# Astronautics Equation Booklet 2021a

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# Nomenclature

**Greek Symbols**

| | |
|---|---|
| $\alpha, \vec{\alpha}$ | Angular acceleration |
| $\gamma$ | Flight path angle |
| $\epsilon$ | Specific energy |
| $\varepsilon$ | Emissivity |
| $\varepsilon_0$ | Vacuum permittivity |
| $\theta$ | Sidereal time |
| $\theta$ | True anomaly |
| $\kappa$ | Curvature |
| $\lambda$ | Longitude |
| $\mu$ | Standard gravitational parameter |
| $\mu_0$ | Vacuum permeability |
| $\rho$ | Density |
| $\rho$ | Range |
| $\sigma$ | Stefan-Boltzmann constant |
| $\tau$ | Orbital period |
| $\phi$ | Latitude |
| $\Omega$ | Right ascension of ascending node |
| $\omega$ | Argument of perigee |
| $\omega, \vec{\omega}$ | Angular velocity |
| $\omega_E$ | Earth sidereal angular velocity |

**Latin Symbols**

| | |
|---|---|
| $A$ | Area |
| $A$ | Azimuth |
| $a$ | Altitude |
| $a$ | Semi-major axis |
| $a, \vec{a}$ | Acceleration |
| $\vec{B}$ | Magnetic field |
| $B$ | Ballistic coefficient |
| $C$ | Circumference |
| $c$ | Speed of light |
| $C_J$ | Jacobi energy |
| $\boldsymbol{E}$ | Identity matrix |
| $\vec{E}$ | Electric field |
| $E$ | Eccentric anomaly |
| $E$ | Energy |
| $e$ | Eccentricity |
| $e$ | Elementary charge |
| $F, \vec{F}$ | Force |
| $G$ | Gravitational constant |
| $g_0$ | Earth gravity |
| $h$ | Planck constant |
| $H, \vec{H}$ | Angular momentum |
| $h, \vec{h}$ | Specific angular momentum |
| $\boldsymbol{I}$ | Inertia tensor |

$i$ Inclination
$J$ Flux
$k_B$ Boltzmann constant
$L$ Luminosity
$m$ Mass
$m$, $\vec{m}$ Magnetic moment
$m_e$ Electron mass
$m_n$ Neutron mass
$m_p$ Proton mass
$n$ Jacobi energy
$N_A$ Avogadro number
$p$ Semi-latus rectum
$r, \vec{r}$ Position
$r_a$ Apogee radius
$R_E$ Earth radius
$r_p$ Perigee radius
$r_{SOI}$ Sphere of influence
$S$ Surface area
$T$ Temperature
$T, \vec{T}$ Torque
$u$ Atomic mass unit
$V$ Volume
$v$, $\vec{v}$ Velocity

**Abbreviations**

AOP Argument of perigee
au Astronomical unit
RAAN Right ascension of ascending node

# Introduction

## Notation

In the remainder of this booklet the following notation is used for mathematical symbols:

**scalars:** upper or lower case latin or greek letters: $x, \Omega$,
**vectors:** an arrow over lower case latin or greek letters: $\vec{x}, \vec{\Omega}$,
**matrices:** bold, upper case latin or greek letters: $\mathbf{X}, \mathbf{\Omega}$.

Unless specified otherwise, vectors are column vectors with their components denoted by the same symbol as the vector and a subscript indicating the index starting at 1 for the first component:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

Similarly, the entries of a matrix are denoted by the same symbol as the matrix itself with a subscript indicating first the row and then the column of the entry starting at 1:

$$\mathbf{X} = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{pmatrix}.$$

Angles in the following equations and as arguments of trigonometric functions are in radians unless otherwise noted.

CHAPTER 1

# Constants

## 1.1. Mathematical Constants

$$\pi = 4 \arctan 1 \approx 3.1415926535897932$$

$$e = \exp 1 \approx 2.7182818284590452$$

$$i = \sqrt{-1}$$

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887498948$$

## 1.2. Physical Constants

$$c = 299\,792\,458\ \frac{\mathrm{m}}{\mathrm{s}} \approx 3 \times 10^{8}\ \frac{\mathrm{m}}{\mathrm{s}} \qquad \text{(Speed of light)}$$

$$g_0 = 9.806\,65\ \frac{\mathrm{m}}{\mathrm{s}^2} \approx 9.81\ \frac{\mathrm{m}}{\mathrm{s}^2} \qquad \text{(Earth gravity)}$$

$$G = 6.674\,30 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2} \qquad \text{(Gravitational constant)}$$

$$h = 6.626\,069\,3 \times 10^{-34}\ \mathrm{J\,s} \qquad \text{(Planck constant)}$$

$$k_B = 1.380\,649 \times 10^{-23}\ \frac{\mathrm{J}}{\mathrm{K}} \qquad \text{(Boltzmann constant)}$$

$$\sigma = 5.670\,367 \times 10^{-8}\ \frac{\mathrm{W}}{\mathrm{m}^2\,\mathrm{K}^4} \qquad \text{(Stefan-Boltzmann constant)}$$

$$N_A = 6.022\,141\,5 \times 10^{23}\,\frac{1}{\text{mol}} \qquad \text{(Avogadro number)}$$

$$u = 1.660\,56 \times 10^{-27}\,\text{kg} \qquad \text{(Atomic mass unit)}$$

$$m_e = 9.109\,389\,7 \times 10^{-31}\,\text{kg} \qquad \text{(Electron mass)}$$

$$m_p = 1.672\,623\,1 \times 10^{-27}\,\text{kg} \qquad \text{(Proton mass)}$$

$$m_n = 1.674\,928\,6 \times 10^{-27}\,\text{kg} \qquad \text{(Neutron mass)}$$

$$e = 1.602\,176\,53 \times 10^{-19}\,\text{C} \qquad \text{(Elementary charge)}$$

$$\mu_0 = 4\pi \times 10^{-7}\,\frac{\text{T}^2\,\text{m}^3}{\text{J}} \approx 12.566\,370\,614 \times 10^{-7}\,\frac{\text{T}^2\,\text{m}^3}{\text{J}} \qquad \text{(Vacuum permeability)}$$

$$\varepsilon_0 = \frac{1}{c^2\mu_0} \approx 8.854\,187\,817 \times 10^{-12}\,\frac{\text{C}^2}{\text{J}\,\text{m}} \qquad \text{(Vacuum permittivity)}$$

$$0\,\text{K} = -273.15\,^\circ\text{C} \qquad \text{(Absolute zero)}$$

## 1.3. Astronomical Constants

$$1\,\text{au} = 149\,597\,870\,700\,\text{m} \approx 1.496 \times 10^8\,\text{km} \qquad \text{(Astronomical unit)}$$

### Time

$$\begin{aligned} 1\,\text{Julian day} = 1\,\text{d} &= 86\,400\,\text{s} & 1\,\text{Julian year} &= 365.25\,\text{d} \\ & & 1\,\text{Gregorian year} &= 365.2425\,\text{d} \\ & & 1\,\text{mean tropical year} &\approx 365.2422\,\text{d} \\ 1\,\text{sidereal day} &\approx 86\,164\,\text{s} & 1\,\text{sidereal year} &\approx 365.26\,\text{d} \end{aligned}$$

### Solar System

Approximate gravitational parameters (in $\text{km}^3\,\text{s}^{-2}$):

$$\begin{aligned} \mu_{Mercury} &= 2.203 \times 10^4 & \mu_{Venus} &= 3.249 \times 10^5 & \mu_{Earth} &= 3.986 \times 10^5 \\ \mu_{Mars} &= 4.283 \times 10^4 & \mu_{Jupiter} &= 1.267 \times 10^8 & \mu_{Saturn} &= 3.793 \times 10^7 \\ \mu_{Uranus} &= 5.794 \times 10^6 & \mu_{Neptune} &= 6.837 \times 10^6 \\ \mu_{Moon} &= 4.905 \times 10^3 & \mu_{Sun} &= 1.327 \times 10^{11} \end{aligned}$$

Approximate mean body radii (in km):

$$\begin{aligned} R_{Mercury} &= 2440 & R_{Venus} &= 6052 & R_{Earth} = R_E &= 6371 \\ R_{Mars} &= 3390 & R_{Jupiter} &= 69\,911 & R_{Saturn} &= 58\,232 \\ R_{Uranus} &= 25\,362 & R_{Neptune} &= 24\,622 \\ R_{Moon} &= 1737 & R_{Sun} &= 695\,700 \end{aligned}$$

Approximate semi-major axes (in au unless otherwise noted):

$$\begin{aligned} r_{Mercury} &= 0.387 & r_{Venus} &= 0.723 & r_{Earth} &= 1.000 \\ r_{Mars} &= 1.523 & r_{Jupiter} &= 5.205 & r_{Saturn} &= 9.582 \\ r_{Uranus} &= 19.20 & r_{Neptune} &= 30.05 \\ r_{Moon} &= 384\,400\,\text{km} \end{aligned}$$

Approximate orbital eccentricities:

$$e_{Mercury} = 0.205 \qquad e_{Venus} = 0.007 \qquad e_{Earth} = 0.017$$
$$e_{Mars} = 0.094 \qquad e_{Jupiter} = 0.049 \qquad e_{Saturn} = 0.057$$
$$e_{Uranus} = 0.046 \qquad e_{Neptune} = 0.011$$
$$e_{Moon} = 0.055$$

### Sun

Approximate luminosity (radiation power):

$$L_{sun} = 3.828 \times 10^{26}\,\mathrm{W}$$

Approximate solar flux at Earth:

$$J_{Earth} = \frac{L_{sun}}{4\pi\,(1\,\mathrm{au})^2} \approx 1361\,\frac{\mathrm{W}}{\mathrm{m}^2}$$

### Earth

Sidereal angular velocity:

$$\omega_E = 2\pi\left(1 + \frac{1}{365.26}\right)\frac{\mathrm{rad}}{\mathrm{d}}$$
$$\approx 72.9217 \times 10^{-6}\,\frac{\mathrm{rad}}{\mathrm{s}}$$

