UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Mathematical Sciences

Local risk minimization analysis for contingent claims using weak derivatives and running infimum processes

by

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ABSTRACT

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In this thesis, we examine the local risk minimization approach and the Föllmer-Schweizer decomposition, in certain credit risk models. We start by extending the model proposed by Okhrati et al. (2014) to the non-smooth case for which the hedging strategies are based on. Assuming that the evolution of the price of the underlying asset is a Lévy process of finite variation, we investigate the local risk minimization for a defaultable claim, whose default time is given through a first passage time (structural framework) where the default barrier is constant. We derive the Kunita-Watanabe (KW) decomposition through a solution of a partial-integro differential equation (PIDE) using non-smooth Itô's formula of Okhrati and Schmock (2015). This allow us to obtain a solution of a PIDE which is continuous but not necessarily smooth.

We also investigate a structural credit risk model using the local risk minimization approach where the default is modelled via a random variable. In this model, the underlying asset is a spectrally positive Lévy process and the compensator technique are used to obtain the Föllmer-Schweizer decomposition for a contingent claim that is prone to default from an investor's point of view. In our analysis, we use a progressive filtration $\mathbb{G}$. We highlight that we do not assume that the $\mathcal{H}$-hypothesis holds, which states that a local martingale under the initial filtration $\mathbb{F}$ remains a local martingale under the expanded filtration $\mathbb{G}$.

Furthermore, we study the local risk minimization for a defaultable contingent claim where the default time is exogenously defined though a hazard rate model depending on both the underlying and its infimum. This allows us to introduce some particularly interesting cases for claims that are subject to both endogenous and exogenous defaults. The endogenous default is determined in a structural framework depending on the infimum process with constant barrier. Similarly to the previous model, our construction is made under a progressive filtration expansion $\mathbb{G}$. In this setup, the underlying asset is modelled through an exponential jump diffusion Lévy process. We aim at determining locally risk minimizing hedging strategies through solutions of either PDEs or PIDEs. We also provide some applications and examples in credit risk modelling for the diffusion and jump diffusion case.

Finally, under the setup of the above models, we provide some credit risk models examples and their associated numerical implementations through solutions of PDEs and PIDEs using finite differences.

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Declaration of Authorship

I, Nikolas Karpathopoulos, declare that the thesis entitled “Local risk minimization analysis for contingent claims using weak derivatives and running infimum processes” and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that

- this work was done wholly or mainly while in my candidature for research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of other, this is always clearly attributed;
- where I have quoted from the work of other, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all the main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission.

Signed:

Date:

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In the memory of my father Sotiris.

To my family.

Nomenclature

FS Föllmer-Schweizer.

KW Kunita-Watanabe.

GKW Galtchouk-Kunita-Watanabe.

MMM minimal martingale measure.

MVT mean variance tradeoff.

LRM locally risk minimizing.

PLRM pseudo locally risk minimizing.

PIDE partial integro-differential equation.

SC structure condition.

The following class of processes is used throughout this thesis.

$\mathcal{M}$, the space of martingales, 12.

$\mathcal{M}_{loc}$, the space of local martingales, 12.

$\mathcal{M}^2$, the space of square integrable martingales, 12.

$\mathcal{M}^2_{loc}$, the space of locally square integrable martingales, 12.

$\mathcal{V}$, the set of all real-valued and adapted càdlàg processes with finite variation paths, 12.

$\mathcal{A}$, class of processes with integrable variation, 12.

$\mathcal{A}_{loc}$, class of processes with locally integrable variation, 12.

$\mathcal{S}(\mathbb{P})$, space of semimartingale, 14.

$\mathcal{S}^2(\mathbb{P})$, space of square integrable semimartingales, 14.

$\mathcal{E}$, the class of Doléans-Dade exponential processes, 52.

The space of the following functions are introduced.

L^p, space of functions whose p moment is Lebesgue integrable, 59.

L^1_{loc}, space of locally integrable functions, 60.

$C^{1,1,1}$, class of continuous differentiable functions with respect to t, x and y, 86.

For the notation bellow please refer to the page where the notations are initially introduced.

$\mathcal{P}$**,** the set of all convex equivalent martingale measures, 54.

$\mathcal{P}_e^2$, 54**.**

$\mathcal{B}(\mathbb{R})$, Borel sigma algebra, 10.

$\mathbb{R}_0^+$ the set of all positive real numbers including zero, 22.

$\mathcal{D}$, 38.

$L^2(X)$, 45.

$I^2(X)$, 45.

Θ, 49.

Θ_{GLP}, 55.

$\bar{X}$, the supremum of X, 25.

$\underline{X}$, the infimum of X, 25.

$[\cdot]$, quadratic variation, 14.

$\langle\cdot\rangle$, predictable quadratic variation, 16.

$\mathscr{O}$, optional σ-field, 11.

$\mathscr{P}$, predictable σ-field, 12.

Chapter 1

Introduction

1.1 Preface

The 2007-2008 financial crisis and its consequences to financial institutions pinpointed the importance of the counterparty credit risk modelling, in particular hedging and pricing of contingent claims. Some of these claims are subject to default (hence called defaultable claims) so their value after a specific time become worthless. A defaultable claim is not fully hedgeable due to the existence of a non-tradable (intrinsic) risk and the aim of a hedging strategy is to minimize investor's risk. Let us start with the definition of credit risk.

Credit risk is the risk of default caused by the failure of a borrower or a counter-party to meet its obligations. The result of a default can lead to a company's bankruptcy, as it may not be able to meet its financial obligations. Defaultable claims are an essential tool for managing this risk partially.

A corporate bond is considered as a debt obligation issued by a firm and sold to investors. Bondholders who buy a corporate bond are actually lending money to a firm contributing to its financial stability. Generally speaking, corporate bonds have a higher probability of default than those issued by a government. Therefore, they consist a typical case of defaultable claims. Mathematically speaking, given a fixed time $T > 0$ which represents the maturity time and assuming that the dividend process is zero, a defaultable claim H is defined by $(K, \tilde{K}, Z_t, \tau)$ where

- K represents the payoff function at the maturity time T i.e. the amount of money returned to investor if no prior default occurs.
- $\tilde{K}$ is the recovery claims representing the recovery payoff function received at the maturity time T, in case that default occurs prior or at the maturity time T.
- $(Z_t)_{t \geq 0}$ is the recovery process, representing the amount of money that bondholders have to pay at time of default if this occurs prior or at the maturity time T.

Assuming that $B_t = \exp(\int_0^t r_s ds)$ then the discounted value of a defaultable claim H has the form

$$H = \frac{K}{B_T} 1_{\{\tau > T\}} + \frac{\tilde{K}}{B_T} 1_{\{\tau \leq T\}} + \frac{Z_\tau}{B_\tau} 1_{\{\tau \leq T\}}.$$

In the simple case when the recovery process $(Z_t)_{t \geq 0}$ is zero and $\tilde{K} = 0$ then the defaultable claim is of the form $H = \frac{K}{B_T} 1_{\{\tau > T\}}$, this is the form of a defaultable zero coupon bond.

Obviously, the default time has a fundamental role in the pricing and hedging of credit risk. There are two major ways under which we can estimate the default probabilities. The structural and the reduced form models.

In structural models the default time is modelled through a first-passage model, i.e. the first time that the underlying asset hits a barrier. The barrier can be a constant or a stochastic process. Thus, the default time has an economic interpretation as it is associated with the underlying asset and it is endogenously defined. In these models, if the underlying asset admits continuous paths, then investors are aware of the arrival of the default event and has all the available information. In this framework, the pricing and hedging of defaultable claims can be obtained through the famous Black-Scholes model. However, these models have their limitations too. Assuming that the underlying asset is modelled by an exponential Brownian motion with a drift term and investors have full information then the default is a predictable stopping time. The predictability of the stopping time implies that the short credit spread is zero which is an unrealistic fact.

On the contrary, in reduced form models the credit event is defined by a completely different perspective. More precisely, the stopping time is modelled through a hazard rate (or intensity) process, therefore the default event is totally inaccessible and so investors are unaware of its arrival. An advantage of this approach is that the short credit spread is non-zero consistent with the market's observations.

Dealing with hedging of contingent claim market completeness is not guaranteed, since most of the times it is impossible to construct a hedging strategy against the occurrence of default. By definition of complete markets and assuming that the market is arbitrage free, there exists a unique martingale measure so that a contingent claim can be perfectly hedged. However, in real markets perfect hedging strategies do not exist. In fact, for models where the underlying asset includes jumps or the volatility term is stochastic, the market completeness property that the the contingent claim is redundant (i.e. the contingent claim can be valued by reference of other contingent claims) is violated. So, in incomplete markets, there are several martingale measures and various market prices of risk. Evidently, hedging contingent claims in incomplete markets is quite challenging and more interesting.

Quadratic hedging methods are quite popular and well-established criterion for hedging defaultable claims in incomplete markets. They are divided into two different approaches: the so-called local risk minimization and the mean variance approach. In

Schweizer (2001), both approaches are thoroughly analysed when the underlying asset is a semimartingale. The mean variance approach is applicable for hedging strategies whose portfolios are self-financing. The method was first introduced by Duffie and Richardson (1991) and later on, it was extended by Schweizer (1992). On the other hand, the local risk minimization is applied for a portfolio that is mean self-financing, and its value at the maturity time T is equal with the contingent claim. In this thesis, we study the local risk minimization, so let us describe the method in more detail.

The local risk minimization was introduced by Föllmer and Sondermann (1986) for the case when the underlying asset is a martingale. Schweizer (2001) extended the approach to the case when the underlying asset is a local martingale, through the Galtchouk-Kunita-Watanabe (GKW) decomposition. Schweizer (1994) applies for the first time the local risk minimization under partial information where the optimal hedging strategy is obtained through predictable projections.

As previously mentioned, the local risk minimization is applied when the portfolio is not self-financing but mean self-financing, meaning that the cost process should be a local martingale. This is the main distinction between the local risk minimization and the mean variance. Another important feature of this method is that the cost process should be strongly orthogonal to the martingale part of the underlying asset.

The determination of the Föllmer-Schweizer (FS) decomposition is crucial in the context of the local risk minimization approach. In certain cases, there are ways to obtain this decomposition explicitly. In general, the FS decomposition can be obtained by determining an equivalent martingale measure, the so-called minimal martingale measure. Moreover, under some appropriate conditions, such as the structure condition (SC), the local risk minimization approach is equivalent to the pseudo-optimal local risk minimizing strategy, which in many circumstances, is more flexible. The existence of the FS decomposition can be proved through the SC, see Schweizer (2001).

1.2 Research aims and outcomes

The main aim of this dissertation is to study the local risk minimization in certain partial information credit risk models when the underlying asset admits jumps. In particular, we focus on the unification of the structural and the reduced form models in credit risk modelling. This is made by extending the relationship between the default time and the underlying asset for hedging contingent claims in incomplete markets. We apply the local risk minimization approach for defaultable claims when there exists dependency between the asset and the default time through intensity and hazard rate models. To achieve this the analysis should be made under an appropriate filtration expansion $\mathbb{G}$ such that $\mathbb{F} \subset \mathbb{G}$, where $\mathbb{F}$ represents the available information for investors.

In what follows, we summarize the aims of this thesis and more specifically:

- determining the canonical decomposition of a stochastic process under an enlarged filtration (shown by $\mathbb{G}$) without using the $\mathcal{H}$-hypothesis,

- deriving the optimal semi-explicit solution for hedging strategies through the local risk minimization method under some circumstances for certain contingent claim, including defaultable ones,

- providing a unification of structural and reduced form models,

- solve appropriate PIDEs or PDEs numerically through finite differences.

In particular, we focus on hedging of contingent claims prone to default of the forms $F(X_T)1_{\{\tau>T\}}$ or more generally $F(X_T, \underline{X}_T)1_{\{\tau>T\}}$, where $(X_t)_{t\geq 0}$ is the underlying asset (or the exponential asset denoted by $(Y_t)_{t\geq 0}$), $(\underline{X}_t)_{t\geq 0}$ is the running infimum process of $(X_t)_{t\geq 0}$ and F is a real valued function. For the case when $F(x) = c$, where c is a constant, then the contingent claim is a defaultable zero coupon bond.

In contrast to the general approach where the determination of the FS decomposition is obtained through the GKW via a minimal martingale measure, in this work, we obtain the KW and GKW decompositions directly without applying any Girsanov's theorem. Instead, we prove the existence of such decomposition directly through the KW decomposition and with the help of PIDEs or PDEs. Therefore, semimartingales and compensator techniques are crucial in this work.

Next, we highlight the main contributions by going through the relevant chapter of the dissertation.

- First, in Chapter 4, we extend the model proposed by Okhrati et al. (2014). We apply the local risk minimization for defaultable claim under which the default time is defined through a structural model i.e. the first hitting time under which the underlying asset becomes strictly negative. In this case, assuming that the underlying asset is modelled through a finite variation Lévy process, Okhrati et al. (2014) proved that the default time admits an intensity. The recovery and interest rate are assumed to be zero. We generalize the approach of Okhrati et al. (2014) in the following way. The hedging strategy has a semi-closed form determined through a PIDE. The solution $f = f(t, x)$ of this PIDE can be determined through the Feyman-Kac formula. However, due to the absence of the diffusion term the solution of the PIDE is not necessarily $C^{1,1}([0, T] \times \mathbb{R})$ causing problems especially in the numerical implementations, see Cont et al. (2004) for a discussion. In this chapter, we aim to fix this by applying the Itô's formula for non-smooth functions introduced in Okhrati and Schmock (2015).

- In Chapter 6, we analyse the local risk minimization for a defaultable claim where the default time is modelled through a structural model whose barrier is a random variable i.e. the true barrier is not observable to investors and all they can infer is the probability distribution of this barrier. In our analysis, we investigate the hedging strategy for a defaultable claim that pays certain amount at the maturity if there is no pre-default event and zero otherwise. The underlying asset is modelled by a spectrally positive Lévy process of finite variation. In Dong and Zheng

(2015), the existence of an intensity process in this structural framework is proved and it admits an explicit form. Under the progressive filtration expansion, we derive the GKW decomposition through a solution of appropriate 3-dimensional (3D) PIDE $f = f(t, x, y)$ which involves the running infimum process.

- Chapter 7 studies further the concept of local risk minimization using the running infimuum process investigating contingent claims that admits two types of defaults. More specifically, our model captures an endogenous default (structural framework) and an exogenous one determined through a hazard rate process. The endogenous structural default time is given by the first hitting time under which the asset will hit a constant barrier; representing the time of liquidation. In this chapter, as an underlying asset we no longer work with finite variation processes. Instead we use exponential jump diffusion processes, assuming that the Lévy measure is absolutely continuous with respect to Lebesgue integral, making our model more flexible and it offers a more realistic model for the asset values. We derive appropriate canonical decompositions that allows us to determine the KW and GKW decompositions under a progressive filtration expansion. The hedging strategy is obtained in a semi-explicit form through appropriate PIDEs or PDEs. More importantly, under some assumptions this unifies structural and reduced form credit risk modelling. We provide examples of the hedging strategies when the exponential underlying asset admits both continuous sample paths (such as an exponential Brownian motion) and a jump diffusion Lévy process.

- Finally, in Chapter 8, we solve the corresponding PIDEs and PDEs numerically through finite differences. For the integral terms, we consider the quadrature trapezoidal rule introduced in Cont and Voltchkova (2005). For the time discretisation we apply the implicit-explicit scheme.

1.3 Thesis outline

In this section, we briefly describe the thesis structure, which contains seven main chapters excluding the introduction, the conclusion and the appendices.

In Chapter 2, we provide the main mathematical tools which will be used throughout this thesis. More specifically, we begin by the definition of semimartingales, the local martingales and the Lévy processes. We also introduce some useful results for the reflected Lévy processes.

A thorough literature review of credit risk modelling is analysed in Chapter 3. We start with the introduction of structural models and the reduced form models. We also study the quadratic hedging approaches where an emphasis is given to the local risk minimization approach.

The reader who is familiar with the content of these two chapters can proceed directly into the following chapter (Chapter 4), which is our first contribution.

In Chapter 4, we apply the local risk minimization for a defaultable claim whose default time admits an intensity. We obtain the KW and GKW decompositions through the canonical decomposition of $(f(t, X_t)1_{\{\tau>t\}})_{t\geq 0}$ assuming that the involved functions are weakly differentiable.

Chapter 5 provides an overview of partial information models. It is possible to obtain a reduced form model that admits an intensity through a structural model so that the default time becomes totally inaccessible. This can be achieved by relaxing the complete information assumption of the structural models. There are two major ways under which we can introduce partial information in a structural model, see Jarrow and Protter (2004). The first approach is to generalize a first-passage model by assuming that the default threshold is a random variable with a known distribution see Giesecke and Goldberg (2004), Giesecke (2006) and Giesecke (2001). Whereas, the second approach is to assume that the underlying asset is partially observed. For more details, we refer to Duffie and Lando (2001), Kusuoka (1999) and Coculescu et al. (2008). In this chapter, we also study the filtration expansion and filtration shrinkage in credit risk modelling. The idea of filtration expansion is developed during 1970's from Yor, Jacod and Jeulin. There are two main kinds of enlargement of filtration: initial and progressive filtration expansion. We analyse both types of expansions and we provide their canonical decomposition.

Once again, the reader who is acquainted with filtration expansions can skip the previous chapter and continue to Chapters 6 and 7.

In Chapter 6, we apply the local risk minimization approach under a partial information, assuming that the barrier in the structural model is a random variable following a negative exponential distribution and as a result the default time admits an intensity process. Since the intensity process involves a reflected Lévy process at the infimum, we start by determining the canonical decomposition of $(f(t, X_t - \underline{X}_t, -\underline{X}_t))_{t\geq 0}$ under the available information to investors $\mathbb{F}$, through the Itô's formula for a smooth function $f(t,x,y)$, i.e. $f(t,x,y) \in C^{1,1,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$. Given a progressive filtration expansion $\mathbb{G}$ and since the default time τ is $\mathbb{G}$ is totally inaccessible stopping time, we derive the canonical decomposition of $(f(\tau \wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}))_{t\geq 0}$. Based on this result, we determine the canonical decomposition of $(f(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}})_{t\geq 0}$ under $\mathbb{G}$, which for the local martingale case can give us the GKW decomposition.

In Chapter 7, we introduce the local risk minimization for a hazard rate model such that its intensity process $(\lambda_t)_{t\geq 0}$ is defined as $\lambda_t := g(t, Y_t, \underline{Y}_t)$ where $(Y_t)_{t\geq 0}$ is the exponential underlying asset $Y_t = \exp(X_t)$ for a continuous and positive function $g(t,x,y) : [0,T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, and the conditional survival probability under $\mathbb{F}$ is $\mathbb{P}(\tau > t \mid \mathcal{F}_t) = e^{-\int_0^t g(s,Y_s,\underline{Y}_s)ds}$. We follow the same technique as in the previous chapter. Assuming that jumps of the underlying asset at the default time is zero, we determine the canonical decomposition of $(f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}})_{t\geq 0}$ under the augmented filtration $\mathbb{G}$. The setup of this chapter leads to interesting credit risk models including structural and reduced form models. We also study contingent claims that are subject to two default events and investigate a unification of structural and reduced form models.

In Chapter 8, we solve the PIDEs and PDEs introduced in Chapters 4, 6 and 7 (for most of them when the underlying asset is a martingale) numerically through the finite differences. We also simulate sample paths of the optimal hedging strategies.

The thesis conclusion along with some future work is presented in Chapter 9. In Appendix A, we provide a useful martingale for the reflected Lévy process at its infimum given that $(X_t)_{t\geq 0}$ is a jump diffusion process. This result we help us to determine the canonical decomposition of $(f(\tau\wedge t, X_{\tau\wedge t}-\underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}))_{t\geq 0}$ under $\mathbb{G}$ in Chapter 7. Also, in Appendix B, we analyse the optimal and predictable projections formulas. Finally, in Appendix C, we introduce some trivial proofs to estimate the derivatives in space and time numerically through the Euler scheme.

Chapter 2

Basic definitions and preliminaries

2.1 Introduction

This chapter provides some necessary mathematical tools for readers who are not familiar with stochastic processes. We briefly describe the main mathematical background which will be used throughout this thesis. Some fundamental properties for semimartingales and Lévy processes are provided. The content of this chapter is mainly based on Jacod and Shiryaev (2003), Protter (2004), Sato (1999), Papapantoleon (2008) and Kyprianou (2014) and the references therein unless otherwise stated.

2.2 Introduction to stochastic calculus

In this section, we present some preliminary results of stochastic calculus in the content of martingales and local martingales. But first let us introduce some preliminary definitions.

Throughout this Chapter we assume that $\mathbb{F}$ is the natural filtration generated by the process $(X_t)_{t\geq 0}$ i.e. $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ where $\mathcal{F}_t = \sigma(X_s)_{0\leq s\leq t}$ with $t \in [0, T]$.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We assume that there exist a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, then we say that $\mathbb{F}$ satisfies the usual hypothesis if and only if

- $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets.
- $\mathbb{F}$ is right continuous, i.e. $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, with $t \in [0, T]$.

Definition 2.2. A stochastic process $(X_t)_{t\geq 0}$ is called $\mathbb{F}$-adapted if and only if the **random variables** X_t are $\mathcal{F}_t$-measurable for all $t \in [0, T]$.

Definition 2.3 (Stopping time)**.** A random variable $\tau : \Omega \to [0, \infty)$ is called a stopping time if $\{\tau < t\} \in \mathcal{F}_t$, for all $t \in [0, T]$.

In general, we classify the stopping times as follows.

Definition 2.4 (Classification of stopping times)**.** There are three types of stopping times:

- A stopping time τ is called **predictable** if there exists an increasing sequence of stopping times $\{\tau_n\}_{n=1}^{+\infty}$, $\tau_n < \tau$, such that $\lim_{n\to\infty} \tau_n = \tau$.
- A stopping time τ is called **accessible** if there exists a localizing sequence of predictable stopping times $\{\tau_n\}_{n=1}^{+\infty}$ such that

$$\mathbb{P}(\bigcup_{n=1}^{+\infty} \{\omega : \tau_n(\omega) = \tau(\omega) < \infty\}) = \mathbb{P}(\tau < \infty).$$

- Similarly, a stopping time τ is called **totally inaccessible** if $\mathbb{P}(\tau = \varsigma < \infty) = 0$, for every predictable stopping time ς.

Remark 2.5. The concept of accessible and totally inaccessible stopping times plays a fundamental role in the theory of credit risk modelling. As we will later see in the rest of this dissertation, a predictable stopping time implies that the short credit spreads are zero which is dreadful as it is inconsistent the market's data. On the other hand, a totally inaccessible stopping time admits an intensity and so the short-credit spread is non zero and consistent with the market's data. Note also that every stopping time can be decomposed into an accessible and totally inaccessible times.

Definition 2.6 (Hitting time)**.** Let A be a Borel set, $A \in \mathcal{B}(\mathbb{R})$, and $(X_t)_{t\geq 0}$ be a stochastic process. Then it yields

$$\tau(\omega) := \inf \{t > 0 : \, X_t \in A\} \, ,$$

is a stopping time, and it is called a hitting time.

Remark 2.7. The hitting time (also called first passage time) is considered one of the most appropriate ways to define the default time in credit risk modelling, since it has an economic interpretation as it connects the default time with the evolution of the price of the underlying asset. As we will see in Chapter 3, if the underlying asset is modelled by a geometric Brownian motion the τ is a predictable stopping time. On the other hand, if $(X_t)_{t\geq 0}$ is a Lévy process of finite variation (see Section 2.4) then following Protter (2004), Chapter III, Theorem 4, the default time τ is totally inaccessible and therefore it admits an intensity. When the underlying asset is a jump diffusion process (Brownian motion plus a jump process), the stopping time is neither predictable nor totally inaccessible. We use these results in Chapters 4, 6 and 7, and under some appropriate conditions we derive the hedging strategy for defaultable claims through the local risk minimization approach.

Definition 2.8. Assume that τ is a stopping time. Then it generates a σ-algebra $\mathcal{F}_\tau$ given by

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : \, A \cap \{\tau \leq t\} \in \mathcal{F}_t, \; \forall t \geq 0\} \, , \; t \in [0, T].$$

Definition 2.9 (Stopped process)**.** Given a stopping time τ and a stochastic process $(X_t)_{t\geq 0}$, the process $(X_t^\tau)_{t\geq 0}$ $X_t^\tau := (X_{\tau\wedge t})_{t\geq 0}$ is called stopped process and it has the following form

$$X_t^\tau = X_{\tau\wedge t} = X_t 1_{\{t<\tau\}} + X_\tau 1_{\{t\geq\tau\}},\ t \in [0, T].$$

Definition 2.10 (Càdlàg and càglàd)**.** A stochastic process $(X_t)_{t\geq 0}$ is said to be càdlàg, if its paths are right continuous with left limits (for every $t > 0$ and $s < t$, $X_{t-} = \lim_{s\to t} X_s$). Equivalently, a process $(X_t)_{t\geq 0}$ is càglàd, if its paths are left continuous with right limits (for all $t \in [0, T]$ and $s > t$, $X_{t+} = \lim_{s\to t} X_s$).

Martingales play fundamental role in stochastic calculus. Martingales have constant expectation. Bellow, we provide their definition.

Definition 2.11 (Martingale)**.** A stochastic process $(X_t)_{t\geq 0}$ adapted to the filtration $\mathbb{F}$ is a martingale (respectively, supermartingale, and submartingale) if and only if :

- $\mathbb{E}[|X_t|] < \infty$, for every $t \in [0, T]$.
- For all $t \geq 0$, and $s \leq t$ then $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ a.s (respectively $\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s$, and $\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s$).

Definition 2.12 (Local martingale)**.** A stochastic process $(X_t)_{t\geq 0}$ is a local martingale if there exists an increasing sequence of stopping times $\{\tau_n\}_{n=1}^{+\infty}$, such that the stopped process $(X_t^{\tau_n})_{t\geq 0}$ remains a local martingale for all n.

Remark 2.13. Following Jacod and Shiryaev (2003), Chapter I, Theorem 4.18, it is possible to show that every local martingale $(M_t)_{t\geq 0}$ can be decomposed uniquely as $M_t = M_0 + M_t^c + M_t^d$, where $(M_t^c)_{t\geq 0}$ and $(M_t^d)_{t\geq 0}$ are respectively the continuous and the discontinuous local martingale parts of $(M_t)_{t\geq 0}$ and $M_0 = M_0^c = M_0^d = 0$

Remark 2.14. Note also that every martingale is a local martingale. However, a local martingale is not necessarily a martingale.

Having defined the càdlàg and càglàd processes, let us introduce the optional and predictable σ-fields. We borrow their definitions from Jacod and Shiryaev (2003), Chapter I. The notion of optional and predictable σ-fields play fundamental role in projections and the theory of filtration expansion, as we will later see in Chapter 5. We start with the definition of the optional σ-field.

Definition 2.15 (Optional σ-field)**.** The optional σ-field $\mathscr{O}$ on $\Omega \times \mathbb{R}_0^+$ is generated by all the non-anticipating, predictable and right continuous processes. Furthermore, a process or a random set which is $\mathscr{O}$-measurable is called optional.

Definition 2.15 does not incorporate the full-capacity of optional processes. The Proposition bellow introduces alternatives but fundamental ways to introduce optional processes. For its proof, we refer to Jacod and Shiryaev (2003) Chapter I, Propositions 1.21, 1.23 and 1.24.

Proposition 2.16. *Let $(X_t)_{t\geq 0}$ be an optional process. Then the following are equivalent*

1. *If τ is a stopping time then the stopped process $(X_{\tau\wedge t})_{t\geq 0}$ is also optional.*
2. *If τ and ς are two stopping times and Y is and $\mathcal{F}_\varsigma$-measurable. Then the processes $Y1_{[\varsigma,\tau]}$, $Y1_{[\varsigma,\tau)}$, $Y1_{(\varsigma,\tau]}$ and $Y1_{(\varsigma,\tau)}$ are optional.*
3. *Every process $(X_t)_{t\geq 0}$ that is adapted and càdlàg is also optional.*

Definition 2.17 (Predictable σ-field)**.** The predictable σ-field $\mathscr{P}$ on $\Omega \times \mathbb{R}_0^+$ is generated by all the non-anticipating and adapted left continuous processes. A process which is measurable with respect to $\mathscr{P}$ is called a predictable.

We introduce the classes of martingales and local martingales, which will be used extensively throughout this thesis.

Definition 2.18 ($\mathcal{M}$ and $\mathcal{M}^2$ space)**.** We define as $\mathcal{M}$ the space of martingales, and $\mathcal{M}_{loc}$ as the space of all the local martingales. We also denote as $\mathcal{M}^2$ the space of all square integrable martingales, that is all martingales $(X_t)_{t\geq 0}$ such that $\sup_{t\in\mathbb{R}_+} \mathbb{E}[X_t^2] < \infty$. Similarly, $\mathcal{M}_{loc}^2$ is the space of all the locally square-integrable martingales.

2.3 Semimartingales

In this section, we study some general properties of semimartingales. A semimartingale is a process which can be expressed as the sum of a local martingale and a finite variation process. The concept of compensators is also introduced.

To formalize our analysis of semimartingales, we first introduce some important classes of predictable processes, see also Jacod and Shiryaev (2003).

We define $Var[X]_t$ to be the variation process of $(X_t)_{t\geq 0}$, that is the process such that $Var[X]_t(\omega)$ is the total variation [1] of the function $s \to X_s(\omega)$ on the interval $[0, t]$.

Definition 2.19. We define $\mathcal{V}$ as the set of all the real valued- càdlàg processes $(\Lambda_t^X)_{t\geq 0}$, with $\Lambda_0^X = 0$ and their paths $t \to \Lambda_t^X(\omega)$ have finite variation over a finite interval $t \in [0, T]$.

Definition 2.20. The set $\mathcal{A}$ is the set of all the processes $(\Lambda_t^X)_{t\geq 0} \in \mathcal{V}$ such that they have integrable variation i.e. $\mathbb{E}[Var(\Lambda^X)_\infty] < \infty$. Similarly, the set $\mathcal{A}_{loc}$ is defined as the localized class of $\mathcal{A}$. In other words, a process $(\Lambda_t^X)_{t\geq 0}$, $(\Lambda_t^X)_{t\geq 0} \in \mathcal{A}_{loc}$, if its paths have locally integrable variation.

Definition 2.21 (Semimartingale)**.** A (càdlag) process $(X_t)_{t\geq 0}$ is a **semimartingale** if and only if it can be decomposed as a sum of two predictable processes i.e.

$$X_t = X_0 + M_t^X + \Lambda_t^X,$$

[1] The total variation of a function g on interval $[a, b]$ is given as $Var[g] = \sup \sum_{i=1}^n |g(t_i) - g(t_{i-1})|$, where the supremum is taken all over the partitions of $[a, b]$ i.e. $a = t_0 < t_1 < \ldots < t_n = b$.

where $(M_t^X)_{t\geq 0}$ is a local martingale, with $M_0^X = \Lambda_0^X = 0$ and $(\Lambda_t^X)_{t\geq 0} \in \mathcal{V}$.

Note that in general the above decomposition is not unique since the predictable part $(\Lambda_t^X)_{t\geq 0}$ may not be predictable. A typical example is the Cauchy process which a pure jump Lévy process (see Section 2.4) and it does not have a finite mean nor finite variation. In order to verify this, note that the Cauchy distribution is $f(x) = \frac{1}{\pi}\frac{1}{x^2+1}$ and so its second moment $\int \frac{x^2}{1+x^2}dx \to 1$ as $x \to \infty$. Thus the decomposition of the Cauchy process into the local martingale and the finite variation part contains strictly large jumps and therefore its finite variation part is not predictable.

For any semimartingale we define its jumps as $\Delta X_t = X_t - X_{t-}$.

Definition 2.22 (Special semimartingale)**.** A semimartingale $(X_t)_{t\geq 0}$ is called **special semimartingale,** if it admits the **canonical decomposition** $X_t = X_0 + M_t^X + \Lambda_t^X$ with $M_0^X = \Lambda_0^X = 0$, where $(M_t^X)_{t\geq 0}$ is a local martingale, $(\Lambda_t^X)_{t\geq 0} \in \mathcal{V}$ and $(\Lambda_t^X)_{t\geq 0}$ is a **predictable and unique** process.

2.3.1 Stochastic Integration for semimartingales and quadratic variation

In this section, we introduce two objects that have a key role in our analysis: the quadratic variation process and stochastic integration. As we will later see in Chapter 3, Section 3.4, the value process of a trading strategy is expressed through a stochastic integral of a predictable process with respect to Lévy process. Generally speaking, in quadratic hedging approach the hedging strategy is in fact an orthogonal projection of the contingent claim onto the linear subspace of hedgeable portfolios. The notion of orthogonality is related with the quadratic covariation, i.e. we say that two semimartingales $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are orthogonal if and only if their covariation process $([X,Y]_t)_{t\geq 0}$ is a uniformly integrable martingale. Therefore quadratic variation and stochastic integration are essential in the theory of the local risk minimization. Note that the quadratic variation process is expressed in a unique way and therefore every semimartingale admits a quadratic variation process.

We start by briefly analysing the concept of stochastic integration for simple processes. For the general theory of stochastic integration for semimartingales, we refer to Protter (2004), Chapter II, or Jacod and Shiryaev (2003), Chapter 1, Section 4.

Let $(H_t)_{t\geq 0}$ be a simple $\mathbb{F}$-predictable process which can be represented as

$$H_t = H_0 1_{t=0} + \sum H_i 1_{(T_i,T_{i+1}]},$$

where $0 = T_0 < T_1 < T_{2\ldots} < T_n = T$ is a partition of $[0,T]$ and each H_i is a random variable $\mathcal{F}_{T_i}$ measurable, then the stochastic integral with respect to a semimartingale $(X_t)_{t\geq 0}$ is given by

$$\int_0^T H_s dX_s = H_0 X_0 + \sum_{i=0}^{n} H_i (X_{T_{i+1}} - H_{T_i}).$$

The same result holds true for a sequence of simple processes $(H^k_t)_{t\geq 0}$ such that $\sup_{(t,\omega)\in[0,T]\times\Omega}|H^k_t(\omega) - H_t(\omega)| \to 0$ as $k \to \infty$ and so $\int_0^T H^k_s dS_s \underset{k\to\infty}{\to} \int_0^T H_s dS_s$. Then we are able to approximate every càdlàg process by a sequence of simple ones and so its stochastic integral is well defined, for more details, we refer to Cont and Tankov (2004), Chapter 8.

A quite useful result which will be used quite often in this thesis is how we can decompose a stochastic integral for a semimartingale. From the Doob-Meyer decomposition we saw that a semimartingale is synthesized from a local-martingale and a finite variation process. Similar result we can get for the stochastic integration for semimartingales. For more details, we refer to Klebaner (2004), Section 8.4.

Let $(X_t)_{t\geq 0}$, be a semimartingale. Let $(C_t)_{t\geq 0}$, be a predictable process satisfying the following conditions

- $\sqrt{\int_0^t C_s^2 d\,[M^X]_s}$, where $[M^X]_t$ is the quadratic variation process of $(M^X_t)_{t\geq 0}$ (see also Definition 2.25).
- $\int_0^t |C_s| d\,Var[\Lambda^X]_s < \infty$, where $Var[\Lambda^X]_t$ is the variation process of $(\Lambda^X_t)_{t\geq 0}$.

Then the stochastic integral for a semimartingale can be decomposed as

$$\int_0^t C_s dX_s = \int_0^t C_s dM^X_s + \int_0^t C_s d\Lambda^X_s,\ t \in [0,T].$$

Remark 2.23. If τ is a stopping time then the stochastic integral for a stopped process is

$$\int_0^{\tau\wedge t} C_s dX_s = \int_0^t C_s 1_{\{s\leq\tau\}} dX_s = \int_0^t C_s dX_{\tau\wedge s}.$$

Definition 2.24 (Space of semimartingales)**.** We define $\mathcal{S}(\mathbb{P})$ to be the space of semimartingales and $\mathcal{S}^2(\mathbb{P})$ the space of square integrable semimartingales.

Definition 2.25 (Quadratic variation)**.** The quadratic variation process $([X_t])_{t\geq 0}$ of a semimartingale $(X_t)_{t\geq 0}$ is the càdlàg process, which is given by

$$[X]_t = (X_t)^2 - 2\int_0^t X_{s-} dX_s,\ \ t \in [0,T].$$

One way to calculate the quadratic covariation is through integration by parts. The following Corollary provides its definition, see also Protter (2004), Chapter II.

Corollary 2.26 (Integration by parts formula)**.** *Let again $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be semimartingales. Then their product $(X_tY_t)_{t\geq 0}$ is also a semimartingale and it yields*

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X,Y]_t,\ t \in [0,T].$$

The next Proposition investigates the properties of quadratic variation, for its proof, we refer to Jacod and Shiryaev (2003), Chapter I, Proposition 4.49.

Proposition 2.27. *Assume that* $(X_t)_{t\geq 0}$ *is a semimartingale and* $(Y_t)_{t\geq 0} \in \mathcal{V}$*, then the following holds*

1. *The quadratic variation process* $([X]_t)_{t\geq 0}$ *will be* $[X]_t = \int_0^t \Delta X_s dY_s$*, and* $X_tY_t = \int_0^t Y_{s-}dX_s + \int_0^t X_s dY_s$*.*

2. *If* $(Y_t)_{t\geq 0}$ *is predictable then the quadratic covariation process* $([X,Y]_t)_{t\geq 0}$ *takes the form* $[X,Y]_t = \int_0^t \Delta Y_s dX_s$*.*

3. *If* $(Y_t)_{t\geq 0}$ *is predictable process and* $(X_t)_{t\geq 0}$ *is a local martingale, then* $([X,Y]_t)_{t\geq 0}$ *is also a local martingale.*

4. *If* $(X_t)_{t\geq 0}$*, or* $(Y_t)_{t\geq 0}$ *is a continuous process then* $([X,Y]_t)_{t\geq 0} = 0$*.*

Definition 2.28. Denote by $([X]_t^c)_{t\geq 0}$ the continuous part of $([X]_t)_{t\geq 0}$. Then it yields

$$\begin{aligned}[X]_t &= [X]_t^c + X_0^2 + \sum_{s\leq t}(\Delta X_s)^2 \\ &= [X]_t^c + \sum_{s\leq t}(\Delta X_s)^2, \quad t\in[0,T].\end{aligned}$$

2.3.2 Compensators and predictable quadratic variation

An another vital tool which will be used throughout this dissertation is the notion of predictable quadratic variation. As we will later see in Chapter 3, Section 3.4, the hedging strategy in the local risk minimization approach can be expressed though the Galtchouk-Kunita-Watanabe (GKW) decomposition using the predictable quadratic covariation process of the value process and the underlying asset i.e. $\theta_t = \frac{d\langle V,X\rangle_t}{d\langle X\rangle_t}$. In contrast to the quadratic variation process which always exists, the predictable quadratic variation is not necessarily exists. However, if we assume that the underlying asset $(X_t)_{t\geq 0}$ and the value of our portfolio $(V_t)_{t\geq 0}$ are a square integrable martingales and the quadratic covariation of the value of the contingent claim and the underlying asset belongs to $\mathcal{A}_{loc}$ i.e. $[V,X]_t \in \mathcal{A}_{loc}$ then the predictable quadratic covariation process exists see Proposition 2.33. This result will be extensively used in Chapters 4, 6 and 7. First let us provide the definition of a compensator.

Definition 2.29 (Compensator)**.** Assume that $(\Lambda_t^X)_{t\geq 0}$ is a finite variation process with $\Lambda_0^X = 0$, and $(\Lambda_t^X)_{t\geq 0} \in \mathcal{A}_{loc}$. A predictable process $(\Lambda_t^p)_{t\geq 0}$ is called compensator of $(\Lambda_t^X)_{t\geq 0}$ if and only if the process $(\Lambda_t^X - \Lambda_t^p)_{t\geq 0}$ is a local martingale.

The following Theorem provides an essential result for the compensators, see Jacod and Shiryaev (2003), Chapter I, Theorem 3.18.

Theorem 2.30. *For each predictable process* $(H_t)_{t\geq 0}$ *such that* $\int_0^t H_s d\Lambda_s^X \in \mathcal{A}_{loc}$ *then* $\int_0^t H_s d\Lambda_s^p \in \mathcal{A}_{loc}$ *and* $\int_0^t H_s d\Lambda_s^p = \left(\int_0^t H_s d\Lambda_s^X\right)^p$*. Furthermore, the process*

$$\left(\int_0^t H_s d\Lambda_s^X - \int_0^t H_s d\Lambda_s^p\right)_{t\geq 0}, \; t\in[0,T],$$

is a local martingale.

Definition 2.31. The predictable quadratic variation process (or sharp bracket process) $(\langle X\rangle_t)_{t\geq 0}$ of a semimartingale $(X_t)_{t\geq 0}$ is the compensator of $([X]_t)_{t\geq 0}$. That is the unique predictable process that makes the process $([X]_t - \langle X\rangle_t)_{t\geq 0}$ into a local martingale.

An equivalent characterization of the predictable quadratic variation is introduced in the next Theorem. For its proof, we refer to Jacod and Shiryaev (2003), Chapter I, Theorem 4.2.

Theorem 2.32. *Let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ are square integrable local martingales. Then the predictable quadratic covariation $(\langle M,N\rangle_t)_{t\geq 0}$ is the unique predictable process such that the process $(M_tN_t - \langle M,N\rangle_t)_{t\geq 0}$ is a local martingale. Moreover,*

$$\langle M,N\rangle_t = \frac{1}{4}\left(\langle M+N,M+N\rangle_t - \langle M-N,M-N\rangle_t\right)$$

Note that if $(X_t)_{t\geq 0}$ is continuous semimartingale (such that $\Delta X_t = 0$) and $X_0 = 0$, then $[X]_t = \langle X^c\rangle_t = \langle X\rangle_t^c = [X^c]_t = [X]_t^c$, with $t\in[0,T]$. To see this, since $(X_t)_{t\geq 0}$ is continuous semimartingale then its conditional quadratic variation is also continuous. Thus $([X]_t)_{t\geq 0}$ is a predictable process and from the Doob-Meyer decomposition the local-martingale part is zero and so $[X]_t = \langle X\rangle_t$.

The next Proposition investigates a case under which the predictable quadratic variations exists, see Jacod and Shiryaev (2003), Chapter I, Proposition 4.50.

Proposition 2.33. *Assume that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are locally square integrable martingales. Then $([X,Y]_t)_{t\geq 0}\in\mathcal{A}_{loc}$ and therefore its compensator $(\langle X,Y\rangle_t)_{t\geq 0}$ exists.*

Example 2.34. Let $(N_t)_{t\geq 0}$ be a Poisson process. Then $(\Delta N_t)^2 = \Delta N_t$ and its quadratic variation

$$[N]_t = \sum_{s\leq t}(\Delta N_s)^2 = \sum_{s\leq t}1 = \sum_{s\leq t}\Delta N_s = N_t.$$

In order to find the predictable quadratic variation process $(\langle N\rangle_t)_{t\geq 0}$ we know that the process $[N]_t - t = N_t - t$ is a martingale. Therefore the compensator of $([N,N]_t)_{t\geq 0}$ is $\langle N,N\rangle_t = t$. More generally, assume that $X_t = X_0 + \mu t + \sum_{i=1}^{N_t} Y_i$ i.e. $(X_t)_{t\geq 0}$ is a compound Poisson process, where Y_i are i.i.d. **random variables** with a given distribution function f_Y. In this case, its quadratic variation $([X]_t)_{t\geq 0}$ is

$$\begin{aligned}[X]_t &= X_0^2 + \sum_{i=1}^{N_t}|Y_i|^2 = X_0^2 + \sum_{s\leq t}|\Delta X_s|^2 \\ &= X_0^2 + \lambda\int_0^t\int_{\mathbb{R}} z^2 f_Y(z)dz, \quad t\in[0,T],\end{aligned}$$

so the process $([X]_t - X_0^2 - \lambda\int_0^t\int_{\mathbb{R}} z^2 f_Y(z)dz)_{t\geq 0}$ is a martingale. Therefore the compensator of $([X]_t)_{t\geq 0}$ i.e. the predictable quadratic variation $(\langle X\rangle_t)_{t\geq 0}$ with $t\in[0,T]$

will be

$$\langle X \rangle_t = X_0^2 + \lambda \int_0^t \int_{\mathbb{R}} z^2 f_Y(z) dz, \ t \in [0,T].$$

Notation 2.35. Throughout this thesis we use $[X]_t^{\mathbb{F}}$ or $[X]_t^{\mathbb{G}}$, with $t \in [0,T]$, (equivalently $\langle X \rangle_t^{\mathbb{F}}$and $\langle X \rangle_t^{\mathbb{G}}$) to be the quadratic (equivalently the predictable quadratic variation) under filtrations $\mathbb{F}$ and $\mathbb{G}$. Since the quadratic variation or covariation is defined path-wise, the superscript could be removed; however, a predictable quadratic variation depends on the filtration, and to avoid ambiguity, a superscript is required in this case. To see why the predictable quadratic variation depends on the filtration, let us consider the case when $(X_t)_{t\geq 0}$ is simply a Brownian motion, i.e. $X_t = W_t$ then we know that

$$[W]_t = \lim_{|\Pi|\to 0} \sum_{i=1}^{N} (W_{t_{i+1}} - W_{t_i})^2 \tag{2.1}$$

where Π is the partition of $t \in [0,T]$ and

$$\langle W \rangle_t = \lim_{|\Pi|\to 0} \sum_{i=1}^{N} \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2 \mid \mathcal{F}_{t_i}] \tag{2.2}$$

We know that as $\Pi \to 0$, then for the Brownian motion we have $[W]_t = \langle W \rangle_t = t$. In general, comparing Equations (2.1) and (2.2) we know that the predictable quadratic variation depends on the filtration.

2.4 Lévy processes

There are several reasons why we need to allow jumps in the underlying asset and not to restrict ourselves in diffusion processes. In real world, market completeness, which simply states that the underlying asset is redundant in pricing and hedging of contingent claims, is not robust from financial point of view. The underlying asset may have abnormal vibrations (jumps) which is due to a new and important information that arrives from the market and it has an prominent effect on the evolution of the underlying asset. Thus there are various risks such that we cannot consider perfect hedging strategies. Moreover, in credit risk the default time can be occurred suddenly and diffusion models are not able to capture this event, since in this case the underlying asset approaches the default barrier continuously. So, Lévy processes seems to be the appropriate way for modelling the underlying asset.

In this section, we provide some fundamental results for Lévy processes. Lévy processes are $\mathbb{R}^d$-valued processes with independent and stationary increments. We provide some important results for Lévy processes along with the corresponding Itô's formula. We start with their definition.

Definition 2.36 (Lévy process)**.** A càdlàg process is a Lévy process if the following conditions are satisfied

1. $(X_t)_{t\geq 0}$ has independent increments i.e. $X_{t+s} - X_s$ is independent of $\mathcal{F}_s$, for all s, t in $[0, T]$ with $t > s$.

2. $(X_t)_{t\geq 0}$ has stationary increments i.e. for any s, t in $[0, T]$ the distribution of $X_{t+s} - X_t$ and X_s have the same law.

3. $(X_t)_{t\geq 0}$ is stochastically continuous i.e. for every $t \in [0, T]$, $s \leq t$ and, $\varepsilon > 0$, $\lim_{s\to t} \mathbb{P}\left[|X_t - X_s| \geq \varepsilon\right] = 0$.

Definition 2.37 (Infinite divisibility)**.** A real-valued **random variable** X has infinitely divisible distribution F if for any $n \geq 1$ there is a sequence of i.i.d. random variables $X_1^{(1/n)}$, $X_2^{(1/n)}, \ldots,$ $X_n^{(1/n)}$ such that $X \stackrel{d}{=} X_1^{(1/n)} + X_2^{(1/n)} + \ldots + X_n^{(1/n)}$, where d represents the equality under the given distribution F.

Another way to define infinite divisibility is through convolutions. Following Sato (1999) given a law P_X the real valued random variable is infinitely divisible if $\forall n$ $P_X = P_{X_1^{(1/n)}} \times P_{X_2^{(1/n)}} \times \ldots \times P_{X_n^{(1/n)}}$ is the n-convolution of P_X. Therefore the infinitely divisible distribution can also be defined as a distribution F, where the n-th convolution remains a distribution. Typical examples of infinitely divisible distributions are Gaussian, Cauchy, Poisson, exponential and geometric distribution.

The following Theorem provides a characterization of infinite divisible distributions through characteristic functions, see Sato (1999), Theorem 8.1.

Theorem 2.38 (Lévy-Khintchine formula)**.** *Let P_X be the law of a* ***random variable*** *X. Then P_X is infinitely divisible if and only if there exists a triplet (b, σ^2, ν) with $b \in \mathbb{R}$, $\sigma \geq 0$ and ν is the measure which satisfies the following condition*

$$\nu(\{0\}) = 0 \ \ \textit{and} \ \ \int_{\mathbb{R}} \left(1 \wedge |z^2|\right) \nu(dz) < \infty,$$

such that

$$\int_{\mathbb{R}} e^{iuz} P_X(dz) = e^{-\Psi(z)}, \ z \in \mathbb{R},$$

where

$$\Psi(u) = \exp\left(ibu - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz1_{\{|z|\leq 1\}})\nu(dz)\right). \tag{2.3}$$

The triplet (b, σ^2, ν) is called Lévy triplet, where ν is the Lévy measure, b is the drift term and $\sigma \geq 0$ is the Gaussian component.

Jump and Lévy measures have a key role for studying Lévy processes. The next Definition introduces the jump measure. See also Papapantoleon (2008).

[Jump measure] Let $A \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ such that $0 \notin \bar{A}$, and let $0 \leq t \leq T$. We define the jump measure of the Lévy process $(X_t)_{t\geq 0}$ as follows

$$\begin{aligned} N(\omega; t, A) &= \#\left\{0 \leq s \leq t : \, (X_s(\omega) - X_{s-}(\omega)) \in A\right\} \\ &= \sum_{s\leq t} 1_A(\Delta X_s(\omega)), \ t \in [0, T]. \end{aligned}$$

We can see that the above equation forms a counting measure and it counts the number of jumps of the process $(X_t)_{t\geq 0}$ on the set A for $t \in [0,T]$.

Definition 2.39 (Lévy measure)**.** Given a Lévy process $(X_t)_{t\geq 0}$, then the Lévy measure ν is given by

$$\begin{aligned}\nu(A) := \mathbb{E}[N(1,A)] &= \int_{\omega\in\Omega} N(\omega;1,A)d\mathbb{P}(\omega)\\ &= \mathbb{E}[\sum_{s\leq t} 1_A(\Delta X_s(\omega))], \;\; \forall A \in \mathcal{B}(\mathbb{R}\setminus\{0\}),\; t\in[0,T].\end{aligned}$$

A fundamental result of Lévy processes is the Lévy-Itô decomposition which states that every Lévy process can be decomposed into a continuous and a jump part. The theorem bellow characterize this, see Papapantoleon (2008), Theorem 6.1.

Theorem 2.40 (Lévy-Itô decomposition)**.** *Consider a triplet (b, σ^2, v) where $b \in \mathbb{R}$, $\sigma \geq 0$ and ν is the Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which four independent processes exist $(X_t^{(1)})_{t\geq 0}$, $(X_t^{(2)})_{t\geq 0}$, $(X_t^{(3)})_{t\geq 0}$, $(X_t^{(4)})_{t\geq 0}$,with $t \in [0,T]$, where $(X_t^{(1)})_{t\geq 0}$ is a constant drift $(X_t^{(2)})_{t\geq 0}$ is a Brownian motion, $(X_t^{(3)})_{t\geq 0}$ is a compound Poisson process and $(X_t^{(4)})_{t\geq 0}$ is a square integrable (pure jump) martingale with an a.s countable number of jumps with magnitudes less than 1, on each finite interval. Taking $X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)} + X_t^{(4)}$ we have that there exists a probability space on which $(X_t)_{t\geq 0}$ is a Lévy process with characteristic exponent*

$$\psi(u) = iub - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}}(e^{iuz} - 1 - iuz1_{\{|z|\leq 1\}})\nu(dz), \;\; \forall u \in \mathbb{R}.$$

In other words, every Lévy process $(X_t)_{t\geq 0}$ can be decomposed to the following form

$$X_t = bt + \sigma W_t + \int_0^t \int_{|z|>1} zN(ds,dz) + \int_0^t \int_{|z|\leq 1} z\tilde{N}(ds,dz), \;\; t\in[0,T], \tag{2.4}$$

where $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is the compensated jump measure, $(W_t)_{t\geq 0}$ represents the Brownian motion and $N(dt,dz)$ is the jump measure on.

Clearly, the Lévy-Itô decomposition verifies that every Lévy process is also a semi-martingale.

The next Theorem investigates whether or not the process $(X_t)_{t\geq 0}$ has a finite activity, see Sato (1999), Theorem 21.3, or Proposition 7.1 of Papapantoleon (2008).

Theorem 2.41. *Let $(X_t)_{t\geq 0}$ be a Lévy process with triplet (b, σ^2, ν)*

- *if $\nu(\mathbb{R}) < \infty$ then the paths of $(X_t)_{t\geq 0}$, with $0 \leq t \leq T$ have a finite number of jumps on every compact interval a.s. In this case, $(X_t)_{t\geq 0}$ has finite activity.*
- *if $\nu(\mathbb{R}) = \infty$ then the paths $(X_t)_{t\geq 0}$ with $0 \leq t \leq T$ have a infinite number of jumps on every compact interval a.s. In this case, $(X_t)_{t\geq 0}$ has infinite activity.*

Similarly, a Lévy process has paths of finite variation if and only if there is no Brownian motion part and the Lévy measure satisfies $\int_{|z|\leq 1}|z|\nu(dz) < \infty$. For the following Theorem we refer to Sato (1999), Theorem 21.9, or Papapantoleon (2008), Proposition 7.2.

Theorem 2.42. *Let $(X_t)_{0\leq t\leq T}$ be a Lévy process with triplet (b,σ^2,ν).*

- *If $\sigma = 0$ and $\int_{|z|\leq 1}|z|\nu(dz) < \infty$, then almost all paths of $(X_t)_{t\geq 0}$ for every $t\in[0,T]$ have finite variation.*
- *If $\sigma \neq 0$ or $\int_{|z|\leq 1}|z|\nu(dz) = \infty$, then almost all paths of $(X_t)_{t\geq 0}$ for every $t\in[0,T]$ have infinite variation.*

An emphasis to some particular types of Lévy processes should be given. We start with the finite variation Lévy process. The following Lemma introduces the finite variation process, see also Kyprianou (2014), Lemma 2.12.

Lemma 2.43 (Finite Variation)**.** *Let $(X_t)_{t\geq 0}$ be a Lévy process of **finite variation** with Lévy triplet $(b,0,\nu)$, $t\in[0,T]$, then its Lévy-Itô decomposition has the form*

$$X_t = \mu t + \int_0^t \int_{\mathbb{R}} zN(ds,dz) = \mu t + \sum_{s\leq t}\Delta X_s,$$

for every $0\leq t\leq T$ with $\mu = b - \int_{-1}^{1} z\nu(dz)$ and its Lévy-Khintchine formula is

$$\mathbb{E}\left[e^{iuX_t}\right] = \exp\left[t\left(iu\mu + \int_{\mathbb{R}}(e^{iuz}-1)\nu(dz)\right)\right].$$

Remark 2.44. In the above Lemma 2.43, note that the Lévy triplet is not given by $(\mu,0,\nu)$ instead we still use $(b,0,\nu)$.

Definition 2.45 (Subordinator)**.** A Lévy process $(X_t)_{t\geq 0}$ is called **subordinator** if and only if $X_t(\omega)$, $\omega\in\Omega$, is an increasing function of $t\in[0,T]$.

An immediate consequence of subordinators is the following Lemma, see Kyprianou (2014), Lemma 2.14.

Lemma 2.46. *A Lévy process $(X_t)_{t\geq 0}$ is a subordinator if and only if $\sigma = 0$, $\nu(-\infty,0) = 0$, $\int_{[0,1]} z\nu(dz) < \infty$ and $\mu = -\left(b + \int_0^1 z\nu(dz)\right) \geq 0$.*

In light of Theorem 2.40 and Corollary 2.43 we provide the following processes. We also refer to Papapantoleon (2008).

- A Lévy process with Lévy triplet (b,σ^2,ν) is called **spectrally negative**, if it has no positive jumps, equivalently $\nu(0,\infty) = 0$. Its Lévy-Itô decomposition has the form

$$X_t = bt + \sigma W_t + \int_0^t \int_{z<-1} zN(ds,dz) + \int_0^t \int_{-1<z<0} z\tilde{N}(ds,dz),\ 0\leq t\leq T,$$

and its Lévy-Khintchine formula has the form

$$\mathbb{E}[e^{iuX_t}] = \exp\left[t\left(iub - \frac{u^2\sigma^2}{2} + \int_{-\infty}^{0}(e^{iuz} - 1 - iuz1_{\{z>-1\}})\nu(dz)\right)\right].$$

If $(X_t)_{t\geq 0}$ is a spectrally negative with paths of finite variation, then it follows that $X_t = \mu t - S_t$, where $\mu = -b + \int_{-1}^{0} z\nu(dz) > 0$ and $(S_t)_{t\geq 0}$ is a driftless subordinator.

- Similarly, a Lévy process is called **spectrally positive**, if $\nu(-\infty, 0) = 0$. In this case the Lévy-Itô decomposition will be

$$X_t = bt + \sigma W_t + \int_0^t \int_{z>1} zN(ds, dz) + \int_0^t \int_{0<z<1} z\tilde{N}(ds, dz),\ 0 \leq t \leq T,$$

and the Lévy-Khintchine formula is

$$\mathbb{E}[e^{iuX_t}] = \exp\left[t\left(iub - \frac{u^2\sigma^2}{2} + \int_{0}^{+\infty}(e^{iuz} - 1 - iuz1_{\{z>1\}})\nu(dz)\right)\right].$$

If $(X_t)_{t\geq 0}$ is a spectrally positive with paths of finite variation, then it follows that $X_t = -\mu t + S_t$, where $\mu = -b + \int_0^1 z\nu(dz) < 0$ and $(S_t)_{t\geq 0}$ is again a driftless subordinator.

- A jump diffusion Lévy process $(X_t)_{t\geq 0}$ with Lévy triplet (b, σ^2, ν) has **jumps of finite variation** if and only if $\int_{|z|\leq 1} |z|\nu(dz) < \infty$. Its Lévy-Itô decomposition resumes the following form

$$X_t = \mu t + \sigma W_t + \int_0^t \int_{\mathbb{R}} zN(ds, dz),\ t \in [0, T],$$

with $\mu = b - \int_{-1}^{1} z\nu(dz) < \infty$, and the Lévy-Khintchine formula takes the form

$$\mathbb{E}[e^{iuX_t}] = \exp\left[t\Big(iu\mu - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}}(e^{iuz} - 1)\nu(dz)\Big)\right].$$

Remark 2.47. Based on the Lemma 2.43 and the definition of the subordinators, then any Lévy process of finite variation $(X_t)_{t\geq 0}$ can be written as the difference of two independent driftless subordinators

$$X_t = \mu t + S_t - S'_t,\ t \in [0, T],$$

where $\mu = \mu^+ + \mu^-$, $\mu \in \mathbb{R}$ and $(S_t)_{t\geq 0}$, $(S'_t)_{t\geq 0}$ are two independent driftless subordinators. Please, see Kyprianou (2014), Exercise 2.8.

2.4.1 Itô's lemma for semimartingales and Lévy processes

In this section, we briefly provide the main results of Itô's formula for semimartingales. We start with a simple version where the process $(X_t)_{t\geq 0}$ has paths of finite variation, see Protter (2004), Chapter II, Theorem 31.

