UNIVERSITY OF SOUTHAMPTON

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Growth of homotopy groups

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Abstract

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This thesis studies the limiting behaviour of the torsion in the homotopy groups $\pi_n(X)$ of a space X as $n \to \infty$. It is a 'three paper thesis', the main body of which consists of the following papers:

[1] G. Boyde, Bounding size of homotopy groups of spheres. *Proceedings of the Edinburgh Mathematical Society*, 63(4):1100–1105, 2020.

[2] G. Boyde, *p-hyperbolicity of homotopy groups via K-theory*, preprint, available at arXiv:2101.04591 [math.AT], 2021.

[3] G. Boyde, *$\mathbb{Z}/p^r$-hyperbolicity via homology*, preprint, available at arXiv:2106.03516 [math.AT], 2021.

In [1], we improve on the best known bound for the size of the homotopy group $\pi_q(S^n)$, using the combinatorics of the EHP sequence.

In [2], we study Huang and Wu's p- and $\mathbb{Z}/p^r$-hyperbolicity for spaces related to the wedge of two spheres $S^n \vee S^m$. We show that $S^n \vee S^m$ is $\mathbb{Z}/p^r$-hyperbolic for all primes p and all $r \in \mathbb{N}$, which implies that various spaces containing $S^n \vee S^m$ as a retract are similarly hyperbolic. We then prove a K-theory criterion for p-hyperbolicity of a finite suspension ΣX, and deduce some examples.

In [3], we study p- and $\mathbb{Z}/p^r$-hyperbolicity for spaces related to the Moore space $P^n(p^r)$. When $p^s \neq 2$, we show that $P^n(p^r)$ is $\mathbb{Z}/p^s$-hyperbolic for $s \leq r$. Combined with Huang and Wu's work, and Neisendorfer's results on homotopy exponents, this completely resolves the question of when such a Moore space is $\mathbb{Z}/p^s$-hyperbolic for $p \geq 5$. We then prove a homological criterion for $\mathbb{Z}/p^r$-hyperbolicity of a space X, and deduce some examples.

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Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
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[1] G. Boyde, Bounding size of homotopy groups of spheres. *Proceedings of the Edinburgh Mathematical Society*, 63(4):1100–1105, 2020.

[2] G. Boyde, *p-hyperbolicity of homotopy groups via K-theory*, preprint, available at arXiv:2101.04591 [math.AT], 2021.

[3] G. Boyde, *$\mathbb{Z}/p^r$-hyperbolicity via homology*, preprint, available at arXiv:2106.03516 [math.AT], 2021.

Signed:.. Date:..................

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Notation

$\mathbb{Z}/\ell$	The cyclic group of order ℓ
S^n	The n-dimensional sphere
$P^n(\ell)$	The mod-ℓ Moore space, with $\widetilde{H}^m(P^n(\ell);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/\ell & m = n, \\ 0 & \text{otherwise.} \end{cases}$
$X \cong Y$	The spaces X and Y are homeomorphic
$X \simeq Y$	The spaces X and Y are homotopy equivalent
ΩX	The space of based loops on X
ΣX	The reduced suspension $S^1 \wedge X$ of X
$\langle t, x \rangle$	The image of $(t, x) \in S^1 \times X$ in the quotient ΣX
η	The unit map $X \longrightarrow \Omega\Sigma X$ of the adjunction $\Sigma \dashv \Omega$
ev	The counit (evaluation) map $\Sigma\Omega X \longrightarrow X$ of the adjunction $\Sigma \dashv \Omega$
X^k	The cartesian product of k copies of the space X
$X^{\vee k}$	The wedge sum of k copies of the space X
$X^{\wedge k}$	The smash product of k copies of the space X
JX	The James construction on X
$J_k(X)$	The k-th stage of the James construction on X

Chapter 0

Background

This introduction provides context for the three papers which comprise this thesis; we will refer to them as Paper 1 [Boy20], Paper 2 [Boy21b], and Paper 3 [Boy21a].