Approximate geopotential coefficients (JGM-3):

| | | |
|---|---|---|
| $J_2 = 1.082\,64 \times 10^{-3}$ | $J_{2,1} = 1.561\,87 \times 10^{-9}$ | $J_{2,2} = 1.815\,53 \times 10^{-6}$ |
| | $\lambda_{2,1} = 1.725\,98$ | $\lambda_{2,2} = -5.211\,23 \times 10^{-1}$ |
| $J_3 = -2.532\,44 \times 10^{-6}$ | $J_{3,1} = 2.209\,12 \times 10^{-6}$ | $J_{3,2} = 3.744\,09 \times 10^{-7}$ |
| | $\lambda_{3,1} = 0.121\,62$ | $\lambda_{3,2} = -0.599\,99$ |
| | $J_{3,3} = 2.213\,60 \times 10^{-7}$ | |
| | $\lambda_{3,3} = 1.099\,24$ | |
| $J_4 = -1.619\,33 \times 10^{-6}$ | $J_{4,1} = 6.788\,34 \times 10^{-7}$ | $J_{4,2} = 1.676\,26 \times 10^{-7}$ |
| | $\lambda_{4,1} = -2.417\,97$ | $\lambda_{4,2} = 1.084\,02$ |
| | $J_{4,3} = 6.042\,16 \times 10^{-8}$ | $J_{4,4} = 7.644\,80 \times 10^{-9}$ |
| | $\lambda_{4,3} = -0.200\,12$ | $\lambda_{4,4} = 2.118\,73$ |

to be used with exactly

$$R_E = 6378.1363\,\mathrm{km} \qquad \mu = 398\,600.4415\,\frac{\mathrm{km}^3}{\mathrm{s}^2}$$

CHAPTER 2

# Mathematics

## 2.1. Geometry

### 2D shapes

Circle of radius $r$:

$$r^2 = x^2 + y^2 \qquad \kappa = \frac{1}{r}$$

$$A = \pi r^2 \qquad C = 2\pi r$$

Arc with radius $r$ and angle $\alpha$:

$$A = \frac{\alpha}{2} r^2 \qquad C = \alpha r$$

Ellipse with semi-major axis $a$ and semi-minor axis $b$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \kappa = \frac{1}{a^2 b^2} \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} \right)^{-3/2}$$

$$A = \pi ab \qquad h = \left( \frac{a-b}{a+b} \right)^2$$

$$\begin{aligned} C &= 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 t}\, dt \\ &\approx \pi (a+b) \left( 1 + \frac{h}{10 + \sqrt{4-h}} \right) \qquad \text{(Ramanujan)} \end{aligned}$$

Triangle with base $b$ and height $h$:

$$A = \frac{1}{2}bh$$

Parallelogram with base $b$ and height $h$:

$$A = bh$$

### 3D shapes

Sphere with radius $r$:

$$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

Right cylinder with base radius $r$ and height $h$:

$$V = \pi r^2 h \qquad S = 2\pi r\,(r + h)$$

Cone with base radius $r$ and height $h$:

$$V = \frac{1}{3}\pi r^2 h \qquad S = \pi r\left(r + \sqrt{h^2 + r^2}\right)$$

Regular tetrahedron with side length $l$:

$$V = \frac{l^3}{6\sqrt{2}} \qquad S = \sqrt{3}l^2 \qquad h = \sqrt{\frac{2}{3}}l$$

Square pyramid with base length $l$ and height $h$:

$$V = \frac{1}{3}l^2 h \qquad S = l^2 + \sqrt{l^2 + 4h^2}$$

## 2.2. Trigonometry

### Identities

Pythagorean theorem:

$$\sin^2\alpha + \cos^2\alpha = 1$$

Symmetries:

$$\sin\left(\alpha \pm \frac{\pi}{2}\right) = \pm\cos\alpha \qquad \cos\left(\alpha \pm \frac{\pi}{2}\right) = \mp\sin\alpha$$
$$\sin(\alpha + \pi) = -\sin\alpha \qquad \cos(\alpha + \pi) = -\cos\alpha$$
$$\sin(-\alpha) = -\sin\alpha \qquad \cos(-\alpha) = \cos\alpha$$
$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha \qquad \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$$
$$\sin(\pi - \alpha) = \sin\alpha \qquad \cos(\pi - \alpha) = -\cos\alpha$$

$$\tan\left(\alpha \pm \frac{\pi}{4}\right) = \frac{\tan\alpha \pm 1}{1 \mp \tan\alpha}$$
$$\tan\left(\alpha + \frac{\pi}{2}\right) = -\frac{1}{\tan\alpha}$$
$$\tan(-\alpha) = -\tan\alpha$$
$$\tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan\alpha}$$
$$\tan(\pi - \alpha) = -\tan\alpha$$

Exponential definition:

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \qquad \arcsin\alpha = -i\ln\left(i\alpha + \sqrt{1-\alpha^2}\right)$$
$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \qquad \arccos\alpha = -i\ln\left(\alpha + \sqrt{\alpha^2-1}\right)$$
$$\tan\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{i\left(e^{i\alpha} + e^{-i\alpha}\right)} \qquad \arctan\alpha = \frac{i}{2}\ln\left(\frac{i+\alpha}{i-\alpha}\right)$$

Angle sum and difference:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Multiple angle:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha \qquad \cos(2\alpha) = 1 - 2\sin^2\alpha$$
$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$
$$\sin(3\alpha) = -4\sin^3\alpha + 3\sin\alpha \qquad \cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha$$
$$\tan(3\alpha) = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha}$$

Power reduction:

$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2} \qquad \cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$
$$\sin^3\alpha = \frac{3\sin\alpha - \sin(3\alpha)}{4} \qquad \cos^3\alpha = \frac{3\cos\alpha + \cos(3\alpha)}{4}$$

Product to sum:

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$
$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

Sum to product:

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Linear combination:

$$a \sin \alpha + b \cos \alpha = c \sin(\alpha + \varphi)$$

$$c = \sqrt{a^2 + b^2} \qquad \varphi = \arctan \frac{b}{a}$$

### Planar trigonometry

Given a planar triangle with sides $a, b, c$ and corresponding angles $A, B, C$.

Law of cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Law of sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

### Spherical trigonometry

Given a spherical triangle with sides $a, b, c$ and corresponding angles $A, B, C$.

Law of cosines:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Complementary law of cosines:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$
$$\cos B = -\cos C \cos A + \sin C \sin A \cos b$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

Law of sines:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

## 2.3. Vector Algebra

### Vector arithmetic

$$\begin{aligned} \vec{x} + \vec{y} &= \vec{y} + \vec{x} \\ a\left(\vec{x} + \vec{y}\right) &= a\vec{x} + a\vec{y} \\ \vec{x} + \left(\vec{y} + \vec{z}\right) &= \left(\vec{x} + \vec{y}\right) + \vec{z} \end{aligned}$$

### Vector norm

$$\begin{aligned} |\vec{x}| &= \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\ |c\vec{x}| &= |c|\,|\vec{x}| \\ |\vec{x} + \vec{y}| &\leqslant |\vec{x}| + |\vec{y}| \qquad \text{(Triangle inequality)} \end{aligned}$$

### Vector inner product (dot product)

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = \sum_i x_i y_i$$

where $\vec{x}^T \vec{y}$ is interpreted as a matrix multiplication.

Properties of the inner product:

$$\begin{aligned} \vec{x} \cdot \vec{y} &= \vec{y} \cdot \vec{x} \\ (a\vec{x}) \cdot \vec{y} &= \vec{x} \cdot (a\vec{y}) = a\left(\vec{x} \cdot \vec{y}\right) \\ (\vec{x} + \vec{y}) \cdot \vec{z} &= \vec{x} \cdot \vec{z} + \vec{y} \cdot \vec{z} \end{aligned}$$

### Vector outer product

$$
\begin{aligned}
\vec{x} \otimes \vec{y} &= \vec{x}\vec{y}^T \\
&= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \begin{pmatrix} y_1 & y_2 & y_3 & \ldots \end{pmatrix} \\
&= \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 & \\ x_2 y_1 & x_2 y_2 & x_2 y_3 & \ldots \\ x_3 y_1 & x_3 y_2 & x_3 y_3 & \\ & \vdots & & \ddots \end{pmatrix}
\end{aligned}
$$

where $\vec{x}\vec{y}^T$ is interpreted as a matrix multiplication.

### Vector cross product

$$
\vec{x} \times \vec{y} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}
$$

As a skew symmetric matrix:

$$
\vec{x} \times \vec{y} = \mathbf{X}_\times \cdot \vec{y}
$$

where

$$
\mathbf{X}_\times = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}
$$

Properties of the cross product:

$$
\begin{aligned}
\vec{x} \times \vec{y} &= -\vec{y} \times \vec{x} \\
(a\vec{x}) \times \vec{y} &= \vec{x} \times (a\vec{y}) = a\,(\vec{x} \times \vec{y}) \\
(\vec{x} + \vec{y}) \times \vec{z} &= \vec{x} \times \vec{z} + \vec{y} \times \vec{z} \\
\vec{x} \times \vec{x} &= 0 \\
\vec{x} \cdot (\vec{x} \times \vec{y}) &= \vec{y} \cdot (\vec{x} \times \vec{y}) = 0
\end{aligned}
$$

**Vector identities**

$$
\begin{aligned}
\vec{x} \cdot (\vec{y} \times \vec{z}) &= \vec{y} \cdot (\vec{z} \times \vec{x}) = \vec{z} \cdot (\vec{x} \times \vec{y}) \\
&= \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \\
\vec{x} \times (\vec{y} \times \vec{z}) &= (\vec{x} \cdot \vec{z})\,\vec{y} - (\vec{x} \cdot \vec{y})\,\vec{z} && \text{(Triple product)} \\
(\vec{w} \times \vec{x}) \cdot (\vec{y} \times \vec{z}) &= (\vec{w} \cdot \vec{y})\,(\vec{x} \cdot \vec{z}) - (\vec{x} \cdot \vec{y})\,(\vec{w} \cdot \vec{z}) \\
& && \text{(Binet-Cauchy identity)} \\
|\vec{x} \times \vec{y}|^2 &= (\vec{x} \times \vec{y}) \cdot (\vec{x} \times \vec{y}) \\
&= (\vec{x} \cdot \vec{x})\,(\vec{y} \cdot \vec{y}) - (\vec{x} \cdot \vec{y})^2 && \text{(Lagrange's identity)}
\end{aligned}
$$

