.6. Note that for Rubidium, the adjacent Hyperfine \( F \)-eigenstates about the third level \( |3⟩ \) are at most a few hundred \( MHz \) away from the excited Fine \( J \)-eigenstate. In comparison, a typical common laser one-photon detuning \( \Delta \) of \( \sim GHz \) is used on the D2 line. We can thereby model a single excited third Fine \( J \)-eigenstate we detune our virtual level from.

Figure 15.6: An atom modelled as a three level system. Two different energy levels \( |1⟩ \) (lower, blue) and \( |2⟩ \) (higher, red). Two optical fields connect each of these two levels to an virtual level detuned from \( |3⟩ \) by \( \Delta \). Here \( \delta \) represents the two-photon detuning from the \( |1⟩ \leftrightarrow |2⟩ \) transition and Laser \( j \) is used to couple \( |j⟩ \) to \( |3⟩ \), \( j \in \{1, 2\} \).

Stimulated Raman transitions can retrieve two-photon Rabi flopping between \( |1⟩ \) and \( |2⟩ \). This modifies the rate of oscillations to the Effective Two-Photon Rabi frequency given by

\[
\Omega_{2R} = \frac{\Omega_{13,1}^* \Omega_{23,2}}{2\Delta}
\]

where for \( j \in \{1, 2\} \), \( \Delta \) is the Common One-Photon Laser Detuning between \( |j⟩ \) and \( |3⟩ \) and \( \Omega_{j,3,j} \) represent the one-photon Rabi frequency for the \( |j⟩ \leftrightarrow |3⟩ - \Delta \) transition by Laser \( j \).

Similar to a single-photon detuning shown in equation 15.16, the effect of a two-photon detuning \( \delta \) can be shown to give a Generalized Off-Resonant Two-Photon Rabi frequency given by

\[
\Omega^2 = |\Omega_{2R}|^2 + (\delta_{ac} - \delta)^2
\]
where the Relative Light-Shift $\delta_{ac}$ is defined as

$$\delta_{ac} = \Omega_{ac}^1 - \Omega_{ac}^2$$

and for $j \in \{1, 2\}$, $\Omega_{j}^{ac}$ defines the Light-Shift of state $|j\rangle$ due to the ac stark effect from the other laser. Both laser’s act on both ground-states with the effect of the One-Photon Transition Detuning $\Delta_{|j\rangle}$, $a$ of the each laser $a \in \{1, 2\}$ introducing a light shift on $|j\rangle$ given by

$$\Omega_{j}^{ac} = \sum_{a} \frac{|\Omega_{|j\rangle}^{a}|^2}{4\Delta_{|j\rangle}^{a}}$$

where $\Omega_{|j\rangle}^{a}$ represents the on-resonance one-photon Rabi frequency for the $|j\rangle \leftrightarrow |3\rangle$ transition by Laser $a$.

If we let the atom move, we can model the two-photon detuning $\delta$ as

$$\delta = -\delta_L + \frac{\mathbf{k}_{\text{eff}} \cdot \mathbf{v}}{2\hbar} + \frac{Mv_R^2}{2\hbar} + g_f \mu_B m_f B$$

where $\delta_L$ is the relative laser frequency between the two Raman beams, $\mathbf{k}_{\text{eff}}$ the effective wave-vector of the two Raman lasers, $\mathbf{v}$ the velocity of the atom, $M$ the mass of the atom and $v_R$ the Raman Recoil Velocity.

The third term in equation 15.19 to be the Doppler shift of the atom relative to the Raman beams (effective\textsuperscript{8}). Likewise, the fourth term comes from the recoil\textsuperscript{9} by the Raman beams with the fifth term from the Zeeman effect under the influence of a magnetic field strength $B$.

**Raman Beam Configuration**

For stimulated Raman transitions there are two configurations regarding the orientation of the Stoke and Anti-Stoke optical fields: co- and counter-propagating. If the wavevector of these two fields is respectively given by $\mathbf{k}_1$ and $\mathbf{k}_2$ then the co- (counter-) configuration has $\mathbf{k}_1 \approx \mathbf{k}_2$ ($\mathbf{k}_1 \approx -\mathbf{k}_2$). The co- (counter-) configurations thus have an effective wavevector $\mathbf{k}_{\text{eff}} = \mathbf{k}_1 - \mathbf{k}_2 \approx 0$ ($\approx 2\mathbf{k}_1$). From equation 15.19 we can infer that the Doppler term almost vanishes in the co-propagating configuration making it mostly insensitive to Doppler shifts. The counter-propagating configuration in comparison is about twice as sensitive to Doppler shifts as a one-photon transitions.

\textsuperscript{8}The superposition of the two Raman beams produce an effective/resultant field with wave-vector $\mathbf{k}_{\text{eff}}$.

\textsuperscript{9}This term is therefore called the Raman Recoil Shift.
Velocity Selection

Depending on the configuration, stimulated Raman transitions either interrogate all atoms (co) or only atoms moving at a selective velocity (counter) due to the Doppler effect. The selected velocity consist of atoms moving at a speed \( v \) along the Raman interrogation (optical) axis that is given by

\[
v = \frac{\omega_{21} - c(k_1 - k_2)}{2(k_1 + k_2)}
\]  

where \( k_1 = |k_1| \) and \( k_2 = |k_2| \) are the wavenumbers of Stoke and Anti-Stoke optical fields respectively.

Due to broadening effects on the stimulated Raman transition there is a range of velocities \( \Delta v \) of atoms along the interrogation axis that induce a light-atom interaction (transition). For an interrogation duration of \( \tau \) by the Raman laser beams of the atoms the selective velocity class \( \Delta v \) can be calculated using a Fourier transform [137] to give

\[
\frac{\Delta v}{2\pi} \simeq (k_1 \tau)^{-1}
\]

For \( \text{Rb} \) atomic experiments the selective velocity class \( \Delta v \) consists of atoms moving at a speed \( 7.8 - 780.0 \text{ mm/s} \) along the interrogation axis for a Raman interrogation duration of 1-100 \( \mu\text{s} \). Atoms with a velocity outside \( \Delta v \) don’t undergo stimulated Raman transitions in the counter configuration [134].
15.2. **THE THREE ’R’S OF ATOM INTERFEROMETRY**

15.2.3 Ramsey Sequence

**Raman Optical Pulses**

Before we can introduce the third ’R’ in atom interferometry, we need to introduce Raman optical pulses. Note that an optical pulse in this sub-thesis is not related to the optical pulse defined in sub-thesis I. Here the pulses of duration \( \tau \) are more akin to switching the Raman laser beams on and off.

In the resonant case \((\delta = \delta_{ac})\), an atom starting in \( |1\rangle \) state is placed in a coherent superposition of the states \( |1\rangle \) and \( |2\rangle \) (with equal amplitude) when the Raman laser beams interrogate the atoms for a duration of

\[
\tau_{\pi/2} = \frac{\pi}{2\Omega}
\]

We will approximate \( \tau \) to be sufficiently short s.t. \( \Delta v \) includes all atoms.

If we observe the atoms after a \( \pi/2 \) (\( \tau_{\pi/2} \)) pulse we find that half occupy \( |2\rangle \) which experienced a momentum kick of \( \hbar k_{\text{eff}} \). For the counter-propagating configuration we thus find the atomic cloud bifurcate into two satellite clouds whose separation depends on the free evolution time\(^{10} \) \( T \) and the difference between the momentum kick \( \hbar k_{\text{eff}} \) and the atom’s initial momentum \( p \). The \( \pi/2 \) pulse thus acts as a beamsplitter of the atom’s matterwave leading to a mixed state over the atom’s velocity distribution and state.

We can similarly define a \( \pi \) pulse as the interrogation of atoms by resonant Raman lasers for a duration of

\[
\tau_{\pi} = \frac{\pi}{\Omega}
\]

This pulse exchanges the amplitudes in a superposition of \( |1\rangle \) and \( |2\rangle \) or starting from either \( |1\rangle \) or \( |2\rangle \) states results in a population inversion. An atom in \( |2\rangle \) with momentum \( p + \hbar k_{\text{eff}} \) is therefore placed in \( |1\rangle \) at momentum \( p \) and vice versa. Applying a \( \tau_{\pi} \) optical pulse to atoms in a superposition thus acts similar to a mirror as depending on the orientation of \( k_{\text{eff}} \) the satellite cloud separation can be made to decrease (combine) or increase (separate\(^{11} \)) with \( T \).

\(^{10}\) The duration in which Raman lasers are off after a Raman pulse.

\(^{11}\) Augmentation pulses defined as having reversed \( k_{\text{eff}} \) \( \pi \) can lead to satellite cloud separation increasing.
The $\pi/2$ and $\pi$ pulses are commonly called beamsplitter and mirror optical pulses respectively and can be used in a sequence to construct atomic versions of optical interferometers. Contrary to an optical interferometer where matter partitions light, in an atomic interferometer the light (optical pulses) 'partitions' matter through state mixing. This can be seen by considering an ensemble of cold-atom quantum systems: if we take an ensemble of atomic clouds, apply an optical pulse (interferometric) sequence to them and observe the atoms at time $t$ during the sequence, we would be able reconstruct a trace in time $t$ that resembles an optical interferometer. This reconstruction would namely show the momentum imparted onto the atoms by an optical pulses either combines, splits, refocuses or defocuses the physical trajectories of the atomic clouds. Over a specific sequence of pulses these trajectories can be shaped to form an interferometer.

**Atom Interferometry**

The Ramsey sequence first introduced in 1950 is an $\pi/2 - \pi/2$ interferometer sequence where $\pi/2$ pulses are separated by a free evolution time window of length $T$ [138]. The effect of this sequence is to give a periodic dependence on the upper state on the pulse spacing $T$ (momentum inclusive detuning $\delta$) for fixed $\delta$ ($T$). This can be explained using a matterwave approach in phase space of the sequence using classical momentum transfer and euclidean geometric arguments. A full quantum mechanical treatment of the $\left| 2 \right>$ state population probability can alternatively be found in [119] and is coarsely in agreement under a close resonance condition.

Figure 15.7 shows the Ramsey sequence being applied to a two-level atom starting in the lower level $\left| 1 \right>$ in phase space. In this picture, the $x$-coordinate evolves with time hence why the $x$-axis can also be viewed as the time coordinate $t$. When an $\pi/2$ pulse is applied the matterwave amplitude of the atom is split equally between the lower level $\left| 1 \right>$ and upper level $\left| 2 \right>$. This partial Rabi flop can in phase space be thought of as an impulse over time hence incrementally changing the momentum of the atom.

---

12For example, aluminium diffraction gratings splitting light beams into multiple paths
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The shaded regions denote the application of a $\frac{\pi}{2}$ pulse whose net effect is to split the population probability amplitude in two, like a 50/50 beam splitter. The population at $|2\rangle$ through the photon’s momentum transfer travel at an angle $\phi$ relative to $|1\rangle$ atoms. The $z$-axis $\hat{p}$ denotes atom’s momentum from the pulse sequence of separation $T$.

The momentum kick from the stimulated Raman transition changes the direction of momentum for the $|2\rangle$ matterwave amplitude by $\phi$. Thus when the second $\frac{\pi}{2}$ pulse is applied there will be a physical separation between matterwave amplitudes in the same level that can interfere with each other. As the levels $|1\rangle$, $|2\rangle$ are orthogonal there cannot be interference between them. The interferometric signal can be detected from either fluorescence/stimulated emission of $|2\rangle$ or from absorption imaging of $|1\rangle$ during an experiment. Hence primarily we measures the total population per level after a pulse-sequence.
For the purpose of calculating the phase shift due to interference, a more common 2D-projection of the phase space will be used. This is shown in figure 15.8. This figure also reveals a similarity to Young’s double slit experiment allowing for the derivation of the phase shift accrued in the Ramsey sequence.

Young’s Double Slit experiment intensity profile for two slits of finite aperture $a$ can be given as [139]

$$I(\psi) = I_0 \cos^2 \left[ \frac{\alpha}{2} \sin \left( \frac{\beta}{2} \right) \right]$$

where the variables $\alpha$ and $\beta$ are defined as

$$\alpha = \frac{2\pi d}{\lambda} \sin (\psi) , \beta = \frac{2\pi a}{\lambda} \sin (\psi)$$

(15.22)

Here $\lambda$ is the wavelength of the light used, $d$ the slit separation and $\psi$ describes the position on the screen.
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The variable $\alpha$ is what generates the interference between the two slits as evident from its proportionality on $d$. The $\beta$ variable describes the diffraction envelope i.e. single slit diffraction from a finite slit. This is again evident from the proportionality to the slit size $a$. To simplify the calculation the diffraction envelope is ignored by setting $\beta = 0$. Later the diffraction envelope will be briefly commented on.

Let a lower level $|1\rangle$ atom of mass $M$ moving along the x-axis have an initial momentum $P_a$. If $\hat{\mathbf{x}}$ is used to denote the x-axis’s unit vector, we have

$$P_a = Mv_a$$

The momentum of the photon $P_\gamma$ can more generally be given as

$$P_\gamma = P_\gamma \{ \cos (\theta) \hat{x} + \sin (\theta) \hat{y} \}$$

$$= \hbar |k_1 - k_2| \{ \cos (\theta) \hat{x} + \sin (\theta) \hat{y} \}$$

where $k_1$ and $k_2$ are the wave vectors of the Raman beams and $\theta$ defines an angle from the x-axis. The sum between these two momenta $P_\Sigma$ returns the momenta of the atom in the upper level $|2\rangle$ and is angled at $\phi$ from the x-axis, as shown in figure 15.9 (a,b). This can be re-expressed as

$$P_\Sigma = Mv_\Sigma$$

$$= \hat{x}(Mv_a + P_\gamma \cos (\theta)) + \hat{y}(P_\gamma \sin (\theta))$$

$$\Rightarrow v_\Sigma = \frac{1}{M} \sqrt{M^2v_a^2 + P_\gamma^2(\cos^2 \theta + \sin^2 \theta) + 2Mv_a \cos (\theta)P_\gamma}$$

$$= \sqrt{v_a^2 + \frac{P_\gamma^2}{M^2} + 2\frac{v_a}{M} \cos (\theta)P_\gamma}$$

where $v_\Sigma$ represents the speed a level $|2\rangle$ atom.

If we let $s_2$ represent the displacement of atoms in $|2\rangle$ after a time $T$ and $\phi$ be the angle at which level $|2\rangle$ atoms move as shown in figure 15.9 (b) & (c), we find that the spatial separation $d$ is given by

$$d = s_2 \sin (\phi) = v_\Sigma \sin (\phi)T$$

This allows for the calculation of $\alpha$ from equation 15.22 as

$$\alpha = \frac{2\pi}{\lambda} \sqrt{v_a^2 + \frac{P_\gamma^2}{M^2} + 2\frac{v_a}{M} \cos (\theta)P_\gamma \sin (\phi) \sin (\psi)T}$$ (15.23)
Figure 15.9: Vectors in the $xy$-plane projected phase space (not to scale).

(a) The wave vectors $k_i$ for laser $i$, $i \in \{1, 2\}$ induce a relative momentum kick $P_\gamma$ onto the atom of momentum $P_a$ which are rotated $\theta$ apart.

(b) The imparted atom acquires a new momentum $P_\Sigma = P_a + P_\gamma$ rotated at $\phi$ from the atom’s initial momentum.

(c) A Ramsey interferometer sequence applied to a 2-level system starting in the ground state $|1\rangle$. Through a double momentum kick from the Raman beams (a,b) the trace of $|2\rangle$ is orientated away from $|1\rangle$ by $\phi$. After a free evolution of $T$, a physical separation of $d$ is observed along $\hat{y}$ between the traces of $|1\rangle$ and $|2\rangle$. The trajectory of state $|i\rangle$ (‘interferometer arm’) is of length $s_i$ for $i \in \{1, 2\}$ for a $T$ durated time interval.

Using the matterwave picture the wavelength $\lambda$ is replaced with the de Broglie wavelength of the upper ground level. This is as only the upper ground level $|2\rangle$ will fluoresce thus contribute to the interference pattern.

$$\lambda = \frac{h}{p} = \frac{2\pi h}{Mv_\Sigma} \approx \frac{2\pi h}{Mv_a}$$

Given $P_a = Mv_a >> P_\gamma$, the expression for $\alpha$ from equation 15.23 can be simplified using a Taylor expansion on the rooted factor giving

$$v_a \sqrt{1 + \frac{P_\gamma^2}{v_a^2 M^2} + 2 \frac{v_a}{v_a^2 M} \cos(\theta) P_\gamma} \approx v_a (1 + \frac{P_\gamma^2}{2v_a^2 M^2} + \frac{v_a}{v_a^2 M} \cos(\theta) P_\gamma)$$

By taking scalar product between vectors we identify two useful relations:

$$v_a \cdot P_\gamma = v_a P_\gamma \cos(\theta), \quad P_\gamma \cdot P_\gamma = P_\gamma^2 = \hbar^2 (k_1 - k_2)^2$$

With these relations we can further simplify $\alpha$ to

$$\alpha \approx \frac{\sin(\phi)T}{\hbar} \left( Mv_a^2 + \frac{P_\gamma^2}{2M} + v_a \cdot P_\gamma \right) \sin(\psi)$$
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Our definition of $\alpha$ assumes photons are moving along the x-axis. The atoms in $|2\rangle$ however due to imparted momentum move at an angle $\phi$. Correcting for this with a y-projection, $(x(\sin \phi)^{-1})$ and of convenience $\psi = \frac{\pi}{2}$ we find

$$\alpha \approx \frac{T}{\hbar} \left( Mv_a^2 + \frac{P^2}{2M} + v_a \cdot P_a \right)$$

Using the half-angle formula for cosine, the expected interference of matter-waves (without diffraction i.e. $\beta = 0$) is given as

$$I(\psi = \pi/2) \approx \frac{I_0}{2} \left[ 1 + \cos (\alpha) \right]$$

$$\approx \frac{I_0}{2} \left[ 1 + \cos (\delta_{\text{classical}}T) \right]$$

where a new variable $\delta_{\text{classical}}$ has been defined as

$$\delta_{\text{classical}} = \frac{Mv_a^2}{\hbar} + \frac{P_a \cdot (k_1 - k_2)}{M} + \frac{\hbar \cdot (k_1 - k_2)^2}{2M}$$

The second term corresponds to a Doppler shift whilst the last term to a two-photon recoil shift with recoil velocity $v_{\text{Recoil}} = \frac{1}{M} P_a$.

The interference pattern can alternatively be calculated more accurately by determining the probability amplitudes $c_1$ and $c_2$ at a time $t$ after an atom-laser(s) interaction. Setting the Raman lasers two-photon detuning to zero and modelling for a Ramsey sequence, it can be shown that the upper level population probability $|c_2(2\tau \pi + T)|^2$ is given by

$$|c_2(2\tau \pi + T)|^2 \approx \frac{1}{2} \left[ 1 + \cos (\delta_{\text{quantum}}T - \phi_{L}^{(\text{rel})}) \right]$$

Here $\phi_{L}^{(\text{rel})}$ is the relative phase between the two $\frac{\pi}{2}$ pulses of $\tau \pi$ long and $\delta_{\text{quantum}}$ is given by

$$\delta_{\text{quantum}} = -\delta_L + \frac{P_a \cdot (k_1 - k_2)}{M} + \frac{\hbar \cdot (k_1 - k_2)^2}{2M}$$

i.e. $\delta_{\text{quantum}}$ is our two-photon detuning $\delta$ in the absence of a magnetic field ($B = 0$) [119, 135].

We highlight the resemblance between equation equations 15.24 and 15.26 confirming the similarity between Young’s double slit experiment and the Ramsey sequence. Comparing equations 15.25 and 15.25 shows how close our matterwave approximation result is to the formal quantum calculation.
The quantum treatment shows that the classical derivation using matter-waves corresponds to the \( \frac{\pi}{2} \) pulses to be in phase \( (\phi_L^{(rel)} = 0) \) i.e. identical slits. If the de Broglie wavelength had not been approximated, the classical equivalent to the laser detuning would have become \( \frac{Mv_a \Sigma}{\hbar} \). This is proportional to \( v\Sigma \), the speed of the atom in the upper level. A higher transfer of momentum from the Raman transition to the atom would classically correspond to an increase in \( v\Sigma \). The laser detuning \( \delta_L \) therefore does seem to have an connection to its classical counterpart\(^{13}\). Finally, the atomic interferometer ends up using fluorescence to measure the populations in each state. Fluorescence is random so over a number of atoms in all directions distributes evenly. This may explain why \( \psi = \pi/2 \) namely that unlike the Young’s double slit experiment for atoms, there is no directional dependence on the intensity distribution.

Returning attention to the diffraction envelope \( \beta \), we can make use of the solutions of the effective two-level system found in \([119, 135]\). For a laser interaction of constant intensity starting at \( t_0 \) lasting \( t \) long, the excited state population \( |c_2(t_0 + t)|^2 \) is given by

\[
|c_2(t_0 + t)|^2 = e^{i(\Omega_1 t + \Omega_2 t)} e^{-i\frac{\pi}{4}} \left\{ c_2(t_0) \left[ \cos \left( \frac{\Omega t}{2} \right) + i \cos \left( \Theta \right) \sin \left( \frac{\Omega t}{2} \right) \right] + c_1(t_0) e^{-i(\delta_0 - \phi_L)} \left[ i \sin \left( \Theta \right) \sin \left( \frac{\Omega t}{2} \right) \right] \right\}
\]

where \( \sin \left( \Theta \right) = \frac{\Omega_2}{\Omega} \) and \( \phi_L \) is the phase-offset at \( t = 0 \). For a single \( \pi \)-pulse with initial conditions \( c_1(t_0) = 1 \), \( c_2(t_0) = 0 \), \( t_0 = 0 \), the population \( |c_2|^2 \) simplifies to the following Lorentzian lineshape (times 1/2 for \( \pi/2 \) pulse)

\[
|c_2(t = \tau\pi)|^2 = \frac{\Omega_2^2 R}{\Omega_2^2 R + (\delta_{ac} - \delta)^2}
\]  \( (15.28) \)

Summary: The Ramsey sequence can be modelled as an atom interferometric equivalent to Young’s double slit light interferometer experiment with the \( \frac{\pi}{2} \) pulses generating two sources of matterwaves. By varying the pulse separation \( T \) or momentum inclusive detuning \( (\delta_{quantum} \text{ or } \delta_{classical}) \) the separation between the sources \( d \) is altered thus producing fringes. The diffraction envelope governing the fringe’s amplitude is a Lorentzian with relative light-shift as offset and two-photon rabi frequency as scaling factor. Note that unlike the double slit experiment however, the intensity distribution is isotropic.

\(^{13}\)Alternatively and more likely this term may represent the phase shift due to the separation of the wavepackets at the output of the interferometer.
15.2. THE THREE ‘R’S OF ATOM INTERFEROMETRY

Bloch Sphere Visualisation

A useful and common tool used to visualise the state of \( N \) atoms uses the density matrix \( \hat{\sigma} \) instead of state-vector \( \Psi \). Letting \(^*\) denote complex conjugation, the density matrix for a two-level system is given by

\[
\hat{\sigma} = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{pmatrix} = \begin{pmatrix}
|c_1|^2 & c_1 c_2^* \\
c_1^* c_2 & |c_2|^2
\end{pmatrix} = |\Psi\rangle \langle \Psi|
\]

The diagonal terms \( \sigma_{11} \) and \( \sigma_{22} \) are called ‘populations’ with the off-diagonal terms \( \sigma_{12} \) and \( \sigma_{21} \) called ‘coherences’. To see why it helps to consider a so-called ‘Pure State’ where the coherences and populations satisfy

\[
\sigma_{11} = \frac{N_1}{N}, \sigma_{22} = \frac{N_2}{N}, \\
\sigma_{11} \sigma_{22} = \sigma_{12} \sigma_{21} = |\sigma_{12}|^2
\]

In both pure and mixed states, the populations \( N_1, N_2 \) of atoms in states \(|1\rangle\) and \(|2\rangle\) respectively can be calculated directly from the density matrix populations. Given the relationship between the coherences and populations require definite phase-relations, the coherences are associated with the coherence of the quantum system.

Typically the evolution of the \( \hat{\sigma} \) is discussed in the ‘rotating frame’ entered by using the following transformation:

\[
\tilde{c}_1 = c_1 e^{-i\frac{\delta}{2}}, \quad \tilde{c}_2 = c_2 e^{i\frac{\delta}{2}}
\]

In the rotating frame, equation 15.29 still holds\(^{14}\) for a pure state. Pure states can thus be expressed using a single state vector.

In practice however, due to factors such as imperfect state preparation, various broadening mechanisms (\(e.g.\) Doppler, Power and Natural) and others, a practical quantum system is better described using a statistical ensemble of multiple state vectors: a Mixed States \([140]\). The elements of the density matrix of these mixed states are related as

\[
\tilde{\sigma}_{11}(t)\tilde{\sigma}_{22}(t) > |\tilde{\sigma}_{12}(t)|^2
\]

Whilst a statistical mixture is more general and more accurate in describing experiments, a pure state visualisation continues to be a helpful tool to coarsely conceptualise our quantum system.

\(^{14}\)Using the transformed counterpart.
To describe the quantum system, instead of using the TDSE on state $|\Psi\rangle$ we can now use the Optical Bloch Equation\textsuperscript{15} giving the evolution of the density matrix as
\[
i\hbar \frac{d\hat{\sigma}}{dt} = [\hat{H}_0(\ell), \hat{\sigma}] - i\hbar \hat{\Gamma} \hat{\sigma}
\]
where the first term involves the commutator operation and the last term describes spontaneous emission: $\Gamma$ is the scattering rate\textsuperscript{16} of $|2\rangle$ \textsuperscript{141}.

The visualisation advantage becomes obvious in the rotating frame when we introduce the following new variables:
\[
\begin{align*}
     u &= c_1c_2^* + c_1^*c_2 \\
     v &= i(c_1c_2^* - c_1^*c_2) \\
     w &= |c_2|^2 + |c_1|^2
\end{align*}
\]
Let the Bloch Vector $\mathbf{R}$ in Cartesian coordinates \{x, y, z\} be defined as
\[
\mathbf{R} = u \hat{x} + v \hat{y} + w \hat{z}
\]
Excluding atom loss, spontaneous emission and other decoherence/damping events, the evolution of the two-level quantum system can be fully described by the motion of this vector. This vector is confined to a surface called the Bloch Sphere, a sphere whose radius is set by the number of atoms $N$ \textsuperscript{142}.

The Bloch sphere visualisation of quantum systems is especially useful for atom interferometry involving two-level systems\textsuperscript{17}. For example, the Ramsey sequence evolution on can be visualised as in figure 15.10

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\textsuperscript{15}This equation is also known as the Liouville variant of the TDSE.

\textsuperscript{16}Inverse of the lifetime of the excited state $|2\rangle$.

\textsuperscript{17}This excludes laser-atom interactions such as Stimulated-Raman Adiabatic Passage (STIRAP) \textsuperscript{143}.

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Figure 15.10: A Bloch sphere representation of the Ramsey sequence for two different pulse separations $T$ (red and blue trajectories) at constant detuning $\delta = 0$. The atom starts in state $|1\rangle$ (green arrow) and ends in the state given by the red (or blue arrow depending on $T$). The first $\pi/2$ pulse leaves the atom in an superposition between $|1\rangle$ and $|2\rangle$ i.e. on the equator of the Bloch sphere. A free evolution of time $T$ rotates the Bloch vector about the z-axis. The final $\pi/2$ pulse rotates the Bloch vector again about the x-axis to its final state. A similar plot is acquired when $T$ is fixed and $\delta$ varies. Figure courtesy of J. Saywell.
Chapter 16

Static Atom Interferometer

Two atomic interferometers partake to an experimental level in this thesis, a 'static' and a 'dynamic' interferometer. The static one is named such as its optics remain static during an experiment. The dynamic interferometer (rotation sensor) discussed in sub-thesis IV by contrast has optical components that moves.

In this chapter we discuss the static atomic interferometer and the experiments we ran on it. The experimental set-up is shown along with $\pi$-pulse Rabi flop data. Note that all error’s in the data figures in this chapter are scaled by ten to make them visible. Also, only $\pi^+ - \pi^-$ Raman interrogation data are shown in this chapter.

To clarify, we inherited the experimental configuration from PhD student M. Carey and previous PhD students of the research group. For completeness we include a description of the experiment in this chapter and quote the measured\(^1\) coherence time & trigger-delay latency. Works by the author in this chapter include setting up\(^2\) the Magneto-Optical Trap (MOT), taking mirror pulse measurements, optimising the \{Raman laser, sensor and AOM switching\} systems and introducing minor modifications\(^3\) to the experimental set-up. The MOT laser system was also assembled by the author as described along with the AOM switching system in sub-thesis IV.

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\(^1\)Unfortunately the raw data was made inaccessible to the author following logistical challenges during candidacy.

\(^2\)Align lasers with the appropriate polarisation with magnetic fields and zero magnetic fields at the center of the vacuum chamber.

\(^3\)These include (not limited to) reducing the optical path of the Raman laser system by 50%, adding a depump laser beam, adding flips to the Raman laser system and securing previously loose components to the vacuum chamber.
16.1 Cooling, Trapping and Pumping

In our single vacuum chamber, *Alkali Metal Dispensers* have been used as an initial\(^4\) source of Rubidium (*Rb*). These dispensers heat *Rb* containing compounds to temperatures of 800 – 1100 \(K\) releasing *Rb* atoms into the chamber. These atoms will need cooling to \(< 3 \ mK\) \([144]\) for coherent laser interactions across a cloud of them. This cooling is accomplished on parts of the tails of the velocity distribution by *laser cooling*.

Laser cooling uses a closed transition cycle to impart momentum onto moving atoms within a velocity capturing range. Radiation pressure from red-detuned light slows down atoms within this range by exciting them into an excited state. The Doppler effect provides velocity selectivity whilst spontaneous emission averages out the initial momentum imparted on the atom during excitation. This process is depicted in figure 16.1.

For the existing static atom interferometer the closed transition cycle is made of the \( |2\rangle = |5S_{1/2}, F = 3\rangle \) ground state with a red-detuning from the \( |3\rangle = |5P_{3/2}, F' = 4\rangle \) excited state. By the selection rules (equation 15.3) this transition is not fully closed with atoms leaking into the \( |1\rangle = |5S_{1/2}, F = 2\rangle \) 'Dark' state. Our experiments therefore use an additional *repump* laser in resonance with the \( |1\rangle \leftrightarrow |5P_{3/2}, F' = 3\rangle \) transition.

![Figure 16.1: A $^{85}$Rb starting in for example the ground state $|2\rangle$ is excited to an excited state $|3\rangle$. Only those atoms that are Doppler blue-shifted into resonance from the red-shifted cooling beam can undergo this excitation inducing a velocity selectivity. After about the lifetime of the excited state the atom relaxes back to the ground state and the cycle continues. Rarely, the atom can leak into a dark state $|1\rangle$ which the cooling laser is not resonant with breaking the transition cycle. To close the cycle a *repumping* laser is used for each dark state. Note that the dark state in some configurations can be situated between $|2\rangle$ and $|3\rangle$.](image)

\(^4\)The Helmholtz coils used for trapping the atoms run at \(\sim 2\ A\) of current naturally heating the old chamber to about 340 \(K\). Continued use of dispensers from previous experiments has left an layer of *Rb* on the chamber walls that act as dispensers. As such the dispensers are rarely used nowadays.
16.1. COOLING, TRAPPING AND PUMPING

16.1.1 Optical Molasses

The first order cooling limit of laser cooling is due to spontaneous\(^5\) relaxations. We can estimate this limit by reviewing the cooling and heating processes during laser cooling. In 1D, a cooling laser incident on an atom applies a Doppler force \(F\) along the optical axis that is given by

\[
F = -\beta v
\]

where \(v\) is the speed of the atom and \(\beta\) a frictional coefficient. This force cools the atom of mass \(M\) by reducing the kinetic energy \(E_{\text{kin}}\) as

\[
\frac{dE_{\text{kin}}}{dt} = -\frac{2\beta}{M}E_{\text{kin}}
\]

The cooling laser induces four stimulated counteracting processes [145]: Two processes involve the momentum of absorbed-emitted photons to be anti-aligned resulting in no momentum shift whilst the other two have the momentum of photons aligned giving a momentum kick of

\[
\Delta P = \pm 2\hbar k
\]

where \(k\) is the photons wavenumber. Assuming \(s \ll 1\) and optimized detuning \(\delta = -\frac{\Gamma}{2}\), the aligned momentum processes heats the atoms by

\[
\frac{dE_{\text{kin}}}{dt} = \frac{\hbar^2 k^2 \Gamma}{m} s
\]

where \(\Gamma\) is the transition linewidth and \(s\) the saturation parameter.

The **Doppler cooling limit** is reached when \(\beta = 2\hbar k^2 s\) and these rates equilibrate [146]. This gives a **Doppler temperature** \(T_D\) of

\[
T_D = \frac{\hbar \Gamma}{2k_B}
\]

with \(k_B\) the Boltzmann constant.

This Doppler limit only depends on the transition linewidth \(\Gamma\) which for Rb gives 145.57 \(\mu\)K on the D2 transition [123] (used in these experiments).

\(\hline\)

\(^5\)A rare example in which spontaneous is favoured over stimulated. The cooling mechanism builds on spontaneous emission being random whilst stimulated emission directionality actually causes heating.
By retro-reflecting the cooling laser along the three Cartesian axes \( \{x, y, z\} \), thermal atoms within a particular velocity class can be cooled to the Doppler limit. These `'optical molasses'` can however be cooled further, and localized for higher phase-space densities.