Theorem 2.48 (Change of variables)**.** *Let $(X_t)_{t\geq 0}$ be a semimartingale with paths of finite variation and $t \in [0,T]$. We assume that the function $f(x) \in C^1(\mathbb{R})$. Then the process $(f(X_t))_{t\geq 0}$ is also a finite variation process and its form is given by*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{s\leq t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\} \\ &= f(X_0) + \int_0^t f'(X_{s-})dX_s^c + \sum_{s\leq t}\{f(X_s) - f(X_{s-})\}, \ \ t \in [0,T], \end{aligned}$$

where $(X_t^c)_{t\geq 0}$ is the continuous martingale part of $(X_t)_{t\geq 0}$.

For a general semimartingale we have the following Theorem, see Protter (2004), Chapter II, Theorem 32.

Theorem 2.49 (Itô's lemma for semimartingales)**.** *Let $(X_t)_{t\geq 0}$ be a semimartingale where $t \in [0,T]$ and let a function $f(x) \in C^2(\mathbb{R})$. Then the process $(f(X_t))_{t\geq 0}$ is also a semimartingale and it has the following form*

$$\begin{aligned} f(X_t) = f(X_0) &+ \int_0^t f(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d[X]_s^c \\ &+ \sum_{s\leq t}\{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\}, \ \ t \in [0,T]. \end{aligned}$$

For the multidimensional Itô's lemma for semimartingales we refer to Klebaner (2004), Section 8.10.

Based on Kyprianou (2014) we will briefly describe the Itô's lemma for Lévy processes.

Assume that $(X_t)_{t\geq 0}$ is a Lévy process of finite variation $X_t = \mu t + \int_0^t \int_{\mathbb{R}} zN(ds,dz), \ \ t \in [0,T]$, where $\mu = b - \int_{-1}^1 z\nu(dz) < \infty$, then we have the following Theorem.

Theorem 2.50. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function differentiable with respect to t and x i.e. $f(t,x) \in C^{1,1}([0,T] \times \mathbb{R})$ then for a finite variation Lévy process $(X_t)_{t\geq 0}$ it yields*

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \mu\int_0^t \frac{\partial f}{\partial x}(s,X_s)ds \\ &+ \int_0^t \int_{\mathbb{R}} \left(f(s,X_{s-}+z) - f(s,X_{s-})\right) N(ds,dz), \ t \in [0,T]. \end{aligned}$$

The above Theorem can be extended for a general Lévy process, see Papapantoleon (2008), or Jeanblanc et al. (2009) Section 11.2.4.

Theorem 2.51. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function differentiable with respect to $t \in [0,T]$ and twice differentiable with respect to x i.e. $f \in C^{1,2}([0,T] \times \mathbb{R})$.*

Then for a general Lévy process $(X_t)_{t\geq 0}$ *with triplet* (b, σ^2, ν) *of the form (2.4), we have*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-})dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-})d\,[X]_s^c$$
$$+ \int_0^t \int_{\mathbb{R}} (f(s, X_{s-} + z) - f(s, X_{s-}) - z\frac{\partial f}{\partial x}(s, X_{s-}))N(ds, dz), \quad t \in [0, T].$$

Equivalently, we get

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \sigma \int_0^t \frac{\partial f}{\partial x}(s, X_{s-})dW_s \\ &+ \int_0^t \int_{|z|\leq 1} (f(s, X_{s-} + z) - f(s, X_{s-}))\tilde{N}(ds, dz) \\ &+ \int_0^t \frac{\partial f}{\partial t}(t, X_s)ds + b\int_0^t \frac{\partial f}{\partial x}(s, X_s)ds + \frac{1}{2}\sigma^2 \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)ds \\ &+ \int_0^t \int_{|z|>1} (f(s, X_{s-} + z) - f(s, X_{s-}))N(ds, dz) \\ &+ \int_0^t \int_{|z|\leq 1} (f(s, X_{s-} + z) - f(s, X_{s-}) - z\frac{\partial f}{\partial x}(s, X_{s-}))\nu(dz)ds, \ t \in [0, T]. \end{aligned}$$

Note that the same result holds true for the multidimensional Itô's formula for Lévy processes, see for instance Jacod and Shiryaev (2003), Chapter 2.

2.4.2 Compensation formula for Lévy integrals

The following Theorem can be found in Kyprianou (2014), Theorem 4.4.

Theorem 2.52. *Suppose that* $\psi : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}_0^+$ *is a function which satisfies*

1. $\psi(t, z)[\omega]$ *is measurable.*
2. $\forall t \in [0, T]$ $\psi(t, z)[\omega]$ *is* $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$*-measurable.*
3. *with probability one* $\forall x \in \mathbb{R}$ $\{\psi(t, x)[\omega] : t \in [0, T]\}$ *is a left continuous process.*

Then

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \psi(s, z)N(ds, dz)\right] = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \psi(s, z)\nu(dz)ds\right].$$

An immediate consequence of Theorem 2.52 is the following result. For its proof, we refer to Kyprianou (2014), Corollaries 4.5 and 4.6.

Corollary 2.53. *Under the same assumptions of Theorem 2.52 then for all* $u \leq t$ *with* $t \in [0, T]$ *we have*

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \psi(s, z)N(ds, dz) \mid \mathcal{F}_u\right] = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \psi(s, z)\nu(dz)ds \mid \mathcal{F}_u\right].$$

If we further assume $\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \psi(s, z)\nu(dz)ds\right] < \infty$*, then the process*

$$\left(\int_0^t \int_{\mathbb{R}} \psi(s, z)N(ds, dz) - \int_0^t \int_{\mathbb{R}} \psi(s, z)\nu(dz)ds\right)_{t\geq 0}, \quad t \in [0, T],$$

is a martingale.

Based on the Itô's-lemma for Lévy processes (see Theorem 2.51) and the compensation formula, we get the following Proposition. For its proof, we refer to Hilber et al. (2013), Chapter10, Proposition 10.3.1.

Proposition 2.54 (Dynkin's formula)**.** *Let $(X_t)_{t\geq 0}$ be a real-valued Lévy process (b, σ^2, ν) of the form (2.4). Assume that there exist a function $f(t,x) \in C^{1,1}([0,T] \times \mathbb{R})$ and we denote by $\mathcal{A}$ the following integro-differential operator*

$$\mathcal{A}f(t,x) = \frac{\partial f}{\partial t}(t,x) + b\frac{\partial f}{\partial x}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x) + \int_{\mathbb{R}} \left(f(t,x+z) - f(t,x) - z\frac{\partial f}{\partial x}(t,x) \right) \nu(dz),$$

for every $t \in [0,T]$. If we assume that the following integrability condition is satisfied

$$\int_{\mathbb{R}} |f(t,x+z) - f(t,x) - z\frac{\partial f}{\partial x}|\nu(dz) < \infty, \quad \forall t \in [0,T] \text{ and } x \in \mathbb{R},$$

then the process

$$\left(f(t,X_t) - f(0,X_0) - \int_0^t \mathcal{A}f(s,X_s)ds \right)_{t\geq 0}, \; t \in [0,T],$$

is a local martingale.

Proposition 2.54 will be used extensively throughout this thesis and especially in Chapters 4, 6 and 7. It allow us to construct appropriate PIDEs using the Itô's formula.

2.4.3 Reflected Lévy processes

When we need to determine the probability of default for a first passage model whose barrier is a random variable with a known prior distribution, we need to take into account the running infimum process of the underlying asset $(X_t)_{t\geq 0}$. When the underlying asset is a diffusion process then the probability of default can be calculated through the reflection principle. But for the general case when the underlying asset admits jumps the situation is slightly more complicated. As we will see in Chapter 6, under the assumption that the underlying asset is a Lévy process of finite variation, when the barrier is random with a given prior distribution, then its intensity process involves the running infimum of the underlying asset. A short of extension of the previous model is made in Chapter 7 under the jump diffusion case, where in this case, the default time is modelled through a hazard rate approach whose intensity is deterministic function with respect to the underlying asset $(X_t)_{t\geq 0}$ and its running infimum $(\underline{X}_t)_{t\geq 0}$.

Considering the discussion above reflected Lévy processes are key parts in the theory of credit risk modelling when investors have incomplete information regarding the default threshold. In this section, we introduce some fundamental results in the theory of the reflected Lévy process.

Let again $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space, and we assume that there exists a Lévy process given in Definition 2.36.

Definition 2.55 (Reflected Lévy process)**.** Let

$$
\begin{aligned}
&\bar{X}_t = \sup_{0\le s\le t} X_s, \quad \underline{X}_t = \inf_{0\le s\le t} X_s, \quad R_t = \bar{X}_t - X_t,\\
&\tilde{X}_t = -X_t, \qquad \tilde{M}_t = \sup_{0\le s\le t} \tilde{X}_s = -\underline{X}_t,\\
&\tilde{N}_t = \inf_{0\le s\le t} \tilde{X}_s = -\bar{X}_t, \quad \tilde{R}_t = \tilde{M}_t - \tilde{X}_t = X_t - \underline{X}_t.
\end{aligned}
$$

The processes $(\bar{X}_t)_{t\ge0}$, $(\underline{X}_t)_{t\ge0}$ and $(R_t)_{t\ge0}$, are respectively, the **supremum process**, the **infimum process** and t**he reflected Lévy process at the supremum**. Similarly, the processes $(\tilde{X}_t)_{t\ge0}$ and $(\tilde{R}_t)_{t\ge0}$ are the dual of $(X_t)_{t\ge0}$ and the **reflected Lévy process at the infimum**.

We can see that the $(\bar{X}_t)_{t\ge0}$ and $(-\underline{X}_t)_{t\ge0}$ are two non-negative and right continuous processes. Also the reflected Lévy process at the infimum $(X_t - \underline{X}_t)_{t\ge0}$, can be viewed as the dual process of the reflected Lévy process at the supremum $(\bar{X}_t - X_t)_{t\ge0}$. Let us investigate some crucial properties of reflected processes. We start with the Duality Lemma, see Kyprianou (2014), Lemma 3.4.

Lemma 2.56 (Duality lemma)**.** *The processes* $\left\{X_{(t-s)} - X_t :\ 0\le s\le t\right\}$ *and* $(\tilde{X}_t)_{t\ge0}$, $\tilde{X}_t = -X_t$ *have the same law under* $\mathbb{P}$.

A useful application of the Duality lemma is that it provides the relationship between the supremum and infimum. The proof of the following Lemma can be found in Kyprianou (2014), Lemma 3.5.

Lemma 2.57. *For fixed* $t\in[0,T]$*, then the pairs* $(\bar{X}_t, \bar{X}_t - X_t)_{t\ge0}$ *and* $(X_t - \underline{X}_t, -\underline{X}_t)_{t\ge0}$*, have the same law under* $\mathbb{P}$.

Assuming that $(X_t)_{t\ge0}$ is a spectrally negative Lévy process of finite variation, then based on the Itô's lemma we get the following result, see Kyprianou (2014), Exercise 4.2.

Corollary 2.58. *Assume that* $(X_t)_{t\ge0}$ *is a spectrally negative Lévy process of finite variation with* $0\le t\le T$ *and* $\bar{X}_t = \sup X_s$*. Since* $(\bar{X}_t)_{t\ge0}$ *is a continuous and non decreasing process, then for a function* $f(y,x)\in C^{1,1}(\mathbb{R}_0^+\times\mathbb{R})$ *we get*

$$
\begin{aligned}
f(\bar{X}_t, X_t) = f(\bar{X}_0, X_0) &+ \mu\int_0^t \frac{\partial f}{\partial x}(\bar{X}_s, X_s)ds + \int_0^t \frac{\partial f}{\partial y}(\bar{X}_s, X_{s-})d\bar{X}_s\\
&+ \int_0^t\int_{-\infty}^0 (f(\bar{X}_s, X_{s-}+z) - f(\bar{X}_s, X_{s-}))N(ds,dz),\ t\in[0,T].
\end{aligned}
$$

Martingales for the reflected Lévy processes

Let us briefly provide some useful martingales for the reflected Lévy process at the supremum. We start with Asmussen-Kella-Whitt martingale, its proof can be found in Jeanblanc et al. (2009), Proposition 11.2.6.7.

Proposition 2.59 (Asmussen-Kella-Whitt martingale)**.** *Assume that $(X_t)_{t\geq 0}$ is a general Lévy process (b, σ^2, ν) of the form*

$$X_t = bt + \sigma W_t + \int_0^t \int_{\{|z|\leq 1\}} z\tilde{N}(ds, dz) + \int_0^t \int_{\{|z|>1\}} zN(ds, dz),\ t \in [0, T],$$

where $\tilde{N}(dt, dz) = N(dt, dz) - dt\,\nu(dz)$ is the compensated jump measure. We also assume that $(R_t)_{t\geq 0}$, $R_t = \bar{X}_t - X_t$ is the reflected Lévy process, and $(\bar{X}_t{}^c)_{t\geq 0}$ be the continuous part of $(\bar{X}_t)_{t\geq 0}$. Let $f(x) \in C^2(\mathbb{R}_0^+)$, then the process

$$\Big(f(R_t) - f(R_0) - f'(0)\bar{X}_t^c - \frac{\sigma^2}{2}\int_0^t f''(R_s)ds + b\int_0^t f'(R_s)ds$$
$$- \int_0^t \int_{\mathbb{R}} \Big(f(R_{s-} + h_s(z)) - f(R_{s-}) + z1_{\{|z|\leq 1\}} f'(R_{s-})\Big)\,\nu(dz)ds\Big)_{t\geq 0},$$

with $t \in [0, T]$, is a local martingale, where $h_t(z) = -(z \wedge R_{t-})$, is a predictable function.

Given a smooth function $f(t, x)$ and a martingale we have the following Proposition, see Nguyen-Ngoc and Yor (2005), Proposition 1.

Proposition 2.60. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, be a $C^{1,1}([0, T] \times \mathbb{R})$ function. Assume that for $0 < c < d$, S is a stopping time given by: $S := \inf_t\{X_t \notin (c, d)\}$. We further assume that the process $(f(S \wedge t, X_{S\wedge t}))_{t\geq 0}$ is a martingale, $\forall x \in (c, d)$. Then the process*

$$\Big(f(t, R_t) + \int_0^t \frac{\partial f}{\partial x}(s, 0)d\bar{X}_s^c + \sum_{s\leq t}\Big(f(s, \Delta\bar{X}_s) - f(s, 0)\Big)\,1_{\{\Delta\bar{X}_s>0\}}\Big)_{t\geq 0},\ t \in [0, T],$$

is a local martingale, for all $x \in (c, d)$, under $\mathbb{P}(\cdot \mid R_0 = x)$, and $\tau = \inf_t\{t : \ R_t \notin (c, d)\}$.

Remark 2.61. Proposition 2.60 can also be applied for the reflected Lévy process at the infimum, by replacing $\bar{X}_t - X_t$ with $X_t - \underline{X}_t$.

An another useful martingale for the reflected Lévy process will be proved in Chapters 6 and 7.

Chapter 3

Introduction to credit risk modelling

3.1 Introduction

The purpose of this chapter is to provide a literature review of credit risk modelling. We start by describing the structural and the reduced form models and finally, we present the concept of quadratic hedging approach.

In general, in credit risk modelling, there are two ways to formulate the default process: reduced form and structural models. As we will later see in the structural models, the default is determined through the underlying asset. These models have an economic interpretation as the default can be calculated through the first passage time. Albeit, for the case when the underlying asset values are modelled by continuous stochastic processes, the default event is predictable and therefore investors can predict its arrival. The predictability of default is strongly related to short credit spreads. Assuming that the firm's value process is modelled by a continuous semimartingale, in a structural model, it can be concluded that for a short period of time, the risk of default does not affect the firm and so short credit spreads are zero. In reality, this is unrealistic, and these models are inconsistent with the market's observations.

On the contrary, reduced form models avoid connecting the default time with the firm's value process directly. In this case, the default event is normally totally inaccessible meaning that investors cannot predict its occurrence. In these models, the short credit spread is non zero. So far based on the literature review, these models are divided into two different approaches: the intensity based models and the hazard rate models. Intensity models describe the default through an intensity process $(\lambda_t)_{t\geq 0}$, such that the process $(1_{\{\tau>t\}} - \int_0^{\tau\wedge t} \lambda_s ds)_{t\geq 0}$ is a local martingale. Hazard rate models are focusing on the conditional survival probability $\mathbb{P}(\tau > t \mid \mathcal{F}_t)$, where $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is an expanded filtration such that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the minimal filtration generated by the underlying asset $(X_t)_{t\geq 0}$ and $\mathbb{N} = (\mathcal{N}_t)_{t\geq 0}$ is the filtration generated by $(1_{\{\tau\leq t\}})_{t\geq 0}$.

The concept of hedging in incomplete markets is an active area of research. By the term hedging, we mean that a future investor tries to countervail potential losses by constructing an appropriate portfolio. Mathematically speaking, there are various methods to formulate an appropriate hedging. Among others are delta hedging, super-hedging, utility maximization and quadratic hedging approaches etc. Typically, in incomplete markets we use the last three methods. Let us briefly describe them.

We start with the delta hedging. Delta is defined as the first derivative of the option price with respect to the corresponding underlying asset. In delta hedging, a trader tries to offset from potential long and short positions that an underlying asset may have for a short period of time. In simple words, we try to keep delta close to zero. A position of a claim that has delta equals to zero is called delta neutral. Delta hedging is strongly connected with the famous Black-Scholes model. In real financial markets, where the underlying asset is geometric Brownian motion, for European contingent claims delta has an analytical form. However, despite its simplicity, it is not very useful in the presence of jumps, see Merton (1976) for more details.

The main idea of a superhedging approach is that we search for a self-financing strategy such that the portfolio's terminal value should be equal or greater than the value of a contingent claim. So, given a contingent claim H then for a self-financing strategy $(\phi_t)_{t\geq 0}$ we have

$$\mathbb{P}\left(V_T(\phi) = V_0 + \int_0^T \phi_s dX_s \geq H\right) = 1.$$

To determine the price of superhedging involves calculating a non-trivial optimal problem: from a set of equivalent martingale measures (absolutely continuous to the physical measure $\mathbb{P}$) obtain the value of the claim H under the "least favourable" measure (see Proposition 10.1 of Cont and Tankov (2004)). When the underlying asset $(X_t)_{t\geq 0}$ admits jumps then the cost of superhedging is quite high, even for a European call option, see Cont and Tankov (2004), Chapter 10.

An another and perhaps a more useful and general approach is the utility maximization. This approach is related to a function $\mathfrak{U} : \mathbb{R} \to \mathbb{R}$, which is concave and increasing, called utility function. A key element of this approach is to specify an appropriate utility function. A choice of a utility function can be $\mathfrak{U}^a(x) = 1 - \exp(-ax)$ for $a > 0$. Unfortunately, with this choice of utility function, the methods provides non-linear pricing rules. The only choice when we can obtain linear pricing rules is by choosing $\mathfrak{U}(x) = -x^2$. With this specific form, we get the quadratic hedging. For a comprehensive review of the method, we refer to Cont and Tankov (2004), Chapter 10 and Kallsen (1999).

So, quadratic hedging specify an appropriate hedging strategy such that its risk is minimized in the mean square sense. A major drawback of this approach is that the future earnings or losses that a contingent claims may have are treated equally. On the other hand, in certain cases quadratic approach provides an explicit form of the hedging strategy. Local risk minimization and mean variance are the two quadratic methods.

The remainder of this chapter is structured as follows. We start by providing the main idea of the structural models. This is described in Section 3.2. In Section 3.3, we present the reduced form models. In Section 3.4, we analyse both quadratic methods. An emphasis is placed on the local risk minimization approach since it is the heart of our research.

3.2 Structural models

In this section, we briefly describe the structural models. For a rigorous description of these models, we refer to Ammann (2001), Bielecki and Rutkowski (2004) and also to Giesecke (2004). Throughout this section and unless otherwise stated, we assume that the underlying asset $(X_t)_{t\geq 0}$ is adapted with respect to the filtration $\mathbb{F}$.

3.2.1 Merton's model

The idea of pricing credit-risky bonds was first developed by Merton (1974). In this model, the firm's underlying asset is the main uncertainty. Under the physical measure the value asset $(X_t)_{t\geq 0}$ follows a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad t \in [0, T],$$

where $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$ are the mean rate and the volatility and $(W_t)_{t\geq 0}$ is a Brownian motion. The firm is financed by a zero coupon bond with face value K, and maturity time T. If the firm does not fulfil its obligation at date T, then defaults. Thus, the default time is given by

$$\tau = \begin{cases} T & \text{if } X_T < K \\ \infty & \text{otherwise} \end{cases},$$

meaning that if $X_T \geq K$ the value of the assets at maturity is greater than its debt, and bondholders receive the amount K as required, whereas shareholders receive the reaming $X_T - K$. On the contrary, if $X_T < K$ then the firm cannot pay back its obligation K. Therefore, bondholders should take over the firm and to receive $K - X_T$, and in this case shareholders do not receive anything. To summarize the bond holders receive

$$B_T^T = \min(K, X_T) = K - \max(0, K - X_T), \tag{3.1}$$

this payoff is equivalent to a portfolio consisting of a non zero coupon bond with face value K and an European put option with strike price K with maturity T. The European put option works as an insurance protecting bondholders from default. Similarly, the shareholders receive

$$E_T = \max(0, X_T - K),$$

which is equivalent to the payoff for an European call option with strike price K. The discounted price of the equity is

$$E_0 = BSCall(\sigma, T, K, r, X_0) = X_0\Phi(d_+) - e^{-rT}K\Phi(d_-),$$

where $BSCall$ refers to the Black-Scholes formula for a call option, Φ is the standard normal distribution function and

$$d_\pm = \frac{(r \pm \frac{1}{2}\sigma^2)T - \log(\frac{K}{X_0})}{\sigma\sqrt{T}}.$$

The probability of default is given by

$$\begin{aligned}\mathbb{P}(X_T < K) &= \mathbb{P}(\sigma W_T < \log(\frac{K}{X_0}) - (\mu - \frac{1}{2}\sigma^2)T) \\ &= \Phi\left(\frac{\log(\frac{K}{X_0}) - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{t}}\right).\end{aligned}$$

The price of the bond at time $t = 0$ whose payoff is given by (3.1) is

$$B_0^T = Ke^{-rT} - BSPut(\sigma, T, K, r, X_0), \tag{3.2}$$

where $BSPut$ is the Black-Scholes formula for a vanilla put option. Equation (3.2) can be rewritten as

$$B_0^T = X_0 - X_0\Phi(d_+) + e^{-rT}K\Phi(d_-).$$

In contrast to Merton's model, the debt capital structure of a firm is usually more complicated than a risky zero-coupon bond. To solve this, Geske (1977, 1979) investigate the case for compound options. At each coupon payment date until the maturity T, the equity holders can pay a coupon payment to the bondholders. Furthermore, Geske (1977, 1979) provides a general pricing formula for a risky bond with arbitrary and finite payments, based on the calculation of multidimensional normally distributed integrals. He also investigates some effects such as sinking funds, payout restrictions or safety covenants in the valuation formula.

Chance (1990) investigates the duration of a zero coupon bond based on Merton's model. He showed that the duration of a bond is a combination of a European put option and a default-free coupon bond. Furthermore, he also proved that the higher is the debt ratio, the shorter the duration will be. Therefore, a default prone zero coupon bond has a shorter duration than a non risky one.

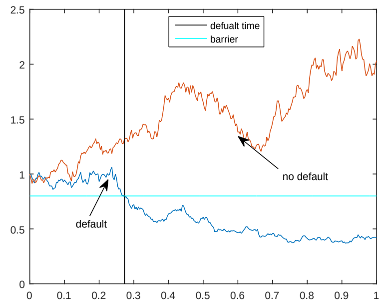


Figure 3.1: Illustration of a first passage model, when the underlying asset is continuous and the barrier is constant.

3.2.2 First passage models

Merton's model has the drawback that the firm's value defaults only at the maturity T. This is considered to be unrealistic, since usually in case of default the firm consumes its assets long before the maturity. Therefore Merton's model cannot capture premature events. First passage time models attempt to solve this problem, where the default occurs as soon as the underlying asset crosses certain barriers.

The idea of the first passage models where introduced by Black and Cox (1976). In a first-passage-model, the default time is given though

$$\tau = \inf\{t : X_t < D\}. \tag{3.3}$$

The barrier D could be constant, a deterministic function, a random variable or a stochastic process. The barrier in the Black and Cox (1976) model is assumed to be an exponential and deterministic function

$$D(t) = Ke^{-k(T-t)}. \tag{3.4}$$

An interesting case of the barrier (3.4) is when we substitute k with r. In this case, the barrier will be the discounted value of the interest rate. Assuming that the barrier is given by (3.4) and by observing that

$$\{X_t < D(t)\} = \{(c-k)t + \sigma W_t < \log(\frac{K}{X_0}) - kT\},$$

where $c = \mu - \frac{1}{2}\sigma^2$, then if we calculate the distribution of the infimum of the Brownian motion, we get

$$\mathbb{P}(0 \leq \tau \leq t) = \Phi\left(\frac{\log(\frac{K}{X_0}) - cT}{\sigma\sqrt{T}}\right) + (\frac{K}{X_0}e^{-kT})^{\frac{2}{\sigma^2}(c-k)}\Phi\left(\frac{\log(\frac{K}{X_0}) + (c-2k)T}{\sigma\sqrt{T}}\right).$$

Similarly to the Merton's model, the equity payoff is a European down and out call option, with strike price K a time dependent barrier $D(t)$ and maturity time T

$$E_T = \max(X_T - K, 0)1_{\{M_T \geq D\}},$$

where $M_t = \inf_{s \leq t} X_0 \exp((c-k)s + \sigma W_s)$ and $D = K\exp(-kT)$. Following Merton (1973), a closed formula for E_0 can be derived. The payoff function of the risky bond at time T is

$$K - \max(K - X_T, 0) + \max(X_T - K, 0)1_{\{M_T < D\}}.$$

The Black-Cox model is implemented for constant interest rates. Longstaff and Schwartz (1995) extended it by assuming that the interest rate is a stochastic process $(r_t)_{t\geq 0}$ following the Vasicek model

$$dr_t = (a - br_t)dt + \sigma_r d\tilde{W}_t, \;\; t \in [0, T],$$

where a and b are constants, and $(\tilde{W}_t)_{t\geq 0}$ is a Brownian motion correlated with W_t i.e. $d\tilde{W}_t dW_t = \rho dt$. In Longstaff-Schwartz model, the default boundary is constant i.e. $D = k$. Another example for stochastic interest rate, Kim et al. (1993) considered the CIR process

$$dr_t = (a - br_t)dt + \sigma_r \sqrt{r_t} d\tilde{W}_t, \;\; t \in [0, T].$$

In the first passage models, bond investors should immediately take over the firm when default occurs. However, following Giesecke (2004), this is unrealistic since in general firms need a period of time to reorganize their operation after default. The reorganization of the firm can be modelled through an excursion approach. Assuming that the barrier D is constant and given a positive and bounded functional f on $[0,\infty)^2$, we introduce a continuous functional $Q(X)$

$$Q(X)_t = \int_0^t f(s,t)1_{\{X_s \leq D\}} ds, \;\; t \in [0, T].$$

The above term measures the risk of firm's liquidation during a stopping time $\tau_L$which is given by

$$\tau_L := \inf\{t > 0 : Q(X_t) > \delta\},$$

for some constant δ. For the case when $\delta = 0$, it is easy to see that $\tau_L = \tau$, where τ is the first passage time introduced in (3.3), when D is constant. For more details, we refer to Giesecke (2004).

Credit spread for structural models

The credit spread is defined as the difference between the yield of a defaultable claim and a default free zero bond. Based on this difference, we can get an idea of the extra interest rate that investors have to pay in order to cover any possible future losses. Let us assume that y_c is the yield of a defaultable zero coupon bond and $(r_t)_{t\geq 0}$ is the yield for the risk free zero coupon bond. Let also $\bar{P}^d(t,T)$ be the discounted price of the

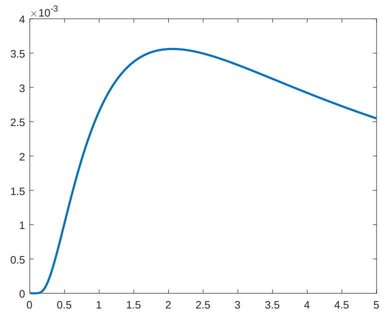


Figure 3.2: Credit spread in Merton's model with maturity $T = 5$, $r = 0.05$ and volatility $\sigma = 0.3$.

default free bond, then the credit spread

$$S(t,T) = \frac{1}{T-t}\int_t^T (y_c(s) - r_t(s))ds = -\frac{1}{T-t}\log(\frac{P^d(t,T)}{\bar{P}^d(t,T)}),$$

which can be rewritten as

$$S(t,T) = -\frac{1}{T-t}\ln \mathbb{P}(\tau > T \mid \mathcal{F}_t),\ t < T,\ \tau > t, \tag{3.5}$$

For the Merton's model, the credit spread has an analytical form, see Giesecke (2004). Figure 3.2 displays the credit spread in the Merton's model with maturity time $T = 5$ and volatility parameter $\sigma = 0.3$.

Equivalently, the short credit spread is just the limit of (3.5) (whenever it exists), as $T \downarrow t$, for a short period of time i.e.

$$\lim_{T\downarrow t} S(t,T) = -\lim_{T\downarrow t}\frac{\ln \mathbb{P}(\tau > T \mid \mathcal{F}_t)}{T-t}.$$

When the underlying asset values are modelled by a continuous process it can be shown that the short credit spread for the structural models is zero. For instance, see Giesecke (2006), Proposition 3.2. Short zero spreads means that for a small period of time, investors do not request any compensation for their losses derived from the bond, which from a financial point of view is not appealing. For applications of the short credit spreads using jump processes, we refer to Okhrati (2014).

3.2.3 Extensions and drawbacks of first passage models

Following the first passage models, we have seen that the default time is a predictable event and therefore there exists a sequence of pre-default events under which the investors can predict the arrival of credit event. For its mathematical meaning we refer to Chapter 2, Definition 2.4. The notion of predictability of τ is strongly related to credit spreads. We have seen that the credit spread for a short period of time tends to zero. Therefore the value of a risky bond will be the same as the non risky one. As consequence, investors have no reason to pay an extra yield for the risky bond. The empirical results derived from the structural models show that these models are not very useful. For example, Eom et al. (2004) compare five different structural models, and they concluded that these models are not very successful in pricing of corporate bonds.

The introduction of jumps in the underlying asset is investigated by Zhou (2001). He assumed that the dynamics of the firm's value process $(X_t)_{t\geq 0}$ follows a jump diffusion process given by

$$\frac{dX_t}{X_t} = (\mu - \lambda q)dt + \sigma W_t + (\Pi - 1)dY_t, \quad t \in [0, T],$$

where dY_t is a Poisson process with intensity $\lambda > 0$, q is a positive constant, Π is the jump magnitude which follows log normal distribution, $\mu \in \mathbb{R}$ is the expected value of returns and $\sigma > 0$ is the volatility. Also, Π and dY are independent. A major advantage of Zhou's model is that by introducing jumps in the underlying asset the credit spread remains positive consistent with the market's data. Also, the model incorporates a recovery rate, which is endogenously determined, linked with the underlying asset. Note that when the evolution of the price of the underlying asset is a jump diffusion process, the stopping time τ modelled though a first passage approach, is neither predictable or totally inaccessible. This can be seen since if we are in the diffusion state then the stopping time can be anticipated. However, when we are in the jump state then it cannot be anticipated. This will be studied further in Chapter 7 .

Schoutens and Cariboni (2009), in Chapter 4, provide a first passage model for Lévy processes, using as an underlying asset the exponential variance gamma process. In their model the barrier D is constant, and they find the value of a down and out option by solving a PIDE. .

3.3 Reduced form models

In contrast to the structural framework, in the reduced form models the default occurs unexpectedly and without warning. There are no pre-default events and thus investors cannot predict the time when a default happens. Relaxing the assumption of full information to investors makes the default event an inaccessible stopping time. In this case, the default is determined externally based on a point process.

Following Jeanblanc and Le Cam (2008), reduced form models are divided into two different approaches. This categorization is based on the available information, equivalently to the reference filtration. The first approach is the intensity based models where a unique flow of information exists and it is given by $\mathbb{F}$. The second approach is the hazard rate models where the computation relies on the introduction of an expanded filtration $\mathbb{G}$, $\mathbb{F} \subseteq \mathbb{G}$.

3.3.1 Intensity based models

In the intensity based models the default time τ is an $\mathbb{F}$-stopping time, where $\mathbb{F}$ represents the full available information to investors. In these models the probability of default is associated with a process $(\lambda_t)_{t\geq 0}$ called intensity or hazard rate. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be our space, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, and $\mathbb{F}$ represents the available information that investors have over time t. We define the point process $(N_t)_{t\geq 0}$

$$N_t = 1_{\{\tau \leq t\}} = \begin{cases} 1 & \text{if } \tau \leq t \\ 0 & \text{else} \end{cases},$$

where $\tau : \Omega \to \mathbb{R}^+$ is a default time, which is an $\mathbb{F}$-stopping time. Since $(N_t)_{t\geq 0}$ is an increasing process and therefore is a submartingale, then from the Doob-Meyer decomposition there exists a process $(A_t)_{t\geq 0}$ such that the difference $(N_t - A_t)_{t\geq 0}$ is a $\mathbb{F}$-local martingale. The process $(A_t)_{t\geq 0}$ neutralizes the effect of $(N_t)_{t\geq 0}$. A fundamental property under which the intensity process exists is if the process $(A_t)_{t\geq 0}$ is absolutely continuous with respect to Lebesgue measure. If so, then the intensity process $(\lambda_t)_{t\geq 0}$, with $A_t = \int_0^{\tau\wedge t} \lambda_s ds$ satisfies

$$\lambda_t = \lim_{h\to 0} \frac{1}{h}\mathbb{E}[A_{t+h} - A_t \mid \mathcal{F}_t] = \lim_{h\to 0} \frac{1}{h}\mathbb{P}(\tau \in (t, t+h) \mid \mathcal{F}_t), \quad t \in [0, T]. \tag{3.6}$$

Assuming that intensity exists we provide some examples of it along with the probability of default. Let us introduce some important cases for the intensity see also Giesecke (2004).

- We start with the most trivial case, under which the intensity process is constant. For example, we can assume that the process $(N_t)_{t\geq 0}$ is a compound Poisson with rate λ. In this case, the probability of default is $\mathbb{P}(\tau \leq T) = 1 - e^{-\lambda T}$.
- Next, we consider the case when the intensity process is just a deterministic function with respect to time $t \geq 0$, i.e. $\lambda = \lambda(t)$. A typical example is when the counting process $(N_t)_{t\geq 0}$ is a inhomogeneous Poisson process whose rate is given by $\lambda(t)$. Then the probability of default is $\mathbb{P}(\tau \leq T) = 1 - e^{-\int_0^T \lambda(s)ds}$.
- The most general case is when the intensity is formulated by a stochastic process $\lambda = (\lambda_t)_{t\geq 0}$. We consider $(N_t)_{t\geq 0}$ to be a Cox-process, which is a generalization of the compound Poisson process, whose rate is given by $(\lambda_t)_{t\geq 0}$. If the increasing process $(A_t)_{t\geq 0}$ is absolutely continuous with respect to Lebesgue measure then, the probability of default is given as $\mathbb{P}(\tau \leq T) = 1 - \mathbb{E}[e^{-\int_0^T \lambda_s ds}]$.

We highlight the importance of continuity of intensity. If the compensator $(A_t)_{t\geq 0}$ of the counting process $(N_t)_{t\geq 0}$ is not continuous then the intensity process does not exist. Let us now state some important result under which we examine the continuity of $(A_t)_{t\geq 0}$. We start with the most fundamental result given by the Theorem bellow, which analyses when the $\mathbb{F}$-stopping time τ is totally inaccessible. For its proof, we refer to Protter (2004), Chapter III, Section 5, Theorem 17.

Theorem 3.1 (Dellacherie's theorem). *Let $\mathbb{F}$ is the minimal filtration that makes τ into a stopping time with $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for each $t > 0$. Let also $F(x) = \mathbb{P}(\tau \leq x)$ be the law of τ, $\forall x \geq 0$. Then the compensator $(A_t)_{t\geq 0}$ of $(N_t)_{t\geq 0}$ exists and it is given by*

$$A_t = \int_0^{\tau\wedge t} \frac{dF(s)}{1 - F(s-)}, \; 0 \leq t \leq T.$$

Furthermore, if $(A_t)_{t\geq 0}$ is continuous, which implies that F is continuous, then τ is totally inaccessible and $A_t = \ln(1 - F(\tau \wedge t))$.

Remark 3.2. Following Theorem 3.1, the process $(A_t)_{t\geq 0}$ is continuous with respect to the Lebesgue measure. Then the intensity process $(\lambda_t)_{t\geq 0}$ is considered as the Radom-Nikodym derivative of $\left(\frac{dA_t}{dt}\right)_{t\geq 0}$. Furthermore, in many applications we need the predictability of the intensity, or equivalently $\int_0^t \lambda_s ds = \int_0^t \lambda_{s-} ds$. Assuming that the intensity process $(\lambda_t)_{t\geq 0}$ is a càdlàg process then the last expression holds true since the process $(A_t)_{t\geq 0}$ is absolutely continuous with respect to Lebesgue measure.

Remark 3.3. Theorem 3.1 determines the compensator of $(N_t)_{t\geq 0}$ when $\mathbb{F} = \sigma(\tau \wedge t)$. Perhaps a more interesting result is to derive the compensator of $(N_t)_{t\geq 0}$ given a filtration $\mathbb{F}$ which is not the minimal one that makes τ into a stopping time and a positive random variable L which is not a stopping time under the filtration $\mathbb{F}$. Then under an augmented filtration $\mathbb{G}$, $\mathbb{F} \subset \mathbb{G}$, that renders L into a stopping time we can derive the compensator of $(N_t)_{t\geq 0}$. This is analysed in Chapter 5.

Apart from Theorem 3.1, in Janson et al. (2011) an the references therein, they studied other cases under which the process $(A_t)_{t\geq 0}$ is continuous. Their results are based on the $(\tilde{A}_t)_{t\geq 0}$ compensator of $(A_t)_{t\geq 0}$. A particularly interesting case is the following one. If $(A_t)_{t\geq 0}$ is a càdlàg process and let $(\tilde{A}_t)_{t\geq 0}$ be its compensator. Then for a given constant K and if

$$\mathbb{E}[A_t - A_s \mid \mathcal{F}_s] \leq K(t - s), \text{ a.s,}$$

for all $0 \leq s \leq t$, then the process $(\tilde{A}_t)_{t\geq 0}$ has paths absolutely continuous with respect to Lebesgue measure such that $\tilde{A}_t = \int_0^t \lambda_s ds$. Zeng (2006) generalized the above result for the case when $K_t = (K_t)_{t\geq 0}$ is an $\mathbb{F}$-predictable process. In Janson et al. (2011), Theorem 2 (this is a result of Zeng (2006)) specified when the compensator $(\tilde{A}_t)_{t\geq 0}$ of $(A_t)_{t\geq 0}$ is absolutely continuous with respect to Lebesgue measure. This is achieved if there exists an integrable process $(Y_t)_{t\geq 0}$ and its compensator $(\tilde{Y}_t)_{t\geq 0}$, such that $d\tilde{Y}_t \ll dt$ i.e. the process $(\tilde{Y}_t)_{t\geq 0}$ is absolutely continuous with respect to Lebesgue measure. In this case, if

$$\mathbb{E}[A_{t+h} - A_t \mid \mathcal{F}_s] \leq \mathbb{E}[Y_{t+h} - Y_t \mid \mathcal{F}_s],$$

then we get $\tilde{A}_t = \tilde{Y}_t$ for every $t \in [0, T]$.

The following Theorem, investigates the intensity for a sub-filtration, see Janson et al. (2011). For the definition of the optional projection and its properties we refer to Appendix B.1.

Theorem 3.4. *Assume that τ is an $\mathbb{F}$-stopping time, with an intensity process $\int_0^t \lambda_s dc(s)$. Let $\mathbb{H}$ be a sub-filtration of $\mathbb{F}$ i.e. $\mathbb{H} \subset \mathbb{F}$, where τ is also $\mathbb{H}$-stopping time. Then the compensator of τ under $\mathbb{H}$ is $\int_0^t \lambda_s^\circ ds$, where $\lambda_s^\circ = \mathbb{E}[\lambda_t \mid \mathcal{F}_s]$ is the optional projection of $(\lambda_t)_{t\geq 0}$. Then we get that if the process*

$$\left(1_{\{\tau\leq t\}} - \int_0^t \lambda_s ds\right)_{t\geq 0}, \quad t \in [0,T],$$

is an $\mathbb{F}$-local martingale, then the process

$$\left(1_{\{\tau\leq t\}} - \int_0^t \lambda_s^\circ ds\right)_{t\geq 0}, \quad t \in [0,T],$$

is a $\mathbb{H}$-local martingale.

An immediate consequence of Theorem 3.4, along with Theorem 3.1, is that if the process $(A_t)_{t\geq 0}$ is absolutely continuous under the initial filtration $\mathbb{F}$, then it will remains absolutely continuous under $\mathbb{H}$, which implies that the stopping time τ is a totally inaccessible stopping time under the sub-filtration $\mathbb{H}$.

Reduced form models were introduced by Jarrow and Turnbull (1995). Later on, they were extended by Duffie and Singleton (1999), Madan and Unal (1998). Jarrow and Turnbull (1995) introduced a simple discrete model. Assuming that the risk-free interest rate follows the Markov property, they construct a lattice for the default and default-free terms. Then they obtained a risk-neutral probability measure, under which they determine the price of a defaultable claim in a recursive form. Moreover, they constructed a continuous time model equivalent to the discrete one. In this continuous model, the credit event is exponentially distributed with intensity λ which is constant and independent of the underlying asset. A major drawback of their model, is that the intensity is constant. Jarrow et al. (1997) introduce a model under which the intensity is no longer constant. This was achieved by relating the default probabilities with the credit ratings.

Intensity calculation

In general, there are two major ways under which we can calculate the intensity process. Before providing them, let us introduce an essential definition.

Definition 3.5 (class $(\mathcal{D})$)**.** Let $(X_t)_{\geq 0}$ be a right continuous $\mathbb{F}$-supermartingale, and let $\mathcal{T}$ be the collection of all finite stopping times (respectively $\mathcal{T}_a$, the set of all stopping times bounded by α). We say that $(X_t)_{t\geq 0}$ belongs to class $\mathcal{D}$ if and only if the collection of random variables $\{X_\tau, \tau < \infty\}$, for all finite stopping times $\tau \in \mathcal{T}$, is uniformly integrable (respectively $\{X_\tau, \tau < \infty\}$, for all the finite stopping times $\mathcal{T}_\alpha$ is uniformly integrable).

Similarly, the process $(X_t)_{t\geq 0}$ belongs to the class $\mathcal{DL}$ or locally to the class $\mathcal{D}$, if $(X_t)_{t\geq 0} \in \mathcal{D}$ on every interval $[0, \alpha]$, $0 \leq \alpha < \infty$.

Perhaps the most important result is the following Theorem, see Meyer (1966), Chapter VII.

Theorem 3.6 (Meyer's approximation)**.** *Let $(X_t)_{t\geq 0}$ be a right continuous potential* [1] *of class $\mathcal{D}$, and let (A_t^h) be an increasing process given by*

$$A_t^h = \int_0^t \frac{X_s - \mathbb{E}[X_{s+h} \mid \mathcal{F}_s]}{h}, \quad t \in [0, T].$$

Moreover, for every stopping time τ we have that

$$A_\tau = \lim_{h\to 0} A_\tau^h,$$

in the sense of the weak topology $\sigma(L^1, L^\infty)$.

We remark that the process $(1_{\tau\leq t})_{t\geq 0}$ belongs to the class $\mathcal{D}$ but it is not a potential, so the above Theorem cannot be used. However, the process $(1_{\{\tau>t\}})_{t\geq 0} \in \mathcal{D}$ and it is a potential therefore by applying Meyer's theorem we get that the process $\left(1_{\{\tau>t\}} + \int_0^t \lambda_s ds\right)_{t\geq 0}$ is a martingale. Since $1_{\{\tau\leq t\}} = 1 - 1_{\{\tau>t\}}$, then we also get that the process $\left(1_{\{\tau\leq t\}} - \int_0^t \lambda_s ds\right)_{t\geq 0}$ is a martingale. In other words, the process $(\int_0^t \lambda_s ds)_{t\geq 0}$, is the compensator of $(1_{\{\tau\leq t\}})_{t\geq 0}$. Similarly to Meyer's theorem we have the following interesting Theorem, see Aven (1985).

Theorem 3.7 (Aven's approximation)**.** *Let $(N_t)_{t\geq 0}$ be a counting process adapted to $\mathbb{F}$, and assume that there exists an intensity process $(\lambda_t)_{t\geq 0}$ such that*

$$\lambda_t = \lim_{h_n\to 0} \frac{\mathbb{E}\left[N_{t+h} - N_t \mid \mathcal{F}_t\right]}{h_n}, \; t \in [0, T].$$

Furthermore, assume that there exist processes $(m_t)_{t\geq 0}$, $(M_t)_{t\geq 0}$. Then the following statements hold for $(\lambda_t)_{t\geq 0}$ and $(m_t)_{t\geq 0}$

1. *For each $t \in [0, T]$*

$$\lim_{n\to\infty} M_t^n = \lambda_t, \quad a.s.$$

2. *For each $t \in [0, T]$ there exists for almost all ω an $n_0 = n_0(t, \omega)$ such that*

$$|M_s^n(\omega) - \lambda_s(\omega)| \leq m_s(\omega), \quad s \leq t, \; n \geq n_0.$$

3. *$\int_0^t m_s ds < \infty a.s.$, for all $t \in [0, T]$.*

[1] An adapted cádlág process $(X_t)_{t\geq 0}$ is a potential if it is a non-negative supermartingale such that $\lim_{t\to\infty} \mathbb{E}[X_t] = 0$.

Then the process $\left(N_t - \int_0^t \lambda_s ds\right)_{t\geq 0}$ *is an* $\mathbb{F}$*-local martingale, i.e. the process* $(\int_0^t \lambda_s ds)_{t\geq 0}$ *is the compensator of* $(N_t)_{t\geq 0}$.

It is worth mentioning that when the underlying asset admits jumps we can construct a structural model under which there exists an intensity process. For example, in Okhrati (2014), Theorem 3.1, the author determined the intensity process when the underlying asset $(X_t)_{t\geq 0}$ is a jump-diffusion process with jumps of finite variation for a structural model whose default time is given by the first time when $(X_t)_{t\geq 0}$ is strictly negative. Based on this result and along with the result of Guo and Zeng (2008) in the next chapter (Chapter 4), we determine the hedging strategy of a defaultable claim through the solution of a PIDE.

Affine intensity models

After proving the existence of the intensity and specifying its model, the next step is to determine the probability of default through the intensity. In general this is not easy task, however, Duffie and Kan (1996) provide a useful tool under which we can calculate the default probabilities. Assume that the intensity process can be expressed as

$$\lambda(y) = p + q \cdot y, \quad y \in \mathbb{R},$$

or using processes,

$$\lambda_t = \lambda(Y_t) = p + q \cdot Y_t,$$

where $Y_t = (Y_t^1, Y_t^2, \dots, Y_t^n)$ is a multidimensional Markov process, $p \in \mathbb{R}^+$, and $q \in \mathbb{R}^{+,n}$ is a n-dimensional positive constant $q = (q_1, q_2, \dots q_n)$. Let Y_0 be the initial state, then the probability of default is given through

$$\mathbb{P}(\tau \leq T) = 1 - \exp(P(T) - Q(T)Y_0),$$

the parameters P, and Q are determined through a Riccati ordinary differential equation, along with some boundary conditions. As an application let us investigate the case when the process $(Y_t)_{t\geq 0}$ follows the CIR process.

Example 3.8. Assume that $d = 1$, and let $(Y_t)_{t\geq 0}$ follows the CIR dynamics

$$dY_t = c(\mu - Y_t) + \sigma\sqrt{Y_t}dW_t, \ t \in [0, T],$$

where $c > 0$, $\sigma > 0$ and $\mu \in \mathbb{R}$. Under some calculations we have

$$\mathbb{E}[Y_t] = Y_0 e^{-ct} + \mu(1 - e^{-ct}),$$

and

$$Var[Y_t] = Y_0 \frac{\sigma^2}{c}(e^{-ct} - e^{-2kt}) + \frac{\mu\sigma^2}{2c}(1 - e^{-ct})^2.$$

If we assume that $2c\mu > \sigma^2$, and $Y_0 > 0$, and by letting $\lambda(y) = y$ we get

$$Q(T) = \frac{2(e^{\gamma T} - 1)}{(\gamma - c)(e^{\gamma T} - 1) + 2\gamma},$$
$$P(T) = \frac{2c\mu}{\sigma^2} \log\left(\frac{2\gamma e^{(\gamma - c)\frac{T}{2}}}{(\gamma - c)(e^{\gamma T} - 1) + 2\gamma}\right),$$

where $\gamma = \sqrt{c^2 + 2\sigma^2}$.

Pricing rule for intensity models

Assume that H is the payoff at the maturity time, and is $\mathcal{F}_T$-measurable, and integrable random variable. We further assume that the interest rate is zero, then in this case we have

$$\mathbb{E}[H1_{\{\tau>T\}} \mid \mathcal{F}_t] = 1_{\{\tau>t\}}(D_t - \mathbb{E}[\Delta D_\tau 1_{\{\tau\le T\}} \mid \mathcal{F}_t]), \quad \forall t \in [0, T], \tag{3.7}$$

where $(D_t)_{t\ge 0}$, $D_t = e^{A_t}\mathbb{E}[He^{-A_T} \mid \mathcal{F}_t] = \mathbb{E}[e^{A_t - A_T} \mid \mathcal{F}_t]$. If we assume that the intensity exists such that $A_t = \int_0^t \lambda_s ds$, then formula (3.7) becomes

$$\mathbb{E}[H1_{\{\tau>T\}} \mid \mathcal{F}_t] = 1_{\{\tau>t\}}\mathbb{E}[He^{-\int_t^T \lambda_s ds} \mid \mathcal{F}_t] \tag{3.8}$$

For the general case, when there exists a recovery process $\mathbb{F}$-predictable $(R_t)_{t\ge 0}$, then from Duffie et al. (1996) the pricing rule (3.7) becomes

$$\mathbb{E}[H1_{\{\tau>t\}}\mathcal{F}_t] = 1_{\{\tau>t\}}(D_t - \mathbb{E}[\exp(-\int_t^\tau r_s ds)\Delta D_\tau \mid \mathcal{F}_t]),$$

where

$$D_t = \mathbb{E}\left[\int_t^T R_u \lambda_u \exp\left(-\int_t^u (r_v + \lambda_v) dv\right) du \mid \mathcal{F}_t\right]$$
$$+ \mathbb{E}\left[H \exp(-\int_t^T (r_v + \lambda_v) dv) \mid \mathcal{F}_t\right], \ t \in [0, T],$$

We have seen that by introducing jumps in a structural model the stopping time becomes totally inaccessible. In general, it is not easy to determined the intensity when we are dealing with jump processes. Guo and Zeng (2008), introduced a first passage model, where the underlying asset is a general Hunt process [2] with Lévy system (U, K) [3]. Then the totally inaccessible part of the stopping time admits an intensity, see also Janson et al. (2011) Corollary 10. We will further analyse their model using a finite variation Lévy process in Chapter 4.

[2]Hunt process is a quasi left continuous process satisfying the strong Markov property.

[3]Lévy systems for Markov processes is the structure that characterizes the jump behaviour of a Hunt process, K represents the kernel on $\mathbb{R}$ and U is a continuous additive functional.

3.3.2 Hazard rate models

The hazard rate approach is based on the calculation of the conditional probability $Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t)$, where $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ represents the augmented filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t$, where $\mathbb{N} = (\mathcal{N}_t)_{t\geq 0}$ is the σ-algebra generated by the default process $(N_t)_{t\geq 0}$ and as always $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by the underlying asset process $(X_t)_{t\geq 0}$.

Obviously, F_t represents the cumulative distribution function of the default time τ. Thus for a given density function $q(t)$ then

$$F_t = \mathbb{P}(\tau \leq t \mid \mathcal{F}_t) = \int_0^t q(s)ds.$$

The survival probability is given by $Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t) = 1 - F_t$. One important assumption that we should make when studying hazard rate models is $Z_t < 1$. If we assume that $Z_t = 1$, then with probability one the default event will never occur.

Definition 3.9 (Hazard process)**.** Let $Z_t < 1$, then the $\mathbb{F}$-hazard process of the stopping time τ, denoted by $(\Gamma_t)_{t\geq 0}$, $\Gamma_t = \int_0^{\tau\wedge t} \lambda_s ds$ and it is defined through the formula $Z_t = 1 - F_t = e^{-\Gamma_t}$ or $\Gamma_t = -\ln(1 - F_t) = -\ln(Z_t)$ $\forall t \in [0, T]$.

Remark 3.10. Similarly to the intensity models, the process $(N_t - \int_0^{\tau\wedge t} \lambda_s ds)_{t\geq 0}$ is a $\mathbb{G}$-martingale.

Hazard rate and filtration expansion

Let us describe the idea of filtration expansion applied in hazard rate models. Note that more detailed analysis for filtration expansion along with the appropriate canonical decomposition will be made in Chapter 5. Since we are dealing with a filtration expansion, it may useful to introduce the $\mathcal{H}'$ and $\mathcal{H}$ hypothesis.

Definition 3.11 ($\mathcal{H}'$ and $\mathcal{H}$ hypothesis)**.** A $\mathbb{F}$-semimartingale we say that satisfies the $\mathcal{H}'$ hypothesis if and only if it remains a semimartingale under the augmented filtration $\mathbb{G}$. Similarly, an $\mathbb{F}$-local martingale satisfies the $\mathcal{H}$ hypothesis if and only if it remains a local martingale under the filtration $\mathbb{G}$.

Note that for the case of progressive enlargement the $\mathcal{H}'$ hypothesis is always valid until the stopping time τ. If $(X_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale and τ is a $\mathbb{G}$-stopping time then $(X_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{G}$-semimartingale. Assume that $(Z_t)_{t\geq 0}$, $Z_t := \mathbb{P}(\tau > t \mid \mathcal{F}_t)$, which from the Doob-Meyer decomposition under $\mathbb{F}$, we have $Z_t = M_t - K_t$, where $(M_t)_{t\geq 0}$ is a local martingale and $(K_t)_{t\geq 0}$ is an increasing and predictable process. Let us also define the process $(\epsilon_t)_{t\geq 0}$, $\epsilon_t = (\Delta X_\tau N_t)_{t\geq 0}$ and the process $(B_t)_{t\geq 0}$ be the $\mathbb{F}$-predictable dual projection of $(\epsilon_t)_{t\geq 0}$. Then using Jeulin's formula, see Jeulin and Yor (1978), under the progressive filtration expansion $\mathbb{G}$ of $\mathbb{F}$ we have that the process

$$\left(X_{\tau\wedge t} - \int_0^{\tau\wedge t} \frac{d\left\langle X, M\right\rangle_s^{\mathbb{F}} + B_s}{Z_{s-}}\right)_{t\geq 0},\ t \in [0, T],$$

is a $\mathbb{G}$-local martingale.

Assuming that the process $(\epsilon_t)_{t\geq 0}$ is zero, which implies that the $\mathbb{G}$-stopping time τ avoids all the $\mathbb{F}$-stopping times, then following Protter (2004)Chapter VI, above formula is simplified

$$\left(X_{\tau\wedge t} - \int_0^{\tau\wedge t} \frac{d\left\langle X, M\right\rangle_s^{\mathbb{F}}}{Z_{s-}}\right)_{t\geq 0}, \ t\in[0,T],$$

is a $\mathbb{G}$-local martingale.

Honest and initial times are the most common applications of the $\mathcal{H}'$ hypothesis. Let us first provide their definitions.

Definition 3.12 (Honest time)**.** A random variable τ is said to be an **honest time** if for every $t<\infty$ there exists an $\mathbb{F}$-measurable random variable $\mathcal{W}_t$, under which $\tau=\mathcal{W}_t$ on the sets $\{\tau\leq t\}$.

For the initial time we borrow its definition from Jeanblanc and Le Cam (2009)

Definition 3.13 (Initial time)**.** Let $P_t(\omega, dx)$ be the regular conditional distribution of τ and assume that η is a σ-finite measure. Then a positive random variable τ is called **initial time** if the regular conditional distribution is absolutely continuous with respect to the σ-finite measure η i.e. $P_t(\omega, dt)\ll\eta(dt)$ with $t\in[0,T]$.

Based on the Definition 3.13 there exists a family of $\mathbb{F}$-adapted processes $(a_t^\theta)_{t\geq 0}$ such that

$$Z_t=\mathbb{P}(\tau>t\mid\mathcal{F}_t)=\int_t^{+\infty} a_t^\theta\eta(du).$$

Now let us briefly provide the canonical decompositions for a progressive filtration expansion, when τ is an honest or an initial time.

- For the case when the default time is an honest time under filtration $\mathbb{G}$ and $(X_t)_{t\geq 0}$ is a $\mathbb{F}$-semimartingale, then under the progressive filtration expansion $\mathbb{G}$ of $\mathbb{F}$ the process

$$\left(X_t - \int_0^{\tau\wedge t} \frac{d\left\langle X, M\right\rangle_s^{\mathbb{F}}+dB_s}{Z_{s-}} + 1_{\{\tau\leq t\}}\int_\tau^t \frac{d\left\langle X, M\right\rangle_s^{\mathbb{F}}+dB_s}{F_{s-}}\right)_{t\geq 0}, \ t\in[0,T],$$

  is a $\mathbb{G}$-local martingale

- If τ is an initial time and a^θ is the density of the conditional law of τ introduced in Definition 3.13, then under the progressive filtration $\mathbb{G}$ of $\mathbb{F}$ we have that the process

$$\left(X_t - \int_0^{\tau\wedge t} \frac{d\left\langle X, G\right\rangle_s^{\mathbb{F}}+dB_s}{Z_{s-}} - \int_{\tau\wedge t}^t \frac{d\left\langle X, a^\theta\right\rangle_s^{\mathbb{F}}}{a_{s-}^\theta}\,|_{\theta=\tau}\right)_{t\geq 0}, \ t\in[0,T],$$

  is also a $\mathbb{G}$-local martingale.

A rigorous study of the initial times under the progressive filtration expansion can be found in Jeanblanc and Le Cam (2009).

Pricing rule for hazard rate models

Regarding the pricing rule for hazard rate models following Bielecki and Rutkowski (2004), Chapter 5, Corollary 5.1.1, we know that for any random variable H $\mathcal{G}$-measurable and let $\mathbb{G} = \mathbb{N} \vee \mathbb{F}$, then we have

$$\mathbb{E}[1_{\{t<\tau\leq s\}} H \mid \mathcal{G}_t] = 1_{\{\tau>s\}} \mathbb{E}[1_{\{t<\tau\leq s\}} e^{\Gamma_t} H \mid \mathcal{F}_t]. \tag{3.9}$$

Comparing the two pricing rules, following (3.8) and Corollary (3.9), we can see that the pricing rule of the intensity based models is more complicated since we need to calculate the jump term of $(D_{\tau\wedge t})_{t\geq 0}$. As an illustration Jeanblanc and Le Cam (2008) calculate the price of a zero coupon bond with constant intensity, using the intensity pricing rule.

Credit spread for reduced form models

For the intensity based models following Proposition 5.10 of Giesecke (2006), we have that for any $\mathbb{F}$-totally inaccessible stopping time

$$\lim_{T\downarrow t} S(t,T) = \lambda_t, \text{ a.s, for all } t \in [0,T].$$

The same result holds true for the hazard rate models under the expanded filtration $\mathbb{G}$.