**Vector calculus**

Nabla operator in Cartesian coordinates:

$$
\nabla = \begin{pmatrix} \frac{d}{dx_1} \\ \frac{d}{dx_2} \\ \frac{d}{dx_3} \end{pmatrix}
$$

Gradient of a scalar function of three variables $V : \mathbb{R}^3 \to \mathbb{R}$:

$$
\nabla V = \begin{pmatrix} \frac{dV}{dx_1} \\ \frac{dV}{dx_2} \\ \frac{dV}{dx_3} \end{pmatrix}
$$

Divergence of a vector field in three dimensions $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\nabla \cdot \vec{F} = \begin{pmatrix} \frac{d}{dx_1} \\ \frac{d}{dx_2} \\ \frac{d}{dx_3} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \frac{dF_1}{dx_1} + \frac{dF_2}{dx_2} + \frac{dF_3}{dx_3}$$

Curl of a vector field in three dimensions $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\nabla \times \vec{F} = \begin{pmatrix} \frac{d}{dx_1} \\ \frac{d}{dx_2} \\ \frac{d}{dx_3} \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{dF_3}{dx_2} - \frac{dF_2}{dx_3} \\ \frac{dF_1}{dx_3} - \frac{dF_3}{dx_1} \\ \frac{dF_2}{dx_1} - \frac{dF_1}{dx_2} \end{pmatrix}$$

## 2.4. Matrices

Given $n \times m$ matrix $\boldsymbol{A}$, transpose $\boldsymbol{A}^T$ and conjugate $\boldsymbol{A}^*$:

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & & \\ \vdots & & \ddots & \\ a_{1m} & & & a_{nm} \end{pmatrix} \qquad \boldsymbol{A}^* = \begin{pmatrix} \overline{a}_{11} & \overline{a}_{21} & \dots & \overline{a}_{n1} \\ \overline{a}_{12} & \overline{a}_{22} & & \\ \vdots & & \ddots & \\ \overline{a}_{1m} & & & \overline{a}_{nm} \end{pmatrix}$$

$$(\boldsymbol{A} \cdot \boldsymbol{B})^T = \boldsymbol{B}^T \cdot \boldsymbol{A}^T \qquad (\boldsymbol{A} \cdot \boldsymbol{B})^* = \boldsymbol{B}^* \cdot \boldsymbol{A}^*$$

$$(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T \qquad (\boldsymbol{A} + \boldsymbol{B})^* = \boldsymbol{A}^* + \boldsymbol{B}^*$$

$$(c\boldsymbol{B})^T = c\boldsymbol{B}^T \qquad (c\boldsymbol{B})^* = c\boldsymbol{B}^*$$

Matrix product of $n \times m$ matrix $\boldsymbol{A}$ with $m \times p$ matrix $\boldsymbol{B}$:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & & \\ \vdots & & \ddots & \\ c_{n1} & & & c_{np} \end{pmatrix} \qquad c_{ij} = \sum_{s=1}^{m} a_{is} b_{sj}$$

Matrix vector product of $n \times m$ matrix $\boldsymbol{A}$ with $m$ vector $\vec{x}$:

$$\boldsymbol{A} \cdot \vec{x} = \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad\qquad y_i = \sum_{s=1}^{m} a_{is} x_s$$

**Square matrices**

Symmetric, skew-symmetric, Hermitian matrix:

$$\boldsymbol{A} = \boldsymbol{A}^T \qquad\qquad \boldsymbol{A} = -\boldsymbol{A}^T \qquad\qquad \boldsymbol{A} = \boldsymbol{A}^*$$

Identity matrix:

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

Trace:

$$\mathrm{tr} \boldsymbol{A} = \sum_{i=1}^{n} a_{ii}$$

$$\mathrm{tr} \boldsymbol{A} = \mathrm{tr} \boldsymbol{A}^T \quad \mathrm{tr}\left(\boldsymbol{A} \cdot \boldsymbol{B}\right) = \mathrm{tr}\left(\boldsymbol{B} \cdot \boldsymbol{A}\right) \quad \mathrm{tr}\left(\boldsymbol{A} + \boldsymbol{B}\right) = \mathrm{tr} \boldsymbol{A} + \mathrm{tr} \boldsymbol{B}$$

Determinant:

$$\det \boldsymbol{A} = |\boldsymbol{A}| \quad \det\left(\boldsymbol{A} \cdot \boldsymbol{B}\right) = \det \boldsymbol{A} \det \boldsymbol{B} \quad \det\left(\boldsymbol{A}^{-1}\right) = \frac{1}{\det \boldsymbol{A}}$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Matrix inverse:

$$\boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{E}$$

$$(\boldsymbol{A} \cdot \boldsymbol{B})^{-1} = \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1} \qquad (c\boldsymbol{A})^{-1} = \frac{1}{c}\boldsymbol{A}^{-1}$$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \boldsymbol{A}^{\boldsymbol{-1}} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & \\ \vdots & & \ddots & \\ 0 & & & a_n \end{pmatrix} \qquad \boldsymbol{A}^{\boldsymbol{-1}} = \begin{pmatrix} \frac{1}{a_1} & 0 & \dots & 0 \\ 0 & \frac{1}{a_2} & & \\ \vdots & & \ddots & \\ 0 & & & \frac{1}{a_n} \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\boldsymbol{A}\vec{v} = \lambda\vec{v} \qquad \det(\boldsymbol{A} - \lambda\boldsymbol{E}) = 0$$

## 2.5. Derivatives

Given functions $f(x), g(x)$ and constants $a, b$.

Linearity:

$$\frac{d}{dx}(af + bg) = a\frac{df}{dx} + b\frac{dg}{dx}$$

Product rule:

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$$

Chain rule:

$$\frac{d}{dx}f(g(x)) = \left.\frac{df}{dx}\right|_{g(x)} \frac{dg}{dx}$$

Common derivatives:

| $f(x)$ | $\frac{d}{dx}f(x)$ | $f(x)$ | $\frac{d}{dx}f(x)$ |
|---|---|---|---|
| $ax+b$ | $a$ | $x^n$ | $nx^{n-1}$ |
| $\exp x$ | $\exp x$ | $a^x$ | $a^x \ln a$ |
| $\ln x$ | $\frac{1}{x}$ | $\log_b x$ | $\frac{1}{x \ln b}$ |
| $\sin x$ | $\cos x$ | $\cos x$ | $-\sin x$ |
| $\tan x$ | $\frac{1}{\cos^2 x}$ | $\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos x$ | $-\frac{1}{\sqrt{1-x^2}}$ | $\arctan x$ | $\frac{1}{1+x^2}$ |
| $\sinh x$ | $\cosh x$ | $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\frac{1}{\cosh^2 x}$ | $\operatorname{arcsinh} x$ | $\frac{1}{\sqrt{1+x^2}}$ |
| $\operatorname{arccosh} x$ | $\frac{1}{\sqrt{1-x^2}}$ | $\operatorname{arctanh} x$ | $\frac{1}{1-x^2}$ |

## 2.6. Integrals

Given functions $f(x), g(x)$, indefinite integrals $F = \int f\,dx, G = \int g\,dx$ and constants $a, b, c$.

Fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Definite integrals:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Linearity:

$$\int af(x) + bg(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

Integration by parts:

$$\int f(x)\frac{d}{dx}g(x)\,dx = f(x)g(x) - \int g(x)\frac{d}{dx}f(x)\,dx$$

Integration by substitution $y = g(x)$:

$$\int_a^b f(g(x))\frac{d}{dx}g(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

Common integrals:

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $ax + b$ | $\frac{1}{2}ax^2 + bx$ |
| $x^n$ | $\frac{1}{n+1}x^{n+1}$ |
| $\exp x$ | $\exp x$ |
| $a^x$ | $\frac{a^x}{\ln a}$ |
| $x \exp x$ | $(x-1)\exp x$ |
| $\ln x$ | $x \ln x - x$ |
| $\frac{1}{x}$ | $\ln \lvert x \rvert$ |
| $\frac{1}{ax+b}$ | $\frac{1}{a}\ln \lvert ax+b \rvert$ |
| $\sin x$ | $-\cos x$ |
| $\cos x$ | $\sin x$ |
| $\tan x$ | $\ln \left\lvert \frac{1}{\cos x} \right\rvert$ |
| $\arcsin x$ | $x \arcsin x + \sqrt{1-x^2}$ |
| $\arccos x$ | $x \arccos x - \sqrt{1-x^2}$ |

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $\arctan x$ | $x\arctan x - \frac{1}{2}\ln\left(1+x^2\right)$ |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\arcsin\frac{x}{a}$ |
| $\frac{1}{a^2+x^2}$ | $\frac{1}{a}\arctan\frac{x}{a}$ |
| $\frac{1}{a^2-x^2}$ | $\frac{1}{2a}\ln\left\|\frac{x+a}{x-a}\right\|$ |
| $\sinh x$ | $\cosh x$ |
| $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\ln\cosh x$ |

## 2.7. Special functions

**Associated Legendre polynomials:**

$$P_{l,m}(x) = \frac{1}{2^l l!}\left(1-x^2\right)^{m/2}\frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l$$

CHAPTER 3

# Newtonian Mechanics

## 3.1. Point Masses

Kinematics of position, velocity and acceleration (inertial frame):

$$\vec{r}(t) \qquad \vec{v}(t) = \frac{d\vec{r}}{dt} \qquad \vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$$

Dynamics (inertial frame):

$$\vec{a}(t) = \frac{\vec{F}(t)}{m}$$

Kinetic energy:

$$E_{kin} = \frac{1}{2}mv^2$$

Angular momentum:

$$\vec{H} = m\vec{r} \times \vec{v}$$

### Newtonian Gravitation

Gravitational potential of point mass $m_1$ at $\vec{r}_1$:

$$V(\vec{r}) = -\frac{Gm_1}{|\vec{r} - \vec{r}_1|} = -\frac{\mu_1}{|\vec{r} - \vec{r}_1|}$$

Acceleration and force on point mass $m_2$ at $\vec{r}_2$:

$$\vec{a}(\vec{r}_2) = -\nabla V(\vec{r}_2) = -\frac{Gm_1}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1)$$
$$\vec{F}(\vec{r}_2) = m_2(-\nabla V(\vec{r}_2)) = -\frac{Gm_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1)$$