16.1.2 Sub–Doppler Cooling

The Doppler limit arises from the equilibrium between spontaneous emission and heating from stimulated emission. Further cooling is possible in certain optical configurations\(^6\) due to formation of a polarization gradients in the retro-reflected cooling laser beam along the optical axis. Spatial variations in potential energy from the optical fields induce optical pumping into dark \( m_f \) states in a process called `'Sisyphus' cooling`\(^7\). The polarisation gradient provides local steepening of the Doppler force from the varying Zeeman sub-level energies. This lowers the temperature limit to thermal equilibrium with recoil-heating from the imparted recoil momentum when a photon is emitted by the atom [147]. This new sub-Doppler temperature limit \( T_R \) is called the `'Recoil'` limit [148] and is given by

\[
T_R = \frac{\hbar^2 k^2}{M}
\]

For \( \text{Rb} \) this limit is 370.47 nK [123]. Cooling below \( T_R \) is possible with novel methods such as Dark MOT’s [149] where cooling and repump laser light is blocked from reaching the centre of the cloud. Alternatives include Transparency schemes [150] where the cloud core is made transparent to the cooling and repump lasers by shifting the atom’s energy levels through the ac stark effect with a third laser. These schemes can be implemented with little modification: a gradient Neutral Density filter in-front of the cooling beams for a dark MOT and an additional laser for a transparency scheme.

16.1.3 Temperature from Velocity Distribution

We can estimate the temperature \( T \) of an ultra-cold atomic cloud using the standard deviation \( \sigma_v \) of its velocity distribution. The temperature \( T \) can then be given as

\[
T = M\sigma_v^2/k_B
\]

\(^6\)The \( \sigma^+ - \sigma^- \) configuration is used by the cooling laser in a MOT.
\(^7\)The name is inspired by the ancient Greek myth of the first murderer Sisyphus and the punishment he was sentenced to.
A better estimate is acquired however by fitting a measured velocity distribution to two summed Gaussians as the sub-Doppler cooling is more effective at the center of the cloud. The core temperature can namely be over 17 times lower then the broader background temperature of the cloud [119].

16.1.4 Magneto-Optical Trap

Optical molasses do not provide high atom number. The atom number can be increased however at the cost of heating the cloud to around the Doppler limit. This is accomplished through the Zeeman effect by applying an anti-Helmholtz coil configuration system. Both the current and new experiment make use of this arrangement.

The anti-Helmholtz coils provide a magnetic field minimum and approximately a linear magnetic field gradient inbetween the coils (placed to overlap with the centre of the vacuum chamber). This provides a spatial dependent energy-shift through the Zeeman effect.

The Zeeman effect also varies with each sublevel providing different energy-shifts to different magnetic sublevels. These sublevels are coupled using circular ($\sigma$) polarised light from the cooling laser. Opposing helicities upon reflection of cooling light thus enables a directional spatial force on the atoms. This forms a Magneto-Optical Trap (MOT) [151] as shown in figure 16.2, the starting point of the experiment.

![Figure 16.2: Two different viewpoints (a) and (b) of the cloud of Rubidium atoms in a magneto-optical trap configuration. (a) uses a Watec 902HC CCD camera whilst (b) a Thorlabs DCC1545M CMOS camera. In (b), the upper spot is the cloud of atoms with the other spots reflections of the laser beams.](image)
Unfortunately, Sisyphus cooling is suppressed due to heating as the potential energy landscape is altered from the magnetic fields. The presence of *Eddy currents* when the MOT coils are turned off further complicates the experiment as they induce magnetic fields that decay slowly \[152\]. As such, after the MOT coils are turned off, we wait 15 ms (with molasses cooling) in the current experiment before interrogating the atoms with optical pulses.

On a final note, the atom number can be increased by scanning the cooling laser offset from resonance, effectively scanning the capturing velocities over the velocity distribution. The miniaturised MOT laser system (\*c.f.\* subsection 22.2.3) intended for the dynamic atomic interferometer is compatible for this providing simple implementation for higher atom number.

### 16.1.5 Optical Pumping

A common practice in atomic interferometry is pumping the atoms into a single magnetic-insensitive Zeeman sublevel. Apart from better state preparation, this also reduces energy-level and initial state inhomogeneities across the atomic ensemble which otherwise cause higher sensitivity to interferometric phase variations induced by background magnetic fields.

For the new dynamic interferometer under construction (\*c.f.\* sub-thesis IV), this can be achieved using *Spin Polarization* where $\pi^o$ light\[8\] is applied onto the MOT cloud. Through the unique vanishing of one of the *Clebsch Gordon Coefficient* \[153\], light resonant to the appropriate transition enables both faster state preparation and optical pumping to $m_f = 0$.

By applying a $\pi$ pulse, the atoms can be placed from the mainly populated $m_f = 0$ sublevel to a $m_f = 0$ sublevel in the lower ground state. This is as Raman interrogations only couple like-indexed $m_f$ states to each other.

Any atom that is not put by the $\pi$-pulse into the ground state is generally pushed away using a laser that acts only on the upper ground state, in this case the cooling laser.

Optical pumping is not currently being used in the existing experiment, but is in progress of being added onto it\[9\]. As such, post composite pulse testing the existing experiment can act as an practice site for optical pumping.

\[8\]Light linearly polarised parallel along the quantization axis.

\[9\]Only the triggering remains to be implemented. The polarization beam for optical pumping is already aligned onto the cloud with the correct polarisation.
16.2 The Experimental Set-up

16.2.1 Raman Interrogation & Detection

The static atomic interferometer partially illustrated in figure 16.3 makes use of a free-running\textsuperscript{10} 'Raman' laser, a DFB laser of the same model as discussed in section 4.1. The DFB uses the same laser mount (custom configuration card) with its characterisation by M. Carey in coarse agreement with the author’s findings in sub-thesis I.

The Raman laser was first driven using a Thorlabs driver, but this was later replaced with a more stable Newport model. For optical stability, two Faraday Optical Isolators (FOI) are used in series. This is followed by a microscope slide that picks off a small portion of light used to measure the wavelength. The rest passes through a Toptica Tapered Amplifier (BoosTA) that couples into a fibre.

On the other end of the fibre, the polarisation is filtered using a quarter/half wave place pair\textsuperscript{11} and a polarising beam splitting cube (PBSC). The laser-light is then split by an 310 MHz Acousto-Optical Modulator\textsuperscript{12} (AOM) driven at $\nu_{AOM}$ which diffracts some light into the first upper sideband $\Omega_{AOM}^+$. Staying on the AOM route: The AOM sideband is coupled into a fibre once more and amplified by an DILAS Tapered Amplifier (TA). Protected from optical feedback by an FOI, the sideband passes through a fibre and given s-polarisation using a waveplate pair and a PBSC. The s-polarised AOM sideband then passes through a second 80 MHz AOM that is trigged by a Personal Computer (PC) for generating pulses.

As both Raman beams must pass through this AOM, the different components are separated by polarisation. Hence another PBSC separates the common optical path so the AOM sideband can be coupled into its own fibre. An additional half wave plate is used for coarse control of the polarisation on the other end of the fiber. There the sideband passes through Beam-Shaping Optics (BSO) detailed in [119] before interrogating the $^{85}$Rb atoms.

\textsuperscript{10}Non-frequency referenced like the cooling and repump lasers discussed in subsection 22.2.3.
\textsuperscript{11}Also used to optimise the BoosTA.
\textsuperscript{12}A good introduction to electro-optical modulators can be found in [154].

The same mount was used for driving the (DFB) as discussed in section 4.1. The PMT is shown in further detail in the centre-left inset. Three additional insets show the effects of RF-modulation by $\nu_{\text{AOM}}$ and $\nu_{\text{EOM}}$ generating from the carrier $\Omega_c$ the sidebands $\Omega_{\text{AOM}}^+$ and $\Omega_{\text{EOM}}^-$ respectively. The counter[co]-propagating [$\ast$] configuration is shown in the $\pi^- - \pi^- [\ast: \sigma^+ - \sigma^-]$ polarisation scheme. Note the MOT laser system, TTL Pulse-Blaster trigger connections and initialisation infrastructure are not shown.
16.2. THE EXPERIMENTAL SET-UP

Returning to the 310 MHz AOM, the undiffracted light passes through an Electro-Optical Modulator (EOM) driven at $\nu_{\text{EOM}}$ with some detuning offset parameter $\delta_L$. To first order this generates two small optical sidebands $\Omega_{\text{EOM}}^-$ and $\Omega_{\text{EOM}}^+$ equidistant from the optical carrier $\Omega_c$. Using a PBSC and a half-wave plate this is then coupled into a Mach-Zehnder Interferometer (MZI) locked using a custom PID servo built by M. Carey.

The servo measures part of the light exiting the MZI\(^{13}\) on a photodiode/detector that is amplified (ZVA-183-S+) and mixed with the EOM driving signal to generate an error signal. This is fed into a resistor R heating one arm of the MZI to suppress the carrier by about $-10$ dB.

A Pellicle Beam Splitter (PBS) partitions the MZI output into two paths, one for locking and the other passing through a DILAS TA. The amplified EOM sidebands then pass through a fiber before joining the optical path of the AOM sideband. By giving the EOM sideband $p$-polarisation, the optical sidebands are combined on a PBSC followed by the Raman switching 80 MHz AOM. The EOM sidebands are separated by the following PBSC and coupled into their own fiber like the AOM sideband.

The EOM sidebands pass through their own BSO which is set-up along the same optical axis aiming towards the $^{85}\text{Rb}$ atoms. This describes counter-propagating interrogation set-up. For co-propagating Raman beams, the fibres pass through a fibre-coupler to combine the beams at one port. Note that additional quarter wave plates are placed between a BSO and MOT chamber for the $\sigma^+ - \sigma^-$ scheme. Otherwise, the PBSC inside the BSO are rotated such as to provide $\pi^+ - \pi^-$.

The measurement of the upper-ground level population $|c_2|^2$ uses detection of the fluorescence by a temperature-stabilised Photo-Multiplier Tube (PMT). In series with a Standford Research amplifier, the signal is detected by an oscilloscope (’Scope’-MSOX3024A) and passed onto the PC\(^{14}\).

A useful modification is the added feature of dual monitoring on a Tektronix oscilloscope the different side-beams using a Fabry-Perot Optical Spectrum Analyser (FP-OSA). The 4\(^{th}\) port on the PBSC used to combine the AOM & EOM sidebands are viewed on the FP-OSA post amplification allowing for the distribution in power of the two EOM sidebands to be measured.

\(^{13}\)A FOI is placed right after the exit as the MZI proved sensitive to optical feedback.
\(^{14}\)A TTL signal generator to trigger the different instruments is connected to the PC.
In practice, up to 10% difference in power between the EOM sidebands have been measured. With the lower EOM sideband as one of the two optical fields driving stimulated Raman transitions, the power difference in EOM sidebands can now be accounted for. The leakage of the carrier post amplification can also be measured and optimised (reduced). The AOM sideband can also be viewed post amplification allowing for its stability to be checked. Using a beam blocker (Flip) the FP-OSA input can be switched to view the unamplified output of the MZI. Thus the suppression of the carrier by the MZI can be measured and optimised (reduced).

To clarify the phase-control, the EOM is driven by an Rohde & Schwarz SMB100A signal generator feeding an I&Q modulator to control its phase. The modulator is programmed using an arbitrary waveform generator for particular phase-shifts between the input and output RF signal. The output of the I&Q then passes through a power splitter that forwards the phase modified RF to the EOM (through an Qubig RF amplifier) and to the RF mixer (in series with an ZX60-6013-S+ amplifier) for generating the error signal for the MZI lock.

A list of equipment for the Raman laser and detection system can be found in tables 16.2 and 16.3. The MOT or vacuum system instruments are not shown in these tables. C.f. section 22.2 for the MOT laser and switching system. Further details on the vacuum system and MOT coils of this experiment can be found in the theses of previous PhD students [86, 118, 119].

16.2.2 Cooling and Repump Switching

Whist we refer the reader to the thesis of previous PhD students regarding the MOT vacuum chamber, and to section 22.2 for the MOT laser system, for completion we include here the a short description of the laser-switching used in the current experimental set-up for the cooling and repump lasers.

Like the Raman laser beams, an AOM is used for rapid switching of the MOT laser beam. The cooling uses a Gooch & Housego M110-1F-GHI1 110 MHz AOM driven by an AA-Opto-electronics MODA-110-B4-33 RF-driver that is triggered by an Keysight 33600A Arbitrary Waveform Generator (AWG). The repump AOM is of the same type as the 80 MHz Raman AOM for switching the Raman beams. It is driven by an 33600A AWG amplified using an RFPA-AP2500-1. To avoid the frequency-shift and reduce switching times, a double-pass configuration is used with both AOMs. A picture of this arrangement can be shown in figure 22.5 in the next sub-thesis IV.
16.2.3 Trigger Sequence

Regarding the triggering of the instruments by the Pulse-Blaster (TTL), a similar sequence\textsuperscript{15} structure was used as described in [119] for preparing, interrogating and detecting the atoms. Briefly, the sequence for a simple optical pulse can coarsely be described in the following seven stages:

1 [Loading the MOT]
   Helmholtz coils, Cooling, Repump AOM’s on to load MOT. (\([1 – 10]\) s)

2 [Sisyphus Cooling]
   Coils off, ramp cooling and repump to ‘molasses level’ \(\in [0, 1]\). (11 ms)

3 [State Preparation]
   Repump off, pump into dark state \(|1\rangle\) by cooling laser. (4 ms)

4 [Raman Sequence]
   Cooling off, switch Raman AOM & EOM per pulse sequence. \((\tau_{\text{pulse}})\)

5 [Readout-1]
   After a delay (10 \(\mu\)s), cooling on for fluorescence signal. (300 \(\mu\)s)

6 [Readout-2]
   Cooling off, repump on for preparing reference measurement. (100 \(\mu\)s)

7 [Readout-3]
   Cooling on, repump off for measuring reference of all atoms. (300 \(\mu\)s)

Table 16.1: Steps of an typical atom interferometric sequence (duration).

This sequence is repeated for different pulse settings\textsuperscript{16} during an experiment. Programmed in Python, the Pulse-Baster uses leading-edge triggers to trigger the respective instruments. Further details might be found in the thesis\textsuperscript{17} of M. Carey, the fellow PhD student who architected the current generation of the experiment. The remaining relevant details for this sub-thesis are the 10 ms delay between step 7 and step 1 between successive runs and the ramp time-scale of 5 ms in step 2.

Apart from the typically 2 s MOT loading time, two time-scales stand out namely steps 2 and 3. The reason for their combined length lasting up to 15 ms is to give time for the eddy currents to dissipate inside the coils and metal vacuum chamber. When the coils are turned off, the dynamic magnetic field from these currents can take up to 15 ms to vanish.

\textsuperscript{15}The only exception was the lack of a depump laser for optical pumping. The author had revived the set-up used by A. Dunning and prior to their departure worked on scripting an optical pumping scheme.

\textsuperscript{16}These include the laser-phase profile \(\phi_L, \phi_{\text{rel}}^L\), laser detunings \(\delta_L, \Delta, \text{etc.}\)

\textsuperscript{17}‘Velocimetry, trapping and optimal coherent manipulation of atomic rubidium’, (2020)
Regarding the molasses level, for the cooling (repump) this was set to 0.65 (1.0). The numbers represent the normalised percentage of the starting power of the respective laser beam. For colder atoms, it is beneficial to use lower powers during step 2 however, we do observe a higher loss of atoms at lower powers. The main postulate is the dynamic magnetic field from the eddy currents.

### 16.2.4 Equipment Tables

<table>
<thead>
<tr>
<th>Brand &amp; Model</th>
<th>Instrument</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MITEQ</strong></td>
<td>RF I&amp; Q Modulator</td>
<td>RF$_{in} \in [1, 4]$ GHz, I/Q $\in [0, 0.5]$ GHz</td>
</tr>
<tr>
<td>SDM0104LC1CDQ</td>
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<td></td>
</tr>
<tr>
<td><strong>AA</strong></td>
<td>Fixed Frequency Driver (Signal + Amp)</td>
<td>For pulse on/off optical sidebands</td>
</tr>
<tr>
<td>MODA80-B4-33</td>
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<td></td>
</tr>
<tr>
<td><strong>DILAS</strong></td>
<td>Tampered Amplifier</td>
<td>2.0 W Max. optical sidebands</td>
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<tr>
<td>TAL-0780-2000</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Toptica</strong></td>
<td>Tampered Amplifier</td>
<td>1.5 W Max. optical carrier</td>
</tr>
<tr>
<td>SYS BoosTA 780L</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>SpinCore</strong></td>
<td>100 MHz TTL Signal Generator 'Pulse-Blaster'</td>
<td>Min. 50 ns pulses 10 ns resolution</td>
</tr>
<tr>
<td>PB24-100-32K</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Hamamatsu</strong></td>
<td>Photo-Multiplier Tube</td>
<td>380-890 nm</td>
</tr>
<tr>
<td>H725-50</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Advantest</strong></td>
<td>Wavemeter</td>
<td>1 pm resolution</td>
</tr>
<tr>
<td>Q8326</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>N/A</strong></td>
<td>Custom PID Locking Box</td>
<td>By M.Carey</td>
</tr>
<tr>
<td>N/A</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>EOTECH</strong></td>
<td>GaAs Photo-Detector</td>
<td>Rise/fall time &lt; 30 ps</td>
</tr>
<tr>
<td>ET-4000</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Standford Research</strong></td>
<td>Low-noise Power Amp.</td>
<td>For fluorescence</td>
</tr>
<tr>
<td>SR570</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 16.2: List of instruments used for the static atomic interferometer. The labels in single quotes refer to the abbreviations used in figure 16.3 showing the coarse layout of the set-up. The double quotes refer to a common alternative name to the instrument.
### 16.2. The Experimental Set-Up

<table>
<thead>
<tr>
<th><strong>Brand &amp; Model</strong></th>
<th><strong>Instrument</strong></th>
<th><strong>Comments</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mini Circuits</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ZVA-183-S+</td>
<td>Large Wideband RF Amp.</td>
<td>0.7-18 GHz</td>
</tr>
<tr>
<td>ZX60-6013-S+</td>
<td>Low Noise Wideband Amp.</td>
<td>0.02-6 GHz</td>
</tr>
<tr>
<td>ZHL-1-2W-S+</td>
<td>High Power Wideband Amp.</td>
<td>5.0-500 MHz</td>
</tr>
<tr>
<td>ZAPD-4-S+</td>
<td>RF Power Splitter</td>
<td>2.0-4.2 GHz</td>
</tr>
<tr>
<td>ZX05-C42-S+</td>
<td>RF Frequency Mixer</td>
<td>1.0-4.2 GHz</td>
</tr>
<tr>
<td><strong>Newport</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model 325</td>
<td>Temperature Controller</td>
<td>'TC'</td>
</tr>
<tr>
<td>Model 500 Series</td>
<td>Laser Driver</td>
<td>'DFB-Driver'</td>
</tr>
<tr>
<td><strong>Qubig</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>QDG8</td>
<td>RF Amplifier</td>
<td>0.2-8 GHz</td>
</tr>
<tr>
<td>EO-T2700M3-VIS</td>
<td>Electro-Optical Modulator</td>
<td>2.38-2.78 GHz</td>
</tr>
<tr>
<td><strong>Agilent/Keysight</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSOX3024A</td>
<td>Mixed Signal Oscilloscope</td>
<td>200 MHz, 4 channel</td>
</tr>
<tr>
<td>33612A</td>
<td>Arbitrary Waveform Generator</td>
<td>80 MHz, 2 channel</td>
</tr>
<tr>
<td><strong>Thorlabs</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ITC502</td>
<td>Dual Diode Laser Driver</td>
<td>'DFB-Driver'</td>
</tr>
<tr>
<td>ITC4005</td>
<td>Combined Laser Driver</td>
<td>'TA-Driver'</td>
</tr>
<tr>
<td><strong>Gooch &amp; Housego</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M080-2B/F-GH2</td>
<td>Acousto-Optical Modulator</td>
<td>For pulse on/off</td>
</tr>
<tr>
<td>FS310-2F-SU4</td>
<td>Acousto-Optical Modulator</td>
<td>For $\Omega_{\text{AOM}}$ sideband</td>
</tr>
<tr>
<td><strong>Tektronix</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DPO4104</td>
<td>Mixed Domain Oscilloscope (RF Spectrum Analyser)</td>
<td>1 GHz Oscilloscope (DC to 6 GHz)</td>
</tr>
<tr>
<td><strong>Rohde &amp; Schwarz</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SMB100A-B106-B1H</td>
<td>RF Signal Generator</td>
<td>9 kHz to 6 GHz, thermally locked</td>
</tr>
<tr>
<td><strong>Marconi</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2022</td>
<td>AM/FM Signal Generator</td>
<td>10 kHz to 1 GHz, synchronised</td>
</tr>
</tbody>
</table>

Table 16.3: Continuation of table 16.2, the list of instruments used for the static atomic interferometer as illustrated in figure 16.3. The labels in single quotes refer to the abbreviations used in figure 16.3 showing the coarse layout of the set-up.
16.3 Rabi-flop Experiments

For atomic physics experiments, we need to know the rate of Rabi oscillations to determine the length of mirror (\( \pi \)) and beamsplitter (\( \pi/2 \)) optical pulses. We can estimate the Rabi frequency by running a Rabi-flop experiment in which we vary the length \( \tau \) of a pulse at a constant laser detuning \( \delta_L \). For ultra-cold atoms, \( \delta_L \) is typically within (or near) the \( kHz \)−\( MHz \) range. With this as starting point, we can iterate the Rabi-flop experiment at different \( \delta_L \) until Rabi-flopping is observed\(^{18}\). Two Rabi-flop trials at different Raman laser powers and \( \delta_L \) are shown in figure 16.4.

![Graphs showing Rabi oscillations at different laser detunings](image)

Figure 16.4: Rabi oscillations at (left) \( \delta_L = -200 \ kHz \) for 100 datapoints of three averaged measurements (right) \( \delta_L = -300 \ kHz \) for 200 datapoints of ten averaged measurements. Lasers operated at 12.6 \( mW \) (cooling), 1.1 \( mW \) (repump) and 50 \( mW \) [32 \( mW \)-right] (Counter Raman-per frequency component). The laser’s carrier operated about 780.265 \( nm \) with \( \nu_{AOM} \) = 340.432 \( MHz \) and \( \nu_{EOM} - \delta_L = \omega_{21} - \nu_{AOM} = 3.035732440 GHz - \nu_{AOM} \).

Given oscillations of period \( T \) operate at a frequency \( \nu = 1/T \), we can take the time-scale of the first inversion \( \tau_1 \) and estimate the measured Rabi frequency \( \Omega_{\text{measured}} \) as

\[
\Omega_{\text{measured}} = \frac{1}{2\tau_1}
\]

From figure 16.4 we estimate a \( \pi \)-pulse has a length of \( \sim 2.2 \ \mu s \) (left) and \( \sim 2.8 \ \mu s \) (right) giving respective measured Rabi frequencies of 227 \( kHz \) and \( \sim 179 \ kHz \). N.B. due to logistical (supervisory) challenges I lost access to most of my data hence the inconsistent parameter usage in these datasets.

\(^{18}\)Alternatively, we can calculate the light-shift using the convoluted form (over the atomic distribution) of equation 15.17 and 15.18.
16.3. RABI-FLOP EXPERIMENTS

The attenuation in figure 16.4 can be attributed to decoherence of the atoms which doesn’t alter $\Omega_{\text{measured}}$. Hence our estimate uses peak detection.

16.3.1 Diffraction-envelope Spectrum

We can use the collective datasets of Rabi flopping at different laser detunings $\delta_L$ to calculate the on-resonance two-photon rabi frequency $\Omega_{2R}$ and the light-shift $\delta_{ac}$. Estimation can be done by fitting the peak inversion $|c_2|^2$ at different $\delta_L$ to

$$|c_2(t = \tau)\rangle^2 = K \frac{\Omega_{2R}^2}{\Omega_{2R}^2 + (\delta_{ac} - \delta)^2}$$  \hspace{1cm} (16.1)$$

where we introduced the scaling constant $K$ into equation 15.28 to account for signal loss. The best fit shown in figure 16.5 produces $K = 0.658 \pm 0.016$, $\Omega_{2R} = 328.4 \pm 29$ kHz, $\delta_{ac} = -314 \pm 11$ kHz.

Whilst we could use these two estimates to calculate $\Omega$ at different $\delta$, it is important to note the actual measured Rabi frequency is the convolved generalised two-photon Rabi frequency $\{\Omega\}_{\text{atom}}$ given by

$$\{\Omega\}_{\text{atom}} = \left\{ \frac{1}{2\Delta N_{m_f}} \sum_{m_f, m_f'} [\Omega_{13,1} \Omega_{23,2}] \right\}_{\text{atom}}$$ \hspace{1cm} (16.2)$$

where the curly brackets indicate a convolution over the atomic detuning distribution and $N_{m_f}$ the number of $|1\rangle$ Zeeman sublevels. In practice, an additional sum would be needed in equation 16.2 over the adjacent $|3\rangle$ states ($F'$) as in our set-up they are separated apart less than $\Delta$. Hence the convolved frequency considers the effect of the Zeeman-sublevel and atomic-detuning distributions.

A related formula to 16.2 is fitted to data in [119] suffering from a systematic error. They demonstrated their calculation also required a more complex computation involving the transition-specific Clebsch-Gordan coefficients. A simpler estimation can thus be made using the fit demonstrated here providing the light-shift and on-resonance two-photon Rabi frequency. Given the analogy between equations 15.28 and 16.1, we can refer to this spectrum as ‘Diffraction-envelope spectrum’: iterated Rabi-flops at different detunings.

\footnote{We avoid including damped oscillations into our fit as our data has few full oscillations.}
\footnote{Primarily the loss of atoms and imperfect population inversion from the detuning distribution of atoms.}
\footnote{The exact form of this formula is beyond the scope of this thesis.}
Figure 16.5: (upper figures) Rabi-flops at three different laser detunings. (bottom) The peak inversion at different laser detunings $\delta_L$ fitted in Matlab R2018a using `fittype` to the Lorentzian given in equation 16.1. Lasers operated at 12.6 mW (cooling), 1.1 mW (repump) and 50 mW (Counter Raman-per frequency component). The laser’s carrier operated about 780.267 nm with $\nu_{AOM} = 340.432 \text{ MHz}$ and $\nu_{EOM} - \delta_L = \omega_{21} - \nu_{AOM} = 3.035732440 \text{ GHz} - \nu_{AOM}$. Each dataset consists of 20 datapoints taken in a random order with each datapoint averaging three separate measurements. Fit values $[x = x’ (95\% \text{ confidence bounds})]: K = 0.658 (0.642, 0.674)$, $\Omega_{2R} = 328.4 (299.4, 357.3) \text{ kHz}$ and $\delta_{ac} = -314.3 (-325.2, -303.5) \text{ kHz}$.

The advantage of taking a Diffraction-envelope spectrum is that it only uses Rabi-flop data and does not require a priori knowledge of the pulse length or the precise value of the light shift. Running Rabi-flop experiments at several laser detuning coarsely around the light-shift suffice to find the parameters of interest. This is unfortunately also the drawback, the need for multiple datasets and a coarse estimate of the light-shift.
16.4 Raman Spectrum Experiments

An alternative method to measure the light-shift is taking a Raman spectrum. If the two-photon Rabi frequency is known on resonance, the population $|c_2(t = \tau_\pi)|^2$ for a $\pi-$pulse at various detunings $\delta_L$ can be measured using the same set-up as for a Diffraction-envelope spectrum. This Raman spectrum can then be fitted to equation 16.1 as shown in figure 16.6.

Figure 16.6: (a) Rabi-flops at $\delta_L = -323 \, kHz$. (b) Raman spectrum for a 2.2 $\mu s$ pulse. (c) The Raman spectrum at different laser detunings $\delta_L$ fitted in Matlab R2018a using fittype to the Lorentzian given in equation 16.1. Lasers operated at 12.6$mW$ (cooling), 1.1$mW$ (repump) and 50$mW$ (Counter Raman-per frequency component). The laser’s carrier operated about 780.265 nm with $\nu_{AOM} = 340.432 \, MHz$ and $\nu_{EOM} - \delta_L = \omega_{21} - \nu_{AOM} = 3.035732440 \, GHz - \nu_{AOM}$. Each dataset consists of 100 datapoints taken in a random order with each datapoint averaging three separate measurements. Fit values $[x = x'$ (95% confidence bounds)]: $K = 0.7038 \, (0.679, 0.7286)$, $\Omega_{2R} = 227.8 \, (215.8, 239.8) \, kHz$ and $\delta_{ac} = -323 \, (-331, -315) \, kHz$. 

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We start of with a Rabi-flop experiment at some initially guessed $\delta_L$. From figure 16.5 namely we observe the light-shift does not significantly affect the oscillation rate but mainly the peak fidelity. Thus as long as we can infer a time-scale of oscillations we can estimate $\{\Omega\}_{\delta_{\text{atom}}}$. Using this estimate we then run a Raman spectrum for a $\pi-$pulse which we fit to extract the light-shift. For completeness, in figure 16.6 (a) we show the Rabi-flop at the fit-estimated light-shift suggesting a $\sim 2.2 \mu s$ $\pi-$pulse.

The disadvantage to this approach is the requirement to know the Rabi frequency. However, even an off-resonance Rabi-flop can be used to estimate the length of a $\pi-$pulse. Hence for a coarse estimate, only two datasets are needed to find $\{\Omega\}_{\delta_{\text{atom}}}$ and $\delta_{\text{ac}}$. Note that we actually measure $\{\delta_{\text{ac}}\}_{\delta_{\text{atom}}}$ here i.e. the convolved light-shift over the atomic distribution (including the distribution over Zeeman sublevels).

16.4.1 Spectrum 'Kink'

There is an asymmetry in figure 16.6 (b,c) between the tails of the Lorentzian lineshape. Our fitted line seems to fit less well namely towards $\delta_L = -800 \ kHz$. We noticed a similar asymmetry near $\delta_L = -1 \ MHz$ when operating our laser previously near 780.267 nm prompting us to change the wavelength to the current 780.265 nm. By equation 4.2, a wavelength shift change of 1pm shifts the carrier frequency by $\sim 500 \ MHz$. We infer therefore that if the 'kink' was caused by the laser, it would have shifted from the Lorentzian lineshap centre by $\sim \pm 1 \ MHz$. The 'kink' didn’t shift suggesting another origin.

The next hypothesis considers the AOM and EOM optical sidebands. Starting with testing the AOM sideband, we kept $\nu_{\text{EOM}}$ constant and increased $\nu_{\text{AOM}}$ by 1MHz. Comparing the spectrum before and after the shift in figure 16.7 we again found the 'kink' had not shifted suggesting the origin is not related to the AOM sideband.

This suggests the Raman spectrum 'kink' around $-1 \ MHz$ away from the Lorentzian centre stems from the EOM. Given the RF driver of the EOM had recently been purchased, we postulate that the EOM is introducing an systematic error\footnote{There could be several causes to this such as a different impedance mismatch for $\delta_L \sim -1 \ MHz$ of the EOM, RF cable, etc.} around $\delta_L = 1 \ MHz$. This was confirmed in later tests on composite pulses where we altered $\nu_{\text{EOM}}$ (and $\nu_{\text{AOM}}$ accordingly) causing the 'kink' to shift (c.f. figure 17.4). Note that the effect of a fit on an envelope spectrum with a 'kink' is reduced by other datapoints.
16.5. MISCELLANEOUS COMMENTS

16.5.1 Calculating Upper Amplitude

N.B. the current signal measured by the PMT is fitted to the exponential decay function $A e^{-t/\tau} + c$ against time $t$. The value of $|c_2|^2$ is calculated as $A_{\text{signal}}/A_{\text{reference}}$ from the two respective fluorescence fits.

16.5.2 Shim Coils

The Helmholtz coils for spatial confinement are not guaranteed to provide a vanishing $B$ at the site of laser cooling. To account for any stray magnetic fields, three pairs of ‘Shim’ coils are used zero the field at the centre of the vacuum chamber. In our experiments, an quantum axis is introduced along the Raman interrogation axis by off-setting the corresponding Shim-coil pair from $B_{\text{Raman-axis}} = 0$.

16.5.3 Zeeman Spectrum

To measure the magnetic field at the site of optical molasses, a co-propagating interrogation is used (non-velocity selective). Using a $\sim 300 \, \mu s$ pulse, the atoms dephase to the steady-state allowing for the Zeeman sublevel populations to be measured by varying $\delta_L$. By iterating the measurements post Shim-coil tuning, Zeeman degeneracy can be achieved at which $B = 0$.

16.5.4 Attenuating ASE

To attenuate the amplified spontaneous emission (ASE) from the TA’s we can place a heated $Rb$ vapour cell for the Raman beams between the Raman AOM and the BSO for each sideband.
16.5.5 Measuring Coherence

We can measure the coherence time of our $^{85}$Rb atoms using a spin-echo experiment. During an spin-echo experiment, the Raman beams are left on for a duration $\tau$ that is scanned. Every $\tau_\pi$ ($\pi$-pulse interval), the laser phase $\phi_L$ is flipped by $\pi$. The effect from changing the laser phase is to flip the field vector in the Bloch sphere about which the Bloch vector precesses. Therefore, by symmetry the effect from the laser detuning is cancelled leaving behind only the dampening by decoherence. A coherence time beyond hundreds of microseconds was estimated (raw data is lost following data access struggles).