For a space X and $n \in \mathbb{N}$, one may define the *homotopy group* $\pi_n(X)$, which captures information about the continuous maps from the n-dimensional sphere S^n to X 'up to deformation'. Formally, $\pi_n(X)$ is the set $[S^n, X]$ of based homotopy classes of continuous maps $S^n \longrightarrow X$.

The homotopy groups of even relatively simple compact spaces like spheres are not known in their entirety, and in many cases are nonzero in arbitrarily high dimension. This provides a stark contrast with homology, which vanishes above the dimension of the space, and is often (reasonably) easy to compute. Our focus is on the asymptotics of the homotopy groups, that is, the behaviour of $\pi_n(X)$ as $n \to \infty$.

In rational homotopy theory, deep results on this subject have been obtained. Papers 2 and 3 fit into a program initiated by Huang and Wu, which investigates the asymptotics of the part of the homotopy groups that cannot be seen rationally. Paper 1 asks an older related question, namely 'can we bound the size of the homotopy groups of spheres?'.

1 Foundations

In this section, we collect the preliminaries necessary for the rest of the thesis. Throughout, we assume a basic knowledge of algebraic topology, notably the theory of singular (co)homology and the Universal Coefficient Theorem. These topics are covered in the book of Hatcher [Hat02]. Neisendorfer's book [Nei10] is an invaluable resource for the parts of homotopy theory which most closely concern us; especially the theory of Moore spaces and homotopy Lie algebras. Other textbooks which the reader may find useful are those of Arkowitz ([Ark11], for general homotopy theory), Hilton-Mislin-Roitberg ([HMR75], for localization) and Félix-Halperin-Thomas ([FHT15], for rational homotopy theory). We will appeal to all of these texts from time to time.

Perhaps most importantly of all, we need to be clear about what is meant by exponential growth.

Definition 1.1 (Exponential growth)**.** A sequence $(\alpha_k \mid k \in \mathbb{N})$ is said to *grow exponentially* if there exists $C > 1$ such that for k large enough we have

$$\alpha_k \geq C^k,$$

or equivalently if

$$\liminf_{k \to \infty} \frac{\ln(\alpha_k)}{k} > 0.$$

Our objects of study are topological spaces, so we now establish some simple conventions for these. We give only those definitions necessary to establish notation; more detail may be found in the textbooks of Arkowitz and Hatcher [Ark11, Hat02].

Throughout, all spaces X will be assumed to come with a basepoint $x_0 \in X$. When there is no ambiguity about which space we are talking about, we will often denote the basepoint by $*$. We will write $X \cong Y$ to mean that the spaces X and Y are homeomorphic as based topological spaces, and we will write $X \simeq Y$ to mean that X and Y are based homotopy equivalent.

Now let X and Y be spaces, with basepoints x_0 and y_0 respectively. The *product* $X \times Y$ is the ordinary cartesian product of topological spaces, with basepoint (x_0, y_0). The *wedge sum* $X \vee Y$ is the subspace of $X \times Y$ given by

$$X \vee Y := \{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\},$$

while the *smash product* of X and Y is the quotient

$$X \wedge Y := {}^{X \times Y}\!/\!_{X \vee Y}.$$

The product $X \times Y$ is the product in the category of based spaces, and the wedge $X \vee Y$ is the coproduct. In practice, this means that a pair of maps $A \longrightarrow X$ and $A \longrightarrow Y$ determine a unique map $A \longrightarrow X \times Y$, and moreover all maps $A \longrightarrow X \times Y$ are obtained in this way. Similarly, a pair of maps $X \longrightarrow Z$ and $Y \longrightarrow Z$ determine a unique map $X \vee Y \longrightarrow Z$, and moreover all maps $X \vee Y \longrightarrow Z$ are obtained in this way.

Let X be a space, and let $k \in \mathbb{N}$. We will write $X^{\vee k}$ for the wedge sum of k copies of X, and $X^{\wedge k}$ for the smash product of k copies of X, reserving the unadorned notation X^k for the ordinary product of k copies of X.