Potential energy of point mass $m_2$ at $\vec{r}_2$:

$$E_{pot} = U(\vec{r}_2) = m_2 V(\vec{r}_2)$$

### Rotating frame

General vector derivatives in inertial and rotating frame:

$$\left.\frac{d}{dt}\vec{x}\right|_{in} = \left.\frac{d}{dt}\vec{x}\right|_{rot} + \vec{\omega} \times \vec{x}$$
$$\left.\frac{d^2}{dt^2}\vec{x}\right|_{in} = \left.\frac{d^2}{dt^2}\vec{x}\right|_{rot} + 2\vec{\omega} \times \left.\frac{d}{dt}\vec{x}\right|_{rot} + \vec{\omega} \times (\vec{\omega} \times \vec{x}) + \dot{\vec{\omega}} \times \vec{x}$$

Angular acceleration:

$$\vec{\alpha} = \left.\frac{d}{dt}\vec{\omega}\right|_{in} = \left.\frac{d}{dt}\vec{\omega}\right|_{rot}$$

Linear velocity and acceleration:

$$\vec{v}_{rot} = \left.\frac{d}{dt}\vec{r}\right|_{rot} = \vec{v}_{in} - \vec{\omega} \times \vec{r}$$
$$\vec{a}_{rot} = \left.\frac{d^2}{dt^2}\vec{r}\right|_{rot} = \underbrace{\vec{a}_{in}}_{external} \underbrace{-2\vec{\omega} \times \vec{v}_{rot}}_{Coriolis} \underbrace{-\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{Centrifugal} \underbrace{-\vec{\alpha} \times \vec{r}}_{Euler}$$

Fictitious forces:

$$\vec{F}_{Coriolis} = -2m(\vec{\omega} \times \vec{v}_{rot}) \qquad \vec{F}_{Centrifugal} = -m(\vec{\omega} \times (\vec{\omega} \times \vec{r}))$$
$$\vec{F}_{Euler} = -m(\vec{\alpha} \times \vec{r})$$

## 3.2. Attitude Representation

Given inertial frame basis $\vec{E}_1, \vec{E}_2, \vec{E}_3$, body fixed basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$, and angular velocity vector $\vec{\omega} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3$ .

Inertial frame to body fixed frame, body fixed frame to inertial frame:

$$\vec{x}_{body} = \boldsymbol{R} \cdot \vec{x}_{rest} \qquad\qquad \vec{x}_{rest} = \boldsymbol{R}^T \cdot \vec{x}_{body}$$

### Direction cosines

$$\boldsymbol{R} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\dot{\boldsymbol{R}} = \begin{pmatrix} \dot{a}_{11} & \dot{a}_{12} & \dot{a}_{13} \\ \dot{a}_{21} & \dot{a}_{22} & \dot{a}_{23} \\ \dot{a}_{31} & \dot{a}_{32} & \dot{a}_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \omega_3 a_{21} - \omega_2 a_{31} & \omega_3 a_{22} - \omega_2 a_{32} & \omega_3 a_{23} - \omega_2 a_{33} \\ \omega_1 a_{31} - \omega_3 a_{11} & \omega_1 a_{32} - \omega_3 a_{12} & \omega_1 a_{33} - \omega_3 a_{13} \\ \omega_2 a_{11} - \omega_1 a_{21} & \omega_2 a_{12} - \omega_1 a_{22} & \omega_2 a_{13} - \omega_1 a_{23} \end{pmatrix}$$

Properties:

$$a_{ij} = \vec{E}_j \cdot \vec{e}_i = \cos \angle \left( \vec{E}_j, \vec{e}_i \right)$$

### Euler rotations

zxz rotations $\psi, \theta, \phi$ (Euler angles):

$$\boldsymbol{R} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \cos\phi\sin\psi + \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ -\sin\phi\cos\psi - \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \cos\phi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{pmatrix} = \frac{1}{\sin\theta}\begin{pmatrix}\sin\phi & \cos\phi & 0\\ \cos\phi\sin\theta & -\sin\phi\sin\theta & 0\\ -\sin\phi\cos\theta & -\cos\phi\cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}$$

### Quaternions

$$\vec{q} = \begin{pmatrix}q_1\\ q_2\\ q_3\\ q_4\end{pmatrix} = \begin{pmatrix}m_1\sin\frac{\mu}{2}\\ m_2\sin\frac{\mu}{2}\\ m_3\sin\frac{\mu}{2}\\ \cos\frac{\mu}{2}\end{pmatrix}$$

$$\boldsymbol{R} = \begin{pmatrix}q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4)\\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_1q_4 + q_2q_3)\\ 2(q_1q_3 + q_2q_4) & 2(-q_1q_4 + q_2q_3) & -q_1^2 - q_2^2 + q_3^2 + q_4^2\end{pmatrix}$$

$$\begin{pmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \dot{q}_4\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0 & \omega_3 & -\omega_2 & \omega_1\\ -\omega_3 & 0 & \omega_1 & \omega_2\\ \omega_2 & -\omega_1 & 0 & \omega_3\\ -\omega_1 & -\omega_2 & -\omega_3 & 0\end{pmatrix}\begin{pmatrix}q_1\\ q_2\\ q_3\\ q_4\end{pmatrix}$$

Rotation by angle $\mu$ around rotation axis $\vec{m}$ (inertial frame):

$$\vec{m} = \begin{pmatrix}m_1\\ m_2\\ m_3\end{pmatrix}$$

Properties:

$$\vec{q}\cdot\vec{q} = |\vec{q}|^2 = 1$$

$$\vec{m} = \frac{q_1\vec{E}_1 + q_2\vec{E}_2 + q_3\vec{E}_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$$

$$\frac{\mu}{2} = \text{arctan2}\left(\sqrt{q_1^2 + q_2^2 + q_3^2}, q_4\right)$$

$$\sin\frac{\mu}{2} = \sqrt{q_1^2 + q_2^2 + q_3^2}$$

## 3.3. Rigid Bodies

Center of mass:

$$\vec{x}_{CM} = \frac{\sum_i m_i \vec{x}_i}{\sum_i m_i} \qquad \vec{x}_{CM} = \frac{\int_V \rho(\vec{x})\vec{x}\, dV}{\int_V \rho(\vec{x})\, dV}$$

Angular momentum vector:

$$\vec{H} = \sum_i m_i \left(\vec{r}_i \times \vec{v}_i\right) \qquad \vec{H} = \int_V \rho(\vec{r}) \left(\vec{r} \times \vec{v}\right) dV$$

### Inertia Tensor

$$\begin{aligned} \boldsymbol{I} &= \int_V \rho(\vec{r}) \left(|\vec{r}|^2 \boldsymbol{E} - \vec{r}\vec{r}^T\right) dV \\ &= \int_V \rho(\vec{r}) \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} dV \end{aligned}$$

Parallel axes theorem:

$$\begin{aligned} \boldsymbol{I}_r &= \boldsymbol{I}_{cm} + m \left(|\vec{r}|^2 \boldsymbol{E} - \vec{r}\vec{r}^T\right) \\ &= \boldsymbol{I}_{cm} + m \begin{pmatrix} r_2^2 + r_3^2 & r_1 r_2 & r_1 r_3 \\ r_1 r_2 & r_1^2 + r_3^2 & r_2 r_3 \\ r_1 r_3 & r_2 r_3 & r_1^2 + r_2^2 \end{pmatrix} \end{aligned}$$

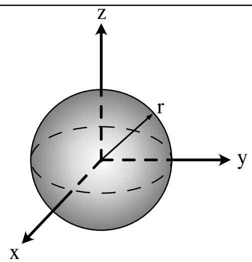
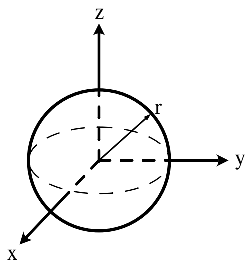
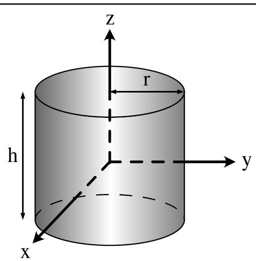
Change of basis from $E$ to $e$ via rotation matrix $\boldsymbol{R}$:

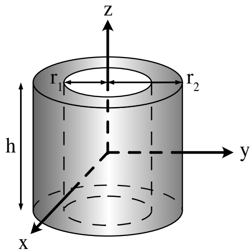
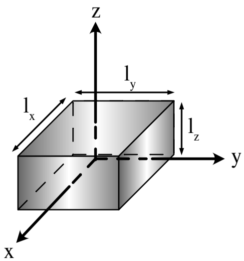
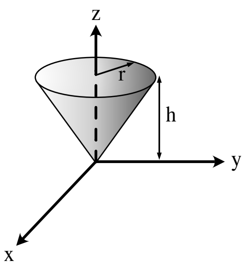
$$\vec{r}_e = \boldsymbol{R} \cdot \vec{r}_E$$

$$\boldsymbol{I}_e = \boldsymbol{R} \cdot \boldsymbol{I}_E \cdot \boldsymbol{R}^T$$

Inertia tensor in coordinates along principal body axes:

$$\boldsymbol{I} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

| Body of mass $m$ | Principal moments |
|---|---|
| <br>uniform solid sphere<br>(center of mass) | $I_x = \frac{2}{5}mr^2$<br>$I_y = \frac{2}{5}mr^2$<br>$I_z = \frac{2}{5}mr^2$ |
| <br>uniform spherical shell<br>(center of mass) | $I_x = \frac{2}{3}mr^2$<br>$I_y = \frac{2}{3}mr^2$<br>$I_z = \frac{2}{3}mr^2$ |
| <br>uniform solid cylinder<br>(center of mass) | $I_x = \frac{1}{12}m\left(3r^2+h^2\right)$<br>$I_y = \frac{1}{12}m\left(3r^2+h^2\right)$<br>$I_z = \frac{1}{2}mr^2$ |

| Body of mass $m$ | Principal moments |
|---|---|
|   uniform hollow cylinder (center of mass) | $I_x = \frac{1}{12}m\left(3\left(r_1^2+r_2^2\right)+h^2\right)$<br>$I_y = \frac{1}{12}m\left(3\left(r_1^2+r_2^2\right)+h^2\right)$<br>$I_z = \frac{1}{2}m\left(r_1^2+r_2^2\right)$ |
|   uniform solid cuboid (center of mass) | $I_x = \frac{1}{12}m\left(l_y^2+l_z^2\right)$<br>$I_y = \frac{1}{12}m\left(l_x^2+l_z^2\right)$<br>$I_z = \frac{1}{12}m\left(l_x^2+l_y^2\right)$ |
|   uniform solid cone (apex) | $I_x = \frac{3}{5}mh^2+\frac{3}{20}mr^2$<br>$I_y = \frac{3}{5}mh^2+\frac{3}{20}mr^2$<br>$I_z = \frac{3}{10}mr^2$ |