16.5.6 Trigger-delay Measurement

There is an trigger latency between the leading edge trigger pulse being sent from the Pulse-Blaster to the triggering reaching the EOM and affecting the laser light. Especially for composite pulses where the temporal phase-profile of a pulse is sensitive to imprecise phase-shifts, it is of interest to compensate this systematic offset by triggering the EOM earlier by this latency. That way, the trigger of the 80 MHz switching Raman AOM can be paired with the EOM trigger.

To measure the trigger delay $\tau_{\text{delay}}$, inspiration from the spin-echo experiment is drawn as shown in figure 16.8. We start with a $2\pi$-pulse to interrogate the $^{85}$Rb atoms. During an iteration, an artificial delay $\tau$ for the EOM RF driver is scanned relative to the ‘on’ trigger of the Raman AOM. For some $\tau'$, the phase $\phi_L$ of the Raman beams are reversed exactly midway in the $2\pi$-pulse giving the effect of a single $\pi$-pulse i.e. highest upper level population. The difference between the temporal length of a $\pi$-pulse and the $\tau$ of highest $|c_2|^2$ are then compared to give $\tau_{\text{delay}}$. The delay was measured to be 715(1) ns which was subtracted from each EOM trigger (raw data was lost).

Figure 16.8: Diagram illustrating the temporal total Raman beam power $P$ and laser phase $\phi_L$ during (a) spin-echo (b) trigger-delay measurement. The upper row shows the Raman beam switching on/off, sharing common x-axis (time $t$) with the phase-profile in the lower row. N.B. $\tau_{nm} = n\tau_\pi$, $n \in \mathbb{N}$ where $\tau_\pi$ is the length of a $\pi$-pulse.
Chapter 17

Composite Pulses

The broadband condition states that the external time-scale $t_{\text{ext}}$ of external atomic variables (velocity, acceleration, etc.) is longer than the internal time-scales $t_{\text{int}}$ of the internal state evolution on the Bloch sphere. This conditions allows us to treat the motion of the atoms classically and treat the optical pulses as instantaneous. With $t_{\text{int}} \sim \Gamma^{-1}$ and taking $t_{\text{ext}}$ as the time-scale of acceleration ($\Gamma v_R$) for escaping the Doppler broadened Raman beams we find

$$k_{\text{eff}}(\Gamma v_R)t_{\text{ext}} = \Gamma$$

where $k_{\text{eff}}$ is the effective wave-vector of the Raman lasers, $\Gamma$ the inverse lifetime of the excited state and $v_R$ the Raman recoil velocity. For stimulated Raman transitions about the D2 line of Rb, $v_R = 5.89 \, \text{mm/s}$ and $\Gamma = 2\pi \cdot (5.89 \, \text{MHz})$ giving $t_{\text{ext}}/t_{\text{int}} \simeq 780$ [141]. As 780 is not particularly large, it shows the broadband condition is not imposed strictly in our experiments. Problems such as the pulse-length errors discussed in section 26.5 are thus expected to be measurable and impose a contrast 'fidelity' limit in our interferometers.

In this chapter we discuss the preliminary work on the development of composite pulses. These pulses aim to provide a general enhancement method to mitigate several perturbations discussed later in chapter 26. We highlight that the computational work regarding composite pulses should be credited primarily to PhD student J. Saywell. The same experimental configuration in chapter 16 which is credited primarily to PhD student M. Carey. The composite pulse dataset in this chapter involved equal contribution from J. Saywell, the author and M. Carey.
17.1 Introduction to Terminology

17.1.1 Pulse Fidelity

In bra-ket notation, a quantum system with Hamiltonian $\hat{H}(t)$ starting at $t_0$ in the state $|\psi(t_0)\rangle$ evolves to $|\psi(t)\rangle$ according to the TDSE as

$$\hat{H}(t) |\psi(t)\rangle = i\hbar \partial_t |\psi(t)\rangle$$

The state evolution can also be described using an evolution operator $\hat{U}$ which rotates the vector to $|\psi(t)\rangle$ given by

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle \quad (17.1)$$

It can be shown [155] that $\hat{U}$ can be given as

$$|\psi(t)\rangle = \hat{T} e^{-i \int_{t_0}^{t} \hat{H}(t') dt'} |\psi(t_0)\rangle \quad (17.2)$$

where $\hat{T}$ represents the time-ordering operator accounting for the case of a non-vanishing commutator of the Hamiltonian at two times $t_1, t_2$ i.e. $[\hat{H}(t_1), \hat{H}(t_2)] \neq 0, \forall t_1, t_2 \in \mathbb{R}$. In our quantum system the commutator vanishes which removes the $\hat{T}$ factor (the operator does’t alter the vector).

Hence with equation 17.2, the state dynamics of the quantum system can be described by a single evolution operator matrix $\hat{U}$. As such, let the quantum system start at $t_0$ in the state $|\psi(t_0)\rangle$. An ideal interferometric sequence can be described by an ‘intended’ evolution operator $\hat{U}$ rotating this state to $|\psi(t)\rangle$ as per equation 17.1. In practice however, experimental perturbations\footnote{Examples include pulse-length errors (c.f. section 26.5), off-resonance errors (c.f. section 26.4) and other are discussed in the next chapter 26.} introduce a deviation in evolution to an ‘actual’ operator $\hat{V}$.

The distinguishability between the intended and actual final state is called the Fidelity and is calculated from the following overlap:

$$F_{|\psi\rangle} = |\langle \psi | \hat{V}^\dagger \hat{U} |\psi\rangle|^2$$

For our experiments, this fidelity can be equivalently ([119]) be defined as

$$F_{|\psi\rangle} = |\langle \psi_{\text{final}} | 2 \rangle|^2 = |c_2|^2, \quad \langle \psi_{\text{final}} \rangle = \langle \psi | \hat{V}^\dagger \hat{U} \& |\psi\rangle = |2\rangle$$

The fidelity of an optical pulse/interferometric sequence described by $\hat{U}$ can thus measured directly from the upper level population $|c_2|^2$ as in our static interferometer experiment.
17.1.2 Visualising the Bloch Sphere

Before describing composite pulses, we comment on the effect of errors/perturbations on the Bloch sphere. To simplify the description, we will consider two particular cross-sections of the Bloch sphere of an $|S\rangle$ and $|P\rangle$ laser coupled quantum two-level system. The pole-to-pole cross section shown in figure 17.1 describes on-resonance laser-atom interactions at constant laser phase. In comparison, the equatorial cross-section in figure 17.2 describes the 'free evolution' of the atomic state in the absence of a laser field interrogating the atoms.

Figure 17.1: Pole-to-pole cross-section of the Bloch sphere compared to clocks. An two-level atom coherently driven by a resonant laser(s) resembles a clock rotating at the Rabi frequency.

The polar Bloch clock suggests Rabi oscillations on resonance can be modelled as the dial of a clock spinning at the Rabi frequency during a laser interaction. Variations in Rabi frequency can therefore be visualised as causing over/under-shooting of the clock’s dial from the set target time. For example, setting the lower level $|S\rangle$ at 6 ‘clock and the upper level $|P\rangle$ at 12 o’clock, an ideal mirror pulse acting on $|S\rangle$ would have a final state $|\psi\rangle = |P\rangle$. If the Rabi frequency shifts however, we find $|\psi\rangle \neq |P\rangle$. Instead, $|\psi\rangle$ would be given by a superposition with $|c_2|^2 > |c_1|^2$ i.e. about $|P\rangle$.

Figure 17.2: Equitorial cross-section of the Bloch sphere compared to a clock. A two-level atom in the absence of a laser field undergoing free-evolution.

Free evolution changes the relative phase of the wavefunction at a rate that depends on the atom’s detuning in the rotating frame picture. From the equitorial Block clock we thus infer the absence of a laser field transfers the motion of the Bloch vector to the equatorial plane. The rate of this clock can be shown to equal $\delta - \delta_{ac}$ [119] i.e. depend on the detuning of the atom. The equitorial Bloch clock thereby spreads states of different detunings.
Note that the polar Bloch clock is 'frozen' on resonance when the laser field is turned off. As such, the clock describes the oscillation of the charge distribution by the laser interrogation. Also, in our experiments we drive transitions between the two ground $|S\rangle$ orbitals. For easier visualisation however we chose to illustrate Bloch clocks using the $|S\rangle \leftrightarrow |P\rangle$ transition. Also, the two Bloch clocks are actually coupled resulting in some perturbations spreading the ensemble of state vectors along the polar axis and in the equitorial planes of the Bloch sphere. Finally, the amplitude coefficients $c_1$ & $c_2$ in full relate to the polar angle $\vartheta$ and the azimuthal angle $\varphi$ of the Bloch sphere as

$$c_1 = \cos (\vartheta/2), \quad c_2 = e^{i\varphi} \sin (\vartheta/2)$$

### 17.1.3 The Composite Pulse

In nuclear magnetic resonance imaging (nMRI), composite pulses have been used since 1981 to compensate for experimental perturbations/imperfections in the RF-pulse interrogation of samples [156]. As the name implies, such pulses comprise of a series of sub-pulses that are concatenated in series. Each sub-pulse $j$ might have a different phase and/or duration each described by their own evolution operator $\hat{U}_j$. The composite pulse then rotates the starting state by the ordered product $\prod_j \hat{U}_j$. If we describe each sub-pulse $j$ to have a duration that induces a rotation $\theta^{(j)}$ on our polar Bloch clock during a laser interaction at laser phase $\phi_j$, a composite pulse of total rotation $\theta_{\text{total}}$ can then be expressed as

$$\left[\theta^{(1)}_{\phi_1} \theta^{(2)}_{\phi_2} \theta^{(3)}_{\phi_3} \ldots \theta^{(j)}_{\phi_j}\right] \{\theta_{\text{total}}\} \text{ e.g. } [180^\circ 240^\circ 180^\circ 210^\circ 180^\circ 300^\circ 180^\circ 210^\circ 180^\circ 240^\circ] \{900^\circ\}$$

Composite pulses fall in terms of their fidelity into two categories, point-to-point (PP) and general rotor (GR). PP composite pulses are designed and optimised for quantum systems starting and terminating in a particular starting state e.g. $|1\rangle \rightarrow |2\rangle$. GR composite pulses in contrast are optimised to introduce a specific total rotation $\theta_{\text{total}}$ [119].

Depending on the role of the optical pulse in an atomic interferometer, either a PP or GR composite pulse could be preferred. Taking any atom interferometric sequence as example, if the previous optical pulse is affected by experimental perturbations, its final state $|\psi_{\text{final}}\rangle$ for an ensemble of atoms spreads out resulting in a wider distribution of initial states for the current sub-pulse. In other words, the initial state $|\psi_{\text{initial}}\rangle$ becomes less defined for the next optical pulse by perturbations. This pulse might then prove to have a better optimisation by an GR composite pulse then a PP pulse which seems better suited for e.g. the interferometric’s first pulse following an effective state preparation into a single state.
Apart from optical pulses in atom interferometers not sharing a common precision level in the final/initial state, they also are not guaranteed to contribute equally to the contrast of the interferometer. Substituting optical pulses with composite pulses could allow for the errors of the primary contrast contributing optical pulses to be compensated by the adjacent composite pulses in the interferometric sequence introducing a higher order correction of perturbations experienced by the quantum system.

17.1.4 Atomic Adaptation

Existing composite pulse adapted and applied to cold atomic systems have demonstrated higher fidelity than rectangular $\pi$/mirror-pulses ($[180^\circ\{180^\circ\}]$) over a larger atomic detuning range. For example, the $[180_{240}180_{210}180_{300}180_{210}180_{240}]\{900^\circ\}$ sequence known as a KNILL composite pulse (GR) has $\leq 38\%$ higher pulse peak fidelity compared to a standard rectangular pulse. Likewise, the PP composite pulses with sequences $[90_{0}180_{180}270_{0}]\{540^\circ\}$ and $[90_{0}180_{90}90_{90}]\{360^\circ\}$ respectively called WALTZ and LEVIT can be shown to provide $\leq 50, 64\%$ higher peak fidelities [119].

From these findings, the natural progression is developing novel specialised composite pulses that optimise pulse fidelity over a finite atomic detuning range. Preliminary, only the laser phase $\phi_L$ is varied per sub-pulse with possible amplitude modulation of the Raman beams left for future experiments.

Matterwave Bandwidth

Composite pulses can be viewed in the matterwave description as having a higher diffraction envelope providing higher quality beamsplitting and mirror pulses for a larger range of matterwaves (velocity distribution). Adapting composite pulses and optimising them for atomic interferometers can thus be thought of as designing optical pulses (beamsplitters and mirrors) of higher matterwave bandwidth.
17.2 SPINACH and GRAPE

For the computational optimisation of a distribution of quantum states, an quantum optimal control algorithm is used called Gradient Ascent Pulse Engineering or GRAPE. Developed in 2005, [157], GRAPE calculates the derivative of the fidelity for a given set of phases \( \{ \phi_1, \phi_2, \ldots, \phi_j, \ldots \} \) averaging over a range of atomic detunings \( \delta \in [\delta_{\text{min}}, \delta_{\text{max}}] \) (weighted velocity distribution) and power inhomogeneities. If we let each phase-set represent a point \( |x\rangle \) in some \( N \)-dimensional space (for \( N \) sub-pulses), the algorithm calculates the gradient along the different axes (phase of sub-pulse \( j \in [1, N] \)) to ascend to the local point of maxima. The algorithm is then iterated over several random initial phases to secure the global maxima.

To calculate the fidelity gradients, the open source spin dynamics simulation library Spinach is used. Spinach comprises of multiple modules including 'Optimal Control' that supports GRAPE algorithms.

An example of the computed phase-profile of an GRAPE optimised composite pulse using Spinach is shown in figure 17.3. Interestingly, the profiles tend to be symmetric which with further work we might be able to model using a novel simple analytical/visual interpretation. Further details however on the theoretical and computational development of composite pulses and their discussion can be found in J. Saywell’s paper: [158].

![Figure 17.3: Temporal phase-profile of a 100 step 12 µs composite mirror pulse superimposed on the associated RF electronic phase-error measured using the HP 8702B Vector Network Analyzer. Phase profile computed and optimised by J. Saywell for a 310 kHz Rabi frequency π-pulse.](image-url)
17.3 Enhancements of Mirror Pulses

This section will discuss preliminary work of the composite pulse tests prior to the departure of the author from the research group. Parts of this work later developed towards a publication in the *Journal of Physics B* [120]. Further details beyond this chapter can therefore be found therein.

The same experimental set-up was used as discussed in section 16.2. A $^{85}\text{Rb}$ 'hot' cloud of $\sim 1$-$10$ $\mu K$ is interrogated by the two counter-propagating Raman beams in the $\sigma^+ - \sigma^-$ configuration such that the relative light-shift spread between Zeeman-sublevels introduces larger off-resonance and pulse-length errors. Also, a weak biasing magnetic field was also introduced along the Raman beams as discussed in subsection 16.5.2. This is to reduce magnetic field sensitivity as discussed in subsection 26.4.

To verify the GRAPE composite pulse fidelity yields a higher peak and larger bandwidth, the PP optimised pulse at two temperatures is compared to a WALTZ and a rectangular $\pi$-pulse as shown in figure 17.4. As a test against systematic errors, all composite pulses were also tested using an varying phase offset $\phi_\epsilon$ such that

$$\phi_j \rightarrow \phi_j + \phi_\epsilon, \ \forall j \in [1, \cdots, N]$$

As expected, the composite pulses didn’t demonstrate a measurable dependence on the phase-offset with any discrepancy well-within the standard error of the mean of the datasets.

A coarse quantification of the level of enhancement in figure 17.4 confirms the WALTZ pulse does not significantly enhance the peak fidelity of a $\pi$-pulse. The GRAPE pulses provide $\sim 20\%$ higher peak fidelity against the rectangular $\pi$-pulse. These preliminary numbers might be a result of a fidelity ceiling-limit being reached in the experiment from sub-interferometers by Raman beam cross-talk (*c.f.* section 26.2). Unfortunately, testing with shorter composite GRAPE pulses proves difficult as our experimental set-up is currently limited to $\leq 100$ $mW$ limiting the length of each sub-pulse. Using fewer sub-pulses is neither an alternative as the fidelity enhancement (control over the atomic ensemble) scales with the number of steps in the composite pulse. Regarding the bandwidth (FWHM) of a Raman spectrum, the WALTZ pulse measures an increases by about 50\% compared to the rectangular $\pi$-pulse. With the GRAPE pulses we find an even larger bandwidth enhancement of $\sim 130\%, 180\%$ for the 40, 100 $\mu K$ optimisation respectively.
Figure 17.4: (upper) Temporal response of a 100 $\mu$K composite pulse computed by J. Saywell, data taken with M. Carey. The experimental data represents the upper level population $|c_2|^2$ against different pulse lengths is shown, analogous to an Rabi-flop experiment. The theory line represents the results a simulation that includes the AOM switching latency at 50 mW Raman power. (lower) Raman spectra of a standard 'naive' square $\pi$-pulse (rect. $\pi$) compared to a standard $\pi$-WALTZ pulse and two custom $\pi$-composite pulses computed by J. Saywell. N.B. the data was taken with M. Carey, figures are courtesy of J. Saywell and M. Carey and the lower simulation is lost.

Figure 17.4 thus shows that the GRAPE mirror composite pulses provide an enhanced matterwave bandwidth scalable in optimisation in temperature (detuning range). The simulation of the theoretical evolution of the ensemble of atomic states when including AOM latency is also in agreement with the data confirming the AOM introduces an error in the start of the phase-sequence. Later generations of the composite pulse tests eliminated this error by rapidly detuning the Raman beams away and near resonance by $\sim 1 \text{GHz}$ as commented in subsection 26.5.2.

Including this improvement, we reaffirm that composite pulses have applications in atom interferometry, in particular Large Momentum Transfer schemes (LMT) which experience degradation from increasing differential Doppler shifts per augmentation pulse$^2$ [159, 160]. This could allow for large area interferometers beyond $18\hbar k_{\text{eff}}$ momentum splitting [161].

Apart from phase optimisation, implementing optimised amplitude-shaped pulses in the future might also provide additional robustness against imperfections at certain frequencies in the experiment as demonstrated in other works e.g. [162].

$^2$A pulse with effective wave vector $k_{\text{eff}}$ reversed. Usually applied in pairs to close the interferometer again.
Chapter 18

Discussion

Laser-atom interaction errors in atomic physics are typically categorised into two types: off-resonance and pulse-length errors [119]. Both errors introduce an offset between the intended and actual final state vector when a light-pulse sequence is applied to an initial state. The lowering of the fidelity is however through two different mechanisms leading to the error dichotomy. Following this naming convention, we classify the sources of perturbations (discussed in more detail in chapter 26) based on their effect on ultra-cold atoms.

The final section of this discussion focuses on the experimental limitations of our set-up. This will set the bound on the effectiveness of enhancement methods against systematic errors by defining the scope of control over our experiments and other similar experimental configuration currently available.
18.1 Off-resonance Errors

Off-resonance errors are errors introduced by a non-zero Raman laser detuning \( i.e. \delta - \delta_{ac} \neq 0 \). Sources can be identified from the general two-photon detuning \( \delta \): the Raman’s relative laser frequency, Zeeman shifts, Doppler shifts, Stark shifts and photon recoil. If any of these sources place the two-photon Raman transition off-resonant, our atomic experiment will harbor an off-resonance error. Note that given the order of significance of these sources is highly dependent on the experimental control variables, we will not order the error sources in magnitude. Instead, we order off-resonance errors in order of work and review in this sub-thesis.

One of the effects from off-resonance errors is to spread out across the equitorial Bloch clock\(^1\) the state-vector of the atomic ensemble comprising the atomic cloud. The mixed state spreads out at a rate proportional to \( \delta - \delta_{ac} \) giving larger detunings larger perturbations. These errors also induce pulse-length errors as the generalised off-resonant two-photon Rabi frequency \( \Omega \) depends on the two-photon detuning \( \delta \) and the light-shift \( \delta_{ac} \).

18.1.1 Zeeman Shifts

Background magnetic fields introduce Zeeman shifts to magnetic sensitive Zeeman sublevels \( (m_f \neq 0) \). This shifts \( \delta \) proportional to \( B \) and \( m_f \), introducing an off-resonance error as discussed further in section 26.4. In our experiments, our state preparation optically pumped atoms into a target Hyperfine level leaving our atoms in a uniform distribution of Zeeman sublevels. Zeeman shifts are thus off-resonance errors that follow from inadequate state preparation that scale with the presence of background magnetic fields.

We identified two magnetic fields that have AC components \( \in [1 \ kHz, 100 \ MHz] \) that induce measurable fidelity reduction from Zeeman shifts between different \( m_f \) sublevels\(^2\): the MOT coils fields (including Eddy currents) and stray fields from instrumentation. These magnetic field sources thus vary during an interferometer cycle leading to a temporal inhomogeneous systematic error meaning the systematic error varies with length of the sequence.

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\(^1\)This is the equitorial plane of the Bloch sphere as defined in subsection 17.1.2.

\(^2\)Assuming the atomic cloud has a uniform \( m_f \) distribution.
A different interferometric sequence of the same total duration can also be subject to different magnitudes of Zeeman shift induced off-resonance errors as the error differs during free evolution and during an optical pulse\(^3\). The error thus has an inhomogeneous component against different interferometer sequences, even if they are of the same length.

Another complication stems from spatial variations in $B$ as provided by the MOT coils. A non point-like cloud would experience a spatial varying Zeeman shift even for atoms in the same\(^4\) magnetic sensitive sublevel (at different positions). This introduces another correlation of this error to an experimental control variable: the atomic cloud’s size. Including the DC $B$ fields without (with) spatial variations, this introduces inhomogeneous (homogeneous) detuning errors. Zeeman shifts thus introduce several correlations with experimental control variables with both homogeneous and inhomogeneous dependencies.

### 18.1.2 Doppler Shifts

The velocity distribution of our atomic cloud introduces inhomogeneous detunings from Doppler shifts by the Doppler effect. The two-photon detuning $\delta$ thus correlates with the velocity distribution thereby the temperature of the atomic cloud. Warmer clouds having a wider velocity distribution experience larger Doppler shift systematic errors. All velocity dependent perturbations introduce off-resonance errors following the inclusion of the velocity distribution of the atomic cloud in $\delta$. These include spatial Raman inhomogeneities (26.1), power problems in the Raman beams (26.2), pulse-length errors (26.5) and impurities (26.9).

Raman power inhomogeneities propagate the variations of Raman power (using beamshaping optics: $\sim 15\%$ spatially, $\sim 1\%$ temporally) into variations of Doppler shifts across the atomic cloud. This is complicated further by the spatial-to-velocity correlation of the atomic cloud as even ideal uniform Raman beam power incident on the cloud introduces inhomogeneous Doppler shifts. The atomic cloud is namely colder at the core than its surface providing a velocity gradient when we measure atoms’s speed closer to the cloud’s core. The use of beamshaping optics reduce the variations in Doppler shift across the cloud making the systematic error more homogeneous.

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\(^3\)Zeeman shifts introduce pulse-length errors (c.f. section 26.5) which give different magnitude phase-shifts during free-evolution then during the application of a pulse.

\(^4\)If the two atoms are in the same velocity class in addition to the same magnetic sensitive sublevel, the Zeeman shift can still vary if $B$ differs for the different positions.
If atoms undergo acceleration as is the case in our experiments (gravity), the Doppler-shift can vary between optical pulses of common length when applied to our atoms at different times. This follows as the mean of the velocity distribution changes due to the acceleration which adds an increasing homogeneous systematic error over time. The mean also changes if we flip the orientation of the effective wavevector of the Raman laser beams as the momentum recoil from photons shift the velocity distribution. This is enhanced when the Raman beams have a component along the axis of acceleration. Thus similar to Zeeman shifts, Doppler shifts also contribute both homogeneous and inhomogeneous errors.

Impurities add additional homogeneous and inhomogeneous systematic errors, in particular the collision induced impurities. Local changes to the velocity class propagate via the spatial-to-velocity correlation into local changes to the Doppler-shift hence magnitude of Raman laser detuning with the atomic cloud. Impurities come from many sources most notably the longitudinal and transverse relaxation processes. Whils it is beyond this thesis to review these, we note Doppler shift through impurities become correlated with the damping time-scales of relaxation processes such as atom-atom collision rates.

18.1.3 Raman’s Relative Laser Frequency

The two-photon detuning $\delta$ depends on the relative laser frequency between the two Raman beams: $\delta_L$. In our experimental set-up (c.f. section 16.2), since both Raman beams are synthesised from optical sidebands we find that $\delta_L$ is insensitive to variations in the carrier frequency of the laser. Drifts, frequency-jitter and other noise in the carrier frequency therefore do not introduce off-resonance errors through $\delta_L$.

The optical modulators used to generate the sidebands do propagate noise, drifts and other RF frequency perturbations into $\delta_L$. The electronic specifications of modulators and their drivers can thus provide insight into the perturbation propagation into perturbations in $\delta_L$ leading to systematic off-resonance errors.

We have not made measurements on the frequency stability of our optical drivers limiting that discussion to the providers specifications. We can however infer that instability in $\delta_L$ will vary with time introducing temporal inhomogeneous systematic errors. The errors are however homogeneous across the atomic cloud at any particular moment of time.
18.1. OFF-RESONANCE ERRORS

18.1.4 Stark Shifts

The ac Stark effect defined in subsection 15.2.2 introduces light-shifts i.e. Stark shifts in atoms when light is incident on them. Whilst we can account for homogeneous light-shifts by setting $\delta_L - \delta_{ac} = 0$, variations in the light-shift introduce inhomogeneous errors that need another mitigation technique.

The power problems in the Raman beams (26.2) cross-talk between the two Raman beams: 1% of one optical sideband (Raman beam) leaking into the other Raman beam, and 8% in return. This 8% consists of two frequency components, one driving Raman transitions and the other one detuned off-resonance. As such, we actually have 1 and 4% of coss-talk between the Raman beams with the remaining 4% unused light introducing a light-shift.

Any drift in power will translate into variations in light-shift. The power problems in the Raman beams thus introduce variations in the light-shift from the cross-talk and main Raman beams during an interferometric sequence. This stacks on top of variations in $\delta_L$ if try to set $\delta_L - \delta_{ac} = 0$ demonstrating light-shift off-resonance errors are inhomogeneous throughout an interferometric cycle, but are uniform across the atomic cloud.

Unlike $\delta_L$, light-shifts are dependent on the Raman’s carrier laser frequency as they depend on one-photon Rabi frequencies and the one-photon transition detuning. Light-shifts are therefore sensitive to laser’s linewidth, drift, jitter, amplified spontaneous emission (ASE) and noise. Sub-thesis I provides insight into the performance of the laser used in our experiments. From that work, apart from ASE we can infer the laser’s carrier field introduces a negligible error to our experiments when its drivers are calibrated properly.

In summary, the systematic errors from light-shifts are proportional to the amplitudes of the different optical components that comprise the Raman and other laser beams. Cross-talk between Raman beams can introduce a homogeneous systematic error proportional to the Raman powers whilst variations in Raman powers introduce inhomogeneous powers across the cloud or throughout the interferometers cycle.

18.1.5 Photon Recoil

The Raman beams impart a recoil momentum when they interact with atoms. The recoil velocity $v_R$ of $^{85}$Rb atoms (D2–transition) is 6.0230(1) mm/s \[123\] i.e. $\sim 12$ mm/s for two-photon Raman transitions as used in our experiments.
If the two Raman beams cross-talk, the atomic cloud can experience non-uniform recoil momenta in different two directions. The leaking of the AOM en EOM optical sidebands into each Raman beam thus introduces inhomogeneity in the photon momentum recoil imparted on the atomic cloud. This inhomogeneity would scale with the power distribution of the different optical components per Raman beam that when paired drive stimulated two-photon Raman transitions.

The photon recoil term in $\delta$ effectively represents a Doppler shift due to momentum recoil: $k_{\text{eff}} \cdot \mathbf{v}_R$. This introduces a dependence on the orientation of the Raman beams between the co- and counter-propagating arrangements.

Higher orders of momentum transfer as in Large Momentum Schemes (LMT) also introduce pulse sequence dependent photon recoil errors. The application of augmentation pulses in LMT namely varies $\delta$ by a constant between consecutive pulses. The recoil term thus shares features with the Doppler shift term in the homogeneity of the induced systematic errors.

### 18.2 Pulse-Length Errors

Pulse-length errors are errors introduced from imperfections by optical pulses such that the actual phase rotation $\theta$ on the polar Bloch clock\(^5\) deviates from the intended rotation by $\Delta \theta$. There are two causes for deviation, perturbations in the Rabi frequency ($\Delta \Omega$) and perturbations in the pulse-shape and timing ($\Delta$). Some sources of pulse-length errors introduce both types of perturbations like those described in section 26.5.

On the polar Block clock the effect of pulse-length error is to spread out the state vectors of the atomic ensemble. The rate of spreading is proportional to $\Delta \theta$ giving larger perturbations larger state mixing. We note that all off-resonance errors introduce pulse-length errors due to detuning $\Delta$ affecting the Rabi frequency $\Omega$. Some pulse-length errors however also introduce off-resonance errors as discussed in the previous subsection 18.1.

We will not order the pulse-length errors quantitatively in magnitude due to the complexity of such analysis. Instead we order the discussion on the level of review and work related to each source in this sub-thesis.

\(^5\)This is the pole-to-pole plane normal to the field vector as defined in subsection 17.1.2.
18.2.1 Rabi Frequency Perturbations ($\Delta \theta = \Delta \Omega \tau$)

Off-resonance errors introduce pulse-length errors through propagating the perturbations in resonance to perturbations in the generalised two-photon Rabi frequency $\Omega$. Other sources include the limited interrogation window (26.3) since the atoms that fall out-of-bound with the Raman beams experience free evolution even when optical pulses are applied. A limited interrogation window that scales with the cross-sectional dimensions of the Raman beam thus introduces inhomogeneous systematic errors across the atomic cloud with $\Delta \Omega = \Omega$ (Between in-bound and out-of-bound atoms).

18.2.2 Pulse-shape&Timing Perturbations ($\Delta \theta = \Omega \Delta \tau$)

Apart from sources of pulse-length errors discussed in section 26.5, we identify one additional source introducing $\Delta \tau$ perturbations: phase-profile inhomogeneities (26.8). Since the Raman beams relative phase $\phi_{L}^{(rel)}$ controls the optical field’s vector orientation along the equator, perturbation in $\phi_{L}^{(rel)}$ propagate into perturbations in the trajectory of state vectors on the Bloch sphere. This trajectory mods from a smooth curve during an ideal error-less pulse to a nutated curve whose deviations from smooth are proportional to the pulse-length error. The nutations introduce additional travel distance resulting in an underestimation of the pulse-length that is inhomogeneous if perturbations in $\phi_{L}^{(rel)}$ are random.

Finally, a pulse-length error that falls into both categories come from Raman beams that do not turn off entirely during free-evolution following diffraction through the unmodulated AOM. This gives a time and pulse-sequence dependent systematic error as some laser interrogation takes place during the state preparation, read-out and between pulses.

18.3 Experimental Limits

For our static interferometer experimental set-up discussed in chapter 16.2 we identify seven limits that impact implementation of novel composite pulses (and even simple rectangular pulses). We include these seven limits into the discussion so they can be compared to other experimental set-ups and to set a reference on experimental limits relevant for attempts at implementing composite pulse schemes. These limits are summarised and listed on the next page in the table 18.1.
CHAPTER 18. DISCUSSION

1. **Phase programming temporal limit**
The arbitrary waveform generator used to program a set phase-profile on the I&Q modulator is limited to $10^8$ samples/s \(i.e.\) 100 MHz or 10 ns steps. With the Pulse-blast having a lower trigger pulse-length limit of 50 ns with 10 ns resolution, the sub-pulses minimum length becomes limited to the 100 ns level.

2. **Sub-pulse temporal limit**
Limited Raman powers limit the Rabi frequency hence lower bound of the optical pulses duration to \(\sim\) \(\mu\)s. By extension, the length of each sub-pulse becomes lower-bound. The time-scale is limited to the number of steps in the composite pulse and the distribution of lengths ([un]even).

3. **Phase programming noise limit**
The I&Q modulator has a measured $10^o$ noise level from over-saturation in previous experiments as per figure 26.7. Electrical damage was caused during previous experiments by using 1 V I,Q voltages whilst the specifications stipulate a limit of 0.3 V set by the Germanium diodes in the full-bridge rectifier mixer circuits.

4. **EOM bandwidth limit**
The experiment has an EOM modulation bandwidth (varying \(\delta L\)) around 2 MHz. Including the 'kink' observed at \(\sim\) -1 MHz, the experimental set-up is limited in EOM driving frequency by the Raman power drop observed for driving frequencies outside the bandwidth.

5. **Perturbations/Imperfections**
The different off-resonance (see section 18.1) and pulse-length errors (see section 18.2) discussed in further detail in chapter 26.

6. **[1.5 mmx1.5 mm square Raman windows]**
Expansion of the atomic cloud with downwards gravitational drop limits interferometric sequences to \(\sim\) 13 ms (simulation) post state preparation. As state-preparation takes near 15 ms from Eddy-current effects, sequences could be extended to \(\sim\) 30 ms by managing background magnetic fields \(B\) (e.g. mitigation methods against Zeeman shifts discussed in chapter 27) and diagonal orientated beam shaping optics (+2 ms, in this set-up).