In contrast to the structural models, in reduced form models the default event occurs as a complete surprise and therefore investors cannot predict the default time. However, a major disadvantage of these models is that the default process has no economic interpretation.

3.4 Quadratic hedging methods

In the previous two sections, we described models for pricing defaultable claims. Let us now describe the idea of hedging in incomplete markets.

In complete markets where arbitrage opportunities are absent, a contingent claim H can be perfectly hedged uniquely. A contingent claim H is said to be attainable if there exists a self-financing strategy $(\phi_t)_{t\geq 0}$ and $V_T(\phi) = H$. An attainable claim can be represented as a stochastic integral of an underlying asset plus a constant that represents the initial cost. The integrand provides sequential hedging that it is self-financing. Through the non-arbitrage assumptions, a market is complete if and only if every contingent claim is attainable. In this case, there exists a unique martingale measure under which we have perfect hedging, see Bingham and Kiesel (2004), Chapter 6, for more details.

On the other hand, in an incomplete market claims fail out to be represented in terms of a stochastic integral with respect to the underlying asset as there is an intrinsic risk. In this case, there is no unique martingale measure. For non attainable claims, it is impossible to assume that $V_T(\phi) = H$ and on the same time the strategy to be self-financing. Quadratic hedging approaches are considered proper methods for hedging defaultable claims in incomplete markets. Their aim is to find a strategy with a minimum hedging cost in the mean square sense. They are divided into two different but quite similar methods. The local risk minimization and the mean variance approaches.

If we prefer to hedge contingent claim such that the terminal condition $V_T(\phi) = H$ but the strategy is not self-financing then we choose the local risk minimization approach. Since the self-financing condition is not valid, in this method we assume that the cost process is a local martingale so that the strategy is a mean-self-financing. We look for a non self-financing hedging strategy $\phi_t = (\theta, \eta)_t$ and given that the cost process is a local martingale and the hedging error is minimized in the following sense

$$\inf_{\theta} \mathbb{E}\left[(C_T(\phi) - C_t(\phi))^2 \mid \mathcal{F}_t\right].$$

The method was introduced by Föllmer and Sondermann (1986) and later Föllmer and Schweizer (1991) extended it to the semimartingale case.

On the other hand, if we prefer a self-financing strategy but the terminal condition $V_T(\phi) = H$ is not valid any more, we choose the mean variance approach. In this method the hedging strategy tries to minimize the risk globally. We look for self-financing strategies $\phi_t = (\theta, \eta)_t$ with initial capital V_0 such that the hedging error is minimized in the mean square sense as follows

$$\inf_{\theta} \mathbb{E}\left[\left(V_0 + \int_0^T \theta_s dX_s - H\right)^2\right].$$

The method was first introduced in a general framework by Schweizer (1992). A disadvantage of this method is that in certain circumstances the hedging strategy is difficult to calculate it. An excellent review analysing both approaches is presented in Schweizer (2001).

When the underlying asset is a (local) martingale the two methods coincide. In this case, the methodology boils down to the determination of the GKW decomposition. For the semimartingale case, we need to introduce appropriate martingale measures and then to determine the GKW decomposition.

3.4.1 Risk minimization approach: martingale case

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ that satisfies the usual hypothesis. We assume that the underlying asset $(X_t)_{t \ge 0}$ is an $\mathbb{F}$-local martingale under the physical measure $\mathbb{P}$.

To formalize the theory we need to introduce some terminology. We start by defining the space $L(X)$ and the notion of self-financing pre-strategies.

Definition 3.14. We denote by $L(X)$ the linear space of all real valued predictable processes $(\theta_t)_{t\geq 0}$ integrable with respect to $(X_t)_{t\geq 0}$ i.e. we say that $(\theta_t)_{t\geq 0} \in L(X)$ if $\int_0^t \theta_s dX_s < \infty$, see alsoDellacherie and Meyer (1982) Chapter VI for more details. A self-financing pre-strategy is any pair (V_0, θ_t) such that V_0 is an $\mathcal{F}_0$-measurable random variable and $(\theta_t)_{t\geq 0} \in L(X)$. We associate the value process of (V_0, θ_t) given by

$$V_t = V_0 + \int_0^t \theta_s dX_s, \quad t \in [0, T].$$

The predictability of the process $(\theta_t)_{t\geq 0}$ in Definition 3.14 is quite essential. This is the informational restriction so that $(\theta_t)_{t\geq 0}$ cannot anticipate fluctuations of $(X_t)_{t\geq 0}$.

Definition 3.15. We define the space $L^2(X)$ to be the space of all real $\mathbb{F}$-predictable processes $(\theta_t)_{t\geq 0}$ such that

$$\|\theta\|_{L^2(X)} := \left(\mathbb{E}\left[\int_0^T \theta_s^2 dX_s\right]\right)^{\frac{1}{2}} < \infty.$$

The following Lemma shows that the stochastic integral of $(\theta_t)_{t\geq 0}$ with respect to $(X_t)_{t\geq 0}$ is well defined given that $(\theta_t)_{t\geq 0} \in L^2(X)$. Its proof can be found in Schweizer (2001).

Lemma 3.16. *Let $(X_t)_{t\geq 0}$ be a $\mathbb{P}$-local martingale. We assume that $(\theta_t)_{t\geq 0} \in L^2(X)$, then the stochastic integrals $\int_0^t \theta_s dX_s$ are well defined. Moreover, the space $I^2(X) := \left\{\int_0^t \theta_s dX_s \mid \theta_t \in L^2(X)\right\}$ is well defined and a subspace of $\mathcal{M}_0^2(\mathbb{P})$ (the space of square integrable martingales null at time zero).*

Definition 3.14 we define the pre-trading strategy. In the theory of risk minimization since we are working with orthogonal martingales, we need further to assume that $(\theta_t)_{t\geq 0} \in L^2(X)$. A pre-strategy in which we have the extra condition that $(\theta_t)_{t\geq 0} \in L^2(X)$ we call it **RM-strategy** (or L^2-strategy). The Definition bellow formulate this.

Definition 3.17 (RM-strategy)**.** An RM-strategy $(\phi_t)_{t\geq 0}$ is any pair $\phi_t = (\theta, \eta)_t$ such that $(\theta_t)_{t\geq 0} \in L^2(X)$ and $\eta = (\eta_t)_{t\geq 0}$ is a $\mathbb{F}$-adapted and real valued process such that the value process $V_t(\phi) = \theta_t X_t + \eta_t$ is a right continuous and square integrable process under $\mathbb{F}$, for all $t \in [0, T]$.

Based on Lemma 3.16, since the stochastic integral $\int_0^t \theta_s dX_s$ is well defined and a local martingale, we are able to provide the definition of the cost process. We also define the risk process.

Definition 3.18 (Cost and risk process)**.** Let $(\phi_t)_{t\geq 0}$ be a RM-strategy, then the cost process is given by

$$C_t(\phi) := V_t(\phi) - \int_0^t \theta_s dX_s, \ \ 0 \leq t \leq T,$$

and the risk process is

$$R_t(\phi) := \mathbb{E}\left[(C_T(\phi) - C_t(\phi))^2 \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T.$$

If the hedging strategy $(\phi_t)_{t\geq 0}$ is self-financing strategy then its cost process is constant which implies that the risk process is zero. However, in incomplete markets this is no longer true. For all the hedging strategies with $V_T(\phi) = H$ we aim to minimize the risk process in a suitable way. The following definition describes this.

Definition 3.19. A RM-strategy ϕ is called risk minimizing strategy if for any other RM-strategy $(\tilde{\phi}_t)_{t\geq 0}$ such that $V_T(\phi) = V_T(\tilde{\phi})$, $\mathbb{P}$ a.s., we have

$$R_t(\phi) \leq R_t(\tilde{\phi}), \quad \mathbb{P} - \text{a.s.}, \ \forall t \in [0, T].$$

The proof of the following Lemma can be found in Schweizer (1994).

Lemma 3.20. *An RM-strategy $(\phi_t)_{t\geq 0}$ is risk-minimizing if and only if*

$$R_t(\phi) \leq R_t(\tilde{\phi}), \quad \mathbb{P} \ a.s,$$

for every $t \in [0, T]$ and for every RM-strategy $(\tilde{\phi}_t)_{t\geq 0}$ which is admissible continuation of $(\phi_t)_{t\geq 0}$, in the sense that $V_T(\phi) = V_T(\tilde{\phi})$, $\mathbb{P}$ a.s. $\theta_s = \tilde{\theta}_s$ for $s \leq t$ and $\eta_s = \tilde{\eta}_s$ for $s < t$.

As we have already mentioned an RM-strategy is not self-financing. However, it turns out that an RM is a mean self-financing. The definition is given as follows, see also Schweizer (2001).

Definition 3.21. An RM-strategy $(\phi_t)_{t\geq 0}$ is mean-self-financing strategy if and only if the cost process $(C_t(\phi))_{t\geq 0}$ is a $\mathbb{P}$-(local) martingale.

Remark 3.22. Remember that a (RM) strategy is called self-financing if and only if the cost process $(C_t)_{t\geq 0}$ has constant paths. Then following Definition3.21, any self-financing strategy is also a mean self-financing.

A quite equivalent result is the following Lemma, for its proof, see Schweizer (1994).

Lemma 3.23. *If $(\phi_t)_{t\geq 0}$ is a risk minimizing RM-strategy, then it is also mean-self-financing.*

An essential tool for studying the KW and GKW decompositions is the orthogonality of two local martingales.

Definition 3.24 (Orthogonality)**.** Two martingales $(X_t)_{t\geq 0}, (Y_t)_{t\geq 0} \in \mathcal{M}^2$ are said to be (strongly) orthogonal if and only if their product $(X_t Y_t)_{t\geq 0}$ is also a martingale.

An important consequence of orthogonality is the following Proposition, see Jacod and Shiryaev (2003), Chapter I, Proposition 4.15.

Proposition 3.25. *Assume that $(X_t)_{t\geq 0}, (Y_t)_{t\geq 0} \in \mathcal{M}^2$. Then there is equivalency between:*

1. *$(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are strongly orthogonal in the sense of Definition 3.24.*
2. *$\langle X, Y \rangle_t^{\mathbb{F}} = 0$.*

Let us provide the definition of KW and GKW decompositions which will be used extensively throughout this thesis. For its proof, we refer to Ansel and Stricker (1993).

Definition 3.26 (Kunita-Watanabe decomposition)**.** Let $(X_t)_{t\geq 0}$ be an $\mathbb{F}$-local martingale. Then any $(Y_t)_{t\geq 0}$ local martingale can be decomposed as

$$Y_t = Y_0 + \int_0^t \theta_s dX_s + L_t,\ 0 \leq t \leq T,$$

where $(\theta_t)_{t\geq 0}$ is an $\mathbb{F}$-predictable and integrable process, and $(L_t)_{t\geq 0}$ is a square integrable martingale orthogonal to the local martingale $(X_t)_{t\geq 0}$. The above decomposition is called Kunita-Watanabe (KW) decomposition.

For the martingale case, the determination of the RM hedging strategy for a contingent claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$ boils down calculating the GKW decomposition.

Definition 3.27 (Galtchouk-Kunita-Watanabe decomposition)**.** Let $(X_t)_{t\geq 0}$ be an $\mathbb{F}$-local martingale and H be **an $\mathcal{F}_T$-measurable random variable** which can be decomposed as

$$H = H_0 + \int_0^T \theta_s dX_s + L_T,$$

where $(\theta_t)_{t\geq 0}$ is an $\mathbb{F}$-measurable and integrable process, and $(L_t)_{t\geq 0}$ is a square integrable $\mathbb{F}$-local martingale, orthogonal to $(X_t)_{t\geq 0}$. The above decomposition is called Galtchouk-Kunita-Watanabe (GKW) decomposition.

Following Föllmer and Sondermann (1986), using Definition 3.27, and since the space $I^2(X) = \{\int_0^t \theta_s dX_s \mid \theta_t \in L^2(X)\}$ is a subspace of $\mathcal{M}_0^2(\mathbb{P})$ then any defaultable claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$ can be uniquely expressed through the GKW decomposition as follows

$$H = \mathbb{E}[H \mid \mathcal{F}_0] + \int_0^T \theta_u^H dX_u + L_T^H,\ \ \mathbb{P} - \text{a.s.}, \tag{3.10}$$

where $(\theta_t^H)_{t\geq 0} \in L^2(X)$ and is local martingale $(L_t^H)_{t\geq 0}$ orthogonal to the local martingale $(X_t)_{t\geq 0}$.

The next Theorem characterizes the uniqueness of the risk minimizing strategy under the martingale case. For its proof, we refer to Schweizer (2001), Theorem 2.4.

Theorem 3.28. *Assume that $(X_t)_{t\geq 0}$, with $0 \leq t \leq T$, is a $\mathbb{P}$-local martingale. Then for any contingent claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$ admits a unique risk minimizing RM* *hedging strategy* $(\phi_t^*)_{t\geq 0}$*, with* $V_T(\phi^*) = H$*,* $\mathbb{P}$ *a.s. Based on (3.10), the hedging strategy* $\phi_t^* = (\theta_t^*, \eta_t^*)_{t\geq 0}$ *can be determined by*

$$\theta_t^* = \theta_t^H, \ \ 0 \leq t \leq T,$$
$$V_t(\phi^*) = \mathbb{E}[H \mid \mathcal{F}_t] := V_t^*, \ \ 0 \leq t \leq T,$$
$$C_t(\phi^*) = \mathbb{E}[H \mid \mathcal{F}_0] + L_t^H \ \ 0 \leq t \leq T.$$

Based on Theorem 3.28 the process $(\theta_t^H)_{t\geq 0}$ can be determined through

$$\theta_t^H = \frac{d\left\langle V, X\right\rangle_t^{\mathbb{F}}}{d\left\langle X\right\rangle_t^{\mathbb{F}}}, \ t \in [0,T].$$

Remark 3.29. We remind the reader that under the local martingale case the two quadratic approaches are the same. Under the martingale case we need to determine the GKW decomposition.

3.4.2 Local risk minimization approach: semimartingale case

In this section, we assume that the process $(X_t)_{t\geq 0}$ is no longer a martingale, but merely a semimartingale under $\mathbb{P}$. In general, the risk minimizing strategies under the semimartingale case do not exist. The following Proposition verifies this. For its proof, we refer to Föllmer and Schweizer (1991).

Proposition 3.30. *If* $(X_t)_{t\geq 0}$ *is semimartingale under the physical measure* $\mathbb{P}$*, a contingent claim* H *admits in general no risk minimizing strategy* $(\phi_t)_{t\geq 0}$ *with* $V_T(\phi) = H$*,* $\mathbb{P}$*a.s.*

Following Proposition 3.30 since the risk-minimizing strategy does not exist in the semimartingale framework under the original pricing measure $\mathbb{P}$, we need to find a less forceful conditions such that the variability of the risk process is determined locally. This is given in Definition 3.36.

Since $(X_t)_{t\geq 0}$ is a semimartingale then from Doob-Meyer decomposition it can be decomposed as

$$X_t = X_0 + M_t^X + \Lambda_t^X, \ \ t \in [0,T],$$

where $(M_t^X)_{t\geq 0}$ is an $\mathbb{F}$-local martingale and $(\Lambda_t^X)_{t\geq 0}$ is an $\mathbb{F}$-predictable process, with paths of finite variation and $\Lambda_0^X = 0$. We assume that $(\Lambda_t^X)_{t\geq 0}$ is absolutely continuous with respect to $(\left\langle M^X\right\rangle_t^{\mathbb{F}})_{t\geq 0}$, such that

$$\Lambda_t^X = \int_0^t a_s d\left\langle M^X\right\rangle_s^{\mathbb{F}}, \ \ 0 \leq t \leq T. \tag{3.11}$$

for some predictable process $(a_t)_{t\geq 0}$.We also introduce the **mean variance tradefoff** (in short MVT) **process**, $(K_t)_{t\geq 0}$

$$K_t := \int_0^t k_s d\left\langle M^X \right\rangle_s^{\mathbb{F}}, \ 0 \leq t \leq T,$$

where $(k_t)_{t\geq 0}$, $k_t = a_t^2$.

A vital and quite technical condition for determining a local risk minimizing strategy is the following.

Definition 3.31 (Structure condition (SC))**.** We say that the process $(X_t)_{t\geq 0}$ satisfies the **structure condition** (SC) if and only if the MVT process $(K_t)_{t\geq 0}$ is finite $\mathbb{P}$ a.s.

Remark. Note that when $(X_t)_{t\geq 0}$ is a continuous semimartingale then the SC is automatically satisfied, for more details, see Schweizer (1995).

The local risk minimization depends on the determination of the GKW decomposition under the minimal martingale measure (MMM) which is equivalent to determining the FS decomposition under the physical measure $\mathbb{P}$. The FS decomposition was first introduced by Föllmer and Schweizer (1991), where they determined the MMM when $(X_t)_{t\geq 0}$ is a continuous semimartingale. An extension is made to the discontinuous case by Ansel and Stricker (1992) assuming that $(X_t)_{t\geq 0}$ is locally bounded. The Definition bellow introduces the FS decomposition.

Definition 3.32 (FS decomposition)**.** A **random variable** U admits the Föllmer Schweizer (in short FS) decomposition if and only if can be written as

$$U = U_0 + \int_0^T \theta_s^U dX_s + L_T^U,$$

where $(L_t^U)_{t\geq 0}$ is a local martingale orthogonal to the martingale part of $(X_t)_{t\geq 0}$ i.e. $(M_t^X)_{t\geq 0}$ and U_0 is a constant.

Definition 3.33 (Θ-space)**.** We denote Θ as the space of all processes $(\theta_t)_{t\geq 0} \in L(X)$ for which the stochastic integral $\int_0^t \theta_s dX_s$ is in the space of square integrable semimartingales. Equivalently $(\theta_t)_{t\geq 0}$ must be predictable with

$$\mathbb{E}\left[\int_0^T \theta_s^2 d[M^X]_s^{\mathbb{F}} + \left(\int_0^T \left|\theta_s d\Lambda_s^X\right|\right)^2\right] < \infty.$$

Definition 3.34. An L^2 strategy is a pair $\phi_t = (\theta, \eta)_t$, where $(\theta_t)_{t\geq 0} \in \Theta$ and $\eta = (\eta_t)_{t\geq 0}$ is a real valued process $\mathbb{F}$-adapted such that the value process $(V_t)_{t\geq 0}$, $V_t(\phi) = \theta_t X_t + \eta_t$ is a right continuous and square integrable for all $t \in [0, T]$.

For the martingale case i.e. the process $(\Lambda_t)_{t\geq 0}$ is zero, we have $\Theta = L(X)$ and therefore the RM-strategy coincides with the L^2 strategy.

The following Definition introduces the notion of small perturbations. This will help us to define the LRM strategies. At the moment, let us stay to the one dimension. Note that similar results has been introduced to the multidimensional case, see Schweizer (2008) for more details.

Let $q = \{t_0, t_1, \ldots\}$ be a partition of $[0, T]$, with mesh size $|q| = \max_{t_i, t_{i+1} \in q}(t_{i+1} - t_i)$. A sequence of partitions (q_n) is called increasing if $q_n \subseteq q_{n+1}$ for all n.

Definition 3.35. A small perturbation is an L^2 strategy $\Delta = (\delta, \varepsilon)$ such that δ is bounded, the variation of the stochastic integral $\int_0^t \delta_s d\Lambda_s$ is uniformly bounded for every t, ω and $\delta_T = \varepsilon_T = 0$. For any subinterval $(s, t]$ of $[0, T]$ we define the small perturbations

$$\Delta \mid_{(s,t]} := (\delta 1_{(s,t]}, \varepsilon 1_{[s,t)}).$$

The main idea of the local risk minimization is that we minimize the risk locally. The strategy is "optimal" even if there exist small perturbations. The Definition bellow describes the LRM strategies.

Definition 3.36 (LRM strategy)**.** For an L^2 strategy $(\phi_t)_{t \geq 0}$, a small perturbation Δ and a partition q of $[0, T]$ we set

$$r^q(\phi, \Delta) := \sum_{t_t, t_{i+1} \in q} \frac{R_{t_i}\left(\phi + \Delta \mid_{(t_i, t_{i+1}]}\right) - R_{t_i}(\phi)}{\mathbb{E}[\langle M^X \rangle^{\mathbb{F}}_{t_{i+1}} - \langle M^X \rangle^{\mathbb{F}}_{t_i} \mid \mathcal{F}_{t_i}]} 1_{(t_i, t_{i+1}]}.$$

The strategy $(\phi_t)_{t \geq 0}$ is called locally risk minimizing (LRM) if

$$\lim_{n \to \infty} \inf r^{q_n}(\phi, \Delta) \geq 0 \quad (\mathbb{P} \otimes \left\langle M^X \right\rangle^{\mathbb{F}}), -\text{a.e on } [0, T]),$$

for every small perturbation Δ and every increasing sequence $(q_n)_{n \in \mathbb{N}}$ of partitions ending to the identity.

The idea of determining LRM hedging strategies is restricted to the class of of self-financing strategies. The following Lemma present this. For the proof see Schweizer (2001).

Lemma 3.37. *Assume that* $(\left\langle M^X \right\rangle^{\mathbb{F}}_t)_{t \geq 0}$ *is* $\mathbb{P}$ *strictly increasing. If an* L^2 *strategy is LRM then it is also mean self-financing.*

The following Theorem provide sufficient conditions such that an LRM strategy exists. For its proof, we refer to Schweizer (2001).

Theorem 3.38. *Assume that* $(X_t)_{t \geq 0}$ *is a semimartingale, and it satisfies the SC,* $(M^X_t)_{t \geq 0} \in \mathcal{M}^2_0(\mathbb{P})$ *and* $(\left\langle M^X \right\rangle^{\mathbb{F}}_t)_{t \geq 0}$ *is* $\mathbb{P}$ *a.s. strictly increasing. Also we further assume that* $(\Lambda^X_t)_{t \geq 0}$ *is* $\mathbb{P}$ *a.s. continuous, and* $\mathbb{E}[K_T] < \infty$*. Let* $H \in L^2(\mathcal{F}_T, \mathbb{P})$ *be a contingent claim, and* ϕ *be an* L^2 *strategy with* $V_T(\phi) = H$*. Then* ϕ *is LRM strategy if* ϕ *is mean self-financing and the martingale* $(C_t(\phi))_{t \geq 0}$ *is strongly orthogonal to the martingale part of* $(X_t)_{t \geq 0}$*,* $(M^X_t)_{t \geq 0}$*.*

Definition 3.39 (PLRM strategy)**.** Let H be a contingent claim, $H \in L^2(\mathcal{F}_T, \mathbb{P})$. An L^2 strategy ϕ with $V_T(\phi) = H$, $\mathbb{P}$ a.s., is called pseudo-locally risk minimizing (PLRM) or pseudo optimal for H if $(\phi_t)_{t\geq 0}$ is mean self-financing and the martingale $C_t(\phi)$ is orthogonal to $(M_t^X)_{t\geq 0}$.

We emphasize here, that if the MVT process is uniformly bounded with respect to t and ω then SC is satisfied and there exists an PLRM hedging strategy.

From Theorem 3.38 and the above Definition the determination of the LRM strategy is given through the PLRM strategy by obtaining the FS decomposition. An important and necessary condition for this is that the underlying asset should satisfy the SC and the defaultable claim must be attainable. The following Proposition describes this, for its proof, see Schweizer (2001).

Proposition 3.40. *A contingent claim $H \in L^2(\mathcal{F}_T, \mathbb{P})$ admits a pseudo optimal L^2 strategy ϕ, with $V_T = H$, $\mathbb{P}$ a.s. if and only if it can be written as*

$$H = H_0 + \int_0^T \theta_u^H dX_u + L_T^H, \ \ \mathbb{P} \ a.s,$$

where $H_0 \in L^2(\mathcal{F}_0, \mathbb{P})$, $(\theta_t^H)_{t\geq 0} \in \Theta$, and $(L_t^H)_{t\geq 0} \in \mathcal{M}_0^2$ orthogonal to $(M_t^X)_{t\geq 0}$. Moreover, the hedging strategy $\phi_t = (\theta, \eta)_t$ is given by

$$\theta_t = \theta_t^H, \ \ 0 \leq t \leq T,$$

the cost process is

$$C_t(\phi) = H_0 + L_t^H, \ \ 0 \leq t \leq T,$$

and its value process

$$V_t(\phi) = C_t(\phi) + \int_0^t \theta_u dX_u = H_0 + \int_0^t \theta_u^H dX_u + L_t^H, \ \ 0 \leq t \leq T. \tag{3.12}$$

3.4.3 Minimal martingale measure

A key element for the relation between the FS and GKW decompositions is the MMM. Through this measure we can determine the FS decomposition explicitly. It was introduced by Föllmer and Schweizer (1991) and Schweizer (1995), when $(X_t)_{t\geq 0}$ is a continuous semimartingale. Choulli et al. (2010) determined it when $(X_t)_{t\geq 0}$ is a Lévy process. Ceci et al. (2015b) determined the MMM for a jump diffusion process when there exists an asymmetric information in the underlying asset through projections. In general, the characterization of the MMM remains an open problem. For the rest of this section, we restrict ourselves to the case when $(X_t)_{t\geq 0}$ is merely a continuous semimartingale. The Definition bellow introduces the MMM.

Definition 3.41. A martingale measure $\hat{\mathbb{P}} \approx \mathbb{P}$ will be called minimal if

$$\hat{\mathbb{P}} = \mathbb{P} \text{ on } \mathcal{F}_0,$$

and if any square integrable $\mathbb{P}$ martingale which is orthogonal to $(M_t^X)_{t\geq 0}$ under $\mathbb{P}$ remains a martingale under $\hat{\mathbb{P}}$:

$$(L_t)_{t\geq 0} \in \mathcal{M}^2 \text{ and } \left\langle L, M^X \right\rangle_t^{\mathbb{F}} = 0 \Longrightarrow (L_t)_{t\geq 0} \text{ is a martingale under } \hat{\mathbb{P}}.$$

In general to determine the MMM, we need to find martingale density. Following Föllmer and Schweizer (1991) when $(X_t)_{t\geq 0}$ is a semimartingale, the MMM can be defined as

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(-\int adM^X)_T.$$

The process $(\hat{G}_t)_{t\geq 0}$ $\hat{G}_t = \mathcal{E}(-\int_0^t a_s dM_s^X) = \exp(-\int_0^t a_s dM_s^X - \frac{1}{2}\int_0^t a_s^2 d\langle M^X\rangle_s)$ is the density, and $\mathcal{E}$ is the Doléans-Dade stochastic exponential, see Protter (2004) Chapter II for its properties. The process $(\tilde{G}_t)_{t\geq 0}$ should be a uniformly integrable martingale. In general, the density $(\hat{G}_t)_{t\geq 0}$ is the unique strong solution of the following stochastic differential equation

$$d\hat{G}_t = -\hat{G}_{t-} a_t dM_t, \ \ \hat{G}_0 = 1.$$

For the case when $(X_t)_{t\geq 0}$ is a continuous semimartingale then $(\hat{G}_t)_{t\geq 0}$ has the following explicit form

$$\hat{G}_t = \exp\left(-\int_0^t a_s dM_s^X - \frac{1}{2}\int_0^t a_s^2 d\left\langle X\right\rangle_s^{\mathbb{F}}\right), \ \ 0 \leq t \leq T,$$

and we have the following useful Theorem, see Föllmer and Schweizer (1991).

Theorem 3.42. *Assume that $(X_t)_{t\geq 0}$ is continuous semimartingale. Then the following statements are valid*

1. *The MMM $\hat{\mathbb{P}}$ is uniquely determined.*
2. *$\hat{\mathbb{P}}$ exists if and only if*

$$\hat{G}_t = \exp\left(-\int_0^t a_s dM_s^X - \frac{1}{2}\int_0^t a_s^2 \left\langle X\right\rangle_s^{\mathbb{F}}\right), \ 0 \leq t \leq T, \tag{3.13}$$

   is a square integrable martingale under $\mathbb{P}$; in that case $\hat{\mathbb{P}}$is given by $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \hat{G}_T$.

3. *The MMM preserves orthogonality: based on Proposition 3.25, we know that any $(L_t)_{t\geq 0} \in \mathcal{M}^2$ with $\left\langle L, M^X \right\rangle_t^{\mathbb{F}} = 0$ under $\mathbb{P}$ satisfies $\left\langle L, M^X \right\rangle_t^{\mathbb{F}} = 0$ under $\hat{\mathbb{P}}$.*

Following Theorem 3.42 then since $(\hat{G}_t)_{t\geq 0}$ is a $\mathbb{P}$ square integrable martingale then

- $(\hat{G}_t X_t)_{t\geq 0}$ is a $\mathbb{P}$ martingale.
- $(\hat{G}_t \int_0^t \theta_s dX_s)_{t\geq 0}$ is a $\mathbb{P}$ martingale, for every $(\theta_t)_{t\geq 0} \in \Theta$.

Based on (3.12), it is easy to see that for a pseudo optimal L^2 strategy ϕ for a contingent claim H, the product $\hat{G}_t V_t(\phi)$ is also a $\hat{\mathbb{P}}$ martingale. Therefore the value process

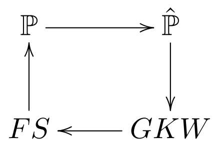


Figure 3.3: Relation between FS decomposition and GKW decomposition in the continuous semimartingale case, through the MMM.

$(V_t(\phi))_{t\geq 0}$ is a $\hat{\mathbb{P}}$-(local) martingale. Let us define the value process under $\hat{\mathbb{P}}$

$$V_t^{H,\hat{\mathbb{P}}} := \mathbb{E}[H \mid \mathcal{F}_t], \;\; 0 \leq t \leq T. \tag{3.14}$$

For the special case when $(X_t)_{t\geq 0}$ is a continuous semimartingale under $\mathbb{P}$, then it will be a local martingale under $\hat{\mathbb{P}}$, and its value process $(V_t^{H,\hat{\mathbb{P}}})_{t\geq 0}$ admits a GKW decomposition, and we get

$$V_t^{H,\hat{\mathbb{P}}} = V_0^{H,\hat{\mathbb{P}}} + \int_0^t \theta_u^{H,\hat{\mathbb{P}}} dX_u + L_t^{H,\hat{\mathbb{P}}}, \;\; 0 \leq t \leq T, \tag{3.15}$$

where $(\theta_t^{H,\hat{\mathbb{P}}})_{t\geq 0} \in L(X)$, and $(L_t^{H,\hat{\mathbb{P}}})_{t\geq 0}$ is a local martingale orthogonal to $(X_t)_{t\geq 0}$ under $\hat{\mathbb{P}}$.

The Theorem bellow introduces the relationship between the MMM and the FS decomposition. For its proof, see Schweizer (2001).

Theorem 3.43. *Assume that $(X_t)_{t\geq 0}$ is continuous and hence it satisfies the SC. We further assume that the strictly positive $\mathbb{P}$-local martingale $(\hat{G}_t)_{t\geq 0}$ given by (3.13), is a square integrable martingale. Define the MMM $\hat{\mathbb{P}}$ as $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \hat{G}_T$ and let the value process under $\hat{\mathbb{P}}$ given by (3.14). Then if either*

$$H \text{ admits a Föllmer Schweizer decomposition}$$

or

$$V_0^{H,\hat{\mathbb{P}}} \in L^2(\mathbb{P}), \; \theta^{H,\hat{\mathbb{P}}} \in \Theta, \; \text{and } L^{H,\hat{\mathbb{P}}} \in \mathcal{M}^2(\mathbb{P}),$$

then for $t = T$ (3.15) becomes

$$V_T^{H,\hat{\mathbb{P}}} = V_0^{H,\hat{\mathbb{P}}} + \int_0^T \theta_u^{H,\hat{\mathbb{P}}} dX_u + L_T^{H,\hat{\mathbb{P}}},$$

and it gives the Föllmer Schweizer decomposition of H and $(\theta_t^{H,\hat{\mathbb{P}}})_{t\geq 0}$ determines a pseudo optimal L^2 strategy for H. A sufficient condition is that the MVT process $(K_t)_{t\geq 0}$ is uniformly bounded.

An intuitive explanation of Theorem 3.43 is that under the assumption that $(X_t)_{t\geq 0}$ is a continuous semimartingale calculating the optimal local risk minimization strategy boils down to the determination of the GKW decomposition, under the MMM $\hat{\mathbb{P}}$. Assuming that the contingent claim H can be expressed as a function of the random variable X_T

then determining the value process (3.15) under the MMM can be reduced into solving a partial differential equation. Similar idea will be used in Chapters 4, 6 and 7 where assuming that $(X_t)_{t\geq 0}$ or $(Y_t)_{t\geq 0}$ with $Y_t = \exp(X_t)$ is a Lévy process , and without using the minimal martingale measure, then the value process will be expressed through a PIDE.

3.4.4 Mean variance approach

We turn now our attention to the mean variance approach. As we have seen the local risk minimization method is applied at the maturity, $V_T = H$, $\mathbb{P}$ a.s. On the contrary, in the mean variance approach we do not need to impose this condition, but instead the self-financing condition is applied. This method minimizes the difference between the contingent claim and the value of the portfolio at the maturity time T.

Let Θ_2 be the space such that all the stochastic integrals of the form $U_t(\theta) := \int_0^t \theta_u^H dX_u$, that satisfies $U_T(\theta) \in L^2(\mathbb{P})$. Given a contingent claim $H \in L^2(\mathbb{P})$ which is $\mathcal{F}_T$-measurable, then if we assume that there exists a linear subspace $\Theta \subset \Theta_2$, the mean variance approach is determined by

$$\underset{\theta\in\Theta}{\text{minimize}}\, \mathbb{E}\left[\left(H - V_0 - \int_0^T \theta_s^H dX_s\right)^2\right].$$

Definition 3.44 (Mean variance optimal strategy)**.** A mean variance strategy is any pair (V_0, θ_t) such that $(\theta_t)_{t\geq 0} \in \Theta$ and $V_0 \in \mathbb{R}$. If $H \in L^2(\mathbb{P})$ is contingent claim then an mean variance strategy $(V^{mvo}, \theta^{mvo})_t$ is called mean variance optimal strategy for H if it minimizes

$$\mathbb{E}\left[(H - V_T(V_0, \theta))^2\right] = \left\| H - V_0 - \int_0^T \theta_u dX_u \right\|^2_{L^2(\mathbb{P})},$$

over all mean variance strategies (V_0, θ_t).

When the underlying asset $(X_t)_{t\geq 0}$ is a martingale we have seen that the optimal mean variance hedging strategy can be determined through the GKW decomposition. However, when we are dealing with semimartingales then the situation is more complicated, and we need to introduce the variance optimal martingale measure.

Let us denote by $\mathcal{P}$ the set of all convex equivalent martingale measures and

$$\mathcal{P}_e^2 := \left\{\mathbb{Q} \in \mathcal{P} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^2(\mathbb{P})\right\} \subseteq \mathcal{P},$$

be the set of all equivalent martingale measures with square integrable density, and we assume that $\mathcal{P}_e^2 \neq \emptyset$

Definition 3.45. The variance optimal martingale measure $\tilde{\mathbb{P}}$ is the unique element of $\mathcal{P}_e^2$ such that minimizes $\left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|_{L^2(\mathbb{P})} = \sqrt{1 + Var_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]}$ for all $\mathbb{Q} \in \mathcal{P}_e^2$.

Definition 3.46. The space Θ_{GLP} consists of all $(\theta_t)_{t\geq 0} \in L(X)$ such that $U_T(\theta)$ is in $L^2(\mathbb{P})$ and the process $U_t(\theta) = \int_0^t \theta_s dX_s$ is uniformly integrable for every $\mathbb{Q}$ martingale measure, $\mathbb{Q} \in \mathcal{P}_e^2(X)$. The space Θ_S consists of all $(\theta_t)_{t\geq 0} \in L(X)$ such that $U(\theta)$ is in the space of square integrable semimartingales.

The space Θ_S is a generalization of the $L^2(X)$ introduced in the martingale space. Following Schweizer (2001), an advantage of the Θ_{GLP} is that in this space duality formulations are more manageable in the theoretical framework. The next Theorem is the section's main result, for its proof, we refer to Schweizer (2001).

Theorem 3.47. *Let $H \in L^2(\mathbb{P})$ be a contingent claim and write the GKW decomposition of H under the variance optimal martingale measure $\tilde{\mathbb{P}}$ with respect to $(X_t)_{t\geq 0}$ as*

$$H = \mathbb{E}^{\tilde{\mathbb{P}}}[H] + \int_0^T \xi_u^{H,\tilde{\mathbb{P}}} dX_u + L_T^{H,\tilde{\mathbb{P}}} = V_T^{H,\tilde{\mathbb{P}}},$$

with

$$V_t^{H,\tilde{\mathbb{P}}} := \mathbb{E}^{\tilde{\mathbb{P}}}[H \mid \mathcal{F}_t] = \mathbb{E}^{\tilde{\mathbb{P}}}[H] + \int_0^t \xi_u^{H,\tilde{\mathbb{P}}} dX_u + L_t^{H,\tilde{\mathbb{P}}}, \quad 0 \leq t \leq T.$$

Then the mean variance optimal strategy for H is given by

$$V_0^{mvo} = \mathbb{E}^{\tilde{\mathbb{P}}}[H],$$

and

$$\begin{aligned} \theta_t^{mvo} &= \xi_t^{H,\tilde{\mathbb{P}}} - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(V_{t-}^{H,\tilde{\mathbb{P}}} - \mathbb{E}^{\tilde{\mathbb{P}}}[H] - \int_0^t \theta_u^{mvo} dX_u\right) \\ &= \xi_t^{H,\tilde{\mathbb{P}}} - \tilde{\zeta}_t \int_0^{t-} \frac{1}{\tilde{Z}_u} dL_u^{H,\tilde{\mathbb{P}}}, \quad 0 \leq t \leq T, \end{aligned}$$

where

$$\tilde{Z}_t := \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \mid \mathcal{F}_t\right] = \tilde{Z}_0 + \int_0^t \tilde{\zeta}_u dX_u, \quad 0 \leq t \leq T,$$

for a process $(\tilde{\zeta}_t)_{t\geq 0} \in \Theta_{GLP}$.

Biagini and Cretarola (2009, 2012), determine a hedging strategy for a continuous semimartingale asset, through the LRM under partial information. More specifically, in their models the default time is determined through the hazard rate. Okhrati (2019) studied the LRM for a defaultable claim whose asset is modelled by a continuous semimartingale and when there exists a delay in the information between investors and a company's management board. Heath et al. (2001), compared the two quadratic approaches and they provide numerical results for a hedging strategy with stochastic volatility. Kohlmann et al. (2010) apply the mean variance approach when the asset is a quasi left continuous semimartingale with bounded jumps.

In general when the asset $(X_t)_{t\geq 0}$ is a jump diffusion process, determining the LRM hedging strategy is not an easy task. More precisely, when $(X_t)_{t\geq 0}$ is a discontinuous process, then the GKW and FS decompositions do not coincide under the minimal martingale measure $\hat{\mathbb{P}}$. The main reason for this is that orthogonality is no preserved between the measures $\mathbb{P}$ and $\hat{\mathbb{P}}$. Given that $(X_t)_{t\geq 0}$ is a discontinuous semimartingale, Choulli et al. (2010) compared the two decompositions and proved their relationship. Ceci et al. (2015b) extended the results of Choulli et al. (2010) by determining the GKW decomposition through an MMM for a semimartingale that admits jumps when investors have a restricted available information in the underlying asset.

Chapter 4

Hedging defaultable claims by Itô's-formula: the case of non-smooth functions

4.1 Introduction

In this chapter, we provide an application of the local risk minimization (LRM) approach under complete information for investors using non-smooth Itô's formula. Our approach is based on Okhrati et al. (2014), who developed a hedging strategy for a defaultable claim, where the default time is modelled through a structural model with constant barrier and the underlying asset is a Lévy process of finite variation. The hedging strategy can be derived through the GKW decomposition based on a solution of a PIDE. However, a drawback of this model is that the solution of this PIDE is not necessarily smooth which brings some limitations especially in the numerical implementations.

In general, if the solution of a PIDE represented as $f(t,x)$ is a smooth function, then it has a probabilistic representation given by the Feynman-Kac formula. However, for models when the asset is modelled by a pure jump process with infinite activity and so the diffusion component does not exists, the smoothness of the conditional expectation derived by the Feynman-Kac formula does not always hold. In this case, the jump components of a PIDE behave as convection terms. For more details, we refer to Cont et al. (2004). Another problem is that sometimes the initial\terminal condition of a PIDE ($f(T,x) = F(x)$ or $f(0,x) = F(x)$) is not continuous or differentiable. Basically, in most cases in quantitative finance $F(x)$ is not smooth, for example exotic options.

Assuming that a function is continuous and it admits weak derivatives, Okhrati and Schmock (2015) extended the Itô's formula for Lévy processes for the non-smooth case. This is a generalization of the Meyer's-Itô formula of Lévy processes. Using the above result, we extend Okhrati et al. (2014) model to the non-smooth case.

The chapter is organized as follows. In Section 4.2, we give some auxiliary results describing the default time which is modelled trough a structural framework. In Section 4.3, we analyse the Itô's formula for non-smooth functions. Also, in Section 4.4, the hedging strategy through the local risk minimization approach is obtained.

4.2 Preliminaries and model description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F}$, $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, and it is the filtration generated by the underlying asset process $(X_t)_{t\geq 0}$.

Assumption 4.1. *We assume that the underlying asset $(X_t)_{t\geq 0}$ is a Lévy process of finite variation, with triplet $(b, 0, \nu)$, where ν is the Lévy measure on $\mathbb{R} \setminus \{0\}$. From Lévy-Itô decomposition the process has the following representation*

$$X_t = u + \mu t + \int_0^t \int_{\mathbb{R}} zN(ds, dz), \quad X_0 = u > 0, \ t \in [0, T], \tag{4.1}$$

where $N(dt, dz)$ is the jump measure with intensity $\nu(dz)dt$, $\int_{\mathbb{R}} z^2\nu(dz) < \infty$ and $\mu = b - \int_{-1}^{1} z\nu(dz)$. The process may have finite activity i.e. $\nu(\mathbb{R}) < \infty$ or not $(\nu(\mathbb{R}) = \infty)$, and we assume that for the case $\nu(\mathbb{R}) < \infty$, the Lévy measure is continuous.

Let τ be a stopping time which is given through

$$\tau = \inf\{t : X_t < 0\}, \tag{4.2}$$

with the convention that $\tau = \infty$, if $X_t \geq 0$. We remark that the available information $\mathbb{F}$, generated by the underlying asset $(X_t)_{t\geq 0}$ is complete, however, the stopping time is a totally inaccessible one.

Remark 4.2. Note that the stopping time introduced in (4.2) is strictly negative. Furthermore, the stopping time (4.2) cannot be positive, since in this case it will not be totally inaccessible (see also Theorem 4.4). The result is a special case of Theorem 1.3 of Guo and Zeng (2008).

In the financial environment, it may be useful to introduce the exponential underlying asset, $(Y_t)_{t\geq 0}$, $Y_t := e^{X_t}$. In this case, the stopping time can also be defined as $\tau := \inf\{t : Y_t < D\}$, where the barrier D is constant $0 < D < e^u$. This is consistent with the first passage models introduced in Section 3.2.2. Note also that in this case, the stopping time can be expressed as $\tau = \inf\{t : X_t < \log(D)\}$ and the model based on (4.2) can incorporate this case as well.

Following Kyprianou (2014), Chapter 7, we provide some general results in to order to justify why the default time, defined in (4.2), is a totally inaccessible stopping time.

Definition 4.3. A Lévy process $(X_t)_{t\geq 0}$ creeps downwards at level $x = 0$ if

$$\mathbb{P}(X_\tau = 0) > 0,$$

where τ is introduced in (4.2).

The following Theorem characterizes when a finite variation Lévy process creeps downwards. For its proof we refer to Kyprianou (2014), Theorem 7.11.

Theorem 4.4. *Let $(X_t)_{t\geq 0}$ be a Lévy process which is not a compound Poisson (i.e. the Lévy measure can also take infinite values). Then $(X_t)_{t\geq 0}$ creeps downwards if and only if $(X_t)_{t\geq 0}$ is of finite variation with Lévy-Knitchine formula*

$$\Psi(\theta) = -i\theta\mu + \int_{\mathbb{R}\{0\}} (1 - e^{i\theta z})\nu(dz), \text{ where } \mu < 0.$$

Since the process $(X_t)_{t\geq 0}$, given by (4.1), is not a compound Poisson since from Assumption 4.1 the process may have infinite activity and its drift μ is positive, then the process never creeps downwards and therefore the stopping time occurs at a jump time of $(X_t)_{t\geq 0}$. Using Theorem 4, Chapter III of Protter (2004), then the stopping time is totally inaccessible. The later implies that the indicator process $(1_{\{\tau\leq t\}})_{t\geq 0}$ admits an intensity $(\lambda_t)_{t\geq 0}$. Since $\mathbb{P}(\tau = 0) = 0$ then by applying Theorem 1.3 of Guo and Zeng (2008), the intensity process is given through

$$\lambda_t = 1_{\{\tau > t\}}\nu(-\infty, -X_t], \quad t \in [0, T], \tag{4.3}$$

and thus the process $\left(1_{\{\tau\leq t\}} - \int_0^t 1_{\{\tau > s\}}\nu(-\infty, -X_s]ds\right)_{t\geq 0}$ is an $\mathbb{F}$-local martingale. Note that the process $(\int_0^t 1_{\{\tau > s\}}\nu(-\infty, -X_s]ds)_{t\geq 0}$ is well defined and finite almost surely, see Okhrati et al. (2014), Lemma A.2.

Our goal here is to find a hedging strategy for a defaultable claim which its payoff is given by $F(X_T)1_{\{\tau > T\}}$, $F(x) \in L^1_{loc}(\mathbb{R})$, using the Itô's formula for the non smooth case, introduced by Okhrati and Schmock (2015), when the default time τ is given by (4.2). We assume that the underlying process $(X_t)_{t\geq 0}$ is a pure jump Lévy process of finite variation, defined in (4.1).

4.3 Itô's lemma for the non-smooth case for Lévy processes of finite variation

We begin with the introduction of some basic results of the Itô's formula for the non -smooth case, when the underlying asset is of finite variation, see Okhrati and Schmock (2015).

Let us first provide the essential definitions of $L^p(A)$ spaces and $L^1_{loc}(A)$, $A \subset \mathbb{R}$.

Definition 4.5 (L^p space)**.** Let $(\Omega, \mathcal{F}, m)$ be a measurable space, and m is the Lebesgue measure. If $f : \Omega \to \mathbb{C}$ is a measurable function on Ω, then we define

$$L^p(A) := \left\{ f : \int_A |f|^p dm \text{ is finite} \right\}, \quad 0 < p < \infty,$$

and its norm is given by

$$\|f\|_p = \left(\int_A |f|^p dm\right)^{\frac{1}{p}}.$$

Definition 4.6. A measurable function $f : \Omega \to \mathbb{C}$ is called locally integrable if for every open set A we have $\int_A |f(x)|dm < \infty$ where m is Lebesgue measure. We denote this space by $L^1_{loc}(\Omega)$.

$$L^1_{loc}(\Omega) = \{f : f|_A \in L^1(A) \; \forall A \subset \Omega, \; A \,\text{compact}\},$$

Similarly, the space $L^p_{loc}(\Omega)$, $p \geq 1$ is

$$L^p_{loc}(\Omega) = \{f : f|_A \in L^p(A) \; \forall A \subset \Omega, \; A \,\text{compact}\}, \; \forall p \geq 1.$$

Definition 4.7 (Weak derivative)**.** Suppose that $a \in \mathbb{N}_0^d$ is a multi-index. Then a function $f \in L^1_{loc}(A)$, $A \subset \mathbb{R}$ is weakly differentiable and its a^{th} weak derivative denoted by $\partial^a f$ is given b

$$\int_A (\partial^a f(x))\varphi(x)dx = (-1)^{|a|} \int_A f(x)(\partial^a \varphi(x))dx, \quad \forall \varphi \in C_c^\infty(A). \tag{4.4}$$

More generally, for a function $f : \mathbb{R}_0^+ \times A \to \mathbb{R}$ then $f \in L^1_{loc}(\mathbb{R}_0^+ \times A)$ has weak derivatives $\partial^a f \in L^1_{loc}(\mathbb{R}_0^+ \times A)$ given by

$$\int_{[0,\infty)\times A} (\partial^a f(x))\varphi(x)dx = (-1)^{|a|} \int_{[0,\infty)\times A} f(x)(\partial^a \varphi(x))dx \;\; \forall \varphi \in C_c^\infty(\mathbb{R}_0^+ \times A).$$

An intuition behind the Definition 4.7 is that weak derivatives behave like ordinary derivatives except on sets with zero measure. Weak derivatives ignore sets of zero measure as integration neglects sets whose measure is zero.

The following Lemma reveals the boundedness of the weak derivative. For the proof, we refer to Okhrati and Schmock (2015).

Lemma 4.8. *Assume that $f \in L^1_{loc}(A)$ has the weak derivative $\partial^a f \in L^1_{loc}(A)$. Suppose that $\varphi \in C^\infty(\mathbb{R}^d)$ is a test function with support K such that $\varphi \geq 0 \; \forall x \in \mathbb{R}^d$. In this case, we have*

$$|\partial^a (f * \varphi)(x)| \leq \sup_{z \in A \cap \Lambda(x)} |\partial^a f(z)|,$$

where $\Lambda(x) = \{y \in \mathbb{R}^d : x - y \in K\}$.

Assumption 4.9. *We assume that the measure $q_t(B) = \mathbb{P}(X_t \in B)$,where $B \subset A$ is Borel measurable set and $t \in [0, T]$, is absolutely continuous with respect to Lebesgue measure.*

Remark 4.10. Note that if $B = [0, t] \times A$ where $A \in \mathcal{B}(\mathbb{R})$ then Assumption 4.9 is the amount of time that the Lévy process spends on the Borel set A. By the absolute continuity of measure $q_t(\cdot)$ and using Proposition 3.1 of Okhrati and Schmock (2015) it is equivalent to assume that both $X_t \in A$ and $X_{t-} \in A$ for each $t \in [0, T]$.

Remark 4.11. Note also that Assumption 4.9 is valid for the finite variation case for the infinity activity case when the Lévy measure is absolutely continuous with respect to Lebesgue measure. For the finite activity case it is not valid. To see this, following Sato (1999) Remark 27.3, when $(X_t)_{t\geq 0}$ is a compound Poisson then the measure $q_t(\cdot)$ is expressed as

$$q_t(A) = e^{-t\nu(A)} \sum_{k=0}^{+\infty} \frac{t^k \nu^k}{k!},$$

which is not continuous as $\mathbb{P}(X_t = 0) > 0$ for $t \in [0, T]$.

The following Theorem establishes the non-smooth Itô formula for infinite activity Lévy processes and it is the main result of Okhrati and Schmock (2015).

Theorem 4.12. *Assume that $f : [0, T] \times A \to \mathbb{R}$ is a continuous function on $\mathbb{R}_0^+ \times A$ such that $f \in L^1_{loc}([0, T] \times A)$, supp$(f) \subset [0, T] \times A$ and A is open set in $\mathbb{R}$. Also, let the weak derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x} \in L^1_{loc}([0, T] \times A)$ be locally bounded defined by (4.4). Suppose that $(X_t)_{t\geq 0}$ is a finite variation Lévy process with triplet $(b, 0, \nu)$ given by (4.1) satisfying Assumption 4.9 such that $\forall t \in [0, T]$ X_t and X_{t-} are in A, then we have almost everywhere*

$$\begin{aligned} f(t, X_t) = f(0, X_0) &+ \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \mu \int_0^t \frac{\partial f}{\partial x}(s, X_s) ds \\ &+ \int_0^t \int_{\mathbb{R}} (f(s, X_{s-} + z) - f(s, X_s) N(ds, dz), \ \ t \in [0, T]. \end{aligned}$$

4.4 LRM for a defaultable claim

In this section, we provide the main results of this chapter. Our aim is to determine the LRM for a defaultable claim of the form $F(X_T)1_{\{\tau>T\}}$ when $F \in L^1_{loc}(A)$, A is an open set of $\mathbb{R}$ and the stopping time τ is defined in (4.2). Note that our analysis is made for a simple defaultable claim where there is no recovery rate in case of default and absent of an interest rate. We derive the LRM through the Kunita-Watanabe (KW) decomposition by determining the canonical decomposition of $(f(t, X_t)1_{\{\tau>t\}})_{t\geq 0}$. In simple words, through a solution of a PIDE which is not necessarily smooth, for the local martingale case we derive the GKW decomposition.

4.4.1 KW and GKW decompositions

Note that all the discussions are arguments of this sections are direct extensions of Okhrati et al. (2014). More precisely, we use the non-smooth version of the Itô's formula developed in Okhrati and Schmock (2015) and extend the results of Okhrati et al. (2014).

Before we start our analysis we impose the following integrability condition.

Assumption 4.13. *Let $f(t,x)$ be function such that $f \in L^1_{loc}([0,T]\times A)$ and $supp(f) \subset [0,T] \times A$, for some open set $A \subset \mathbb{R}$. Then we impose the following integrability condition*

$$\int_{\mathbb{R}} |f(t,x+z) - f(t,x)|\nu(dz) < \infty, \quad \forall t \in [0,T] \text{ and } x \in A.$$

As already mentioned, our methodology is primarily based on the solution of an appropriate PIDE. The Assumption bellow provides the general form of the PIDE.

Assumption 4.14. *A function $F = F(x)$ belongs to class $(\star)$ if there exist a function $f \in L^1_{loc}([0,T] \times A)$, $A \in \mathbb{R}$, $f = f(t,x)$ which is the solution of the following PIDE*

$$\mathcal{A}f(t,x) = \frac{(\mathcal{A}P(t,x) - x\mathcal{A}f(t,x) - \alpha f(t,x))}{\int_{\mathbb{R}} z^2\nu(dz)}\alpha, \quad \forall 0 \leq t \leq T \text{ and } x \in A,$$

and $f(T,x) = F(x)$ $\forall x \in \mathbb{R}$, where $P(t,x) = xf(t,x)$, $\alpha = \mu + \int_{\mathbb{R}} z\nu(dz)$ and the operator $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{A}f(t,x) = \frac{\partial f}{\partial t}(t,x) + \mu\frac{\partial f}{\partial x}(t,x) - \int_{(-\infty,-x]} f(t,x+z)\nu(dz)\\ + \int_{\mathbb{R}}(f(t,x+z) - f(t,x))\nu(dz), \ \forall x \in \mathbb{R}, \, t \in [0,T].\end{aligned} \tag{4.5}$$

We proceed to find the canonical decomposition of $(f(t,X_t)1_{\{\tau>t\}})_{t\geq 0}$.

Proposition 4.15. *Let $(X_t)_{t\geq 0}$ be a finite variation Lévy process given by (4.1) satisfying Assumption 4.9 or equivalently X_t, X_{t-} belong in $A \subset \mathbb{R}$. Let $f : [0,T]\times A \to \mathbb{R}$ be $L^1_{loc}([0,T] \times A)$ continuous function, such that $supp(f) \subset [0,T] \times A$ and its derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t} \in L^1_{loc}([0,T] \times A)$ are locally bounded, satisfying (4.4). We further assume that $f(t,x)$ satisfies Assumption 4.13. Then the process $(U_t)_{t\geq 0}$, $U_t = f(t,X_t)1_{\{\tau>t\}}$ is a $\mathbb{F}$-semimartingale and it admits the following decomposition*

$$U_t = U_0 + O_t + \int_0^t \mathcal{A}f(s,X_s)1_{\{\tau>s\}}ds, \ t \in [0,T], \tag{4.6}$$

where $(O_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale and the operator $\mathcal{A}$ is given by (4.5).

Proof. Since $f \in L^1_{loc}(\mathbb{R}^+_0 \times A)$ with $A \subset \mathbb{R}$, then $(f(t,X_t))_{t\geq 0}$ is an $\mathbb{F}$-special semimartingale and using the integration by parts formula (see Definition 2.26) the process $(f(t,X_t)1_{\{\tau\leq t\}})_{t\geq 0}$ can be decomposed as follows

$$f(t,X_t)1_{\{\tau\leq t\}} = \int_0^t 1_{\{\tau<s\}}df(s,X_s) + \int_0^t f(s,X_{s-})d(1_{\{\tau<s\}}) + [f(\cdot,x), 1_{\{\tau\leq\cdot\}}]^{\mathbb{F}}_t. \tag{4.7}$$

Note that the process $Z_t = (f(t,X_t)1_{\tau\leq t})_{t\geq 0}$ is an $\mathbb{F}$-semimartingale, since the three terms given in (4.7) are also semimartingales. Since the weak derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$ are locally bounded, using Theorem 4.12 we have that

$$
\begin{aligned}
f(t,X_t) = f(0,X_0) &+ \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \mu\int_0^t \frac{\partial f}{\partial x}(s,X_s)ds \\
&+ \int_0^t\int_{\mathbb{R}} (f(s,X_{s-}+z) - f(s,X_{s-}))N(ds,dz),\ t\in[0,T].
\end{aligned}
\tag{4.8}
$$

The rest of the proof follows in a similar way to the proof of Proposition of Okhrati et al. (2014), the only difference is that instead of classical Itô's formula for smooth functions, we use the version given by (4.8). □

Having determined the canonical decomposition of $(f(t,X_t)1_{\{\tau>T\}})_{t\geq 0}$ we are able to determine the KW decomposition, see Chapter 3, Section 3.4.

The next Theorem determines the process $(\theta_t)_{t\geq 0}$. For the definition of the orthogonality we refer to Definition 3.24.