### Dynamics

Angular momentum vector:

$$\vec{H} = \boldsymbol{I}\vec{\omega}$$

Torque:

$$\vec{T} = \vec{r} \times \vec{F}$$

Dynamics (inertial frame):

$$\begin{aligned} \frac{d}{dt}\vec{H}\bigg|_{rest} &= \frac{d}{dt}\left(\boldsymbol{I}\vec{\omega}\right)\bigg|_{rest} \\ &= \frac{d}{dt}\boldsymbol{I}\bigg|_{rest}\vec{\omega} + \boldsymbol{I}\vec{\alpha} \\ &= \vec{T} \end{aligned}$$

Dynamics (body fixed frame):

$$\begin{aligned} \frac{d}{dt}\vec{H}\bigg|_{rest} &= \frac{d}{dt}\vec{H}\bigg|_{rot} + \vec{\omega} \times \vec{H} \\ &= \boldsymbol{I}\vec{\alpha} + \vec{\omega} \times \vec{H} && \text{(Euler's equation)} \\ &= \vec{T} \end{aligned}$$

Rotational kinetic energy:

$$E_{rot} = \frac{1}{2}\vec{\omega}^T\boldsymbol{I}\vec{\omega} = \frac{1}{2}\vec{\omega} \cdot \vec{H}$$

### Torque-free motion

General case:

$$\begin{pmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{pmatrix} = \begin{pmatrix} (I_2 - I_3)\omega_2\omega_3 \\ (I_3 - I_1)\omega_1\omega_3 \\ (I_1 - I_2)\omega_1\omega_2 \end{pmatrix}$$

Body of rotation with $I_1 = I_2$:

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix}\omega_{12}\cos(\lambda t + \lambda_0)\\ \omega_{12}\sin(\lambda t + \lambda_0)\\ \omega_0\end{pmatrix} \qquad \begin{pmatrix}H_1\\ H_2\\ H_3\end{pmatrix} = \begin{pmatrix}I_1\omega_{12}\cos(\lambda t + \lambda_0)\\ I_1\omega_{12}\sin(\lambda t + \lambda_0)\\ I_3\omega_0\end{pmatrix}$$

Angles $\theta = \angle(\vec{H}, \vec{z})$, $\gamma = \angle(\vec{\omega}, \vec{z})$:

$$\tan\gamma = \frac{\omega_{12}}{\omega_0} \qquad \tan\theta = \frac{I_1\omega_{12}}{I_3\omega_0} = \frac{I_1}{I_3}\tan\gamma$$

Precession and spin rates:

$$\omega_p = \frac{H}{I_1} = \frac{\omega_{12}}{\sin\theta}$$

$$\omega_s = \left(1 - \frac{I_3}{I_1}\right)\omega_0 = \omega_0 - \frac{\omega_{12}}{\tan\theta}$$

CHAPTER 4

# Electromagnetism

Lorentz force:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

## 4.1. Magnetic Dipole

Dipole magnetic field:

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\vec{r}\,(\vec{m}\cdot\vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right]$$

$$B_r = \frac{2m\mu_0\cos\theta}{4\pi r^3} \qquad B_\theta = \frac{m\mu_0\sin\theta}{4\pi r^3}$$

### Magnetic dipole in external field

Potential:

$$U = -\vec{m}\cdot\vec{B}$$

Torque:

$$\vec{T} = \vec{m} \times \vec{B}$$

Force (current loop or bar magnet model):

$$\vec{F}_{loop} = -\nabla U = \nabla\left(\vec{m}\cdot\vec{B}\right)$$

$$\vec{F}_{bar} = (\vec{m}\cdot\nabla)\,\vec{B} = \vec{F}_{loop} - \vec{m} \times \left(\nabla \times \vec{B}\right)$$

## 4.2. Radiation & Optics

Diffraction limit:

$$\theta = \frac{1.22\lambda}{d} \qquad \text{(Rayleigh limit)}$$

### Black Body Radiation

Wien's displacement law:

$$\lambda_{max} = \frac{b}{T} \qquad b \approx 2.897\,772 \times 10^{-3}\,\mathrm{m\,K}$$

Planck's law:

$$P_\lambda = \frac{2\pi hc^2}{\lambda^5 \left(e^{\frac{hc}{\lambda k_B T}} - 1\right)}$$

Stefan-Boltzmann law:

$$E_\lambda = \sigma T^4$$

CHAPTER 5

# Orbital Mechanics

General two-body problem:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \ddot{\vec{r}}_{cm} = 0$$

$$\vec{d} = \vec{r}_2 - \vec{r}_1 \qquad \ddot{\vec{d}} = -\frac{G(m_1 + m_2)}{|\vec{d}|^3}\vec{d} = -\frac{\mu}{|\vec{d}|^3}\vec{d}$$

$$\vec{r}_1 = \vec{r}_{cm} - \frac{m_2}{m_1 + m_2}\vec{d} \qquad \vec{r}_2 = \vec{r}_{cm} + \frac{m_1}{m_1 + m_2}\vec{d}$$

Restricted two-body problem $m_1 \gg m_2$:

$$\vec{r}_1 = 0 \qquad \ddot{\vec{r}}_2 = -\frac{Gm_1}{|\vec{r}_2|^3}\vec{r}_2 = -\frac{\mu}{|\vec{r}_2|^3}\vec{r}_2$$

## 5.1. Keplerian Orbits

Orbit equation:

$$r = \frac{p}{1 + e\cos\theta} \qquad p = \frac{h^2}{\mu} = a\left(1 - e^2\right)$$

Flight path angle:

$$\gamma = \arctan\frac{v_r}{v_\perp}$$

$$v_r = \frac{\mu}{h}e\sin\theta \qquad v_\perp = \frac{h}{r} \qquad v = \sqrt{v_r^2 + v_\perp^2}$$

Time along orbit:

$$t - t_0 = \int_{\theta_0}^{\theta} \frac{h^3}{\mu^2(1 + e\cos\theta)^2} d\theta$$

### Constants of motion

Specific orbital angular momentum:

$$\vec{h} = \frac{\vec{H}}{m} = \vec{r} \times \dot{\vec{r}} = \vec{r} \times \vec{v} \qquad h = \left|\vec{h}\right| = r^2\dot{\theta}$$

Specific orbital energy:

$$\epsilon = \frac{E}{m} = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} = -\frac{\mu^2}{2h^2}(1 - e^2) \qquad \text{(Vis-viva equation)}$$

Eccentricity vector:

$$\vec{e} = \frac{\dot{\vec{r}} \times \vec{h}}{\mu} - \frac{\vec{r}}{r} \qquad e = |\vec{e}|$$

### Circular orbits ($e = 0$)

$$a = r_p = r_a = r \qquad v = \sqrt{\frac{\mu}{r}}$$

Orbital period and mean motion:

$$\tau = 2\pi\sqrt{\frac{r^3}{\mu}} \qquad n = \frac{2\pi}{\tau} = \sqrt{\frac{\mu}{r^3}}$$

Time to cover angle $\theta$:

$$t = \frac{\theta}{n}$$

**Elliptic orbits ($0 < e < 1$)**

$$r_a = a(1+e) \qquad r_p = a(1-e)$$

$$\frac{r_a}{r_p} = \frac{1+e}{1-e} \qquad e = \frac{r_a - r_p}{r_a + r_p}$$

Orbital period and mean motion:

$$\tau = 2\pi\sqrt{\frac{a^3}{\mu}} \qquad n = \frac{2\pi}{\tau} = \sqrt{\frac{\mu}{a^3}}$$

Eccentric anomaly:

$$\sin E = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \qquad \cos E = \frac{e+\cos\theta}{1+e\cos\theta}$$

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}$$

Elliptic mean anomaly:

$$M_e = E - e\sin E \qquad \text{(Kepler's equation)}$$

Time since perigee passage:

$$\begin{aligned} t &= \frac{1}{n}M_e = \frac{T}{2\pi}M_e = \sqrt{\frac{a^3}{\mu}}M_e \\ &= \frac{h^3}{\mu^2(1-e^2)^{3/2}}\left[2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right] \end{aligned}$$

**Parabolic orbits ($e = 1$)**

$$a \to \infty \qquad r_p = \frac{p}{2}$$

$$v = \sqrt{\frac{2\mu}{r}} \qquad v_\infty = 0$$

Barker parameter:

$$B = \tan\frac{\theta}{2}$$

Parabolic mean anomaly:

$$M_p = \frac{1}{2}B + \frac{1}{6}B^3 \qquad \text{(Barker's equation)}$$

$$B = \left(3M_p + \sqrt{9M_p^2 + 1}\right)^{1/3} - \left(3M_p + \sqrt{9M_p^2 + 1}\right)^{-1/3}$$

Time since perigee passage:

$$\begin{aligned} t &= \sqrt{\frac{p^3}{\mu}} M_p = \frac{h^3}{\mu^2} M_p \\ &= \frac{h^3}{\mu^2}\left(\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}\right) \end{aligned}$$

**Hyperbolic orbits ($e > 1$)**

$$r_p = \frac{p}{1+e} \qquad e = \frac{r_p v_\infty^2}{\mu} + 1$$

$$v_\infty = \sqrt{\frac{\mu}{-a}} \qquad \theta_\infty = \arccos\left(-\frac{1}{e}\right)$$