7. **Temporal coherence length limit**
A spin-echo experiment suggested a coherence length of hundreds of \(\mu\)s as discussed in section 16.5. This estimate came from the time-scale measurement at which the spin-echo contrast drops below the \(-3\) \(dB\) line. Regrettably the raw data is lost following logistical challenges.

Table 18.1: List (Arabic numerals) of the identified [sources] regarding the limits on preliminary testing composite pulses in our experimental set-up.
Chapter 19

Conclusion

We demonstrated that the Ramsey $\pi/2 - \pi/2$ interferometric sequence can be viewed as the quantum equivalent to the classical Young’s double slit experiment. Each beam splitter optical $\pi/2$ pulse partitioning the matterwave is equivalent to a slit diffracting the light. The measured atomic interference pattern bar a quantum ad hoc requirement are equivalent with the diffraction envelope. The diffraction envelope is a Lorentzian lineshape of on-resonance two-photon Rabi frequency $\Omega_{2R}$ in half width half maximum against atomic detuning $\delta$ (centred around the light-shift $\delta_{ac}$).

The Lorentzian diffraction envelope can be fitted to multiple Rabi flop data or to a Raman spectrum for finding the generalised Rabi frequency $\Omega$ and light shift. Fitting on the dataset of highest measured Rabi frequency we found $\Omega = 328 \pm 29 \ kHz$ at 50 $mW$ with light-shift $\delta_{ac} = -314 \pm 11 \ kHz$ at 780.267 nm carrier wavelength. Whilst not measured, previous estimates from the previous generation of the experiment suggest our $^{85}Rb$ atomic cloud has a temperature of a few $\mu K$, diameter on tens-hundreds of $\mu m$ and $\sim 10^6 - 10^8$ atoms. Note these measurements were taken in the $\pi^+ - \pi^-$ Raman configuration.

These Rabi-flop and Raman spectrum measurements are subject to errors from experimental imperfections/perturbations which impose several limits on coherence, contrast and precision. We classify these errors formally into two classes: off-resonance and pulse-length errors and review their sources in a general setting. Experimentally we outline the preliminary results on composite pulses, the main mitigation method against errors we investigate. For reference we review the experimental limitations in our set-up that can navigate other experimentalists in implementing composite pulse schemes and act as a reference on technical challenges when implementing such schemes.
CHAPTER 19. CONCLUSION

The preliminary data on testing composite pulses suggests up to 180% enhanced matterwave bandwidth (atomic distribution $\delta$) using Gradient Ascent Pulse Engineered (GRAPE) optimised optical pulses with a multiphase-step profile in the $\sigma^+ - \sigma^-$ Raman configuration. A fidelity ceiling seems to have been reached however which we hypothesise to originate from the Raman beam imperfections in the experimental set-up. In particular, the cross-leaks between the two Raman beams of $\sim 1, 8\%$ introducing three simultaneous sub-interferometers of unequal Raman beam powers in addition to the Raman beams not turning off fully during free-evolution.

The temporal evolution of the GRAPE composite pulse could however be fitted suggesting the computer simulations describe the undergoing physics (two error classes) to a sufficient extend. The enhanced bandwidth and $\sim 20\%$ peak fidelity increase by GRAPE composite pulses thus appear to successfully re-focus the trajectories of atom’s state vectors on the Bloch sphere, reducing the spread of state mixing that accumulates during an interferometric sequence by systematic errors.

Composite pulses thus provide better mirror pulses for atomic physics experiments by reducing sensitivity to experimental imperfections and perturbations. This is of interest for maintaining high fidelity augmentation pulses in large momentum transfer schemes and virtually any atom interferometric sequence if enhancement hold for beamsplitter composite pulses. Further details on the progress on GRAPE pulses can be found in the following research paper: [120] including the choice of testing with $\sigma^+ - \sigma^-$ (larger errors).
Sub-thesis IV

Hardware Enhancements
Chapter 20

Introduction

In this sub-thesis we discuss the work towards raising the technology readiness level of atomic rotation sensors. Capable of measuring rotation, acceleration and velocity, these sensors have potential applications to internal navigation systems to a commercial level.

For navigation-grade gyroscopic applications a rotation sensors needs a short-term rotation sensitivity $\alpha \leq 1 \text{ mdeg}/\sqrt{\text{h}}$, a bias stability $\beta_1 \leq 1 \text{ mdeg}/\text{h}$ and a scale-factor stability $\beta_2 \leq 1\cdot10^{-6}$ [163]. Here $\alpha$ quantifies the effect of a random rotation walk at short integration times (short-term), $\beta_1$ the deviation/drift in bias from its mean and $\beta_2$ the amount of deviation/drift in scale-factor of the sensor from its mean.

Atomic sensors have demonstrated short-term (1 s) sensitivity of $\alpha = 0.78 \text{ mdeg}/\sqrt{\text{h}} (2.04\cdot10^{-3} \text{ mdeg}/\sqrt{\text{h}})$ and long-term sensitivity$^3$ of $123.77 \text{ mdeg}/\text{h}$ (30.89 $\text{ mdeg}/\text{h}$) using a MOT (atomic beam) atomic source [26]. They have also demonstrated a scale-factor stability $\beta_2$ of $0.1\cdot10^{-6}$ at research level in laboratory and $(1-100)\cdot10^{-6}$ outside laboratory [164].

A typical air-core fiber gyroscope has a long-term sensitivity of $1044 \text{ mdeg}/\text{h}$ and a short term (100 $\mu$s to 10 s) sensitivity $\alpha = 15 \text{ mdeg}/\sqrt{\text{h}}$ [165]. Not only is $\alpha$ of fiber gyroscopes too large for navigation-grade applications but atom interferometers also have 8-35 times better long-term sensitivity. A bias stability $\beta_1 \leq 1 \text{ mdeg}/\text{h}$ has also been reported, smaller by a factor of $(1-10)\cdot10^3$ to commercial devices [164].

$^1$The offset between sensor input (actual) and sensor output (measured).
$^2$A multiplier which maps sensor input using calibration data to the sensor’s output.
$^3$Bias stability $\beta_1$ after long integration times (in this case of 15 min).
In this work the particular atomic sensor we use is a Point Source Interferometry (PSI), an atom Mach-Zehnder light-pulse interferometric sequence that utilises the classical ballistic expansion of a cloud of ultra-cold atoms. The sub-thesis assumes familiarity terminology covered in the previous sub-thesis (III) such as cooling and repump laser, Raman transitions, etc. Reviewing relevant sections of chapters 15 and 16 in particular is therefore advised.

20.1 Sub-thesis Structure

The sub-thesis is divided into three core chapters: an theory chapter (21) describing the PSI principle and experimental implementation, an experimental progress chapter (22) describing the level of progress prior to the departure of the author from the group, and a final future considerations chapter (23) on the upcoming tasks for the first prototype. A summary of the progress are found at the end in the conclusion chapter 24.

20.2 Disclaimer

This sub-thesis involves the work of several researchers and members of staff. In particular, Dr. M. Belal (postdoc) should primarily be credited towards the experimental progress and design of the new novel Raman laser system, the design of the new interferometer by Dr. T. Freegarde (supervisor), the manufacturing of the designed PSI flanges and mounts to Mark Bampton (mechanical workshop staff) and the custom temperature stabilisation unit to Mr. Gareth Savage (electronic workshop staff).

It should be noted that the development of the new interferometer was limited by years of delay of equipment and designing of the interferometer along with previously mentioned logistical and supervisory challenges. Certain parts continue to be place holders or have not arrived yet along with the design being delayed until early 2018. This has unfortunately delayed the development of the new interferometer thereby limiting the scope of this sub-thesis. Uncertainty also laid at access to the laboratory, an issue not yet resolved. This sub-thesis therefore instead focuses on the design, progress (by the author) and future tasks that remain to be completed.
Chapter 21

Mach-Zehnder Atomic Interferometer

Unlike the static atom interferometer as discussed in sub-thesis III, parts of the vacuum and optical system for the new interferometer discussed here move in space over time. Having previously referred to such an interferometer as dynamic, in this chapter we discuss the details of the specific interferometer under construction, its principle of operation along with the design of the new set-up.

21.1 Light Rotation Sensor

The rotation of a classical Mach-Zehnder interferometer can be used for rotation sensing through the Sagnac effect. The difference in optical path due to a rotation $\Omega$ introduces a Sagnac phase $\Phi_{\text{Sagnac}}$ given by

$$\Phi_{\text{Sagnac}} = \frac{2E}{\hbar c^2} \Omega \cdot A$$

where $E$ is the energy of the photons (or atoms if atomic), $\hbar$ the reduced Planck’s constant, $c$ the speed of light in vacuum and $A$ the area enclosed by the interferometer [166].

For photons, $E$ is inversely proportional to the wavelength with shorter wavelength photons producing finer phase-shifts (higher precision). The de Broglie wavelength of atoms serve a similar function in addition to the effective larger interferometer area when slowed (cooled). The ability to control the speed of the particle therefore provides an additional advantage to the atomic version.
21.2 Atomic Rotation Sensor

The rotation sensor under construction will make use of the so called (Atomic) Mach-Zehnder sequence which is an $\frac{\pi}{2} - \pi - \frac{\pi}{2}$ sequence with each pulse separation of time $T$. This sequence visualized on the Bloch sphere in figure 21.1 produces fringes that are affected by rotational motion as well as linear motion allowing for the use as an internal guidance system. In this section the fringe generation is explained.

![Bloch sphere representation of the Mach-Zehnder sequence](image)

Figure 21.1: A Bloch sphere representation of the Mach-Zehnder sequence for $T$ (red and blue trajectories) at constant detuning $\delta = 0$. (Left figure) The atom starts at state $|1\rangle$ given by the green arrow and ends in the state given by the blue arrow. First $\frac{\pi}{2}$ pulse leaves the atom in an superposition between $|1\rangle$ and $|2\rangle$ i.e. on the equator of the Bloch sphere. A free evolution of time $T$ rotates the Bloch vector about the z-axis. (Centre figure) A $\pi$ pulse flips the initial Bloch vector (green) about the z-axis to a new position given by the blue arrow. (Right figure) A free evolution of time $T$ starting from the green arrow position. This is followed by the final $\frac{\pi}{2}$ pulse leaving the atom in $|2\rangle$. Figure courtesy of J. Saywell.

Let $\mathbf{x}(t)$ be the 'position' vector of a matterwaves relative to our coordinate origin time $t$. Hence $\mathbf{x}(t)$ describes the 'displacement' of our matterwaves at time $t$. The phase difference $\phi(t)$ accrued by the matterwaves at time $t$ after a $\mathbf{x}(t)$ displacement is given by

$$\phi(t) = k_{\text{eff}} \cdot \mathbf{x}(t)$$

where $k_{\text{eff}}$ is the wave vector difference between matterwaves in $|2\rangle$ and $|1\rangle$ i.e. the wave vector acquired from the Raman beam interactions. Hence $k_{\text{eff}} = k_1 - k_2$ where $k_j$ is the wave vector of the $j^{th}$ Raman beam.
The vector $\mathbf{x}$ can be viewed in two different frames of reference: the *inertial* frame of reference the atoms ‘experience’ and the *rotating-accelerating* frame which the sensor ‘experiences’. The imaging detector of the fringes moves with the sensor *i.e.* observe the vector $\mathbf{x}_{\text{moving}}$ in the rotating-accelerating frame. This vector can be derived by *Lorentz-boosting* from the displacement vector $\mathbf{x}_{\text{inertial}}$ in the inertial frame by an acceleration and rotation.

Changing from the inertial to an constant accelerating frame can be described using the linear equations of motion

$$\mathbf{x}_{\text{inertial}}(t) \mapsto \mathbf{x}_{\text{accelerating}}(t) = \mathbf{x}(t = 0)_{\text{inertial}} + ut + \frac{1}{2}at^2$$  \hspace{1cm} (21.1)

where $\mathbf{v}(t)$ and $\mathbf{a}$ represent the linear velocity and acceleration of the sensor respectively and $u = \mathbf{v}(t = 0)$. By the initial condition $\mathbf{x}(t = 0) = 0$, the first term vanishes. From equation 21.1 it is evident that the following holds

$$\frac{d\mathbf{x}_{\text{accelerating}}(t)}{dt} = \mathbf{u} + \mathbf{a}t = \mathbf{v}(t)$$  \hspace{1cm} (21.2)

To enter the dual constant accelerating-rotating frame, we boost from the accelerating frame to a frame that rotates at $\Omega$. By considering the motion of the coordinate vectors [167], we can infer the following relation between the non-rotating frame $\{\mathbf{x}_{\text{accelerating}}(t)\}$ and rotating frame $\{\mathbf{x}_{\text{moving}}(t)\}$ as

$$\frac{d\mathbf{x}_{\text{accelerating}}(t)}{dt} = \frac{d\mathbf{x}_{\text{moving}}(t)}{dt} + \Omega \times \mathbf{x}_{\text{accelerating}}(t)$$  \hspace{1cm} (21.3)

By ignoring the smallest $\Omega a$ order terms, we find equation 21.3 becomes

$$\mathbf{v}(t) = \mathbf{u} + \mathbf{a}t \simeq \frac{d\mathbf{x}_{\text{moving}}(t)}{dt} + \Omega \times \mathbf{u}(t)t$$  \hspace{1cm} (21.4)

By taking the integral of equation 21.4 over a time window $t' \in [0, t]$ or $t' \in [t, 2t]$, the vector $\mathbf{x}_{\text{moving}}(t')$ at the end of each window, we find the approximate forms

$$\mathbf{x}_{\text{moving}}(t) \approx ut + \frac{1}{2}at^2 - \frac{1}{2}\Omega \times ut^2, \text{ for } t' \in [0, t]$$

$$\mathbf{x}_{\text{moving}}(2t) \approx ut - \frac{3}{2}at^2 - \frac{3}{2}\Omega \times ut^2, \text{ for } t' \in [t, 2t]$$  \hspace{1cm} (21.5)

The last term in equation 21.5 describes the *Coriolis acceleration* due to the *Coriolis effect* [168]. The *Centrifugal effect* is not included, contributing as a higher order correction [169].

---

1The magnitude $\Omega$ of $\Omega$ represents the angular speed of rotation and the direction of $\Omega$ direction is given by the right hand screw rule.
To simplify this the rest of the interferometer’s phase-shift $\Phi$ we will make use of figure 21.2 showing a projected Mach-Zehnder sequence in $xy$-projected phase space. We introduce several important markers and auxiliary phases in this figure.

From figure 21.2 we infer $\Phi$ can be calculated from the phase-shift of the labelled excited arm $|2\rangle$. Tracing its trajectory, we see the phase is given by

$$\Phi = -(\phi(t' \in [0, T]) + \phi(t' \in [T, 2T])) = -(\phi_1 + \phi_\alpha) \quad (21.6)$$
where the $-ve$ sign stems from the mirror pulse phase reversal. We can calculate the phases $\phi_1$ and $\phi_\alpha$ using equation 21.5 as

$$
\phi_1 \approx k_{\text{eff}} \cdot (ut + \frac{1}{2}ut^2 - \frac{1}{2}\Omega \times ut^2), \text{for } t' \in [0, t] \\
\phi_\alpha \approx k_{\text{eff}} \cdot (ut - \frac{3}{2}ut^2 - \frac{3}{2}\Omega \times ut^2), \text{for } t' \in [t, 2t]
$$

(21.7)

which approximates the Mach-Zehnder atomic interferometric phase to

$$
\Phi = k_{\text{eff}} \cdot (2\Omega \times u + \bar{a})T^2
$$

(21.8)

This is the same phase found in [170]. We thus provide an alternative method involving a matterwave description in calculating the interferometer’s phase. Our method also allows to find phases $\phi_2$ and $\phi_2$ in figure 21.2 describing the atom’s not affected by the mirror pulse (lost matterwaves) thus their effects on the total interferometer’s phase-shift.

### 21.2.1 Implementation Principle

To implement the atomic rotation sensor, we aim to use an $\sim 100 \mu m$ diameter $\sim 10 \mu K$ ultra-cold Rb gas contained within a glass vacuum cell that is made to rotate with the interrogating Raman lasers inducing stimulated Raman transitions [171]. This forms a Point Source Interferometer (PSI) that takes advantage of the ballistic expansion of the atomic cloud [144]. This rotating interferometric sequence is illustrated in figure 21.3 and can be thought of as an atom number interferometers running simultaneously that interfere to produce fringes.

The different interferometric pulses apply momentum along different directions due to the rotating quantisation axis$^2$. This gives a spatial matterwave phase-response as described by equation 21.8. Essentially, the ballistic expansion of the atomic cloud which provides a direct map between the atom’s velocity and the atom’s spatial position producing fringes. The position of the fringes correlate to the atoms acceleration, whilst the fringe separation correlate to the continuous rotation rate of the interrogating Raman laser beams [26] as per equation 21.8. Dual imaging along multiple axes through phase-shearing [172] provide inertial sensing along multiple axes in a single measurement.

$^2$Defined by the Raman interrogating beams that are parallel to the magnetic bias field provided by one of the compensating Shim-coils.
Figure 21.3: An atomic Mach-Zehnder interferometric sequence applied onto a cloud of atoms (purple). The Raman interrogating beams (yellow) rotate with the apparatus about the atomic cloud. Throughout the sequence, the cloud undergoes ballistic expansion providing a map between the atoms spatial and velocity distribution. Each atom described by Bloch clocks runs at a different rates depending on the detuning. The sequence therefore also provides a spatial dependent Bloch clock rate whose evolution is affected by the rotation rate, acceleration and interferometric sequence of the interrogating Raman lasers.
21.2.2 Implementation Hardware

The PSI has its rotation axis parallel to the Earth’s rotation axis which is rotated at Southampton’s (UK) latitude angle as shown in figure 21.4. This is to avoid the centrifugal high order term from Earth’s rotation [173] when we test the PSI in Southampton, UK. This through phase-shearing [172] allows for simultaneous measurement of acceleration due to gravity. If acceleration alone is of interest, the rotation rate is simply set to zero.

![Figure 21.4: 3D design of the PSI vacuum system. Including view-ports, the core custom Ti flange connects the glass vacuum cell, dispenser sources, ultra-high vacuum pump (red) and a valve (closed). This core is connected to Ti plates onto which the beam shaping optics can be mounted, hosting the free-space to fibre collimators. PCB printed coils occupy the space between the plates and glass cell for generating the magnetic fields. The set-up is placed on a precision rotation stage, itself tilted at the latitude angle of Southampton, UK. Figure courtesy of main supervisor Dr. T. Freegarde.](image-url)
For calibration and testing, the PSI is placed on an Aerotech APR260DR-160-RE-AS-A-NOTT direct-drive precision rotation stage. The stage is driven by an Aerotech ENSEMBLE HLE10-40-B-MXH Ensemble HLe Controller and a linear digital drive providing fine control over the rate of rotation. The measured rotation rate can thus be compared to the set rotation rate.

To retrieve information on the velocity and displacement, the measured acceleration can be integrated up to the appropriate order with respect to time. Limited to the integration error and accuracy in the acceleration error, the PSI rotation can be used as an internal navigation system.

Further details on the hardware of the PSI are discussed in the next chapter.
Chapter 22

Experimental Progression

Prior to the departure of the author from the research group, several steps of progress were made towards the early development of a prototype of the Point Source Interferometer (PSI). In this chapter, we highlight which steps have been taken giving a coarse insight into the upcoming interferometer. In particular, we introduce and describe the various laser systems designated for the PSI and provide a ‘progress-report’ on their current level of development regarding portability applications. The vacuum system preparations are discussed at then end highlighting the preparation procedure and layout of parts.

This chapter contains work which involved collaboration with other researchers. These include the design and construction of the vacuum system’s custom flange, design and implementation of the temperature stabilisation system and the design/assembly of the novel Raman laser system. The cleaning and assembly of vacuum components, pumping of the vacuum system, work on the MOT laser systems, work on locking modulators (not included in this thesis) and designing the portable laser shelves are however by the author along with the suggestions of future steps in the next chapter (23).
22.1 Novel Raman System

For the new dynamic point source interferometer (PSI) we propose a novel high-power fibre-based Raman laser system for interrogating the $^{85}\text{Rb}$ atoms. Starting with a stable 1560 nm telecom seed laser coupled into a 50—50 fibre-splitter, per arm the light gets modulated by a nested Mach-Zehnder waveguide (SSBM) with an optional phase modulator (PM) in series for phase control. The optical sidebands are amplified using an Erbium Doped Fibre Amplifier (EDFA) which passes through an high-power AOM for switching the beams on/off. The 1560 nm seed sidebands are frequency-doubled using Second Harmonic Generation (SHG) before fibre coupled to beam shaping optics attached to the vacuum chamber mount. Apart from the beam shaping optics, the proposed scheme’s layout is shown in figure 22.1.

Figure 22.1: Proposal for a novel high-power fibre-based Raman laser system for driving stimulated two-photon Raman transitions for ultra-cold $^{85}\text{Rb}$ experiments. The stable free-running seed laser has light send through polarisation maintaining fibres (PMF) partitioned into two channels/arms. The SSBM’s (nested Mach-Zehnder waveguides) driven by a RF source, I&Q drivers and an Arbitrary Waveform Generator (AWG) generate the up/down 750 MHz shifted frequency components depending on the arm, both arms amplified through an Erbium Doped Fibre Amplifier (EDFA) in gain saturation mode. With an Acousto-Optical Modulator (AOM) in series acting as switches for pulse generation, the light is frequency doubled (SHG) using a Periodically Poled Lithium Niobate (PPLN) crystal with the PMF then connected to the Beam Shaping Optics (BSO) on the vacuum chamber mount. An optional PM can be used for additional phase modulation control (PM–$\phi$) for composite pulses, spin-echo and other phase-profile related interrogations. Otherwise, the SSBM’s should be programmable using the $\phi$ terminals. Finally, 1700 MHz RF could be used for the $^{87}\text{Rb}$ isotope.
22.1. NOVEL RAMAN SYSTEM

22.1.1 Raman Laser System Progress

A Laser 2000 Rock fibre-laser with kHz linewidth has been installed as seed laser. It has an output power up to 125 mW and 1530 – 1565 nm bandwidth, sufficient for the anticipated levels of attenuation towards the EDFA’s.

For optical modulation (SSBM) we selected the iXblue MXIQER-LN-30. For laser frequency control of the Raman beams, these modulators require a locking servo for to counter the slow electronic and thermal drifts. Unfortunately the modulators didn’t come with such a servo hence a custom system has to be considered. The main proposal is an analog dither lock.

The lock would mix the signal from an local oscillator with the photodetector pins. Using an appropriate filter and low-noise amplifier, the mixed signal can be forwarded to a Max 32 Chipkit microcontroller using an Analog to Digital Converter (ADC). Setting up a peak-detection servo, an appropriate error signal can be fed to the bias ports using an Digital to Analogue Converter (DAC) added onto the local oscillator voltage.

The advantage of using a microcontroller is the ability to freeze the voltages during the application of the sequence, removing the minor power fluctuations during the interrogation of Rb atoms. In principle, operating the EDFA’s in gain saturation should filter effects from the dither lock to some degree making this concern of second order.

As shown in subsection 22.2.5, progress on the new Raman system includes installed free-space optics. Per arm, the high-power output of the EDFA passes an Isomet M1099-T50L-1550 AOM for switching the beams on/off. This is followed by a temperature controlled Covesion MSHG1550-1.0-40 Periodically Poled Lithium Niobate (PPLN) crystal for SHG.

To separate the different frequencies, a short-pass dichroic mirror is used with the transmitted light coupled into a polarisation maintaining fibre connected to the Holo-or TH-227-M-Y-A 3 mm top hat Beam Shaping Optics (BSO) attached to the mount surrounding the vacuum chamber.

1C.f. the engineering notes by Photline Technologies regarding their principle of operation. They specify the modulators have a 1-18 GHz bandwidth in the 1520-1620 nm with 35-50 dB spectral extinction ratios [174].

2A Covesion PV40 encapsulates the SHG crystal, temperature stabilised using an OC1 temperature controller.

3The free space and fibre coupling use Schäfer + Kirchhoff 60FC-L-4-M20L-02 (5MA18-45-S) 780 nm (1560 nm) fibre-free space collimators.
So far, preliminary tests by the groups post-doctoral researcher Dr. M. Belal on the frequency-doubled light suggests Raman powers can be attained up to $5 \, \text{W}$ with the place-holder EDFA’s. The proposed laser system therefore allows for advanced and custom composite pulses at high powers. This is novel compared to [175, 176] which cannot generate a wide variety of optical pulses with complex phase-profiles.

If we consider a single-photon detuning $\Delta$ of about $10 \, \text{GHz}$, the current Raman powers would give us a two-photon Rabi frequency on the order of tens of $\text{MHz}$ [119]. This is around a hundred times larger then the current attainable Rabi frequencies in the static atom interferometer set-up.

After completing the remaining elements of the laser system, future work involves a performance characterisation focused on the linewidth, frequency/phase noise levels, output power, AOM switching speeds and levels of carrier suppression. For portability a further test on sensitivity against noise from the environment needs to be completed.
22.2 Portability Design

22.2.1 Portable Server Rack

To provide a portable system, a double \(^4\) 19” test server rack is being populated with the parts needed to drive the PSI experiment. With a volume of 870 \(L\), the rack fits all necessary parts other than the sensor head (vacuum chamber with \(Rb\) atoms, rotary mechanics and mounts).

These PSI parts populating the front-half of the rack are:

- Magneto-Optical Trap (MOT) laser system shelf.
- MOT AOM switching optics shelf.
- 2x Raman AOM switching optics shelves.
- 2x EDFA’s.
- Raman seed laser shelf.
- Custom Temperature control shelf.
- Pulse-Blaster.

The back-half of the rack is populated with:

- MOT servo and power supply.
- Raman/AOM RF electronics/drivers.
- SSBM’s custom locking circuit.
- Power supplies.
- Camera electronics.
- Raman seed laser drivers.
- Master control computer.

The server rack is mounted with four top cooling fans and a three-phase power feed that can be connected to mains.

\(^4\)The front half primarily has the optical connections and optical components filled shelf whilst the parts such as drivers are left behind the shelves in the back half of the rack.
22.2.2 Optics Rack Shelf

The MOT laser system, AOM switching optics and Raman seed laser shelves are in progress of being miniaturised to a 300x300 mm \textit{Thorlabs} MS12B/M double density micro aluminium (\textit{Al}) breadboard thermally isolated with a 400x400 mm acrylic box temperature stabilised with heating resistors at 25 °C\textsuperscript{5} to avoid drift in optics. Additional sorbothane feet under the breadboard-supporting acrylic plate provide vibrational damping with future acoustic foam around the breadboard being considered for acoustic isolation\textsuperscript{6} from external sounds.

The two primary candidates for acoustic isolation foams are the \textit{Steinbach} M1148/4-10-2.1 and S 3705a polyurethane self-adhesive foams with respective sound insulation factors\textsuperscript{7} of 25-45 dB (125 Hz – 8 kHz) and 35-45 dB (63 Hz – 4 kHz). These foams can be placed onto the acrylic plates and should provide some additional thermal dampening.

The different optics shelves need to be tested for robustness against temperature, acoustic and vibrational variations. Regarding thermal, this could involve simultaneous monitoring the shelf’s internal temperature, output optical frequencies and powers during external temperature ramps of different magnitudes and time-scales. For acoustic and vibrational reliability tests, a similar measurement could be made involving transmission loss spectrum measurements\textsuperscript{8}.

The custom temperature stabilisation details is built by Gareth Savage from the School of Physics’s electronic workshop. Capable of temperature stabilising four separate shelves, the electronics fit to occupy a 3U shelf where U = 44.45 mm is an industrial standard rack unit.

\textsuperscript{5}The casing of the lasers are individually temperature stabilised to 25 °C. A few degrees warmer could provide a better stabilisation.

\textsuperscript{6}The primary interest is blocking sound from external sources affecting the optical shelves \textit{i.e.} prevent their further propagation into the shelf. This is separate from absorption which is useful for reducing echo.

\textsuperscript{7}Estimates from their respective datasheets. Note this factor is not the propagation loss through a material as it factors in the area of the foam. Finally these estimates are limited to flanking sounds, those that propagate from source to the detector from bypassing the foam.

\textsuperscript{8}Measuring the sound or vibration inside the shelf relative to the external source at different frequencies to get a spectrum of attenuation.
22.2.3 MOT Laser System Shelf

For the first two prototypes, a Schroff 4U shelf was dedicated to the MOT laser system. The first prototype didn’t feature a temperature stabilisation of the breadboard. It mainly used free-space optics using 1/2” optics. A picture of the prototype is shown in figure 22.2 with a schematic shown in figure 22.3. Note that without stacking the lasers in the future, the MOT shelf’s components could be reshuffled to place them in a 3U shelf.

Figure 22.2: The first prototype of a miniaturised Magneto-Optical Trap laser system using Vescent electro-optical components. One laser (top, repump) is locked with a saturated absorption spectroscopic module using a peak servo loop. The remaining laser (bottom, cooling) is locked to the repump beat signal detection and an optical phase locking loop. For comparison, common lab tools such as a fibre-scope and fibre-cleaning tape are shown for comparison. Finally, this prototype didn’t feature shelf-temperature stabilisation.

The first prototype halved the previously daily mW-level drifts in power. To increase the shelf’s stability, the second prototyped involved substitution of several free-space optics with fibres and introduced temperature stabilisation. The stability from the previous q.o.d. drifts was found to have increased to over 3 months. Accounting for qualitative power drifts in the MOT shelf output power, the second prototype shown in figure 22.4 thus has months of stable operation within 1 mW level. This prototype therefore could and was used for testing composite pulse (e.g. [120]).
Figure 22.3: Schematic of the MOT laser shelf. The MOT laser shelf comprises of two D2-100-HP1 DFB for the repump and cooling seed lasers. The repump laser is frequency-referenced using a D2-210-Rb saturated absorption spectroscopy module ([177]) with the cooling laser referenced relative to the repump. A heterodyne D2-250 module with a D2-160 beatnote photodiode generates an error signal forwarded to the Integrated Control Electronics (ICE) unit hosting amongst other things the D2-135 offset phase lock servo. Using this servo, the cooling laser can be detuned with control from the repump. The output of this MOT laser shelf are sent to the MOT AOM switching shelf.

Currently the latest prototype can output a maximum of about 15 mW cooling and 5 mW repump light. However, this can be enhanced further by optimising the fibre coupling, accounting for the slight elliptical shape of the laser beams and selecting a more appropriate lens.
22.2. PORTABILITY DESIGN

22.2.4 MOT AOM Switching Shelf

To switch the cooling and repump lasers on/off, the output light from the MOT shelf is fibre-coupled into the MOT AOM switching shelf. This shelf has not been built yet, but will use the same configuration as the existing free-space MOT switching set-up used by the static interferometer from sub-thesis III. There namely both repump and cooling beams undergo a double-pass\(^9\) through an Acousto-Optical Modulator (AOM) with the depump beam being generated as in [119]. The existing layout is shown in figure 22.5.

![Figure 22.5: The free-space (Gooch Housego) AOM configuration used for switching the cooling, repump and depump beams. The cooling light (top right) passes through a single pass M110-1F-GHI1 110 MHz AOM (driven at 130 MHz) whose $-ve$ first order provides a depump beam. The zeroth order passes further through a M080-2B/F-GH2 80 MHz AOM in double pass to avoid a net frequency shift. The repump light performs an identical double pass on an identical 80 MHz AOM.](image)

The double-pass is achieve by placing a Polarising Beam Splitting Cube (PBSC) in-front of the AOM and letting the $p$-polarisation pass through the AOM. The first order is then reflected through a quarter waveplate ($\frac{\lambda}{4}$) using a D pick-off mirror so the $s$-polarised light can be separated from the incident light. An additional half wave plate is used to set up the $p$-polarisation for the input light.

Finally, the shelf-version of both cooling, repump and depump AOM switches is planned to occupy a single 3U shelf. This shelf will lie in series with the MOT laser system shelf and the beam shaping optics adjacent to the vacuum chamber hosting the $Rb$ atoms.

\(^9\)This gives a zero frequency-shift and higher switching rates.
22.2.5 Raman AOM Switching Shelf

The proposed Raman system from section 22.1 contains some free-space optics namely the AOM and SHG optics. Due to the uncertainty in the EDFA delivery and the problems with higher order modes and output power in their temporary place-holders, the layout is likely subject to change. The optical head for instance along with the particular high-power AOM and short-pass dichroic mirror (SPDM) have needed substitutions on several occasions. As such, early prototypes by Dr. M. Belal as shown in figure 22.6 are allocated 4U shelves in case we find they do not fit in 3U shelves.

Figure 22.6: The miniaturised Raman Second Harmonic Generation (SHG) stage and AOM switching shelf. An optical head (upper-left) transfers the EDFA’s high power 1550 nm light into an Isomet M1099-T50L-1550 AOM (upper right) who’s first order is coupled into the Covesion MSHG 1550-1.0-40 PPLN crystal (bottom-left) for SHG with lenses L. The output is filtered by wavelength using a short-pass dichroic mirror (SPDM) with the 780 nm light coupled into a fibre in the picture. Also, BD–beam dump, PBSC–Polarising Beam Splitting Cube.

Each arm of the new Raman system will have the 1560 nm light undergo a single-pass through the AOM to maximise powers. By taking the same diffraction order and driving the AOM’s at the same frequency, the need to account for this frequency-shift is eliminated for PSI when using counter propagating beams.