Theorem 4.16. *Assume that $(X_t)_{t\geq 0}$ satisfies Assumptions 4.1 and 4.9. We further assume that $f(t,x): [0,T]\times\mathbb{R}\to\mathbb{R}$ is a non smooth function, such that $f\in L^1_{loc}([0,T]\times A)$ and $supp(f)\subset[0,T]\times A$. Its weak derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ are locally bounded satisfying (4.4). Moreover, we presume that $[U,X]^{\mathbb{F}}_t\in\mathcal{A}_{loc}$. Then the process $U_t = f(t,X_t)1_{\{\tau>t\}}$, whose canonical decomposition is given in Theorem 4.15, has the following form*

$$
U_t = f(t,X_t)1_{\{\tau>t\}} = U_0 + \int_0^t \theta_s dX_s + L_t,\ t\in[0,T] \tag{4.9}
$$

where the process $(L_t)_{t\geq 0}$ is orthogonal to the martingale part of $(X_t)_{t\geq 0}$ and the process $(\theta_t)_{t\geq 0}$ is given through

$$
\theta_t = \frac{\mathcal{K}f(t,X_{t-})}{\int_{\mathbb{R}} z\nu(dz)}1_{\{\tau>t\}},\ t\in[0,T], \tag{4.10}
$$

for some operator $\mathcal{K}f(t,x) = \mathcal{A}P(t,x) - x\mathcal{A}f(t,x) - \alpha f(t,x)$, and $\alpha = \mu + \int_{\mathbb{R}} z\nu(dz)$. Finally, for $t=T$ and for $F\in L^1_{loc}$ we have

$$
U_T = F(X_T)1_{\{\tau>T\}} = U_0 + \int_0^T \theta_s dX_s + L_T. \tag{4.11}
$$

Proof. From the canonical decomposition of $(U_t)_{t\geq 0}$, in Proposition 4.15, we have that

$$
O_t := (f(t,X_t)1_{\{\tau>t\}} - \int_0^t \mathcal{A}f(s,X_s)1_{\{\tau>s\}}ds)_{t\geq 0},\ t\in[0,T],
$$

and similarly for $P(t,x) = xf(t,x)$

$$
O^{(1)}_t := (P(t,X_t)1_{\{\tau>t\}} - \int_0^t \mathcal{A}P(s,X_s)1_{\{\tau>s\}}ds)_{t\geq 0},\ t\in[0,T],
$$

are $\mathbb{F}$-local martingales with $0\leq t\leq T$. We need to determine the KW decomposition, see Definition 3.26, of $(O_t)_{t\geq 0}$ with respect to $(M^X_t)_{t\geq 0}$. By assumption, we know that

$[U,X]^{\mathbb{F}}_t \in \mathcal{A}_{loc}$ and following Proposition 2.27, we get that $[U,X]^{\mathbb{F}}_t = [O, M^X]^{\mathbb{F}}_t$. Let $Q^{(1)}_t := \int_0^t \mathcal{A}f(s,X_s)1_{\{\tau>s\}}ds$ and $Q^{(2)}_t := \int_0^t \mathcal{A}P(s,X_s)1_{\{\tau>s\}}ds$. From integration by parts formula for semimartingales, see Corollary 2.26, we have

$$U_tX_t = U_0X_0 + \int_0^t U_{s-}dX_s + \int_0^t X_{s-}dU_s + [U,X]^{\mathbb{F}}_t. \tag{4.12}$$

Since $U_tX_t = U_0X_0 + O^{(1)}_t + Q^{(2)}_t$ then from (4.12) we get that the process

$$\left([U,X]^{\mathbb{F}}_t - \left(Q^{(2)}_t - \int_0^t X_{s-}dQ^{(1)}_s - \int_0^t U_{s-}d\Lambda_s\right)\right)_{t\geq 0},\ t\in[0,T],$$

is an $\mathbb{F}$-local martingale. Therefore the process $(\langle U,X\rangle^{\mathbb{F}}_t)_{t\geq 0}$ is given through

$$\begin{aligned}\langle U,X\rangle^{\mathbb{F}}_t &= Q^{(2)}_t - \int_0^t X_{s-}dQ^{(1)}_s - \int_0^t U_{s-}d\Lambda_s\\ &= \int_0^t \left(\mathcal{A}P(s,X_s) - X_{s-}\mathcal{A}f(s,X_s) - f(s,X_s)\alpha\right)1_{\{\tau>s\}}ds,\ t\in[0,T],\end{aligned}$$

where $\alpha = \mu + \int_{\mathbb{R}} z\nu(dz)$. We proceed with the calculation of $(\langle X\rangle^{\mathbb{F}}_t)_{t\geq 0}$. Since $[X]^{\mathbb{F}}_t \in \mathcal{A}_{loc}$ and $[M^X]^{\mathbb{F}}_t = [X]^{\mathbb{F}}_t$, then

$$\langle X\rangle^{\mathbb{F}}_t = \int_0^t\int_{\mathbb{R}} z^2\nu(dz)ds,\ t\in[0,T]. \tag{4.13}$$

The above argument shows that $(\langle U,X\rangle^{\mathbb{F}}_t)_{t\geq 0}$ is absolutely continuous with respect to $(\langle X\rangle^{\mathbb{F}}_t)_{t\geq 0}$ and the Radon-Nikodym derivative is in fact equal to $(\theta_t)_{t\geq 0}$ and $t\in[0,T]$. Next we define $(L_t)_{t\geq 0}$ as follows

$$L_t = O_t - U_0 - \int_0^t \theta_s dM^X_s,\ t\in[0,T].$$

Breaking down the last decomposition leads to the following equation:

$U_t - \int_0^t \mathcal{A}f(s,X_s)1_{\{s<\tau\}}ds = U_0 + \int_0^t \theta_s d(X-\Lambda^X)_s + L_t,$

where $\Lambda^X_t = \mu t + \int_0^t\int_{\mathbb{R}} z\nu(dz)ds$. It is easy to show that $\langle L, M^X\rangle^{\mathbb{F}}_t = 0$ and hence the $(L_t)_{t\geq 0}$ is orthogonal to the martingale part of $(X_t)_{t\geq 0}$ i.e. $(M^X_t)_{t\geq 0}$. Since $f(t,x)$ satisfies Assumption 4.14 from the above decomposition we obtain (4.9).

Finally, by letting $t = T$ and since $f(T,x) = F(x)$, for a function $F \in L^1_{loc}$ then the decomposition (4.11) is well defined. □

Remark 4.17. In light of Theorem 4.16 the process $(\theta_t)_{t\geq 0}$ has the following semi-explicit form

$$\theta_t = \frac{\int_{\mathbb{R}} zf(t,X_{t-}+z)\nu(dz) - \int_{(-\infty,-X_{t-}]} zf(t,X_{t-})\nu(dz) - f(t,X_{t-})\int_{\mathbb{R}} z\nu(dz)}{\int_{\mathbb{R}} z^2\nu(dz)}1_{\{\tau\geq t\}}.$$

The following Corollary investigates the case when $(X_t)_{t\geq 0}$ is a local martingale.

Corollary 4.18. *Assume that $(X_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale and we further assume that Assumptions 4.1 and 4.9 are satisfied. Let $f(t,x) : [0,t] \times \mathbb{R} \to \mathbb{R}$ be the solution of the following PIDE*

$$\mathcal{A}f(t,x) = 0, \ \ t \in [0,T], \ x \in A,$$

where A is an open subset of $\mathbb{R}$, with a terminal condition

$$f(T,x) = F(x),$$

where $F \in L^1_{loc}(\mathbb{R})$, belongs to class $(\star)$. We also assume that $[U,X]^{\mathbb{F}}_t \in \mathcal{A}_{loc}$. Then the GKW decomposition has the form

$$U_T = F(X_T)1_{\{\tau>T\}} = U_0 + \int_0^T \frac{\mathcal{A}P(s,X_{s-})}{\int_{\mathbb{R}} z^2\nu(dz)} 1_{\{\tau\geq s\}} dX_s + L_T,$$

where the process $(L_t)_{t\geq 0}$ is a $\mathbb{F}$-local martingale orthogonal to $(X_t)_{t\geq 0}$.

Proof. Since $(X_t)_{t\geq 0}$ is a local martingale then $\alpha = 0$, and the operator $\mathcal{A}f(t,x)$ is also zero. Then the process $(\theta_t)_{t\geq 0}$ is reduced to $\theta_t = \frac{\mathcal{A}P(t,X_{t-})}{\int_{\mathbb{R}} z^2\nu(dz)} 1_{\{\tau\geq t\}}$. □

For the rest of this subsection, we are working with finite activity Lévy processes and we further assume that the Lévy measure is absolutely continuous with respect to Lebesgue measure. In this case, under the assumption that the terminal condition of the PIDE $f(T,x) = F(x)$ is smooth, the use of non-smooth Itô's formula is redundant. We use these results primarily in Chapter 8 , in which we solve the PIDEs numerically through finite differences.

In light of Corollary 4.18, let us present an Example specifying the number invested in the risky asset $(X_t)_{t\geq 0}$ which is the process $(\theta_t)_{t\geq 0}$, see also Okhrati et al. (2014).

Example 4.19. Let $(X_t)_{t\geq 0}$ be a spectrally negative Lévy process of finite variation i.e.

$$X_t = X_0 + \mu t + \int_0^t \int_{-\infty}^0 zN(ds,dz), \ \mu > 0, \ t \in [0,T].$$

For the case when $(X_t)_{t\geq 0}$ has finite activity, then it can be represented as a compound Poisson process with intensity λ, $X_t = X_0 + \mu t + \sum_{j=1}^{N_t} Y_j$, where the i.i.d random variables Y_j follow a given prior distribution. For example, we can assume that Y_j follow negative exponential distribution with parameter δ, i.e. $h(z) = \delta \exp(\delta z)1_{\{z\leq 0\}}$. In order $(X_t)_{t\geq 0}$ to be a martingale, we can choose $\mu = \frac{\lambda}{\delta}$. If we further assume that the terminal condition $F(x) \in L^1_{loc}(A)$ is constant given by K, from Corollary 4.18, the process $(\theta_t)_{t\geq 0}$ is given by

$$\theta_t = \frac{\delta^2 \int_{-X_{t-}}^0 zf(t,X_{t-}+z)h(z)dz + \delta f(t,X_{t-})}{2} 1_{\{\tau\geq t\}}, \ t \in [0,T], \tag{4.14}$$

where $f(t,x)$ is the solution of the PIDE introduced in Corollary 4.18 with terminal condition $f(T,x) = K$.

For the special case when $K = 1$, Example 4.19 represents the indicator process $1_{\{\tau>T\}}$ and the solution of the PIDE introduced in Corollary 4.18 has an analytical representation. In this case, it is easy to see that $\mathbb{P}(\tau > T) = f(0, X_0)$. In Chapter 8 we solve this PIDE numerically through finite differences and we provide its error.

Furthermore, the result can be extended to the case when $(X_t)_{t\geq 0}$ is not necessarily a spectrally negative. In this case we can obtain the structure of the default time. Note that from Assumption 4.1 since $\int_{\mathbb{R}} z^2\nu(dz) < \infty$ is square integrable and $[1_{\{\tau>t\}}, X_t]_t \in \mathcal{A}_{loc}$, we may derive the distribution of the default time, see Propositions 5.1 and 5.2 in Okhrati et al. (2014).

Remark 4.20. Its not hard to see that Example 4.19 can also be applied when $F(x)$ is a real valued function and not just a constant, e.g. the payoff of a European call option, $F(x) = \max(x - K, 0)$. In this case the process $(\theta_t)_{t\geq 0}$ has the same form as in (4.14).

4.4.2 PLRM strategies

The above construction leads to the determination of the PLRM and not of the LRM. However, following Chapter 3, Theorem 3.38, if we assume that $(X_t)_{t\geq 0} \in \mathcal{M}^2$, the $\mathbb{F}$-predictable quadratic variation process $(\langle X_t\rangle_t^{\mathbb{F}})_{t\geq 0}$ is strictly increasing, and if the structure condition (SC) holds, then the PLRM leads to the LRM, which yields that the FS decomposition exists, see Chapter 3, Proposition 3.40. Based on Assumption 4.1 since $\int_{\mathbb{R}} z^2\nu(dz) < \infty$ then we know that $(X_t)_{t\geq 0}$ is square integrable.

It remains to prove that the SC for the process $(X_t)_{t\geq 0}$ is uniformly bounded. Note that based on Assumption 4.1 $(X_t)_{t\geq 0}$ is square integrable, i.e. $X_t \in \mathcal{S}^2(\mathbb{P})$ (see Definition 2.24). Once again, from Doob-Meyer decomposition we know that the process $(X_t)_{t\geq 0}$ can be written as $X_t = X_0 + M_t^X + \Lambda_t^X$, where the process $(\Lambda_t^X)_{t\geq 0}$ is given by

$$\Lambda_t^X = \mu t + \int_0^t \int_{\mathbb{R}} z\nu(dz),\ t \in [0, T].$$

Since $\langle M^X\rangle^{\mathbb{F}} = \langle X\rangle^{\mathbb{F}}$, where the process $(\langle X\rangle_t^{\mathbb{F}})_{t\geq 0}$ is given through (4.13) which is increasing, then $(\Lambda_t^X)_{t\geq 0}$ can be rewritten as

$$\Lambda_t^X = \int_0^t \frac{\mu + \int_{\mathbb{R}} z\nu(dz)}{\int_{\mathbb{R}} z^2\nu(dz)} d\,\langle X\rangle_s^{\mathbb{F}},\ t \in [0, T].$$

Following Section 3.4.2, we know that the MVT process $(K_t)_{t\geq 0}$ is

$$K_t = \int_0^t \left(\frac{\mu + \int_{\mathbb{R}} z\nu(dz)}{\int_{\mathbb{R}} z^2\nu(dz)}\right)^2 d\,\langle X\rangle_s^{\mathbb{F}},\ t \in [0, T],$$

and it is uniformly bounded for all $\omega \in \Omega$, $t \in [0, T]$. Thus we showed that all the sufficient conditions, in order a PLRM strategy to exist, are met.

Until now, we have characterized the process $(\theta_t)_{t\geq 0}$ which represents the amount of money invested in the risky asset $(X_t)_{t\geq 0}$. Using Proposition 4.1 of Okhrati et al.

(2014), which basically an immediate result of Proposition 3.40, we can determine the number of shares invested in the non risky asset $(\eta_t)_{t\geq 0}$ along with the value process $(V_t)_{t\geq 0}$ of our portfolio $\phi_t = (\theta, \eta)_t$ and its cost process $(C_t)_{t\geq 0}$.

If we further assume that the solution of the PIDE, introduced in Assumption 4.14, $f(t, x) \in L^2(\mathbb{R}_0^+ \times A)$ for an open set A of $\mathbb{R}$ and $(\theta_t)_{t\geq 0}$ belongs to the Θ-space (see Definition 3.33) then by applying Proposition 4.1 of Okhrati et al. (2014) there exists an L^2-strategy (see Definition 3.34) $\phi_t = (\theta_t, \eta_t)$ such that the process $(L_t)_{t\geq 0}$, which is strongly orthogonal to the martingale part of $(X_t)_{t\geq 0}$ i.e. $(M_t^X)_{t\geq 0}$, is

$$L_t = U_t - U_0 - \int_0^t \theta_s dX_s,$$

the value process $(V_t)_{t\geq 0}$ for the optimal portfolio $\phi_t = (\theta_t, \eta_t)_t$ is

$$V_t(\phi) = U_0 - \int_0^t \theta_s dX_s,$$

the amount of money invested in the non risky asset at time $t \geq 0$ $(\eta_t)_{t\geq 0}$ is

$$\eta_t = V_t - \theta_t X_t,$$

and finally the cost process is given by

$$C_t = U_0 + L_t.$$

Chapter 5

Partial information

5.1 Introduction

The main objective of this chapter is to provide the appropriate literature review for the partial information models. As we have seen in Chapter 3, the structural models assume that investors have complete information about the underlying asset value and its barrier, whereas in the reduced form models the information is less detailed. Although both approaches seem to be disjoint actually, they are not. For example, in Chapter 4, we saw a model that although it is a structural it admits an intensity like reduced form ones. In this chapter, we briefly describe these models.

The chapter is organized as follows. Section 5.2 provides the types of partial information that investors may have. Section 5.3 describes the types of filtration expansion: initial and progressive. We analyse both of them. Finally, Section 5.4 describes the concept of filtration shrinkage.

5.2 Types of partial information

Jarrow and Protter (2004) tried to bridge the gap in these two approaches by providing a unified approach based on the information that a modeller has. It seems natural to assume that market investors are not privileged, as they do not have the same information as equity holders. Usually, the management board of a firm has precise knowledge of the firm's economic state, whereas bond holders have just a snapshot of it. This is the main idea of partial information models.

Generally speaking, there are two major ways under which we can build a hybrid model. The first method is for given a first passage model, its barrier D is random. The second method assumes that the underlying asset values modelled by process $(X_t)_{t\geq 0}$ are partially observed. Alternatively, we may assume that investors cannot observe neither the barrier nor the underlying asset, see Giesecke (2004). Both approaches we will explain them in this section.

Let us briefly describe some advantages of incomplete information models, see also Giesecke and Goldberg (2004, 2008). First, these models preserve the endogenous property that a structural model has plus the short term uncertainty that reduced models have i.e. the default event is a totally inaccessible stopping time usually defined through an intensity process which implies positive short credit spreads. Therefore, from an economic perspective, incomplete information models are more flexible and reasonable, since investors may not have a precise and meticulous aspect of the firm's value. Moreover, following Giesecke and Goldberg (2004), these models provide tractable formulas for pricing contingent claims. Lastly, they try to present a unified perspective which integrates the best aspects of reduced form models and the structural models.

Duffie and Lando (2001) are among the first ones who introduced a model with incomplete information. In their model, investors cannot observe the underlying asset $(X_t)_{t\geq 0}$ directly. Instead, investors receive a noisy periodic accounting report of the asset. That is given a partition $0 = t_1, t_2 \dots, t_n = T$ of $[0, T]$ investors observe a discrete process $(Y_t)_{t\geq 0}$, $Y_{t_k} = X_{t_k} + U_{t_k}$, where $U_{t_k}$are normally distributed random variables independent of X_{t_k}. Kusuoka (1999) extended Duffie's and Lando's model to the continuous case. In his model, investors observe the barrier, but they receive continuous noisy reports for a process $(Y_t)_{t\geq 0}$ with drift $\mu = f(t, X_t)$.

Coculescu et al. (2008) introduced a structural model with a noisy asset, where the market observes a continuous process which is correlated with the underlying asset. In this model, the default threshold is a continuous function of time. Similarly, to Duffie and Lando's model in this model, the default process admits an intensity. A study of a structural model with unobservable barrier is also made by Hillairet and Jiao (2012). In this paper, they determine the default probabilities for the cases when there exists a random barrier under a filtration expansion. They also provide numerical results for these probabilities.

Giesecke and Goldberg (2004) introduced a structural model where investors cannot observe its barrier. In their model (I^2-model), the barrier is independent of the underlying asset. Although their model does not admit an intensity, it has a flexible reduced form pricing formula. Since the barrier is unobservable to investors, a prior distribution for the barrier should be given. They used a scaled beta distribution for the barrier. Finally, they provide a comparison between their model, Merton model and the Black-Cox model. In contrast to the structural models, their model provides positive short spreads which is consistent with the empirical observations. Giesecke and Goldberg (2008) further examines the I^2 model by investigating the risk premium. The risk premium is a mapping that connects the default probabilities of a martingale measure and the original pricing measure.Giesecke (2006) investigate a structural model whose barrier D is a random variable, but he connected the prior distribution of the barrier with the infimum of the underlying asset. If the prior distribution of D exists then the default time is totally inaccessible, however, the intensity process does not exist. On the contrary, if the underlying asset is partially observed then the default time is totally inaccessible and an intensity process exists. Finally, if we have incomplete information about both the barrier and the underlying asset then again an intensity process exists. His construction is primarily based on the theory of enlargement of filtrations which will be analysed in Section 5.3.

All the previous works are formed when the underlying asset is continuous. Dong and Zheng (2015) construct a structural model where the barrier is a random variable and the underlying asset is a Lévy process of finite variation. We thoroughly describe their model in Chapter 6 and we apply the local risk minimization for defaultable claims.

5.3 Filtration expansion

Based on the discussion above, it is evident that the concept of filtration expansion plays a fundamental role in the incomplete information models. If we choose to analyse a model from an investor's perspective, then we need to work under smaller or an enlarged filtration to determine the default time which should be totally inaccessible.

The main objective of filtration expansion is to find an appropriate enlarged filtration $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ such that $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \geq 0$, that makes a positive random variable L a stopping time under $\mathbb{G}$. There are some important questions that should be addressed when we are dealing with the theory of filtration expansion. First and foremost, what are the appropriate conditions under which an $\mathbb{F}$-local martingale remains a $\mathbb{G}$-semimartingale? Can we determine the canonical decomposition under $\mathbb{G}$? In this section, we try to answer these questions.

Roughly speaking, there are two main kinds of enlargement of filtration: initial filtration expansion and progressive filtration expansion. The initial filtration expansion is defined as the filtration $\mathbb{G}$ generated by the $\mathbb{F}$ and the filtration generated by the positive random variable L. On the other hand, the progressive filtration expansion is defined as the smallest right continuous filtration $\mathbb{G}$ which includes $\mathbb{F}$ and the natural filtration generated by the process $(1_{\{t\geq L\}})_{t\geq 0}$. Another interesting filtration expansion is introduced by Guo and Zeng (2008), which is equivalent of progressive filtration expansion. Their result plays fundamental role studying the canonical decomposition of semimartingales stopped at τ, see for instance Kchia et al. (2013). Given a positive random variable L which is a $\mathbb{G}$-stopping time, then for a given filtration $\mathbb{F}$ its expansion will be

$$\mathcal{F}_t \wedge \{t < L\} = \mathcal{G}_t \wedge \{t < L\}, \ t \in [0, T].$$

Among many we mention the following surveys regarding filtration expansion: Jeulin and Yor (1978), Jeulin (1980), Yor (1978) and Jacod (1985) which are not available in English. Protter (2004) has devoted a chapter in both initial and progressive filtration expansions (Chapter VI). Kchia et al. (2013) establish a relationship between the canonical decomposition of initial and progressive filtrations expansions. In particular, in their Lemma 3, they showed that both types of filtration expansions coincide after L, see their Definition 1 for the precise definition of how two filtrations coincide. Quite recently, Aksamit and Jeanblanc (2017) provide a good survey describing both expansions with a financial point of view. In this section, we briefly describe the initial and progressive filtration expansions along with their canonical decomposition.

5.3.1 Initial expansion

The initial enlargement of the reference filtration $\mathbb{F}$ is defined as

$$\mathbb{G} = \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(L) \right).$$

In initial filtration expansion there is no general results so that $\mathcal{H}'$ hypothesis is applied. However, following Jacod (1985), we know that if the law of a positive random variable L is continuous then $\mathcal{H}'$ hypothesis holds. Condition 5.1 describe this.

We assume that $(E, \mathcal{E})$ is a standard Borel space and $\mathcal{E}$ its Borel sets.

Condition 5.1 (Jacod's condition)**.** For each t there exists a positive σ-finite measure η_t on $(E, \mathcal{E})$ such that $P_t(\omega, \cdot) \ll \eta_t(\cdot)$ a.s on ω; where $P_t(\omega, dx)$ refers to the family of the regular conditional distributions with respect to $\mathbb{F}$ such that $P_t(:, A)$ is a version of $\mathbb{P}(L \in A \mid \mathcal{F}_t)$, and η_t is the corresponding family of laws of L.

We refer as $\mathscr{O}(\mathbb{F})$ and $\mathscr{P}(\mathbb{F})$ to be the space of the $\mathbb{F}$-optional and $\mathbb{F}$-predictable fields introduced in Definitions 2.15 and 2.17. The following Theorem plays fundamental role when we study canonical decompositions under the initial filtration expansion. It proposes that the $\mathcal{H}'$-hypothesis, i.e. an $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale (see also Definition 3.11), given that Condition 5.1 holds. For its proof we refer to Jacod (1985).

Theorem 5.2. *Under the Condition 5.1 any $\mathbb{F}$-semimartingale is a $\mathbb{G}$-semimartingale.*

An equivalent result of condition 5.1 is the following:

Condition 5.3. The Condition 5.1 is equivalent to the following condition: there is a positive σ-finite measure η on $(E, \mathcal{E})$ such that $P_t(\omega, \cdot) \ll \eta(\cdot)$ for each $t > 0$ and ω.

Given now condition 5.3, we can also assert that $\mathcal{H}'$ is again applied. The following Proposition describe this. For its proof, we refer to Jacod (1985).

Proposition 5.4. *Assume that Condition 5.3 is satisfied. In this case, every $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale.*

In order to have simple statements, let us introduce an auxiliary space $\hat{\Omega}$ equipped with a filtration $\hat{\mathbb{F}} = \left(\hat{\mathcal{F}}_t\right)_{t\geq 0}$ with $t \in [0, T]$, such that

$$\hat{\Omega} := \Omega \times E, \qquad \hat{\mathbb{F}} := \bigcap_{s>t} (\mathcal{F}_s \times \mathcal{E}),$$

and note that in this space we have

$$\mathscr{P}(\hat{\mathbb{F}}) = \mathscr{P}(\mathbb{F}) \times \mathcal{E}, \qquad \mathscr{O}(\hat{\mathbb{F}}) \supset \mathscr{O}(\mathbb{F}) \times \mathcal{E}.$$

Jacod (1985) in Lemma 1.8 formulates the existence of an $\mathcal{O}(\hat{\mathbb{F}})$-measurable càdlàg process $(p_t^x)_{t\geq 0}$, with $x \in E$, which is considered as a version of a density process $p_t^x(\omega) = \frac{P_t(\omega,dx)}{\eta(dx)}$ in the sense:

- For each $x \in E$ p^x is a $\mathbb{F}$-(local) martingale. Furthermore, if we denote as $T^X = \inf\{t \geq 0 : q_{t-}^x = 0\}$, then $p_t^x > 0$ and $p_{t-}^x > 0$ on $t \in [0, T^X)$ and p_t^x for $t \in [T^x, +\infty)$.
- For each t the measure $\eta(dx)p_t^x(\omega)$ on E is a version of $P_t(\omega, dx)$

Establishing the notion of the version of a density we are ready the $\mathbb{F}$-predictable projection of $(Y_t)_{t\geq 0}$, which is given in the following Lemma. See also Jacod (1985), Lemma 1.10. The definition of the predictable projection can be found in Appendix B.

Let the function $(x, \omega, t) \to Y_t(\omega)$ be a $\mathscr{P}(\hat{\mathbb{F}})$-measurable which is positive or bounded. The $\mathbb{F}$-predictable projection of the process $(Y_t(L))_{t\geq 0}$ is

$$^p(Y(L))_t = \int_{\mathbb{R}} p_{t-}(u) Y_t(u) \eta(du), \; t \in [0, T].$$

The next Theorem provides the canonical decomposition under $\mathbb{G}$, for its proof, we refer to Jacod (1985).

Theorem 5.5. *Assume that $(M_t)_{t\geq 0}$ is a continuous $\mathbb{F}$-local martingale. Then there exists a $\mathscr{P}(\mathbb{F})$-measurable function $(x, \omega, t) \to k_t(\omega)$ such that*

$$\langle p, M \rangle_t^{\mathbb{F}} = \int_0^t k_s p_{s-} d\, \langle M \rangle_s^{\mathbb{F}}. \tag{5.1}$$

Furthermore, for the function k_t defined above satisfying (5.1) we have

1. $$\int_0^t |k_s(L)| d\, \langle M \rangle_s^{\mathbb{F}} < \infty \;\; a.s\; \forall t \in [0, T],$$
2. *and the following process is a $\mathbb{G}$-local martingale*

$$\left(M_t - \int_0^t k_s(L) d\, \langle M \rangle_s^{\mathbb{F}} \right)_{t\geq 0}, \;\; t \in [0, T].$$

Theorem 5.5 assumed that the process $(M_t)_{t\geq 0}$ is a continuous local martingale. We proceed to the general case, when $(M_t)_{t\geq 0}$ is just a local martingale. In particular, we consider the $\mathbb{F}$-stopping times

$$R_n := \inf_t \left(q_{t-} \leq \frac{1}{n} \right),$$

and in this case we get $\cup_n[0, R_n] = \{q_- > 0\}$.

The following theorem is the main result of this section, its proof can be found in Jacod (1985), Theorem 2.5.

Theorem 5.6. *Let $(M_t)_{t\geq 0}$ be an $\mathbb{F}$-local martingale.*

1. *For all x that do not belong to set B dependent on $(M_t)_{t\geq 0}$-η negligible, and for all integers n the stopped process $[q, M]_t^{R_n}$ has paths of locally integrable variation (i.e. belongs to $\mathcal{A}_{loc}$). Therefore it's compensator, under $\mathbb{F}$, exists and is equal to the predictable quadratic covariation process $\langle p, M\rangle_t^{R_n}$ on the set $\bigcup_n [0, R_n]$.*

2. *There exists a predictable process $(A_t)_{t\geq 0}$ and an $\hat{\mathbb{F}}$-predictable function $(x, \omega, t) \to k_t(\omega)$ such that for all $x \notin B$*

$$\langle p, M\rangle_t^{\mathbb{F}} = \int_0^t k_s p_{s-} dA_s, \ \text{a.s on} \ \cup_n \{t \leq R_n\}, \ \ t \in [0, T].$$

   Also if $(M_t)_{t\geq 0}$ is square locally integrable, then we can take $A_t = \langle M, M\rangle_t$.

3. *If the process $(A_t)_{t\geq 0}$ and the function k_t verify condition 2 we have*

$$\int_0^t |k_s(L)| dA_s < \infty, a.s, \ \ t \in [0, T],$$

   and the following process is a $\mathbb{G}$-local martingale,

$$\Big(M_t - \int_0^t k_s(L) dA_s\Big)_{t\geq 0}, \ \ t \in [0, T].$$

5.3.2 Progressive expansion

Let us proceed with the progressive filtration expansion of the reference filtration $\mathbb{F}$. We assume that a given positive random variable $L : \Omega \to \mathbb{R}^+$ is honest, see Definition 3.12. Then following Protter (2004), Chapter VI, Section 3, the filtration

$$\mathbb{G}^L = \bigcap_{u>t} \mathcal{G}_u^0, \ \ \text{where } \mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(L \wedge t), \tag{5.2}$$

is a progressive expansion of $\mathbb{F}$ that makes L a stopping time. Under the assumption that L is an honest time (see Definition 3.12) or equivalently L is the end of an optional set, then as we will see, it has been shown in Yor (1978) that if $(X_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale is still a $\mathbb{G}^L$ semimartingale. In this section, we will determine the canonical decomposition under $\mathbb{G}^L$.

At the moment we will not make any assumption about the random variable L. For this reason, following Jeulin and Yor (1978), we work under a bigger filtration that includes $\mathbb{G}^L$ such that

$$\Gamma \in \mathbb{G} \Leftrightarrow \{\Gamma \in \mathcal{G} \text{ and } \exists \Gamma_t \in \mathcal{F}_t : \ \Gamma \cap \{L > t\} = \Gamma_t\{L > t\}\}.$$

Remark 5.7. Note that for all $t \geq 0$ then $\mathcal{G}_t^L \subset \mathcal{G}_t$, and the random variable L is a stopping time under both filtrations $\mathbb{G}$ and $\mathbb{G}^L$. Moreover, for any $\mathbb{G}$-stopping time τ, there exists an $\mathbb{F}$-stopping time S such that $\tau \wedge L = S \wedge L$.

There exists a useful property that $\mathbb{G}^L$ inherits, which is given in the Lemma bellow. For its proof, see Jeulin and Yor (1978), Lemma 1.

Lemma 5.8. *Let $(H_t)_{t\geq 0}$ be a $\mathbb{G}$-predictable process. Then there exists a $\mathbb{F}$-predictable process $(J_t)_{t\geq 0}$ such that*

$$H_t 1_{\{t\leq L\}} = J_t 1_{\{t\leq L\}},\ t \in [0,T].$$

The process $(Z_t)_{t\geq 0}$ which is the $\mathbb{F}$-optional projection of $(1_{\{L>t\}})_{t\geq 0}$ plays fundamental role. It is also known as the Azema $\mathbb{F}$ supermartingale. For its properties we refer to Appendix B. Let $Z_t =^{\circ} 1_{\{L>t\}}$ then its canonical decomposition will be

$$Z_t = M_t - A_t,\ \ t \in [0,T], \tag{5.3}$$

where $(A_t)_{t\geq 0}$ is an increasing $\mathbb{F}$-adapted and integrable process which is the $\mathbb{F}$-predictable compensator of $(Z_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$ is a $\mathbb{F}$-local martingale. Yor (1978) proved that $\mathbb{P}(Z_{L-} > 0) = 1$. A consequence of Lemma 5.8 is the following Proposition. For its proof see Jeulin and Yor (1978), Proposition 2.

Proposition 5.9. *Let $(H_t)_{t\geq 0}$ be a $\mathbb{G}$-predictable process and $(Z_t)_{t\geq 0}$ be the Azema $\mathbb{F}$-supermartingale with canonical decomposition given by (5.3). Then the process*

$$\left(H_L 1_{\{L\leq t\}} - \int_0^{t\wedge L} \frac{H_s}{Z_{s-}} dA_s\right)_{t\geq 0},\ \ t \in [0,T],$$

is a $\mathbb{G}$-local martingale.

The next Theorem characterizes the canonical decomposition in $\mathbb{G}$ of the stopped process $(X_{L\wedge t})$. For its proof, we refer to Jeulin and Yor (1978), Theorem 1. We remind the reader that we do not make any hypothesis about L except that it is a positive random variable with values in $(0, +\infty)$.

Theorem 5.10. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{F}$-local martingale, and $(Z_t)_{t\geq 0}$ be the Azema $\mathbb{F}$-supermartingale with canonical decomposition given by (5.3). We denote by $(B_t)_{t\geq 0}$ the $\mathbb{F}$-predictable dual projection of the $\mathbb{G}$-adapted process $(\epsilon_t)_{t\geq 0}$, $\epsilon_t = (1_{\{L\leq t\}} \Delta X_L)_{t\geq 0}$ and $C_t = \langle X, M\rangle_t^{\mathbb{F}} + B_t$. Then the processes*

$$\left(X_t 1_{\{t<L\}} + \int_0^{L\wedge t} \frac{dB_s - d\,\langle X, M\rangle_s^{\mathbb{F}}}{Z_{s-}}\right)_{t\geq 0},\ t \in [0,T],$$

and

$$\left(X_{L\wedge t} - \int_0^{L\wedge t} \frac{1}{Z_{s-}} 1_{\{Z_{s-}<1\}} dC_s\right)_{t\geq 0},\ t \in [0,T],$$

are $\mathbb{G}$-local martingales.

Protter (2004) under the assumption that the stopping times in $\mathbb{G}$ avoids all the stopping times in $\mathbb{F}$ [1] simplified the above Theorem since in this case the process $(B_t)_{t\geq 0}$ is zero.

To determine the canonical decomposition for the case after the default $X_t 1_{\{L\geq t\}}$we need to make an assumption which is often quite natural: we assume that L is the end of an optional set. As we will see, a random variable is the end of an optional set if and only if is an honest time.

The Theorem bellow characterizes the connection between an honest random variable and the optional set. For its proof, we refer to Protter (2004), Chapter VI, Theorem 14.

Theorem 5.11. *The random variable L is an honest time if and only if there exists an optional set $\Lambda \subset [0,\infty] \times \Omega$ such that $L(\omega) = \sup\{t \leq \infty : (t,\omega) \in \Lambda\}$.*

Let us investigate the case when the random variable L is a honest time. In this case, the progressive filtration expansion $\mathbb{G}^L$ is given by

$$\mathcal{G}_t^L = \{A \in \mathcal{G},\ \exists A_t, B_t \text{ and } A = \{A_t \cap (t < L)\} \cup \{B_t \cap (L \leq t)\}\}\,. \tag{5.4}$$

Remark 5.12. Note that for all $t \geq 0$ then $\mathcal{F}_t \subset \mathcal{G}_t^L$ and so the positive random variable L is a stopping time in $\mathbb{G}^L$. Furthermore, if S is an $\mathbb{F}$ stopping time then

$$\mathcal{G}_S^L = \{A \in \mathcal{G} \mid \exists A_S \in \mathcal{F}_S,\ A \cap \{t < L\} = A_S \cap \{S < L\}\}\,.$$

Yor (1978) proved the following fundamental result.

Theorem 5.13. *If $(X_t)_{t\geq 0}$ is a $\mathbb{F}$-semimartingale, where $t \in [0,T]$, then the processes $(X_t 1_{\{t<L\}})_{t\geq 0}$ and $(X_{L\wedge t})_{t\geq 0}$ are semimartingales under $\mathbb{G}^L$.*

A stronger result of Theorem 5.13 is the following Theorem, see Yor (1978), Theorem 2.

Theorem 5.14. *If $(X_t)_{t\geq 0}$ is an square integrable $\mathbb{F}$-local martingale, $t \in [0,T]$, then the process $\tilde{X}_t = X_t 1_{\{t<L\}}$ and $\bar{X}_t = X_{L\wedge t}$ are quasimartingales [2] under $\mathbb{G}^L$.*

The next Lemma connects the filtrations $\mathbb{G}$ and $\mathbb{G}^L$. For its proof, we refer to Jeulin and Yor (1978), Lemma 2.

Lemma 5.15. *Let L be the end of an $\mathbb{F}$ optional set (honest time) and assume that $\mathbb{F}$, $\mathbb{G}^L$ and $\mathbb{G}$ are the filtrations defined above satisfying the usual hypothesis such that $\mathbb{F} \subset \mathbb{G}^L$ and $\mathbb{G}^L \subset \mathbb{G}$. Then the following assertions are equivalent*

[1] We say that a $\mathbb{G}$ stopping time L avoids all the stopping times in $\mathbb{F}$ if and only if $\mathbb{P}(L = S) = 0$ for all the $\mathbb{F}$ stopping times S.

[2] An $\mathbb{G}^L$-adapted càdlàg process $(X_t)_{t\geq 0}$ is called quasimartingale if for each $t \geq 0$ we have $\mathbb{E}[X_t] < \infty$ and $Var[X_t] < \infty$.

1. *If $(X_t^L)_{t\geq 0}$ is $\mathbb{G}$-local martingale stopped at L, $0 \leq t \leq T$; then $(X_t)_{t\geq 0}$ is also an $\mathbb{F}$-local martingale.*

2. *Assume that (Y_t) is a $\mathbb{G}$-local martingale stopped at L, where $0 \leq t \leq T$. We further assume that the random variable Y_L is $\mathcal{G}_t^L$ with $t = L$ measurable. Then $(Y_t)_{t\geq 0}$ is also $\mathbb{G}^L$-local martingale.*

Similarly to Lemma 5.8, we derive an equivalent result for the filtration $\mathbb{G}^L$, see also Protter (2004), Chapter VI, Theorem 17 along with its proof.

Lemma 5.16. *Assume that L is an honest time and let $(H_t)_{t\geq 0}$ be a $\mathbb{G}^L$-predictable process. Then there are two $\mathbb{F}$-predictable processes $(J_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ such that*

$$H_t = J_t 1_{\{t\leq L\}} + K_t 1_{\{L<t\}},\ t \in [0, T].$$

In light of Proposition 5.9 and assuming that the random variable L is an honest time, let us determine the canonical decomposition of the stopped process $(X_t^L)_{t\geq 0}$ under $\mathbb{G}^L$, for its proof, we refer to Jeulin and Yor (1978).

Proposition 5.17. *Assume now that L is an honest time and let $(H_t)_{t\geq 0}$ be a $\mathbb{G}^L$-adapted process, where $\mathbb{G}^L := (\mathcal{G}_t^L)$ is introduced in (5.4) and $(Z_t)_{t\geq 0}$ is the Azema $\mathbb{F}$-supermartingale whose canonical decomposition is given in (5.3). Then the process*

$$\Big(H_L 1_{\{L\leq t\}} - \int_0^{L\wedge t} \frac{H_s}{Z_{s-}} dA_s\Big)_{t\geq 0},\ t \in [0, T],$$

is a $\mathbb{G}^L$-local martingale.

The next Theorem, provides the canonical decomposition of an $\mathbb{F}$-local martingale for the general case, not only for a stopped process. For its proof, we refer to Jeulin and Yor (1978).

Theorem 5.18. *Let $(X_t)_{t\geq 0}$ be an $\mathbb{F}$-local martingale with $t \in [0, T]$, and we assume that L is an honest time. Once again, we denote by $(B_t)_{t\geq 0}$ the $\mathbb{F}$-predictable dual projection of the $\mathbb{G}^L$-adapted process $(\epsilon_t)_{t\geq 0}$, $\epsilon_t = (1_{\{L\leq t\}} \Delta X_L)_{t\geq 0}$ and $C_t = \langle X, M\rangle_t^{\mathbb{F}} + B_t$. Then the process*

$$\Big(X_t + \int_0^t 1_{\{L<s\}} \frac{1}{1 - Z_{s-}} dC_s - \int_0^{L\wedge t} \frac{1}{Z_{s-}} 1_{\{Z_{s-}<1\}} dC_s\Big)_{t\geq 0},\ t \in [0, T],$$

is a $\mathbb{G}^L$-local martingale.

Guo et al. (2009) introduced the concept of delay information. Investors may have full information but they receive with a delay. An example of a delay information is formulated as $(\mathcal{F}_{t-\delta})_{t\geq 0}$ for some $\delta > 0$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated from the underlying asset [3]. By applying a progressive filtration expansion on the delay

[3] For the proper definition of delay filtration in continuous and discrete case please see Definitions 1 and 3 of Guo et al. (2009)

filtration, they determine the intensity process. They also made a comparison between the continuous and discrete delay case.

5.4 Filtration shrinkage

In contrast to Duffie and Lando model described above, Cetin et al. (2004) provide an alternative model based on filtration shrinkage. They obtained a reduced form model which admits an intensity through a structural model. Let us briefly describe their model.

Assume that the underlying asset is a Brownian motion of the following form

$$dX_t = \sigma dW_t, \; X_0 = x, \quad t \in [0, T],$$

and the barrier in the structural model is zero

$$g(t) := \sup \{s \leq t : X_s = 0\},$$

the random function $g(t)$ describes the last time when the underlying asset $(X_t)_{t\geq 0}$ hits the zero level. Let

$$\tau_a = \inf \left\{ t > 0 : \; t - g(t) \geq \frac{a^2}{2}, \; \text{where } X_s < 0 \text{ for } s \in (g(t-), t) \right\}.$$

The stopping time τ_a is the first time when the underlying asset is strictly negative. Based on Cetin et al. (2004), investors cannot observe the underlying asset directly. Instead, an investor only observes when the underlying asset may have positive or negative cash flows. Given that the firm has negative cash balances for an $\frac{a^2}{2}$ units of time, then after this period, the default time for the structural model is defined as the first time when the firm will reproduce a positive cash balance and more precisely, it is occurred as soon as cash balances double their size. In this case, the process $(Y_t)_{t\geq 0}$ for the reduced form model is given by

$$Y_t = \begin{cases} X_t & t < \tau_a, \\ 2X_{\tau_a} & t \geq \tau_a, \end{cases}$$

with $0 \leq t \leq T$ and the default time will be

$$\tau = \inf \{t \geq \tau_a : \; Y_t = 0\} \tag{5.5}$$

Let

$$sign(x) = \begin{cases} 1 & x > 0, \\ -1 & x \leq 0. \end{cases}$$

Set $\tilde{\mathbb{F}} := (\tilde{\mathcal{F}}_t)_{0\leq t\leq T}$ the filtration generated by the process $(sign(Y_t))_{t\geq 0}$, and let $\mathbb{F}$ be its right continuous version. Then clearly we have $\mathbb{F} \subset \mathbb{G}$. Moreover, Cetin et al.

(2004) proved that the stopping time defined in (5.5) admits an intensity $(\lambda_t)_{t\geq 0}$, $\lambda_t = 1_{\{t>\tau_a\}}\frac{1}{2(t-\bar{g}_{t-})}$, where $\bar{g}_t := \sup\{s \leq t;\ Y_s = 0\}$.

Cetin (2012) discussed the pricing of defaultable claims in a structural framework assuming that the underlying asset, which is a general continuous diffusion process, is unobservable. Although the default time for the underlying asset is predictable, using an observable diffusion process and under filtration shrinkage he provided an explicit representation of the intensity. His framework is based on the non linear filtering theory.

Perhaps the most known result in filtration shrinkage is the Stricker's theorem. The following Theorem provides it. For its proof, we refer to Stricker (1977), Theorem 3.1.

Theorem 5.19 (Stricker's theorem)**.** *Assume that $(X_t)_{t\geq 0}$ is a a semimartingale under the filtration $\mathbb{G}$. Let also $\mathbb{F}$ be a sub-filtration of $\mathbb{G}$, $\mathbb{F} \subset \mathbb{G}$, such that $(X_t)_{t\geq 0}$ is $\mathbb{F}$-adapted. Then $(X_t)_{t\geq 0}$ remains a semimartingale under $\mathbb{F}$.*

A disadvantage of Stricker's theorem is that the process $(X_t)_{t\geq 0}$ should be also adapted to the sub-filtration $\mathbb{F}$. Föllmer and Protter (2011) investigated the case when the process $(X_t)_{t\geq 0}$ is no longer $\mathbb{F}$-adapted. Semimartingales and quasimartingales maintain their form into a smaller filtration and they are semimartingales in the smaller filtration, although their canonical decomposition may be changed. This is also true for martingales. On the contrary the situation for local martingales is more complicated. For example, if $(X_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale and $\mathbb{F}$-adapted, then it may not be a local martingale under $\mathbb{F}$. The optional projection of a local martingale onto a filtration which is not adapted may not remain a local martingale. We provide their main results.

For the rest of this section, we assume that $(X_t)_{t\geq 0} \in L^1$ for all $t \in [0, T]$.

We start investigating the martingale case. In fact the optional projection of a $\mathbb{G}$-martingale is martingale in $\mathbb{F}$. The Theorem bellow analyses this, for its proof, see Föllmer and Protter (2011).

Theorem 5.20. *Assume that $(X_t)_{t\geq 0}$ is a $\mathbb{G}$-martingale. Then the optional projection of $(X_t)_{t\geq 0}$ onto $\mathbb{F}$ is again a martingale for the filtration $\mathbb{F}$.*

When the process is supermartingale (decreasing process) we have the following result, see Föllmer and Protter (2011) for its proof.

Theorem 5.21. *Let $(X_t)_{t\geq 0}$ be a supermartingale for $\mathbb{G}$. Then ${}^{\circ}(X_t)_{t\geq 0}$ is a supermartingale for $\mathbb{F}$.*

Let $(X_t)_{t\geq 0}$ be a $\mathbb{G}$ measurable process, and suppose that the optional projection of $(|X|_t)_{t\geq 0}$, ${}^{\circ}(X_t)_{t\geq 0}$ is indistinguishable from a finite valued process, where the optional projection is taken on $\mathbb{F} \subset \mathbb{G}$. If we let $Y_t =^{\circ} (X_t^+) -^{\circ} (X_t^-)$, then $(Y_t)_{t\geq 0}$ defines an optional projection of $(X_t)_{t\geq 0}$.

The following Theorem shows that if the optional projection of a $\mathbb{G}$-semimartingale exists and is finite, then it also an $\mathbb{F}$-semimartingale. For its proof, we refer to Föllmer and Protter (2011).

Theorem 5.22. *Let $(X_t)_{t\geq 0}$ be a semimartingale for $\mathbb{G}$ such that $^\circ(|X|_t)_{t\geq 0}$ exists and is a finite valued process, where $^\circ(|X|_t)_{t\geq 0}$ is the optional projection of $(|X|_t)_{t\geq 0}$ onto the filtration $\mathbb{F} \subset \mathbb{G}$. Then from Theorem 5.21 $^\circ(X_t)_{t\geq 0}$ exists and it is an $\mathbb{F}$-semimartingale.*

Let us investigate the case when $(X_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale.

For the special case when $(X_t)_{t\geq 0}$ is positive $\mathbb{G}$-local martingale and adapted to the sub-filtration $\mathbb{F}$, then it remains a local martingale under $\mathbb{F}$. The following result describes this. For the proof, we refer to Stricker (1977).

Theorem 5.23. *Let $(X_t)_{t\geq 0}$ be a positive $\mathbb{G}$-local martingale and we assume that is adapted to the sub-filtration $\mathbb{F}$. Then $(X_t)_{t\geq 0}$ is also an $\mathbb{F}$-local martingale .*

We proceed with the case when $(X_t)_{t\geq 0}$ is no longer a positive local martingale.

If there exists a decreasing sequence of $\mathbb{G}$ stopping times that remains stopping times in $\mathbb{F}$ then its optional projection $^\circ(X_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale. The next Theorem illustrate this. For the case when the process $(X_t)_{t\geq 0}$ is not $\mathbb{F}$-adapted, then its local martingale property is violated. This was shown through a three-dimensional Bessel process, see Föllmer and Protter (2011), for more details.

Theorem 5.24. *Let $(X_t)_{t\geq 0}$ be a $\mathbb{G}$-local martingale, and $^\circ(X_t)_{t\geq 0}$ be the optional projection of $(X_t)_{t\geq 0}$ onto the sub-filtration $\mathbb{F}$. Then $^\circ(X_t)_{t\geq 0}$ is a $\mathbb{F}$-local martingale if there exists a reduced sequence of stopping times $(\tau_n)_{n\geq 1}$ for $(X_t)_{t\geq 0}$ in $\mathbb{G}$ which are also $\mathbb{F}$ stopping times. Conversely, if $(X_t)_{t\geq 0}$ is positive and $^\circ(X_t)_{t\geq 0}$ is a $\mathbb{F}$-local martingale, then a reducing sequence of stopping times for $^\circ(X_t)_{t\geq 0}$ in $\mathbb{F}$ is also a reducing sequence in $\mathbb{G}$.*

Chapter 6

Locally risk minimizing hedging strategies under a random barrier

6.1 Introduction

In Chapter 4, a hedging strategy is obtained under a unique flow of information for a defaultable claim of the form $F(X_T)1_{\{\tau>T\}}$, through the LRM approach. In this chapter, an extension is made assuming that we have partial information. More specifically, we assume that the default time time is defined through a structural model whose barrier is a random variable. Then, we obtain a semi-explicit hedging strategy through a PIDE.

In Chapter 5, we saw that there are two major ways to introduce partial information in credit risk modelling. One approach is to assume that the barrier on the first passage time is random. Giesecke (2006) was the first who introduced incompleteness on the barrier, where the underlying asset is a geometric Brownian motion. Based on this framework Dong and Zheng (2015) obtained a structural model with a random barrier when the underlying asset is a finite variation Lévy process. The second method is to assume that the underlying asset is partially observed but the barrier is constant. Duffie and Lando (2001) discussed this through a discrete noisy asset process and Kusuoka (1999) extended it to the continuous case. Alternatively, we may assume that neither the barrier nor the underlying asset can be observed. Finally, we can also assume that there exists a delay in the available information. Guo et al. (2009) investigated and compared continuous and discrete delay filtrations. In Okhrati (2019), semi-explicit solutions of hedging strategies of defaultable claims are obtained under delayed data through solving PDEs.

Regarding applications of hedging strategies under partial information, Ceci et al. (2015b, 2017) obtained a LRM approach assuming that the asset, which is a jump diffusion process, is partially observed. Ceci et al. (2014) provided the GKW decomposition under delayed information.

In this chapter, we obtain semi-explicit solution of LRM hedging strategies for a spectrally positive Lévy process of finite variation, assuming that the barrier follows a negative exponential distribution. Dong and Zheng (2015) proved that the intensity of such a structural model exists and it has an explicit form. Based on this result and in the spirit of Okhrati et al. (2014), a hedging strategy for a defaultable claim of the form $F(X_T - \underline{X}_T, -\underline{X}_T)1_{\{\tau>T\}}$, is obtained through PIDE given that F is a smooth function and $(\underline{X}_t)_{t\geq 0}$ is the infimum process of $(X_t)_{t\geq 0}$.

There is an indicator process in the intensity of Dong and Zheng (2015) which brings some regularity problems for the existence of smooth solutions for our PIDEs. This will make parts of our proofs heuristic as we cannot establish the existence of such smooth solution or it may not even exist though the numerical solutions look reasonable. However, in the next chapter (Chapter 7), we will not have this regularity problems in our approach.

An application of Lévy spectrally positive processes on option valuation is given in Chan (2005). Spectrally positive processes are also used to manage the insurance risk see Klüppelberg et al. (2004) for more details. The general properties of the spectrally positive processes have been analysed in Bertoin (2000).

The structure of this chapter is organized as follows. In Section 6.2, we briefly describe the model for the default time. In Section 6.3, we prove some useful martingales for the reflected Lévy process at its supremum and infimum. In Section 6.4, we derive the canonical decomposition of $(f(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}})_{t\geq 0}$ under the progressive expanded filtration $\mathbb{G}$. Finally, in Section 6.5, we derive the hedging strategy through the LRM in $\mathbb{G}$.

6.2 Preliminaries and model description

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F}$, where $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the available information for investors generated by a spectrally positive Lévy process of finite variation i.e. $\mathbb{F} = \sigma(X_s)_{0\leq s\leq t}$, for all $t \in [0, T]$, with Lévy triplet $(b, 0, \nu)$, and $X_0 = u$, $u > 0$. The process $(X_t)_{t\geq 0}$ has the following Lévy-Itô decomposition

$$X_t = X_0 + \mu t + S_t, \quad t \in [0, T], \tag{6.1}$$

where $\mu < 0$, $\mu = -b - \int_0^1 z\nu(dz)$ and S_t is a subordinator. Alternatively, (6.1) can be written as

$$X_t = X_0 + \mu t + \int_0^t \int_0^{+\infty} zN(ds, dz), \quad t \in [0, T],$$

where $N(dt, dz)$ is the jump measure with intensity $\nu(dz)dt$.

We impose the following Assumption.

Assumption 6.1. *The Lévy measure ν is absolutely continuous with respect to Lebesgue measure, and it satisfies $\int_0^{\infty} z\nu(dz) < \infty$ (or equivalently $\mathbb{E}[X_t] < \infty$) and $\int_0^{+\infty} z^2\nu(dz) < \infty$ (equivalently $Var[X_t] < \infty$).*

We assume that the default time is defined as

$$\tau = \inf \{t > 0 : X_t \leq D\}. \tag{6.2}$$

For the case when $(X_t)_{t\geq 0}$ is greater than D for every $t \in [0,T]$, we assume that $\tau = \infty$.

We assume that the random barrier D follows a negative exponential distribution i.e. $\mathbb{P}(D \leq x) = e^x \ \forall x < 0$. Also, D is assumed to be independent of $(X_t)_{t\geq 0}$.

Note that the prior distribution of the random barrier D must have a support on $(-\infty, X_0)$ since it allows us the default time τ to be totally inaccessible. Moreover, we use negative exponential distribution for the barrier following the same setup as in Giesecke (2006).

Let $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ be the enlarged progressive filtration expansion of $(\mathcal{F}_t)_{0\leq t\leq T}$ and τ, which includes $\mathbb{G}^\tau$ i.e. $\mathbb{G}^\tau \subset \mathbb{G}$. Following Section 5.3.2, it is given by

$$\mathcal{G}^\tau_t = \bigcap_{u>t} \mathcal{G}^0_t, \quad \mathcal{G}^0_t = \mathcal{F}_t \vee \sigma(\tau \wedge t), \quad t \in [0,T].$$

Based on Section 5.3.2, the form of $\mathbb{G}$ is

$$\mathcal{G}_t = \{B \in \mathcal{G} : \exists B_t \in \mathcal{F}_t, \ B \cap \{\tau > t\} = B_t \cap \{\tau > t\}\}.$$

Remark 6.2. Recall that both filtrations $\mathbb{G}$ and $\mathbb{G}^\tau$ coincide before τ. Also, in Guo and Zeng (2008) it is shown that the compensator of τ under both filtration expansions $\mathbb{G}$ and $\mathbb{G}^\tau$are identical.

We assume that the filtration $\mathbb{G}$ is complete and satisfies the usual hypothesis that makes τ a stopping time. Since τ is a $\mathbb{G}$-totally inaccessible stopping time, then Dong and Zheng (2015), under Assumption 6.1, obtained the $\mathbb{G}$ compensator of $(N_t)_{t\geq 0}$, $N_t = 1_{\{\tau\leq t\}}$ i.e. $A_t = \int_0^t \lambda_s ds$, where the intensity $(\lambda_t)_{t\geq 0}$ is given through

$$\lambda_t = 1_{\{\tau>t\}}(-\mu 1_{\{X_s - \underline{X}_s = 0\}} 1_{\{\mu<0\}} + \Pi(X_t - \underline{X}_t)), \tag{6.3}$$

where $(\underline{X}_t)_{t\geq 0}$, $\underline{X}_t := \inf_{s\leq t} X_s$, and the process

$$\left(N_t - \int_0^t \lambda_s ds\right)_{t\geq 0}, \quad t \in [0,T], \tag{6.4}$$

is $\mathbb{G}$-local martingale, where

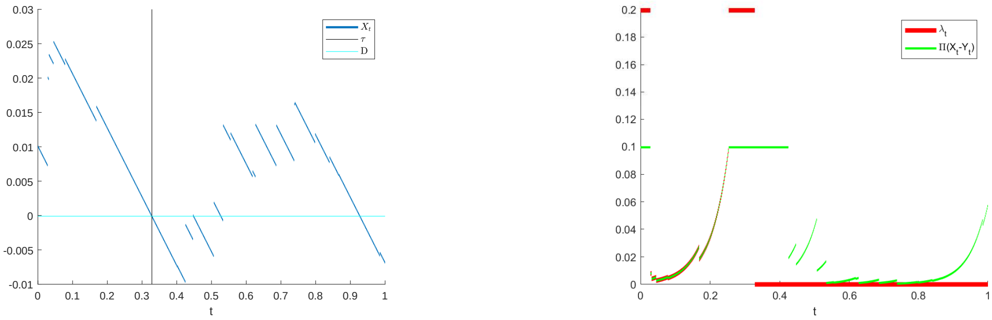
$$\Pi(x) := \int_0^\infty (1 - e^{-z})\nu(x + dz), \quad x \geq 0, \tag{6.5}$$

and

$$\nu(x + dz) := \nu((x + z, x + z + dz]).$$

If the Lévy measure ν admits a density π then $\nu(x + dz) := \pi(x + z)dz$.

Next, we provide two examples illustrating the intensity $(\lambda_t)_{t\geq 0}$. The first example is



(a) A sample path of $(X_t)_{t\geq 0}$

(b) A sample path of intensity process, and $\Pi(X_t - \underline{X}_t)$.

Figure 6.1: Simulation results for Example 6.3, when $(X_t)_{t\geq 0}$ is a compound Poisson, where the jumps are exponentially distributed, with parameters $\lambda = 20$, $q = 200$, $T = 1$ and $\mu = -\frac{\lambda}{q}$.

a compound Poisson with exponential distribution, which is a finite variation process with finite activity, while the second one is a variance gamma Lévy process (finite variation process with infinite activity). Both of these applications can be found in Dong and Zheng (2015).