Hyperbolic eccentric anomaly:

$$\sinh F = \frac{\sqrt{e^2-1}\sin\theta}{1+e\cos\theta} \qquad \cosh F = \frac{\cos\theta + e}{1+e\cos\theta}$$

$$\tanh\frac{F}{2} = \sqrt{\frac{e-1}{e+1}}\tan\frac{\theta}{2}$$

Hyperbolic mean anomaly:

$$M_h = e\sinh F - F$$

Time since perigee passage (hyperbolic):

$$t = M_h \sqrt{-\frac{a^3}{\mu}}$$
$$= \frac{h^3}{\mu^2(e^2-1)} \left[ \frac{e \sin\theta}{1 + e\cos\theta} - \frac{1}{\sqrt{e^2-1}} \ln\left( \frac{\sqrt{e+1} + \sqrt{e-1}\tan\frac{\theta}{2}}{\sqrt{e+1} - \sqrt{e-1}\tan\frac{\theta}{2}} \right) \right]$$

## 5.2. Coordinates & Observations

Perifocal frame:

$$\vec{r} = r \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \qquad \vec{v} = \frac{\mu}{h} \begin{pmatrix} -\sin\theta \\ e + \cos\theta \\ 0 \end{pmatrix}$$

Keplerian elements $a, e, i, \omega, \Omega, \theta$ to Cartesian $\vec{r}, \vec{v}$:

$$\boldsymbol{Q} = \begin{pmatrix} \cos\Omega\cos\omega - \cos i \sin\Omega\sin\omega & -\cos\Omega\sin\omega - \cos i\cos\omega\sin\Omega & \sin\Omega\sin i \\ \cos\omega\sin\Omega + \cos\Omega\cos i\sin\omega & \cos\Omega\cos i\cos\omega - \sin\Omega\sin\omega & -\cos\Omega\sin i \\ \sin i\sin\omega & \cos\omega\sin i & \cos i \end{pmatrix}$$

$$\vec{r} = \boldsymbol{Q} \cdot \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix} \qquad \vec{v} = \frac{\mu}{h}\boldsymbol{Q} \cdot \begin{pmatrix} -\sin\theta \\ e + \cos\theta \\ 0 \end{pmatrix}$$

Cartesian $\vec{r}, \vec{v}$ to Keplerian elements $a, e, i, \omega, \Omega, \theta$ :

$$a = \frac{h^2}{\mu(1-e^2)}$$
$$e = |\vec{e}|$$
$$i = \arccos\frac{h_z}{h}$$
$$\Omega = \operatorname{atan2}(h_x, -h_y) \mod 2\pi$$
$$\omega = \operatorname{atan2}\left(\frac{e_z}{\sin i}, e_y \sin\Omega + e_x \cos\Omega\right) \mod 2\pi$$
$$\theta = \begin{cases} \arccos\frac{\vec{r}\cdot\vec{e}}{re} & \vec{r}\cdot\vec{v} \geqslant 0 \\ 2\pi - \arccos\frac{\vec{r}\cdot\vec{e}}{re} & \vec{r}\cdot\vec{v} < 0 \end{cases}$$

Modified equinoctial elements:

$$p = a(1-e^2)$$
$$f = e\cos(\omega+\Omega)$$
$$g = e\sin(\omega+\Omega)$$
$$h = \tan\frac{i}{2}\cos\Omega$$
$$k = \tan\frac{i}{2}\sin\Omega$$
$$L = \Omega+\omega+\theta$$

### Topocentric horizon

Azimuth $A$, altitude $a$, and range $\rho$ at sidereal time $\theta$ and latitude $\phi$ to Cartesian ECI:

$$\delta = \arcsin(\cos\phi\cos A\cos a + \sin\phi\sin a)$$
$$h = \begin{cases} 2\pi - \arccos\left(\frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) & 0 \leqslant A < \pi \\ \arccos\left(\frac{\cos\phi\sin a - \sin\phi\cos A\cos a}{\cos\delta}\right) & \pi \leqslant A < 2\pi \end{cases}$$
$$\alpha = \theta - h$$

$$\dot{\delta} = \frac{-\dot{A}\cos\phi\sin A\cos a + \dot{a}\left(\sin\phi\cos a - \cos\phi\cos A\sin a\right)}{\cos\delta}$$

$$\begin{aligned}\dot{\alpha} &= \dot{\theta} - \dot{h} \\ &= \omega_E + \frac{\dot{A}\cos A\cos a - \dot{a}\sin A\sin a + \dot{\delta}\sin A\cos a\tan\delta}{\cos\phi\sin a - \sin\phi\cos A\cos a}\end{aligned}$$

$$\vec{\hat{\rho}} = \begin{pmatrix}\cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta\end{pmatrix} \qquad \dot{\vec{\hat{\rho}}} = \begin{pmatrix}-\dot{\delta}\sin\delta\cos\alpha - \dot{\alpha}\cos\delta\sin\alpha, \\ -\dot{\delta}\sin\delta\sin\alpha + \dot{\alpha}\cos\delta\cos\alpha \\ \dot{\delta}\cos\delta\end{pmatrix}$$

$$\vec{\Omega} = \begin{pmatrix}0 \\ 0 \\ \omega_E\end{pmatrix} \qquad \vec{R}_{obs} = R_E(\phi)\begin{pmatrix}\cos\phi\cos\theta \\ \cos\phi\sin\theta \\ \sin\phi\end{pmatrix}$$

$$\begin{aligned}\vec{r} &= \vec{R}_{obs} + \vec{\rho} \\ \vec{v} &= \dot{\vec{R}}_{obs} + \dot{\vec{\rho}} = \vec{\Omega}\times\vec{R}_{obs} + \dot{\rho}\vec{\hat{\rho}} + \rho\dot{\vec{\hat{\rho}}}\end{aligned}$$

### Gibbs' method

Given co-planar observations $\vec{r}_1, \vec{r}_2, \vec{r}_3$:

$$r_1 = |\vec{r}_1| \qquad r_2 = |\vec{r}_2| \qquad r_3 = |\vec{r}_3|$$

$$\begin{aligned}\vec{N} &= r_1(\vec{r}_2\times\vec{r}_3) + r_2(\vec{r}_3\times\vec{r}_1) + r_3(\vec{r}_1\times\vec{r}_2) \\ \vec{D} &= \vec{r}_1\times\vec{r}_2 + \vec{r}_2\times\vec{r}_3 + \vec{r}_3\times\vec{r}_1 \\ \vec{S} &= (r_2 - r_3)\vec{r}_1 + (r_3 - r_1)\vec{r}_2 + (r_1 - r_2)\vec{r}_3 \\ \vec{v}_2 &= \sqrt{\frac{\mu}{\left|\vec{N}\right|\left|\vec{D}\right|}}\left(\frac{\vec{D}\times\vec{r}_2}{r_2} + \vec{S}\right)\end{aligned}$$

**Epochs**

Julian Day $JD$: noon January 1, 4713 BC.

Modified Julian Day $MJD$: midnight November 17, 1858.

$$MJD = JD - 2400000.5$$

Modified Julian Day 2000 $MJD2000$: noon January 1, 2000.

$$MJD2000 = JD - 2451545.0$$

## 5.3. Maneuvers

Rocket equation:

$$\frac{m_f}{m_i} = \exp\left(-\frac{\Delta v}{I_{sp} g_0}\right) \qquad \text{(Tsiolkovsky)}$$

**Hohmann transfers**

Planar circular Hohmann transfer:

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) \qquad \Delta v_2 = \sqrt{\frac{\mu}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

General planar Hohmann transfer:

$$h = \sqrt{2\mu \frac{r_a r_p}{r_a + r_p}}$$

$$v_a = \frac{h}{r_a} \qquad v_p = \frac{h}{r_p}$$

Planar bi-elliptic circular Hohmann transfer $r_A \to r_B \to r_C$:

$$v_0 = \sqrt{\frac{\mu}{r_A}} \qquad \alpha = \frac{r_C}{r_A} \qquad \beta = \frac{r_B}{r_A}$$

$$\Delta v = v_0\left(\sqrt{\frac{2(\alpha + \beta)}{\alpha\beta}} - \frac{1 + \sqrt{\alpha}}{\sqrt{\alpha}} - \sqrt{\frac{2}{\beta(1 + \beta)}}(1 - \beta)\right)$$

Phasing maneuver:

$$a = \left( \frac{T_{phasing}\sqrt{\mu}}{2\pi} \right)^{2/3} \qquad 2a = r_a + r_p$$

### Single impulse maneuvers

Planar orbit intersection points for coaxial orbits:

$$\theta_1 = \theta_2 = \arccos\left( \frac{h_1^2 - h_2^2}{e_1 h_2^2 - e_2 h_1^2} \right)$$

Planar orbit intersection points for apse line angle $\eta = \theta_1 - \theta_2$:

$$a = e_1 h_2^2 - e_2 h_1^2 \cos\eta \qquad b = -e_2 h_1^2 \sin\eta \qquad c = h_1^2 - h_2^2$$

$$\phi = \arctan\frac{b}{a}$$

$$\theta_1 = \phi \pm \arccos\left( \frac{c}{a}\cos\phi \right) \qquad \theta_2 = \theta_1 - \eta$$

Planar orbit change:

$$\Delta v = \sqrt{v_2^2 + v_1^2 - 2v_2 v_1 \cos(\gamma_2 - \gamma_1)}$$

New orbit from planar maneuver:

$$h^* = h + r\Delta v_\perp \qquad v_\perp^* = v_\perp + \Delta v_\perp \qquad v_r^* = v_r + \Delta v_r$$

$$\tan\theta^* = \frac{v_\perp^* v_r^*}{v_\perp^{*2} e\cos\theta + (v_\perp + v_\perp^*)\,\Delta v_\perp}\,\frac{v_\perp^2}{\mu/r}$$

$$e^* = v_r^* \frac{h^*}{\mu \sin\theta^*}$$

### Inclination change

Orbital plane rotation by $\delta$ around $\vec{r}$ (no shape change):

$$\Delta v = 2v_\perp \sin\frac{\delta}{2}$$

Orbital plane rotation by $\delta$ around $\vec{r}$ with velocity change to $v_r^*, v_\perp^*$:

$$\begin{aligned}\Delta v &= \sqrt{(v_r^* - v_r)^2 + v_\perp^{*2} + v_\perp^2 - 2v_\perp^* v_\perp \cos\delta} \\ &= \sqrt{v^{*2} + v^2 - 2v^* v\left[\cos(\gamma^* - \gamma) - \cos\gamma^* \cos\gamma(1 - \cos\delta)\right]}\end{aligned}$$