The *n-dimensional sphere* S^n is the subspace of the Euclidean space $\mathbb{R}^{n+1}$ given by

$$S^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} {x_i}^2 = 1\}.$$

We take the basepoint to be the 'North Pole' $(1, 0, 0, \ldots, 0)$. There is then a natural inclusion $S^{n-1} \hookrightarrow S^n$ obtained by restriction of the inclusion

$$\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1},$$

$$(x_0, x_1, \ldots, x_{n-1}) \longmapsto (x_0, x_1, \ldots, x_{n-1}, 0).$$

The sphere provides perhaps the most important example of the smash product, as follows.

Proposition 1.2. *We have a homeomorphism $S^n \wedge S^m \cong S^{n+m}$.*

Proof. The proposition will follow from the observation that (as unbased spaces) we have a homeomorphism $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$. Let X^c denote the one-point compactification of a space X [Rot88], and regard one-point compactification as a functor from unbased spaces to based spaces, by taking the point at infinity as the basepoint. There is then a homeomorphism $(\mathbb{R}^n)^c \cong S^n$.

Then, for unbased noncompact locally compact Hausdorff spaces X and Y we have $(X \times Y)^c \cong X^c \wedge Y^c$ [Rot88, Chapter 11]. Thus,

$$S^n \wedge S^m \cong (\mathbb{R}^n)^c \wedge (\mathbb{R}^m)^c \cong (\mathbb{R}^n \times \mathbb{R}^m)^c \cong (\mathbb{R}^{n+m})^c \cong S^{n+m},$$

as required. □

1.1 Finitely generated abelian groups

We depend heavily on the simplicity of the structure theory of finitely generated abelian groups. The key theorem is the following classical result, which can be found in for example [Lan02].

Theorem 1.3. *Let A be a finitely generated abelian group. Then A is isomorphic to a finite direct sum of (finite and infinite) cyclic groups.* □

That is, given a finitely generated abelian group A, we may write

$$A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{n} \mathbb{Z}/\ell_i.$$

We can go further; the Chinese Remainder Theorem says that if a and b are coprime, then $\mathbb{Z}/a \oplus \mathbb{Z}/b \cong \mathbb{Z}/ab$. This implies that the finite cyclic groups $\mathbb{Z}/\ell_i$ in the above decomposition of A may be decomposed as a direct sum of cyclic groups according to the prime factors of ℓ_i. In particular, in Theorem 1.3, we may assume that each finite cyclic group has prime power order.

We claim that if A is written as a direct sum of cyclic groups of infinite and prime power order, then the number of summands of given isomorphism type is independent of the choice of decomposition. To see this, note first that the number of $\mathbb{Z}$-summands is equal to the dimension of the rational vector space $A \otimes \mathbb{Q}$, so is certainly independent of the choice of decomposition. Next, fix p prime and $r \in \mathbb{N}$ and construct the $\mathbb{Z}/p$-vector space ${p^{r-1}A}/{p^r A}$. The dimension of this vector space is equal to the sum of

- the number of $\mathbb{Z}$-summands appearing in the decomposition, and
- the number of summands isomorphic to $\mathbb{Z}/p^s$ for some $s \geq r$.

It follows that the number of $\mathbb{Z}/p^r$-summands is equal to the difference $\dim_{\mathbb{Z}/p}({p^{r-1}A}/{p^r A}) - \dim_{\mathbb{Z}/p}({p^r A}/{p^{r+1} A})$, so is also independent of the choice of decomposition. All in all, we have the following well-known improvement on Theorem 1.3.

Theorem 1.4. *Let A be a finitely generated abelian group. Then A is isomorphic to a finite direct sum of infinite cyclic groups and finite cyclic groups of prime power order. Furthermore, the number of summands of given isomorphism type is well-defined independent of the choice of decomposition.* □

This theorem justifies the following definition, which is foundational in what follows.