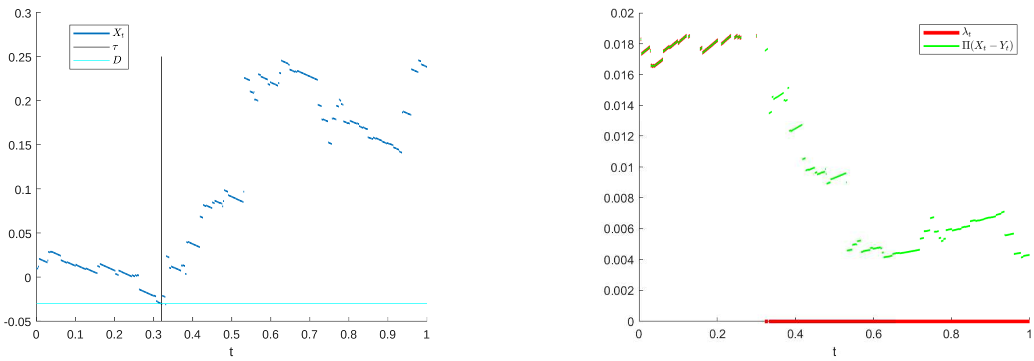
Example 6.3. Let $(X_t)_{t\geq 0}$ defined by

$$X_t = X_0 + \mu t + \sum_{i=1}^{N_t} e_i,\ t \in [0, T],$$

where μ is a constant, $\mu < 0$, $(N_t)_{t\geq 0}$ is Poisson process with intensity λ and e_i are exponentially distributed random variables with parameter q,. The Lévy density is defined as $\pi(z) = qe^{-qz}$. In this case the intensity $(\lambda_t)_{t\geq 0}$ (6.3) becomes

$$\begin{aligned}
\lambda_t &= 1_{\{\tau > t\}} \left(-\mu 1_{\{X_t - \underline{X}_t = 0\}} + \int_0^{+\infty} (1 - e^{-z}) \nu(X_t - \underline{X}_t + z) dz \right) \\
&= 1_{\{\tau > t\}} \left(-\mu 1_{\{X_t - \underline{X}_t = 0\}} + \int_0^{+\infty} (1 - e^{-z}) \lambda q e^{-q(X_t - \underline{X}_t + z)} dz \right) \\
&= 1_{\{\tau > t\}} \left(-\mu 1_{\{X_t - \underline{X}_t = 0\}} + \frac{\lambda}{q+1} e^{-q(X_t - \underline{X}_t)} \right).
\end{aligned}$$

Example 6.4. Let $(X_t)_{t\geq 0}$ be a variance gamma process $VG(t; \sigma, q, \theta)$, of finite variation where θ is the drift term of a Brownian motion, σ is the volatility of the Brownian motion and q the rate of a time changed gamma process. Following Madan et al. (1998), the variance gamma process can be obtained through time-changed (given by a gamma process) Brownian motion whose drift term can also be derived based on a gamma process i.e. $X_t = \theta\Gamma(t; 1, q) + \sigma W(\Gamma(t; 1, q))$. Also, they showed that the process can be expressed as the difference of two independent gamma processes $X(t; \sigma, q, \theta) = \Gamma(t; c_+, q_+) - \Gamma(t; c_-, q_-)$ and in this case the process $(X_t)_{t\geq 0}$ with the



(a) A sample path of $(X_t)_{t\geq 0}$ (b) A sample path of $(\lambda_t)_{t\geq 0}$ and $\Pi(X_t - \underline{X}_t)$

Figure 6.2: The underlying asset and the intensity process for the variance gamma case presented in Example 6.4, with $\theta = 0.01$, $\sigma = 0.2$, $q = 0.5$, $T = 1$ and $\mu = -0.2$.

additional drift term μ with $\mu < 0$ has the form

$$X_t = X_0 + \mu t + \Gamma(c_+, q_+) - \Gamma(c_-, q_-),$$

where $c_\pm = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{q}} \pm \frac{\theta}{2}$ and $q_\pm = c_\pm^2 q$, and its infinite activity Lévy measure has the form

$$\nu(dz) = \begin{cases} \frac{c_-^2}{q_-}\frac{\exp\left(-\frac{c_-}{q_-}|z|\right)}{|z|}dz, & \text{for } z < 0, \\ \frac{c_+^2}{q_+}\frac{\exp\left(-\frac{c_+}{q_+}\right)}{z}dz, & \text{for } z > 0. \end{cases}$$

In this case the intensity $(\lambda_t)_{t\geq 0}$ (6.3) is given by

$$\lambda_t = 1_{\{\tau > t\}}\left(-\mu 1_{\{X_t - \underline{X}_t = 0\}} + \int_0^{+\infty}(1 - e^{-z})\frac{c_-^2}{q_-}e^{-\frac{c_-}{q_-}(z + X_t - \underline{X}_t)}\frac{1}{(z + X_t - \underline{X}_t)}dz\right).$$

Since the default time is given by (6.2) and the default event is given via $\{\tau \leq t\} = \{\underline{X}_t \leq D\}$, the conditional survival probability at time t under the filtration $\mathbb{F}$ is given by

$$Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{P}(D > X_t) = F_D(\underline{X}_t), \tag{6.6}$$

where $F_D(x)$ is the prior distribution of D. Note that since D follows negative exponential distribution, then $F_D(\underline{X}_t) = e^{\underline{X}_t}$. Following Giesecke (2006), Definition 5.1, the unique non decreasing $\mathbb{F}$-predictable process $(A_t)_{t\geq 0}$, $A_t = \int_0^t \lambda_s ds$ is given through

$$A_t = \int_0^t \frac{dK_s}{Z_{s-}}, \quad 0 \leq t \leq T,$$

where $(K_t)_{t\geq 0}$ is the unique $\mathbb{F}$-predictable compensator of $1 - Z_t = \mathbb{E}[1_{\{\tau\leq t\}} \mid \mathcal{F}_t] = \mathbb{P}(\tau \leq t \mid \mathcal{F}_t)$.

We need to determine the process $(K_t)_{t\geq 0}$. According to Giesecke (2001) if $(Z_t)_{t\geq 0}$ is continuous then $K_t = -Z_t$ and $A_t = -\ln(Z_t)$. However, if $(Z_t)_{t\geq 0}$ is discontinuous then calculating $(K_t)_{t\geq 0}$ is non-trivial. Dong and Zheng (2015), in Lemma 3.2 and

Proposition 3.11, characterized the likelihood process under the reference filtration $\mathbb{F}$ as follows

Lemma 6.5. *If $h > 0$ and $k_t^h := \frac{1}{h}\mathbb{E}[K_{t+h} - K_t \mid \mathcal{F}_t]$, then*

$$k_t^h = e^{\underline{X}_t}\frac{1}{h}\mathbb{E}\left[1 - e^{-(m-\underline{X}_h)}\right]\bigg|_{m=X_t-\underline{X}_t}, \quad t \in [0,T],$$

if $(X_t)_{t\geq 0}$ is spectrally positive Lévy process, then

$$\tilde{k}_t = \lim_{h\downarrow 0} k_t^h = e^{\underline{X}_t}(-\mu 1_{\{X_t-\underline{X}_t\}}1_{\{\mu<0\}} + \Pi(X_t - \underline{X}_t)), \quad \forall\, t \in [0,T], \ \ a.s.$$

Based on Remark 3.15 of Dong and Zheng (2015) and using the Meyer's approximation theorem, see Theorem 3.6, we conclude that the $\mathbb{F}$-compensator of $\mathbb{P}(\tau > t \mid \mathcal{F}_t)$ is given by $K_t = \int_0^t \tilde{k}_s ds$ for all $t \in [0,T]$ $\mathbb{P}$-almost surely.

6.3 Martingales associated with reflected Lévy process

In this section, we present some martingales related to the reflected Lévy process at supremmun, and infimum. Our results and proofs are based on Nguyen-Ngoc and Yor (2005), who apply the Kennedy martingales, see Kennedy (1976), for Lévy processes. More precisely, given a Lévy process of finite variation, and $C^{1,1,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$ function, under a boundary condition, we manage to obtain the canonical decomposition of the reflected Lévy process at supremmum $(f(t, \bar{X}_t - X_t, \bar{X}_t))_{t\geq 0}$ and at infimum $(f(t, X_t - \underline{X}_t, -\underline{X}_t))_{t\geq 0}$, where $(\bar{X}_t)_{t\geq 0}$, and $(\underline{X}_t)_{t\geq 0}$ are the supremmum and the infimum of $(X_t)_{t\geq 0}$. First, let us introduce the d-dimensional compensation formula, see Jeanblanc et al. (2009), Proposition 11.2.3.

Lemma 6.6 (d-dimensional compensation formula)**.** *Let $(Z_t)_{t\geq 0}$ be a d-dimensional Lévy process, and ν its Lévy measure. We assume that $H : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}_0^+$ be a positive Borel measurable function where for every $t \in [0,T]$ $\int_0^t ds \int_{\mathbb{R}^d\setminus\{0\}} H_s(\omega, z)\nu(dz) < \infty$, and $H_t(\omega, 0) = 0$. Then*

$$\mathbb{E}\left[\sum_{s\leq t} H_s(\omega, \Delta Z_s)1_{\{\Delta Z_s\neq 0\}}\right] = \mathbb{E}\left[\int_0^t ds \int_{\mathbb{R}^d\setminus\{0\}} H_s(\omega, z)\nu(dz)\right].$$

Equivalently, the process

$$\left(\sum_{s\leq t} H(s, \Delta Z_s)1_{\{\Delta Z\neq 0\}} - \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} H(s,z)\nu(dz)\right)_{t\geq 0}, \tag{6.7}$$

is an $\mathbb{F}$-local martingale.

We also need the following integrability condition.

Assumption 6.7. *Assume $f(t,x,y) : [0,T] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,1,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$. Let the function $f(t,x,y)$ satisfy the following integrability conditions*

$$\int_{-\infty}^{x} |f(t,x+z,y) - f(t,x,y)|\nu(dz) < \infty, \quad \forall t \in [0,T], \ x \geq 0 \text{ and } y \in \mathbb{R},$$

and

$$\int_{x}^{+\infty} |f(t,0,y+z-x) - f(t,x,y)|\nu(dz) < \infty, \quad \forall t \in [0,T], \ x \geq 0 \text{ and } y \in \mathbb{R}.$$

The following Proposition is an application of Nguyen-Ngoc and Yor (2005), Proposition 4.

Proposition 6.8. *Let $(X_t)_{t\geq 0}$ be a finite variation Lévy process, $X_t = X_0 + \mu t + \sum_{s\leq t} \Delta X_s$, and its Lévy measure ν is absolutely continuous with respect to Lebesgue measure. Assuming that $f(t,x,y) : [0,T] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,1,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$ function that satisfies Assumption 6.7 and*

$$\begin{aligned}\mathcal{L}f(t,x,y) &= \frac{\partial f}{\partial t} - \mu\frac{\partial f}{\partial x} + \int_{-\infty}^{x} (f(t,x-z,y) - f(t,x,y))\nu(dz) \\ &\quad + \int_{x}^{+\infty} (f(t,0,y+z-x) - f(t,x,y))\nu(dz), \ \forall t \in [0,T], x \geq 0, \ y \in \mathbb{R},\end{aligned}$$

where $\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0$, then the process

$$\left(f(t, \bar{X}_t - X_t, \bar{X}_t) - f(0, \bar{X}_0 - X_0, \bar{X}_0) - \int_0^t \mathcal{L}f(s, \bar{X}_s - X_s, \bar{X}_s)ds \right)_{t\geq 0}, \ t \in [0,T],$$

is an $\mathbb{F}$-local martingale.

Proof. Since $(X_t)_{t\geq 0}$ is a finite variation process by applying the change of variable formula, see Theorem 2.48, it yields

$$\begin{aligned}f(t, \bar{X}_t - X_t, \bar{X}_t) &= f(0, \bar{X}_0 - X_0, \bar{X}_0) + \int_0^t \frac{\partial f}{\partial s}(s, \bar{X}_s - X_s, \bar{X}_s)ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})d\bar{X}_s \\ &\quad - \int_0^t \frac{\partial f}{\partial x}(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})dX_s \\ &\quad + \int_0^t \frac{\partial f}{\partial y}(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})d\bar{X}_s \\ &\quad + \sum_{s\leq t}\{f(s, \bar{X}_s - X_s, \bar{X}_s) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-}) \\ &\quad - \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})\Delta\bar{X}_s \\ &\quad + \frac{\partial f}{\partial x}(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})\Delta X_s\}, \ t \in [0,T],\end{aligned}$$

where $\Delta X_s = X_s - X_{s-}$, and $\Delta\bar{X}_s = \bar{X}_s - \bar{X}_s$.

Since $(X_t)_{t\geq 0}$ and $(\bar{X}_t)_{t\geq 0}$ are of finite variation then $dX_t^c = dX_t - \Delta X_t$ and $d\bar{X}_t^c = d\bar{X}_t - \Delta \bar{X}_t$, where $(X_t^c)_{t\geq 0}$, $(\bar{X}_t^c)_{t\geq 0}$ are the continuous local martingale part of $(X_t)_{t\geq 0}$ and the path-by-path continuous part of $(\bar{X}_t)_{t\geq 0}$ respectively. Note that since $(\bar{X}_t)_{t\geq 0}$ is of finite variation it yields $[\bar{X}^c]_t = [\bar{X}]_t^c = [X - \bar{X}, \bar{X}]_t^c = 0$ with $t \in [0, T]$. The above result can be rewritten as

$$
\begin{aligned}
f(t, \bar{X}_t - X_t, \bar{X}_t) = f(0, \bar{X}_0 - X_0, \bar{X}_0) &+ \int_0^t \frac{\partial f}{\partial s}(s, \bar{X}_s - X_s, \bar{X}_s) ds \\
&+ \int_0^t (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y})(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-}) d\bar{X}_s^c \\
&- \int_0^t \frac{\partial f}{\partial x}(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-}) dX_s^c \\
&+ \sum_{s\leq t} \{f(s, \bar{X}_s - X_s, \bar{X}_s) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})\},\ t \in [0, T],
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
f(t, \bar{X}_t - X_t, \bar{X}_t) = f(0, \bar{X}_0 - X_0, \bar{X}_0) &+ \int_0^t \frac{\partial f}{\partial s}(s, \bar{X}_s - X_s, \bar{X}_s) ds \\
&+ \int_0^t (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y})(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-}) d\bar{X}_s^c \\
&- \mu \int_0^t \frac{\partial f}{\partial x}(s, \bar{X}_s - X_s, \bar{X}_s) ds \\
&+ \sum_{s\leq t} \{f(s, \bar{X}_s - X_s, \bar{X}_s) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})\},\ t \in [0, T].
\end{aligned}
$$

Using the support property $X_{t-} = X_t = \bar{X}_{t-} = \bar{X}_t$ on $supp(d\bar{X}^c)$, the term

$$
\int_0^t \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)(s, 0, \bar{X}_s) d(\bar{X})^c,
$$

is eliminated using the boundary condition,

$$
\frac{\partial f}{\partial x}(t, 0, y) + \frac{\partial f}{\partial y}(t, 0, y) = 0,
$$

therefore it yields

$$
\begin{aligned}
f(t, \bar{X}_t - X_t, \bar{X}_t) = f(0, \bar{X}_0 - X_0, \bar{X}_0) &+ \int_0^t \frac{\partial f}{\partial s}(s, \bar{X}_s - X_s, \bar{X}_s) ds \\
&- \mu \int_0^t \frac{\partial f}{\partial x}(s, \bar{X}_s - X_s, \bar{X}_s) ds \\
&+ \sum_{s\leq t} \{f(s, \bar{X}_s - X_s, \bar{X}_s) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})\}.
\end{aligned}
$$

Suppose that z defines the jumps of $(X_t)_{t\geq 0}$ at time t and let $(R_t)_{t\geq 0}$, $R_t = \bar{X}_t - X_t$ be the reflected process of $(X_t)_{t\geq 0}$. We note that $(\bar{X}_t)_{t\geq 0}$ can be written as

$$
\bar{X}_t = \max\left(\bar{X}_{t-},\ X_t\right),\ \ t \in [0, T], \tag{6.8}
$$

and we investigate the following cases

- If $z < R_{t-}$or $X_t < \bar{X}_{t-}$ then from (6.8), $\max(\bar{X}_{t-}, X_t) = \bar{X}_{t-}$, which implies that $\bar{X}_t = \bar{X}_{t-}$. In this case the reflected process becomes $R_t = \bar{X}_{t-} - X_t$ and if we add and subtract X_{t-} then $R_t = \bar{X}_t - X_{t-} - (X_t - X_{t-}) = R_{t-} - z$.

- If $z \geq R_{t-}$or $X_t \geq \bar{X}_{t-}$ and from (6.8), we have that $\max(\bar{X}_{t-}, X_t) = X_t$ and thus $\bar{X}_t = X_t$, which yields $R_t = 0$.

The above summation can be expressed as an integral using jump measure $N(dt, dz)$. That is

$$
\begin{aligned}
f(t, \bar{X}_t - X_t, \bar{X}_t) &= f(0, \bar{X}_0 - X_0, \bar{X}_0) \\
&\quad + \int_0^t \frac{\partial f}{\partial s}(s, \bar{X}_s - X_s, \bar{X}_s)ds - \mu \int_0^t \frac{\partial f}{\partial x}(s, \bar{X}_s - X_s, \bar{X}_s)ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} \Big(f(s, \bar{X}_{s-} - X_{s-} - z, \bar{X}_{s-}) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-}) \Big) 1_{\{z < R_{s-}\}} N(ds, dz) \\
&+ \int_0^t \int_{-\infty}^{+\infty} \Big(f(s, 0, \bar{X}_{s-} - (\bar{X}_{s-} - X_{s-}) + z) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-}) \Big) 1_{\{z \geq R_{s-}\}} N(ds, dz).
\end{aligned}
$$

Then using Lemma 6.6 and assuming that $\tilde{N}(dt, dz)$ is the compensated jump measure, $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$, we have

$$
\begin{aligned}
f(t, \bar{X}_t - X_t, \bar{X}_t) &= f(0, \bar{X}_0 - X_0, \bar{X}_0) \\
&\quad + \int_0^t \frac{\partial f}{\partial s}(s, \bar{X}_s - X_s, \bar{X}_s)ds - \mu \int_0^t \frac{\partial f}{\partial x}(s, \bar{X}_s - X_s, \bar{X}_s)ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} (f(s, \bar{X}_{s-} - X_{s-} - z, \bar{X}_{s-}) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})) 1_{\{z < R_{s-}\}} \nu(dz)ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} (f(s, \bar{X}_{s-} - X_{s-} - z, \bar{X}_{s-}) - f(s, \bar{X}_s - X_s, -\bar{X}_s)) 1_{\{z < R_s\}} \tilde{N}(ds, dz) \\
&+ \int_0^t \int_{-\infty}^{+\infty} (f(s, 0, \bar{X}_{s-} - (\bar{X}_{s-} - X_{s-}) + z) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})) 1_{\{z \geq R_{s-}\}} \nu(dz)ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} (f(s, 0, \bar{X}_{s-} - (\bar{X}_{s-} - X_{s-}) + z) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_s)) 1_{\{z \geq R_{s-}\}} \tilde{N}(ds, dz).
\end{aligned}
$$

Since $\int_0^t \int_{-\infty}^{+\infty} (f(s, \bar{X}_{s-} - X_{s-} - z, \bar{X}_{s-}) - f(s, \bar{X}_{s-} - X_{s-}, \bar{X}_{s-})) 1_{\{z < R_{s-}\}} \tilde{N}(ds, dz)$ and $\int_0^t \int_{-\infty}^{+\infty} (f(s, 0, \bar{X}_{s-} - (\bar{X}_{s-} - X_s) + z,) - f(s, 0, \bar{X}_{s-} - X_{s-}, \bar{X}_s)) 1_{\{z < R_{s-}\}} \tilde{N}(ds, dz)$ are $\mathbb{F}$-local martingales, then by the continuity of the Lévy measure and if we define the operator

$$
\begin{aligned}
\mathcal{L}f(t, x, y) &= \frac{\partial f}{\partial t}(t, x, y) - \mu \frac{\partial f}{\partial x}(t, x, y) + \int_{-\infty}^{x} (f(t, x - z, y) - f(t, x, y)) \nu(dz) \\
&\quad + \int_{x}^{+\infty} (f(t, 0, y - x + z) - f(t, x, y)) \nu(dz), \ \forall t \in [0, T].
\end{aligned}
$$

Given that Assumption 6.7 is satisfied, it follows that the process

$$\left(f(t,\bar{X}_t - X_t, \bar{X}_t) - f(0,\bar{X}_0 - X_0, \bar{X}_0) - \int_0^t \mathcal{L}f(s,\bar{X}_s - X_s, \bar{X}_s)ds\right)_{t\geq 0},$$

with $t \in [0,T]$, is an $\mathbb{F}$-local martingale. □

Let $(\underline{X}_t)_{t\geq 0}$, $\underline{X}_t := \inf_{0\leq s\leq t} X_s$ be the infimum process of $(X_t)_{t\geq 0}$. It is known that $(Q_t)_{t\geq 0}$, $Q_t = \sup_{0\leq s\leq t}(-X_s) = -\underline{X}_t$ and from Lemma 2.56 and Proposition 6.8 we obtain the following result.

Proposition 6.9. *Assume that $(X_t)_{t\geq 0}$ is a finite variation Lévy process i.e. $X_t = X_0 + \mu t + \sum_{s\leq t}\Delta X_s$ and its Lévy measure ν is absolutely continuous with respect to Lebesgue measure. Let also $f(t,x,y) : [0,T]\times\mathbb{R}_0^+\times\mathbb{R}\to\mathbb{R}$ be a $C^{1,1,1}([0,T]\times\mathbb{R}_0^+\times\mathbb{R})$ function*

$$\begin{aligned}\mathcal{L}f(t,x,y) &= \frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \int_{-x}^{+\infty}(f(t,x+z,y) - f(t,x,y))\nu(dz)\\ &\quad + \int_{-\infty}^{-x}(f(t,0,y-z-x) - f(t,x,y))\nu(dz),\ \forall t\in[0,T],\ x\geq 0,\ y\in\mathbb{R},\end{aligned}$$

that satisfies

$$\int_{-x}^{+\infty}|f(t,x+z,y) - f(t,x,y)|\nu(dz) < \infty,\ \ \forall t\in[0,T], x\geq 0,\ y\in\mathbb{R},$$

and

$$\int_{-\infty}^{-x}|f(t,0,y-z-x) - f(t,x,y)|\nu(dz) < \infty,\ \ \forall t\in[0,T],\ x\geq 0\ y\in\mathbb{R},$$

such that the boundary condition

$$\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0,$$

is also satisfied, then the process

$$\left(f(t,X_t - \underline{X}_t, -\underline{X}_t) - f(0,X_0 - \underline{X}_0, -\underline{X}_0) - \int_0^t \mathcal{L}f(s,X_s - \underline{X}_s, -\underline{X}_s)ds\right)_{t\geq 0},$$

with $t\in[0,T]$, is an $\mathbb{F}$-local martingale.

Proof. First, since $(X_t)_{t\geq 0}$ is a process of finite variation then its Lévy-Itô decomposition has the form

$$X_t = X_0 + \mu t + \int_0^t\int_{\mathbb{R}} zN(ds,dz),\ \ t\in[0,T].$$

If $\int_0^t \int_{\mathbb{R}} |z| N(ds,dz) < \infty$ then the process

$$\left(\int_0^t \int_{\mathbb{R}} zN(ds,dz) - t\int_{\mathbb{R}} z\nu(dz)\right)_{t\geq 0}, \ t \in [0,T],$$

is an $\mathbb{F}$-local martingale. Moreover, since the Lévy measure ν is continuous, then it admits a density m, such that $\nu(z) = \int_{-\infty}^z m(y)dy$. Now, let $Y_t = -X_t$, $Y_t = Y_0 - \mu t + \sum_{s\leq t}(-\Delta X_s)$ and $\hat{\nu}$ its Lévy measure. We claim that $\hat{\nu}$ can be written as

$$\hat{\nu}(z) = \int_{-z}^{+\infty} m(y)dy. \tag{6.9}$$

It is easy to verify (6.9) for the case when the process is a compound Poisson. For example, if $X_t = \sum_{i=1}^{N_t} U_i$, where U_i are i.i.d. random variables, and its Lévy measure ν will be $\nu(z) = \int_{-\infty}^z \lambda m_U(y)dy$. For the process $(Y_t)_{t\geq 0}$, $Y_t = -X_t = \sum_{i=1}^{N_t}(-U_i)$, its cumulative distribution function is

$$\mathbb{P}(-U \leq y) = \mathbb{P}(U \geq -y) = 1 - F_U(-y),$$

and its density will be $m_{-U}(y) = F'_U(-y) = m_U(-y)$. Thus the Lévy measure $\hat{\nu}$ has the following for

$$\begin{aligned}\hat{\nu}(z) &= \int_{-\infty}^z m_U(-y)\lambda dy \\ &\underset{-y=u}{=} -\int_{+\infty}^{-z} m_U(u)\lambda du \\ &= \int_{-z}^{+\infty} m_U(u)\lambda du.\end{aligned}$$

The same hold true for the general case due to Assumption 6.1. Since Proposition 6.8 is valid for any Lévy process of finite variation and observing that the reflected Lévy process $R_t = \bar{Y}_t - Y_t = \overline{(-X_t)} + X_t = X_t - \underline{X}_t$, we get that the process

$$\left(f(t,\bar{Y}_t - Y_t, \bar{Y}_t) - f(t,\bar{Y}_0 - Y_0, \bar{Y}_0) - \int_0^t \mathcal{L}f(s,\bar{Y}_s - Y_s, \bar{Y}_s)ds\right)_{t\geq 0},$$

with $t \in [0,T]$, is an $\mathbb{F}$-local martingale, where

$$\begin{aligned}\mathcal{L}f(t,x,y) &= \frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \int_{-\infty}^x (f(t,x-z,y) - f(t,x,y))\hat{\nu}(dz) \\ &+ \int_x^{+\infty}(f(t,0,y+z-x) - f(t,x,y))\hat{\nu}(dz) \\ &= \frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \int_{-\infty}^x (f(t,x-z,y) - f(t,x,y))q(-z)dz \\ &+ \int_x^{+\infty}(f(t,0,y-x+z) - f(t,x,y))q(-z)dz.\end{aligned}$$

Setting $u = -z$, we obtain

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} - \int_{+\infty}^{-x} (f(t, x+u, y) - f(t, x, y)) q(u) du \\
&\qquad - \int_{-x}^{-\infty} (f(t, 0, y-u-x) - f(t, x, y)) q(u) du \\
&= \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \int_{-x}^{+\infty} (f(t, x+u, y) - f(t, x, y)) q(u) du \\
&\qquad + \int_{-\infty}^{-x} (f(t, 0, y-x-u) - f(t, x, y)) q(u) du.
\end{aligned}$$

Therefore, under the operator

$$\begin{aligned}
\mathcal{L} f(t, x, y) = \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} &+ \int_{-x}^{+\infty} (f(t, x+u, y) - f(t, x, y)) \nu(du) \\
&+ \int_{-\infty}^{-x} (f(t, 0, y-u-x) - f(t, x, y)) \nu(du),
\end{aligned}$$

and subject to the boundary condition $\frac{\partial f}{\partial x}(t, 0, y) + \frac{\partial f}{\partial y}(t, 0, y) = 0$, the process

$$\left(f(t, X_t - \underline{X}_t, -\underline{X}_t) - f(0, X_0 - \underline{X}_0, -\underline{X}_0) - \int_0^t \mathcal{L} f(s, X_s - \underline{X}_s, -\underline{X}_s) ds \right)_{t \geq 0},$$

with $t \in [0, T]$, is an $\mathbb{F}$-local martingale. □

An immediate consequence of Proposition 6.9 is the next Lemma. We stress that for the spectrally positive case the running infimum process $(\underline{X}_t)_{t \geq 0}$ is strictly decreasing.

Lemma 6.10. *Let $(X_t)_{t \geq 0}$ be a spectrally positive of finite variation Lévy process i.e. $X_t = X_0 + \mu t + S_t$, $\mu < 0$ and its Lévy measure ν is absolutely continuous with respect to Lebesgue measure. We assume that $f(t, x, y) : [0, T] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,1,1}([0, T] \times \mathbb{R}_0^+ \times \mathbb{R})$ function, and for every $t \in [0, T]$, $x \geq 0$ and $y \in \mathbb{R}$ let*

$$\mathcal{L} f(t, x, y) = \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \int_0^{+\infty} (f(t, x+z, y) - f(t, x, y)) \nu(dz), \; t \in [0, T],$$

and if

$$\frac{\partial f}{\partial x}(t, 0, y) + \frac{\partial f}{\partial y}(t, 0, y) = 0,$$

then the process

$$\left(f(t, X_t - \underline{X}_t, -\underline{X}_t) - f(0, X_0 - \underline{X}_0, -\underline{X}_0) - \int_0^t \mathcal{L} f(s, X_s - \underline{X}_s, -\underline{X}_s) ds \right)_{t \geq 0},$$

with $t \in [0, T]$ is an $\mathbb{F}$-local martingale.

6.4 Canonical decompositions under the progressive filtration expansion $\mathbb{G}$

In this section, given a $C^{1,1,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$ function $f : [0,T] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$, we provide the canonical decomposition of $(f(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}})_{t\geq 0}$ when $(X_t)_{t\geq 0}$ is a spectrally positive Lévy process of finite variation, under filtration $\mathbb{G}$. This result will be used in Section 6.5 to help us to determine the hedging strategy $\phi_t = (\theta_t, \eta_t)$, using the LRM approach.

We first need to extend the canonical decomposition introduced in the Lemma 6.10 to the expanded filtration $\mathbb{G}$. In Chapter 5, we saw that a local martingale under filtration $\mathbb{F}$, is not always $\mathbb{G}$-local martingale. However, given a $\mathbb{F}$-local martingale $(X_t)_{t\geq 0}$ and a $\mathcal{G}$-measurable stopping time τ, then the process $(X_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{G}$-semimartingale. From Theorem 5.10 the process $(X_{\tau\wedge t})_{t\geq 0}$ has the following canonical form under $\mathbb{G}$

$$\left(X_{\tau\wedge t} - \int_0^{\tau\wedge t} \frac{(\langle X, M\rangle_s^{\mathbb{F}} + dB_s)}{Z_{s-}}\right)_{t\geq 0}, \quad t \in [0,T],$$

where $(Z_t)_{t\geq 0}$ is the Azema supermartingale under $\mathbb{F}$, $Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t)$, and $(M_t)_{t\geq 0}$ is the martingale part of $(Z_t)_{t\geq 0}$. The process $(B_t)_{t\geq 0}$ is the optional dual projection of the process $(1_{\{\tau\leq t\}}\Delta X_\tau)_{t\geq 0}$. Under the assumption that $\mathbb{G}$-stopping time τ avoids all the $\mathbb{F}$-stopping times i.e. $\mathbb{P}(\tau = \varsigma) = 0$, for all $\mathbb{F}$-stopping times ς, Protter (2004), in Chapter VI, simplified the above decomposition with $B_t = 0$. Alternatively, we can assume that $\Delta X_\tau = 0$. Since the underlying asset is a spectrally positive process, in Theorem 6.13 we will also prove that $\Delta X_\tau = 0$.

Our first goal is to find the canonical decomposition of $f(\tau \wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t})$ under the filtration $\mathbb{G}$, given that the Lévy process $(X_t)_{t\geq 0}$ is spectrally positive and of finite variation, with $X_0 = u$, $u > 0$.

Assumption 6.11. *Let $f(t,x,y) : [0,T] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ be a $C^{1,1,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$ function. Then $f(t,x,y)$ satisfies the integrability condition*

$$\int_0^{+\infty} |f(t, x+z, y) - f(t,x,y)|\nu(dz) < \infty,$$

for every $t \in [0,T]$, $x \geq 0$ and $y \in \mathbb{R}$.

The following Lemma is a direct application of Protter (2004), Chapter VI, Theorem 15.

Lemma 6.12. *Assuming that $f(t,x,y) : [0,T]\times\mathbb{R}_0^+\times\mathbb{R} \to \mathbb{R}$ is a $C^{1,1,1}([0,T]\times\mathbb{R}_0^+\times\mathbb{R})$ function and $(X_t)_{t\geq 0}$ is a spectrally positive of finite variation Lévy process, given by (6.1) with $X_0 = u$. We also assume that $f(t,x,y)$ satisfies Assumption 6.11, also the Lévy measure ν satisfies Assumption 6.1. Then the process*

$$\Big(f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}) - f(0, X_0 - \underline{X}_0, -\underline{X}_0) - \int_0^{\tau\wedge t} \mathcal{L}f(s, X_s - \underline{X}_s, -\underline{X}_s)ds\Big)_{t\geq 0},$$

is a $\mathbb{G}$-local martingale,where τ is a $\mathbb{G}$-totally inaccessible stopping time and the operator $\mathcal{L}f(t,x,y)$ is defined in 6.10.

Proof. Let

$$M_t = \left(f(t, X_t - \underline{X}_t, -\underline{X}_t) - f(0, X_0 - \underline{X}_0, -\underline{X}_0) - \int_0^t \mathcal{L}f(s, X_s - \underline{X}_s, -\underline{X}_s)ds \right)_{t\geq 0},$$

be the $\mathbb{F}$-local martingale defined in Proposition 6.10. From Protter (2004), Chapter VI, Theorem 15, it yields that the stopped martingale $(M_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{G}$-semimartingale and the process

$$\left(M_{\tau\wedge t} - \int_0^{\tau\wedge t} \frac{1}{Z_{s-}} d\left\langle Z, f\right\rangle_s^{\mathbb{F}} \right)_{t\geq 0}, \quad t \in [0,T],$$

is a $\mathbb{G}$-local martingale. First note that since $(X_t)_{t\geq 0}$ is a spectrally positive process its running infimum is continuous. Therefore the process $(Z_t)_{t\geq 0}$, $Z_t = e^{\underline{X}_t}$ is a continuous one. Since both $(X_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are of finite variation then $[f(\cdot, X_{\cdot} - \underline{X}_{\cdot}, -\underline{X}_{\cdot}), Z_{\cdot}]_t^{\mathbb{F}} = 0$, which implies that $\langle f(\cdot, X_{\cdot}, -\underline{X}_{\cdot}, -\underline{X}_{\cdot}), Z_{\cdot}\rangle_t^{\mathbb{F}} = 0$. So the process $(M_{\tau\wedge t})_{t\geq 0}$ is an $\mathbb{G}$-local martingale, and the result is straightforward. □

As a result we get the following Theorem, which is the main part of this section.

Theorem 6.13. *Assume that the Lévy measure ν satisfies Assumption 6.1 and $f(t,x,y)$ is $C^{1,1,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$, which satisfies Assumption 6.11. Let also $(X_t)_{t\geq 0}$ be a spectrally positive Lévy process of finite variation introduced in (6.1). Then the process $(U_t)_{t\geq 0}$, $U_t = f(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}}$ has the following canonical decomposition*

$$O_t = \left(U_t - U_0 - \int_0^{\tau\wedge t} \mathcal{A}f(s, X_s - \underline{X}_s, -\underline{X}_s)ds \right)_{t\geq 0}, \tag{6.10}$$

is a $\mathbb{G}$-local martingale, where the operator $\mathcal{A}f(t,x,y)$ is given as

$$\begin{aligned} \mathcal{A}f(t,x,y) :&= \frac{\partial f}{\partial t}(t,x,y) + \mu \frac{\partial f}{\partial x}(t,x,y) + \mu f(t,x,y)1_{\{x=0\}} \\ &+ \int_0^{+\infty} (f(t,x+z,y) - f(t,x,y))\nu(dz) \\ &- \int_0^{+\infty} f(t,x,y)(1-e^{-z})\nu(x+dz), \ t \in [0,T], \ x \geq 0, \ y \in \mathbb{R}, \end{aligned} \tag{6.11}$$

subject to the mixed boundary condition

$$\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0.$$

Proof. Let $U_t = f(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}}$ and we know that

$$f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}) = U_t + f(\tau, X_\tau - \underline{X}_\tau, -\underline{X}_\tau)1_{\{\tau\leq t\}}, \ t \in [0,T],$$

which implies

$$U_t = f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}) - f(\tau, X_\tau - \underline{X}_\tau, -\underline{X}_\tau)1_{\{\tau\leq t\}}.$$

Therefore we need to calculate the compensators of $f(\tau \wedge X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t})$ and $f(\tau, X_\tau - \underline{X}_\tau, -\underline{X}_\tau)1_{\{\tau\le t\}}$. We start with the first one. From Lemma 6.12, we have that the process

$$M^{(1)}_{\tau\wedge t} = \Big(f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}) - f(0, X_0 - \underline{X}_0, -\underline{X}_0) - \int_0^{\tau\wedge t} \mathcal{L}f(s, X_s - \underline{X}_s, -\underline{X}_s)ds \Big)_{t\ge 0}, \; t\in[0,T],$$

is a $\mathbb{G}$-local martingale. Then based on the Assumption 6.11, and using the compensation formula, Lemma 6.6, the process $(f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}))_{t\ge 0}$ can be written as $f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}) = M^{(1)}_{\tau\wedge t} + \Lambda^f_{\tau\wedge t}$, where $\Lambda^f_{\tau\wedge t} = \int_0^{\tau\wedge t}\mathcal{L}f(s, X_s - \underline{X}_s, -\underline{X}_s)ds$. Thus the compensator of $f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t})$ is

$$\int_0^{\tau\wedge t}\mathcal{L}f(s, X_s - \underline{X}_s, -\underline{X}_s)ds, \quad t\in[0,T].$$

We proceed with the second term. From Proposition 2.27 we know that

$$\left[f(\cdot, X_\cdot - \underline{X}_\cdot, -\underline{X}_\cdot),\, 1_{\{\tau\le\cdot\}}\right]^{\mathbb{G}}_t = \int_0^t \Delta f_s d(1_{\{\tau\le s\}}).$$

We will prove that $\Delta f(\tau, X_\tau - \underline{X}_\tau, -\underline{X}_\tau) = 0$ through contradiction.

Let $\tau = \inf\{t\ge 0 : X_t < 0\}$. From the definition of creeping, we know that the process creeping downwards and therefore $X_\tau \le 0$. Let $\omega\in\Omega$ fixed and assume that $\Delta X_\tau(\omega)\ge 0$, where $\Delta X_\tau = X_\tau - X_{\tau-}$and $X_{\tau-} = \lim_{s\to\tau-} X_s$.On the other hand, since the process, since the process $(X_t)_{t\ge 0}$ is spectrally positive then $\Delta X_\tau \ge 0$ i.e. $X_\tau(\omega)\ge X_{\tau-}(\omega)$. Also for each $s\le\tau$ we have that $X_s(\omega)\ge 0$ which implies $X_{\tau-}(\omega)\ge 0$. If $X_\tau(\omega) > X_\tau(\omega)$, since $X_{\tau-}(\omega)\ge 0$, thus we obtain $X_\tau(\omega) > 0$ which is a contradiction. Thus we must have $\Delta X_\tau(\omega) = 0$ for all $\omega\in\Omega$. This can be also verified from Figure 6.1a, where the process $(X_t)_{t\ge 0}$ cross the default barrier continuously.

Moreover, following Kyprianou (2014), Section 7.5, since the process creeps downwards, and $(X_t)_{t\ge 0}$ has negative jumps, then it must hit 0, i.e. $\tau^{\{0\}} = \inf\{t>0 : X_t = 0\}$, which suggests that the above argument is valid for any stopping time of the form $\tau = \inf\{t>0 : X_t\le 0\}$. This can be generalized for any barrier $D\le 0$, $\tau = \inf\{t>0 : X_t \le D\}$. We note that the above result holds for any filtration $\mathbb{F}$ or the augmented $\mathbb{G}\subset\mathbb{F}$. Finally, since the process $(\underline{X}_t)_{t\ge 0}$, and the function $f(t,x,y)$ are continuous then we obtain that $\Delta f(\tau\wedge t, X_{\tau\wedge t} - \underline{X}_{\tau\wedge t}, -\underline{X}_{\tau\wedge t}) = 0$. Therefore we get that

$$[f(\cdot, X_\cdot - \underline{X}_\cdot, -\underline{X}_\cdot),\, 1_{\{\tau\le\cdot\}}]^{\mathbb{G}}_t = \int_0^{\tau\wedge t}\Delta f(s, X_s - \underline{X}_s, -\underline{X}_s)d(1_{\{\tau\le s\}}) = 0,$$

which implies that

$$\int_0^{\tau\wedge t} f(s, X_s - \underline{X}_s, -\underline{X}_s)d(1_{\{\tau\le s\}}) = \int_0^{\tau\wedge t} f(s-, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})d(1_{\{\tau\le s\}}). \tag{6.12}$$

It is easy to see that

$$f(\tau, X_\tau - \underline{X}_\tau, -\underline{X}_\tau)1_{\{\tau\le t\}} = \int_0^t f(s, X_s - \underline{X}_s, -\underline{X}_s)d(1_{\{\tau\le s\}}),\ t\in[0,T],$$

note that from (6.12) the process $M_t^{(2)} := (1_{\{\tau\le t\}} - \int_0^{\tau\wedge t}\lambda_s ds)_{t\ge 0}$ is a $\mathbb{G}$-local martingale and it yields

$$\begin{aligned}
f(\tau, X_\tau - \underline{X}_\tau, -\underline{X}_\tau)1_{\{\tau\le t\}} &= \int_0^t f(s, X_s - \underline{X}_s, -\underline{X}_s)d(1_{\{\tau\le s\}})\\
&= \int_0^t f(s-, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})d(1_{\{\tau\le s\}})\\
&= \int_0^t f(s-, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})dM_s^{(2)}\\
&+ \int_0^{\tau\wedge t} f(s, X_s - \underline{X}_s, -\underline{X}_s)\tilde{\lambda}_s ds,
\end{aligned}$$

where $\tilde{\lambda}_t = -\mu 1_{\{X_t - \underline{X}_t = 0\}} + \Pi(X_t - \underline{X}_t)$. Therefore the compensator of $(U_t)_{t\ge 0}$ is

$$\int_0^{\tau\wedge t} \mathcal{L}f(s, X_s - \underline{X}_s, -\underline{X}_s) - f(s, X_s - \underline{X}_s, -\underline{X}_s)(-\mu 1_{\{X_s - \underline{X}_s = 0\}} + \Pi(X_s - \underline{X}_s))ds,$$

where $\Pi(x)$ is the function introduced in (6.5). □

The following Lemma simplifies the above Theorem in the 2-dimensional case.

Lemma 6.14. *Assume that the Lévy measure satisfies Assumption 6.1 and* $(X_t)_{t\ge 0}$ *be a spectrally positive Lévy process of finite variation introduced in (6.1). Let* $g(t,x)$ *be a* $C^{1,1}([0,T]\times\mathbb{R}_0^+)$ *function satisfying the following integrability condition:*

$$\int_0^{+\infty} |g(t, x+z) - g(t,x)|\nu(dz) < \infty,\ \forall t\in[0,T] \textit{and } x\ge 0.$$

Then the process $(U_t)_{t\ge 0}$, $U_t = g(t, X_t - \underline{X}_t)1_{\{\tau > t\}}$ *has the following decomposition*

$$\left(U_t - U_0 - \int_0^{\tau\wedge t} \mathcal{A}g(s, X_s - \underline{X}_s)ds\right)_{t\ge 0},\ t\in[0,T],$$

is a $\mathbb{G}$*-local martingale and the operator* $\mathcal{A}g(t,x)$ *is given by*

$$\begin{aligned}
\mathcal{A}g(t,x) :&= \frac{\partial g}{\partial t}(t,x) + \mu\frac{\partial g}{\partial x}(t,x) + \mu g(t,x)1_{\{x=0\}} + \int_0^{+\infty}(g(t,x+z) - g(t,x))\nu(dz)\\
&- \int_0^{+\infty} g(t,x)(1-e^{-z})\nu(x+dz),\ \ \forall t\in[0,T],\ x\ge 0,
\end{aligned}$$

subject to the Neumann boundary condition

$$\frac{\partial g}{\partial x}(t,0) = 0.$$

Proof. The proof is direct application of Theorem 6.13, for a continuous and differentiable function $g(t,x)$, with $f(t,x,y) := g(t,x)$. □

6.5 Hedging strategy for a defaultable claim under partial information

The main part of this section is devoted to the determination of the hedging strategy for a defaultable claim with a payoff of the from $F(X_T - \underline{X}_T, -\underline{X}_T)1_{\{\tau>T\}}$, for a continuous function $F : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$. Calculating the canonical decomposition of $(f(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}})_{t\geq 0}$ under filtration $\mathbb{G}$, is a crucial step to determine $(\theta_t)_{t\geq 0}$. We next turn our attention to the determination of $(\theta_t)_{t\geq 0}$ without applying the $\mathcal{H}$-hypothesis. The following Lemma provides the form of $(X_t^\tau)_{t\geq 0}$ under the expanded filtration $\mathbb{G}$.

Lemma 6.15. *Under the Assumption 6.1, the stopped process $(X_t^\tau)_{t\geq 0}$ is a square integrable $\mathbb{G}$-special semimartingale of the form*

$$X_t^\tau = X_0 + M_{\tau\wedge t}^X + \int_0^{\tau\wedge t} \mathcal{L}f_1(s, X_s - \underline{X}_s, -\underline{X}_s)ds, \ t \in [0, T], \tag{6.13}$$

where $(M_{\tau\wedge t}^X)_{t\geq 0}$ is a $\mathbb{G}$-local martingale, the operator $\mathcal{L}f(t, x, y)$ is given by Lemma 6.10, and

$$f_1(t, x, y) = (x - y), \ \ t \in [0, T], \ x \geq 0, \ y \in \mathbb{R}. \tag{6.14}$$

Simplifying (6.13), we get $X_{\tau\wedge t} = X_0 + M_{\tau\wedge t}^X + \Lambda_{\tau\wedge t}^X$, where $(\Lambda_{\tau\wedge t}^X)_{t\geq 0}$ is a $\mathbb{G}$-adapted càdlàg process of finite variation with the following form

$$\Lambda_{\tau\wedge t}^X = \mu\tau \wedge t + \int_0^{\tau\wedge t} \int_0^{+\infty} z\nu(dz)ds, \quad t \in [0, T].$$

The $\mathbb{G}$-predictable quadratic variation is given by

$$\langle X^\tau \rangle_t^{\mathbb{G}} = \int_0^{\tau\wedge t} \int_0^{+\infty} z^2\nu(dz)ds, \quad t \in [0, T]. \tag{6.15}$$

Proof. Let $f_1(t, x, y) = (x - y)$. Then $f_1(t, X_t - \underline{X}_t, -\underline{X}_t) = (X_t - \underline{X}_t - (-\underline{X}_t)) = X_t$. If we apply Lemma 6.12, then then process

$$M_{\tau\wedge t}^X := \left(X_t^\tau - X_0 - \int_0^{\tau\wedge t} \mathcal{L}f_1(s, X_s - \underline{X}_s, -\underline{X}_s)ds \right)_{t\geq 0}, \ t \in [0, T],$$

is a $\mathbb{G}$-local martingale. Thus the process $(X_t^\tau)_{t\geq 0}$, under $\mathbb{G}$, has the form

$$X_t^\tau = X_0 + M_{\tau\wedge t}^X + \int_0^{\tau\wedge t} \mathcal{L}f_1(s, X_s - \underline{X}_s, -\underline{X}_s)ds, \ t \in [0, T].$$

Next, we calculate $\langle X^\tau \rangle_t^{\mathbb{G}}$. Once again, following Assumption 6.1 since $\int_0^{+\infty} z^2\nu(dz) < \infty$, we know that $(X_t)_{t\geq 0}$ is a square integrable and $\mathbb{G}$-adapted. From Proposition 2.27 $[X^\tau]_t^{\mathbb{G}} \in \mathcal{A}_{loc}$, which yields that the $\mathbb{G}$-predictable quadratic variation $\langle X^\tau \rangle_t^{\mathbb{G}}$ exists.

First, we calculate the quadratic variation $[X^\tau]_t^{\mathbb{G}}$. Following Definition 2.25, we know

$$[X^\tau]_t^{\mathbb{G}} = X_{\tau\wedge t}^2 - 2\int_0^{\tau\wedge t} X_{s-}dX_s, \quad t \in [0, T]. \tag{6.16}$$

Note that

$$\int_0^{\tau\wedge t} X_{s-}dX_s = \int_0^{\tau\wedge t} X_{s-}dM_s^X + \int_0^{\tau\wedge t} X_{s-}\mathcal{L}f_1(s, X_s - \underline{X}_s, -\underline{X}_s)ds,\ t\in[0,T]. \tag{6.17}$$

Let $f_2(t,x,y)=(x-y)^2$, then $f_2(t,X_t-\underline{X}_t,-\underline{X})=X_t^2$. Again, by applying Lemma 6.12, it follows that

$$\hat{M}_{\tau\wedge t} := \left(X_{\tau\wedge t}^2 - X_0^2 - \int_0^{\tau\wedge t}\mathcal{L}f_2(s, X_s-\underline{X}_s, -\underline{X}_s)ds\right)_{t\geq 0},\quad t\in[0,T], \tag{6.18}$$

is a $\mathbb{G}$-local martingale. Plugging (6.17), and (6.18) into (6.16) then

$$\begin{aligned}[X^\tau]_t^{\mathbb{G}} &= \hat{M}_{\tau\wedge t} + \int_0^{\tau\wedge t}\mathcal{L}f_2(s, X_s-\underline{X}_s, -\underline{X}_s)ds \\ &\quad - 2\left(\int_0^{\tau\wedge t} f_1(s, X_{s-}, -\underline{X}_{s-}, -\underline{X}_{s-})d\hat{M}_s\right) \\ &\quad - 2\left(\int_0^{\tau\wedge t} f_1(s, X_s-\underline{X}_s, -\underline{X}_s)\mathcal{L}f_1(s, X_s-\underline{X}_s, -\underline{X}_s)ds\right).\end{aligned}$$

Thus, the compensator of $([X^\tau]_t^{\mathbb{G}})_{t\geq 0}$ i.e. $(\langle X^\tau\rangle_t^{\mathbb{G}})_{t\geq 0}$ is given by

$$\begin{aligned}\langle X^\tau\rangle_t^{\mathbb{G}} = \int_0^{\tau\wedge t}\Big(&\mathcal{L}f_2(s, X_s-\underline{X}_s, -\underline{X}_s) \\ &- 2f_1(s, X_s-\underline{X}_s, -\underline{X}_s)\mathcal{L}f_1(s, X_s-\underline{X}_s, -\underline{X}_s)\Big)ds.\end{aligned}$$

□

Assumption 6.16. *Let $f(t,x,y)$ be $C^{1,1,1}([0,T]\times\mathbb{R}_0^+\times\mathbb{R})$ function which is the solution of the following PIDE*

$$\mathcal{A}f(t,x,y) := \frac{(\mathcal{A}P(t,x,y)-(x-y)\mathcal{A}f(t,x,y)-f(t,x,y)\alpha)}{\int_0^{+\infty}z^2\nu(dz)}\alpha,\ \ 0\leq t\leq T,$$

where $\alpha=\mu+\int_0^{+\infty}z\nu(dz)$, $P(t,x,y)=(x-y)f(t,x,y)$, with $t\in[0,T]$, $x\geq 0$ and $y\in\mathbb{R}$ given a terminal condition

$$f(T,x,y)=F(x,y),\ x\geq 0,\ y\in\mathbb{R},$$

and subject to the boundary condition

$$\frac{\partial f}{\partial x}(t,0,y)+\frac{\partial f}{\partial y}(t,0,y)=0.$$

Remark 6.17. Note that due to the existence of the indicator function in operator $\mathcal{A}f(t,x,y)$ given by (6.11), the existence of a smooth solution in the above PIDE is not guaranteed or it may not even exist. One way of fixing this problem could be to use the non-smooth version of Itô's formula and investigate the existence of a solution that is just weakly differentiable. As we assume that Assumption 6.16 holds, this makes the proof of the following results a bit heuristics though the numerical results look reasonable. In the next chapter (Chapter 7), the intensity of our models are continuous, and hence irregular terms (such as indicator function in the operator $\mathcal{A}f(t,x,y)$ introduced in (6.11)) would not exist in the PIDEs.

We conclude this section with the determination of the hedging strategy $(\theta_t)_{t\geq 0}$.

Theorem 6.18. *Let the process $(X_t^\tau)_{t\geq 0}$ given by (6.13) and we assume that Assumption 6.1 holds. Furthermore, let $[U, X^\tau]_t \in \mathcal{A}_{loc}$. We also assume that $f(t,x,y)$ is the solution of the PIDE given in Assumption 6.16. Then for all $t \in [0,T]$ we have the following decomposition*

$$U_t = f(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}} = U_0 + \int_0^t \theta_s dX_s^\tau + L_{\tau\wedge t}$$
$$= U_0 + \int_0^t \theta_s 1_{\{\tau\geq s\}} dX_s + L_{\tau\wedge T},\ t \in [0,T],$$

and for $t = T$

$$U_T = F(X_T - \underline{X}_T, -\underline{X}_T)1_{\{\tau>T\}} = U_0 + \int_0^T \theta_s dX_s^\tau + L_{\tau\wedge T}$$
$$= U_0 + \int_0^T \theta_s 1_{\{\tau\geq s\}} dX_s + L_{\tau\wedge T},$$

where $F(x,y)$ is the PIDE's terminal condition, and the process $(\theta_t)_{t\geq 0}$ is given by

$$\theta_t = \frac{\mathcal{K}f(t, X_t - \underline{X}_t, -\underline{X}_t)}{\int_0^{+\infty} z^2\nu(dz)},\ \ t \in [0,T],$$

for some operator

$$\mathcal{K}f(t,x,y) = \mathcal{A}P(t,x,y) - (x-y)\mathcal{A}f(t,x,y) - f(t,x,y)\alpha,$$

where $P(t,x,y) = (x-y)f(t,x,y)$, $\mathcal{A}f(t,x,y)$ is given from (6.11) and $\alpha = \mu + \int_0^{+\infty} z\nu(dz)$. The process $(L_{\tau\wedge t})_{t\geq 0}$ is orthogonal to the martingale part of $(X_t^\tau)_{t\geq 0}$ i.e. $(M^X_{\tau\wedge t})_{t\geq 0}$.

Proof. Let $(U_t)_{t\geq 0}$ be a process as given in Theorem 6.13, and $f_1(t,x,y) = (x-y)$ is a smooth function. From Theorem 6.13 we know that the process

$$O_t = \left(U_t - U_0 - \int_0^{\tau\wedge t} \mathcal{A}f(s, X_s - \underline{X}_s, -\underline{X}_s)ds\right)_{t\geq 0},\ t \in [0,T],$$

is a $\mathbb{G}$-local martingale. Then the process

$$O_t^{(1)} := \Big(P(t, X_t - \underline{X}_t, -\underline{X}_t)1_{\{\tau>t\}} - P(0, X_0 - \underline{X}_0, -\underline{X}_0)$$
$$- \int_0^{\tau\wedge t} \mathcal{A}P(s, X_s - \underline{X}_s, -\underline{X}_s)ds\Big)_{t\geq 0},\ t \in [0,T],$$

is also a $\mathbb{G}$-local martingale, where $P(t, X_t - \underline{X}_t, -\underline{X}_t)$ is defined in Theorem 6.13 and the operator $\mathcal{A}$ is given by (6.11). Since $[U, X]^{\mathbb{G}}_{\tau\wedge t} = [O, M^X]^{\mathbb{G}}_{\tau\wedge t}$, then the two processes have the same compensator. Once again, we obtain the KW decomposition of $(O_t)_{t\geq 0}$ with respect to $(M^X_{\tau\wedge t})_{t\geq 0}$. Since $[O, \hat{M}]^{\mathbb{G}}_{\tau\wedge t} \in \mathcal{A}_{loc}$, its compensator exists and is given by $\langle O, M^X\rangle^{\mathbb{G}}_{\tau\wedge t}$. From the integration by parts formula with $X_0 = u$, $u > 0$, we have

$$U_t X^\tau_t = U_0 X_0 + \int_0^t U_{s-} dX^\tau_s + \int_0^t X^\tau_{s-} dU_s + [U, X^\tau]^{\mathbb{G}}_t.$$

Let again define $F^{(1)}_t := \int_0^{\tau\wedge t} \mathcal{A} f(s, X_s - \underline{X}_s, -\underline{X}_s) ds$, and $F^{(2)}_t := \int_0^{\tau\wedge t} \mathcal{A} P(s, X_s - \underline{X}_s, -\underline{X}_s) ds$. From the above integration by parts formula, we have

$$\begin{aligned}
[U, X^\tau]^{\mathbb{G}}_t - \left(F^{(2)}_t - \int_0^{\tau\wedge t} U_{s-} \mathcal{U} f_1(s, X_s - \underline{X}_s, -\underline{X}_s) ds - \int_0^{\tau\wedge t} X_{s-} dF^{(1)}_s \right) \\
= \left(O^{(1)}_t - \int_0^{\tau\wedge t} U_{s-} dM^X_s - \int_0^{\tau\wedge t} X_{s-} dO_s \right).
\end{aligned}$$

Therefore the $\mathbb{G}$-predictable quadratic covariation is given by

$$\begin{aligned}
\left\langle O, \hat{M}^\tau \right\rangle^{\mathbb{G}}_t = & \int_0^{\tau\wedge t} \Big(\mathcal{A} P(s, X_s - \underline{X}_s, -\underline{X}_s) \\
& - f(s, X_s - \underline{X}_s, -\underline{X}_s) \mathcal{L} f_1(s, X_s - \underline{X}_s, -\underline{X}_s) \\
& - X_s \mathcal{A} f(s, X_s - \underline{X}_s, -\underline{X}_s) \Big) ds \\
= & \int_0^{\tau\wedge t} \Big(\mathcal{A} P(s, X_s - \underline{X}_s, -\underline{X}_s) \\
& - f(s, X_s - \underline{X}_s, -\underline{X}_s) \mathcal{L} f_1(s, X_s - \underline{X}_s, -\underline{X}_s) \\
& - f_1(s, X_s - \underline{X}_s, -\underline{X}_s) \mathcal{A} f(s, X_s - \underline{X}_s, -\underline{X}_s \Big) ds, \ t \in [0, T].
\end{aligned}$$

Since the $\mathbb{G}$-predictable quadratic variation $\langle X^\tau \rangle^{\mathbb{G}}_t$ is given by (6.15), then the process $(\theta_t)_{t\geq 0}$ is given by

$$\theta_t = \frac{\mathcal{K} f(t, X_{t-} - \underline{X}_{t-}, -\underline{X}_{t-})}{\int_0^{+\infty} z^2 \nu(dz)}, \ t \in [0, T],$$

where $\mathcal{K} f(t, x, y) = \mathcal{A} P(t, x, y) - f_1(t, x, y) \mathcal{A} f(t, x, y) - f(t, x, y) \mathcal{L} f_1(t, x, y)$. Next, we can define the $\mathbb{G}$local martingale $(L_t)_{t\geq 0}$ which represents the hedging error as

$$L_t = O_t - U_0 - \int_0^{\tau\wedge t} \theta_s dM^X_s,$$

and the rest of the proof follows very similar steps as in Theorem 4.16 under the progressive filtration $\mathbb{G}$. Also, by letting $t = T$ along with the boundary condition introduced in Assumption 6.16 the theorem is proved. □

Remark 6.19. It's not hard to see that the process $(\theta_t)_{t\geq 0}$ introduced in Theorem 6.18

takes the following form

$$\theta_t = \frac{\int_0^{+\infty} z f(t, X_{t-} - \underline{X}_{t-} + z, -\underline{X}_{t-})\nu(dz) - f(t, X_{t-} - \underline{X}_{t-} - \underline{X}_{t-}) \int_0^{+\infty} z\nu(dz)}{\int_0^{+\infty} z^2\nu(dz)}, \tag{6.19}$$

with $t \in [0, T]$.

6.5.1 Local risk minimization under $\mathbb{G}$

In Theorem 6.18, the optimal number of shares $(\theta_t)_{t\geq 0}$ through the KW decomposition is obtained. However, we still need to provide the appropriate conditions such that the FS decomposition exists. This is achieved by calculating the MVT process, see Chapter 3, Section 3.4.2.

The next corollary investigates the special case where $(X_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{G}$-local martingale.

Corollary 6.20. *Let $(X_t^\tau)_{t\geq 0}$ be a $\mathbb{G}$-square integrable local martingale and $f(t,x,y)$ be the solution of the following PIDE*

$$\mathcal{A}f(t,x,y) = 0,\ \forall\, 0 \leq t \leq T,\ x \geq 0 \text{ and } y \in \mathbb{R},$$

with a terminal condition $f(T,x,y) = F(x,y)$ subject to the boundary condition

$$\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0,$$

then in this case, the GKW decomposition is

$$U_T = F(X_T - \underline{X}_T, -\underline{X}_T)1_{\{\tau > T\}} = U_0 + \int_0^T \frac{\mathcal{A}P(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})}{\int_0^{+\infty} z^2\nu(dz)} dX_s^\tau + L_{\tau\wedge T},$$

where the operator $\mathcal{A}f(t,x,y)$ is introduced in Theorem 6.13 and $P(t,x,y) = (x - y)f(t,x,y)$.

Proof. If the operator $\mathcal{L}f_1(t,x,y)$, where $f_1 = (x-y)$, is zero then from Theorem 6.15 the process $(X_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{G}$-local martingale. In this case, from the PIDE introduced in Assumption 6.16, it implies that $\mathcal{A}f(t,x,y) = 0$. □

Following Example 4.19 and assuming that the underlying asset has the form $X_t = X_0 + \mu t + \sum Y_j$, where the i.i.d. random variables follow exponential distribution with parameter δ, $h(z) = \delta \exp(\delta z)$ for $z \geq 0$ and $\mu = -\frac{\lambda}{\delta}$, for the simple case when the payoff function $F(x,y)$ is constant and equal to one and $f(t,x,y)$ is the solution of the PIDE introduced in Corollary 6.20, the process $(\theta_t)_{t\geq 0}$ has the following representation

$$\theta_t = \frac{\delta^2 \int_0^{+\infty} z f(t, X_t - \underline{X}_t + z, -\underline{X}_t)h(z)dz - \delta f(t, X_t - \underline{X}_t, -\underline{X}_t)}{2} 1_{\{\tau\geq t\}}.$$

I would also point out the following when implementing the PIDE in Chapter 8. Despite the irregularities (the indicator function in the operator (6.11)) in our PIDE for the above example, in Section 8.4, we provide a heuristic implementation for its solution.