Rotation around common apse line with radial velocity change:

$$\Delta v = \sqrt{v^{*2} + v^2 - 2v^* v \cos\delta}$$

3-impulse plane change of circular orbit with radius $r_1$ at $r_2$:

$$\rho = \frac{r_2}{r_1} \qquad \Delta v = \Delta v_1 + \Delta v_2 + \Delta v_3$$

$$\begin{aligned}\Delta v_1 = \Delta v_3 &= \left(\sqrt{\frac{2\rho}{1+\rho}} - 1\right)\sqrt{\frac{\mu}{r_1}} \\ \Delta v_2 &= 2\sqrt{\frac{2}{\rho(1+\rho)}}\sin\frac{\delta}{2}\sqrt{\frac{\mu}{r_1}}\end{aligned}$$

### Patched conics

Planetary arrival / departure / gravity assist:

$$v_p = \sqrt{v_\infty^2 + \frac{2\mu}{r_p}} \qquad \Delta = r_p\sqrt{1 + \frac{2\mu}{r_p v_\infty^2}}$$

$$e = \frac{r_p v_\infty^2}{\mu} + 1$$

$$\beta = \arccos\frac{1}{e} \qquad \delta = 2\arcsin\frac{1}{e}$$

### Interplanetary phasing

For Hohmann transfer from coplanar circular orbit 1 to coplanar circular orbit 2 in $t_{12}$.

Synodic period:

$$\tau_{syn} = \frac{2\pi}{|n_1 - n_2|}$$

Phase angles:

$$\phi_0 = \pi - n_2 t_{12} \qquad \phi_f = \pi - n_1 t_{12}$$
$$\phi_0' = -\pi + n_1 t_{12} \qquad \phi_f' = -\pi + n_2 t_{12}$$

Wait time:

$$t_{wait} = \frac{2\phi_0' \pm 2k\pi}{n_2 - n_1}$$

### Attitude maneuvers

Single coning maneuver covering $\theta$:

$$\Delta H = 2\left|\vec{H}_0\right| \tan\frac{\theta}{2}$$

Continuous coning maneuver covering $\theta$:

$$\Delta H = \left|\vec{H}_0\right| \theta$$

Yoyo de-spin from $\omega_0$ to $\omega$ with two masses of $m/2$ at radius $R$:

$$K = 1 + \frac{I}{mR^2}$$
$$l = R\sqrt{K\frac{\omega_0 - \omega}{\omega_0 + \omega}}$$

Gravity gradient:

$$\vec{T} = \frac{3\mu}{r^7}\vec{r} \times \boldsymbol{I} \cdot \vec{r}$$

## 5.4. Perturbations

### Aerodynamic drag

$$\vec{F}_{drag} = \frac{1}{2}\rho(r)C_d v^2 A\left(-\frac{\vec{v}}{v}\right) \qquad \vec{a}_{drag} = \frac{1}{2}\rho(r)C_d v^2 \frac{A}{m}\left(-\frac{\vec{v}}{v}\right)$$

Atmospheric density model for isothermal ideal gas:

$$\rho(h) = \rho_0 \exp(-\alpha h) = \rho_0 \exp(-h/H)$$

Approximate values for planetary atmospheres:

| | | |
|---|---|---|
| Earth | $\rho_0 \approx 1.225\,\mathrm{kg\,m^{-3}}$ | $\alpha \approx 0.1378\,\mathrm{km^{-1}}$ |
| Mars (0 km – 25 km) | $\rho_0 \approx 0.0159\,\mathrm{kg\,m^{-3}}$ | $\alpha \approx 0.090\,51\,\mathrm{km^{-1}}$ |
| Mars (25 km – 125 km) | $\rho_0 \approx 0.0525\,\mathrm{kg\,m^{-3}}$ | $\alpha \approx 0.1371\,\mathrm{km^{-1}}$ |
| Venus | $\rho_0 \approx 65\,\mathrm{kg\,m^{-3}}$ | $\alpha \approx 0.062\,89\,\mathrm{km^{-1}}$ |

### Third body

$$\vec{F}_{body} = m\mu_{body}\left(\frac{\vec{r}_{body} - \vec{r}}{|\vec{r}_{body} - \vec{r}|^3} - \frac{\vec{r}_{body}}{|\vec{r}_{body}|^3}\right)$$
$$\vec{a}_{body} = \mu_{body}\left(\frac{\vec{r}_{body} - \vec{r}}{|\vec{r}_{body} - \vec{r}|^3} - \frac{\vec{r}_{body}}{|\vec{r}_{body}|^3}\right)$$

Sphere of influence for planet ($m_p$) at radius $R$ from Sun ($m_s$):

$$r_{SOI} = R\left(\frac{m_p}{m_s}\right)^{\frac{2}{5}}$$

### Solar radiation pressure

$$F_{SRP} = PA \qquad a_{SRP} = P\frac{A}{m}$$

$$P_{absorbtion} = \frac{J_{Earth}}{c(r/1\,\mathrm{au})^2}\cos\alpha \qquad P_{reflection} = 2\frac{J_{Earth}}{c(r/1\,\mathrm{au})^2}\cos^2\alpha$$

### Geopotential

Potential expansion in spherical harmonics with radius $r$, latitude $\phi$, and longitude $\lambda$ and Earth equatorial radius $R_E$:

$$\vec{F}_{geo} = -m\nabla U(\vec{r}) \qquad \vec{a}_{geo} = -\nabla U(\vec{r})$$

$$J_{l,m} = \sqrt{C_{l,m}^2 + S_{l,m}^2} \qquad \lambda_{l,m} = \arctan\frac{C_{l,m}}{S_{l,m}}$$

$$\begin{aligned}
U(r,\phi,\lambda) &= -\frac{\mu}{r}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\left(\frac{R_E}{r}\right)^l P_{l,m}(\sin\phi)\left(C_{l,m}\cos m\lambda + S_{l,m}\sin m\lambda\right) \\
&= -\frac{\mu}{r}\Bigg[1 - \sum_{l=2}^{\infty} J_l\left(\frac{R_E}{r}\right)^l P_{l,0}(\sin\phi) + \\
&\qquad \sum_{l=2}^{\infty}\sum_{m=1}^{l}\left(\frac{R_E}{r}\right)^l P_{l,m}(\sin\phi)\left(C_{l,m}\cos m\lambda + S_{l,m}\sin m\lambda\right)\Bigg] \\
&= -\frac{\mu}{r}\Bigg[1 - \sum_{l=2}^{\infty} J_l\left(\frac{R_E}{r}\right)^l P_{l,0}(\sin\phi) + \\
&\qquad \sum_{l=2}^{\infty}\sum_{m=1}^{l}\left(\frac{R_E}{r}\right)^l P_{l,m}(\sin\phi) J_{l,m}\sin\left(m\lambda + \lambda_{l,m}\right)\Bigg]
\end{aligned}$$

## 5.5. Propagation

### Gauss equations

Given radial, horizontal, and out of plane accelerations $a_r, a_s, a_w$:

$$\begin{aligned}
\frac{da}{dt} &= \frac{2e\sin\theta}{n\sqrt{1-e^2}}a_r + \frac{2a\sqrt{1-e^2}}{nr}a_s \\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}\sin\theta}{na}a_r + \frac{\sqrt{1-e^2}}{na^2e}\left(\frac{a^2(1-e^2)}{r} - r\right)a_s \\
\frac{di}{dt} &= \frac{r\cos(\theta+\omega)}{na^2\sqrt{1-e^2}}a_w \\
\frac{d\Omega}{dt} &= \frac{r\sin(\theta+\omega)}{na^2\sqrt{1-e^2}\sin i}a_w \\
\frac{d\omega}{dt} &= -\frac{\sqrt{1-e^2}\cos\theta}{nae}a_r + \frac{a(1-e^2)}{eh}\left[\sin\theta\left(1+\frac{1}{1+e\cos\theta}\right)\right]a_s \\
&\quad - \frac{r\cot i\sin(\theta+\omega)}{na^2\sqrt{1-e^2}}a_w \\
\frac{dM}{dt} &= n - \frac{1}{na}\left(\frac{2r}{a} - \frac{1-e^2}{e}\cos\theta\right)a_r \\
&\quad - \frac{(1-e^2)\sin\theta}{nae}\left(1+\frac{r}{a(1-e^2)}\right)a_s
\end{aligned}$$

### Secular rates

Secular rates in rad s$^{-1}$ for Earth radius $R_E$ due to oblateness:

$$\begin{aligned}
\dot{\Omega} &= -\frac{3}{2}\left(\frac{J_2\sqrt{\mu}R_E^2}{(1-e^2)^2a^{7/2}}\right)\cos i \\
\dot{\omega} &= -\frac{3}{2}\left(\frac{J_2\sqrt{\mu}R_E^2}{(1-e^2)^2a^{7/2}}\right)\left(\frac{5}{2}\sin^2 i - 2\right)
\end{aligned}$$

See also Section 6.4 on page 58.