22.2.6 Remaining Rack Parts

Regarding the remaining front-half of the rack, the place-holder EDFA’s take up 3U shelves along with the pulse-blaster (SpinCore PBUSB-RM-24-100-4k) and likely the seed-laser also. The back-half of the rack have been fitted with custom trays on which the various electronics can rest.
22.3 Experimental Vacuum Chamber

A custom Ti flange with mount parts designed by Dr. T Freegarde has been machined by Mr. Mark Bampton from the School of Physics mechanical workshop. A preliminary test-fit prior to the vacuum preparation is shown in figure 22.7 including the glass vacuum chamber, beam shaping optics, prototype PCB magnetic coils and precision rotation test stage.

Figure 22.7: Progress on the designed PSI science chamber from figure 21.4. The left image shows one of the Ti plates removed to expose the inner Glass Cell connection and PCB-mountable MOT (Helmholts and Shim) coils. The right images show the complete rotation head at different angles on a test rotary stage. The extruding shafts host the beamsheaping and steering optics.

The experimental vacuum chamber along with the imaging system will be placed on an optical bench for testing, occupying itself a rack in the future. With the rest of the PSI inside a 19” rack, the current prototype is expected to occupy two racks as a whole. A possible future re-design using field programmable gate arrays could allow for the PSI to occupy a single rack as half of the current 19” rack consists of electronics.
22.3.1 Vacuum System Preparation

For a good vacuum, the custom $Ti$ flange (and other flanges) was carefully cleaned. The process starts chemically with three-steps:

1. A 45 min ultra-sonic bath ($Kerry$ $Pulsator$ KS850) using a 2% $Galvex$ 18.01 solution dissolved using de-ionised water.

2. Ten single-stroke swipes using lint-free\textsuperscript{10} wipes doped with acetone, per stroke surface area

3. Ten single-stroke swipes using lint-free wipes doped with isopropanol, per stroke surface area.

Step 1 primarily removes bio-degradable surfactants coarsely such as polishing compounds used for providing a smooth finish to the flange post machining. The second step provides a finer cleaning step, mainly removing residue from step one. The third step similar, removing the evaporated acetone residue from step two providing the finest cleaning step by solution.

Post the chemical preparation, the vacuum system assembly involved four additional steps:

4. Pre-baking of the Ti-flange separately at 200 $^\circ C$ using blank flanges.

5. Assembling the vacuum system in a cleanroom.

6. Final baking at 150 $^\circ C$ of pump-connected assembled vacuum system.

7. Transferring active pumps and relocating vacuum to the test-site.

The pre-baking step (step four) aims to remove any moisture and other embedded/trapped particles within the vacuum accessible outer layer of the vacuum system. The custom $Ti$ flange is sealed with $LewVac$ FL-40CF blank flanges and a $LewVac$ FL-H1000-40CF 1 m hose. Using $Thorlabs$ HW-KTT 2/M M6 steel screws (and washers), these are secured using $Pfeiffer$ $Vacuum$ 490DFL040-S10 annealed copper gaskets and a ‘skip-every-two’ bolts star pattern with 1/16 rotation per bolt visit\textsuperscript{11}. A $Wera$ 7000A torque wrench was used to reliably set the torque of the M6 screws to $\sim 25$ $Nm$.

\textsuperscript{10}We use the $Kimtech$ branded lint-free wipes which are non-abrasive and leave few particles behind after a swipe.

\textsuperscript{11}For the turbo-pump mount of twenty-bolts, a 'skip-ten-five' bolt pattern is used.
This open-end vacuum system was then placed inside a Binder ED400 oven for the pre-bake. The other end of the vacuum system’s hose (outside the oven) was connected to a water-cooled Varian Turbo-V 81-M turbo-pump in series with an Agilent Leybold SH-110 scroll dry-pump\(^{12}\) (forepump) by a LewVac FL-H250-25KF flexible hose. An Oerlikon Ionivac Sensor ITR 90 was used to monitor the pressure down to \(5 \cdot 10^{-10}\) mbar through a T-shaped flange between the turbo-pump and oven-connected hose.

Using a temperature heating ramp of \(1 \, ^\circ C/min\), the vacuum system was heated to 200 \(^\circ C\) for a few days. Throughout, the turbo-pump capable of pressures down to \(5 \cdot 10^{-10}\) mbar (using the dry-pump) was eventually turned on giving a final pressure of \(\leq 5 \cdot 10^{-9}\) mbar i.e. near or below the limit of our sensor. The oven was then ramped down at \(1 \, ^\circ C/min\), completing the pre-baking step.

For step five, we visited\(^{13}\) the School of Physics’s NanoMaterials Rapid Prototyping Facility. Under a fume cupboard, using the same gaskets, screws and torque the pre-baked custom \(Ti\) flange is connected to

- 1x LewVac FL-40CF blank flange
- 1x MDC DN40CF fused silica view-port
- 1x Vacuum Generators ZCR40R all-metal valve
- 1x Saes NEXTorr D 100-5 dual ion & getter pump
- 1x Precision Glassblowing custom octagonal cell\(^{14}\)
- 1x Dispenser-mounted MDC DN40CF IFM7-C40 feedthrough

The open-end vacuum system is then placed inside the same oven used during the pre-bake (step four) with the same 1 m hose connecting the metal valve to the previous pump configuration. To avoid the build-up for thermal stress, a \(1 \, ^\circ C/min\), ramp limit is imposed along with heating limited to 150 \(^\circ C\) (from the Saes NEXTorr D 100-5 pump).

\(^{12}\)Following previous use of the dry-pump by other researchers, to restore the pumping efficiency and vacuum limit of \(6.6 \cdot 10^{-2}\) mbar a tip-seal replacement kit(SHO110TS) was used along with an acetone and isopropanol cleaning of the pump.

\(^{13}\)As conventional, all terminals of the vacuum parts outside their packaging are stored in aluminium wrapping during transport with all handling by latex gloves.

\(^{14}\)Cleaned in factory.
Following the departure of the author from the research group, the experimental vacuum preparations terminated at this step reaching similar pressures inside the vacuum system as during the pre-baking step at nominal temperature. The next step (seven) will involve interfacing the Saes NEX-Torr N10PS-03PSU dual pump driver on an interruptible power supply unit. The active pump can then be set to the dual pump (after closing off the metal valve) without a concern of a power shortage eliminating the vacuum. Following the valve closure, the 1 m hose can then also be disconnected allowing for the vacuum assembly to be relocated from the oven to an optical bench test-site.

### 22.4 Dispensers

The PSI will use the same source of \( Rb \) atoms in the vacuum chamber namely \( Saes \) RB/NF/7/25 FT10+10 atomic dispensers. Six such atomic dispensers are connected\(^{15} \) using M1.6 steel screws\(^{16} \) to the seven-pin \( MDC \) DN40CF IFM7-C40 feedthrough with the centre pin designated as common ground. By exceeding the activation current, we can promote the release of \( Rb \) atoms from the Rubidium Chromate active ingredient through a redox reaction with the Zirconium powder reducing agent. \( Rb \) also has the tendency to stick to the surface of glass as observed in the existing static interferometer. In that set-up, the dispensers are rarely used as the MOT coil provide sufficient heating to thermally release the \( Rb \) from the glass walls. For the PSI, an alternative method worth investigating involves the use of UV-light \(^{[178]} \) to serve the same function, avoiding the need for high-current (hot) coils. It should be noted however that these dispensers are amongst many hazards both carcinogenic and irritants. There is also a concern for fire in moist environments from self-heating ingredients along with environmental toxicity when considering disposal. For further details on the hazard, c.f. the BSDSE020521 Rev. 4 safety datasheet (or the latest recommended revision).

Finally, the discussed vacuum system should be able to support Bose-Einstein Condensation (BEC) of Rubidium atoms as demonstrated in a similar set-up using \(^{87}Rb \) \(^{[179]} \). The advantage of BEC over a MOT (or atomic beam) as atomic source include a larger spatial coherence and novel engineered optical probing techniques \(^{[180]} \)). The PSI therefore provides a large temperature range for ultracold Rubidium atoms, well into quantum degeneracy (BEC).

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\(^{15}\) We confirmed the bridge using an electrical conductivity test.

\(^{16}\) The advantage of using screws over spot welding is future replacement the dispensers.
Chapter 23

Future Consideration

Including the completion of the vacuum and laser systems, the fully operational PSI requires additional steps before any considerations can be taken for commercial applications. A coarse preliminary preparation has therefore been prepared regarding the upcoming tasks required for completing a PSI prototype. In this chapter, we highlight the main future considerations that are needed for building and testing this prototype.
23.1 Proposed Sequence

Assuming the vacuum and laser systems are operational, they are interfaced with a master control computer that can trigger the various parts of the interferometer. We can inherit several technologies and techniques used in the static interferometer set-up from section 16.2 to accelerate this step. In particular, regarding the trigger sequence we propose the following:

First a MOT is loaded in the glass cell vacuum chamber. The atomic cloud is cooled further, closer to the recoil limit by switching off the Anti-Helmholtz MOT coils. This is followed by an optical pumping stage for state preparation before the state is purified with a $\pi$-pulse. Atoms in the undesired state are then pushed out of the way with a pushing beam so the atomic sequence can begin. Post the sequence, readout takes place by imaging the cloud. It may prove useful to separate the clouds with a pushing beam prior to imaging.

The proposed sequence for the PSI experiment is analogous to the trigger sequence summary shown in table 16.1. As initial conditions, similar time-scales could be used later fine-tuned to optimise performance.

23.2 Characterisation

Part of the future characterisation of the atom interferometer will involve atom number estimation, atomic cloud size measurement and quantifying the velocity distribution (temperature) post Sisyphus cooling. There are several established techniques to complete such characterisation such as time of flight measurements of the ballistic expansion of the cloud [181, 182], release and recapture of the atomic cloud [183] for temperature measurements, CCD imaging for the cloud’s shape and dimensions and fluorescent measurements using the cooling laser for the atom number estimation [184].
23.3 Imaging of Fringes

Regarding the imaging of fringes, a *Princeton Instruments* Excelon ProEM HS 512 X 512 (PROHS512BX3-SH-CM-Q-F-BB) camera will be used. Whilst the exact configuration has not been sorted out yet, several techniques are considered including:

- Mach-Zehnder Phase Imaging
- Light-Sheet Imaging
- Two-photon (Blue) Imaging
- Absorption Imaging

Different techniques are expected to provide different advantages. For instance, light-sheet provides shorter imaging times, thus less averaging out of fringes at higher rotation rates during imaging along with less image heating from the imaging laser. Absorption imaging however uses a far simpler set-up, of interest for rapid prototyping. The different imaging proposals are reviewed below.

23.3.1 Mach-Zehnder Phase Imaging

The atomic cloud can be imaged to $\sim \mu$m resolution [185] using an optical Mach-Zehnder interferometer (MZI). This technique provides a non-destructive image of the cloud through off-resonant optical phase-shift measurements [186]. This might allow for the averaging of the fringes at higher rotation rates to be studied through continuous imaging of the cloud or study the heating by imaging.

By adapting the technique to the design shown in figure 23.1, the imaging can be made compatible with both $\sigma^+ - \sigma^-$ and $\pi^+ - \pi^-$ polarization configurations of the interrogation Raman laser-fields. The proposed arrangement has the un-diffracted order of each Raman beam (not shown) coupled into a fiber and past through a common secondary AOM. Then, when the interrogating Raman beams are switched off, for imaging the secondary AOM is turned on sending the light through the dual Raman port of the MZI. We can detune one Raman laser beam/arm far-off resonant to the D2 $^{85}$Rb transition the second/other Raman beam can provide a reference for locking the optical MZI. This stabilises the phase-measurements from optical drifts.
Figure 23.1: Non-destructive Mach-Zehnder Interferometer (MZI) Imaging of the atomic cloud. For the $\sigma^+ - \sigma^-$ configuration, the * marked quarter-wave plates ($\lambda/4$) present. Only for the $\pi^+ - \pi^-$ arrangement is the reference reflection at the right-bottom Polarising Beam Splitting Cube (PBSC) not ensured hence a double-pass configuration needed as shown. During imaging (***), the reference beam (raman – 2) detuned to a different wavelength beats onto a photodiode (PD) for phase-stabilising the MZI by locking the piezo that steers a movable MZI mirror using a PID (Proportional Integral Derivative Servo). Imaging and Raman interrogations (**) can be optimised for contrast using half-wave plates ($\lambda/2$). A low/high pass dichroic mirror (DM) can be used to separate the different Raman beams from each other.

23.3.2 Light Sheet Imaging

Light Sheet Microscopy is a popular method in biology that uses a highly focused laser in one axis along the focal plane to image the intended object [187]. Light by fluorescence can be collected selectively along the focal plane giving high contrast images.

A cylindrical lens or AOM can be used to focus the light in only one axis generating a light sheet of $\sim \mu m$ thick and $\sim mm$ wide. Unfortunately this introduces a problem: the light-sheet will not be uniform in thickness across. As such, the outer fringes will not be imaged as well (resolution) as the inner fringes where the sheet is thinnest. Another problem is orientating the sheet as fluorescent imaging\(^1\) does not provide enough light. Hence absorption imaging is needed along the Raman interrogating beams for imaging thus an angled light sheet of some sorts.

\(^1\)Fluorescent imaging whilst effective for $\geq$ biological cell-sized structures i.e. $\sim 10 \mu m$, it is subject to a low signal-to-noise ratio for our atomic Rb cloud of far fewer atoms.
23.3.3 Two-photon (Blue) Imaging

We consider two-stage excitation imaging[188]. Here, the atoms in the upper ground-level $|2\rangle = |5S_{1/2}, F = 3\rangle$ are pumped into the $5P_{3/2}$ state using the cooling laser followed by a 776 nm excitation to the $5D_{5/2}$ level. The atoms decay to the lower ground level in two stages, a 5 $\mu$m transition to the $6P_{3/2}$ intermediate level followed by a blue 420 nm decay to $|1\rangle |5S_{1/2}, F = 3\rangle$.

The advantage of this scheme is two-fold, compatibility with high-power Raman beams at a different wavelength from the sensitive imaging system along with the removal of background scattering. Implementation of the blue laser is also simplified with the hyperfine splitting of the $5P_{3/2}$ state known to at least 20 kHz level [189].

23.3.4 Absorption Imaging

The simplest measurement of the fringes by far is absorption imaging along the interrogating axis. This technique is commonly used in other PSI and works well, though they often have a larger enclosed area such as the Kasevich group tower that spans over 10 meters [144]. Due to the miniaturisation and portability impositions, our atomic cloud will at most have $\sim$ cm levels of expansion hence will benefit from a more sophisticated imaging technique.

Absorption imaging also requires longer imaging exposure times which in a rotating system results in fringe averaging. The Mach-Zehnder phase imaging scheme in contrast reduces the required exposure time by 100-1000. As such, we propose the absorption imaging could play a role beyond characterising the MOT, using the same set-up to accelerate the PSI prototype with a substitution of the imaging for higher resolution modifications and upgrades.

23.4 Statistical Fringe Analysis

Extracting the rotation rate and acceleration can be done in multiple ways. Two are under consideration: Principle Component Analysis (PCA) and Analysis of Variance (ANOVA). PCA reduces the correlated data to a subset of uncorrelated data whilst ANOVA provides a variance to the shot-to-shot rotation and acceleration values [190]. As such, PCA can be used for studying shot-to-shot phase-noise correlations whilst ANOVA a tool for rotation and acceleration single estimation.
Sensitivity to acceleration and rotation can be extracted by measuring the phase noise correlations in single-shot measurements. As per [144] the images are partitioned into $s$ parts with their complementary segments amplitudes compared. For acceleration sensitivity, the rotation is turned off whilst for rotation sensitivity, the increase phase-noise correlation from the acceleration can be used for first order approximation.

### 23.5 Outside the Laboratory

The prototype PSI under construction is intended to function as an pilot scale project for possible future commercial internal navigation system. To raise the technology readiness level from bench/pilot scale beyond active commissioning, an demonstration of the performance of the PSI outside the laboratory is needed.

Assuming basic measurements of rotation, acceleration and linear motion have provided the levels of reliability and precision, an experiment could be performed outside the laboratory. One possible option is making a comparative measurement between an existing navigation system and the PSI prototype. The exact details however have been left for when the bridge of in-lab results has been crossed.
Chapter 24

Conclusion

We report the development towards a new Point Source Interferometer (PSI) from an rotating atomic Mach-Zehnder interferometer that utilises the Sagnac effect. Using the matterwave description and Lorentz frame boosts, we derived the Sagnac phase in agreement with the literature. This shows the PSI over an atomic ensemble will produce fringes whose position correlate with the atom’s acceleration, and whose separation correlate with the atom’s relative rotation (to the Raman interrogating lasers).

Progress has been made in designing and populating a 870 L 19” rack that will host the non-vacuum parts of the PSI such as the laser systems and their drivers. This includes a novel fibre-based Raman laser system that whilst in its infancy, preliminary outputs several watts of power. The new laser system is compatible with complex phase-profiles allowing for composite pulses to be applied in sequences in the future.

The cooling and repump laser system has been miniaturised to fit inside a 4U rack shelf (U = 44.45 mm). Used for testing composite pulses [120], the laser system is temperature stabilised providing month’s of stability in power compared to optics alignment requirements on every other day. The existing temperature stabilisation system is also accessible to the remaining optical shelves of the rack.

The vacuum system has been prepared and awaits transfer of active pump to a dual ion & getter pump. With the $Rb$ atomic dispenser sources installed, the vacuum chamber is ready to be placed onto the precision rotation stage on the optical bench. Upon completion of the rack, the next step involve interfacing all parts of the PSI, charactering the Magneto-Optical Trap and selecting a final imaging system.
We propose to use absorption imaging for a test of concept of the fringes, moving onto either Mach-Zehnder phase imaging (with global [Sagnac] phase stabilised during measurement [rotation]), Light-sheet imaging or Blue imaging. The imaged fringes can then be processed using Analysis of Variance and/or Principle Component Analysis for extracting rotation rate of the rotation rate (or acceleration in later generation of the set-up).

Finally, with regards to internal navigation applications, we highlight the interest for some comparative test using an existing navigation system. This could raise the technology readiness level from six/seven up to nine or higher by demonstrating commercial potential. The in-lab testing itself already raises the level up to five as a minimum.
Part V

Summary of Enhancements
Chapter 25

Introduction

This sub-thesis serves to summarise and conclude all the significant findings on enhancing the performance of atom interferometers. We include relevant results from all sub-theses I, II, III & IV and aim to classify them into three categories: minor, modest and major. Their classification of enhancement level is somewhat arbitrary, but coarsely takes into consideration the effect on improving the interferometer, the complexity/novelty of the work, implementation difficulty and how likely an experiment will benefit from the enhancement (enhancement efficiency).

25.1 Sub-thesis Structure

To enable the classification of mitigation methods into classes of enhancement, we review the sources of perturbations from the experiments in sub-thesis III and anticipated perturbations for the set-up in sub-thesis IV in chapter 26. This provides a more nuanced description of the experimental sources to the off-resonance and pulse-length errors discussed previously in sections 18.1 and 18.2 respectively.

This is followed with a discussion (chapter 27) that focuses on mitigation methods we identify from the collective work of the sub-theses. In this chapter we look at every mitigation method and describe the errors they mitigate against, primarily qualitatively though with quantification where this is possible.

We conclude the findings of this thesis with a conclusion (chapter 28) on the levels of enhancements by coarsely dividing the discussed mitigation methods into three categories/levels.
25.2 Disclaimer

Our work includes both proposed (not tested) and implemented (tested) enhancement proposals to atom interferometers. Due to logistical challenges such as data-access and lab-access restriction during candidacy, limited quantification is available when analysing perturbation sources and effectiveness of mitigation methods. The scope of the thesis is regrettably impacted (limited) by problems in supervision and the challenges they brought during the time of research. We nevertheless hope our work can map out the landscape of perturbations and errors whilst providing quantification on a few islands of mitigation methods from a sea of mitigation proposals that future experimentalists can explore.

Some enhancements we review in this work are known/common practice in other research groups thus not all listed enhancements are novel. The work of this sub-thesis should also be viewed as a continuation of the work from the previous sub-theses I, II, III & IV which involved collaboration with other researchers. The works and analysis in this sub-thesis however involve no further collaboration or involvement with other researchers.
Chapter 26

Sources of Perturbations

There are several sources of perturbation which limit the performance of light-pulse interactions with atoms from the ideal case. Some of the significant sources along with mitigation methods are discussed in this chapter. The perturbation sources included in this chapter are:

- 26.1–Spatial Raman Inhomogeneities
- 26.2–Power Problems in Raman Beams
- 26.3–Limited Interrogation Window
- 26.4–Zeeman Shifts from Magnetic Fields
- 26.5–Pulse-length Errors
- 26.6–Fringe Averaging
- 26.7–Initial Cloud Size
- 26.8–Phase-profile Inhomogeneities
- 26.9–Impurities

Some of these perturbations will affect both the existing static, and under construction dynamic atom interferometer. Note that the mitigation methods discussed here along with the causes of perturbations are not mutually exclusive as will be discussed below and further in chapter 27. Finally, solved systematic errors like the trigger-delay of the EOM are not included in this chapter but can be found in section 16.5 or more generally in chapter 27.
26.1 Spatial Raman Inhomogeneities

The power of the Raman beams determines the Rabi frequency and by extension the time-scales of a mirror and beam-splitter pulse. The output beam of a laser or optical fibre can be well modelled as having an Gaussian cross-sectional distribution resulting in an non-flat spatial distribution in power transverse to the optical axis. If such light is used without modification on a cloud of ultra-cold atoms, the Rabi frequency ceases to be common to all atoms which lowers the interferometer’s signal-to-noise ratio. As the atoms fall by gravity, each atom will for sufficiently long interferometric sequence also acquire a time-dependent Rabi frequency even if the total power of the Raman lasers remaining the same.

To provide a uniform interrogation power across the cloud, in figure 16.3 we use Topag Lasertechnik GTH-4.2.2 beamshaping optics that convert a Gaussian-shaped beam to a flat top-head beam. An image of the Raman beam at the focal place post the beamshaper is shown in figure 26.1.

Figure 26.1: One of the Raman interrogation beams imaged by the Thorlabs DCC1545M CMOS camera at low powers to avoid CMOS saturation. The image is taken at the focal plane of a Topag Lasertechnik GTH-4.2.2 Gaussian to flat intensity beamshaper. It convert the cross-sectional circular beam to a square beam of \( \sim 1.5 \text{ mm} \) in size (at focus). An aliasing or CMOS etalon effect produces the visible periodic fringes that depend on the CMOS orientation to the beam.

Whilst this provides a more uniform power across the cloud, the optical phase gets spatially ‘scrambled’ during the beamshaping process. As such, an cloud falling across the beamshaped light (we henceforth shall refer to as ‘window’) will experience a non-uniform spatial dependent phase-shift across the cloud. Essentially, we succeeded in eliminating the spatial and temporal dependence of the Rabi frequency at the cost of introducing a systematic error: inhomogeneous cross-sectional phase-profile by beamshaping.
Regarding fringe detection in the atomic cloud (relevant to the dynamic interferometer, \textit{c.f.} sub-thesis IV for further details), the expected response on the imaged fringes is a phase-imprint that hopefully can be circumvented by digital processing, or might be negligible to the phase-shift of the interferometer/camera. Digital image processing is the main mitigation under investigation for this systematic error.

26.2 Power Problems in Raman Beams

There are two types of power problems with regards to the interrogating Raman beams. The first type are fluctuations in the power of the frequency components in the Raman beams. The second is the leaking of other optical frequency components in our Raman beams when interrogating the $^{85}$Rb atoms.

26.2.1 Power Fluctuations

Fluctuations in the Raman beam powers mainly stem from drift in optics, lasers (including TA’s) and imperfect beamshaping optics. In its entirety, using the current beamshaping optics for the static interferometer we observe $\sim 15\%$ Raman power fluctuations across the beamshaping window. Better beamshaping optics for the new dynamic interferometer could aid reducing these spatial power fluctuations for the dynamic interferometer.

Temporal power fluctuations ($\sim 1\%$ in our set-up) by drifts in optics can be reduced by converting parts of the new laser systems into fibre systems that are less sensitive to thermal drifts or vibrational or acoustical noise. Temperature stabilisation, vibrational and acoustical isolation of the remaining free-space optics can reduce sensitivity from environmental noise even further.

For example, the MOT laser system used for the static interferometer has already shown improvement from temperature stabilisation and free-space to fibre replacements. Previous daily re-alignments have so far not been needed ever since the MOT laser system got transferred over 6 months earlier into a temperature stabilised shelf. In section 22.2, further details can be found regarding the optical drift reduction.
26.2.2 Leaking of Frequency Components

When measuring the power of the Raman beams, apart from accounting for the extra upper EOM sideband $\Omega^{+}_{EOM}$ and its power contribution using the FP-OSA (c.f. section 16.2), there is also the problem of the presence of the other sidebands in each Raman beam. About 1\% of the AOM sideband $\Omega^{+}_{AOM}$ leaks into the EOM Raman beam ($\Omega^{-}_{EOM}$ & $\Omega^{-}_{EOM}$). Likewise, about 8\% of the EOM sidebands find their way into the AOM Raman beam. Higher polarisation rejection optics should reduce these leakages.

In practice our experiments run four simultaneous interferometer, two co- and two counter-propagating interferometers with Raman beams of unequal sideband and total power. This could explain why in [119] a constant offset was observed when comparing Rabi flop data at different powers to the unconvolved form of equation 16.2. The other three interferometers proportionate to the prime interferometer were not accounted for thus lowering the observed Rabi frequency.

26.3 Limited Interrogation Window

For the application of optical pulses, the expansion of the atomic cloud is limited to the size of the interrogation window of the Raman laser beams (figure 26.2 a). If the cloud expands beyond the window or falls out of it, the out-of-bound atoms no longer take part in the interferometric sequence. This atom-loss over time induces a dichotomy between the fringe structure in the state populations of the out-of-bound atoms compared to the in-bound atoms. If the same interrogating Raman lasers are also used for imaging, than a cropped image forms with lower atom number. Thus in general, atoms, fringes and contrast are lost limiting the overall time-scale over which an interferometric sequence can take place.

Figure 26.2: a) The Raman beam interrogation window in the normal and diagonal orientations. An atomic cloud is drawn at the centre falling at a time later downwards due to gravity and undergoing ballistic expansion. b) The atomic cloud (exaggerated) when exiting the Raman window in both normal and diagonal orientations.
26.3. LIMITED INTERROGATION WINDOW

With regards to the loss of atoms we consider two orientations of the Raman window and compare them using a simple simulation. Our aim will be to demonstrate another example of an minor enhancement to our experiment, and acquire a way to estimate the time-scale of supremum during which an interferometric sequences can be applied to our atoms.

26.3.1 Simulation of Atom-loss

To model the loss of atoms from a limited Raman window size we make use of a convolution integral. Apart from a deconvolution factor we can thus compute the time-scales which follow from a size-limited window. By symmetry it suffices to model half of the system along axis of drop. Setting this axis as horizontal allows us to use single-valued functions. The computation is normalised to provide an overlap integral (convolution) that estimates the percentage of overlap between the cloud and window.

Let $f_\square$ and $f_\circ$ represent the Raman window simulation functions in the normal and diagonal orientations as illustrated in figure 26.2 a). Setting the window as $2ax2a$ ($2a \sim 1.5 \text{ mm}$) and using the auxiliary variable $t$ defines our functions as

$$f_\square = \begin{cases} b, & t \in [-a,a] \\ 0, & t \notin [-a,a] \end{cases}$$

$$f_\circ = \begin{cases} t + b, & t \in [-\sqrt{2}a,0] \\ b - t, & t \in [0,\sqrt{2}a] \\ 0, & t \notin [-\sqrt{2}a,\sqrt{2}a] \end{cases}$$

To describe the atomic cloud we simplify\(^1\) to a uniform density within a well-defined boundary tracing a circle of radius $R$ centred at $c$ on the horizontal axis. Calling this function $g_o$ we model it as

$$g_o = \begin{cases} R, & t \in [-R,R] \\ 0, & t \notin [-R,R] \end{cases}$$

\(^1\)A real MOT would realistically not have a uniform density distribution nor a cliff-edge boundary. Our assumption as listed in subsection 26.3.3 is that the Gaussian-like density and edge MOT behaves sufficiently similar to our simple model of the MOT.
To horizontally shift $g_o$ relative to $f_{\Box,\Diamond}$ in time $\tau$ due to a gravitational drop of the cloud we set

$$c(\tau) = \frac{1}{2} g \tau^2$$

with $g = 9.81 \text{ m/s}^2$. To describe the expansion of the cloud of initial radius $R_0$ we set

$$R(\tau) = R_0 + v \tau$$

with $v \sim 5 \text{ mm/s}$; the typical measured average velocity of the atomic ensemble along the interrogation axis in our experiments [191].

Normalisation of these functions is set by

$$f_{\Box} \rightarrow \tilde{f}_{\Box} = \frac{1}{b} f_{\Box}, \quad b = a$$
$$f_{\Diamond} \rightarrow \tilde{f}_{\Diamond} = \frac{1}{b} f_{\Diamond}, \quad b = \sqrt{2}a$$
$$g_o \rightarrow \tilde{g}_o = \frac{2}{\pi R^2} g_o$$

where $\sim$ indicates normalisation.

As long as the boundary of the atomic cloud has not crossed an edge of the Raman window, we require the convolution to equal 1. Otherwise the convolution does not represent the ratio of atoms bound within the Raman window. Whilst the convolution between normalised normal window $\tilde{f}_{\Box}$ and the atomic cloud $\tilde{g}_o$ satisfies this condition, the convolution of the cloud with the diagonal Raman window $\tilde{f}_{\Diamond}$ starts to decrease earlier\(^2\). We therefore need to modify this convolution integral of the diagonal window. We do this by splitting the convolution into two parts, upto and after the crossing of boundary and window edge. This crossing\(^3\) takes place at time-stamp $l$ when

$$R(l) = \frac{\sqrt{2}}{2} (b - c)$$

If we set $\tilde{f}_{\Diamond} = 1$ for $-b \leq \tau \leq l$, we ensure the convolution equals 1 upto the crossing as required. After the crossing we can return using our previous definition of $\tilde{f}_{\Diamond}$ to model the loss of atoms from a decreasing overlap of the cloud with the window.

\(^2\)We forgot to offset the time before the boundary has touched the edge.

\(^3\)By symmetry in time a similar crossing takes place at $-l$ however we only concern ourselves with the downwards drop by gravity and not its time-reverse.
As such the convolution integrals are given by

\[
(\tilde{f} \square \star \tilde{g})_o(\tau) = \int_{-b}^{b} \tilde{g}_o dt \\
(\tilde{f} \circ \star \tilde{g})_o(\tau) = \int_{-b}^{c(l)} \tilde{g}_o dt + \frac{1}{b} \int_{c(l)}^{b} (b - t) \tilde{g}_o dt
\]

Solving the integrals we find

\[
(\tilde{f} \square \star \tilde{g})_o(\tau) = \frac{1}{\pi} \left( (\theta_2 - \theta_1) + \frac{1}{2} (\sin 2\theta_2 - \sin 2\theta_1) \right)
\]

where \( \sin \theta_1 = \frac{1}{R} (-a - c) \), \( \sin \theta_2 = \frac{1}{R} (a - c) \)

and

\[
(\tilde{f} \circ \star \tilde{g})_o(\tau) = \frac{1}{\pi} \left( (\theta_2 - \theta_1) + \frac{1}{2} (\sin 2\theta_2 - \sin 2\theta_1) \right) \\
+ \frac{(b - c)}{b\pi} \left( (\theta_3 - \theta_2) + \frac{1}{2} (\sin 2\theta_3 - \sin 2\theta_2) \right) \\
+ \frac{2}{3b\pi} R \left( (\cos \theta_3)^3 - (\cos \theta_2)^3 \right)
\]

where \( \sin \theta_1 = \frac{1}{R} (-b - c) \), \( \sin \theta_2 = \frac{1}{R} (-c(l) - c) \)

and \( \sin \theta_3 = \frac{1}{R} (b - c) \)

These equations are plotted in *Matlab* v.2018a by making use of the built-in \textit{vpasolve} function to calculate \( c(l) \). For the integration limits \( \theta_{1,2,3} \) the built-in \textit{asin} function is used which is given by

\[
\sin^{-1}(z) = -i \ln iz + \sqrt{1 - z^2}
\]

\textit{i.e.} uses the definition of the complex logarithm. Only the real part of the convolution integrals are sub-sequentially plotted in figures 26.3 & 26.4. All simulations set \( R_0 = 0 \) \textit{i.e.} ideal point-like for simplicity.
26.3.2 Simulation Outcome

The percentile of atoms lost as a function of time in the both configurations per our simulation is shown in figure 26.3. We expect $\sim 12 \text{ ms}$ to $\sim 14 \text{ ms}$ after the state preparation before we start to observe a loss of atoms in our experiments. Rotating the Raman window provides us with an additional $\sim 2 \text{ ms}$ demonstrating the advantage of the diagonal Raman window orientation. As such, the beam-shaping optics was set to the diagonal orientation during and post the testing of all composite pulses including in [120].