Based on Theorem 6.18, let us further introduce a particular interesting example.

Example 6.21. Let $(X_t^{\tau})_{t\geq 0}$ with $0 \leq t \leq T$ be a $\mathbb{G}$-semimartingale of the form (6.13) and $f(t,x,y)$ is the solution of the PIDE introduced in Assumption 6.16, but let now the terminal condition be given as

$$f(T,x,y) = F(x-y) := \max((x-y)-K, 0).$$

Then the optimal number of shares $(\theta_t)_{t\geq 0}$ with $t \in [0,T]$ is given by (6.19).

We remark that the above decomposition determines the PLRM and not the LRM. However, following Assumption 6.1 since $\int_0^{+\infty} z^2\nu(dz) < \infty$ then the process $(X_t^{\tau})_{t\geq 0}$ is square integrable and if MVT process is uniformly bounded, we can find the LRM through the PLRM. So the FS decomposition exists.

Following the same argument as in Chapter 4, it is not hard to see that the MVT process is uniformly bounded and so the SC holds true. In order to determine the optimal hedging strategy $\phi = (\theta_t, \eta_t)_t$ apart from $(\theta_t)_{t\geq 0}$, we need to certify the processes: the amount of money invested in the non-risky asset $(\eta_t)_{t\geq 0}$, the value process $(V_t)_{t\geq 0}$ and finally the cost process $(C_t)_{t\geq 0}$. Note that the definitions of Θ-space and of an L^2-strategy, introduced in Definition 3.33 and 3.34, can be formulated in $\mathbb{G}$ as well. Once again, if we assume that the solution $f(t,x,y)$ of the PIDE introduced in Assumption 6.16 is square integrable i.e. $f(t,x,y) \in L^2(\mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R})$, then all these processes exist by applying directly Proposition 4.1 of Okhrati et al. (2014).

Chapter 7

Local risk minimization and the running infimum process

7.1 Introduction

In this chapter, we analyse the LRM approach of certain contingent claims that might be prone to two types of default events. The first is an endogenous (or structural) default event defined by the first hitting time of the underlying asset to a barrier, and the second default event is caused exogenously which is modelled by a hazard rate process. The first type of default is arising due to specific risk and the second one is because of systematic risk. In our model, the market is not necessarily complete, i.e. the existence of a unique martingale measure (under which any contingent claim can be perfectly hedged and uniquely priced) is not guaranteed. In our study, the underlying asset is modelled by a jump-diffusion Lévy process with finite variation jumps.

Regarding hedging defaultable claims using the LRM approach, Biagini and Cretarola (2009) and Biagini and Cretarola (2012) study defaultable markets through the FS decomposition based on hazard rate models where the default intensity is modelled by a hazard rate process and underlying asset is modelled using a geometric Brownian motion. However, none of this approach can explain neither the effect of internal and external default events simultaneously nor they apply PDE (or PIDE) approaches.

In what follows, we specifically discuss the contribution of our work. First, we obtain semi-explicit solution of hedging strategies in the LRM framework, when the defaultable claim is subject to both structural (caused specifically by the underlying assets) and exogenous default events (caused systematically by external risk factors). This is an improvement over the existing credit risk models where the default event is linked to either a structural default event (normally modelled by a predictable stopping time), like Merton's model Merton (1974) (see Chapter 3), or completely unpredictable modelled via totally inaccessible stopping times such as hazard rate based models in Duffie and Singleton (1999), Jarrow and Turnbull (1995) and Jeanblanc and Le Cam (2008), (see again Chapter 3). In the context of jump processes, it is also possible to model structural credit risk events using totally inaccessible stopping times, see Okhrati et al. (2014).

To understand this further, consider a default time which is modelled by the first hitting time a jump-diffusion process to a certain barrier. This hitting time is a stopping time but it is neither predictable nor totally inaccessible this is because the default can happen in two fashions, either through a sudden jump or via a continuous crossing of the barrier. Under diffusion process, the stopping time is predictable and under pure jump process, it is totally inaccessible, but it is neither one under jump-diffusion processes.

This is achieved using the running infimum process as an auxiliary process in addition to the underlying asset. In other words, the strategies are determined by using not only the underlying process but also its running infimum. As we mentioned earlier, we do not use the minimal martingale measure, instead we determine hedging strategies through either PDEs or PIDEs depending on whether or not the underlying asset is continuous.

However, this extra dimension would allow us to manage the risk of more complicated defaultable claims. For instance, we can extend models such as Okhrati et al. (2014) where the default time is totally inaccessible and the underlying asset is finite variation Lévy process. More specifically, we improve their work by letting the underlying asset to be a jump-diffusion Lévy process and a general hitting time.

In addition, we allow for correlation between the endogenous and exogenous risk factors. This is done through modelling default rate dependent on both the underlying asset and the running infimum process. In order to model the default event via τ, we assume that $\mathbb{P}(\tau > t \mid \mathcal{F}_t) = e^{-\int_0^t g(s,Y_s,\underline{Y}_s)ds}$, $t \geq 0$, where $g(t,x,y)$ is a non-negative continuous function.

In our methodology, we first derive the canonical decomposition of $f(\tau\wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t})_{t\geq 0}$ under the expanded filtration $\mathbb{G}$, where $(Y_t)_{t\geq 0}$ models the underlying asset and $(\underline{Y}_t)_{t\geq 0}$, $\underline{Y}_t = \inf_{0\leq s\leq t}(Y_t)$ assuming that the default time τ is a $\mathbb{G}$ totally inaccessible stopping time. Following an equivalent method as in Chapter 6, under the progressive filtration expansion a hedging strategy for a defaultable claim of the form $F(Y_T, \underline{Y}_T)1_{\{\tau>T\}}$, is obtained given a measurable function F.

Finally, we must point out a limitation of our model which is that for certain models, we need to assume that $\Delta Y_\tau = 0$. This means that the default time does not coincide with any jump of the underlying asset.

The chapter is structured as follows. In Section 7.2, the model and some preliminary results are introduced. In Section 7.2.1, we analyse the canonical decomposition of $(f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}})_{t\geq 0}$ under $\mathbb{G}$. The hedging strategy through the FS decomposition is obtained in Section 7.3. In Section 7.4, we focus on models based on diffusion processes, and section 7.5 is devoted to models based on jump-diffusion processes.

7.2 Model description and preliminary results

Suppose that $(\Omega, \mathcal{G}, \mathbb{P})$ is a complete probability space equipped with the filtration $\mathbb{F}$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the available information to investors generated by a jump-diffusion Lévy process with jumps of finite variation, i.e. $\mathcal{F}_t = \sigma(\{X_s; 0 \leq s \leq t\})$, for all $t \geq 0$, with Lévy triplet (b, σ^2, ν), and $X_0 = u$, $u > 0$. Without any loss in generality, we can assume that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{G}$, then Theorem 31 of Chapter I of Protter (2004) shows that filtration $\mathbb{F}$ is a right-continuous filtration, i.e. this filtration satisfies the usual hypotheses.

If the jump component of $(X_t)_{t\geq 0}$ is of finite variation then the process $(X_t)_{t\geq 0}$ admits the following Lévy-Itô decomposition

$$X_t = X_0 + \mu t + \sigma W_t + \int_0^t \int_{\mathbb{R}} z N(ds, dz), \quad t \in [0, T], \tag{7.1}$$

where $\mu \in \mathbb{R}$, $\mu = b - \int_{[-1,1]} z\nu(dz)$, $\sigma \geq 0$, $W_t = (W_t)_{t\geq 0}$ is a standard Brownian motion, and $N(dt, dz)$ is the jump measure of the process $(X_t)_{t\geq 0}$ with intensity $\nu(dz)dt$.

Assumption 7.1. *The Lévy measure ν is absolutely continuous with respect to Lebesgue measure and $\int_{|z|\geq 1} z^2 \nu(dz) < \infty$ and $\int_{|z|\geq 1} e^{2z}\nu(dz) < \infty$.*

We suppose that the default-free market is composed of two assets, a risky asset modelled by the stochastic process $Y_t = (Y_t)_{t\geq 0}$, defined by $Y_t = e^{X_t}$, $t \in [0, T]$, and a risk-free asset. We further assume that the interest rate is zero, and so the risk-free asset admits the value of one at all times. Note that under Assumption 7.1, Propositions 3.13 and 3.14 of Cont and Tankov (2004) imply that for all $t \geq 0$, X_t and $Y_t = e^{X_t}$ are square-integrable.

Next, we need to specify a defaultable market (hence defaultable claims) within our setup, in order to do so, first, we need to model the arrival rate of default as follows.

Assumption 7.2. *Suppose that τ is a non-negative $\mathcal{G}$-measurable random time modelling the default time of a firm with the asset values modelled by $(Y_t)_{t\geq 0}$ such that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$, for all $0 \leq t \leq T$. Furthermore, we assume that it admits a hazard rate process, i.e. there is a stochastic process $(\lambda_t)_{t\geq 0}$ of the following form*

$$\lambda_t = g(t, Y_t, \underline{Y}_t), \quad t \in [0, T], \tag{7.2}$$

where for $t \geq 0$, $Y_t = e^{X_t}$, $\underline{Y}_t := \inf_{s\leq t} Y_s$ is the running infimum process of $(Y_t)_{t\geq 0}$, and $g : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function, such that

$$Z_t := \mathbb{P}(\tau > t \mid \mathcal{F}_t) = e^{-\int_0^t g(s, Y_s, \underline{Y}_s)\, ds}.$$

Note that $\underline{Y}_t = e^{\underline{X}_t}$, where $\underline{X}_t = \inf_{s\leq t} X_s$, and so $(\underline{Y}_t)_{t\geq 0}$ inherits some sample path properties of $(\underline{X}_t)_{t\geq 0}$.

Remark 7.3. In Assumption 7.2, if $Z_t = 1$ for all $t \in [0, T]$ (for instance when $g(t, x, y)$ is identically zero), then $\tau > t$, $\mathbb{P}$-almost surely for all $t \in [0, T]$, i.e. the market is default-free and we let $\tau = \infty$.

We model a defaultable claim by the triplet (H, τ, T), where H is a non-negative $\mathcal{F}_T$-measurable random variable, τ is as in Assumption 7.2, and T is the maturity of the claim. More precisely, the holder of this claim claim receives H, if $\tau > T$ and nothing otherwise, i.e. the payoff is $H1_{\{\tau > T\}}$; this means that the recovery process is considered zero here. Furthermore, we let $H = F(Y_T, \underline{Y}_T)$, where $F : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a real valued function; hence, the defaultable claims admits the following form

$$F(Y_T, \underline{Y}_T)1_{\{\tau > T\}}. \tag{7.3}$$

In order to motivate our study further, we have a closer look at (7.3) through providing an interesting example. Consider a firm whose equity is modelled by $(Y_t)_{t \geq 0}$ with $t \in [0, T]$. This firm has just issued a debt modelled by $R(Y_T)$ where $R(x)$ is a real-valued function, $R : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. The debt is subject to two types of default, one structural (we also called endogenous or internal) and the other is exogenous caused by external factors. More precisely, the debt will be settled if the firm does not go through liquidation before the maturity. The liquidation time which is a structural default time can be modelled by $\zeta = \inf\{t : Y_t \leq b\}$ where $0 < b < Y_0$ is a specified constant. We assume that if the firm's values fall under b, the equity holders liquidate it.

The firm could be also subject to an exogenous default event modelled by τ. Therefore, the debt will be settled if $\zeta > T$ and $\tau > T$. However, $\{\zeta > T\} = \{\underline{Y}_T > b\}$, then the firm's debt is a special case of (7.3) for $F(x, y) = R(x)1_{\{y > b\}}$.

Since τ is not necessarily an $\mathbb{F}$-stopping time, an expansion of this reference filtration is required. Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be the right continuous progressive filtration expansion of $\mathbb{F}$ by τ given by

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}_t, \quad t \geq 0, \tag{7.4}$$

where $\mathcal{N}_t$, $t \geq 0$ is the sigma-algebra generated by the default indicator $1_{\{\tau \leq t\}}$, $t \geq 0$. Recall that based on the definition of the progressive filtration expansion defined in (5.2) since we have already assumed that the filtration $\mathbb{G}$ is right continuous then the intersection is redundant and so $\mathbb{G}^0 = \mathbb{G}$. Following Kchia et al. (2013), we further assume that $\mathbb{G}$ satisfies

$$\mathcal{G}_t \cap \{\tau > t\} = \mathcal{F}_t \cap \{\tau > t\}. \tag{7.5}$$

Equation (7.5) is also the filtration expansion introduced in Guo and Zeng (2008). Recall that (7.5) is automatically satisfied for the progressive filtration expansion, see Guo and Zeng (2008) for a discussion. Under this assumption, Kchia et al. (2013) proved that (7.4) determine the canonical decomposition of a semimartingale.

It is easy to show that $\mathcal{N}_t = \sigma(\{\tau \leq u; u \leq t\})$, $t \geq 0$. By Theorem I.25 of Protter (2004), $\mathcal{N}$ is a right continuous filtration; hence we can assume that $\mathbb{G}$ is completed and satisfies the usual hypotheses.

If $\lambda_t > 0$, for all $t \geq 0$, then τ is a $\mathbb{G}$-totally inaccessible stopping time. Furthermore, by Proposition 5.1.3 of Bielecki and Rutkowski (2004), the process

$$\left(1_{\{\tau \leq t\}} - \int_0^{\tau \wedge t} \lambda_s ds\right)_{t \geq 0},$$

is a $\mathbb{G}$-local martingale, see also Li and Rutkowski (2014).

Remark 7.4. As it is pointed out earlier, this framework could incorporate both features of structural and reduced form credit risk modelling. For example, if $\tau = \infty$ and $F(x,y) = R(x)$ for all $x \geq 0$ and $y \geq 0$ for a real-valued function $R(x)$, then there is no default present in (7.3) and so the claim is default-free. If $F(x,y) = R(x)$ for all $x \geq 0$, $y \geq 0$ and $Z_t < 1$ then we are in the context of a reduced form model. If $Z_t = 1$ for all $t \geq 0$ then $\tau = \infty$, $\mathcal{N}$ is the trivial filtration and so $\mathbb{F} = \mathbb{G}$. In this case, the model reduces to a structural one. We provide examples of structural credit risk models in Sections 7.4 and 7.5.

Moreover, we need an additional technical assumption which is crucial in some of the jump models that we study.

Assumption 7.5. *We assume that* $\Delta X_\tau = 0$.

Under Assumption 7.5, we have $\Delta \underline{X}_\tau = \Delta Y_\tau = \Delta \underline{Y}_\tau = 0$. A sufficient condition for this assumption to hold is that the $\mathbb{G}$-stopping time τ avoids all the $\mathbb{F}$-stopping times, i.e. $\mathbb{P}(\tau = \varsigma) = 0$, for all $\mathbb{F}$-stopping times ς. Assumption 7.5 is automatically satisfied for diffusion models.

7.2.1 Closed form formulas of some canonical decompositions

In this section for a given finite horizon $0 < S \leq \infty$ we obtain explicit forms of certain canonical decompositions. These results are applied in the next section for finite time horizons. A function $f : [0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is called $C^{1,2,1}([0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+)$ if it is $C^{1,2,1}$ on $[0,S] \times \mathbb{R}^+ \times \mathbb{R}^+$ and the indicated derivatives admit continuous extensions to $[0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$. Sometimes, we use the notation $f \in C^{1,2,1}([0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+)$ to show this. Other relevant notations are interpreted similarly. For instance, $h \in C^{1,2,1}([0,S] \times \mathbb{R}_0^+ \times \mathbb{R})$ indicates that $h(y,x,y)$ is $C^{1,2,1}$on $[0,S] \times \mathbb{R}^+ \times \mathbb{R}$.

Given a function $f \in C^{1,2,1}([0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+)$, we provide a closed-form formula of the canonical decomposition of $(U_t)_{t \geq 0}$, defined by $U_t = f(t, Y_t, \underline{Y}_t)1_{\{\tau > t\}}$, $0 \leq t \leq S$, in $\mathbb{G}$; we remind that $Y_t = e^{X_t}$, where $X_t = (X_t)_{t \geq 0}$ is the Lévy process (7.1). This canonical decomposition is used in Section 7.3 to determine the hedging strategies using the LRM approach. Let us first introduce some integrability conditions.

Assumption 7.6. *Let* $0 < S \leq \infty$ *and* $f : [0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ *such that* $f \in C^{1,2,1}([0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+)$. *It is assumed that function* $f(t,x,y)$ *satisfies the following* *integrability conditions*

$$\int_{-\ln(\frac{x}{y})}^{+\infty} |(f(t,xe^z,y) - f(t,x,y) - zx\frac{\partial f}{\partial x}(t,x,y))|\nu(dz) < \infty,$$

and

$$\int_{-\infty}^{-\ln(\frac{x}{y})} |(f(t,xe^z,xe^z) - f(t,x,y) - zx\frac{\partial f}{\partial x}(t,x,y))|\nu(dz) < \infty,$$

for all $0 \leq t \leq S$*,* $x \geq 0$ *and* $y \geq 0$*.*

To start with, we obtain the canonical decomposition of the stopped process

$$(f(\tau \wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t}))_{0\leq t\leq S}, \quad 0 < S \leq \infty$$

under the augmented filtration $\mathbb{G}$, which is analysed in the following Lemma.

Lemma 7.7. *Assume that* $0 \leq S \leq \infty$ $f : [0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ *and* $g : [0,S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *are respectively* $C^{1,2,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+)$ *and continuous functions on their respective domains. Furthermore, suppose that the function* $f(t,x,y)$ *satisfies Assumption 7.6, for the given* S*,* τ *follows Assumption 7.2, and* $(X_t)_{t\geq 0}$ *is a Lévy process satisfying Assumptions 7.1 and 7.5. If the condition* $\frac{\partial f}{\partial y}(t,y,y) = 0$*, holds for all* $t \in [0,T]$*,* $y \geq 0$*, then the process*

$$\left(f(t\wedge\tau, Y_{t\wedge\tau}, \underline{Y}_{t\wedge\tau}) - f(0, Y_0, \underline{Y}_0) - \int_0^{\tau\wedge t} \mathcal{L}f(s, Y_s, \underline{Y}_s)ds\right)_{t\geq 0}$$

is a $\mathbb{G}$*-local martingale, where for all* $0 \leq t \leq T$*,* $x \geq 0$*, and* $y \geq 0$*, the operator* $\mathcal{L}f(t,x,y)$ *is defined by*

$$\begin{aligned}\mathcal{L}f(t,x,y) \quad &= \quad \frac{\partial f}{\partial t}(t,x,y) + \beta x\frac{\partial f}{\partial x}(t,x,y) + \frac{\sigma^2}{2}(x^2\frac{\partial^2 f}{\partial x^2}(t,x,y) + x\frac{\partial f}{\partial x}(t,x,y)) \\ &\quad + \int_{-\ln(\frac{x}{y})}^{+\infty}(f(t,xe^z,y) - f(t,x,y) - zx\frac{\partial f}{\partial x}(t,x,y))\nu(dz) \\ &\quad + \int_{-\infty}^{-\ln(\frac{x}{y})}(f(t,xe^z,xe^z) - f(t,x,y) - zx\frac{\partial f}{\partial x}(t,x,y))\nu(dz),\end{aligned}$$

for $\beta = \mathbb{E}[X_1 - X_0]$*.*

Proof. Let the function $h : [0,S]\times\mathbb{R}_0^+\times\mathbb{R}_0^+ \to \mathbb{R}$ be defined by $h(t,x,y) = f(t,e^{x-y},e^{-y})$. Then $h(t,X_t - \underline{X}_t, -\underline{X}_t) = f(t,Y_t,\underline{Y}_t)$ and $\frac{\partial h}{\partial x}(t,0,y) + \frac{\partial h}{\partial y}(t,0,y) = 0$. By Proposition A.2, the following process

$$\left(h(t,X_t - \underline{X}_t, -\underline{X}_t) - h(0, X_0 - \underline{X}_0, -\underline{X}_0) - \int_0^t \mathcal{L}^*h(s, X_s - \underline{X}_s, -\underline{X}_s)ds\right)_{t\geq 0},$$

is an $\mathbb{F}$-local martingale where the operator $\mathcal{L}^*$ is defined in Proposition A.2. After simplifying $\mathcal{L}^*h$, this leads to the $\mathbb{F}$-local martingale $(M_t)_{t\geq 0}$, defined by

$$M_t = f(t, Y_t, \underline{Y}_t) - f(0, Y_0, \underline{Y}_0) - \int_0^t \mathcal{L}f(s, Y_s, \underline{Y}_s)ds, \quad 0 \leq t \leq S.$$

Following Assumption 7.5, since $\Delta Y_\tau = 0$, based on Protter (2004), Chapter VI, Theorem 15, the stopped process $(M_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{G}$-semimartingale, and the process

$$\left(M_{\tau\wedge t} - \int_0^{\tau\wedge t} \frac{1}{Z_{s-}} d\left\langle Z, f(\cdot, Y, \underline{Y})\right\rangle_s^{\mathbb{F}}\right)_{t\geq 0}, \ 0 \leq t \leq S,$$

is a $\mathbb{G}$-local martingale, where $Z_t = \mathbb{P}(\tau > t \mid \mathcal{F}_t) = e^{-\int_0^t g(s,Y_s,\underline{Y}_s)ds}$, $0 \leq t \leq T$. Moreover, the process $(Z_t)_{t\geq 0}$ is continuous and of finite variation and so $[f(\cdot, X_\cdot - \underline{X}_\cdot, -\underline{X}_\cdot), Z_\cdot]_t^{\mathbb{F}} = 0$, which implies that $\langle f(\cdot, X - \underline{X}, -\underline{X}), Z\rangle^{\mathbb{F}} = 0$. Therefore the process $(M_{\tau\wedge t})_{t\geq 0}$ is a $\mathbb{G}$-local martingale. □

Proposition 7.8. *Let $(X_t)_{t\geq 0}$ satisfy Assumptions 7.1 and 7.5. We also assume that $f : [0, S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is a $C^{1,2,1}([0, S] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+)$ function satisfying Assumption 7.6 for a given S and $Z_t < 1$ for all $0 \leq t \leq S$. If the default time τ satisfies Assumption 7.2, then the process $(U_t)_{t\geq 0}$ defined by $U_t = f(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}}$, $0 \leq t \leq S$ admits the following canonical decomposition:*

$$U_t = U_0 + \int_0^{\tau\wedge t} \mathcal{A}f(s, Y_s, \underline{Y}_s)ds + O_t, \ 0 \leq t \leq S,$$

where $O = (O_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale, and the operator $\mathcal{A}f(t, x, y)$ is given by

$$\mathcal{A}f(t, x, y) = \mathcal{L}f(t, x, y) - f(t, x, y)g(t, x, y).$$

Proof. It is easy to see that $U_t = f(\tau \wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t}) - f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}}$. By Lemma 7.7, the process $(M_{\tau\wedge t}^f)_{t\geq 0}$, where M_t^f, $0 \leq t \leq S$ is defined by

$$M_t^f = f(t, Y_t, \underline{Y}_t) - f(0, Y_0, \underline{Y}_0) - \int_0^t \mathcal{L}f(s, Y_s, \underline{Y}_s)ds, \quad 0 \leq t \leq T,$$

is a $\mathbb{G}$-local martingale. So the process $(f(\tau \wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t}))_{t\geq 0}$ can be rewritten as $f(\tau \wedge t, Y_{\tau\wedge t}, \underline{Y}_{\tau\wedge t}) = f(0, Y_0, \underline{Y}_0) + M_{\tau\wedge t} + \Lambda_{\tau\wedge t}^f$, where $(\Lambda_{\tau\wedge t}^f)_{t\geq 0}$ is a $\mathbb{G}$-predictable process given by $\Lambda_{\tau\wedge t}^f = \int_0^{\tau\wedge t} \mathcal{L}f(s, Y_s, \underline{Y}_s)ds$, with $t \in [0, S]$.

Next, we proceed with the process $(f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}})_{t\geq 0}$. Since functions $f(t, x, y)$ and $g(t, x, y)$ are continuous, based on the Assumption 7.5, we have $\Delta f(\tau, Y_\tau, \underline{Y}_\tau) = 0$ and so

$$\int_0^t \Delta f(s, Y_s, \underline{Y}_s)d(1_{\{\tau\leq s\}}) = \Delta f(\tau, Y_\tau, \underline{Y}_\tau)1_{\{\tau\leq t\}} = 0, \quad 0 \leq t \leq S. \tag{7.6}$$

Note that the process $(M_t^{(1)})_{t\geq 0}$ defined by $M_t^{(1)} = 1_{\{\tau\leq t\}} - \int_0^{\tau\wedge t} \lambda_s ds$, $t \geq 0$, is a

$\mathbb{G}$-local martingale. So for $0 \le t \le S$, we get

$$f(\tau, Y_\tau, \underline{Y}_\tau) 1_{\{\tau \le t\}} = \int_0^t f(s, Y_s, \underline{Y}_s) d(1_{\{\tau \le s\}}) = \int_0^t f(s, Y_{s-}, \underline{Y}_{s-}) d(1_{\{\tau \le s\}})$$
$$= \int_0^t f(s, Y_{s-}, \underline{Y}_{s-}) dM_s^{(1)} + \int_0^{\tau \wedge t} f(s, Y_s, \underline{Y}_s) g(s, Y_s, \underline{Y}_s) ds,$$

where the second equality is due to (7.6). The result is then proved by defining the operator $\mathcal{A} f(t,x,y) := \mathcal{L} f(t,x,y) - f(t,x,y) g(t,x,y)$. □

7.3 LRM hedging strategies of claims dependent on the running infimum process

In this section, we determine semi-closed-form formulas for the LRM hedging strategies of defaultable claim $F(Y_T, \underline{Y}_T) 1_{\{\tau > T\}}$.

Assumption 7.9. *Suppose that the Lévy measure ν satisfies:*

$$\int_{-\infty}^{\infty} (e^z - z - 1)\, \nu(dz) < \infty, \qquad \textit{and} \qquad \int_{-\infty}^{\infty} (e^{2z} - 2z - 1)\, \nu(dz) < \infty.$$

Remark 7.10. In case of the finite variation Lévy processes where $\int_{\mathbb{R}} |z| \nu(dz) < \infty$ the conditions of the Assumption 7.9 are equivalent to $\int_{|z| \le 1} (e^z - 1) \nu(dz) < \infty$ and $\int_{|z| \le 1} (e^{2z} - 1) \nu(dz) < \infty$.

Lemma 7.11. *Under Assumptions 7.1, 7.5, and 7.9, the stopped process $(Y_{\tau \wedge t})_{t \ge 0}$ is a square integrable $\mathbb{G}$-special semimartingale with the following canonical decomposition:*

$$Y_t^\tau = X_0 + M_{\tau \wedge t}^Y + \Lambda_{\tau \wedge t}^Y,\ t \in [0, S], \tag{7.7}$$

where $\Lambda_t^Y = \alpha \int_0^t Y_s ds$, $\alpha = \mu + \dfrac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1)\ \nu(dz)$, and $(M_{\tau \wedge t}^Y)_{t \ge 0}$ is a $\mathbb{G}$-local martingale. The $\mathbb{G}$-predictable quadratic variation process $(\langle Y^\tau \rangle_t^{\mathbb{G}})_{t \ge 0}$ is equal to

$$\langle Y^\tau \rangle_t^{\mathbb{G}} = \gamma \int_0^{\tau \wedge t} Y_s^2\ ds,\ t \in [0, S], \tag{7.8}$$

where $\gamma = \sigma^2 + \int_{-\infty}^{\infty} (e^{2z} - 2e^z + 1)\ \nu(dz)$.

Proof. Let $f_1 : [0,T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ be defined by $f_1(t,x,y) = x$. Then $f_1(t, Y_t, \underline{Y}_t) = Y_t$, and by applying Lemma 7.7 for function $f_1(t,x,y)$, we obtain

$$Y_{\tau \wedge t} = Y_0 + M_{\tau \wedge t}^Y + \int_0^{\tau \wedge t} \mathcal{L} f_1(s, Y_s, \underline{Y}_s) ds,\ 0 \le t \le T,$$

where $(M_{\tau \wedge t}^Y)_{t \ge 0}$ is a $\mathbb{G}$-local martingale. The canonical decomposition of $(Y_t^\tau)_{t \ge 0}$ is then followed by simplifying $\mathcal{L} f_1(s, Y_s, \underline{Y}_s)$, $s \ge 0$, and letting $\Lambda_t^Y = \int_0^t \mathcal{L} f_1(s, Y_s, \underline{Y}_s) ds$, $t \in [0,T]$.

Next, we calculate its $\mathbb{G}$-predictable quadratic variation process $(\langle X^\tau\rangle_t^{\mathbb{G}})_{t\geq 0}$. From the definition of quadratic variation, we know that $[Y^\tau]_t^{\mathbb{G}} = Y_{\tau\wedge t}^2 - 2\int_0^{\tau\wedge t} Y_{s-}dY_s$. The canonical decomposition of the integral term in the above equation is easily obtained by noting that

$$\int_0^{\tau\wedge t} Y_{s-}dY_s = \int_0^{\tau\wedge t} Y_{s-}dM_s^Y + \alpha\int_0^{\tau\wedge t} Y_s^2 ds,\ 0\leq t\leq T.$$

In order to obtain the decomposition of $(Y_{\tau\wedge t}^2)_{t\geq 0}$, let $f_2 := [0,T]\times\mathbb{R}_0^+\times\mathbb{R}_0^+ \to \mathbb{R}$, $f_2(t,x,y) = x^2$ then $f_2(t,Y_t,\underline{Y}_t) = Y_t^2$, $t\geq 0$. By applying Lemma 7.7 one more time, it follows that the process $\hat{M}_t = (\hat{M}_{\tau\wedge t})_{t\geq 0}$ defined by $\hat{M}_t = Y_t^2 - \int_0^t \mathcal{L}f_2(s,Y_s,\underline{Y}_s)ds,\ t\geq 0$, is a $\mathbb{G}$-local martingale. So, we obtain

$$[Y^\tau]_t^{\mathbb{G}} = \hat{M}_{\tau\wedge t} + \int_0^{\tau\wedge t}\mathcal{L}f_2(s,Y_s,\underline{Y}_s)ds - 2\int_0^{\tau\wedge t} Y_{s-}dM_s^Y - 2\alpha\int_0^{\tau\wedge t} Y_s^2 ds,\ t\geq 0,$$

hence $[Y^\tau]_t^{\mathbb{G}}$ is locally of integrable variation and its compensator, i.e. $(\langle Y^\tau\rangle_t^{\mathbb{G}})_{t\geq 0}$ is given by

$$\langle Y^\tau\rangle_t^{\mathbb{G}} = \int_0^{\tau\wedge t}\left(\mathcal{L}f_2(s,Y_s,\underline{Y}_s) - 2\alpha Y_s^2\right)ds,\ t\in[0,T].$$

The result is proved, once we simplify $\mathcal{L}f_2(t,Y_t,\underline{Y}_t) - 2\alpha Y_t^2$, $t\in[0,T]$. □

Our semi-closed-form solutions of LRM hedging strategies are based on solutions of PIDEs specified in the following Assumption.

Assumption 7.12. *Suppose that $f:[0,T]\times\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}$ (here $S=T$) is a measurable function such that $f\in C^{1,2,1}([0,T)\times\mathbb{R}_0^+\times\mathbb{R}_0^+)$, and it is the solution of the following PIDE*

$$\mathcal{A}f(t,x,y) = \frac{(\mathcal{A}P(t,x,y) - x\mathcal{A}f(t,x,y) - f(t,x,y)x\alpha)}{x\gamma}\alpha,\ t\in[0,T), x\geq 0, y\geq 0,$$

where α and γ are defined in Lemma 7.11, for $(t,x,y)\in[0,T)\times\mathbb{R}_0^+\times\mathbb{R}_0^+$, $\mathcal{A}f(t,x,y)$ is defined by

$$\mathcal{A}f(t,x,y) = \mathcal{L}f(t,x,y) - f(t,x,y)g(t,x,y),$$

the operator $\mathcal{L}$ is the same as in Lemma 7.7, $P(t,x,y) = xf(t,x,y)$, $\mathcal{A}P(t,x,y)$ is defined like $\mathcal{A}f(t,x,y)$, and the following conditions are satisfied: first for all $t\geq 0$, $x\geq 0$, and $y\geq 0$, we have $\frac{\partial f}{\partial y}(t,y,y) = 0$, and second

$$f(t,x,y)\to F(x,y),\quad \text{as } t\to T,\quad \text{point-wise for all}\quad x\geq 0 \text{ and } y\geq 0,$$

where $F(x,y)$ is introduced in Equation 7.3.

In the next Theorem we determine the canonical decomposition of $(f(t\wedge T, Y_{t\wedge T}, \underline{Y}_{t\wedge T})1_{\{\tau>t\wedge T\}})_{t\geq 0}$ in $\mathbb{G}$.

Before we start describing the theorem let us briefly describe some key points that will be used in its proof.

Let us recall ourselves that our model encompasses discontinuous payoffs, e.g. barrier options whose barrier depends on the running infimum process. In fact, the terminal condition $F(x,y)$ in the PIDE introduced in Assumption 7.12 might not be even smooth. However, the solution $f(t,x,y)$ of the PIDE is defined on the entire domain $[0,T]$. One way to fix this problem is to apply Itô's lemma for non-smooth functions for the jump diffusion case. Extending our construction using non-smooth functions including local times its left as a future work.

Alternatively, we may apply the same technique introduced in Okhrati (2019), Proposition 4.1. Their construction is made under a delayed information whose asset is modelled by continuous semimartingale, whereas ours is made under a progressive filtration expansion and when the underlying asset is a jump diffusion process, making our calculations more perplexing. In our proof we use convergence results in uniformly on compacts in probability (abbreviated as u.c.p.)[1]. For more details about u.c.p. convergence, we refer to Protter (2004), Chapter II. One rather technical condition is that for a sequence of local martingales $\{(M_t^n)_{t\geq 0}\}_{n=1,2\ldots}$ converges to a local martingale $(M_t)_{t\geq 0}$ in u.c.p., if and only if $\sup_n \sup_{s\leq t} |\Delta M_s|$ is locally integrable. Now, let us state the following Theorem.

Theorem 7.13. *Suppose that $(X_t)_{t\geq 0}$ is given by Equation (7.1) and $Z_t < 1$. Furthermore, let Assumptions 7.1, 7.2, 7.5 (with $S = T$) and 7.9 be in force. We also assume that $f : [0,T) \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is a continuous function, it satisfies Assumption 7.12, and $([U, X^\tau]_t^{\mathbb{G}})_{t\geq 0}$ belongs to $\mathcal{A}_{loc}$. Then, we have the following decompositions*

$$\begin{aligned} U_{t\wedge T} = f(t\wedge T, Y_{t\wedge T}, \underline{Y}_{t\wedge T})1_{\{\tau > t\wedge T\}} &= U_0 + \int_0^t \theta_{s^-} 1_{\{s<T\}} dY_s^\tau + L_t \\ &= U_0 + \int_0^t \theta_{s^-} 1_{\{s\leq \tau\}} 1_{\{s<T\}} dY_s + L_t, \ \mathbb{P}\text{-a.s.}, \end{aligned}$$

where for $0 \leq t < T$, the process $(\theta_t)_{t\geq 0}$ is given by

$$\theta_t = \frac{\mathcal{K} f(t, Y_t, \underline{Y}_t)}{\gamma Y_t^2},$$

for $(t,x,y) \in [0,T) \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, the operator $\mathcal{K}$ is defined by

$$\mathcal{K} f(t,x,y) = \mathcal{A} P(t,x,y) - x\mathcal{A} f(t,x,y) - f(t,x,y)x\alpha,$$

$P(t,x,y) = xf(t,x,y)$, the operator $\mathcal{A}$ is defined in Proposition 7.8, and the process $(L_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale orthogonal to the local martingale part of $(Y_t^\tau)_{t\geq 0}$ (i.e. $(M_{\tau\wedge t}^Y)_{t\geq 0}$) in $\mathbb{G}$.

In particular, for $t = T$ we obtain

$$U_T = F(Y_T, \underline{Y}_T)1_{\{\tau > T\}} = U_0 + \int_0^T \theta_{s^-} 1_{\{s<T\}} dY_s^\tau + L_T, \tag{7.9}$$

[1] We say that a sequence of of processes $(X_t^n)_{t\geq 0}$ converges to a process $(X_t)_{t\geq 0}$ in u.c.p. if for each $t \geq 0$ $\sup_{0\leq s\leq t} |X_s^n - X_s|$ converges to 0 in probability.

where $F(x, y)$ is the PIDE's terminal condition,

Proof. Take an integer $n \geq 1$ and let $f^{(n)} : [0, a(n)] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ be the restriction of f on to the set $[0, a(n)] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, where $a(n) = T - \frac{1}{n}$. From Proposition 7.8 for $S = a(n)$, we know that the processes $O^{(n)} = (O_t^{(n)})_{t\geq 0}$ and $(O_t^{(1,n)})_{t\geq 0}$ with $t \in [0, a(n)]$ defined by

$$O_t^{(n)} = U_t^{(n)} - U_0^{(n)} - \int_0^{\tau\wedge t} \mathcal{A}f^{(n)}(s, Y_s, \underline{Y}_s)ds, \quad U^{(n)}(t, Y_t, \underline{Y}_t) = f^{(n)}(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}},$$

and

$$O_t^{(1,n)} = P^{(n)}(t, Y_t, \underline{Y}_t)1_{\{\tau>t\}} - P^{(n)}(0, Y_0, \underline{Y}_0) - \int_0^{\tau\wedge t} \mathcal{A}P^{(n)}(s, Y_s, \underline{Y}_s)ds, \ t \in [0, a(n)],$$

are $\mathbb{G}$-local martingale where $P^{(n)}(t, x, y) = xf^{(n)}(t, x, y)$, $0 \leq t \leq a(n)$, $x \geq 0$, and $y \geq 0$. We determine the KW decomposition of $(O_t^{(n)})_{t\geq 0}$ with respect to $(M_t^Y)_{0\leq t\leq a(n)}$. More precisely, we show that $O_t^{(n)} = \int_0^t \theta_s^{(n)} dM_s^Y + L_s^{(n)}$, with $0 \leq t \leq a(n)$, where $(\theta_t^{(n)})_{t\geq 0}$ is a $\mathbb{G}$-predictable process (that we determine in closed form) and $(L_t^{(n)})_{t\geq 0}$, where $0 \leq t \leq a(n)$, is a $\mathbb{G}$-local martingale orthogonal to $(M_t^Y)_{t\geq 0}$ with $L_0^{(n)} = 0$. Note that since $[O^{(n)}, M^Y]_t^{\mathbb{G}}$ is locally of integrable variation, its compensator in $\mathbb{G}$ exists and given by $\left\langle O^{(n)}, M^Y \right\rangle_t^{\mathbb{G}}$ with $0 \leq t \leq a(n)$. From the integration by parts formula, we have

$$U_t^{(n)} Y_t^\tau = U_0^{(n)} Y_0 + \int_0^t U_{s-}^{(n)} dY_s^\tau + \int_0^t Y_{s-}^\tau dU_s^{(n)} + [U^{(n)}, Y^\tau]_t, \ t \in [0, a(n)].$$

Let $F_t^{(1,n)} = \int_0^{\tau\wedge t} \mathcal{A}f^{(n)}(s, Y_s, \underline{Y}_s)ds$ and $F_t^{(2,n)} = \int_0^{\tau\wedge t} \mathcal{A}P^{(n)}(s, Y_s, \underline{Y}_s)ds$ for $0 \leq t \leq a(n)$. From the above integration by parts formula, we obtain

$$\begin{aligned} [U^{(n)}, Y^\tau]_t^{\mathbb{G}} - \left(F_t^{(2,n)} - \int_0^{\tau\wedge t} U_{s-}^{(n)} \mathcal{L}f_1^{(n)}(s, Y_s, \underline{Y}_s)ds - \int_0^{\tau\wedge t} Y_{s-} dF_s^{(1,n)} \right) \\ = \left(O_t^{(1,n)} - \int_0^{\tau\wedge t} U_{s-}^{(n)} dM_s^Y - \int_0^{\tau\wedge t} Y_s dO_s^{(n)} \right), \ 0 \leq t \leq a(n). \end{aligned} \tag{7.10}$$

The right-hand side of the last equation is locally of integrable variation by Lemma 3.11, Chapter I of Jacod and Shiryaev (2003). The bracket on the left hand side of (7.10) is also locally of integrable variation by Lemma 3.10, Chapter 1, of Jacod and Shiryaev (2003). Therefore, $([U^{(n)}, Y^\tau]_t^{\mathbb{G}})_{0\leq t\leq a(n)}$ is locally of integrable variation. By Theorem 2.30, $([U^{(n)}, Y^\tau]_t^{\mathbb{G}})_{0\leq t\leq a(n)}$ admits a compensator in $\mathbb{G}$which is basically the $\mathbb{G}$-predictable quadratic covariation process of $(O_t^{(n)})_{t\geq 0}$ and $(M_t^Y)_{t\geq 0}$with $t \in [0, a(n)]$ and from (7.10) we have

$$\left\langle O^{(n)}, M \right\rangle_t^{\mathbb{G}} = \int_0^{\tau\wedge t} \Big(\mathcal{A}P^{(n)}(s, Y_s, \underline{Y}_s) - f^{(n)}(s, Y_s, \underline{Y}_s) Y_s \alpha - Y_s \mathcal{A}f^{(n)}(s, Y_s, \underline{Y}_s) \Big) ds.$$

From here, we observe that the measure defined by $\left\langle O_\cdot^{(n)}, M_\cdot^Y \right\rangle_t^{\mathbb{G}}$ is absolutely continuous with respect to the measure defined by $\langle Y_\cdot^\tau \rangle_t^{\mathbb{G}}$ with the Radon-Nikodym derivative

$(\theta_t^{(n)})_{t\geq 0}$ given by $\theta_t^{(n)} = \frac{\mathcal{K}f^{(n)}(t,Y_t,\underline{Y}_t)}{\gamma Y_t^2}$, $0 \leq t \leq a(n)$. Therefore, the KW decomposition of $(O_t^{(n)})_{t\geq 0}$ with respect to $(M_t^Y)_{t\geq 0}$ is equal to

$$O_t^{(n)} = \int_0^{\tau\wedge t} \theta_s^{(n)}\, dM_s^Y + L_{\tau\wedge t}^{(n)},\ t \in [0, a(n)], \tag{7.11}$$

where $(L_{\tau\wedge t}^{(n)})_{t\geq 0}$ is orthogonal to $(M_{\tau\wedge t}^Y)_{t\geq 0}$.

On the other hand, by breaking down KW decomposition (7.11), we obtain

$$U_t^{(n)} - \int_0^{\tau\wedge t} \mathcal{A}f^{(n)}(s,Y_s,\underline{Y}_s)\, ds = U_0^{(n)} + \int_0^{\tau\wedge t} \theta_s^{(n)}\, dY_s - \int_0^{\tau\wedge t} \alpha\theta_s^{(n)} Y_s ds + L_t^{(n)},$$

with $t \in [0, a(n)]$. Consider the above decomposition at $s_n(t) = t \wedge a(n)$ for $t \in [0, a(n)]$ and $n \geq 1$. Since $f^{(n)}$ is the restriction of f on $[0, a(n)] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, from the previous decomposition, for $t \in [0, a(n)]$, we obtain

$$\begin{aligned} f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)}) 1_{\{\tau > s_n(t)\}} - \int_0^{\tau\wedge s_n(t)} \mathcal{A}f(s,Y_s,\underline{Y}_s) ds = \\ U_0 + \int_0^{\tau\wedge s_n(t)} \theta_s\, dY_s - \int_0^{\tau\wedge s_n(t)} \alpha\theta_s Y_s ds + L_{s_n(t)}^{(n)}. \end{aligned}$$

However, since $f(t, x, y)$ satisfies Assumption 7.12, we have $\mathcal{A}f(s, Y_s, \underline{Y}_s) = \alpha\theta_s Y_s$ on $[0, \tau \wedge s_n(t)]$, hence we get

$$f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)}) 1_{\{\tau > s_n(t)\}} = U_0 + \int_0^{\tau\wedge s_n(t)} \theta_s\, dY_s + L_{s_n(t)}^{(n)},\ t \in [0, a(n)]. \tag{7.12}$$

Next we take the limit as $n \to \infty$. Note that $(f(s_n(t), Y_{s_n(t)}, \underline{Y}_{s_n(t)}) 1_{\{\tau > s_n(t)\}})_{t\geq 0}$ converges to $(f(t\wedge T, Y_{t\wedge T}, \underline{Y}_{t\wedge T}) 1_{\{\tau > t\wedge T\}})_{t\geq 0}$ in u.c.p. as $n \to \infty$. Since $f(t, x, y)$ is not necessarily partially differentiable on $\{T\} \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, the process $(\theta_t)_{t\geq 0}$ might not be well defined when $t = T$. So in taking the limit of the integral term of (7.12), one should consider this. Obviously, we have $\int_0^{\tau\wedge s_n(t)} \theta_s dY_s = \int_0^{\tau\wedge t} \theta_s 1_{\{s<T\}} 1_{\{s\leq T-\frac{1}{n}\}} dY_s$, and so by part iii of Theorem 4.31, Chapter I, of Jacod and Shiryaev (2003), $(\int_0^{\tau\wedge s_n(t)} \theta_s 1_{\{s<T\}}$ $dY_s)_{t\geq 0}$ converges to $\int_0^{\tau\wedge t} \theta_s 1_{\{s<T\}} dY_s$ in u.c.p. Therefore, $L_{s_n(\cdot)}^{(n)}$ converges to a process $(L_t)_{t\geq 0}$ in u.c.p.

We show that $(L_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale. Note that for all $t \in [0, a(n)]$ we have

$$\begin{aligned} \sup_{u\leq t} |\Delta L_{s_n(u)}^{(n)}| \leq \sup_{u\leq t} |\Delta f(u\wedge T, Y_{u\wedge T}, \underline{Y}_{u\wedge T})| + \sup_{u\leq t} |f(u\wedge T, Y_{u\wedge T}, \underline{Y}_{u\wedge T})| \\ + \sup_{u\leq t} |\theta_{u\wedge T}||\Delta Y_{u\wedge T}^{\tau}|. \end{aligned} \tag{7.13}$$

The process $(L_t)_{t\geq 0}$ is the limit of $\mathbb{G}$-local martingales, and it can be shown that $(L_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale, if $\sup_{n\geq 1}\sup_{u\leq t} |\Delta L_{s_n(u)}^{(n)}|$ is locally integrable[2]. From (7.13), it is enough to show that the right-hand side of this equation is locally integrable. Since

[2]This is a result from George Lowther's blog (see Theorem 6 of `https://almostsure.wordpress.com/2009/12/24/local-martingales/`).

$(f(t\wedge T, Y_{t\wedge T}, \underline{Y}_{t\wedge T}))_{t\geq 0}$ admits a canonical decomposition and since local martingales are locally integrable, then the second term on the right-hand side of (7.13) is locally integrable.

The first term on the right-hand side of (7.13) is also locally integrable because a càdlàg adapted process is locally integrable if and only if its jumps are locally integrable. Since $(\theta_t)_{t\geq 0}$ is $\mathbb{G}$-predictable, càglàd, it is locally bounded and hence we may assume that it is uniformly bounded. Also, note that $(Y_t)_{t\geq 0}$ admits a canonical decomposition, and so $(Y_t^\tau)_{t\geq 0}$ and ΔY_t^τ are locally integrable. Therefore, $\sup_{n\geq 1}\sup_{u\leq t}|\Delta L^{(n)}_{s_n(u)}|$ is locally integrable and hence $(L_t)_{t\geq 0}$ is a local martingale. Therefore, by taking the limit of (7.12) in u.c.p., we obtain

$$f(\cdot\wedge T, Y_{\cdot\wedge T}, \underline{Y}_{\cdot\wedge T})1_{\{\tau>\cdot\wedge T\}} = U_0 + \int_0^{\cdot\wedge T}\theta 1_{\{s<T\}}\, dY^\tau + L_\cdot, \tag{7.14}$$

where L is a $\mathbb{G}$-local martingale. Note that $L^{(n)}_{s_n(t)} = L_{s_n(t)} = L_t^{T-\frac{1}{n}}$ and so $\langle L^{(n)}_{s_n(\cdot)}, M^Y\rangle_t^{\mathbb{G}} = \langle L, M^Y\rangle_t^{T-\frac{1}{n}} = 0$, $t\geq 0$ in $\mathbb{G}$, because $(L^{(n)}_{s_n(t)})_{t\geq 0}$ is orthogonal to $(M_t^Y)_{t\geq 0}$. We also note that $\langle L, M^Y\rangle^{T-\frac{1}{n}}$ increases to $\left\langle L, M^Y\right\rangle^{\mathbb{G}}_{\cdot\wedge T}$, hence $\left\langle L, M^Y\right\rangle^{\mathbb{G}}_{\cdot T} = 0$ and so $(L_t)_{t\geq 0}$ is orthogonal to $(M_t^Y)_{t\geq 0}$. This proves the first part of the theorem.

Finally, by letting $t = T$, and considering the boundary condition for the PIDE of the Assumption 7.12 the next part of the theorem is proved. □

Certain payoff functions such as binary are not continuous. The following result covers also this case.

Theorem 7.14. *Suppose that $(X_t)_{t\geq 0}$ is given by (7.1), $Z_t < 1$, and let Assumptions 7.1, 7.2, 7.5, and 7.9 are in force. We also assume that $f : [0,T]\times\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}$ satisfies Assumption 7.12 and $([U, X^\tau]_t^{\mathbb{G}})_{t\geq 0}\in\mathcal{A}_{loc}$. Then we have the following decompositions*

$$F(Y_T, \underline{Y}_T)1_{\{\tau>T\}} = f(0, X_0) + \int_0^T \theta_{s-}1_{\{s<T\}}dY_s^\tau + L_T, \tag{7.15}$$

for $0\leq t<T$ and the process $(\theta_t)_{t\geq 0}$ is given by

$$\theta_t = \frac{\mathcal{K}f(t, Y_t, \underline{Y}_t)}{\gamma Y_t^2},\ t\in[0,T),$$

for $(t,x,y)\in[0,T)\times\mathbb{R}_0^+\times\mathbb{R}_0^+$ and the operator $\mathcal{K}$is given by

$$\mathcal{K}f(t,x,y) = \mathcal{A}P(t,x,y) - x\mathcal{A}f(t,x,y) - f(t,x,y)x\alpha,$$

where $P(t,x,y) = xf(t,x,y)$ the operator $\mathcal{A}$ is defined in Proposition 7.8 and the process $(L_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale orthogonal to the martingale part of $(Y_t^\tau)_{t\geq 0}$ (i.e. $(M_t^Y)_{t\geq 0}$) in $\mathbb{G}$.

Proof. Define $T_n = a(n) = T - \frac{1}{n}$, $n\geq 1$. If we apply Theorem 7.13 for $T := T_n$ we obtain

$$U_{t\wedge T_n} = f(t\wedge T_n, Y_{t\wedge T_n}, \underline{Y}_{t\wedge T_n})1_{\{\tau>t\wedge T_n\}} = U_0 + \int_0^t \theta_{s-}1_{\{s\leq T_n\}}dY_s^\tau + L_t^{(n)},$$

$\mathbb{P}$-a.s. where $(L_t^{(n)})_{t\geq 0}$ is a $\mathbb{G}$-local martingale orthogonal to the local martingale part of $(Y_t^\tau)_{t\geq 0}$ i.e. $((M_t^Y)_{t\geq 0})$ in $\mathbb{G}$. Then by letting $n\to\infty$ and using similar argument as in the proof of Theorem of 7.13, the result is proved. □

Remark 7.15. Note that in the non-martingale case, i.e. when α is non-zero we have

$$\theta_t = \frac{\mathcal{A}f(t, Y_t, \underline{Y}_t)}{\alpha Y_t}, \quad t\in[0,T)$$

Corollary 7.16. *Consider the same setup as Theorem 7.13, and in addition assume that $(Y_t^\tau)_{t\geq 0}$ is a $\mathbb{G}$-square integrable martingale. Suppose that $f:[0,T]\times\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}$ satisfies the following PIDE:*

$$\mathcal{A}f(t,x,y) = 0, \quad 0\leq t<T, x\geq 0, y\geq 0,$$

$f(t,x,y)\to F(x,y)$ point-wise as $t\to T$, and $\frac{\partial f}{\partial y}(t,y,y) = 0$ for all $0\leq t<T$, $x\geq 0$, and $y\geq 0$. Then in this case we have

$$F(Y_T, \underline{Y}_T)1_{\{\tau>T\}} = f(0, X_0, X_0) + \int_0^T \frac{\mathcal{A}P(s, Y_{s-}, \underline{Y}_{s-})}{\gamma Y_{s-}^2}1_{\{s<T\}}dY_s^\tau + L_T,$$

where $(L_t)_{t\geq 0}$ is a $\mathbb{G}$-local martingale orthogonal to $(M_{\tau\wedge t}^Y)_{t\geq 0}$ the operator $\mathcal{A}$ is introduced in Proposition 7.8, and $P(t,x,y) = xf(t,x,y)$.

Proof. Since $(Y_t^\tau)_{t\geq 0}$ is a martingale, we have $\alpha = 0$. Since $f(t,x,y)$ is the solution of the PIDE of Assumption 7.12 then $\mathcal{A}f(t,x,y) = 0$, and the result easily follows from Theorem 7.13. □

The orthogonal decompositions (7.12) and (7.15) resemble that of FS decomposition, as it admits the same format. Nevertheless, further integrability conditions are required to turn it to an FS decomposition, and hence PLRM and LRM strategies. Note, that we can introduce the spaces Θ and L^2-strategies under filtration $\mathbb{G}$. See also Definitions 2.24 and 3.34.

Suppose that we are given the payoff $H = F(Y_T, \underline{Y}_T)$ for which we have the orthogonal decomposition

$$F(Y_T, \underline{Y}_T)1_{\{\tau>T\}} = U_0 + \int_0^T \theta_{s-}dY_s^\tau + L_T, \tag{7.16}$$

as in (7.9) of Theorem 7.13. Following Proposition 3.40, the existence of the PLRM hedging strategy Φ is equivalent to $U_0\in L^2(\Omega,\mathcal{G}_0,\mathbb{P})$, $\theta\in\Theta$, $(L_t)_{t\geq 0}$, $L_0 = 0$ and $(M_t^Y)_{t\geq 0}$, $M_0^Y = 0$ are square integrable martingales, and $(L_t)_{t\geq 0}$ is strongly orthogonal to $(M_t^Y)_{t\geq 0}$, i.e. $(L_tM_t^Y)_{t\geq 0}$ is a $\mathbb{G}$-martingale process.

Equation (7.16) is known as the FS decomposition, and as pointed out its existence is equivalent to the existence of PLRM hedging strategies. However, PLRM hedging strategies might not be the same as LRM ones. For PLRM hedging strategies to coincide with the LRM ones, certain conditions must hold for the first one being the SC condition. We say that the SC condition holds, if there is a $\mathbb{G}$-predictable process $(\zeta_t)_{t\geq 0}$ such that $\Lambda_t^Y = \int_0^t \zeta_s \, d\left\langle M^Y \right\rangle_s^{\mathbb{G}}$, and the MVT process defined by $\hat{K}_t = \int_0^t \zeta_s^2 \, d\left\langle M^Y \right\rangle_s^{\mathbb{G}}$ is finite.

From Theorem 3.38 we know that LRM and PLRM hedging strategies coincide if there conditions are met: i) the SC condition holds, ii) $\left\langle M^Y \right\rangle_t^{\mathbb{G}}$ is strictly increasing, iii) $\mathbb{E}[\hat{K}_t] < \infty$. In our model we have that $\zeta_t = \frac{a}{\gamma Y_{t-}}$and $\hat{K}_t = \frac{\alpha^2}{\gamma} t$, which is uniformly bounded for every $t \in [0, T]$. Therefore all these conditions are satisfied if $\gamma \neq 0$.

So far we have determined the process $(\theta_t)_{\geq 0}$ representing the amount of money invested in the risky asset $(Y_t^\tau)_{t\geq 0}$ in $\mathbb{G}$ whose canonical decomposition is introduced in Lemma 7.11. By applying Proposition 4.1 of Okhrati et al. (2014) in $\mathbb{G}$ we can formulate the processes: $(\eta_t)_{t\geq 0}$, the value process $(V_t)_{t\geq 0}$ of a hedging strategy $\phi = (\theta_t, \eta_t)$ along with its cost process $(C_t)_{t\geq 0}$.

Remark 7.17. In the next section, we apply Theorem 7.13 to obtain semi-explicit solutions for hedging strategies in our framework by specifying $g(t, x, y)$ explicitly. For instance, one can consider the following choices:

- The simplest example is when $g(t, x, y)$ is constant.
- A company's asset value that is far from its historical infimum should be less prone to default for $t \geq 0$, this rate should decrease as $Y_t - \underline{Y}_t$ increases, for instance for $t \in [0, T]$ and $\alpha > 0$ one can choose $g(t, Y_t, \underline{Y}_t) = e^{-a(Y_t - \underline{Y}_t)}$ or $g(t, Y_t, \underline{Y}_t) = \frac{1}{Y_t - \underline{Y}_t + \alpha}$. Alternatively, one can consider $g(t, Y_t, \underline{Y}_t) = e^{-aY_t}$ which indicates that as the asset values decreases the default rate increases as well.
- Suppose that the default time is independent from the underlying asset and admit a probability density function, $g(t, x, y) = \frac{d_\tau(t)}{1 - D_\tau(t)}$, $t \in [0, T]$, $x \geq 0$ and $y \geq 0$ where d_τ and D_τ are respectively the probability density and distribution function of τ.

The results of this section are in the context of reduced form models as $Z_t < 1$ for all $t \geq 0$. Consider a structural credit risk model in which $g(t, x, y)$ is identically zero which means that $Z_t = 1$ for all $t \geq 0$ and hence $\tau = \infty$. In order to obtain similar results for this structural model we cannot simply let $g(t, x, y) = 0$ and use Theorem 7.13 as this supposes the assumption $Z_t < 1$ for all $t \geq 0$. Nevertheless, starting from Proposition 7.8 and following the same arguments of this section (which we skip it here), we can obtain similar results. Since $g(t, x, y) = 0$ the operator $\mathcal{A}$ is the same as $\mathcal{L}$, no filtration is required i.e. $\mathbb{F} = \mathbb{G}$ and the assumption $\Delta X_\tau = 0$ is no longer required. For illustration purposes we provide the following main result for the case when $g(t,x,y) = 0$. The proof is omitted as it is almost identical to the proof of Theorem 7.13.

Theorem 7.18. *Suppose that $(X_t)_{t\geq 0}$ is given by (7.1), $g(t,x,y) = 0$ for all $t \in [0,T]$, $x \geq 0$, $y \geq 0$, and Assumptions 7.1 and 7.9 are in force. We also assume that $f : [0,T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies Assumption 7.12 and $[X]_t^{\mathbb{F}} \in \mathcal{A}_{loc}$. Then we have the following decomposition*

$$F(Y_T, \underline{Y}_Y) = f(0, Y_0, \underline{Y}_0) + \int_0^T \theta_{s-} 1_{\{s<T\}} dY_s + L_T, \tag{7.17}$$

where for $0 \leq t < T$ the process $(\theta_t)_{t\geq 0}$ is given by

$$\theta_t = \frac{\mathcal{K}^* f(t, Y_t, \underline{Y}_t)}{\gamma Y_t^2}, \quad t \in [0,T)$$

for all $(t,x,y) \in [0,T) \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, the operator $\mathcal{K}^*$ is defined by

$$\mathcal{K}^* f(t,x,y) = \mathcal{L}P(t,x,y) - x\mathcal{L}f(t,x,y) - x\alpha f(t,x,y),$$

where $P(t,x,y) = xf(t,x,y)$ and the process $(L_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale orthogonal to the martingale part of $(Y_t)_{t\geq 0}$ (i.e.$(M_t)_{t\geq 0}$) in $\mathbb{F}$.