## 5.6. Relative Motion

With specific angular momentum, position, and velocity $h, \vec{R}, \vec{V}$ of target in LVLH and $\delta x, \delta y, \delta z$ relative position of chaser in LVLH:

$$\delta \ddot{x} - \left( \frac{2\mu}{R^3} + \frac{h^2}{R^4} \right) \delta x + \frac{2\left(\vec{V} \cdot \vec{R}\right) h}{R^4} \delta y - 2\frac{h}{R^2} \delta \dot{y} = 0$$

$$\delta \ddot{y} + \left( \frac{\mu}{R^3} - \frac{h^2}{R^4} \right) \delta y - \frac{2\left(\vec{V} \cdot \vec{R}\right) h}{R^4} \delta x + 2\frac{h}{R^2} \delta \dot{x} = 0$$

$$\delta \ddot{z} + \frac{\mu}{R^3} \delta z = 0$$

Clohessy-Wiltshire equations:

$$\delta \ddot{x} - 3n^2 \delta x - 2n \delta \dot{y} = 0$$

$$\delta \ddot{y} + 2n \delta \dot{x} = 0$$

$$\delta \ddot{z} + n^2 \delta z = 0$$

$$\delta \vec{r}(t) = \mathbf{\Phi}_{rr}(t) \cdot \delta \vec{r}_0 + \mathbf{\Phi}_{rv}(t) \cdot \delta \vec{v}_0$$

$$\delta \vec{v}(t) = \mathbf{\Phi}_{vr}(t) \cdot \delta \vec{r}_0 + \mathbf{\Phi}_{vv}(t) \cdot \delta \vec{v}_0$$

$$\mathbf{\Phi}_{rr}(t) = \begin{pmatrix} 4 - 3\cos nt & 0 & 0 \\ 6(\sin nt - nt) & 1 & 0 \\ 0 & 0 & \cos nt \end{pmatrix}$$

$$\mathbf{\Phi}_{rv}(t) = \begin{pmatrix} \frac{1}{n}\sin nt & \frac{2}{n}(1 - \cos nt) & 0 \\ \frac{2}{n}(\cos nt - 1) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \frac{1}{n}\sin nt \end{pmatrix}$$

$$\mathbf{\Phi}_{vr}(t) = \begin{pmatrix} 3n\sin nt & 0 & 0 \\ 6n(\cos nt - 1) & 0 & 0 \\ 0 & 0 & -n\sin nt \end{pmatrix}$$

$$\mathbf{\Phi}_{vv}(t) = \begin{pmatrix} \cos nt & 2\sin nt & 0 \\ -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & \cos nt \end{pmatrix}$$

## 5.7. Circular Restricted Three Body Problem

Non-dimensional units:

$$1\,\mathrm{LU} = |\vec{r}_1 - \vec{r}_2| \qquad 1\,\mathrm{TU} = \frac{\tau}{2\pi} \qquad 1\,\mathrm{MU} = m_1 + m_2$$

$$G = 1\,\mathrm{LU}^3/\mathrm{MU}/\mathrm{TU}^2 \qquad \Omega = 1\,\mathrm{rad}/\mathrm{TU}$$

Synodic frame:

$$\mu = \frac{m_2}{m_1 + m_2}$$

$$\vec{R}_1 = -\mu\vec{x} \qquad \vec{R}_2 = (1-\mu)\vec{x}$$

$$\vec{r}_1 = \vec{r} - \vec{R}_1 \qquad \vec{r}_2 = \vec{r} - \vec{R}_2$$

Equations of motion in synodic frame:

$$U = -\frac{1-\mu}{|\vec{r}_1|} - \frac{\mu}{|\vec{r}_2|} - \frac{1}{2}\left(r_1^2 + r_2^2\right)$$

$$\ddot{\vec{r}} = -\frac{1-\mu}{|\vec{r}_1|^3}\vec{r}_1 - \frac{\mu}{|\vec{r}_2|^3}\vec{r}_2 + \begin{pmatrix} r_1 \\ r_2 \\ 0 \end{pmatrix} + 2\begin{pmatrix} \dot{r}_2 \\ -\dot{r}_1 \\ 0 \end{pmatrix}$$

$$= -\nabla U + 2\begin{pmatrix} \dot{r}_2 \\ -\dot{r}_1 \\ 0 \end{pmatrix}$$

Jacobi energy:

$$C_J = -2U - \left|\dot{\vec{r}}\right|^2$$

$$= 2\left(\frac{1-\mu}{|\vec{r}_1|} + \frac{\mu}{|\vec{r}_2|}\right) + r_1^2 + r_2^2 - \dot{r}_1^2 - \dot{r}_2^2 - \dot{r}_3^2$$

Lagrange points:

$$\nabla U(\vec{r}) = 0$$

CHAPTER 6

# Astronautics

## 6.1. Remote Sensing

Ground-projected sample interval:

$$GSI = w\frac{H}{f}$$

Instantaneous field of view:

$$IFOV = 2\arctan\left(\frac{w}{2f}\right)$$

Field of view:

$$FOV \cong N \cdot IFOV$$

Ground-projected field of view (swath width):

$$GFOV = 2H\tan\left(\frac{FOV}{2}\right)$$

Data rate for pixel sample time $t_s$:

$$R_b = \frac{N \cdot Q}{t_s}$$

Ground speed of satellite:

$$v_{gd} = v_{orb}\frac{R_E}{r_{orb}}$$

Slant range at edge of coverage area:

$$R = \frac{R_E \sin\alpha}{\sin(\pi/2 - E - \alpha)}$$

Coverage angle for S/C elevation at edge of coverage area $E$:

$$\alpha = \arccos\left(\frac{R_E}{R_E + h}\cos E\right) - E$$

Antenna beam half-width:

$$\theta = \pi/2 - E - \alpha$$

Coverage diameter:

$$B = 2R_E\alpha$$

## 6.2. Subsystems

### Radiation

Solar flux:

$$J_{sun} = \frac{L_{sun}}{4\pi D^2}$$

Planetary radiation:

$$Q_p = \varepsilon J_p A_{proj-p} F_{s-p}$$

Albedo:

$$Q_a = \rho_a J_{sun} A_{proj-p} F_{s-p} \alpha_s \cos\phi$$

Spacecraft radiation:

$$Q_{s/c} = \varepsilon\sigma T^4 A_{s/c}$$

Thermal balance:

$$Q_{in} + Q_{dis} = Q_{out}$$

### Heat Transfer

$$\vec{Q} = A\vec{q}$$

Conductive heat transfer:

$$\vec{q} = -k\nabla T \qquad \text{(Fourier's law)}$$

Convective heat transfer:

$$q = h_c(T_s - T_\infty) \qquad \text{(Newton's cooling law)}$$

Radiative heat transfer:

$$q = \varepsilon\sigma T^4 \qquad \text{(Stefan-Boltzmann law)}$$

### Solar cells

Solar energy at surface for solar zenith angle $\theta_z$:

$$I_{sc} = J_{sun}A_s = J_{sun}A\cos\theta_z$$

Required solar array power:

$$P_{SA} = P_l\left(1 + \frac{V_{ch}}{t_{ch}}\frac{t_{dis}}{V_{dis}}RF\right)$$

Temperature effect on solar cell:

$$V = V_{ref} + \gamma_V(T - T_{ref}) \qquad I = I_{ref} + \gamma_I(T - T_{ref})$$

Solar cell efficiencies:

$$\eta_{sun} = \cos\theta \qquad \eta = \eta_{temp}\cdot\eta_{rad}\cdot\eta_{sun}$$

Battery capacity:

$$C_r = \frac{P_e \cdot t_e}{DoD\cdot\eta}$$

## 6.3. Reentry

Ballistic coefficient:

$$B = \frac{m}{C_D A}$$

### Planar flight

$$\frac{dv}{dt} = -\frac{D}{m} - g_0 \sin\gamma \qquad v\frac{d\gamma}{dt} = \frac{L}{m} - \left(g_0 - \frac{v^2}{r}\right)\cos\gamma$$

$$\frac{dr}{dt} = \frac{dh}{dt} = v\sin\gamma \qquad \frac{ds}{dt} = \frac{R}{r}v\cos\gamma$$

### Ballistic reentry

$$\beta = -2B\alpha\sin\gamma_0 \qquad \frac{v}{v_0} = \exp\left(-\frac{\rho}{\beta}\right)$$

$$t = -\frac{1}{\alpha v_0 \sin\gamma_0}\left(\ln\frac{\rho}{\rho_0} + \frac{\rho-\rho_0}{\beta} + \frac{\left(\frac{\rho-\rho_0}{\beta}\right)^2}{4} + \dots\right)$$

Maximum deceleration:

$$\ln\frac{v}{v_0} = -\frac{1}{2} \qquad \left.\frac{\dot{v}}{g}\right|_{max} = -\frac{v_0^2\sin\gamma_0\alpha\exp(-1)}{2g_0}$$

Peak heating:

$$\left.\frac{v}{v_0}\right|_{peak} = \exp\left(-\frac{1}{6}\right)$$

### Lift reentry

$$\frac{1}{2}\rho v^2\frac{C_L A}{W} = 1 - \frac{v^2}{v_c^2}$$

$$t_{land} = \frac{1}{2}\frac{L}{D}\frac{v_c}{g_0}\ln\left(\frac{1+\frac{v^2}{v_c^2}}{1-\frac{v^2}{v_c^2}}\right) \qquad S_{land} = -\frac{\frac{L}{D}}{2g_0}v_c^2\ln\left(1-\frac{v_0^2}{v_c^2}\right)$$

Maximum deceleration:

$$\left.\frac{\dot{v}}{g_0}\right|_{max} = \frac{1}{\frac{L}{D}}$$

Peak heating:

$$\left.\frac{v}{v_c}\right|_{peak} = \sqrt{\frac{2}{3}} \qquad Y_{max} = \frac{v_c^2}{5.2 g_0}\frac{L}{D}\frac{1}{\sqrt{1+0.106\left(\frac{L}{D}\right)^2}}$$

## 6.4. Space Debris

Estimated orbital life-time in LEO for ballistic coefficient $B$:

$$t_L = \frac{HB\tau}{2000\pi a^2 \rho} \qquad H = 266\,\mathrm{km}$$

### Magnitudes of secular rates

Luni-solar gravity in near-equatorial Earth orbit ($K_1 \ll 10^{-14}$):

$$\begin{aligned}\frac{de}{dt} =& K_1 e\sqrt{1-e^2}\,[0.089\sin(2\omega+\Omega) - \\ & 0.158\sin(2(\omega+\Omega)) - 0.1842\sin(2\omega)]\end{aligned}$$

Solar radiation pressure in near-equatorial Earth orbit ($K_2 \ll 10^{-10}$):

$$\frac{de}{dt} = -K_2 a^2\sqrt{1-e^2}\sin(\lambda_{sun}-\omega-\Omega)$$

Geopotential in near-equatorial Earth orbit ($K_3 \ll 10^{-3}$):

$$\frac{d\Omega}{dt} = -\frac{K_3 R_E^2}{a^2\left(1-e^2\right)^2} \qquad \frac{d\omega}{dt} = \frac{2K_3 R_E^2}{a^2\left(1-e^2\right)^2}$$

See also Section 5.5 on page 51.

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