Definition 1.5. Let A be a finitely generated abelian group. For p prime, and $r \in \mathbb{N}$, the *number of $\mathbb{Z}/p^r$-summands* in A is the number of $\mathbb{Z}/p^r$-summands occurring in some choice of decomposition of A into cyclic summands of infinite and prime power order. Similarly, we may speak of the *number of $\mathbb{Z}$-summands* in A.

1.2 Suspensions and loop spaces

In this subsection we introduce suspensions and loop spaces, establishing notation and laying the groundwork for the next subsection, where we will introduce homotopy groups, which are the central objects of this thesis. Much of what we will say about loop spaces and suspensions holds in the more general setting of H- and co-H-spaces [Ark11], but for the sake of brevity we will generally ignore this.

Let X and Y be based spaces. Write $C(X,Y)$ for the set of continuous based maps from X to Y and write $[X,Y]$ for the set of based homotopy classes of map from X to Y. It is immediate that as sets $[X,Y]$ is the quotient of $C(X,Y)$ by the equivalence relation defined by based homotopy. Write $[f] \in [X,Y]$ for the homotopy class of some map $f \in C(X,Y)$.

Fixing $f \in C(X,Y)$, we obtain, for any space W, a map

$$f_* : [W,X] \longrightarrow [W,Y],$$

$$[g] \longmapsto [f \circ g].$$

Similarly, for any space Z, f induces a map

$$f^* : [Y,Z] \longrightarrow [X,Z],$$

$$[g] \longmapsto [g \circ f].$$

Both maps depend only on the homotopy class $[f]$.

It will frequently be convenient to regard the circle S^1 as a quotient of the interval $I = [0,1] \subset \mathbb{R}$. Recall that we have defined S^1 to be the unit circle in $\mathbb{R}^2$, with basepoint $(1,0)$. Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ in the usual way, the map

$$I \longrightarrow \mathbb{C}$$

$$t \longmapsto e^{i2\pi t}$$

realises S^1 as the quotient ${}^{I}\!/\!_{(0 \sim 1)}$. When we refer to $t \in S^1$, we mean the point $e^{i2\pi t} \in S^1 \subset \mathbb{R}^2$.

For a space X, the *reduced suspension* ΣX is the smash product $S^1 \wedge X$. Write $\langle t,x \rangle$ for the image of $(t,x) \in S^1 \times X$ under the quotient map $S^1 \times X \longrightarrow S^1 \wedge X =: \Sigma X$. Following Arkowitz [Ark11], we think of $\langle t,x \rangle$ as giving 'coordinates' on ΣX, even though $t \in S^1$ and $x \in X$ are not in general uniquely determined by $\langle t,x \rangle \in \Sigma X$. These 'almost-coordinates' are often useful for writing down maps explicitly; for example, the *suspension* of a map $f : X \longrightarrow Y$ is the map

$$\Sigma f : \Sigma X \longrightarrow \Sigma Y,$$

$$\langle t,x \rangle \longmapsto \langle t, f(x) \rangle.$$

We immediately have the following corollary of Proposition 1.2.

Corollary 1.6. *We have* $\Sigma S^n \cong S^{n+1}$. □

Recall that we regard the wedge $A \vee B$ as a subspace of the product $A \times B$, so it naturally inherits coordinates from the product; a point of $A \vee B$ is precisely a point of $A \times B$ which takes either the form $(a, *)$ or $(*, b)$. With this notation, there is a map

$$c : \Sigma X \longrightarrow \Sigma X \vee \Sigma X,$$

$$\langle t, x \rangle \longmapsto \begin{cases} (\langle 2t, x \rangle, *) & \text{if } t \leq \frac{1}{2}, \text{ and} \\ (*, \langle 2t-1, x \rangle) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

This map is called the *suspension comultiplication* on ΣX, and makes ΣX into a co-H-space [Ark11, Chapter 2]. The comultiplication gives a group structure on the homotopy set $[\Sigma X, Y]$ for any space Y: the product of homotopy classes $[f]$ and $[g]$ is defined to be the homotopy class of the map

$$\Sigma X \xrightarrow{c} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y,$$

where $\nabla : Y \vee Y \longrightarrow Y$ is the *fold map*, which is defined to be the identity on each wedge summand.