7.4 Diffusion models and running infimum process

In this section, we focus on underlying processes with continuous sample paths, i.e. $Y_t = e^{X_0+\mu t+\sigma W_t}$, $\mu \in \mathbb{R}$, $\sigma > 0$, $t \in [0,T]$. Our main goal in this section is to obtain, the FS decomposition with the understanding that under the hypotheses of Proposition 4.1 of Okhrati et al. (2014), this orthogonal decomposition leads to the FS decomposition and hence LRM hedging strategies.

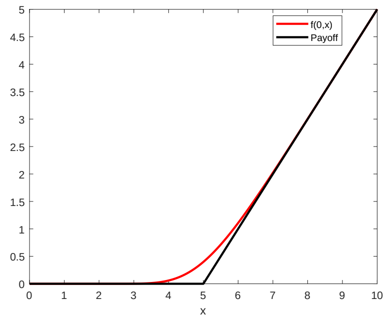
Proposition 7.19. *Let the underlying process $(Y_t)_{t\geq 0}$ follows $Y_t = e^{X_0+\mu t+\sigma W_t}$, $\sigma > 0$, $t \geq 0$. Suppose that Assumption 7.2 holds, and there is a continuous function $f : [0,T) \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ such that for all $(t,x,y) \in [0,T) \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$, f satisfies the following PDE*

$$\frac{\partial f}{\partial t}(t,x,y) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t,x,y) - f(t,x,y)g(t,x,y) = 0, \tag{7.18}$$

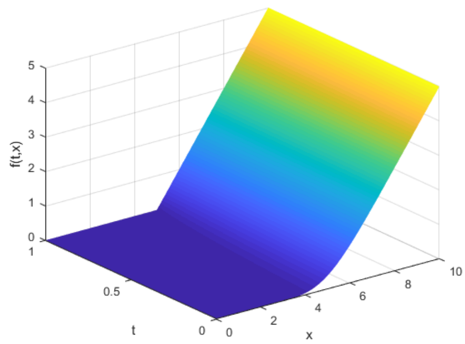
together with the boundary conditions $f(T,x,y) = F(x,y)$ and $\dfrac{\partial f}{\partial y}(t,y,y) = 0$, for all $(t,x,y) \in [0,T] \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$. Then, we obtain

$$F(Y_T, \underline{Y}_T) 1_{\{\tau > T\}} = U_0 + \int_0^T \theta_s 1_{\{s\leq\tau\}} 1_{\{s<T\}} dY_s + L_T, \tag{7.19}$$

where $F(x,y)$ is the PIDE's terminal condition, for all $0 \leq t < T$, $(\theta_t)_{t\geq 0}$ where



(a) Payoff function $f(T, x) = \max(x - K, 0)$ and $f(0, x)$.



(b) 3D representation of $f(t, x)$ on a domain $[0, 1] \times [0, 10]$.

Figure 7.1: Solution of the PDE (7.20) when $K = 5$ $T = 1$ $\sigma = 0.2$ on an interval $x \in [0, 10]$.

$$\theta_t = \frac{\partial f(t, Y_t, \underline{Y}_t)}{\partial x}, \quad t \in [0, T).$$

The process $(L_t)_{t \geq 0}$ is a local martingale orthogonal to the local martingale part of $(Y_t^\tau)_{t \geq 0}$ i.e. $(M^Y_{\tau \wedge t})_{t \geq 0}$. Furthermore, suppose that $(M^Y_{\tau \wedge t})_{t \geq 0}$ and $(L_t)_{t \geq 0}$ are square integrable martingales, and $(\theta_t)_{t \geq 0}$ belongs to Θ space (see Definition 3.33) for all $0 \leq t < T$. Then the FS decomposition of the claim $F(Y_T, \underline{Y}_T)1_{\{\tau > T\}}$ is given by (7.19).

Proof. Since there are no jumps in $(Y_t)_{t \geq 0}$, then from Lemma 7.11, $\alpha = \mu + \frac{\sigma^2}{2}$ and $\gamma = \sigma^2$, and all the integral terms disappears based on the Lévy measure ν disappear. Then the PIDE of Assumption 7.12 reduces to the PDE (7.18). The result then follows from Theorem 7.13. □

In light of Proposition 7.19, we can discuss three cases:

- We start with the simplest case in which we want to hedge a claim H that only depends on Y, for instance $H = \max(Y_T - K, 0)$, $K > 0$. This claim is default free, but it can be still analysed using the previous proposition by letting $g(t, x, y) = 0$, $F(x, y) = \max(x - K, 0)$. Since $g(t, x, y) = 0$, $1_{\{\tau > t\}} = 1$, for all $t \in [0, T]$, and so the sigma-algebra $\mathcal{N}$ is trivial, hence $\mathbb{F} \vee \mathcal{N} = \mathbb{F}$. We are in fact in the setup of Black-Scholes model, the three dimensional PDE reduces to two dimensional, and the hedging strategies are obtained through solving the following PDE

$$\frac{\partial f}{\partial t}(t, x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 f}{\partial x^2}(t, x) = 0, \tag{7.20}$$

  together with the boundary conditions $f(T, x) = \max(x - K, 0)$ for all $x \geq 0$.

- Next, we consider a structural credit risk model in which the default time $\mathcal{T}$ is defined by $\mathcal{T} = \inf\{t \geq 0; Y_t < b\}$ where $0 < b < Y_0$ is a pre-specified constant barrier. Note that $\mathcal{T}$ is a predictable time and does not admit an intensity.

Suppose that we want to hedge the claim $\max(Y_T - K, 0)1_{\{\mathcal{T}>T\}}$, $K > 0$. We can use Proposition 7.19 for this case by letting $g(t,x,y) = 0$ (since there is no exogenous default) and $F(T,x,y) = \max(x - K, 0)1_{\{y\geq b\}}$. Then we can easily observe that $F(T, Y_T, \underline{Y}_T) = \max(Y_T - K, 0)1_{\{\mathcal{T}>T\}}$, and the hedging strategies are determined through solving the following PDE:

$$\frac{\partial f}{\partial t}(t,x,y) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x,y) = 0,$$

together with the boundary conditions $f(t,x,y) \to F(x,y) = \max(x-K,0)1_{\{y\geq b\}}$ as $t \to T$ and $\frac{\partial f}{\partial y}(t,y,y) = 0$ for all $t \in [0,T)$, $x \geq 0$, and $y \geq 0$.

- We can study a reduced form credit risk model in which the default time in which the default time τ follows Assumption 3.40 with an intensity $g(t, Y_t)$, $t \geq 0$, and assume that $Z_t < 1$ for all $t \geq 0$. Note that τ is totally inaccessible stopping time. Suppose that we want to hedge the claim $\max(Y_T - K, 0)1_{\{\tau>T\}}$, $K > 0$. We can use Proposition 7.19 for this case by letting $F(x,y) = \max(x - K, 0)$. Then the three dimensional PDE reduces to two dimensions, and the hedging strategies are determined through solving the following PDE

$$\frac{\partial f}{\partial t}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x) - f(t,x)g(t,x) = 0,\ 0 \leq t \leq T,\ x \geq 0,$$

along with the boundary condition $f(t,x,y) \to F(x,y) = \max(x - K, 0)$ as $\to T$, for all $x \geq 0$.

- Finally, let us consider the most interesting case in which a claim is subject to both internal and exogenous defaults (a double default model). More specifically, let say that a payoff $\max(Y_T - K, 0)$ is paid if $\mathcal{T} > T$ and $\tau > T$, where $\mathcal{T} = \inf\{t \geq 0; Y_t < b\}$ (for a fixed known barrier $0 < b < Y_0$) and τ satisfies Assumption 7.2, in other words, we ant to hedge the defaultable claim $\max(Y_T - K, 0)1_{\mathcal{T}>T}1_{\tau>T}$.

  We can use Proposition 7.19 for this case by letting $F(T,x,y) = \max(x - K, 0)1_{\{y\geq b\}}$. Then we can easily observe that $F(T, Y_T, \underline{Y}_T) = \max(Y_T-K, 0)1_{\{\mathcal{T}>T\}}$, and the hedging strategies are determined through solving the following PDE:

$$\frac{\partial f}{\partial t}(t,x,y) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t,x,y) - f(t,x,y)g(t,x,y) = 0, \tag{7.21}$$

  together with the boundary conditions $f(T,x,y) \to F(x,y) = \max(x-K,0)1_{\{y\geq b\}}$ as $t \to T$ and $\frac{\partial f}{\partial y}(t,y,y) = 0$ for all $t \in [0,T)$, $x \geq 0$, and $y \geq 0$.

  For numerical purposes, it might be easier to use the change of variable $h(t,x,y) = f(t, e^{x-y}, e^{-y})$.Thus the PDE changes to

$$\frac{\partial h}{\partial t}(t,x,y) + \frac{\sigma^2}{2}(\frac{\partial^2 h}{\partial x^2}(t,x,y) - \frac{\partial h}{\partial x}(t,x,y)) - h(t,x,y)g(t,e^{x-y},e^{-y}) = 0, \tag{7.22}$$

  with the conditions $\frac{\partial h}{\partial x}(t,0,y) + \frac{\partial h}{\partial y}(t,0,y) = 0$ and $h(t,x,y) \to \max(e^{x-y} -$

$K, 0)1_{\{e^{-y} \geq b\}}$ as $t \to T$.

7.5 Jump-diffusion models and running infimum process

First, we consider a structural credit risk modelling under a jump-diffusion process.

Proposition 7.20. *Let $(X_t)_{t\geq 0}$ be given by Equation (7.1), and let Assumptions 7.1 and 7.2 hold. We also assume that $f(t,x,y)$ is the solution of the following PIDE:*

$$\mathcal{L}f(t,x,y) = \frac{(\mathcal{L}P(t,x,y) - x\mathcal{L}f(t,x,y) - f(t,x,y)x\alpha)}{x\gamma}\alpha, \quad t \in [0,T], x \geq 0, y \geq 0, \tag{7.23}$$

where α and γ are given in Lemma 7.11, the operator $\mathcal{L}f(t,x,y)$ is defined in Lemma 7.7, $P(t,x,y) = xf(t,x,y)$, and the following conditions are satisfied for all $x \geq 0$ and $y \geq 0$:

$$f(t,x,y) \to F(x,y) \text{ as } t \to T, \qquad \text{and} \qquad \frac{\partial f}{\partial y}(t,y,y) = 0.$$

Moreover, we further assume that $([U, Y^\tau]_t^{\mathbb{G}})_{t\geq 0} \in \mathcal{A}_{loc}$. Then we obtain

$$F(Y_T, \underline{Y}_T) = U_0 + \int_0^T \theta_{s^-} 1_{\{s<T\}} dY_s + L_T,$$

where $F(x,y)$ is the PIDE's terminal condition, the process $(\theta_t)_{t\geq 0}$ $\theta_t = B(t, Y_t, \underline{Y}_t)$, where $B(t,x,y)$ is given by

$$\begin{aligned} &\frac{\sigma^2 \frac{\partial f}{\partial x}(t,x,y)}{\gamma} + \frac{1}{2}\frac{(\sigma^2 - 2\alpha + 2\beta)f(t,x,y)}{\gamma x} \\ &+ \frac{\int_{-\ln(\frac{x}{y})}^{+\infty}(e^z f(t,xe^z,y) - zf(t,x,y) - f(t,xe^z,y))\nu(dz)}{\gamma x} \\ &+ \frac{\int_{-\infty}^{-\ln(\frac{x}{y})}(e^z f(t,xe^z,xe^z) - zf(t,x,y) - f(t,xe^z,xe^z))\nu(dz)}{\gamma x}, \end{aligned} \tag{7.24}$$

and the process $(L_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale orthogonal to the martingale part of $(Y_t)_{t\geq 0}$ in $\mathbb{F}$.

Proof. Since the claim $F(Y_T, \underline{Y}_T)$ is default-free, we can assume that $g(t,x,y) = 0$, for all $(t,x,y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ which means that $\tau = \infty$ and so $1_{\{\tau > t\}} = 1$ for all $t \in [0,T)$. Also, the two operators $\mathcal{A}$ and $\mathcal{L}$ coincide and so $f(t,x,y)$ satisfies the PIDE in Assumption 7.12. The result is then a direct application of Theorem 7.13. □

Remark 7.21. In the previous proposition the assumption $\Delta X_\tau = 0$ is not required. Note that how the strategies (7.24) in the jump diffusion model differ from those of diffusion models in Proposition 7.19.

Example 7.22. Suppose that we are in the setup of the previous proposition, and let $\zeta = \inf\{t : Y_t < b\}$ where $0 < b < Y_0$ is a fixed constant. Note that ζ is an $\mathbb{F}$-stopping time, but it is not necessarily an $\mathbb{F}$-predictable or an $\mathbb{F}$-totally inaccessible stopping time. Suppose that for example we want to find LRM hedging strategies of the claim $\max(Y_T - K, 0)1_{\{\zeta > T\}}$, $K > 0$. Since $\{\zeta > T\} = \{\underline{Y}_t \geq b\}$, these hedging strategies can be found by using the previous proposition and just adjusting the terminal condition, i.e. $F(x, y) = \max(x - K, 0)1_{\{y \geq b\}}$.

This has improved the result of Okhrati et al. (2014) in two main folds, first, the underlying process can now be an exponential jump-diffusion process rather than a finite variation Lévy process, second the internal default time ζ does not need to be totally inaccessible, and hence it does not need to admit an intensity.

Obtaining these hedging strategies, would of course depend on solving the PIDE in (7.23). For instance, if $(Y_t)_{t \geq 0}$ is an $\mathbb{F}$-local martingale, then $\alpha = 0$, and the PIDE (7.23) reduces to

$$\begin{aligned}\frac{\partial f}{\partial t}(t, x, y) + x(\beta + \frac{\sigma^2}{2})\frac{\partial f}{\partial x}(t, x, y) + \frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}(t, x, y)\\ + \int_{-\ln(\frac{x}{y})}^{+\infty}(f(t, xe^z, y) - f(t, x, y) - zx\frac{\partial f}{\partial x}(t, x, y))\nu(dz)\\ + \int_{-\infty}^{-\ln(\frac{x}{y})}(f(t, xe^z, xe^z) - f(t, x, y) - zx\frac{\partial f}{\partial x}(t, x, y))\nu(dz) = 0\end{aligned} \tag{7.25}$$

with the condition $\frac{\partial f}{\partial y}(t, y, y) = 0$, for all $y \geq 0$, and $f(t, x, y) \to \max(x - K, 0)1_{\{y \geq b\}}$ as $t \to T$ for all $x \geq 0$ and $y \geq 0$.

We also point out that if the payoff to hedge does not depend on $(\underline{Y}_t)_{t \geq 0}$, for instance $\max(Y_T - K, 0)$, then the three dimensional PIDE reduces to two dimension, and the boundary condition $\frac{\partial f}{\partial y}(t, y, y) = 0$ is redundant. More precisely, in this case, we have $b = 0$ and $\zeta = \infty$, hence $1_{\{y \geq b\}} = 1$.

Furthermore, for numerical implementations it might be easier to use the change of variable $h(t, x, y) = f(t, e^{x-y}, e^{-y})$. Then the PIDE (7.25) can be further simplified to:

$$\begin{aligned}\frac{\partial h}{\partial t}(t, x, y) + (\beta + \frac{\sigma^2}{2})\frac{\partial h}{\partial x}(t, x, y) + \frac{\sigma^2}{2}\frac{\partial^2 h(t, x, y)}{\partial x^2}\\ + \int_{-x}^{+\infty}(h(t, x + z, y) - h(t, x, y) - z\frac{\partial h}{\partial x}(t, x, y))\nu(dz)\\ + \int_{-\infty}^{-x}(h(t, x + z, x + z) - h(t, x, y) - z\frac{\partial h}{\partial x}(t, x, y))\nu(dz) = 0,\end{aligned} \tag{7.26}$$

with the conditions $\frac{\partial h}{\partial x}(t, 0, y) + \frac{\partial h}{\partial y}(t, 0, y) = 0$ and $h(t, x, y) \to \max(e^{x-y} - K, 0)1_{\{e^{-y} \geq b\}}$ as $t \to T$.

The Example bellow provides specific forms for the hazard rate function $g(t, x, y)$.

Example 7.23. Let the process $(X_t)_{t \geq 0}$ be given by Equation (7.1), and let Assumptions 7.1 and 7.9 hold. Suppose that the default time is independent of the underlying asset. We also assume that $f(t, x, y)$ is the solution of the PIDE given in Assumption 7.12 where $g(t, x, y)$ is for example can be defined by either one of a constant,

$g(t,x,y) = -\frac{-D'_\tau(t)}{1-D_\tau(t)}$, $g(t,Y_t,\underline{Y}_t) = e^{-a(Y_t-\underline{Y}_t)}$, and $g(t,Y_t,\underline{Y}_t) = \frac{1}{Y_t-\underline{Y}_t+a}$ where $a>0$ is a constant. Moreover, we further assume that $([U,Y^\tau]^{\mathbb{G}}_t)_{t\geq 0}$ is of locally integrable variation. Suppose that $F(x,y) = \max(x-K,0)1_{\{y\geq b\}}$. A similar interpretation of the diffusion case shows that in this example, the claim is subject to types of default, one indigenous and the other exogenous. Then for all $0\leq t\leq T$, we have the following decomposition

$$F(Y_T,\underline{Y}_T)1_{\{\tau>T\}} = U_0 + \int_0^T \theta_{s-}1_{\{s\leq\tau\}}1_{\{s<T\}}dY_s + L_T,$$

where $F(x,y)$ is the PIDE's terminal condition, the process $(\theta_t)_{t\geq 0}$, $\theta_t = B(t,Y_t,\underline{Y}_t)$, where $B(t,x,y)$ is given by (7.24). Moreover, the process $(L_t)_{t\geq 0}$ is orthogonal to the martingale part of $(Y^\tau_t)_{t\geq 0}$ i.e. $(M^Y_{\tau\wedge t})_{t\geq 0}$.

For example, if $(Y_t)_{t\geq 0}$ is an $\mathbb{F}$-martingale, then through the transformation $h(t,x,y) = f(t,e^{x-y},e^{-y})$, the PIDE of Assumption 7.12 reduces to

$$\begin{aligned}\frac{\partial h}{\partial t}(t,x,y) + (\beta+\frac{\sigma^2}{2})\frac{\partial h}{\partial x}(t,x,y) + \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}(t,x,y) - h(t,x,y)g(t,e^{x-y},e^{-y})\\ + \int_{-x}^{\infty}(h(t,z+x,y) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\nu(dz)\\ + \int_{-\infty}^{-x}(h(t,z+x,x+z) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\,\nu(dz) = 0,\end{aligned}$$

with the conditions $\frac{\partial h}{\partial x}(t,0,y) + \frac{\partial h}{\partial y}(t,0,y) = 0$ and for $h(t,x,y) \to \max(e^{x-y} - K,0)1_{\{e^{-y}\geq b\}}$ for $t\to T$.

Chapter 8

Numerical results

8.1 Introduction

In this chapter we provide the simulation results for the PIDE's and the PDE along with the hedging strategies given in Chapters 4, 6 and 7. The discretisation of the PIDEs is accomplished via finite differences. The discretisation of the integral terms follows the methodology proposed by Cont and Voltchkova (2005) by introducing the trapezoidal quadrature rule. For the models introduced in Chapters 4 and 6, the absence of the Brownian motion in our models leads to the fact that the PIDE are hyperbolic. To treat their PDE part we apply the Lax-Wendroff approach. The majority of the simulations are made in Matlab R-2017 on a personal computer with Intel i7-4510U 2.6 GHz and 8 GB RAM.

The chapter is organized as follows. We start by introducing the finite differences for the advection equation. This is described in Section 8.2 and we compare the main methods used to descritize the advection PDE. In Section 8.3, we solve numerically the PIDE introduced in Chapter 4. We also simulate a trajectory of the optimal hedging strategy. Moreover, in Section 8.4, we descritize the PIDE proposed in Chapter 6 along with a sample path of the optimal hedging strategy. Finally, in Section 8.5, we descritize the corresponding parabolic PDE and PIDE introduced in Chapter 7 through finite differences.

8.2 Finite differences for advection equation

This section is focused on the investigation of the finite differences for hyperbolic PDEs. We begin by providing some examples of hyperbolic PDEs.

Hyperbolic PDEs

- Advection equation (or one way wave equation)

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x) + \mu \frac{\partial f}{\partial x}(t,x) = 0, \\ f(0,x) = F(x),\ x \in \mathbb{R} \end{aligned} \tag{8.1}$$

- Wave equation

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2}(t,x) = \mu \frac{\partial^2 f}{\partial x^2}, \\ f(0,x) = F(x),\ x \in \mathbb{R}. \end{aligned}$$

Emphasis will be placed on the discretisation of the advection equation (8.1) using finite differences. This is the simplest hyperbolic PDE and its analytical solution given the initial data is

$$f(t,x) = F(x - \mu t).$$

8.2.1 Various types of finite differences for the advection equation

Central differences

A typical approach that someone may consider to descritize the advection equation is to apply central differences in space and forward differences in time. That is

$$\frac{\partial f}{\partial x} = \frac{f(t,x+dx) - f(t,x-dx)}{2dx} + \mathcal{O}(dx^2), \tag{8.2}$$

and for the time discretisation by taking explicit scheme

$$\frac{\partial f}{\partial t}(t,x) = \frac{f(t+dt,x) - f(t,x)}{dt} + \mathcal{O}(dt). \tag{8.3}$$

If we consider the approximated numerical solution given as $f_i^n = f(t_n, x_i)$, where $x_i = x_0 + idx$ and $t_n = ndt$ then plugging (8.2) and (8.3) into (8.1) leads to the following scheme

$$f_i^{n+1} = f_i^n - \mu \frac{dt}{2dx}(f_{i+1}^n - f_{i-1}^n). \tag{8.4}$$

However, as we will later see, in an illustrated application, the above scheme is unstable as spurious oscillations occurred. This can be verified by investigating the Von Neumann stability analysis. Consider $k = \mu \frac{dt}{dx}$ and assume that the solution has the form $f_i^n = g^n(\xi) e^{\nu i dx \xi}$, where $\nu = \sqrt{-1}$. In order for the above approach to be stable, we have to find appropriate conditions such that $|g| < 1$. If we substitute the solution into (8.4) we get

$$g^{n+1}(\xi) e^{\nu i dx \xi} = g^n(\xi) e^{\nu i dx \xi} - \frac{k}{2}\left(g^n(\xi) e^{\nu(i+1)dx\xi} - g^n(\xi) e^{\nu(i-1)dx\xi} \right).$$

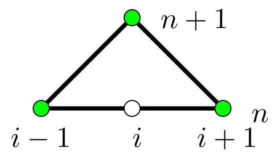


Figure 8.1: Schematic representation of the Lax-Friedrich.

If we divide by $g^n(\xi)e^{\nu i dx\xi}$, it yields

$$g(\xi) = 1 - \frac{k}{2}(e^{\nu dx\xi} - e^{-\nu dx\xi}) = 1 - \nu k \sin(dx\xi).$$

By introducing the variable $\theta = \xi dx$ gives $g(\theta) = 1 - \nu k \sin(\theta)$ and $|g(\theta)| = \sqrt{1 + k^2 \sin(\theta)}$ ≥ 1 for all k and θ, which proves that the approach is unconditionally unstable. Thus, alternative approaches should be considered. There are various numerical methods for hyperbolic PDEs. We begin describing the simplest one, the Lax Friedrich approach.

Lax-Friedrichs scheme

In the above approach, given in (8.4), if we substitute f_i^n by taking its average using f_{i+1}^n and f_{i-1}^n then we get

$$f_i^{n+1} = \frac{1}{2}(f_{i+1}^n + f_{i-1}^n) - \mu \frac{dt}{2dx}(f_{i+1}^n - f_{i-1}^n),$$

or in a matrix form $f^{n+1} = Af^n$

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ b & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & a \\ 0 & 0 & b & 0 \end{bmatrix},$$

where $a = \frac{1}{2} - \mu\frac{dt}{2dx}$ and $b = \frac{1}{2} + \mu\frac{dt}{2dx}$. The method is stable with error convergence $\mathcal{O}(dt + dx)$. Again, by applying the Von Neumann stability analysis, doing the same calculations as above, it yields

$$\begin{aligned} g(\xi) &= \frac{1}{2}(e^{\nu i dx\xi} + e^{\nu i dx\xi}) - \frac{k}{2}(e^{\nu i dx\xi} - e^{\nu i dx\xi}) \\ &= \cos(dx\xi) - 2k\sin(dx\xi). \end{aligned}$$

After some algebra we get that $|g(\theta)| \leq 1$, $\theta = \xi dx$, if and only if $1 - (\frac{\mu dt}{dx})^2 \geq 0$, which leads to $|\mu|\frac{dt}{dx} \leq 1$. This is called Courant-Friedrichs and Lewy condition (CFL) and it is crucial to be valid, otherwise the scheme is unstable. This is the significant difference in the discretisation between the hyperbolic and parabolic partial differential equations.This means that in the hyperbolic equations the time-step dt should be at least smaller than the space-step dx. It is worth mentioning that due to its slow accuracy, the above approach is not commonly used.

Figure 8.2: Schematic representation of the Upwind scheme for $\mu < 0$ and $\mu > 0$.

Upwind method

One main reason why the central differences fail is that at each time $n+1$ f_i^{n+1} depends on f_{i+1}^n and f_{i-1}^n. One way to fix this, is to take the space derivatives with respect to where the flow of information comes. In other words, you examine if the parameter μ is positive or negative. If $\mu > 0$ then the space discretisation is made through backward differences, otherwise we use forward ones. Having this in our mind, the upwind method is defined as

$$\frac{f_i^{n+1} - f_i^n}{dt} = \begin{cases} -\frac{\mu}{dx}(f_i^n - f_{i-1}^n), & \text{if } \mu > 0 \\ -\frac{\mu}{dx}(f_{i+1}^n - f_i^n), & \text{if } \mu \leq 0 \end{cases}. \tag{8.5}$$

The method is first order in time and space stable $\mathcal{O}(dt, dx)$. In order the method to be stable, once again the CFL condition needs to be satisfied.

Lax-Wendroff scheme

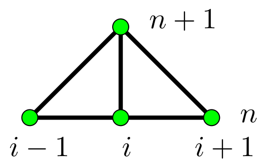


Figure 8.3: Schematic representation of the Lax-Wendroff scheme.

The Lax-Wendroff scheme is an essential extension of the Lax-Friedrich approach. This method is very useful when the initial data $F(x)$ is a continuous function. Applying Taylor expansion of $f(t + dt, x)$ leads to

$$f(t + dt, x) = f(t, x) + dt\frac{\partial f}{\partial t}(t, x) + \frac{dt^2}{2}\frac{\partial^2 f}{\partial t^2}(t, x). \tag{8.6}$$

Based on (8.1) we can see that

$$\frac{\partial f}{\partial t} = -\mu\frac{\partial f}{\partial x}(t, x). \tag{8.7}$$

$$\frac{\partial^2 f}{\partial t^2} = -\mu\frac{\partial^2 f}{\partial x \partial t}(t, x) = \mu^2\frac{\partial^2 f}{\partial x^2}(t, x). \tag{8.8}$$

Plugging (8.7), (8.8) into (8.6), it yields

$$f(t+dt,x) = f(t,x) - \mu dt\frac{\partial f}{\partial x}(t,x) + \mu^2\frac{dt^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x).$$

Then if we apply finite differences in space and time we get

$$f_i^{n+1} = f_i^n - \mu dt\frac{f_{i+1}^n - f_{i-1}^n}{2dx} + \mu^2\frac{dt^2}{2}(\frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{dx^2}). \tag{8.9}$$

The extra term $\mu^2\frac{dt^2}{2}\frac{\partial^2 f}{\partial x^2}$ is an artificial diffusion, and it cancels any oscillations made by the central differences. It is obvious that the approach is a second order accurate in space and time $\mathcal{O}(dt^2, dx^2)$. We can write (8.9) in a matrix form $f^{n+1} = Af^n$, for a matrix A

$$A = \begin{bmatrix} a & b & 0 & 0 \\ c & a & \ddots & 0 \\ 0 & \ddots & \ddots & b \\ 0 & 0 & c & a \end{bmatrix}, \tag{8.10}$$

where $a = -\mu^2\frac{dt^2}{dx^2}$, $b = -\frac{\mu}{2dx} + \mu^2\frac{dt^2}{2dx^2}$ and $c = \frac{\mu}{2dx} + \mu^2\frac{dt^2}{2dx^2}$. Regarding the Von Neumann stability analysis, once again $f_i^n = g^n(\xi)e^{\nu idx\xi}$ and plugging it to (8.9) then

$$\begin{aligned} g(\theta) &= 1 - \frac{k}{2}(e^{\nu dx\xi} - e^{-\nu dx\xi}) + \frac{k^2}{2}(e^{-kdx\xi} - 2 + e^{kdx\xi}) \\ &= 1 - k\nu\sin(\theta) - 2k^2\sin^2(\frac{\theta}{2}), \end{aligned}$$

and

$$|g(\theta)|^2 = 1 - 4k^2(1-k^2)\sin^4(\frac{\theta}{2}),$$

which implies $g(\theta) < 1$ if and only if the CFL condition is satisfied. Alternate methods such as the Leap-frog approach, or the Beam-Warming method can be found in Li et al. (2017), Chapter 3, and LeVeque (2007), Chapter 10.

8.2.2 Numerical example

Example 8.1. We consider the following PDE

$$\frac{\partial f}{\partial x}(t,x) + \frac{\partial f}{\partial x}(t,x) = 0, \;\; 0 \le x \le 1, \; 0 \le t \le 1, \tag{8.11}$$

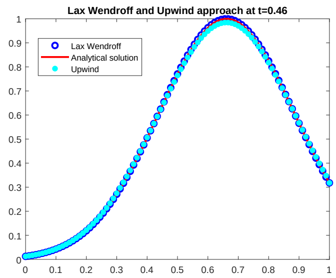
with initial condition be a Gaussian peak given by

$$f(0,x) = F(x) = \exp(-100(x-0.5)^2). \tag{8.12}$$

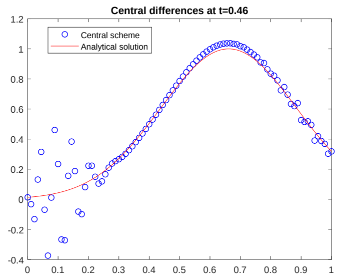
Figure 8.4 shows the numerical results for the three methods discussed in the previous section. Figure 8.4a provides the results for the Lax-Wendroff approach and the upwind, compared with the analytic solution $f(t,x) = \exp(-100(x-t-0.5)^2)$. Figure 8.4b gives the central differences scheme. The grid spacing is $dt = 0.005$ and the time step

$dx = 0.01$. Note that CFL condition is satisfied since the Courant number $\frac{dt}{dx} = 0.5$.

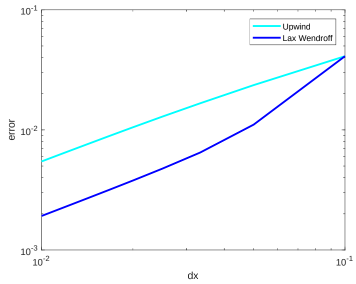
From Figure 8.4b, we can see that the central differences are unstable as expected. Moreover, from Figure 8.4c, we observe that the Lax-Wendroff approach is more accurate than the upwind scheme.



(a) Upwind and Lax-Wendroff approach for the advection equation (8.11), at time $t = 0.46$.



(b) Central differences scheme for the advection equation (8.11), at time $t = 0.46$.



(c) Error of upwind and Lax-Wendroff.

Figure 8.4: Numerical results for the advection equation given by (8.11), with $dt = 0.005$ and $dx = 0.01$.

8.3 Finite differences for the advection PIDE

The PIDEs can be treated using finite differences or finite elements method. In finite differences, we descritize the PIDE on an equidistant grid on an appropriate domain. The integral terms can be approximated either through Fast Fourier transforms (FFT), see Carr and Mayo (2007), Andersen and Andreasen (2000) or using quadrature methods such as trapezoidal rule, see Cont and Voltchkova (2005). On the other hand, in finite elements instead of discretizing the PIDE directly, we have first to choose an appropriate Galerkin basis like wavelet functions or regular triangulations. The discretisation of PIDEs using wavelet basis has been studied in Matache et al. (2005) and Hilber et al. (2013) Chapter 12. For a comprehensive review and comparison of both approaches we refer to Hilber et al. (2009).

In this section, we descritize the model in Chapter 4 through finite differences. The integral term is approximated through trapezoidal rule. Since the PIDE is hyperbolic, we use the Lax-Wendroff approach. Application of the Lax-Wendroff approach in PIDEs has already been investigated in Almendral and Oosterlee (2007), where they found the price of an American option under the variance gamma process (infinite activity). We focus on the martingale case i.e. $\beta = 0$. In this case we have the following PIDE

$$\begin{aligned}\mathcal{A}f(t,x) =& \frac{\partial f}{\partial t}(t,x) + \mu\frac{\partial f}{\partial x}(t,x) - \int_{(-\infty,-x]} f(t,x+z)\nu(dz) \\ &+ \int_{\mathbb{R}} (f(t,x+z) - f(t,x))\nu(dz),\ 0 \le t \le T,\ \text{and } x \in \mathbb{R},\end{aligned} \tag{8.13}$$

with terminal condition $f(T,x) = F(x) = 1$.

Assuming that $\nu(\mathbb{R}) < \infty$ (finite activity), then the Lévy measure admits a density $h(z)$, $\nu(dz) = \lambda h(z)dz$, where $h(z) = \delta e^{\delta z}1_{\{z\le 0\}}$. We rewrite (8.13) as

$$\frac{\partial f}{\partial t} = \mathcal{L}f(t,x),\ t \in [0,T],\ x \in \mathbb{R},$$

where $\mathcal{L}f(t,x)$ is the infinitesimal operator given as

$$\begin{aligned}\mathcal{L}f(t,x) =& -\mu\frac{\partial f}{\partial x}(t,x) + \lambda\int_{-\infty}^{-x} f(t,x+z)h(z)dz \\ &-\lambda\int_{\mathbb{R}} f(t,x+z)h(z)dz + \lambda f(t,x).\end{aligned} \tag{8.14}$$

Following Cont and Voltchkova (2005), in order to solve the PIDE we have to apply an appropriate boundary condition $f(t,x) = g(t,x)$, $\forall x \notin \Omega$.

As in Example 4.19, we assume that the underlying asset $(X_t)_{t\ge 0}$ has the form

$$X_t = u + \mu t + \sum_{j=1}^{N_t} Y_j \quad \mu,\ u > 0,$$

where Y_j are i.i.d. random variables such that $-Y_j \sim \textit{exponential}(\delta)$. Note that in order $(X_t)_{t\ge 0}$ to be a martingale, we take $\mu = \frac{\lambda}{\delta}$.

Suppose that $\Omega = [x_{min}, x_{xmax}]$, and $t = [0,T]$ then the localized problem is

$$\begin{aligned}&\frac{\partial f}{\partial t}(t,x) = \mathcal{L}f(t,x) \text{ on } [0,T]\times\Omega,\quad \Omega = [x_{min}, x_{max}] \\ &f(T,x) = 1, \\ &f(t,x) = g(t,x),\quad \forall x \notin \Omega.\end{aligned}$$

8.3.1 Truncation of integrals

We can truncate the integral terms by replacing the infinite domain to a proper bounded domain. Thus, we have to find appropriate bounds B_L and B_R for the integrals. These bounds correspond to truncating the large jumps of the Lévy process $(X_t)_{t\ge 0}$. Since Y_j follows negative exponential distribution with parameter δ and mean $\frac{1}{\delta}$, we can estimate B_L and B_R. The Lévy density can be written as $\nu(dz) = \lambda h(z)dz$, where

$h(z) = \delta \exp(\delta z) 1_{\{z \leq 0\}}$. Having this result then clearly $B_R = 0$, since the support of this distribution is $(-\infty, 0]$. Now we proceed to find the lower bound B_L. Following Cont and Voltchkova (2005), if the given function $f(t, x)$ satisfies the Lipschitz condition

$$|f(t, x_1) - f(t, x_2)| \leq L \, |x_1 - x_2| \, ,$$

then

$$\left| \int_{-\infty}^{0} (f(x+z) - f(z)) h(z) dz - \int_{B_L}^{0} (f(x+z) - f(z)) h(z) dz \right| < \epsilon,$$

which implies

$$\begin{aligned}
&\leq \left| \int_{-\infty}^{0} (x+z-x) h(z) dz - \int_{B_L}^{0} (x+z-x) dz \right| \\
&\leq L \int_{-\infty}^{B_L} |z| h(z) dz = L \int_{-\infty}^{B_L} |z| \delta e^{\delta z} dz \\
&= -\delta e^{-B_L} \left(\frac{1}{\delta^2} + \frac{B_L}{\delta} \right) \\
&\leq e^{B_L(1-\delta)} = \epsilon.
\end{aligned}$$

Thus, by solving with respect to B_L we obtain $B_L = -\frac{\log(\epsilon)}{(1-\delta)}$. We choose K_L such that $B_L \subseteq (K_L - \frac{1}{2}) dx$ where $\epsilon = 10^{-6}$.

8.3.2 Finite differences

We proceed now to the numerical discretisation with finite differences. Let $f_i^n = f(t_n, x_i)$, where $t_n = ndt$ and $x_i = x_{min} + idx$, $i = 0, 1 \ldots N_x$, for $dt = \frac{T}{N_t}$ and $dx = \frac{(x_{max} - x_{min})}{N_x - 1}$. We split the operator $\mathcal{L}$ (8.14) into $\mathcal{L} = \mathcal{D} + \mathcal{J}$, where $\mathcal{D}$ is the space derivative $\frac{\partial f}{\partial x}$ and $\mathcal{J}$ is the operator for the jump integrals. Since the PIDE is hyperbolic, we apply the Lax-Wendroff approach for the operator.

$$(\mathcal{D} f)_i = -\mu \frac{f_{i+1} - f_{i-1}}{2dx} + \mu^2 dt \frac{f_{i+1} - 2f_i + f_{i-1}}{2dx^2}, \quad \mu > 0,$$

or in a matrix form, assuming that $\mathcal{D} f^n = A$, whose form has already been introduced in (8.10). Regarding the jump terms $\mathcal{J}$, based on Cont and Voltchkova (2005), we apply the trapezoidal quadrature rule with the same space step dx.

$$\begin{aligned}
(\mathcal{J} f)_i &= -\lambda \int_{-\infty}^{0} f(t, x_i + z) h(z) dz + \lambda \int_{-\infty}^{-x_i} f(t, x_i + z) h(z) dz + \lambda f(t, x) \\
&= -\lambda \mathcal{J}_1 f(t, x_i) + \lambda \mathcal{J}_2 f(t, x_i) + \lambda f(t, x_i).
\end{aligned}$$

It remains to calculate the two integrals. For the first integral following Cont and Voltchkova (2005), it yields

$$\begin{aligned}\mathcal{J}_1 f(t,x_i) &= \int_{-\infty}^{0} f(t,x_i+z)h(z)dz \approx \int_{B_L}^{0} f(t,x_i+z)h(z)dz \\ &\approx \sum_{k=K_L}^{k_0} f_{i+k}^n h_k,\end{aligned}$$

where k_0 is the index representing the value at $x = 0$ and the cumulative density function (PDF) is approximated by

$$h_k \approx \int_{(k-\frac{1}{2})dx}^{(k+\frac{1}{2})dx} h(z)dz = \frac{dx}{2}(h(k-\frac{1}{2}dx)+h(k+\frac{1}{2}dx)).$$

For the second integral, we obtain

$$\begin{aligned}\mathcal{J} f_2(t,x) &= \int_{-\infty}^{-x} f(t,x+z)h(z)dz = \int_{-\infty}^{0} f(t,x+z)1_{(-\infty,-x]}h(z)dz \\ &\approx \int_{B_L}^{0} f(t,x_i+z)1_{[B_L,-x_i]}h(z)dz \approx \sum_{k=K_L}^{k_0} f_{i+k}1_{[K_L,-x_i]}h_k.\end{aligned}$$

Calculating the operator $(\mathcal{J} f)_i$ is the most expensive part in our discretisation. Regarding the points which lie outside the domain, we use the boundary condition $f(t^n, x_i) = g(t^n, x_i) = 1$ for all $x_i \notin \Omega$.

For the time discretisation we apply the implicit-explicit scheme, see Cont and Voltchkova (2005), with the following form

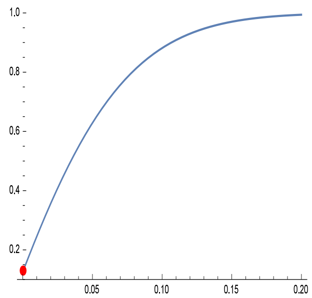
$$\frac{f^{n+1} - f^n}{dt} = \mathcal{D} f^{n+1} + \mathcal{J} f^n, \tag{8.15}$$

after rearranging the terms it yields

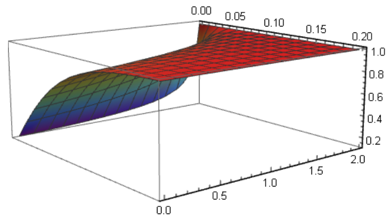
$$\begin{aligned}(I - dt\mathcal{D})f^{n+1} &= (I + dt\mathcal{J})f^n, \\ f^n \mid_{n=T} &= 1.\end{aligned} \tag{8.16}$$

If we set $B = (I + dt\mathcal{J})f^n$ and $b = (I - dt\mathcal{D})f^{n+1}$, where f^n, f^{n+1} are time dependent vectors, then we have to solve a linear system $B\bar{f} = b$ for $\bar{f} = (f^1, f^2, \ldots, f^{N_t})$. Regarding the convergence and stability of the above approach, we refer to Cont and Voltchkova (2005).

The linear system introduced in (8.16) can be solved using iterative methods. The main idea of iterative methods, is to find a proper sequence of solutions such that $\lim_{u\to\infty} \bar{f}^{(u)} \to \bar{f}$. There are various iterative methods for solving linear systems. Typical examples are the Gauss-Seidel, the Jacobi and the successive over-relaxation (SOR) methods. In this thesis, for the linear system generated by (8.16), we use the generalized minimum residual method (GMRES). For more details about the algorithm and its convergence, we refer to LeVeque (2007), Chapter 4.



(a) Analytic solution of $f(0, x)$.



(b) Analytic solution of $f(t, x)$.

Figure 8.5: Analytic solution of $f(t, x)$ for the martingale case with parameters $T = 2$, $\mu = 0.1$, $\lambda = 10$ and $\delta = 100$.

The implicit explicit scheme given in (8.15) provides an error of the form $\mathcal{O}(dt, dx^2)$. In Almendral and Oosterlee (2007) to improve the accuracy, they proposed a second order Backward Difference formula (BDF2) with an error $\mathcal{O}(dt^2, dx^2)$ as follows

$$\frac{\frac{3}{2}f^{n+1} - 2f^n + \frac{1}{2}f^{n-1}}{dt} = \mathcal{D}f^{n+1} + \mathcal{J}(2f^n - f^{n-1}).$$

We proceed providing the numerical results for the aforementioned PIDE. From the Feynman-Kac formula it is known that the solution of the PIDE (8.13) has the form

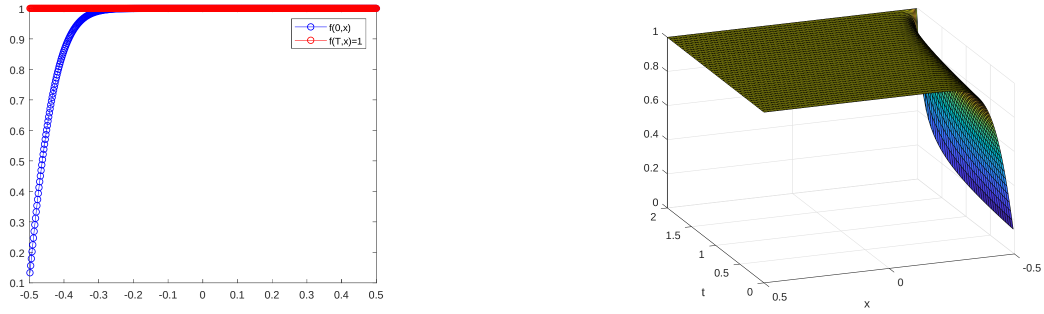
$$f(t, x) = 1 - \mathbb{P}\left(\tau \leq T - t \mid X_0 = u\right).$$

The above formula is valid for any distribution of jumps. For the case when the jumps of $(X_t)_{t \geq 0}$ are exponentially distributed (also negative exponential), then $f(t, x)$ has an analytic form. The solution can be found in Rolski et al. (2008), Theorem 5.6.3. Figure 8.5, provides the graphs for $f(0, x)$ and $f(t, x)$ respectively, using Mathematica.

To validate our numerical results, we focus on the value $f(0, 0)$. Using the parameters $T = 2$, $\mu = 0.1$, $\lambda = 10$ and $\delta = 100$, then analytic solution at time $t = 0$ has value $f(0, 0) = 0.125761$, whose value is represented by a red dot on Figure 8.5a.

We continue with the numerical results using finite differences. Figure 8.6 displays the solution obtained by the finite differences, on a domain $x \in [-0.5, 0.5]$, $t \in [0, 2]$, when the Lévy density follows negative exponential distribution.

Figure 8.7 simulates the error for the numerical value $f(t = 0, x = 0)$, given that the analytic value is 0.125761.



(a) Numerical solution of $f(0,x)$ (b) Numerical solution of $f(t,x)$

Figure 8.6: Numerical solution of $f(t,x)$, for the martingale case, using finite differences, with parameters $T=2$, $\lambda=10$, $\delta=100$, $\mu=0.1$ and $N_x=N_t=500$.

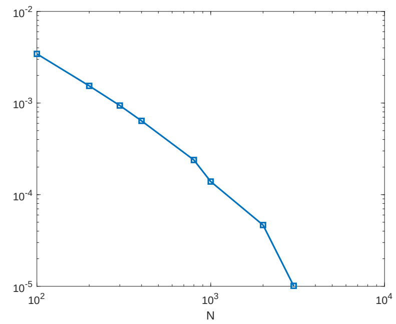


Figure 8.7: Error of the numerical solution given the analytical value $f(0,0)=0.125761$ for various space steps with $dt=\frac{dx}{\mu}$.

8.3.3 Simulation results for the local risk minimization

In this section, we simulate the hedging strategy $\phi_t = (\theta_t,\eta_t)_{t\geq 0}$. Let us remind ourselves that the optimal number of shares is given by

$$\theta_t = \frac{\mathcal{K}f(t,X_{t-})}{\int_{\mathbb{R}} z^2\nu(dz)} 1_{\{\tau>t\}},\ t\in[0,T]. \tag{8.17}$$

Since the underlying asset is a spectrally negative process with exponential Lévy density $k(z)$ defined above, then from Example 4.19, we know that the optimal number of shares is

$$\theta_t = \frac{\delta^2\int_{[-X_{t-},0)} f(t,X_{t-}+z)h(z)dz + \delta f(t,X_{t-})}{2} 1_{\{\tau\geq t\}},\ t\in[0,T],$$

the martingale $(L_t)_{t\geq 0}$ which is the hedging error is

$$L_t = f(t,X_t)1_{\{\tau>t\}} - f(0,X_0) - \int_0^t \theta_s dX_s, \tag{8.18}$$

the value process is given by

$$V_t = f(0,X_0) + \int_0^t \theta_s dX_s,\ t\in[0,T], \tag{8.19}$$

the amount of money invested in the non-risky asset is given by

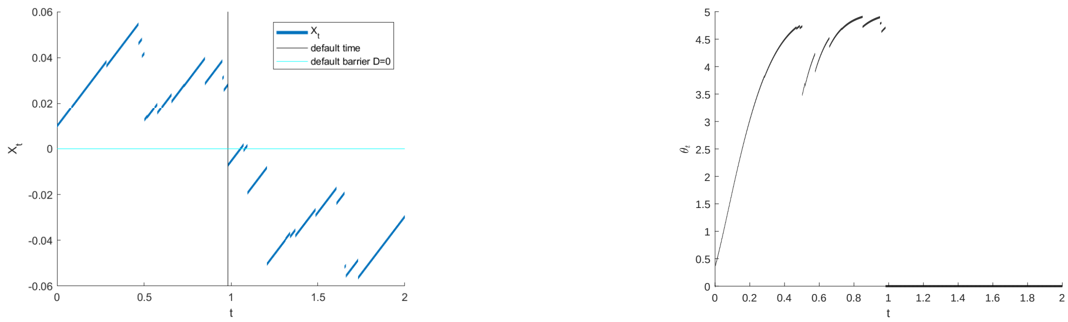
$$n_t = V_t - \theta_t X_t, \ t \in [0, T], \tag{8.20}$$

and the cost process

$$C_t = f(0, X_0) + L_t, \ t \in [0, T]. \tag{8.21}$$

Since we have already derived the numerical solution of $f(t, x)$, it remains to simulate a sample path of the underlying asset. For this reason, we use Monte-Carlo methods, see Cont and Tankov (2004), Chapter 8. Then the process $(f(t, X_t))_{t \geq 0}$ can be obtained through interpolation.

In Figure 8.8, a path of the underlying asset $(X_t)_{t \geq 0}$ along with the optimal number of shares $(\theta_t)_{t \geq 0}$ invested in the underlying asset are shown. The default time is $\tau = 0.9820$, and the probability of default, after 1000 iterations, is $\mathbb{P}(\tau \leq 2) = 1 - \mathbb{P}(\tau > 2) = 0.729$. Figure 8.9 displays a sample path of the cost process $(C_t)_{t \geq 0}$, the value process $(V_t)_{t \geq 0}$, the amount of money invested in the non-risky asset $(\eta_t)_{t \geq 0}$, the process $(L_t)_{t \geq 0}$, which orthogonal to the martingale part of $(X_t)_{t \geq 0}$ and the process $(K_t)_{t \geq 0}$, $K_t = \int_0^t \theta_s dX_s$, which is used to simulate L. Note that all these processes remain constant after the default.



(a) The underlying asset $(X_t)_{t \geq 0}$.

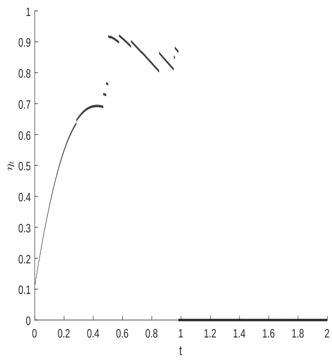
(b) The number of shares $(\theta_t)_{t \geq 0}$.

Figure 8.8: The underlying asset X_t, and the number of shares θ_t, with default time $\tau = 0.9820$ and maturity time $T = 2$.

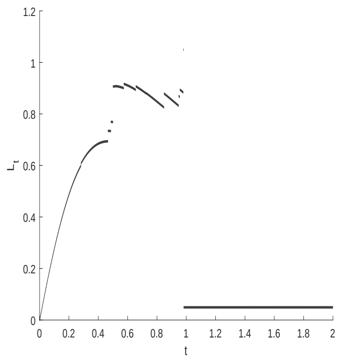
8.4 Finite differences for a 3D PIDE

We proceed to the discretisation of the PIDE given in Chapter 6, for the martingale case. We remind the reader that we need to solve numerically the following PIDE

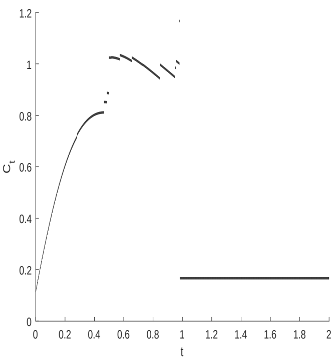
$$\begin{aligned} &\frac{\partial f}{\partial t}(t, x, y) + \mu \frac{\partial f}{\partial x}(t, x, y) + \mu f(t, 0, y) \\ &+ \lambda \int_0^{+\infty} (f(t, x + z, y) - f(t, x, y)) h(z) dz - \lambda \int_0^{\infty} f(t, x, y) h(x + z) dz = 0, \ t \in [0, T], \end{aligned} \tag{8.22}$$



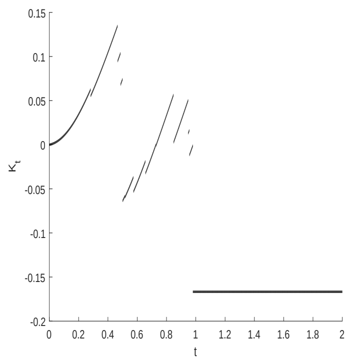
(a) The process $(\eta_t)_{t\geq 0}$.



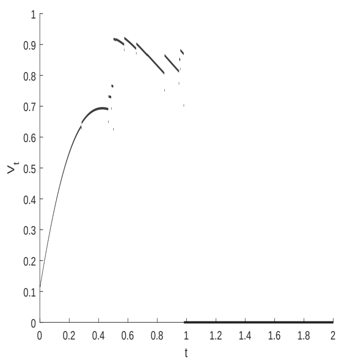
(b) The process $(L_t)_{t\geq 0}$.



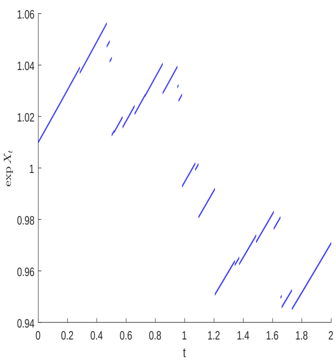
(c) The cost process $(C_t)_{t\geq 0}$.



(d) The process $(K_t)_{t\geq 0}$, $K_t = \int_0^t \theta_s dX_s$.



(e) The value process $(V_t)_{t\geq 0}$



(f) The exponential asset $(e^{X_t})_{t\geq 0}$

Figure 8.9: Simulation results for the non risky asset, the cost process, the value process and the exponential asset with default time $\tau = 0.9820$ and maturity time $T = 2$.

where $h(z)$ is an exponential Lévy density with parameter δ with mean $\frac{1}{\delta}$, $h(z) = \delta \exp(-z\delta)1_{\{x\geq 0\}}$ and a terminal condition

$$f(T, x, y) = F(x, y) = 1,$$

subject to the following boundary condition

$$\frac{\partial f}{\partial x}(t, 0, y) + \frac{\partial f}{\partial y}(t, 0, y) = 0, \tag{8.23}$$

and a Dirichlet boundary condition

$$f(t, x, y) = g(t, x, y), \ \forall t \in [0, T], x, y \notin \Omega,$$

where $\Omega = [x_{min}, x_{max}] \times [y_{min}, y_{max}]$. First, let us introduce the Kronecker product of two matrices.

Definition 8.2. The Kronecker product of two matrices $A \in \mathbb{R}^{n_1 \times m_1}$, $B \in \mathbb{R}^{n_2 \times m_2}$ is defined as follows

$$C := A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m_1}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m_1}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{n_1,1}B & a_{n_1,2}B & \cdots & a_{n_1,m_1}B \end{bmatrix} \in \mathbb{R}^{n_1 n_2 \times m_1 m_2}.$$

We remark that if A and B are square matrices $m \times m$ then their Kronecker product $A \otimes B$ will also be a square matrix $m^2 \times m^2$.

We define

$$\frac{\partial f}{\partial t} = \mathcal{D}f(t, x, y) + \mathcal{J}f(t, x, y),$$

where

$$\mathcal{D}f(t, x, y) = -\mu \frac{\partial f}{\partial x}(t, x, y) - \mu f(t, 0, y), \ \ \mu < 0, \tag{8.24}$$

and

$$\begin{aligned} \mathcal{J}f(t, x, y) &= -\lambda \int_0^{+\infty} f(t, x+z, y) h(z) dz + \lambda f(t, x, y) \\ &+ \lambda \int_0^{+\infty} f(t, x, y) h(x + dz) dz \\ &= -\mathcal{J}_1 f(t, x, y) + \lambda I_{x,y} + \mathcal{J}_2 f(t, x, y). \end{aligned} \tag{8.25}$$

We start by discretizing (8.24) using finite differences. Note that apart from the boundary condition there is no dependency of the variable y. Let us consider $f(t, x, y) = f(t_n, x_i, y_j) := f^n_{i,j}$. Then

$$\mathcal{D}f_{i,j} = -\mu \frac{f^n_{i+1,j} - f^n_{i-1,j}}{2dx} + \mu^2 \frac{dt}{2dx^2}(f^n_{i+1,j} - 2f^n_{i,j} + f^n_{i-1,j}) - \mu f^n_{0,j},$$

by rewriting the above equation in a matrix form, we have that

$$
\begin{aligned}
A_x(i, i+1) &= -\frac{\mu}{2dx} + \frac{\mu^2 dt}{2dx^2}, \\
A_x(i, i) &= -\frac{\mu^2 dt}{dx^2}, \\
A_x(i, i-1) &= \frac{\mu}{2dx} + \frac{\mu^2 dt}{2dx^2},
\end{aligned}
$$

$$
A_x = \begin{bmatrix} a_0 & b & 0 & 0 \\ c & a & \ddots & 0 \\ 0 & \ddots & \ddots & b \\ 0 & 0 & c & a \end{bmatrix},
$$

where $a_0 = -\mu - \frac{\mu^2 dt}{dx^2}$, $a = -\frac{\mu^2 dt}{dx^2}$, $b = \frac{-\mu}{2dx} + \frac{\mu^2 dt}{2dx^2}$, and $c = \frac{\mu}{2dx} + \frac{\mu^2 dt}{2dx^2}$. The matrix $\mathcal{D} f^n_{i,j}$ can be easily obtained through the Kronecker product, $\mathcal{D} f^n_{i,j} := A$ where $A = I_y \otimes A_x$, and I_y is the identity matrix with respect to the variable y.

Let us proceed with the discretisation of $\mathcal{J} f^n_{i,j}$. Following the same methodology applied above, we get

$$
(\mathcal{J}_1 f)_{i,j} \approx \sum_{k=0}^{K_R} f^n_{i+k,j} h_k,
$$

where $h_k := \int_{(k-\frac{1}{2})dx}^{(k+\frac{1}{2})dx} h(z)dz$. For the grid points outside our domain we apply the boundary condition $f(t, x, y) = g(t, x, y) = 1 \ \forall x, y \notin \Omega$.

Regarding the second integral term $\mathcal{J}_2 f(t, x, y)$, since in the integral there is no dependency on the function $f(t, x, y)$ and if we define $\hat{\lambda} \approx \lambda \int_0^\infty h(x+z)dz$, we obtain $\mathcal{J}_2 f_{i,j} \approx \hat{\lambda} I_x \otimes I_y$.