![Figure 26.3: Temporal simulation of the loss of atoms using the normal ($\tilde{f}^\Box \ast \tilde{g}_o$) and diagonal ($\tilde{f}^\Diamond \ast \tilde{g}_o$) Raman interrogation window orientation. The overlap integral of the Raman window and atomic cloud is normalised on the y-axis and plotted over time (x-axis).](image1)

To check the physics of our simulation we like to comment on the atom-loss vs relative drop dependence. We expect namely that whilst the diagonal window orientation experiences a loss of atoms $2\text{ms}$ later, the rate of drop should be faster as seen in figure 26.2 b) compared to the normal window orientation. Figure 26.4 demonstrates that our simulation agrees with this intuitive behaviour.

![Figure 26.4: Spatial simulation of the loss of atoms using the normal ($\tilde{f}^\Box \ast \tilde{g}_o$) and diagonal ($\tilde{f}^\Diamond \ast \tilde{g}_o$) Raman interrogation window orientation. The overlap integral of the Raman window and atomic cloud is normalised on the y-axis and plotted against the relative drop $c(\tau)/b$ (x-axis). N.B. the two curves use their corresponding value for $b$.](image2)
26.3.3 Simulation Imperfections

Our model makes use of several assumptions which oversimplify the problem. We like to quickly remark on them to provide a balanced view on the simulation outcome and any possible discrepancy from measurements.

1. *Symmetry along axis of drop*
   - Our Raman windows are rectangular, not perfectly square: the height and width differ by $\sim 0.1$ mm per beamshaper.

2. *Uniform distribution of atoms in cloud*
   - Our atomic cloud has a density that resembles more like a Gaussian centred at $c(\tau)$.

3. *Discrete boundary of atomic cloud*
   - Because of the Gaussian density, the MOT has a gradual boundary instead of a clear cut-off edge.

4. *Perfectly centered initial cloud*
   - In practice, $c(\tau) = c_0 + \frac{1}{2}gt^2$ where $c_0 \neq 0$.

5. *Multiple velocity classes*
   - The cloud has atoms expanding at a range of velocities $v$. Hence $R(\tau)$ is more complex.

6. *Small initial cloud size*
   - We set $R_0 = 0$ as the effect of the initial cloud-size proved negligible in multiple simulations.
   - When adding the other assumptions this may not hold and $R_0$ establish itself as a more relevant parameter.

7. *Convolution description*
   - For the diagonal convolution we have not calculated the deconvolution factor which could affect the integral from $l$ to $b$.
   - Our convolution integral effectively uses a Heaviside step-function and calculates the ‘turning on/off’ of our cloud. This description does not distort the shape except for the $l$ to $b$ integral for which deeper analysis is needed.

The time-scale at which loss of atoms is first observed seem to agree with measurements taken previously on our experiment [119].
26.4 Zeeman Shifts from Magnetic Fields

26.4.1 Sources of Magnetic Fields

Any laboratory is subject to background magnetic fields which alter the energy-levels of atoms. For a typical atomic physics laboratory, the highest order magnetic fields are ordered below in table 26.1:

<table>
<thead>
<tr>
<th>Source</th>
<th>Magnitude/[$\mu T$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sputter-ion pump</td>
<td>$\sim 1 \cdot 10^5$</td>
</tr>
<tr>
<td>MOT-coils</td>
<td>$\leq 3.5 \cdot 10^3$</td>
</tr>
<tr>
<td>Stray</td>
<td>$\leq 100$</td>
</tr>
<tr>
<td>Earth’s core</td>
<td>$\in [20,70]$</td>
</tr>
<tr>
<td>Lithosphere</td>
<td>$\leq 300 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>Ionosphere (primary)</td>
<td>$\leq 80 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 26.1: For a typical atomic physics laboratory, we tabulated six of magnetic fields at Earth’s surface in order of highest magnitude. We either specify their upper limit or their range of magnitude near the atoms.

From table 26.1 we infer that the largest magnetic field stem typically from the sputter-ion pump used to acquire the ultra-high vacuum. These pumps make use of a homogeneous field generated by a permanent magnet often placed on the outside of the pump’s casing [192].

The second order fields are generated by the MOT–coils used for spatially confining the atoms. Whilst their strength and direction depend on the specific coils dimensions, configuration and current throughput, our estimate in table 26.1 are taken from Sunil Patel’s thesis [118]. We expect typical values in other experimental set-up’s to be around our quoted estimates.

We estimate stray fields to occupy the third order magnetic field contributions. Whilst it is difficult to quantify the magnitude and direction of stray fields in general, we can define stray fields as low amplitude DC or AC (at 50–60 Hz) magnetic fields generated by electrical appliances. Assuming the general laboratory uses modern equipment, we can infer frequent adherence to the World Health Organisation guideline: a magnetic field strength upto $100 \, \mu T$ at a distance of 30 cm from the appliance [193].

---

S. Patel is a former PhD student who assembled and characterised the MOT chamber including the coil used in this sub-theses experiments at 2 A coil current.
Depending on the specific appliances used and their placement relative to the atomic cloud and each other, we thus expect stray fields from appliances to be capped to third order. In principle however, stray fields can easily be comparable to or weaker than the magnetic field generated by Earth’s core. Its estimate in table 26.1 uses the European Space Agency (ESA) data taken by Swarm sattelite constellation over Earth’s surface [194].

For completeness, we include in table 26.1 the measured magnetic field at the Earth’s surface from the lithosphere\(^5\) and the more dynamic primary\(^6\) ionosphere. Measurements also originate from the ESA Swarm project [195] confirming previous observations that the Earth’s average various magnetic fields decrease in strength upto a few percent each decade [196].

### 26.4.2 Qualitative Effects of Zeeman Shifts

Post molasses cooling, the ensemble of atoms comprising the ultra-cold gas cloud are typically distributed in different Zeeman \(m_f\) sublevels. As multiple sublevels are populated, a reduction in interferometric contrast is observed as a background magnetic field shifts the energies of different \(m_f\) states by different amounts as per equation 15.1.4. Hence atoms in different \(m_f\) acquire a \(B\)-dependent laser detuning \(\delta_L\). This becomes even more problematic if the background field is dynamic, spatially inhomogeneous or both as the light-pulse can no longer be resonant with all atoms.

As example, in the static interferometer we experience up to 15 ms eddy currents induced magnetic field when we turn the MOT coils off. Likewise, changing field strengths from the Earth or the movement of appliances producing stray fields around the vacuum chamber. Thus the atoms experience varying magnetic field conditions when used for example in an moving environment such as an atomic internal navigation sensor.

This perturbation manifests itself more formally as the detuning distribution of atoms introducing so-called off-resonance errors. We observe the spreading of states of all atoms across the equator on the Bloch sphere [197] throughout the interferometric sequence. This error scales with stray magnetic fields and the shape of the distribution of atoms in various \(m_f\) levels.

\(^5\)Primarily from minerals e.g. magnetites, vertical component quoted.
\(^6\)To clarify, the radial component is quoted of the primary ionospheric magnetic field caused by internal motion of charged particles relative to each other. The secondary field in comparison is caused by the Earth’s rotation relative to the charged particles thereby is significantly weaker [198].
26.4.3 Quantitative Effects of Zeeman Shifts

To quantify the effects of different magnetic fields, we use the measured fidelity of the tested pulses from figure 17.4. Previous measurements by A. Dunning [119] demonstrated the atoms in our cloud can be well approximated as being evenly distributed amongst the five possible Zeeman sublevels: $m_f \in \{-2, -1, 0, 1, 2\}$. Using this distribution and by computing the Zeeman shift per sublevel (defined by equation 15.1.4) we model the drop in pulse fidelity against rising magnetic fields for: a naive rectangular pulse, WALTZ pulse, and two GRAPE composite pulses optimised for two different temperature distributions ($40 \mu K$ & $100 \mu K$). Our simulation results are shown below:

Figure 26.5: Modelling the effect of a magnetic field $B$ during the interaction of our ultra-cold $^{85}Rb$ atomic cloud with an optical $\pi$ (mirror) pulses:

[upper left] – naive rectangular pulse  
[upper right] – WALTZ  
[lower left] – $40 \mu K$ GRAPE  
[lower right] – $100 \mu K$ GRAPE

In figure 26.5 we include the laser resonance (y-axis) to the atomic cloud’s light-shift. This demonstrates the effect of a widening resonance from using more sophisticated optical pulses: to reduce the sensitivity against $B$. 
By fixing the laser resonance to optimum fidelity (at $\delta_L - \delta_{ac} = 0$), we can compare the fidelity curves against magnetic fields for different pulses in a single figure as demonstrated below:

![Graph showing fidelity curves versus magnetic fields for different pulses.](image)

Figure 26.6: Correcting the laser to the atomic cloud light-shift, we plot a cross-section of the optimised fidelity curve ($\delta_L - \delta_{ac} = 0$) versus magnetic fields for the four different tested mirror pulses from our model. Cross-sections are taken from data in figure 26.5. This optimisation at higher $B$ also corresponds to atoms in the magnetic insensitive state $m_f = 0$.

Note that we fitted the rectangular mirror pulse with the Lorentzian distribution described in equation 16.1, and two summed Gaussian distributions for the other three pulse data-sets in figure 26.5. The squared error of the Lorentzian fit was of the same proportion as fitting a narrow and a broad Gaussian distribution of different heights summed together.

Our model thus assumes the errors from the summed Gaussian fit, the limited datasets and the $\sim \pm 1.5 \text{ MHz}$ in laser detuning from peak fidelity are negligible. To clarify, in shifting the fidelity lineshape per sublevel we have made three additional assumptions.
CHAPTER 26. SOURCES OF PERTURBATIONS

1) – That each $m_f$ sublevel contributes equally to the overall fidelity curve.
2) – That the magnetic field does not distort the fidelity-line shape.
3) – That the finite datasets limit the maximum magnetic field that we can model for reliably before the outermost sublevels ($m_f = \pm 2$) stop having overlapping datasets (we fall outside the $\sim \pm 1.5 \, MHz$ dataset-regime).

As such, for $B \geq 200 \, \mu T$ our model becomes less reliable, especially for the WALTZ and GRAPE simulations where the summed Gaussian fit has not been confirmed to be sufficiently representative of the datasets. The error in our model also increases at very high magnetic field strengths where equation 15.1.4 acquires an additional correction. Fortunately, the values we test here continue to fall within the weak field approximation.

From figure 26.6 we can infer by how much the resonances of the different sublevels overlap at higher magnetic fields when using more sophisticated optical pulses. By looking at the Half Width Half Maximum (HWHM) of each curve namely we can enumerate the magnetic field sensitivity reduction of novel optical pulses.

In increasing order, the HWHM values are: $\text{HWHM}_{\text{rect. } \pi} = 36.5(1) \, \mu T$, $\text{HWHM}_{\text{WALTZ}} = 71.7(1) \, \mu T$, $\text{HWHM}_{40 \, \mu K \text{ GRAPE}} = 129.3(1) \, \mu T$ and $\text{HWHM}_{100 \, \mu K \text{ GRAPE}} = 170.2(1) \, \mu T$. Hence relative to the naive rectangular mirror pulse used more commonly by researchers, a WALTZ pulse nearly doubles the HWHM, the $40 \, \mu K$ GRAPE pulse enhances by a factor of $\sim 3.5$ and the $100 \, \mu K$ GRAPE pulse by $\sim 4.7$.

If we compare our fidelity plots 26.5 and 26.6 to the highest order magnetic field strengths listed in table 26.1, we find our experiments are insensitive to the primary ionosphere and the lithosphere magnetic fields at the Earth’s surface. The Earth’s core magnetic field at the surface along with stray magnetic fields are sufficiently substantial however to affect our experiments, especially if stray fields stack from multiple electronic devices in the vicinity of the atomic cloud. Using novel composite pulses however, or to lesser effect nMR adapted pulses (WALTZ) we can reduce the sensitivity against magnetic fields up to factors of 4.7 by widening the interaction resonance between the laser and the atomic cloud. In particular, GRAPE composite pulses can make experiments less sensitive, possibly insensitive against stray and surface-level Earth’s core magnetic fields thus removing the need for magnetic shielding against such sources. Fields from sputter-ion pumps and from the MOT coils however are strong enough if persisting throughout the experiment, even if they have decayed and regardless if static or dynamic.
26.4.4 Alternative Mitigation Procedures

Besides from nMR or specialised composite pulses, the off-resonance errors from magnetic fields can be mitigated against using alternative methods. Our literature review found four alternative procedures that when implemented can reduce magnetic-induced off-resonance errors: Optical Pumping, using a Glass Vacuum Chamber, using Magnetic Shielding and introducing a Quantisation Axis. Compensation schemes are also commented on.

Optical Pumping

This procedure is already commonly implemented in sophisticated atom interferometry set-ups such as the spin polarisation methods as described in sub-section 16.1.5. Ultracold atoms are pumped into the $m_f = 0$ Zeeman sublevel which is insensitive to magnetic fields thus avoiding magnetic-induced off-resonance errors. Pumping efficiencies upto 98(1)% have been reported in the literature [199] making this procedure the most efficient and robust against magnetic fields.

By selectively altering the distribution of atoms in the possible $n$-sublevels, optical pumping reduces the percentage (of a uniformly $m_f$-distributed cloud) of lost atoms that can contribute to the signal from $\frac{n-1}{n} \cdot 100\%$ to under 3% as the bifurcation of the cloud-laser resonance is proportional to the populations in each sub-level. These two figures do not include atom loss, detection problems and other imperfections that affect the final interferometric signal. It is because of such signal loss mechanisms that despite modelling an even distribution of atoms in the five sublevels, the tested pulses in figure 26.6 still dips below the expected asymptote levels.

Glass Vacuum Chamber

One significant way magnetic fields continue to affect atoms that are successfully optically pumped into $m_f = 0$ is by affecting the spatial confinement. Residual magnetic fields from the MOT-coils can have substantial magnitude that add spatial forces on the atomic cloud even after the coils are switched off. This effect is enhanced by the presence of a metal vacuum chamber as is the case in our experiments.

---

7In our experiments as we use $^{85}$Rb. Therefore, $n = 5$ giving 80% maximum loss of atoms contributing to the signal. This does not include false signals from the other off-resonance atoms either which can lower the visibility of the remaining 20% signal, nor does it include the other various imperfections in the experiment.
We measured eddy current induced DC residual magnetic fields undergoing exponential decay from a magnitude comparable to the MOT-coils when ‘on’, that need time-scales upto 15 ms to decay [119]. Measurements of the AC component of magnetic fields in similar set-ups have demonstrated that AC attenuation of the root mean squared fluctuation down to 30 nT is possible [153], though our experiment has not made use of any compensation/attenuation method against magnetic field fluctuations nor have we measured them. We thus suspect that stronger AC magnetic fields might further be contributing to the noise-floor of our experiment.

We can reduce the magnitude and temporal persistence of residual fields not only by using active compensation schemes, but also by using a glass vacuum chamber over a metal-based vacuum chamber. Eddy currents are then restricted to only the coils onto which a compensation scheme can be applied. Quantification of the loss of residual fields from using a glass vacuum chamber is beyond the scope of this thesis as it depends on the dimensions of the vacuum chamber, the metal used in comparative measurements, stability and method of driving the coils and the efficiency of the feedback system used in any compensation scheme against magnetic field fluctuations.

**Magnetic Shielding**

Given the time-scales of a typical atom interferometric cycle as shown in table 16.1, we can infer the switching of the MOT-coils from iteration of cycles is limited at best to $\sim 1 \, kHz^8$. The on/off MOT-coil triggering during a cycle however by the pulse-blaster is limited by the 10 ns resolution of each trigger pulse. We thus expect magnetic field fluctuations from cycle-to-cycle switching are limited to 1 kHz whilst the faster fluctuations during a cycle range upto 100 MHz which coincidentally matches the clock frequency of the used TTL signal generator.

Several commercially available magnetic shields easily operate into the GHz-domain with supression reported from $-60 \, dB$ upto $-120 \, dB$ within the 0 – 100 MHz range. By selecting the appropriate material(s) and dimensions for shields, magnetic fields can thus be reduced by $10^{-44\%}$ for lower frequencies to $10^{-10\%}$ for higher frequencies [200]. Unfortunately, as we have not measured the spectrum of magnetic field fluctuations in our experiment, we cannot comment further on the effects shields have in reducing spatial forces or off-resonance errors. Such analysis is beyond the scope of the thesis.

---

8Assuming we utilise a better MOT-loading mechanism such as re-capture of atoms and optimise our set-up further.
Quantisation Axis

We define the orientations of different polarisations of light (\(\{\pi^\pm, \pi^0 \& \sigma^\pm\}\)) in our experiment relative to the atoms by constructing an quantisation axis from introducing a weak bias magnetic field as described in 16.5.2. The biasing magnetic field quantises the component of the total angular momentum vector along the chosen axis. This establishes a basis on which the state vectors are defined, thereby the polarisation configurations.

Apart from spatially quantising the \(m_f\) sublevel, the use of a bias magnetic field also reduces sensitivites to magnetic field fluctuations by placing atoms in the magnetic sensitive sublevels (\(m_f \neq 0\)) out of resonance with the Raman interrogating lasers [168]. The interferometric signal thus stems primarily from atoms in the insensitive sublevel, an effect similar to that seen with optical pumping. Hence introducing a quantisation axis avoids the bifurcation of the resonances by cutting those atoms from the interferometric signal. Note however that for atoms with \(n\)-sublevels, the signal reduces to \(\frac{1}{n} \cdot 100\%\) (uniform \(m_f\)). Thus a compromise is made between signal-strength and sensitivity against magnetic fields (noise). The quantification is beyond the scope of this thesis as new measurements would be needed.

26.5  Pulse-length Errors

Laser pulses have none-zero temporal lengths hence the atomic states phase evolve during a pulse [191]. Different atomic detunings acquire different phase evolution rates that uncontrollably spreads out the distribution of states occupied by the atoms comprising the cloud. This can be seen by looking at the matterwave’s phase \(\phi(t)\) at time \(t\):

\[
\phi(t) = k_{\text{eff}} \cdot x(t) - \Omega t
\]

(26.1)

Here \(k_{\text{eff}}\) is the effective wave-vector by the Raman beams, \(x\) the matterwave’s displacement vector and \(\tau\) the length of an optical pulse. Whilst slow, the atoms move giving a temporal dependence \(t\) on \(x\).

What equation 26.1 shows is that the interferometric phase is affected by the generalised off-resonance two-photon Rabi frequency \(\Omega\) which is different for different atoms. Hence a dephasing of matterwaves manifests itself from the atomic distribution in \(\delta\) hence \(\Omega\). In the quantum picture, instead of having \(N\) atoms tracing the same trajectory on the Bloch sphere they spread out reducing the contrast of the interferometer.
Higher Raman powers reduce pulse-length errors by decreasing the fractional phase-shift induced during a $\tau$-durated pulse starting at time $t$ ($\delta\phi(\tau, t)$) to the interferometer’s ideal phase-shift ($\Phi$) defined as having zero pulse-length errors. Using the Ramsey sequence reviewed in subsection 15.2.3 of pulse separation $T$ as example, we infer this fractional error to be

$$
\frac{\delta \phi(\tau, t)}{\Phi_{\text{Ramsey}}} = \frac{k_{\text{eff}} \cdot (x(t + \tau) - x(t)) - \Omega \tau}{\delta_{\text{quantum}} T - \phi_{L}^{(\text{rel})}}
$$

(26.2)

The effect of higher Raman powers such as proposed by the novel Raman laser system in figure 22.1 is to decrease the duration $\tau$ of mirror and beamsplitter pulses leading to a decrease of $\delta\phi$. We also infer from equation 26.2 that longer pulse separations $T$ in interferometers also reduce fractional errors which can be explained from the greater dominance of the phase-shift induced during free evolution to the constant $\delta\phi$ at larger $T$.

Equation 26.2 also helps clarify the effect of spatial and temporal power fluctuations in the Raman beams by analysing the $\Omega \tau$ term of $\delta\phi$. For spatial power fluctuations, the generalised two-photon Rabi frequency $\Omega$ (and relative phase $\phi_{L}^{(\text{rel})}$) will vary across the cloud due to the 15% spatial variations in Raman power from the beamshaping optics. Atom’s within the same velocity class can therefore acquire a different phase depending on their position in the cloud. The gravitational drop of the atomic cloud further complicates the picture as $\Omega$ starts to vary with time also.

### 26.5.1 Power Fluctuations Revisited

During temporal power fluctuations, $\Omega$ varies with time even for adjacent atoms in the same velocity class. Slow fluctuations in $\Omega$ induce varying phase-shift between consecutive interferometers run at the same experimental control variables. This introduces a shot noise into our discrete datasets from variations in the final mixed state vector between consecutive runs. Rapid temporal fluctuations shorter than an interferometric cycle in comparison have the added complication of affecting the phase-shift non-uniformly between pulses, increasing variations in the mixed state vectors of the atomic cloud per dataset (per interferometer run).

Both terms of $\delta\phi$ are also velocity dependent making pulse-length errors sensitive to the atomic cloud’s velocity distribution. This adds thermal noise to our interferometric measurements with the level of state mixing correlated to the velocity distribution (temperature and mean velocity) of the cloud.
26.5.2 Finite Raman Switching Speed

The AOM’s used to switch laser-fields ON/OFF rapidly experience $\sim 100\text{ns}$ switching times between their full 'ON' and full 'OFF' states. This introduces an effective rapid chirp in the Rabi frequency across the cloud that affects the evolution of atoms of different two-photon detunings (e.g. $m_f$ sublevel, velocity class, etc.) differently. By focusing the laser beam into the AOM we can optimise the overlap of the optical beam to the acoustical wave yielding faster optical switching times to $\sim 10\text{ ns}$.

Further reduction is possible by switching the Raman beams very far off-resonance when the AOM switches 'ON' (rising Raman pulse edge), only setting the Raman beams back to near-resonance when the AOM has fully switched. For the falling edge of a Raman pulse the laser is placed off-resonant at the desired time-stamp where we want the pulse to terminate. The AOM switching limit can be reduced to $\sim 1\text{ nsec}$ with a shorter duration chirp in the Rabi frequency using this method.

26.6 Fringe Averaging

This perturbation applies specifically to the dynamic interferometer from sub-thesis IV, a Mach-Zehnder atom interferometer sequence where the rate of rotation and acceleration of the Raman lasers relative to an ultra-cold atomic cloud can be calculated from the imaged spatial fringes of the cloud. Details on the Mach-Zehnder interferometer can be found in chapter 21.

Higher rotation rates and larger accelerations introduce larger image blurring which reduces the visibility of the spatial fringes [163]. The effect of blurring include larger errors on the calculated rotation rate and acceleration estimates and an upper bound on the rotation and acceleration we can resolve. The imaging apparatus has a minimum exposure time which averages out a moving feature and lowers contrast. The loss of higher spatial frequencies might be overcome by short exposure imaging techniques at low powers would benefits from reduced heating by imaging.

\footnote{This chirp adds to the interferometer’s phase as per equation 26.2, the $\Omega \tau$ term.}

\footnote{The atoms see an increasing and maximising Raman power whilst the AOM switches that cannot induce a transition as the lasers are too off-resonant. A rapid return to $\sim \text{GHz}$ detuning re-enables Raman transitions thus starts the pulse. The EOM controlling the laser detuning in our experiment is modulated providing rapid feedback and detuning control.}
26.7 Initial cloud size

A MOT is typically $\sim 100\ \mu m$ across hence is not point-like. Whilst a point-like initial atomic cloud undergoing ballistic expansion places atoms of different velocity classes at different positions, a finite-sized initial cloud allows for atoms of different velocity classes to travel to adjacent positions at the end of the interferometer. Atoms of different velocity classes could namely at appropriate time-scales travel to adjacent positions if they start from different origins. For the dynamic interferometer (c.f. sub-thesis IV), this results in a smearing/washing-out of the superimposed fringes that scales with the initial cloud size. The initial cloud size thus lowers the contrast of the interferometer when imaging fringes.

The correlation (map) between the atoms position and velocity distribution used for reconstructing rotation rate/acceleration is no longer well defined for small atomic cloud expansions. Only for final clouds $\gg$ initial cloud is the point source approximation useful again [163]. Larger cloud expansion and smaller initial cloud size are the countering measures. Though, a finite Raman interrogating window and size of sensor limit the final cloud expansion factor.

Apart from reducing the space of expansion when starting at an non-point cloud, for atom interferometers the initial cloud size also affects the contrast. This follows as the average intensity of the Raman laser beams drop with larger cloud diameters resulting in a loss in contrast over 10% [201].

26.8 Phase-profile Inhomogeneities

A RF I&Q modulator provides a controllable relative phase between the Raman laser beams in our experimental set-up. A delayed trigger or an incorrect input-to-output phase-map would during a Raman optical pulse reduce the precision of the pulse target state. Phase noise on the modulator also limits the control of relative laser phase along with relative phase noise built up between the Raman beam separate fibres.

Characterisation of our modulator revealed high noise levels in the RF output phase ($\sim 10^o$) and non-linearities at higher peak-to-peak input voltages of the modulator. The non-linearity follows from exceeding the specifications of the modulator that recommend a 0.6 peak to peak volt as maximum. The measured phase response is shown in figure 26.7 for 3 amplitudes (Vpp).
26.8. PHASE-PROFILE INHOMOGENEITIES

Figure 26.7: Measured MITEQ SDM0104LC1CDQ RF I&Q phase modulator response at three different peak-to-peak voltages Vpp = 0.6 (a), 1.0 (b) & 2.0 (c). Measurements taken using a HP 8702B Vector Network Analyzer at 2.7135 GHz and 14.01 dBm RF power.

Whilst non-linearities in the modulators in-to-out RF phase are corrected for using a calibration curve, its phase noise is propagated into the Raman beams optical phase. Any phase-shift between the two Raman beams accrued in the optical fibers adds an additional systematic error in the relative phase \( \phi_{(rel)} \) during Raman transitions. For composite pulses like those discussed in chapter 17, these errors are further complicated by the resolution in triggering and implementing a phase-change in \( \phi_{(rel)} \).

Polarisation maintaining fibres (PMF) over previous non-PMF fibres offer mitigation against relative phase noise between the Raman beams. This was implemented for the [120] experiments proving to improve the enhancement of composite pulses. The relative phase-shift between the Raman beams however from different optical path lengths is harder to combat, especially for the dynamic interferometer\(^{11}\) discussed in further detail in sub-thesis IV.

\(^{11}\)If a fibre experiences tension or stress, the carried optical field undergoes an amplitude and/or phase-shift depending on fibre vibration, bending angle and bending speed.
26.9 Impurities

Several impurities will lower the performance of light-pulses from the ideal case. These can be divided into three different categories:

- Velocity class impurities
- Population impurities
- Phase impurities

26.9.1 Velocity class impurities

Collisions between atoms changes their velocity leading to a velocity class impurity. This effect scales with density and size (smaller volume) of the atomic cloud. Impurities also arise from a non point-like initial cloud as discussed in section 26.7. Note that the thermal bath of atoms by the background pressure contribute negligible velocity class impurities during cloud expansion and gravitational drop following the low collision rate between cloud atoms and background pressure atoms.

26.9.2 Population impurity

Two atoms undergoing a collision not only introduces a velocity class impurity, but can exchange their spins [202]. Given the Hyperfine ground states of Rb have opposite spin, a collision can induce a state exchange $|1\rangle \leftrightarrow |2\rangle$. The spatial dependence of states suffers under this impurity which impacts atom interferometers such as the dynamic interferometer discussed in sub-thesis IV. Longitudinal and transverse relaxation processes in general induce population impurities with collisions being an example of a relaxation processes. Reducing the cloud’s density and/or limiting the interferometer’s duration within Rabi flop damping time-scales reduces population impurity.

26.9.3 Phase impurity

Transverse relaxation processes induce non-uniform phase impurities across the atomic cloud throughout the interferometer’s run. Along with the spatial Raman inhomogeneities (cross-sectional) discussed in section 26.1, pulse-length errors (section 26.5), phase-profile inhomogeneities (section 26.8) and both velocity class and population impurities, the phase of the atoms thus experience noise from multiple sources.
Chapter 27

Discussion on Mitigation Methods

We identified several techniques from the previous sub-theses which our research suggests can be implemented to counter the errors induced from sources of perturbations. We list and discuss these identified mitigation methods per sub-thesis and order them coarsely in level of our review & their effectiveness. Our discussion thus involves four different types of mitigation methods:

(Sub-thesis)

- 27.1–Improving Laser Control and Output \((I)\)
- 27.2–Resolving Frequencies Better \((II)\)
- 27.3–Mitigating Systematic Errors \((III)\)
- 27.4–Improving Measurements & Reliability \((IV)\)

Please note that limited lab-access, data-access and other logistical challenges during candidacy limits the depth of discussion on mitigation methods against systematic errors in atom interferometers. The analysis on enhancements of atom interferometers thus comprises mostly of qualitative discussion of each error, providing a review and map of experimental perturbation to systematic error per control variable and their effect along with a review/map of mitigation methods to errors. Where quantitative analysis is available however it is incorporated.
27.1 Improving Laser Control and Output

27.1.1 Cooling Diode Laser Cavities

The output optical power of a diode laser increases by cooling the laser’s cavity to lower temperatures. The level of enhancement depends on the external quantum efficiency and other characteristics of the laser system which due to the time-constraints have not been measured or quantified otherwise. Our work in section 4.1 does offer insight into the temperature dependence on optical power as we recorded up to $\sim 5\,\text{mW}$ power increase over a $\sim 5\,\text{oC}$ temperature change (at common injection currents). Note that the $1\,\text{mW/K}$ estimate depends on the diode laser material, dimensions and other variables of the model. It may not be linear over larger temperature differences as indicated by the observed non-linearities in the power-current measurements and might also depend on the injection current as we observe larger benefits of cooling (more power) at higher currents. We also note that if any optical amplifiers are used as in the static interferometer’s set-up (c.f. section 16.2), a few $\text{mW}$ additional laser power translates into a few $100\,\text{mW}$ or more additional laser power as is the case in the tapered amplifiers that amplified the two Raman laser beams.

The advantage of driving diode lasers at colder temperatures are two-fold:

1. Reducing systematic (e.g. pulse-length) errors that accrue during an atom-optical pulse interaction by reducing the pulses’s duration and
2. Increasing the error signal used by thermal feedback servos thus providing more reliable temperature stabilisation.

There are however four noteworthy drawbacks to this technique;

(a) The laser’s cavity should avoid the nominal temperature of the laboratory for a large error signal which can be problematic if the lab operates at colder temperatures;
(b) At very cold temperatures, measures against condensation should be taken like placing the laser in an enclosed chamber flushed with Nitrogen;
(c) The temperature specifications such as heating/cooling rate and lowest operational temperature of the diode laser from the manufacturer are adhered to and
(d) The temperature tuning parameter is several times larger then the current tuning parameter which limits the operational range for frequency-selective atom-laser interactions.
27.1.2 Drive Diode Lasers Near Thermal-rollover

We can further optimise the optical output power of diode lasers by measuring the currents at which thermal-rollover becomes prevalent as shown in section 4.2. Optimum output power can then be achieved by setting the laser’s injection current right below the thermal-rollover starting current. Combining this technique with the previous cavity cooling technique offers higher optical powers as cooling up-shifting the current at which rollover takes place. This enhances the systematic error mitigation that cavity cooling offers. The level of enhancement will again depend on the laser model and laser system in use hence is not quantified here.

To reduce degradation of the laser we propose this technique when implemented is applied in short periods of time by rapid tuning of the injection current. By limiting the high injection current to short times the diode laser can be made to last longer before dropping in performance. Rapid current tuning could affect the stability of the laser if the stabilisation feedback servo responds slower than the current tuning speeds leading to the next suggestion on impulse testing.

27.1.3 Impulse Testing of Laser Systems

Given the sub-$MHz$ linewidth and sub-$MHz$ drift during an interferometric sequence requirements for atomic physics experiments, lasers need their cavity’s temperature and injection current stabilised using servos. We recommend these servos are manually\(^1\) optimised to reduce sensitivity against environmental changes by reducing their responds time and accuracy: the discrepancy between the actual and set parameters (temperature, current, etc.). An impulse test like the one discussed in subsection 4.1.6 offers a good start to optimise the servo’s feedback. For atomic physics experiments this helps making the lasers more reliable and offer greater control.

27.1.4 Frequency Reference by Mode Locking

The experimental set-up of the static interferometer as described in section 16.2 provides no control over the single transition detuning as the Raman laser is left free-running. This allows off-resonance errors from drift, jitter and other imperfections in the laser to propagate into the laser-atom interactions as systematic errors lowering performance of atom interferometers.

\(^1\)Note that commercial optical amplifier/laser systems with built-in servos might already be optimised by the manufacturer or need specialised servo calibration.
Whilst spectroscopic methods like saturated absorption or a Dichroic-Atomic-Vapor Laser Lock can be used as a frequency reference for our lasers, several advantages arise from using a mode-locked laser (MLL). As discussed subsection 15.1.7, our MLL from sub-thesis I offers comparable linewidth and stabilisation against drift over time to spectroscopic methods. And given the MLL was operating about 3.5 times above the time-bandwidth limit for a Gaussian pulse and by noting the use of MLL’s by other research groups for atomic physics experiments, we expect MLL’s to offer a useful frequency reference for our lasers.

The set-up described in chapter 7 describes how a MLL can be referenced to a well-defined atomic transition and simultaneously offer a frequency/phase reference over its entire bandwidth by using an optical phase-locking loop whose bandwidth exceeds the comb-line spacing of the MLL. A transform limited MLL can easily have a bandwidth a hundred times or more larger than the bandwidth of DAVELL spectroscopy systems, increasing the range of frequency and phase control. The chapter 7 set-up therefore offers a frequency reference for multiple transition lines of atoms and all intermediate frequencies. Such a set-up could also reduce the space occupied on the optical bench by reducing the number of spectroscopy reference optics as multiple lasers can then be frequency/phase referenced from a common MLL.