We have the following first example, from Corollary 1.6.

Example 1.7. *For $n \geq 1$, the sphere S^n is homeomorphic to the suspension of S^{n-1}, and so, for any space X, the homotopy set $[S^n, Y]$ is a group.*

The *loop space* of X, denoted ΩX, is the set $C(S^1, X)$ of continuous based maps $S^1 \longrightarrow X$, which is made a topological space via the compact-open topology (see for example [Hat02]). For a map $f : X \longrightarrow Y$, and a loop $\gamma \in \Omega X$, the composite $f \circ \gamma$ is a loop in Y. We may therefore define a map, called the *loops* on f, to be the map

$$\Omega f : \Omega X \longrightarrow \Omega Y,$$

$$\gamma \longmapsto f \circ \gamma.$$

For $\gamma_1, \gamma_2 \in \Omega X$, define the *concatention* $\gamma_1 \# \gamma_2 \in \Omega X$ by setting

$$\gamma_1 \# \gamma_2(t) \longmapsto \begin{cases} \gamma_1(2t) & \text{if } t \leq \frac{1}{2}, \text{ and} \\ \gamma_2(2t-1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

There is then a map

$$\mu : \Omega X \times \Omega X \longrightarrow \Omega X,$$

$$(\gamma_1, \gamma_2) \longmapsto \gamma_1 \# \gamma_2.$$

The map μ will be referred to as the *concatenation multiplication*, or just the *multiplication* on ΩX, and makes ΩX into an H-group [Ark11, Chapter 2]. When there is no ambiguity, we will omit the # and write the multiplication as a concatenation, setting $\gamma_1\gamma_2 = \gamma_1\#\gamma_2$. For $\gamma \in \Omega X$, the *inverse* of γ, denoted γ^{-1}, is defined by $\gamma^{-1}(t) = \gamma(1-t)$. The H-group inverse is then the map

$$i : \Omega X \longrightarrow \Omega X,$$

$$\gamma \longmapsto \gamma^{-1}.$$

The multiplication gives a group structure on the homotopy set $[X, \Omega Y]$, for any space X, as follows. Let $\Delta : X \longrightarrow X \times X$ be the *diagonal map*, defined by $\Delta(x) = (x, x)$. The product of homotopy classes $[f]$ and $[g]$ in $[X, \Omega Y]$ is then defined to be the homotopy class of the map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} \Omega Y \times \Omega Y \xrightarrow{\mu} \Omega Y.$$

Let $\kappa : C(\Sigma X, Y) \longrightarrow C(X, \Omega Y)$ be defined by $\kappa(f)(x)(t) = f(\langle t, x\rangle)$. The map κ descends to give a map $[\Sigma X, Y] \longrightarrow [X, \Omega Y]$, which by abuse of notation we also call κ. The key proposition is the following.

Proposition 1.8. *[Ark11, Proposition 2.3.5] The map $\kappa : [\Sigma X, Y] \longrightarrow [X, \Omega Y]$ is an isomorphism of groups.* □

The image of a homotopy class f under this isomorphism (in either direction) will be called the *adjoint* of f and denoted $\overline{f}$.

1.3 Homotopy groups and Moore spaces

We are now ready to define the homotopy groups of a space, which are the topic of this thesis.

Definition 1.9. Let X be a based space, and let $n \in \mathbb{N}$. The *n-th (integral) homotopy group* of X, denoted $\pi_n(X)$ or $\pi_n(X; \mathbb{Z})$, is the homotopy set $[S^n, X]$, with group structure induced by the fact that $S^n \cong \Sigma S^{n-1}$, as in Example 1.7.

Recall the definitions of the homotopy fibre and homotopy cofibre from [Ark11]. For $\ell \in \mathbb{N}$, we will write $\ell : S^n \longrightarrow S^n$ for the degree ℓ map on the n-sphere.