Again, for the time discretisation, we apply the explicit implicit scheme which leads

$$
\begin{aligned}
(I - dt\mathcal{D}) f^{n+1} &= (I + dt\mathcal{J}) f^n, \\
f^n \mid_{n=T} &= 1.
\end{aligned}
$$

As for the boundary condition, rewriting (8.23), using central differences in space we obtain

$$
\frac{f_{i+1,j} - f_{i-1,j}}{2dx} + \frac{f_{i,j+1} - f_{i,j-1}}{2dy} = 0,
$$

thus for $i = 0$ we have

$$
\frac{f_{1,j} - f_{-1,j}}{2dx} + \frac{f_{0,j+1} - f_{0,j-1}}{2dy} = 0.
$$

The value $f_{-1,j}$ is called "ghost point" and it yields that

$$
f_{-1,j} = f_{1,j} + \frac{dx}{dy}(f_{0,j+1} - f_{0,j-1}). \tag{8.26}
$$

The boundary condition should satisfy the PIDE. Then at $(t_n, x_i = 0, y_j)$ the differences looks like

$$\frac{f_{0,j}^{n+1} - f_{0,j}^{n}}{dt} = -\mu \frac{f_{1,j}^{n+1} - f_{-1,j}^{n+1}}{2dx} + \frac{\mu^2 dt}{2dx^2}(f_{1,j}^{n+1} - 2f_{0,j}^{n+1} + f_{-1,j}^{n+1}) - \mu f_{0,j}^{n+1} + rhs.$$

Substituting (8.26), we obtain

$$\begin{aligned}\frac{f_{0,j}^{n+1} - f_{0,j}^{n}}{dt} &= -\mu \frac{(f_{0,j+1}^{n+1} - f_{0,j-1}^{n+1})}{2dy} + \frac{\mu^2 dt}{2dx^2}\Big(2f_{1,j}^{n+1} - 2f_{0,j}^{n+1} \\ &+ \frac{dx}{dy}(f_{0,j+1}^{n+1} - f_{0,j-1}^{n+1})\Big) - \mu f_{0,j}^{n+1} + rhs.\end{aligned}$$

Thus before we apply the implicit explicit scheme for the time discretisation, the derivatives $\frac{\partial f}{\partial x}\mid_{x=0}$ and $\frac{\partial^2 f}{\partial x^2}\mid_{x=0}$ becomes

$$\mu \frac{\partial f}{\partial x}\Big|_{x=0} \approx \mu \frac{(f_{0,j+1} - f_{0,j-1})}{2dy},$$

and due to Lax-Wendroff approach we have

$$\mu^2 dt \frac{\partial^2 f}{\partial x^2}\Big|_{x=0} \approx \frac{\mu^2 dt}{2dx^2}\left(2f_{1,j} - 2f_{0,j} + \frac{dx}{dy}(f_{0,j+1} - f_{0,j-1})\right).$$

Figure 8.10, provides the graph of the solution of the above PIDE at $t = 0$ zoomed at $x = 1$, with parameters $\mu = -0.1$, $Nx = 150$, $Ny = 150$, $Nt = 375$, $x \in [0, 4]$, $y \in [-2, 2]$, and $T = 1$. Please note that in contrast to the PIDE in the previous section, there is no closed form to investigate the error.

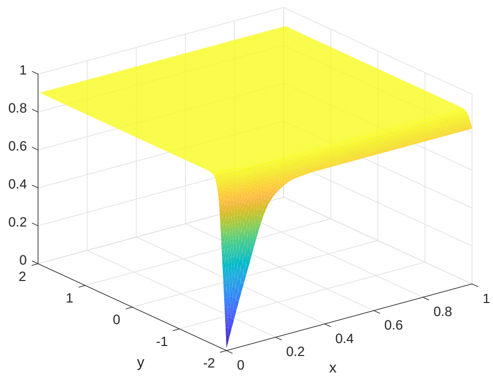
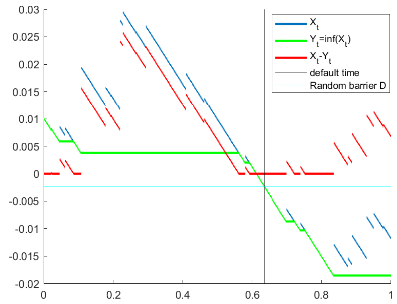


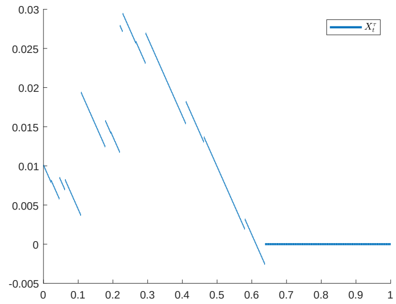
Figure 8.10: Solution of the 3D PIDE (8.22) at $t = 0$ with $\delta = 200$, $\lambda = 20$, $T = 1$ and $\mu = -0.1$.

8.4.1 Simulation results for the local risk minimization approach in $\mathbb{G}$

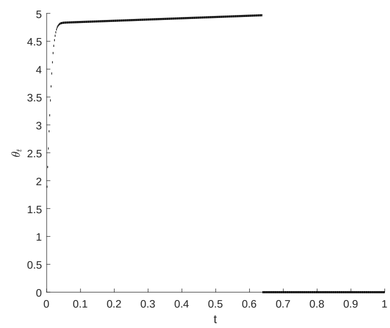
In this section, we provide the simulations for the hedging strategy. Assuming that $(X_t)_{t\geq 0}$ is spectrally positive of finite variation and given that the Lévy measure admits



(a) The underlying asset $(X_t)_{t\geq 0}$, along with its infimum process $(Y_t)_{t\geq 0} = \inf_{s\leq t} X_s$, and the reflected process $(R_t)_{t\geq 0}$, $R_t = X_t - Y_t$, under filtration $\mathbb{F}$.



(b) The underlying asset $(X_t^\tau)_{t\geq 0}$ under filtration $\mathbb{G}$.



(c) The number of shares $(\theta_t)_{t\geq 0}$ under filtration $\mathbb{G}$.

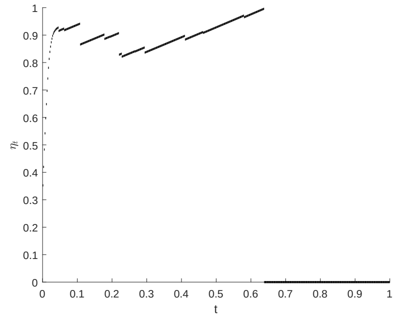
Figure 8.11: A sample path of the underlying asset under $\mathbb{F}$ and $\mathbb{G}$ along with number of assets under $\mathbb{G}$, with default time $\tau = 0.6360$ and $T = 1$.

an exponential Lévy density i.e. $\nu(dz) = \lambda h(z)dz$ such that $h(z) = \delta \exp(-z\delta)1_{\{z\geq 0\}}$, then we have seen in Chapter 6 that the number of shares is given by

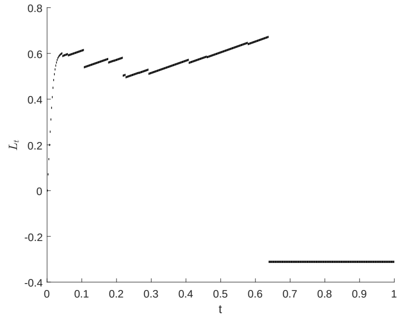
$$\theta_t = \frac{\delta^2 \int_0^{+\infty} z f(t, X_{t-} - \underline{X}_{t-} + z, -\underline{X}_{t-})h(z)dz - \delta f(t, X_{t-} - \underline{X}_{t-}, -\underline{X}_{t-})}{2} 1_{\{\tau\geq t\}}.$$

Figure 8.11 depicts a sample path of the underlying asset $(X_t)_{t\geq 0}$ and its reflected process $R_t = X_t - \underline{X}_t$ under $\mathbb{F}$, along with the underlying asset $(X_t^\tau)_{t\geq 0}$ and the number of shares $(\theta_t)_{t\geq 0}$ under $\mathbb{G}$ with maturity time $T = 1$ and default time $\tau = 0.6360$.

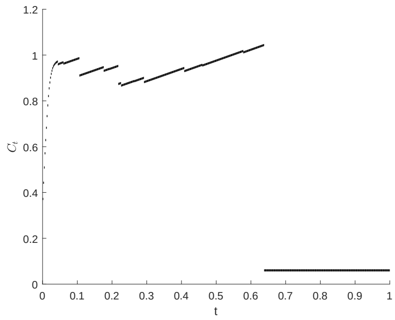
The random barrier D follows an exponential distribution independent of $(X_t)_{t\geq 0}$ with parameter $q = 100$ and mean $\frac{1}{q}$. The probability of default is $\mathbb{P}(\tau \leq 1) = 0.682$. Finally, Figure 8.12 simulates a sample path for the cost process, the portfolio's value, and the amount of money invested in the non-risky asset.



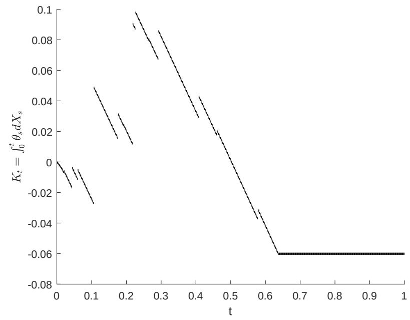
(a) The process $(\eta_t)_{t\geq 0}$.



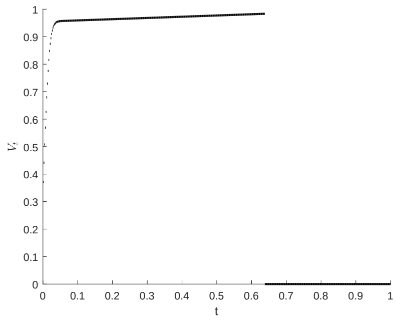
(b) The process $(L_{\tau\wedge t})_{t\geq 0}$.



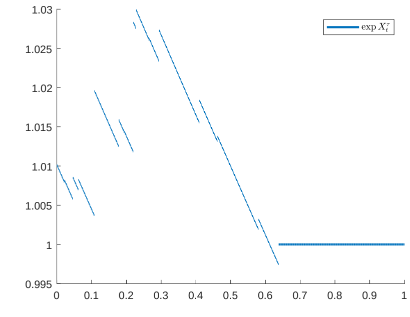
(c) The cost process $(C_t)_{t\geq 0}$.



(d) The process $(K_t)_{t\geq 0}$, $K_t = \int_0^t \theta_s dX_s^\tau$.



(e) The value process $(V_t)_{t\geq 0}$



(f) The exponential asset $\exp(X_t^\tau)_{t\geq 0}$.

Figure 8.12: Sample paths of the cost process, the non risky assets, the value process and the exponential asset with default time $\tau = 0.6360$ and $T = 1$ under filtration $\mathbb{G}$.

8.5 PIDE and PDE for the hazard rate model

In this section, we provide the numerical solution of the PDE and the PIDE introduced in Chapter 7. We begin by solving the corresponding PDE.

8.5.1 Diffusion model

We want to numerically solve the following PDE

$$\frac{\partial h}{\partial t} = \mathcal{A}f(t,x,y), \;\; \forall t \in [0,T] \text{ and } x, y \in \mathbb{R},$$

for an operator $\mathcal{L}f(t,x,y)$ defined as follows

$$\mathcal{A}f(t,x,y) := -\frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}(t,x,y) + \frac{\sigma^2}{2}\frac{\partial h}{\partial x}(t,x,y) + h(t,x,y)g(t,e^{x-y},e^{-y}),$$

along with a terminal condition $h(T,x,y) = \max(e^{x-y} - K, 0)1_{\{e^{-y} \geq b\}}$ and

$$\frac{\partial h}{\partial x}(t,0,x) + \frac{\partial h}{\partial y}(t,0,y) = 0.$$

To descritize the above PDE we also need some additional boundary conditions. In our discretisation, we apply the following boundary conditions

$$h(t,x,y) = \max(\exp(x) - K, 0), \text{ as } y \to 0, \tag{8.27}$$

$$\frac{\partial h}{\partial y}(t,x,y) = 0, \text{ as } y \to -\infty, \;\; \frac{\partial h}{\partial y}(t,x,y) = 0, \text{ as } y \to +\infty, \tag{8.28}$$

$$\frac{\partial h}{\partial x}(t,x,y) = 0, \text{ as } x \to -\infty, \;\; \frac{\partial h}{\partial x}(t,x,y) = 0, \text{ as } x \to +\infty. \tag{8.29}$$

As always the function $h(t,x,y)$ is approximated by $h^n_{i,j} = h(t^n, x_i, y_j)$. For simplicity, we assume that $g(t, e^{x-y}, e^{-y})$ is a positive constant given by g. First, to approximate $\frac{\partial h}{\partial x}(t,x,y)$ and $\frac{\partial^2 h}{\partial x^2}(t,x,y)$ we apply central differences. So at each time level we get $\frac{\partial h}{\partial t} = \mathcal{A}h_{i,j}$ where

$$\mathcal{A}h_{i,j} = -\frac{\sigma^2}{2}\frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j}}{dx^2} + \frac{\sigma^2}{2}\frac{h_{i+1,j} - h_{i-1,j}}{2dx} + gh_{i,j}. \tag{8.30}$$

Alternatively, (8.30) can be rewritten as

$$\mathcal{A}h_{i,j} = ah_{i,j} + bh_{i-1,j} + ch_{i+1,j},$$

where

$$a = \frac{\sigma^2}{dx^2} + g, \;\; b = -\frac{\sigma^2}{2dx^2} - \frac{\sigma^2}{4dx}, \;\; c = -\frac{\sigma^2}{2dx^2} + \frac{\sigma^2}{4dx}.$$

Given an (N_x, N_y) grid point we can construct a matrix A of size $(N_x \times N_y, N_x \times N_y)$ with elements a, b, c introduced above. Note that in our case before we apply the boundary condition, the movements for the variable y is zero.

For the time discretisation we apply the θ-scheme, and so at each time level t^n we have

$$\frac{h^{n+1}_{i,j} - h^n_{i,j}}{dt} = (1-\theta)\mathcal{L}h^{n+1}_{i,j} + \theta\mathcal{L}h^n_{i,j}, \tag{8.31}$$

where for $\theta = 1$ we obtain the explicit scheme for $\theta = 0$ the implicit scheme and finally for $\theta = 0.5$ the Crank-Nicolson method, see Li et al. (2017), Chapter 4 for more details. In this thesis, we use $\theta = 0.5$. Equivalently, (8.31) is formulated in a matrix scheme as

$$(I - dt(1-\theta)A)h^{n+1} = (I + dt\theta A)h^n,$$
$$h^n \mid_{n=T} = \max(\exp(x_i - y_j) - K, 0)1_{\{\exp(-y_j \geq b)\}}.$$

To treat the boundary condition $\frac{\partial h}{\partial x}(t,0,y) + \frac{\partial h}{\partial y}(t,0,y) = 0$ we follow the same steps as in Section 8.4. Therefore at $x = 0$ the partial derivatives $\frac{\partial h}{\partial x} \mid_{x=0}$ and $\frac{\partial^2 h}{\partial x^2} \mid_{x=0}$ can be approximated as

$$\frac{\sigma^2}{2}\frac{\partial h}{\partial x} \mid_{x=0} \approx \frac{\sigma^2}{2}\frac{\frac{dx}{dy}(h_{k_0,j+1} - h_{k_0,j-1})}{2dx},$$

and

$$\frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2} \mid_{x=0} \approx \frac{\sigma^2}{2}\frac{2h_{k_1,j} - 2h_{k_0,j} + \frac{dx}{dy}(h_{k_0,j+1} - h_{k_0,j-1})}{dx^2},$$

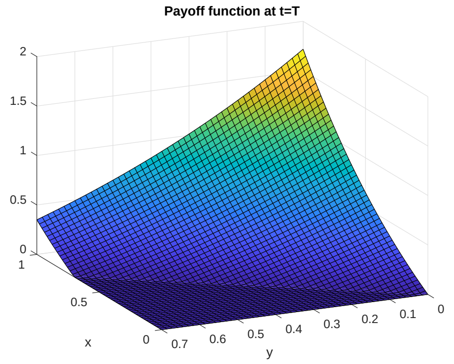
where k_0 is the index representing the value at $x = 0$ and k_1 is the subsequent (from right) index of k_0.

In the same way, we handle the Neumann boundary conditions (8.28) and (8.29). The discretisation of the Dirichlet boundary condition (8.27) is straightforward.

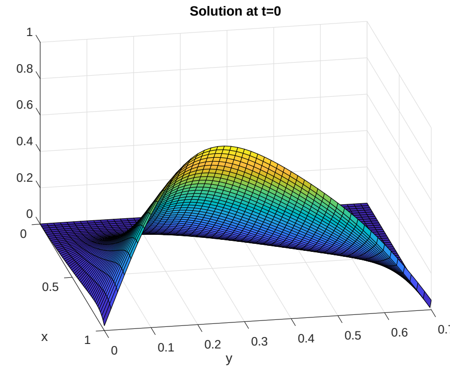
To obtain the solution of the PDE (7.21) for $f(t,x,y)$ we interpolate the solution of $h(t,x,y)$. First, we solve the PDE for $h(t,x,y)$ introduced above when $x \in [-5,5]$, $y \in [-2.4, 4.6]$ and then we apply interpolation $f(t,\tilde{x},\tilde{y}) = h(t, \log(\frac{x}{y}), -\log(y))$ with $\tilde{x} = \tilde{y} \in [0,9]$. Figure 8.14 illustrates the payoff function $f(T,x,y) = \max(x-K,0)1_{\{y \geq b\}}$ along with the solution of the PDE (7.21) at $t = 0$ when $b = 4$ and $K = 5$. We proceed determining the process $(\theta_t)_{t\geq 0}$ for the diffusion model. For simplicity, we assume that there is no default i.e. $g(t,x,y) = 0$. Then $(\theta_t)_{t\geq 0}$ can be obtained based on Theorem 7.18 $\theta_t = \frac{\partial f}{\partial x}(t, Y_t, \underline{Y}_t)$. Using central differences and linear interpolation we obtain:

$$\theta_t = \frac{\partial f}{\partial x}(t, Y_t, \underline{Y}_t) = \frac{f^n_{i+1,j} - f^n_{i-1,j}}{2dx}.$$

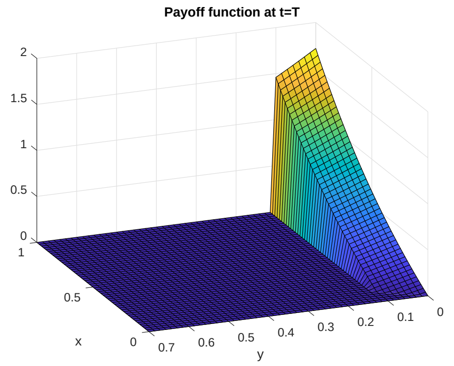
In Figure 8.15, we display the sample paths of the exponential Brownian motion with a drift term μ and the optimal number of shares $(\theta_t)_{t\geq 0}$ for the diffusion PDE when $X_0 = 0.01$, $\mu = 0.1$, $\sigma = 0.2$, $b = 4.0$, maturity time $T = 1$ and a strike price $K = 5.0$. Similar graphs for the value process $(V_t)_{t\geq 0}$ the cost process $(C_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ can be obtained, following the same procedure as we did in Sections 8.3.3 and 8.4.1.



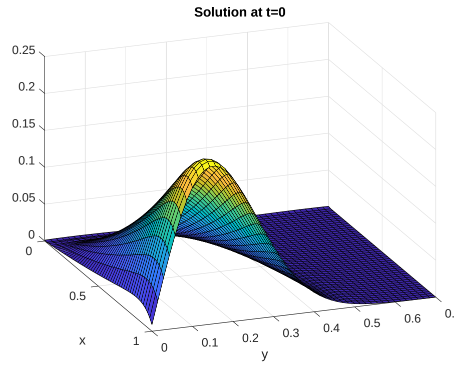
(a) Payoff function $h(T, x, y)$ when $b = 0.2$.



(b) Solution of $h(t, x, y)$ at $t = 0$ when $b = 0.2$.

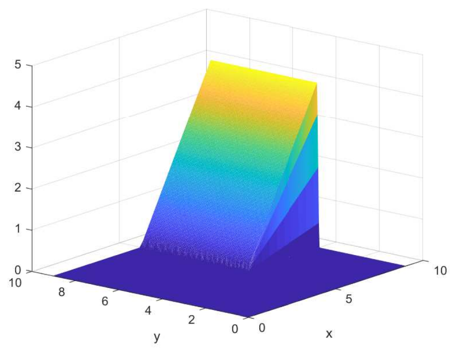


(c) Payoff function $h(T, x, y)$ when $b = 0.9$.

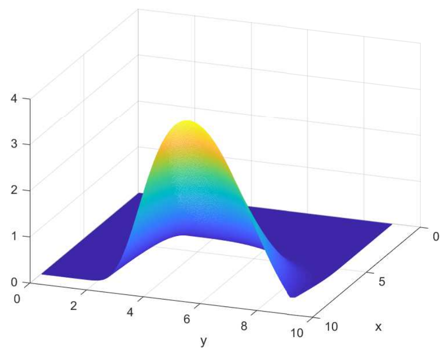


(d) Solution of $h(t, x, y)$ at $t = 0$ when $b = 0.9$.

Figure 8.13: Simulation results for the PDE (7.22) when $g = 0.05$, $x_{min} = y_{min} = 0.0$, $x_{max} = 1.0$, $y_{max} = 0.7$, $\sigma = 0.2$, $N_x = N_y = 70$, $N_t = 200$, $T = 1$ with strike price $K = 1.0$ when $b = 0.2$ and $b = 0.9$.

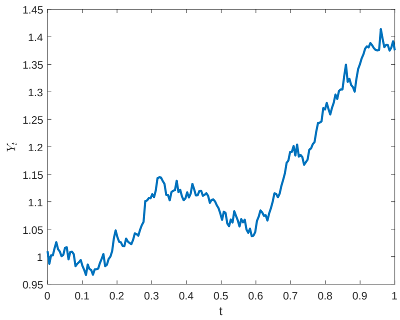


(a) Payoff function $f(T, x, y)$ when $b = 4.0$.

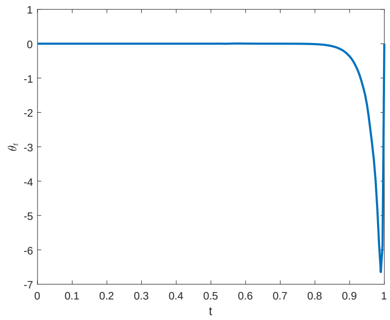


(b) Solution for $f(t, x, y)$ at $t = 0$ when $b = 4.0$.

Figure 8.14: Graphical representation of the PDE (7.21) through interpolation of $h(t, x, y)$ when $x_{max} = y_{max} = 9.0$, $g = 0.05$, $T = 1$, $\sigma = 0.2$ $N_x = N_y = 70$, $N_t = 200$ with strike price $K = 5.0$ when $b = 4.0$.



(a) A sample path of the exponential underlying asset $(Y_t)_{t\geq 0}$ when $\mu = 0.1$ and $\sigma = 0.2$.



(b) A sample of $(\theta_t)_{t\geq 0}$

Figure 8.15: The underlying asset along with the optimal number of shares for the diffusion model when $g(t,x,y) = 0$.

8.5.2 PIDE Jump diffusion case

Once again, we would like to solve numerically the PIDE for $h(t,x,y)$ introduced in Example 8.11

$$\begin{aligned}
&\frac{\partial h}{\partial t}(t,x,y) = \mathcal{L}h(t,x,y), \quad t \in [0,T), x \in \mathbb{R} \text{ and } y \in \mathbb{R}, \\
&h(T,x,y) \to F(x,y) = \max(x-K,0)1_{\{e^{-y}\geq b\}} \text{ as } t \to T,
\end{aligned}$$

along with the boundary condition $\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0$, where

$$\begin{aligned}
&\mathcal{L}h(t,x,y) = -(\beta + \frac{\sigma^2}{2})\frac{\partial h}{\partial x}(t,x,y) - \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}(t,x,y) + h(t,x,y)g(t,e^{x-y},e^{-y}) \\
&- \int_{-x}^{+\infty}(h(t,x+z,y) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\nu(dz) \\
&- \int_{-\infty}^{-x}(h(t,x+z,x+z) - h(t,x,y) - z\frac{\partial h}{\partial x}(t,x,y))\nu(dz).
\end{aligned} \tag{8.32}$$

As always we assume that $\nu(dz) < \infty$, therefore the Lévy measure admits a Lévy density of the form $\nu(dz) = \lambda q(z)dz$, where

$$q(z \mid \mu_J, \sigma_J) = \frac{1}{\sqrt{2\pi\sigma_J^2}} e^{-\frac{(x-\mu_J)^2}{2\sigma_J^2}}.$$

Therefore (8.32) becomes

$$\begin{aligned}
&\mathcal{L}h(t,x,y) = -(\beta + \frac{\sigma^2}{2} + a_1(x) + a_2(x))\frac{\partial h}{\partial x}(t,x,y) - \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}(t,x,y) \\
&+ h(t,x,y)(g(t,e^{x-y},e^{-y}) + c_1(x) + c_2(x)) \\
&- \lambda \int_{-x}^{+\infty} h(t,x+z,y)q(z)dz - \lambda \int_{-\infty}^{-x} h(t,x+z,x+z)q(z)dz,
\end{aligned} \tag{8.33}$$

where

$$a_1(x) = \lambda \int_{-x}^{+\infty} zq(z)dz, \quad a_2(x) = \lambda \int_{-\infty}^{-x} zq(z)dz,$$
$$c_1(x) = \lambda \int_{-x}^{+\infty} q(z)dz, \quad c_2(x) = \lambda \int_{-\infty}^{-x} q(z)dz,$$

along with $h(t,x,y) = p(t,x,y)$ for all $x, y \notin \Omega$, with $\Omega = [x_{min}, x_{max}] \times [y_{min}, y_{max}]$. Since our payoff is a knock out barrier option, following Cont and Voltchkova (2005), we assume that $p(t,x,y) = 0$ for all $x, y \notin \Omega$.

We split the above operator $\mathcal{L}h(t,x,y)$ in (8.33) as $\mathcal{L}h(t,x,y) = \mathcal{D}h(t,x,y) + \mathcal{J}h(t,x,y)$ where

$$\begin{aligned}\mathcal{D}h(t,x,y) &= -(\beta + \frac{\sigma^2}{2} + a_1(x) + a_2(x))\frac{\partial h}{\partial x}(t,x,y) - \frac{\sigma^2}{2}\frac{\partial^2 h}{\partial x^2}(t,x,y) \\ &+ h(t,x,y)(g(t,e^{x-y},e^{-y}) + c_1(x) + c_2(x)),\end{aligned} \tag{8.34}$$

and

$$\begin{aligned}\mathcal{J}h(t,x,y) = -\lambda \int_{-x}^{+\infty} h(t,x+z,y)q(z)dz - \lambda \int_{-\infty}^{-x} h(t,x+z,x+z)q(z)dz \\ = -\lambda \mathcal{J}_1 h(t,x,y) - \lambda \mathcal{J}_2 h(t,x,y).\end{aligned} \tag{8.35}$$

For the discretisation of the operator $\mathcal{D}h(t,x,y)$ in (8.34), we apply the same methodology as in Section 8.5.1. It remains to descritize the integral terms in (8.35). Following Section 8.3.2, it yields

$$\begin{aligned}\mathcal{J}_1 h(t,x,y) &= \int_{-x}^{+\infty} h(t,x+z,y)q(z)dz = \int_{\mathbb{R}} h(t,x+z,y)1_{[-x,+\infty)}q(z)dz \\ &\approx \int_{B_L}^{B_R} h(t,x_i+z,y)1_{[-x_i,B_R]}q(z)dz \approx \sum_{k=K_L}^{K_R} h_{i+k,j}1_{[-x_i,K_R]}q_k,\end{aligned}$$

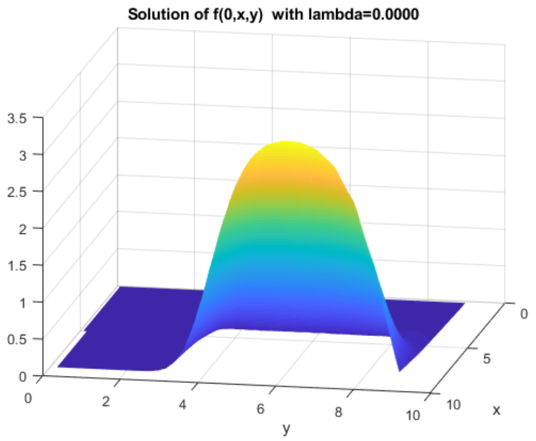
Similarly, for the second integral term we get that

$$\begin{aligned}\mathcal{J}_2 h(t,x,y) &= \int_{-\infty}^{-x} h(t,x+z,x+z)q(z)dz \approx \int_{B_L}^{B_R} h(t,x_i+z,x_i+z)1_{[B_L,-x_i]}q(z)dz \\ &\approx \sum_{k=K_L}^{K_R} h_{i+k,i+k}1_{[K_L,-x_i]}q_k,\end{aligned}$$

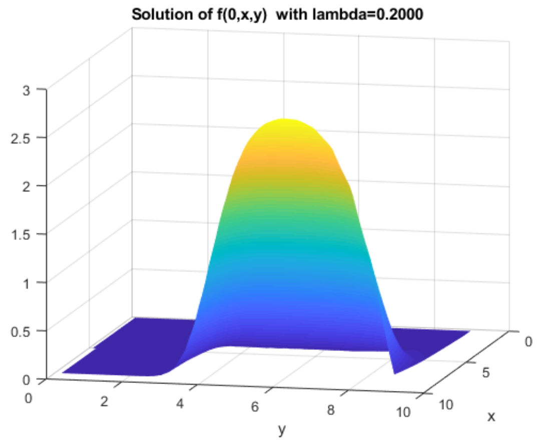
with $q_k = \int_{(k-\frac{1}{2})dx}^{(k+\frac{1}{2})dx} h(z)dz$. The bounds B_L and B_R can be derived following similar calculations of Section 8.3.1. More precisely,

$$q(z) \geq \epsilon \Leftrightarrow -\mu_J - \sqrt{-2\sigma_J^2 \log(\epsilon\sigma_J\sqrt{2\pi}/2)} \leq z \leq \mu_J + \sqrt{-2\sigma_J^2 \log(\epsilon\sigma_J\sqrt{2\pi}/2)}.$$

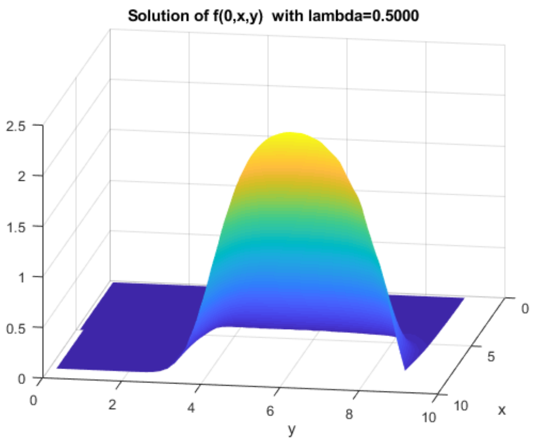
Thus $B_L = -\mu_J - \sqrt{-2\sigma_J^2 \log(\epsilon\sigma_J\sqrt{2\pi}/2)}$, and $B_R = -B_L$. Also K_L and K_R are such that $[B_L, B_R] \subseteq [(K_L - 0.5)dx, (K_R + 0.5)dx]$. See also Chan and Hubbert (2010).



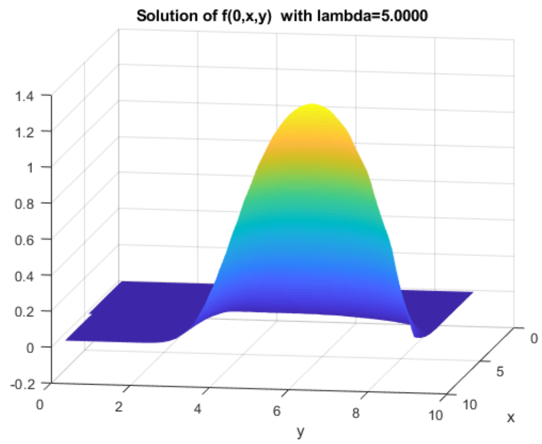
(a) Solution of $f(t, x, y)$ at $t = 0$ when $\lambda = 0$.



(b) Solution of $f(t, x, y)$ at $t = 0$ when $\lambda = 0.2$.



(c) Solution of $f(t, x, y)$ at $t = 0$ when $\lambda = 0.5$.



(d) Solution of $f(t, x, y)$ at $t = 0$ when $\lambda = 5$.

Figure 8.16: The solution of the PIDE of $f(t, x, y)$ at $t = 0$ with various values of λ.

The boundary condition $\frac{\partial h}{\partial x}(t, 0, y) + \frac{\partial h}{\partial y}(t, 0, y) = 0$ is descritized in the same way as in the diffusion case. For the time discretisation, we apply the implicit explicit scheme as in the previous sections.

In Figure 8.16, we provide the solution of the PIDE for $f(t, x, y)$ by solving the above PIDE of $h(t, x, y)$ for various values of λ and then applying linear interpolation with the following data $\beta = 0.1$, $\sigma = 0.2$, $b = 4$, $\mu_J = 0.05$, $\sigma_J = 0.1$ with maturity time $T = 1$ and strike price $K = 5$.

A trajectory of the number of shares $(\theta_t)_{t \geq 0}$ can be derived following similar results as in Section 8.4.1.

Chapter 9

Conclusion and future work

The last chapter of this thesis is devoted on providing a thesis summary along with some future work. We describe our main contributions along with their limitations.

9.1 Thesis summary

The main purpose of this dissertation is to provide semi-explicit solutions of LRM hedging strategies in defaultable markets under some circumstances including partial information to some extent. In this work, the LRM is analysed when the underlying asset is a Lévy process or its exponential. Let us highlight the main parts of this thesis.

In Chapter 4, we discussed the LRM when the default time is modelled through a structural model whose barrier is constant. In our analysis we used finite variation Lévy processes. The hedging strategy is determined by applying the Itô's formula for non-smooth functions. This provides us the flexibility to work with PIDEs whose solutions are continuous but not necessarily smooth in the strong sense.

In Chapter 6, we extend the model discussed in Chapter 4. More specifically, we determine the GKW decomposition given that the default time is modelled through a structural model whose barrier follows a negative exponential distribution. In this case, the underlying asset is a spectrally positive and of finite variation Lévy process.

In Chapter 7, we developed a model through a different technique based on hazard rate approach. Given that the underlying asset is an exponential jump diffusion process, we proposed a new model whose default time is defined through a hazard rate model that involves the running infimum process. Furthermore, we provided examples where claims are prone both to structural and exogenous defaults, and hence a unified framework for both structural and reduced form modelling. Then we determined the hedging strategy which has a semi-explicit form based on the LRM approach through PDEs and PIDEs.

Our analysis is made under the physical measure. We determine the KW decomposition and equivalently the GKW decomposition through appropriate PIDEs. In Chapter 8 we solved numerically the PIDE's through finite differences. The integral terms are descritized through the trapezoidal rule.

9.2 Future work

In this thesis, our future work is going to be focused on Chapters 4, 6, 7 and 8.

The models introduced in Chapters 4, 6 can be extended in various ways. Firstly, in our analysis we develop a hedging strategy for a defaultable claim with a simple payoff function. We need to apply our methodology for more general contingent claims, such as a European contingent claim, whose payoff at the maturity depends on the underlying asset and a strike price. Furthermore, our analysis in these chapters is made with Lévy processes of finite variation. It would be interesting to extend it to general case for the jump diffusion processes. Chapter 7 can be served as a platform to tackle these problems.

Our work in Chapter 6 is based on spectrally positive processes of finite variation. Based on $\Delta X_\tau = 0$ this allow us to obtain a PIDE which can be solved easily. However, working with these processes in finance is not always very popular though there are some applications already. It is worthwhile considering to extend our analysis for spectrally negative processes. The main difficulty here is how we will obtain the process $\left(\int_0^{\tau\wedge t} \Delta f(s, X_s - \underline{X}_s, -\underline{X}_s) d(1_{\{\tau \leq s\}})\right)_{t\geq 0}$. Having this result and combining the spectrally positive and negative processes, we are able to derive the general finite variation case.

The hazard rate model introduced in Chapter 7 can be used to analyse financial-insurance derivatives, see Vandaele and Vanmaele (2009) and Ceci et al. (2015a). In these models, the default is defined through human mortality and our model could be implemented, as one can assume that human mortality is independent of market asset values.

The partial information models that we analysed concern the default time. Alternatively, we may assume that the underlying asset is partially observed. In this setup, the non-linear filtering theory plays fundamental role. Generally speaking, in the filtering theory approach the process $(X_t)_{t\geq 0}$ is not observed directly, however there exists a process $(Y_t)_{t\geq 0}$ correlated to $(X_t)_{t\geq 0}$. The filtration generated by the process $(Y_t)_{t\geq 0}$, $\mathbb{F}^Y = \sigma(Y_s)_{0\leq s\leq t}$ provides the available information.

The main goal of the filtering approach is to specify the conditional distribution π_t of the unobservable process $(X_t)_{t\geq 0}$ given $\mathbb{F}^Y$. Ceci and Colaneri (2012) investigated the non-linear filtering problem when $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are jump diffusion processes with common jumps. In their analysis, the determination of the conditional distribution π_t is made through the solution of the Kushner-Stratonovich stochastic differential equation. By solving an appropriate PIDE, we believe that their model can be used to determine the hedging strategy through the LRM approach when the underlying asset is partially observed.

From a theoretical point of view, we saw that the intensity process of a reduced form model describes the credit spread of a defaultable claim. It is important if we can compare our intensity models introduced in this thesis using a real data and to investigate their fitness.

In the present work, the FS decomposition is obtained without using minimal martingale measures. Instead we obtain it by solving appropriate PIDEs. Determining hedging strategies through a minimal martingale measure is worth investigating. Under a unique flow of information the GKW decomposition analysed in Chapter 4 can be obtained through the minimal martingale measure introduced in Choulli et al. (2010). Furthermore, based on Ceci et al. (2015b) the partial information models introduced in Chapters 6 and 7 can also be analysed through a minimal martingale measure.

One major drawback of the LRM is that we are working with mean self-financing i.e. the cost process is a local martingale. An intuitive explanation of this is that a contingent claim might have intermediate and unexpected profits or losses. On the other hand, the mean variance approach does not have this restriction, but we no longer work with admissible strategies. It is worth considering, whether we are able to apply the mean variance approach in our models by introducing an appropriate mean variance optimal measure, see Kohlmann et al. (2010). Note that under the martingale case both approaches are the same.

Our analysis is devoted to a single (one-dimensional) underlying asset. A useful extension can be made to the multidimensional setting. This can be achieved based on Schweizer (2008), who has already developed the theory of LRM for multidimensional assets. A quite interesting result is to establish our construction when there is a correlation between the default times or in the underlying assets. In this framework, Lévy copulas should be applied.

More complicated and useful numerical methods studying the PIDEs need further developing. Our simulations are made by using compound Poisson processes i.e. when the Lévy measure has finite activity. Dealing with infinite activity Lévy processes such as variance gamma processes is more challenging and financially more interesting. The algorithm for this case is well established and already known see Cont and Tankov (2004) Chapter 12, Cont and Voltchkova (2005) and Almendral and Oosterlee (2007).

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Appendix A

A martingale associated with the reflected Lévy process at its infimum: jump diffusion case.

This chapter consists a basic and a quite useful result for the reflected Lévy process at its infimum.This result will be used in Chapter 7. More precisely, for $0 \leq S \leq \infty$ and a $C^{1,2,1}([0,S] \times \mathbb{R}_0^+ \times \mathbb{R})$ function under some circumstances we obtain the canonical decomposition of $(f(t, X_t - \underline{X}_t, -\underline{X}_t))_{0 \leq t \leq S}$.

Assumption A.1. *Let $0 \leq S \leq \infty$ and $f : [0,S] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ be a $C^{1,2,1}([0,S] \times \mathbb{R}_0^+ \times \mathbb{R})$ function. Then $f(t,x,y)$ satisfies the following integrability conditions*

$$\int_{-x}^{+\infty} |f(t, x+z, y) - f(t,x,y) - z\frac{\partial f}{\partial x}(t,x,y)|\nu(dz) < \infty,$$

and

$$\int_{-\infty}^{-x} |f(t, 0, y-x-z) - f(t,x,y) - z\frac{\partial f}{\partial x}(t,x,y)|\nu(dz) < \infty,$$

for every $0 \leq t \leq S$, $x \geq 0$ and $y \in \mathbb{R}$.

Proposition A.2. *Let $(X_t)_{t\geq 0}$ be a Lévy process $X_t = X_0 + \mu t + \sigma W_t + \sum_{s\leq t} \Delta X_s$ with $\beta = \mathbb{E}[X_1 - X_0]$ and its Lévy measure ν is absolutely continuous with respect to Lebesgue measure. Let $f(t,x,y) : [0,S] \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ be a $C^{1,2,1}([0,T] \times \mathbb{R}_0^+ \times \mathbb{R})$ function satisfying Assumption A.1, with $0 \leq S \leq \infty$ then for an operator $\mathcal{L}^* f(t,x,y)$ is given as*

$$\begin{aligned} \mathcal{L}^* f(t,x,y) &= \frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{-x}^{+\infty} (f(t,x+z,y) - f(t,x,y) - z\frac{\partial f}{\partial x})\nu(dz) \\ &\quad + \int_{-\infty}^{-x} (f(t,0,y-z-x) - f(t,x,y) - z\frac{\partial f}{\partial x})\nu(dz), \end{aligned}$$

for all $t \in [0,S]$, $x \geq 0$ and $y \in \mathbb{R}$, and if the boundary condition

$$\frac{\partial f}{\partial x}(t,0,y) + \frac{\partial f}{\partial y}(t,0,y) = 0,$$

is satisfied, then the process

$$\left(f(t,X_t-\underline{X}_t,-\underline{X}_t)-f(0,X_0-\underline{X}_0,-\underline{X}_0)-\int_0^t\mathcal{L}^*f(s,X_s-\underline{X}_s,-\underline{X}_s)ds\right)_{t\geq 0},$$

is an $\mathbb{F}-$local martingale.

Proof. By Theorem 2 of Whitney (1934), we can smoothly extend f to $\mathbb{R}_0^+\times\mathbb{R}\times\mathbb{R}$. Since $(X_t)_{t\geq 0}$ is a Lévy process by applying Itô's formula for semimartingales with $0\leq t\leq S$, it yields

$$\begin{aligned}f(t,X_t-\underline{X}_t,-\underline{X}_t)&=f(0,X_0-\underline{X}_0,-\underline{X}_0)+\int_0^t\frac{\partial f}{\partial s}(s,X_s-\underline{X}_s,-\underline{X}_s)ds\\&+\int_0^t\frac{\partial f}{\partial x}(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})d\underline{X}_s+\frac{1}{2}\int_0^t\frac{\partial^2 f}{\partial x^2}(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})d[X-\underline{X}]_s^c\\&-\frac{1}{2}\int_0^t\frac{\partial^2 f}{\partial y^2}(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})d[\underline{X}]_s^c+\frac{1}{2}\int_0^t\frac{\partial^2 f}{\partial x\partial y}(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})d[X-\underline{X}]_s^c\\&+\sum_{s\leq t}\Big\{f(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})-f(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})\\&+(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y})(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})\Delta\underline{X}_s-\frac{\partial f}{\partial x}(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})\Delta X_s\Big\},\end{aligned}$$

where $\Delta X_s = X_s - X_{s-}$, $\Delta\underline{X}_s = \underline{X}_s-\underline{X}_{s-}$ and $(X_t^c)_{t\geq 0}$, $(\underline{X}_t^c)_{t\geq 0}$ are respectively the continuous local martingale part of $(X_t)_{t\geq 0}$ and the path-by-path continuous part of $(\underline{X}_t)_{t\geq 0}$. Since $(\underline{X}_t)_{t\geq 0}$ is of finite variation then $[\underline{X}]_t^c=[X-\underline{X},-\underline{X}]_t^c=0$. Using the support property $X_{t-}=X_t=\underline{X}_{t-}=\underline{X}_t$ on $supp(d\underline{X}^c)$, the term $\int_0^t\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)(s,0,-\underline{X}_s)d(\underline{X})_s^c$ is eliminated using the boundary condition $\frac{\partial f}{\partial x}(t,0,y)+\frac{\partial f}{\partial y}(t,0,y)=0$,. Based on this and since $[X]_t^c=\sigma^2$, the above result can be rewritten as follows

$$\begin{aligned}f(t,X_t-\underline{X}_t,-\underline{X}_t)&=f(0,X_0-\underline{X}_0,-\underline{X}_0)+\int_{0+}^t\frac{\partial f}{\partial s}(s,X_s-\underline{X}_s,-\underline{X}_s)ds\\&+\frac{\sigma^2}{2}\int_0^t\frac{\partial^2 f}{\partial x^2}(s,X_s-\underline{X}_s,-\underline{X}_s)ds+\beta\int_0^t\frac{\partial f}{\partial x}(s,X_s-\underline{X}_s,-\underline{X}_s)ds\\&+\sum_{s\leq t}\Big\{f(s,X_s-\underline{X}_s,-\underline{X}_s)-f(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})-\frac{\partial f}{\partial x}(s,X_{s-}-\underline{X}_{s-},-\underline{X}_{s-})\Delta X_s\Big\}.\end{aligned}$$

Suppose that z defines the jumps of $(X_t)_{t\geq 0}$ at time t i.e. $z=\Delta X_t$ and let $(R_t)_{t\geq 0}$, $R_t=\underline{X}_t-X_t$ We note that $(\underline{X}_t)_{t\geq 0}$ can be written as

$$\underline{X}_t=\min\left(X_t,\underline{X}_{t-}\right), \tag{A.1}$$

and we investigate the following cases

- If $z\leq R_{t-}$or $X_t-X_{t-}\leq\underline{X}_{t-}-X_{t-}\implies X_t\leq\underline{X}_{t-}$ for all t, then from Equation (A.1), $\min(X_t,\underline{X}_{t-})=X_t$, which implies that $X_t=\underline{X}_t$ and so $R_t=0$.
- If $z>R_{t-}$or $X_t-X_{t-}\geq\underline{X}_{t-}-X_{t-}\implies X_t\geq\underline{X}_{t-}$ and from Equation (A.1), we

have that $\min(X_t, \underline{X}_{t-}) = \underline{X}_{t-}$ and thus $\underline{X}_t = \underline{X}_{t-}.\bar{X}_t = \bar{X}_{t-}$. In this case the process $(R_t)_{t\geq 0}$ becomes $R_t = \underline{X}_{t-} - X_t$, and if we add and subtract X_{t-} then $R_t = \underline{X}_{t-} - X_{t-} - (X_t - X_{t-}) = R_{t-} - z$.

The above summation can be expressed as an integral using jump measure $N(dt, dz)$. That is

$$
\begin{aligned}
f(t, X_t - \underline{X}_t, -\underline{X}_t) &= f(0, X_0, -\underline{X}_0, -\underline{X}_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s - \underline{X}_{s-}, -\underline{X}_s)ds \\
&+\frac{\sigma^2}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s - \underline{X}_s, -\underline{X}_s)ds + \beta\int_0^t \frac{\partial f}{\partial x}(s, X_s - \underline{X}_s, -\underline{X}_s)ds \\
&+\int_0^t\int_{-\infty}^{+\infty}\Big(f(s, 0, X_{s-} - z) - f(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \\
&-z\frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -X_{s-})\Big)1_{\{z\leq R_{s-}\}}N(ds, dz) \\
&+\int_0^t\int_{-\infty}^{+\infty}\Big(f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \\
&-z\frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})\Big)1_{\{z>R_{s-}\}}N(ds, dz).
\end{aligned}
$$

Then using Lemma 6.6 and assuming that $\tilde{N}(dt, dz)$ is the compensated jump measure, $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$, we have

$$
\begin{aligned}
f(t, X_t - \underline{X}_t, -\underline{X}_t) &= f(0, X_0 - \underline{X}_0, -\underline{X}_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s - \underline{X}_s, -\underline{X}_s)ds \\
&+\frac{\sigma^2}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s - \underline{X}_s, -\underline{X}_s)ds + \beta\int_0^t \frac{\partial f}{\partial x}(s, X_s - \underline{X}_s, -\underline{X}_s)ds \\
&+\int_0^t\int_{-\infty}^{+\infty}\Big(f(s, 0, X_{s-} - z) - f(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \\
&-z\frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})\Big)1_{\{z\leq R_{s-}\}}\nu(dz)ds \\
&+\int_0^t\int_{-\infty}^{+\infty}\Big(\Big(f(s, 0, X_{s-} - z) - f(s, X_s - \underline{X}_{s-}, -\underline{X}_s) \\
&-z\frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})\Big)1_{\{z\leq R_s\}}\tilde{N}(ds, dz) \\
&+\int_0^t\int_{-\infty}^{+\infty}\Big(f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \\
&+z\frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})\Big)1_{\{z>R_{s-}\}}\nu(dz)ds \\
&+\int_0^t\int_{-\infty}^{+\infty}\Big(f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \\
&-z\frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-})\Big)1_{\{z>R_{s-}\}}\tilde{N}(ds, dz).
\end{aligned}
$$

Since the terms

$$\int_0^t \int_{-\infty}^{+\infty} \Big(f(s, 0, X_{s-} - z) - f(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \\ -z \frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \Big) 1_{\{z \le R_{s-}\}} \tilde{N}(ds, dz),$$

and

$$\int_0^t \int_{-\infty}^{+\infty} \Big(f(s, X_{s-} - \underline{X}_{s-} + z, -\underline{X}_{s-}) - f(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \\ -z \frac{\partial f}{\partial x}(s, X_{s-} - \underline{X}_{s-}, -\underline{X}_{s-}) \Big) 1_{\{z > R_{s-}\}} \tilde{N}(ds, dz),$$

are $\mathbb{F}$-local martingales, then by the continuity of the Lévy measure and if we define the operator

$$\mathcal{L}^* f(t, x, y) := \frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} + \int_{-x}^{+\infty} (f(t, x + z, y) - f(t, x, y) - z \frac{\partial f}{\partial x}) \nu(dz) \\ + \int_{-\infty}^{-x} (f(t, 0, y - z - x) - f(t, x, y) - z \frac{\partial f}{\partial x}) \nu(dz),$$

for all $t \in [0, S]$, $x \ge 0$ and $y \in \mathbb{R}$, it follows that the process

$$\left(f(t, X_t - \underline{X}_t, -\underline{X}_t) - f(X_0 - \underline{X}_0, -\underline{X}_{0-}) - \int_0^t \mathcal{L}^* f(s, X_s - \underline{X}_s, -\underline{X}_s) ds \right)_{t \ge 0},$$

with $t \in [0, S]$, is an $\mathbb{F}$-local martingale. □

Appendix B

Projection formulas

In this chapter, we briefly analyze the optional and predictable projections. The following Theorem describes the optional and predictable projection. For its proof, we refer to Dellacherie and Meyer (1982), Chapter VI.

Theorem B.1 (Optional and predictable projection). *Let $(X_t)_{t\geq 0}$ be a positive or bounded measurable process. Then there exists an optional process $(Y_t)_{t\geq 0}$ and a predictable process $(Z_t)_{t\geq 0}$ such that*

$$\mathbb{E}[X_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_\tau] = Y_\tau 1_{\{\tau<\infty\}} \text{ a.s for every stopping time } \tau, \tag{B.1}$$

$$\mathbb{E}[X_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_{\tau-}] = Z_\tau 1_{\{\tau<\infty\}} \text{ a.s for every predictable stopping time } \tau. \tag{B.2}$$

The processes $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq}$ are unique within an evanescent set. They are called the optional projection and predictable projection of $(X_t)_{t\geq 0}$ and they are denoted by $(^\circ X_t)_{t\geq 0}$ and $(^p X_t)_{t\geq 0}$.

Let us investigate some fundamental properties of the predictable and optional projection. Following Dellacherie and Meyer (1982) we have the following Remark.

Remark B.2. The optional and predictable projections have the following properties:

1. Let $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ be positive or bounded measurable processes with optional projections $(Y_t)_{t\geq 0}$, $(\tilde{Y}_t)_{t\geq 0}$ and predictable projections $(Z_t)_{t\geq 0}$,$(\tilde{Z}_t)_{t\geq 0}$. Assuming that $X_t \leq \tilde{X}_t$ a.s for all t, then in this case we have $Y_t \leq \tilde{Y}_t$ and $Z_t \leq \tilde{Z}_t$ a.s for all t. Moreover, given a positive constant K and if $|X_t| \leq K$ uniformly for all t then $|Y_t| \leq K$, and $|Z_t| \leq K$.

   Finally, if $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ are two positive or bounded processes introduced above, then for their linear combination $(aX_t + b\tilde{X}_t)_{t\geq 0}$ admits an optional projection $(aY_t + b\tilde{Y}_t)_{t\geq 0}$ and its predictable projection is $(aZ_t + b\tilde{Z}_t)_{t\geq 0}$, for all a, $b \in \mathbb{R}$.

2. In discrete case with the convection that $\mathcal{F}_{-1} = \mathcal{F}_0$, the optional projection is $Y_n = \mathbb{E}[X_n \mid \mathcal{F}_n]$ and the predictable projection $Z_n = \mathbb{E}[X_n \mid \mathcal{F}_{n-1}]$, $n \geq 0$.

3. In order to prove B.1 and B.2, it suffices to show without conditioning $\mathbb{E}[X_\tau 1_{\{\tau<\infty\}}] = \mathbb{E}[Y_\tau 1_{\{\tau<\infty\}}]$ for every stopping time τ. Similarly, for the predictable projection $\mathbb{E}[X_\tau 1_{\{\tau<\infty\}}] = \mathbb{E}[Z_\tau 1_{\{\tau<\infty\}}]$ for every predictable stopping time τ.

4. If $(X_t)_{t\geq 0}$ is a measurable process and $(H_t)_{t\geq 0}$ is a bounded and optional process (respectively predictable) then

$$^\circ(HX)_t = H_t \, {}^\circ X_t \ \ (\text{respectively } {}^p(HX)_t = H_t \, {}^p X_t).$$

5. For the case when $(X_t)_{t\geq 0}$ is not bounded or positive we may define the optional projection as follows: Assume that $(X_t)_{t\geq 0}$ is a measurable process. Then its optional projection exists, if the optional projection of the positive measurable process $(|X|_t)_{t\geq 0}$ is indistinguishable [1] from a finite process. In this case we have

$$Y_t =^\circ X_t =^\circ X_t^+ -^\circ X_t^- .$$

6. Let $(X_t)_{t\geq 0}$ be an adapted measurable process. Then there exists an optional process $(Y_t)_{t\geq 0}$ such that it is a modification [2]of $(X_t)_{t\geq 0}$.

The next Theorem provides the relation between the two projection. For its proof we refer to Dellacherie and Meyer (1982), Chapter VI.

Theorem B.3. *Let (X_t) be a positive or bounded measurable process. The set of (t,ω) such that $^\circ X_t(\omega) \neq^p X_t(\omega)$ is a countable union of graphs of stopping times.*

The next Theorem is essential for the definition of the dual projection. For its proof ,see Dellacherie and Meyer (1982), Chapter VI, Theorem 57.

Theorem B.4. *Assume that $(X_t)_{t\geq 0}$ is a positive and measurable process and let $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ be its optional and predictable projection. Let $(A_t)_{t\geq 0}$ be an increasing process, then*

$$\mathbb{E}\left[\int_{[0,\infty)} X_s dA_s\right] = \mathbb{E}\left[\int_{[0,\infty)} Y_s dA_s\right],$$

and if $(A_t)_{t\geq 0}$ is predictable then

[1] We say that two stochastic process $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are indistinguishable if and only if

$$\mathbb{P}(X_t = Y_t \text{ for all t} \in \mathbb{R}_+) = 1.$$

[2] Two processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are a modification of each other if

$$\mathbb{P}(X_t = Y_t) = 1 \text{ for all } t \in \mathbb{R}_+.$$

$$\mathbb{E}\left[\int_{[0,\infty)} X_s dA_s\right] = \mathbb{E}\left[\int_{[0,\infty)} Z_s dA_s\right].$$

Having Theorem B.4 we can define the dual projection as follows.

Definition B.5 (Dual projection)**.** Let $(A_t)_{t\geq 0}$ be a raw [3] integrable increasing process. Then the optional (respectively predictable) dual projection of $(A_t)_{t\geq 0}$ is the optional (respectively predictable) increasing process $(B_t)_{t\geq 0}$ defined by

$$\mathbb{E}\left[\int_{[0,\infty)} X_s dB_s\right] = \mathbb{E}\left[\int_{[0,\infty)} {}^{\circ}X_s dB_s\right],$$

and for the case of predictable projection

$$\mathbb{E}\left[\int_{[0,\infty)} X_s dB_s\right] = \mathbb{E}\left[\int_{[0,\infty)} {}^{p}X_s dB_s\right].$$

[3] An increasing process not necessarily $\mathbb{F}$-adapted but whose paths are positive increasing and right continuous is called raw process.

Appendix C

Derivatives approximation through the Euler scheme

In this chapter, we express the derivatives in space using the Euler method.

Forward difference

From Taylor series we know that

$$f(x) = f(x_0) + (x - x_0)\frac{df}{dx} + \frac{(x - x_0)^2}{2!}\left(\frac{d^2 f}{dx^2}\right)\Big|_{x_0}.$$

Assume that we want to compute the derivative $\frac{df}{dx}$ at a given point $x = x_i$. For simplicity, we assume that $f(x) = f_i$ at $x = x_i$. Therefore, for

$$x = x_i,$$
$$x - x_0 = dx,$$
$$f(x) = f(x_0 + dx) = f_{i+1}.$$

Then the value of f_{i+1} using the Taylor scheme can be written as

$$f_{i+1} = f_i + dx\left(\frac{df}{dx}\right)_i + \frac{(dx)^2}{2!}\left(\frac{d^2 f}{dx^2}\right)_i + \frac{(dx)^3}{3!}\left(\frac{d^3 f}{dx^3}\right)_i. \quad \text{(C.1)}$$

Solving with respect to $\frac{df}{dx}$ we have

$$\left(\frac{df}{dx}\right) = \frac{f_{i+1} - f_i}{dx} - \underbrace{\frac{dx}{2!}\left(\frac{d^2 f}{dx^2}\right) - \frac{dx^3}{3!}\left(\frac{d^3 f}{dx^3}\right)}_{\mathcal{O}(dx)},$$

and therefore we get

$$\left(\frac{df}{dx}\right)_i = \frac{f_{i+1} - f_i}{dx} + \mathcal{O}(dx).$$

Backward difference

Similarly to the forward difference, backward difference can be obtained by using the Taylor series expansion of $f(x)$ at $x = x_{i-1}$ which yields

$$f_{i-1} = f_i - dx\left(\frac{df}{dx}\right)_i + \frac{dx^2}{2!}\left(\frac{d^2 f}{dx^2}\right)_i - \frac{dx^3}{3!}\left(\frac{d^3 f}{dx^3}\right), \tag{C.2}$$

or

$$\left(\frac{df}{dx}\right)_i = \frac{f_i - f_{i-1}}{dx} + \underbrace{\frac{dx}{2}\left(\frac{d^2 f}{dx^2}\right)_i - \frac{dx^2}{3!}\left(\frac{d^3 f}{dx^3}\right)_i}_{\mathcal{O}(dx)}.$$

Therefore

$$\left(\frac{df}{dx}\right)_i = \frac{f_i - f_{i-1}}{dx} + \mathcal{O}(dx).$$

Central difference

The forward and backward differences are first order accurate approximations of the first derivative. On the other hand, central difference is a second order accurate method. In order to obtain this formula, by subtracting (C.1) from (C.2), we have

$$f_{i+1} - f_{i-1} = 2dx\left(\frac{df}{dx}\right)_i + 2\frac{dx^3}{3!}\left(\frac{d^3 f}{dx^3}\right)_i.$$

Hence

$$\left(\frac{df}{dx}\right)_i = \frac{f_{i+1} - f_{i-1}}{2dx} + \underbrace{2\frac{dx^2}{3!}\left(\frac{d^3 f}{dx^3}\right)_i}_{\mathcal{O}(dx^2)},$$

or equivalently,

$$\left(\frac{df}{dx}\right)_i = \frac{f_{i+1} - f_{i-1}}{2dx} + \mathcal{O}(dx^2).$$