A MLL can thereby extend control over the frequency and phase of multiple laser systems to greater resolution and range depending on the MLL’s performance (e.g. its time-bandwidth product) than individual spectroscopy set-ups per laser.

### 27.2 Resolving Frequencies Better

#### 27.2.1 Imaging Aperture

In subsection 10.3.2 regarding speckle patterns generated by an Ulbricht sphere we noted the convention of placing a square aperture in front of the camera. The aperture defines an upper spatial frequency limit $\nu_s^+$ in features of the image which can distort the image through the aliasing effect. By varying the aperture-camera separation and the aperture size (length) for light of a particular wavelength we can avoid undersampling. This is achieved by setting $\nu_s^+$ less than or equal to the Nyquist spatial frequency defined by the spacing between individual sensors of the camera.
27.2. **RESOLVING FREQUENCIES BETTER**

An appropriate aperture can thus eliminate image distortions like those seen in figure 26.1. This is relevant to the dynamic interferometer discussed in sub-thesis IV as aliasing can introduce false measurements or disrupt the calculation of the rate of rotation and acceleration. The imaging of high spatial frequency fringes across an atomic cloud is thus made more reliable by using an appropriate sized and positioned aperture in front of the camera.

Finally we note that this mitigation method helps clarify the upper limit in fringes thereby rotation/acceleration that can be measured by the dynamic interferometer. The upper limit of the rotation sensor in development is therefore not only set by the acceleration/rotation-induced image blur and impact of off-resonance and pulse-length errors but also by the aliasing effect.

### 27.2.2 Speckle-based Spectrometer

Characterisation of the laser systems plays an important role in troubleshooting atom interferometers. For atomic physics applications there is a high demand on resolution in laser frequency, down to 1 MHz levels or lower. Whilst commercial grating/prism spectrometers offer large bandwidth of thousands of nm, they do not offer the necessary resolution required for characterising atomic physics laser systems. Spectrometers like scanning Fabry-Perots however whilst offering the necessary resolution are strongly selective to wavelength and have narrow bandwidths typically of a few GHz. Analysing multiple laser systems therefore require multiple Fabry-Perots and struggle with analysing modulations in excess of the spectrometers free spectral range. An alternative we therefore propose for atomic physics experiments are speckle-based spectrometers.

Our review in section 10.3.5 highlights recent advances in speckle-based spectrometers: sub-MHz resolution available over a 480-1080 nm range of wavelength. A self-referenced optical frequency comb as defined in section 5.3 spanning a few hundred nm can using a speckle-spectrometer have all its comb-lines resolved to the required level of resolution. Characterising lasers in atom interferometers that use atoms of different elements, highly modulated laser systems or a MLL frequency reference as suggested by mitigation method 27.1.4 thus benefit by using a speckle-based spectrometer. Note however that there is a risk of false measurements from the reconstruction method of speckle to wavelength. It is unclear if the speckle pattern of two wavelengths when superimposed can resemble the speckle pattern of a third wavelength and under which conditions this is possible. Our work in sub-thesis II focuses on this question and might be of further interest.
27.3 Mitigating Systematic Errors

27.3.1 Composite Pulses

The composite pulses tested in chapter 17 are optimised to mitigate against perturbations in two-photon detuning \( \delta \) and inhomogeneities in Raman power \( i.e. \Delta \Omega \). Off-resonance errors\(^2\) and pulse-length errors by Rabi frequency perturbations \( (\Delta \theta = \Delta \Omega \tau) \) are thereby mitigated by composite pulses. Note that our work is limited to mirror composite pulses, beamsplitter composite pulses are beyond the scope of this thesis. Future work on pulse-shape and timing perturbation based pulse-length errors \( (\Delta \theta = \Omega \Delta \tau) \) and beamsplitter composite pulses would offer novel extensions on our work.

Composite pulses reduce the spread of state mixing on the Bloch sphere about the intended rotation angle. Higher fidelity is obtained by reducing state mixing across the atomic ensemble during the interferometric sequence which theoretically enhances the contrast and visibility of atomic interferometers. Our work on mirror pulses provide insight on how such enhancements can propagate into interferometric sequences.

In chapter 17 we recorded an increase in the full width half maximum of fidelity against detuning \( \delta \) by 50\% for a WALTZ pulse up to 180\% for GRAPE optimised composite pulses against a rectangular mirror pulse. The effects by off-resonance errors (and the pulse-length errors they induce) is therefore greatly reduced when using composite pulses In particular, the Zeeman shift off-resonance errors we computed in section 26.4 are reduced by a factor upto \( \sim 4.7 \) compared to rectangular optical mirror pulses.

Further work is needed to analyse which perturbation source composite pulses compensate by what level beyond Zeeman shifts. Our two tested GRAPE pulses optimised for two different temperatures do provide insight into how GRAPE pulses mitigate against Doppler shifts. With only two GRAPE pulses however further analysis is not available. We note however that the time-scale after which relaxation-induced damping becomes noticeable is over sex times larger using composite pulses \( (+12 \, \mu s) \) than in our rectangular pulse Rabi flopping \( (\sim 2 \, \mu s) \) from chapter 16. The rapid loss of Rabi flop visibility for rectangular pulses in our experimental set-up is therefore caused by one or more systematic errors that composite pulses mitigate against. Which error(s) cannot however be identified without further work.

\(^2\)And by extension the pulse-length errors that off-resonance errors induce.
27.3. MITIGATING SYSTEMATIC ERRORS

Regarding anomalies in our measurements, our composite mirror pulses appear to experience a fidelity ceiling when we compare simulation results to data. Our main hypothesis over decoherence and atomic loss is the role of the sub-interferometers from the Raman beam cross-talk that run simultaneously during the Raman beam interrogations. Those sub-interferometers would apply the same composite phase-profile (we enhance the error they induce) and scale proportionally with the Raman laser power. This might account for the discrepancy in simulated and measured peak fidelity.

27.3.2 Beamshaping Optics

To counter the cross-sectional Raman beam power variations from the transverse modeshapes of optical fibre we utilise beamshaping optics (BSO) as discussed in section 26.1. This reduces the cross-sectional power variations from 50% (half width half maximum) of the Raman beam and caps it to $\sim 15\%$ at the cost of cross-sectional phase inhomogeneities in the Raman beams. This also reduces off-resonance and pulse-length errors along with spatial Raman power inhomogeneities that otherwise affect atom interferometric measurements. Quantifying the level of error enhancement from spatial Raman phase inhomogeneities should accompany the use of BSO.

27.3.3 Rotating Beamshaping Optics

For flat top-head BSO with a square cross-section our simulations demonstrate that rotating the BSO to a diagonal orientation\(^3\) provides $\sim 2\, ms$ additional time for Raman laser interrogation (c.f. section 26.3 for details). The 2\,ms estimate uses the parameters of our experiment and might therefore vary under different experimental control variables or set-ups. Colder atoms and smaller photon recoils increase the amount of additional Raman interrogation time this enhancement yields thereby the mitigation against the limited interrogation window error.

Different experimental settings can therefore offer over 2\,ms additional interrogation time allowing for larger matterwave separations in the interferometer which for the dynamic interferometer in sub-thesis IV increases the sensor’s sensitivity to rotation and acceleration. The resolution of the rotation sensor is enhanced by running longer interferometers as this increases the interferometer’s enclosed area. From section 21.1 (e.g. equation 21.8) we infer that this increases the sensitivity against rotations and acceleration.

\(^3\)Defined as atoms dropping along a diagonal line of the BSO square.
27.3.4 Using Polarisation Maintaining Fibres

In our Raman laser system (c.f. chapter 16) we qualitatively observed improvements when substituting single mode fibers (SMF) with polarisation maintaining fibers (PMF). Whilst not quantified, we can infer the sensitivity to laboratory environmental fluctuations (temperature, vibration, etc.) is reduced when substituting SMF with PMF. The stability in the Raman beams power is thereby less sensitive to the laboratory environment resulting in smaller pulse-length errors from Rabi frequency and Raman phase variations. Note that Raman phase variations from an electronic modulator as discussed in section 26.8 are not mitigated using SMF as they are not generated during propagation in optical fibers. Only relative phase noise between the two Raman beams accrued during propagation in the fibers are attenuated using PMF.

27.3.5 Rapid Raman EOM Detuning Technique

In subsection 26.5.2 we describe a technique that involves detuning the EOM rapidly off-resonance during the time the AOM switches on/off (leading/falling edges). This reduces the pulse-length error introduced by the Rabi frequency chirp from the leading and falling edges of optical pulses. We estimate the pulse-length error accumulates for $\sim 1 \text{ ns}$ compared to $\sim 100 \text{ ns}$ of an unfocused laser beam onto an AOM.

27.3.6 Optical Pumping

Our literature review in section 26.4 showed efficiencies upto 98(1)% are possible when optically pumping atoms into a magnetic insensitive Zeeman sublevel. This would remove off-resonance errors and pulse-length errors induced by inhomogeneous Zeeman shifts across the atomic cloud at the cost of a few percent atom loss.

27.3.7 Glass Vacuum Chamber

The advantage of using a glass vacuum chamber over a metallic glass chamber (c.f. section 26.4) are hard to quantify but we can infer that the $\sim 15 \text{ ms}$ magnetic fields induced by Eddy currents when the MOT coils are turned off in our set-up (c.f. chapter 16) are attenuated when using a glass vacuum chamber. We can thus start our sequence earlier when the atoms have higher coherence. Our 100 $\mu$s measured coherence time post 15 ms can thus be increased up to 15 ms with Zeeman shifts induced errors also attenuated.
27.3.8 Magnetic Field Feedback

We have not explored possible feedback mechanisms in 26.4 to stabilise the magnetic field inside the vacuum chamber other than the Shim coils used to zero or bias the magnetic field. We do however propose further investigations into stabilisation schemes of the magnetic field to counter AC components of the magnetic field we estimate to have a frequency of $1 \text{kHz}$ to $100 \text{MHz}$ in typical experimental set-ups.

Typically these compensation schemes involve additional coils around the vacuum chamber that detect magnetic fields and utilise feedback servos to control the field strength per Cartesian axis. The DC magnetic component can also be controlled more accurately and measured throughout an interferometric sequence providing useful data for further analysis on the performance of the atom interferometer.

27.3.9 Magnetic Shielding

For magnetic fields from DC to $100 \text{MHz}$ our literature review in section 26.4 shows attenuation upto -120 dB is available. This attenuates off-resonance errors from Zeeman shifts by constant or fluctuating background magnetic fields. Such high magnetic isolation would leave almost all magnetic field sources as insignificant sources of systematic errors. The exception is the MOT-coils, we cannot fully isolate these fields as the MOT magnetic fields provide the spatial confinement forces when loading atoms into the MOT. A magnetic field feedback to counter the MOT-coils induced errors (see subsection 27.3.8) in combination with magnetic shielding and optical pumping (see subsection 27.3.6) provide a good combination to counter the off-resonance errors induced by Zeeman shifts.

27.3.10 Quantisation Axis

Another mitigation method against Zeeman shifts we identify in section 26.4 is the use of a weak bias magnetic field to introduce a quantisation axis. Atoms in the magnetic insensitive state ($m_f \neq 0$) are then placed off-resonance with the Raman beams through the Zeeman shift which however results in a loss of atoms involved in the interferometer. The remaining atoms ($m_f = 0$) remain magnetic insensitive demonstrating that a quantisation axis provides additional state preparation. For high signal however we recommend the use of a quantisation axis is used in conjunction with optical pumping (see subsection 27.3.6) to ensure the atom loss remains low.
27.3.11 Enhanced Laser Cooling

In section 16.1.2 we briefly reviewed sub-Doppler cooling methods other than Sisyphys cooling. Methods like dark MOT and transparency schemes offer further cooling leading to the narrowing of the velocity distribution. The off-resonance errors like Doppler shifts that scale with the width of the atomic cloud’s velocity distribution are therefore reduced when using colder atoms. Enhanced laser cooling can thus offer mitigation against atomic cloud’s velocity distribution dependent systematic errors.

27.3.12 Heated Vapour Cells Filters

The use of laser amplifiers as discussed in section 16.2 introduce additional noise in the Raman’s laser namely amplified spontaneous emission (ASE). Any high power Raman laser system is likely to suffer from ASE hence a filter is of interest to reduce Rabi frequency pulse-length errors and off-resonance errors from ac stark shifts from the ASE light. One such filter are heated vapour cells though alternatives are commercially available.

27.4 Improving Measurements & Reliability

27.4.1 High Power Raman Laser System

To reduce errors accrued during an optical pulse (e.g. pulse-length errors) we propose using higher optical Raman powers for atom interrogations. Whilst cavity cooling (see subsection 27.1.1) and driving near thermal-rollover threshold (see subsection 27.1.2) increase the output power of diode lasers, even higher powers can be achieved using a different design. One proposal is discussed in further detail in section 22.1 and offers over a hundred times higher Raman powers (50 mW to 5 W). This novel Raman laser system also shares the ability to program the relative laser phase between the Raman beams enabling composite pulses to be implemented.

Comparing the new Raman laser system to the static interferometer’s Raman laser system (see section 16.2) we see that the on top of higher powers that the drift is reduced, larger modulation bandwidths are available and that the carrier and unwanted sidebands experience greater suppression. Both off-resonance and pulse-length errors are therefore reduced using the high power Raman laser system from sub-thesis IV. N.B. quantifying the level of enhancement by higher Raman powers experimentally is beyond the thesis’s scope unfortunately due to lack of characterisation data for comparison.
Another effect of higher Raman powers is the gain in time that can now be allocated to larger pulse-separations \( T \). As discussed in subsection 27.3.3 this increases the sensitivity of the atom interferometers which enhances the resolution of atom interferometer based sensors like the dynamic interferometer in sub-thesis IV. More pulses can also be squeezed in the same time allowing for more complex composite pulses and more complex interferometers consisting of more pulses such as large momentum transfer schemes.

### 27.4.2 Miniaturise Laser Systems

In section 22.2 we describe our work towards miniaturising the MOT and new Raman laser systems of the new dynamic interferometer\(^4\). Apart from reducing the volume of space a miniaturised system uses and increasing its portability, the laser system also became more reliable. Before miniaturisation the MOT laser system for example the optical components of the laser system drifted substantially in power during a day. After miniaturisation however with the laser system placed at the same position on the same optical bench the rate of drifts in optics collectively reduced by half. This estimate stems from the drop in need to re-aligning optics which decreased from daily to every two days after miniaturisation.

Miniaturisation improves reliability of the interferometer by reducing the effect of drift in optics. The same drift now introduces a smaller arc-length as the optical paths has decreased. This reduces the propagation of thermal and vibrational/acoustic perturbations into systematic errors that reduce the performance of atom interferometers. Maintenance of interferometers is also lowered from a reduction in the need to re-align optics. Quantifying the reduction in errors and maintenance is difficult as it depends on multiple variables such as the decrease in dimensions of the laser systems, the drift per optics component, etc. No further quantification is therefore available to miniaturisation of laser systems and the enhancement it offers.

### 27.4.3 Temperature Stabilisation of Laser Systems

The thermal expansion of materials comprising the optic’s mounts introduce additional sensitivity to temperature in laser systems. Even ‘small’ variations (\( \pm 1 \) °C) in temperature therefore misalign optical systems over time which reduces the performance of the interferometer if not corrected.

\(^4\)The MOT laser system was also used for experiments of the static interferometer as the dynamic interferometer was still under construction.
A simple mitigation against this is discussed further in section 22.2: temperature stabilisation. By placing the laser system inside an enclosed shelf that is lined with an acrylic inner layer and is temperature stabilised to a tenth of a degree, we found our MOT laser system to maintain the same power (within $\pm 1 \text{ mW}$) for over three months. The reliability of the MOT laser system therefore increased from every two days (after miniaturisation, see subsection 27.4.2 for further details) to three months. This demonstrates that temperature variations in the laboratory affected our laser systems substantially.

### 27.4.4 Vibrational Isolation of Laser Systems

Apart from sensitivity to thermal variations, laser systems are also prone to vibrations including sound. Vibrations affect atom interferometers in a multitude of ways from imaging to the reliability and stability of light emitted by the laser’s cavity. Acoustical sensitivity can be highlighted easier from the use of AOMs used to generate optical pulses. External sound can disrupt the photon-phonon interaction inside an AOM which propagate further into systematic errors. Reducing the sensitivity is therefore of interest for making the interferometer more reliable.

Similar to thermal stabilisation (see subsection 27.4.3) we mitigate against vibration induced systematic errors by isolating the laser system from vibrations. As discussed in further details in section 22.2, vibrations from jiggling are attenuated using sorbothane feet placed under the miniaturised breadboard whilst acoustical vibrations upto $8 \text{ kHz}$ can be attenuated upto $45 \text{ dB}$ by isolation foam enclosing the laser system. This ensemble is then placed inside a temperature stabilised shelf to reduce sensitivity against temperature variations (see subsection 27.4.3).

Vibrational isolation increases the reliability of atom interferometers similar to thermal stabilisation: by reducing the propagation of perturbation into systematic errors. For portable application like the dynamic interferometer discussed in sub-thesis IV this hardware enhancement improves reliability during employment of the rotation sensor. Quantifying the level of improvement is hard as each laser system would need to be characterised under different vibrations and sound, at amplitudes and frequencies we anticipate the interferometer to be subject to. As these measurements were not made, no further quantification can be made on the level of enhancement.

\footnote{Depending on the specifications of the foam used. Different figures might hold for different foams with these numbers holding for one particular model under consideration.}
27.4.5 Free-space to Fiber Optics Substitution

The improvement in laser system reliability discussed in subsection 27.4.3 is not solely due to temperature stabilisation. The increase in stability of the output laser power was also impart due to substitution of free-space to fiber optics as discussed in section 22.2. The temperature stabilisation upgrade was namely introduced simulatenously with the substitution of several free-space optics with fiber optics.

Substitution with fiber optics increases reliability of laser systems as they are often less sensitive to thermal drifts than metallic free-space optical mounts. This reduces the systematic errors by damping the propagation of perturbations in laser systems. The coupling and insertion loss of optical fibers however does typically reduce optical powers compared to free-space optics. Reducing the number of free-space to fiber connections is therefore recommended when using this mitigation method.

27.4.6 Mach-Zehnder Interferometer Imaging

In subsection 23.3.1 we propose a novel imaging technique that uses a none-destructive atom-laser interaction through off-resonant optical phase-shift measurements. These measurements are made using an optical Mach-Zehnder interferometer (MZI) along the Raman interrogation axis that is phase-stabilised as shown in figure 23.1.

For the dynamic interferometer discussed in sub-theses IV the fringes are subject to motion blur resulting in the imaged fringes being averaged over during the exposure time (see section 26.6). This reduces the visibility of the fringes proportional to the rate of relative motion between the atomic cloud and the camera. Whilst shorter exposure times reduce the fringe averaging, they also reduce the signal to noise ratio making the rotation and acceleration reconstructions from fringes less reliable.

The advantage of this novel imaging technique lies in reducing the fringe averaging using its non-destructive nature. Destructive imaging techniques like absorption imaging (see section 23.3.4) collapses the wavefunction of the atoms which prevents succesive images being taken of the atomic cloud with fringe information. An optical MZI image in comparison does not collapse the wavefunction therefore can retain fringe information after succesive images.
The principle is therefore that shorter image exposure times by factors from a hundred to a thousand become attainable by correlating successive images of the atomic cloud akin to a video. The 100 to 1000 factors are estimates from the literature when optical MZI imaging is applied to imaging a MOT. Further work is therefore needed both on the theory and experimental side to apply the imaging technique for imaging fringes across an atomic cloud. Quantification on the level of enhancement is therefore not included. Such work could however led to a useful imaging technique that offers insight of ballistic expansion (temperature estimate), centre of mass motion and higher resolution fringe detection per interferometer cycle.

27.4.7 Light-sheet Imaging

Another novel imaging technique for the dynamic interferometer from subthesis IV is to uses a light-sheet generated by focusing a laser using an AOM or a cylindrical lens. The light-sheet can be set to image a slice of the atomic cloud thereby increasing image contrast and reducing heating. This technique can be adapted from Biology for atomic physics applications in a few ways as discussed in section 23.3.2. The main challenge are signal strength and a drop of resolution of outer fringes where the light-sheet expands. Further review is therefore needed to determine the effectiveness of this imaging technique.

27.4.8 Two-photon (Blue) Imaging

An alternative imaging technique that has been implemented in atomic physics experience is two-photon imaging. For Rubidium this involves a two-stage excitation of atoms in the upper ground level $|2\rangle$ to a higher energy level $|3\rangle$ using an additional laser (for details see 23.3.3). Atoms in $|3\rangle$ than decay in two steps, one which involves a 420 nm (blue) transition that can be imaged. The blue light can easily be filtered from background scattering of Raman laser light which improves signal to noise ratio therebye reliability of the rotation sensor of sub-thesis IV. The quality of the filter and level of background light detuned from 420 nm determines the level of improvement of this technique.
Chapter 28

Conclusion on Enhancements

From sub-theses I, II, III & IV we have identified and reviewed several mitigation methods that to varying degrees enhance the performance of atom interferometers by mitigating errors or imperfections. In this chapter we attempt to classify these mitigation methods into three categories: *minor*, *modest* and *major*. The motivation is to coarsely rank effectiveness of the mitigation methods based on their novelty, effectiveness (when quantified) and difficulty of implementation so that the cost and return of using an enhancement technique for atomic physics experiment can be gauged.

Please note that not all suggested enhancements listed in this chapter have been tested in this work and that both proposed and implemented/tested enhancements are shown here. Also, some enhancements listed here are known/common practice in the literature and are included for completeness.

Each mitigation method will be classified using the following format:

- **Subsection of method in chapter 27–Mitigation Method Title** (Brief description of the mitigation method)
  
  *Short explanation for the classification based on effect of the mitigation method and implementation difficulty.*

Further details on each mitigation method such as quantification can be found in chapter 27 by navigating to the corresponding subsection.
28.1 Minor Enhancements

Minor enhancements are coarsely defined as mitigation methods that cost little effort to implement whilst providing small enhancements in performance in return. The following mitigation methods provide minor enhancements:

- **27.2.1–Imaging Aperture**
  (Place a square aperture in front of imaging camera)
  Simple to implement but only enhances interferometers from possible aliasing during characterisation of the Magneto-Optical Trap (MOT) or when the measurement involves imaging the atomic cloud such as the dynamic interferometer from sub-thesis IV.

- **27.3.2–Beamshaping Optics**
  (Using beamshaping optics for the Raman laser beams)
  Easy to append beamshaping optics (BSO) to atom interferometers which reduce systematic errors from inhomogeneities throughout the atomic cloud or spatially across the Raman beams. Enhancement is small as BSO introduce Raman phase variations which partially counter the reduced Raman power variation induced systematic errors.

- **27.3.3–Rotating Beamshaping Optics**
  (Rotating the beamshaping optics used for the Raman laser beams)
  Easy to implement but the gained interferometer time might not be usable if coherence times are shorter than the original interferometer length limit like in our set-up. Otherwise this mitigation method can be classed as modest if the gained time is used.

- **27.3.4–Using Polarisation Maintaining Fibres**
  (Substitute single mode fibres with polarisation maintaining fibres)
  Simple substitution providing a small enhancement in the reliability of the Raman laser system by reducing sensitivity to the environment such as temperature variations and vibrations.

- **27.3.5–Rapid Raman EOM Detuning Technique**
  (Detune Raman lasers during leading/falling edges of optical pulses)
  Easy to implement like in our static interferometer from sub-thesis III. We detuned the electro-optical modulator (EOM) that modulates our Raman lasers from resonance to reduce systematic errors from the chirps in Rabi frequency at the start and end of optical pulses.
27.3.9–Magnetic Shielding

(Place magnetic shielding around the vacuum chamber)

Easy to implement but Shim coils already mitigate against static background magnetic fields. Useful for changes in static background magnetic fields (e.g. a portable atom interferometer) or attenuating AC magnetic fields which Shim coils do not compensate against. The MOT coils however cannot be attenuated by magnetic shielding as they are needed for loading the MOT. Off-resonance errors from MOT coils can therefore persist, and most AC magnetic fields we estimate do originate from the MOT coils. Therefore a small enhancement using this method.

27.3.10–Quantisation Axis

(Introduce a weak bias field along Raman laser interrogation axis)

Easy to implement using the Shim coils but can lower signal depending on the distribution of atoms in the Zeeman sublevels. This method reduces the sensitivity against background magnetic fields which increase reliability of the interferometer giving a net small but positive enhancement.

27.3.12–Heated Vapour Cells Filters

(Place a filter intront of the Raman beams such as heated vapour cells)

Easy to implement and gives a small enhancement from filtering amplified spontaneous emission otherwise part of the Raman laser beam.

27.4.5–Free-space to Fiber Optics Substitution

(Substituting free-space optics with fibre alternatives where possible)

Easy to implement giving a small enhancement in reliability of the laser system. Bulky optical mounts are namely more sensitive to perturbations than secured fibres.
28.2 Modest Enhancements

Modest enhancement we coarsely define as mitigation methods that require some restructuring of the atom interferometer in return for an modest level of enhancement. We identify the following modest enhancements for atom interferometers:

- **27.1.1–Cooling Diode Laser Cavities**
  (Cool the cavity of diode lasers)
  Cooling the cavity increases the output power of the Raman beams at constant current. Thus reduces the duration of optical powers thereby errors accrued during an optical pulse. Depending on the laser system, a substantial power increase can be acquired. Condensation and the challenge in maintaining a constant wavelength are drawbacks however.

- **27.1.2–Drive Diode Lasers Near Thermal-rollover**
  (Operate diode lasers near thermal-rollover currents)
  The upper limit in output power is set by the thermal-rollover in diode lasers. Operating the laser near thermal-rollover pushes the laser’s output power to maximum which reduces errors accrued during optical pulses. To reduce degradation this method should be used intermittently.

- **27.1.3–Impulse Testing of Laser Systems**
  (Perform impulse tests on laser systems)
  Impulse tests offer a useful insight into the reliability of the stabilisation feedback of laser systems. This includes current/temperature stabilisation or an optical feedback loop used to stabilise the relative phase between the two Raman laser beams (to reduce random walk phase noise). This method allows for quantifying and optimising the speed and accuracy of feedback systems which increases the reliability of laser systems and offer greater control.

- **27.3.6–Optical Pumping**
  (Optically pump atoms into the magnetic insensitive Zeeman sublevel)
  Apart from the loss of a few percent of atoms, this technique removes sensitivity to background magnetic fields therefore eliminating off-resonance errors from Zeeman shifts. Implementation needs some care though as the scheme must be programmed into interferometric sequences and needs an appropriately frequenced laser that can be switched on and off when appropriate.
28.2. MODEST ENHANCEMENTS

- **27.3.8–Magnetic Field Feedback**
  (Install a magnetic field feedback system for the vacuum chamber)
  Requires the installation of additional coils around the vacuum systems
  and a servo for stabilising the magnetic field. It does provide measure-
  ments and active control over the impact of background magnetic
  fields. This reduces off-resonance errors from background magnetic field
  including from MOT coils.

- **27.3.11–Enhanced Laser Cooling**
  (Adopting novel sub-Doppler cooling methods for colder atoms)
  Implementation of novel cooling schemes like dark MOT or
  transparency schemes require adapted laser systems. The colder atoms
  introduce smaller off-resonance errors using this method from narrow-
  ing the velocity distribution which reduces the range Doppler shifts
  across the atomic cloud.

- **27.4.2–Miniaturise Laser Systems**
  (Replace components of laser systems with smaller substitutes)
  Requires re-configuring and re-assembling laser systems. Reduces the
  effect of drift in optical components from shorter optical paths making
  the laser system more reliable and reducing maintenance requirements
  (realignments).

- **27.4.3–Temperature Stabilisation of Laser Systems**
  (Place laser system in enclosed and temperature stabilised shelves)
  Requires designing and assembling a shelf that can host the laser system
  along with work into temperature stabilisation and thermal isolation on
  the inner lining of the shelf. This methods provides substantial reduc-
  tion in drifts which improves reliability of the laser system and reduces
  the frequency of maintenance demands.

- **27.4.4–Vibrational Isolation of Laser Systems**
  (Placing laser systems in a shelf that is isolated against vibrations)
  The layout of a shelf needs to be designed to fit a laser system and the
  sorbothane feet and foam used to isolate vibrations and sound respec-
  tively. This reduces the propagation of vibrations into the laser system
  which increases the reliability of the laser system.
• **27.4.7–Light-sheet Imaging**

(Image atomic clouds using a light-sheet)

*Using a cylindrical lens or an acousto-optical modulator to focus light into a sheet that is used to image the atomic cloud. The reduced imaging place enhances image contrast at the cost of signal strength and non-uniform image resolution across the cloud.*

• **27.4.8–Two-photon (Blue) Imaging**

(Image atomic clouds after a two-photon excitation step)

*An additional laser system is needed for the two-photon excitation step along with a filter to remove the background light from the laser systems during imaging. With the image light having different wavelength from the lasers the background noise can be removed from the image leading to higher signal to noise ratios.*
28.3 Major Enhancements

Mitigation methods that are more challenging to implement but give larger enhancements are classified as major enhancements. The following major enhancements have been discussed:

- **27.1.4–Frequency Reference by Mode Locking**
  (Use a mode-locked laser and optical phase locking loops as frequency reference)
  *This method offers greater control over the frequency of all laser systems over larger bandwidths compared to common spectroscopy techniques like saturated absorption. Harder to implement as it requires an overhaul of laser systems and installation of optical phase locking loops that need optimisation.*

- **27.2.2–Speckle-based Spectrometer**
  (Characterise laser systems using a speckle-based spectrometer)
  *Offers a stronger characterisation tool of all laser systems compared to conventional spectrometers. Unless commercially purchased the implementation is hard given the complicated reconstruction of speckles to wavelength and the uncertainty in reconstructing multiple wavelengths. If a commercial speckle spectrometers is available then this mitigation method can be placed in the minor enhancement class as implementation then becomes simple.*

- **27.3.1–Composite Pulses**
  (Substituting rectangular light pulses with novel composite GRAPE pulses)
  *Mitigates against many off-resonance errors (and their induced pulse-length errors) giving smaller sensitivity to perturbations from multiple sources and increasing the peak fidelity (contrast) of the interferometer. The damping time-scale is also reduced. Implementation however is difficult due to the computation requirement of specific pulses and the requirement for programmable phase-control in the Raman laser system.*

- **27.3.7–Glass Vacuum Chamber**
  (Replacing metallic vacuum chambers with all-glass vacuum chambers)
  *Harder to implement as it requires an overhaul of the vacuum system and its cleaning. The reduction in eddy currents does however allow for the interferometric sequence to start earlier when coherence is highest. This also allows for substantially longer interferometers which increase resolution in sensing applications.*
• **27.4.1**–High Power Raman Laser System
(Use our proposed high power Raman laser system)

*Hard to implement following the requirement to construct a new novel laser system but greatly reduces systematic errors accrued during an optical pulse such as pulse-length errors. This mitigation method also allows for composite pulses to be implemented.*

• **27.4.6**–Mach-Zehnder Interferometer Imaging
(Image atomic clouds using an optical Mach-Zehnder interferometer)

*Requires theoretical modelling and a new optical system that can interface with the other parts of the interferometer. Implementation is therefore difficult but the enhancement we suspect is substantial from the shorter exposure times and reduction in motion blur when imaging fringes across an atomic cloud. Also useful for characterising the magneto-optical trap.*

**28.4 Summary**

We have identified and ordered several techniques which mitigate different systematic errors that lower the performance of atom interferometers. Different mitigation methods enhance the performance of the interferometer in different ways: reduce the propagation of perturbations (sensitivity reduction), increase the visibility/contrast of interferometers (larger contrast or signal to noise ration), *etc.* To index these errors we defined three classes of errors that coarsely consider the implementation difficulty and level of enhancement to the atom interferometer. These levels are *minor*, *modest* and *major* with nine, eleven and six major methods per class respectively.

Mitigation methods range from simple suggestions and substitution of electro-optical components to recommendations for using computationally optimised optical pulses. In total find twenty-six enhancements to atom interferometers and offer a review the systematic errors they compensate and common sources of perturbations/imperfections.

The classification of mitigation methods into enhancement levels is qualitative due to the limited scope of the thesis from supervision challenges that generated logistical problems during candidacy (*e.g.* lab and data access). Several mitigation methods are also not novel but are included for completeness following our review on the topic.
Part VI

Appendices
Appendix A

Events and Posters

In this appendix I summarise the events I attended and other works such as posters or presentations that pertain to those events.

A.1 Attended Events

During the MPhil, I attended the following lecture series on atomic physics:

- (11-13/07/17) Universität of Innsbruck Introductory Course on Ultra-cold Quantum Gases
- (24-28/07/17) Onassis Lecture Series on Quantum Physics Frontiers explored with Cold Atoms, Molecules and Photons
- (02-13/10/17) Les Houches Predoc School on Cold Atoms and Quantum Transport

Apart from these lecture series, I attended one atomic physics conference:

- (22-27/07/18) International Conference on Atomic Physics (ICAP)

The only other event I attended not related to atomic physics was the GRAD-net Summer School, 2-5 July 2018 at Herstmonceux Castle, England.
A.1.1 Organised Events

I organised the following three events during the MPhil:

- (08/03/2017) *International Women’s Day Event*
- (30/08-01/09 2017) *Annual Quantum Light and Matter Summer School*
- (08/03/2018) *International Women’s Day Event*

For reference, the annual QLM summer school was held at the University of Bournemouth and whilst open to other graduate students, primarily hosts graduate students from the School of Physics at the University of Southampton. Both International Women’s Day events were held at the School of Physics in Southampton.