Definition 1.10. Let $\ell \in \mathbb{N}$, and let $n \geq 2$. The *mod-ℓ Moore space $P^n(\ell)$* is defined by the homotopy cofibration

$$S^{n-1} \xrightarrow{\ell} S^{n-1} \longrightarrow P^n(\ell).$$

It is immediate that $P^{n+1}(\ell) \cong \Sigma P^n(\ell)$. Moore spaces are used to define homotopy groups with coefficients.

Definition 1.11. Let X be a based space, let $\ell \in \mathbb{N}$, and let $n \geq 2$. The *n-th homotopy group of X with coefficients in $\mathbb{Z}/\ell$*, denoted $\pi_n(X;\mathbb{Z}/\ell)$, is the homotopy set $[P^n(\ell), X]$, with group structure induced by the fact that $P^n(\ell) \cong \Sigma P^{n-1}(\ell)$.

A priori, the homotopy groups with coefficients are only $\mathbb{Z}$-modules, although we will see in Proposition 1.36 that unless $p^r = 2$, the homotopy group $\pi_n(X;\mathbb{Z}/p^r)$ is a $\mathbb{Z}/p^r$-module. We will write $\pi_*(X)$ for the graded abelian group whose degree-n component is $\pi_n(X)$, and similarly for the homotopy groups with coefficients.

Proposition 1.12. *Let $R = \mathbb{Z}$ or $\mathbb{Z}/\ell$. For any space X, and any $n \in \mathbb{N}$, there is an isomorphism*

$$\pi_{n+1}(X;R) \cong \pi_n(\Omega X;R).$$

Proof. For the integral homotopy groups, we have, by Proposition 1.8, that

$$\pi_{n+1}(X) \cong [S^{n+1}, X] \cong [\Sigma S^n, X] \cong [S^n, \Omega X] \cong \pi_n(\Omega X).$$

The case with coefficients in $\mathbb{Z}/p^r$ is identical. □

A space is said to be *n-connected* if $\pi_i(X) = 0$ for all $i \leq n$. In the case $n = 1$ we say that X is *simply connected*. By the *higher homotopy groups*, we mean the homotopy groups $\pi_n(X)$ for $n \geq 2$.

Proposition 1.13. *[Ark11, Proposition 2.3.8] For $n \geq 2$, and for any space X, the homotopy groups $\pi_n(X)$ are abelian. Similarly, for $n \geq 3$, and for any space X, the homotopy groups $\pi_n(X;\mathbb{Z}/\ell)$ are abelian.* □

By Proposition 1.12, $\pi_2(X) \cong \pi_1(\Omega X)$, so a loop space will always have abelian fundamental group.

From our point of view, the most important foundational results on the structure of the homotopy groups are the above proposition, and the next theorem, which is due to Serre. First recall that for any space X, the *Hurewicz map* is a homomorphism $h : \pi_n(X) \longrightarrow H_n(X)$, natural in maps of spaces, and defined as follows. Notice that an element of $\pi_n(X)$ is the homotopy class of some map $f : S^n \longrightarrow X$, and recall that

$$\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } i = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Choose a generator ξ_n of $\widetilde{H}_n(S^n)$, and define the Hurewicz map by setting $h([f]) = f_*(\xi_n) \in H_n(X)$. This is well defined, since the induced map f_* depends only on the homotopy class $[f]$. In some circumstances, we will require some particular good behaviour from the Hurewicz map, and this will be achieved by careful choice of the generators ξ_n.

In [Ser53b], Serre develops the notion of a class $\mathscr{C}$ of abelian groups. A collection of groups $\mathscr{C}$ is said to be a *class* if $\mathscr{C}$ is closed under isomorphism and

1. $\mathscr{C}$ contains the trivial group,
2. $\mathscr{C}$ is closed under quotients and subgroups, and
3. $\mathscr{C}$ is closed under extensions.

Serre proves a 'Hurewicz theorem mod-$\mathscr{C}$', which is as follows.