A.2 Presented Work

For the ICAP conference I published an abstract titled ’Developing a Transportable Ultra-cold Atomic Rotation Sensor’. This was accompanied by an poster of the same name which I presented at the event. Both are appended to the end of this appendix, other presented work (at other attended events) are available upon request.

Finally, as part of the doctoral program I delivered a presentation to the Quantum, Light and Matter group from the University of Southampton titled ’A Transportable Ultra-cold Atomic Rotation Sensor’ on 09/07/18. The presentation slides of this presentation are available upon request.
A.2. PRESENTED WORK

Developing a Transportable Ultra-cold Atomic Rotation Sensor

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Point Source Interferometry (PSI) has demonstrated sufficient inertial sensitivity to be useful for navigation applications [1]. The technique first developed by Dickerson et al. builds on the ballistic expansion of atomic clouds, which provide a direct map of the atom’s velocity to the atom’s spatial position [2]. Using a Mach-Zehnder light-pulse interferometric sequence, this map ensures a spatial-dependent state population across the cloud’s cross-section. Absorption imaging of the expanded cloud provides the rotation rate and acceleration of the interferometer.

In the PSI, a expected cloud of $\sim 100\mu m$ width, $\sim 10\mu K ^{85}$Rb atoms rotates (fig.1) relative to the rotating apparatus constructing a phase-shearing interferometer [3]. The new PSI (fig.2) is aimed having a short term rotation sensitivity of $100\mu rad/\sqrt{Hz}$.

Transportability of the sensor is provided through populating a 870L rack with the laser systems and driving electronics. One such shelf of the control rack is shown in fig.3.

To improve the PSI performance, computational generated composite pulses [4] are being tested through thermal average fluorescence capturing. The current pulses under test make use of modified phase profiles during a pulse to counter the lowering diffraction envelope experienced by off-resonant matter-waves.

In this paper the preliminary experimental results from composite mirror pulses are shown along with progress on the new PSI.

Developing a Transportable Ultra-cold Atomic Rotation Sensor

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New Coherent Manipulation

Novel Light Pulse

Polarisation

Experimental Implementation

Temporal

Response

Frequency

Reference

Transportable Laser Systems

Prototype TLS 1.0 and for the driving laser systems and electronics
accompanied the rotation sensor.

The expected time for the first full rotation sensor prototype at TRL 6 in 2018, TRL 7 in 2020.

Frequency of realignment needed

-1 day drift

Faster temperature stabilisation

- Reduce 4U to 3U shelf

- Polarisation maintaining fibers

- Additional phase control is provided by a PM followed by an EDFA and an AOM for amplitude

- Augmentation pulses (AP) in Large Momentum Transfer (LMT) use inverted effective-momentum to enlarge the enclosed interferometric area.

- Thermal drift far slower.

- Laser pola-

- Over 5% power drop

- New vacuum chamber is being fabricated and final designs finalised.

- Next system chamber is being fabricated and the cold
designs finished.

- LMT with GRAPE pulses testing in August 2018.

- The expected time for the first full rotation sensor prototype at TRL 6 is mid 2019, TRL 7 in 2020.

- The expected time for the first full rotation sensor prototype at TRL 6 in 2018, TRL 7 in 2020.

- New system chamber is being fabricated and the cold designs finished.

- LMT with GRAPE pulses testing in August 2018.
Appendix B

Cleaved Coupled Cavity Lasers

A cleaved-coupled-cavity laser consists of two Fabry-Perot cavities of lengths \( L_1 \) & \( L_2 \) separated by an air-filled cavity of length \( d < \text{sup}(L_1, L_2) \) as shown in figure B.1. This laser can be thought of as an derivative of a single Fabry-Perot cavity laser by cleaving a single cavity into two parts of disproportionate length, separated with an air gap [49]. Except for adaptations, this laser typically has no dispersive element inside its cavity.

\[ \omega_{\text{FSR},i} = \frac{\pi c}{L_i}, i \in \{1, 2\} \]

Hence as long as \( \frac{L_2}{L_1} \neq n \), the laser can be operated. Otherwise, cavity two has either all of its mode overlap with cavity one \(^1\) or no overlap i.e. no lasing. Neither cases are of interest for generating single-mode lasing.

\(^1\)Creating an optical logic buffer and resulting the \( C^3 \) laser to have an spectrum equal to cavity one.

\[ \omega_{\text{FSR},1} = n \omega_{\text{FSR},2}, n \in \mathbb{N} \]

Figure B.1: General structure of a cleaved-coupled-cavity laser. Two cavities of length \( L_i, i \in \{1, 2\} \) separated by an air-gap of length \( d \).

B.1 Operation and Spectra

The principle of lasing utilises the incommensurate lengths of the two coupled cavities meaning the longitudinal modes of each cavity do not align regardless of the frequency offset of either cavity. In other terms, the cavities spectra defined by their free spectral range \( \omega_{\text{FSR},i} = \frac{\pi c}{L_i}, i \in \{1, 2\} \) and frequency offset \( \omega_{\text{offset},i}, i \in \{1, 2\} \) are not harmonics of each other. Assuming \( \omega_{\text{FSR},1} > \omega_{\text{FSR},2} (L_2 > L_1) \), the harmonic condition is satisfied when

\[ \omega_{\text{FSR},1} = n \omega_{\text{FSR},2}, n \in \mathbb{N} \]
A typical gain medium covers multiple longitudinal modes of a cavity resulting in CW multimode operation. By coupling the two non-harmonic cleaved-cavities together, the longitudinal modes $\omega_m$ of the resultant $C^3$ laser becomes dependent on the alignment of the modes of the individual cleaved-cavities. We can calculate the $C^3$ laser longitudinal modes by matching the modes of the cleaved cavities as shown in figure B.2.

As $\omega_{\text{FSR,1}} > \omega_{\text{FSR,2}}$ we infer that

$$\omega_{\text{FSR,1}} = \beta \omega_{\text{FSR,2}}, \beta > 1 \notin \mathbb{N} \quad (B.1.1)$$

Let $\omega_{p,i}$ be the $p^{th}$ harmonic of cavity $i$ defined as

$$\omega_{p,i} = p\omega_{\text{FSR,i}} + \omega_{\text{offset,i}}$$

We first consider the alignment of the modes at some frequency $\omega_{p,1} = \omega_{q,2}$ which solves as

$$\beta p - q = \Delta \omega_{\text{offset}}/\omega_{\text{FSR,2}} \quad (B.1.2)$$

where $\omega_{\text{offset}} := \omega_{\text{offset,2}} - \omega_{\text{offset,1}}$. In general, the adjacent matching cavity mode takes place at $\omega_{p+n,1} = \omega_{q+m,2}$ for some integers $n$ and $m$. Solving this relation reveals

$$n\beta = m \quad (B.1.3)$$

Consequently, $\omega_{\text{FSR}} = n\omega_{\text{FSR,1}}$ thus defines the free spectral range of the $C^3$ laser.

An useful relation to involve can be derived from considering the recurring of the resonances of the $C^3$ lasers from overlapping longitudinal modes of its sub-cleaved cavities. The mode number increment needed to reach the next overlapping longitudinal mode for cavity two ($m$) is always off by a constant compared to cavity 1 ($n$) i.e.

$$m = \tilde{\Delta} + n, \quad \tilde{\Delta} \in \mathbb{N} \quad (B.1.4)$$
The value of $\tilde{\Delta}$ depends on the position of the first overlapping longitudinal mode indices $p_0$ and $q_0$ which in turn depends on $\Delta \omega_{\text{offset}}$. In general, we find

$$\tilde{\Delta} = \Delta + q_0 - p_0 \quad \text{(B.1.5)}$$

Proceeding from equation B.1.4, substituting with B.1.1 and B.1.3 reveals

$$n = \frac{\omega_{\text{FSR},2}}{\omega_{\text{FSR},1} - \omega_{\text{FSR},2}}$$

from which we conclude a free spectral range of the $C^3$ laser to take the form

$$\omega_{\text{FSR}} = \frac{\omega_{\text{FSR},1} \omega_{\text{FSR},2}}{\omega_{\text{FSR},1} - \omega_{\text{FSR},2}} \tilde{\Delta} \quad \text{(B.1.6)}$$

This form is in agreement with an estimate of the $C^3$ laser spectrum from [203] and a deeper analysis found in [204].

The dependence on $\Delta \omega_{\text{offset}}$ is made obvious when substituting equations B.1.2 and B.1.4 into equation B.1.6 to give

$$\omega_{\text{FSR}} = \Delta \frac{\omega_{\text{FSR},1} \omega_{\text{FSR},2}}{\omega_{\text{FSR},1} - \omega_{\text{FSR},2}} - p_0 \omega_{\text{FSR},1} + \frac{\Delta \omega_{\text{offset}}}{\omega_{\text{FSR},1} - \omega_{\text{FSR},2}} \quad \text{(B.1.7)}$$

Calculating the dependence of $p_0$ on $\Delta \omega_{\text{offset}}$ requires a deeper analysis and analytical fancy footwork going beyond the scope and interest of this appendix. We thus conclude this calculation with three observations:

1. Integers $p_0$ and $q_0$ can be determined from $\beta$ and $\omega_{\text{offset},i}$, $i \in \{1, 2\}$.

2. The simplest case when $\Delta \omega_{\text{offset}} = 0$ yields $p_0 = q_0 = 0$. This agrees with our analysis when comparing equations B.1.5, B.1.6 and B.1.7.

3. Only when $\beta$ and $\Delta \omega_{\text{offset}}$ take specific discrete values can the $C^3$ laser operate from the overlap of its sub-cavity longitudinal modes.
B.2 Cavity Instability

With some confidence in the basics of the $C^3$ laser, we briefly comment on the stability of laser cavities. Our calculation of the spectral modes of the $C^3$ laser as given by equation B.1.6 follows from a steady-state analysis. As such, our calculation does not reveal the physics associated with the stability of these modes. This holds true in general, longitudinal and transverse mode analysis does not encapsulate the physics of the stability of the laser and it is thus of interest to consider the example of the $C^3$ laser for which this analysis has been completed in [47].

A lack of sufficient competition between modes will cause any cavity to operate in multimode typically due to spontaneous emission. We can note that unlike a mode-locked laser however, due to the random nature of spontaneous emission the modes will not have a well-defined phase relationship explaining why a frequency comb is not the default lasing condition.

A dispersive element inside a cavity is one method to engineer a high mode competition and ensure single-mode lasing. Because despite the selectivity on the values of $\beta$ and $\Delta\omega_{\text{offset}}$ for the $C^3$ to lase, there exists bimodal (two simultaneous lasing modes) and bistable (two accessible single modes) regions in the current plane as shown in figure B.3. The bimodal lasing state was found to exist regardless of the level of spontaneous emission, demonstrating the intensity of side-modes is not necessarily dependent on the spontaneous emission rate but for the $C^3$ laser inherent to the cavity itself [47].

Whilst not identical or substantiated to have characteristics like an External Cavity Diode Laser (ECDL), the similarity in the structures of ECDLs and $C^3$ lasers intuitively suggests common behaviour. Whilst a deeper analysis regarding stability in ECDLs beyond the scope of this project is needed to understand the possible causes for unstable ECDL lasing, the $C^3$ laser provided a solved platform to make suggestions on lasing phenomena seen in ECDLs.
Figure B.3: The current plane of a $C^3$ laser demonstrating regions of single mode, and multimode lasing. The currents densities $J_1$ and $J_2$ of the first and second sub-cavity do not result in lasing below a threshold value. At sufficient densities, single mode (single-line shaded) regions are supported. They are bounded by stable (unstable) multimode regions, represented as not-shaded (double-shaded) regions. Bistable transitions take place at the edge of the unstable multimode (bistable), region not present for the stable multimode (bimodal) region. This figure stems combining two figures in courtesy of [47].
B.3 CW Instability of ECDL

There are several possible sources for instability during CW lasing. The point this appendix tries to project is the need for such instabilities to be analysed in a *dynamic* environment, not only steady-state. The $C^3$ laser demonstrates namely how a simple set-up similar to an ECDL can demonstrate complex lasing behaviour not visible under the common steady-state analysis. Indeed, our derivation of equation B.1.7 does not even predict an multimode lasing, yet it has been observed and derived using rigorous dynamic analysis.

If we look at figure B.3, we can hypothesise\(^2\) the current densities to be representative of the total loss-gain balance inside the cavity. An ECDL like the mode-locked laser discussed in sub-thesis I (c.f. section 4.3) can therefore be thought of as a $C^3$ laser with negligible cavity spacing (from the anti-reflection coating) and with the external cavity operating at a constant loss-gain value. In this analogy, the equivalent of the current plane would become a line that bisects figure B.3 either vertically or horizontally depending on the labelling convention. It is not immediately obvious whether or not this line can be made to bisect multimode lasing regimes. This could therefore be a source for current-specific laser-output instabilities seen in ECDLs.

\(^2\)This is a speculation, subject to the hasty generalisation fallacy.
Appendix C

Frequency-Jitter Model

Progress was made towards the developments of a more representative simulation of the Mode-Locked Laser’s (MLL’s) spectrum through the mode-matched Fabry-Perot Optical Spectrum Analyser (OSA). In this appendix we outline where this development got to, its approximations and what additional information we hoped the model could provide.

C.1 Modelling the Comb-lines

We model the spectrum of the MLL as semi-equidistant\(^1\) Gaussians of linewidth \(\Delta \nu_q\) whose center is perturbed by a ‘jitter-shift’ \(\delta \nu_q\).\(^2\) Hence continuing from equation 5.4 we model the spectrum from an infinite train of pulses as

\[
\text{Spectrum}[\text{MLL}](\nu) = \tilde{A}(\nu - \nu_c) \sum_{q=0}^{\infty} \delta(\nu - q \nu_m - \nu_{CO})
\]

\[
= \sum_{q=0}^{\infty} \tilde{A}_q(\nu - q \nu_m - \nu_{CO}) e^{-(\Delta \nu_l)^2(\nu - q \nu_m - \nu_{CO} - \delta \nu_q)^2}
\]

The coefficients \(\tilde{A}_q\) help describe the spectral-envelope of width \(\Delta \nu\) centred around the centre comb-line of index \(\bar{q}\) and can be modelled as

\[
\tilde{A}_q(\nu - q \nu_m - \nu_{CO}) \propto e^{-(\Delta \nu_l)^2(\nu - q \nu_m - \nu_{CO})^2}
\]

Simulations would be run for different jitter profiles \{\(\delta \nu_q\)\}_q over \(q\) specific comb-lines. E.g. \{\(\delta \nu_q\)\}_q = \{\Delta \nu_{\bar{q}}\}_{\bar{q} \in \{100,110\}} \cup \{2 \Delta \nu_{\bar{q}}\}_{\bar{q} \in \{111,120\}}\) for example.

\(^1\)The separation is defined by the comb-equation 5.3 with a perturbation term.

\(^2\)We choose the sign convention that introduces an upwards frequency-shifts for \(\delta \nu_q > 0\). This gives us the intuitive relationship between the actual \(\nu_{\text{actual}}\) and ideal frequency-coherent \(\nu_{\text{ideal}}\) shift as \(\delta \nu_q = \nu_{\text{actual}} - \nu_{\text{ideal}}\).
APPENDIX C. FREQUENCY-JITTER MODEL

C.2 Modelling the Fabry-Perot

The spectral-transmission \( T_{\nu}(\nu) \) of the Fabry-Perot as per equation 6.1 complicates the analytical expression of the transmission through the MLL. We therefore opted for another Gaussian simplification. Similar to the spectrum of the MLL, we can simplify the spectrum of the OSA as semi-equidistant\(^3\) Gaussians whose width is defined by equation 6.2 \( i.e. \) the cavity’s finesse \((F)\) and FSR \((\Delta\nu_{fp|FSR})\). Mathematically this can be expressed as

\[
T_{\nu}^{\text{model}}(\nu) \propto \sum_{p=0}^{\infty} e^{-\left(\frac{\nu-\nu_{fp|FSR}}{\Delta\nu_{fp|FSR}}\right)^2\left[1+(L-\bar{L})/L\right]-p\Delta\nu_{fp|FSR}}^2
\]

Perturbing a cavity to length \( L \) from \( \bar{L} \) induces a spectral shift \( \nu_{\text{shear}} \) in the transmission peaks as given by equation 6.1. We calculated this shift by approximating the effect of the cavity’s perturbation\(^4\) \( \delta L = L - \bar{L} \) to shift the spectrum by \( \nu_{\text{shear}} \) using the spectral-transmission function \( T_{\nu}(\nu) \) in equation 6.1 \( i.e. \)

\[
T_{\nu|L+\delta L}(\nu) \simeq T_{\nu|L}(\nu + \nu_{\text{shear}})
\]

This approximation holds for \( \delta L \ll \bar{L} \ i.e. \) small cavity perturbations as is typical in a Fabry-Perot OSA. A more exact model can be made by accounting for the FWHM changes introduced by \( \delta L \). For our purpose, as \( \delta L/L \sim 10^{-3} \) we make use of this approximation giving us

\[
\nu_{\text{shear}} = \frac{\delta L}{L} \nu
\]  

\( (C.2.1) \)

Unlike the comb-lines of the MLL’s spectrum, we take each transmission ‘window’ to have an identical linewidth. The mode-spacing we do model as semi-equidistant, accounting for the accordion-like compression/expansion during a cavity scan by set-delaysing the cavity’s FSR and the frequency-shearing term \( \nu_{\text{shear}} \).

Given a scan of the OSA results from the perturbation in the cavity’s length \( L \), by set-delaying \( L \) over a typical scanning range we can simulate a scan. To convert a scan’s axis from \( L \) into frequency, we can use equation \( C.2.1 \) by setting \( \nu \to \nu_c \), the centre comb-line. This approximation holds well for MLL’s with a narrow spectra.

---

\(^3\)Apart from a frequency-dependent shearing, the mode-spacing is given by the FSR of the cavity \( i.e. \) equation 5.2.

\(^4\)Using the same intuitive sign-convention as \( \delta \nu_q \). An increase in \( L \) decreases the cavity’s FSR namely giving a negative shear as per equation 6.1.
C.3 Modelling the Total Transmission

To model the total transmission, we start with the spectral density function \( \rho_\nu(\nu) \) defined by the product of Spectrum[MLL](\( \nu \)) and \( T_{\text{model}}^\nu \). Because the cavity modes (\( \nu_p \)) of the OSA and the comb-lines (\( \nu_q \)) of the MLL can be paired however, we can eliminate one summing index using the Kronecker delta quantity \( \delta_{pq} \) to give a spectral density function of

\[
\rho_\nu(\nu) = (\delta_{pq} \cdot T_{\text{model}}^\nu \cdot \text{Spectrum[MLL]})(\nu)
\]

By integrating \( \rho_\nu(\nu) \) over all modes/comb-lines we acquire the total transmission \( T_{\text{total}}(L) \) which we can simulate and compared/fitted to our data. The trace measured by the mode-matched OSA is thus be given by

\[
T_{\text{total}}(L) = \int_0^\infty \rho_\nu(\nu)d\nu
\]

C.3.1 Analytical Form

Whilst the integral could be computeted numerically using Runge-Kutta or Gaussian integration methods, we sought to simplify \( T_{\text{total}}(L) \). The product of Gaussians namely produce another Gaussian which can be integrated using a standard integral.

The argument \( Z \) of the exponential factor of \( \rho_\nu(\nu) \) can be calculated as

\[
Z = - [(\Delta \nu)^{-2}(\nu - \bar{\nu} m - \nu_{\text{CO}})^2 + (\frac{\sqrt{2} F}{\Delta \nu_{\text{FSR}} p})^2(\nu + \nu_{\text{shear}} - p\Delta \nu_{\text{FSR}})]
\]

\[
+ (\Delta \nu_q)^{-2}(\nu - q\nu_m - \nu_{\text{CO}} - \Delta \nu_q)^2 \delta_{pq}
\]

\[
= - \delta_{pq}[(\frac{\sqrt{2} F}{\Delta \nu_{\text{FSR}}})^2(\nu + \nu_{\alpha})^2 + (\Delta \nu_q)^{-2}(\nu + \nu_{\beta})^2 + (\Delta \nu)^{-2}(\nu + \nu_{\gamma})^2]
\]

\[
= - \nu^2[(\frac{\sqrt{2} F}{\Delta \nu_{\text{FSR}}})^2 + (\Delta \nu_q)^{-2} + (\Delta \nu)^{-2}] \delta_{pq} - \nu[(\frac{2 F}{\Delta \nu_{\text{FSR}}})^2 \nu_{\alpha} + 2 \frac{\nu_{\beta}}{(\Delta \nu)^2}]
\]

\[
+ 2 \frac{\nu_{\gamma}}{(\Delta \nu)^2} \delta_{pq} - \delta_{pq}[(\frac{\sqrt{2} F \nu_{\alpha}}{\Delta \nu_{\text{FSR}}})^2 - \delta_{pq}(\frac{\nu_{\beta}}{\Delta \nu_q})^2 - \delta_{pq}(\frac{\nu_{\gamma}}{\Delta \nu})^2]
\]

\[
= - \alpha \nu^2 + \beta \nu + \gamma = -\alpha(\nu^2 - \frac{\beta}{\alpha} \nu) + \gamma
\]

\[
= - \alpha(\nu - \frac{\beta}{2\alpha})^2 - (\frac{\beta}{2\alpha})^2 + \gamma
\]

where we introduced six auxilary variables defined as
\[\nu_\alpha = \nu_{\text{bare}} - p\Delta\nu_{p|\text{FSR}}, \nu_\beta = -q\nu_m - \nu_{\text{CO}} - \delta\nu_q, \nu_\gamma = -\bar{q}\nu_m - \nu_{\text{CO}}\]

\[\alpha = \delta_{pq}[(\frac{\sqrt{2} F}{\Delta\nu_{p|\text{FSR}}})^2 + (\Delta\nu_q)^2 + (\Delta\nu)^2]\]

\[\beta = -\delta_{pq}[(\frac{2F}{\Delta\nu_{p|\text{FSR}}})^2\nu_\alpha + 2\nu_\beta + 2\nu_\gamma] \]

\[\gamma = -\delta_{pq}[(\frac{\sqrt{2} F\nu_\alpha}{\Delta\nu_{p|\text{FSR}}})^2 + (\nu_\beta)^2 + (\nu_\gamma)^2]\]

This simplifies \(\rho_\nu(\nu)\) to

\[\rho_\nu(\nu) = \sum_{q=0}^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2} = \sum_{q=0}^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2-\frac{\gamma}{2\alpha}}\]

giving a total transmission of

\[T_{\text{total}}(L) \propto \sum_{q=0}^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2} = \sum_{q=0}^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2} \int_0^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2} d\nu\]

As the comb-lines and OSA cavity modes operate at positive frequencies, the integrant is negligible for \(\nu < 0\) such that

\[\int_0^{\infty} \rho_\nu(\nu)d\nu \simeq \int_{-\infty}^{\infty} \rho_\nu(\nu)d\nu\]

And given \(\frac{\beta}{2\alpha} \gg 0\) (since \(\nu_\beta < 0\)), the Gaussian function is shifted into the positive frequency domain. Given the integration limits, the area can thus be approximated as common to an origin centred Gaussian such that

\[T_{\text{total}}(L) \propto \sum_{q=0}^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2} \int_0^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2} d\nu = \sum_{q=0}^{\infty} e^{-\alpha(\nu-\frac{\beta}{2\alpha})^2} \sqrt{\frac{\pi}{\alpha}}\]

Alternatively, a more rigorous standard integral can be used such as

\[\int_0^{\infty} e^{a(x-b)^2} dx = \frac{1}{2\sqrt{\pi}} \sqrt{\frac{\pi}{a}} \text{erf}(\sqrt{\alpha b}) + 1\]

where we approximate the error function \(\text{erf}(y)\) to converges to 1 for large \(y\), as is the case for \(a = \alpha, b = \frac{\beta}{2\alpha}\).
Appendix D

Globally Independent Speckles

In this appendix we show further detail on the statistical independence inference between the optical field’s global amplitude $A$ and global phase $\theta^A$ of an first-order speckle pattern. This result is used when discussing higher order speckle patterns in subsection 11.4.3. This calculation follows in the footsteps of section 11.3 hence is therefore left outside sub-thesis II.
D.1 Amplitude-Phase Probability Densities

To demonstrated the claim of statistical independence, we shall derive the relevant probability density functions \((\rho)\). Recall the central limit theorem provides an amplitude probability density function given by

\[
\rho_{r,i}(A^{(r)},A^{(i)}) = \frac{1}{2\pi\sigma^2}e^{-[(A^{(r)})^2+(A^{(i)})^2]/(2\sigma^2)}, \quad \sigma^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \frac{|a_k|^2}{2}
\]

We can take a change of variables by calculating the Jacobian determinant:

\[
||J|| = ||\left(\frac{\partial A^{(r)}}{\partial |A|}, \frac{\partial A^{(r)}}{\partial \theta}, \frac{\partial A^{(i)}}{\partial |A|}, \frac{\partial A^{(i)}}{\partial \theta}\right)||
\]

The partial derivatives can be solved by recalling the following relations:

\[
A^{(r)} = \sqrt{I}\cos\theta \\
A^{(i)} = \sqrt{I}\sin\theta
\]

Our Jacobian determinant thus simplifies to

\[
||J|| = ||\left(\cos\theta, |A|\sin\theta, \sin\theta, -|A|\cos\theta\right)|| = |A|
\]

which transform the probability density to

\[
\rho_{r,i}(|A|\cos\theta^A, |A|\sin\theta^A)||J|| = \begin{cases} 
\frac{|A|}{2\pi\sigma^2}e^{-|A|^2/2\sigma^2}, & I \geq 0 \land \theta \in [-\pi, \pi] \\
0, & \text{otherwise}
\end{cases}
\]

By integrating equation D.1.1 over the appropriate variable, we can derive the single variable phase/amplitude probability densities. Doing so we find

\[
\rho_{|A|}(|A|) = \int_{-\pi}^{\pi} \rho_{I,\theta}(I,\theta)d\theta = \begin{cases} 
\frac{|A|}{\sigma^2}e^{-|A|^2/2\sigma^2}, & I \geq 0 \\
0, & \text{otherwise}
\end{cases}
\]

\[
\rho_{\theta}(\theta) = \int_{0}^{\infty} \rho_{I,\theta}(I,\theta)dI = \begin{cases} 
\frac{1}{2\pi}, & \theta \in [-\pi, \pi] \\
0, & \text{otherwise}
\end{cases}
\]

from which we can deduce that

\[
\rho_{|A|,\theta^A}(|A|, \theta^A) = \rho_{|A|}(|A|)\rho_{\theta^A}(\theta^A)
\]

i.e. \(|A| \perp \theta^A\) as desired.
Appendix E

Two-field Selection Rules

In subsection 15.1.6 we reviewed the selection rules for a single optical field interrogating an atom. These rules prevent Rabi oscillations between the two ground levels of Rubidium using electric dipole transitions as those levels have the same parity $\chi$. In our atom interferometer experiments we therefore use stimulated Raman (two-photon) transitions to observe Rabi flopping.

In this appendix we demonstrate how the usage of a second optical field does not alters the selection rules. This explains why stimulated Raman transitions are needed for electric-dipole transitions between the two levels. As will become evident below, the demonstration can easily be adapted to include arbitrary phase-shifts of the two laser fields.
APPENDIX E. TWO-FIELD SELECTION RULES

E.1 No Transition Rule Exemption

The energy perturbation introduced to by two optical field \( \mathbf{E}_p(t) \) & \( \mathbf{E}_q(t) \) can be modelled as before, equalling the work done by the induced electric dipole \( \hat{V}(t) \) given by

\[
\hat{V}(t) = -e \mathbf{r} \cdot (\mathbf{E}_p(t) + \mathbf{E}_q(t))
\]

We set the two optical fields operate at separate angular frequencies \( \omega_1 \) & \( \omega_2 \). For simplicity we also set the amplitudes of the electric fields to be common at \( E_0 \). This simplifies the form of the energy perturbation to

\[
\hat{V}(t) = -eE_0 \mathbf{r} \cdot \left( \hat{\epsilon}_p \frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2} + \hat{\epsilon}_q \frac{e^{i\omega_2 t} + e^{-i\omega_2 t}}{2} \right)
\]

where \( \hat{\epsilon}_{p,q} \) are the spherical unit vectors that describe the polarisation of the two optical fields. For circular \((p,q = \pm 1)\) or linear \((p,q = 0)\) polarised light they are given by

\[
\hat{\epsilon}_p = \begin{cases} 
-\frac{(\hat{\mathbf{x}} + i\hat{\mathbf{y}})}{\sqrt{2}}, & p = 1 \\
\frac{\hat{\mathbf{z}}}{\sqrt{2}}, & p = 0 \\
\frac{(\hat{\mathbf{x}} - i\hat{\mathbf{y}})}{\sqrt{2}}, & p = -1 
\end{cases}
\]

If \( l \) represent the azimuthal quantum number of the electron/atom with \( m_l \) the z-component of \( l \), we can express the energy perturbation using the spherical harmonic functions \( Y_{l,m_l} \) [148]. Letting \( \theta \) & \( \phi \) represent the polar and azimuthal angles in spherical coordinates, we can use

\[
\hat{\epsilon}_p \cdot \mathbf{r} = \sqrt{\frac{4\pi}{3}} r Y_{1,p}(\theta, \phi)
\]

to establish that

\[
\hat{V}(t) = -e \sqrt{\frac{4\pi}{3}} E_0 r(Y_{1,p}(\theta, \phi) \frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2} + Y_{1,q}(\theta, \phi) \frac{e^{i\omega_2 t} + e^{-i\omega_2 t}}{2})
\]

According to Fermi’s golden rule (equation 15.2), the transition rate \( W \) for the \( |a\rangle \rightarrow |b\rangle \) transition becomes

\[
W_{|\psi_a\rangle \rightarrow |\psi_b\rangle} = \frac{2\pi}{\hbar} |\langle \psi_b | \hat{V} | \psi_a \rangle|^2 \rho_b
\]

If we assume the density of states \( \rho_b \) is non-degenerate, we can express \( W \) as

\[
W_{|\psi_a\rangle \rightarrow |\psi_b\rangle} = \frac{2\pi}{\hbar} |\int \int \psi^*_b(r, \theta, \phi) \hat{V}(t) \psi_a(r, \theta, \phi) d\mathbf{r}|^2 \quad (E.1.1)
\]
To demonstrate the single field selection rule hold when using two optical fields, it suffices to show that the integral in equation E.1.1 vanish except when satisfying the selection rules from equation 15.3. To simplify the calculation, we start with dividing the three spherical coordinates \( \{r, \theta, \phi\} \) using \( dV = r^2 \sin(\theta)drd\theta d\phi \). We find the volume integral of interest becomes

\[
\int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{2\pi} \psi^*_a(r, \theta, \phi) \hat{V}(t) \psi_a(r, \theta, \phi) r^2 \sin(\theta)drd\theta d\phi
\]

The analytical form of the wavefunctions \( \psi_{a,b} \) can be divided into a radial \( R_{n,l}(r) \) and angular \( Y_{l,m}(\theta, \phi) \) part. As \( \hat{V}(t) \) is independent of \( r \), the integral term simplifies to

\[
\int_{0}^{\infty} R_{n+l}^* R_{n,l} dr^3 \int_{0}^{\pi} \int_{0}^{2\pi} Y_{l+m+\Delta m}^* Y_{l,m} \frac{\hat{V}}{r} \sin(\theta)drd\theta d\phi
\]

The first integral has no roots (does not vanish) hence can be ignored [205] leaving the angular integrals to introduce forbidden transitions. Substituting \( \hat{V}/r \) into these we find the angular integrals take the form

\[
\left[ e^{i\omega_1 t} + e^{-i\omega_1 t} \right] \int_{0}^{\pi} \int_{0}^{2\pi} (AY_{l+\Delta l,m+\Delta m}^* Y_{l+1,m+1+p}) \sin(\theta)drd\phi
+ BY_{l+\Delta l,m+\Delta m}^* Y_{l-1,m+1+p} \sin(\theta)drd\phi
+ e^{i\omega_2 t} + e^{-i\omega_2 t} \int_{0}^{\pi} \int_{0}^{2\pi} (AY_{l+\Delta l,m+\Delta m}^* Y_{l+1,m+q}) \sin(\theta)drd\phi
+ BY_{l+\Delta l,m+\Delta m}^* Y_{l-1,m+1+q} \sin(\theta)drd\phi \]

where we made use of the following relationship

\[
Y_{l,m} Y_{l',m'} = AY_{l+1,m+m'} + BY_{l-1,m+m'}
\]

We can infer the selection rules from equation E.1.2 by invoking the orthogonality property of the spherical harmonics, insisting the quantum numbers per term are common. Doing so we find the selection rules to be

\[
\Delta l = \pm 1
\]
\[
\Delta m = \{p, q\} = 0, \pm 1
\]

where upon extending the calculation to other quantum numbers we find the selection rules for two fields (equation E.1.3) do not alter the selection rules for a single field (equation 15.3).
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