Theorem 1.14. *[Ser53b] Let X be a simply connected space. If the homology groups $H_i(X)$ belong to $\mathscr{C}$ for $0 < i < n$, then the homotopy groups $\pi_i(X)$ belong to $\mathscr{C}$ for $0 < i < n$, and the kernel and cokernel of the Hurewicz map $\pi_n(X) \longrightarrow H_n(X)$ belong to $\mathscr{C}$.* □

One recovers the ordinary Hurewicz theorem by taking $\mathscr{C}$ to be the class containing only the trivial group. We are concerned only with the following corollary.

Corollary 1.15. *[Ser53b] Let X be a simply connected finite CW-complex. Then the homotopy groups $\pi_i(X)$ are finitely generated for each $i \in \mathbb{N}$.*

Proof. Since X is a finite CW-complex, all homology groups of X are finitely generated. The result then follows from Theorem 1.14 with $\mathscr{C}$ equal to the class of finitely generated abelian groups. □

By Proposition 1.13 and Corollary 1.15, given a simply connected finite CW-complex X, the homotopy groups $\pi_n(X)$ are all finitely generated abelian. For p prime, and $r \in \mathbb{N}$, we may therefore speak of 'the number of $\mathbb{Z}/p^r$-summands' occurring in some homotopy group $\pi_n(X)$ (Definition 1.5) and this number will be well-defined and finite.

Another important result is Whitehead's so-called second theorem, which we now state.

Theorem 1.16. *[Whi49] Let $f : X \longrightarrow Y$ be a map between simply connected CW-complexes. If $f_* : H_i(X) \longrightarrow H_i(Y)$ is an isomorphism for each i, then f is a homotopy equivalence.* □

As an example, we have the following proposition, which gives a kind of 'prime power factorization' for Moore spaces.

Proposition 1.17. *Let $n \geq 3$. If $\ell \in \mathbb{N}$ has a prime power factorization $\ell = p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}$ then*
$$P^n(\ell) \simeq P^n(p_1^{r_1}) \vee P^n(p_2^{r_2}) \vee \dots \vee P^n(p_m^{r_m}).$$

Proof. Define a map $f : P^n(p_1^{r_1}) \vee P^n(p_2^{r_2}) \vee \dots \vee P^n(p_m^{r_m}) \longrightarrow P^n(\ell)$ which is given on the wedge summand $P^n(p_i^{r_i})$ as degree 1 on the top cell and degree $\frac{\ell}{p_i^{r_i}}$ on the bottom cell; that is, according to the following diagram of defining cofibrations.

$$\begin{tikzcd}
S^{n-1} \arrow[r, "p_i^{r_i}"] \arrow[d, equal] & S^{n-1} \arrow[r] \arrow[d, "\frac{\ell}{p_i^{r_i}}"] & P^n(p_i^{r_i}) \arrow[d, "f|_{P^n(p_i^{r_i})}"] \\
S^{n-1} \arrow[r, "\ell"] & S^{n-1} \arrow[r] & P^n(\ell)
\end{tikzcd}$$

By the Chinese Remainder Theorem, f induces an isomorphism on integral homology. Thus, by Whitehead's theorem (Theorem 1.16), f is a homotopy equivalence. □

In both Papers 2 and 3 we will make use of the so-called stable homotopy groups. The existence of these groups is justified by the Freudenthal Suspension Theorem, which we now state. First, note that by [Ark11, Lemma 5.6.3] suspension gives a homomorphism $\Sigma : \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$ for all spaces X.

Theorem 1.18. *[Fre38] Let X be an $(r-1)$-connected space for $r \geq 2$. The suspension homomorphism*
$$\Sigma : \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$$
is an isomorphism for $i < 2r - 1$ and an epimorphism for $i = 2r - 1$. □

Consider the sequence $\pi_{n+k}(\Sigma^k X)$ as $k \to \infty$. Letting $r - 1$ be the connectivity of X, we have that the connectivity of $\Sigma^k X$ is $(r + k) - 1$. Thus, the suspension map $\pi_{n+k}(\Sigma^k X) \longrightarrow \pi_{n+k+1}(\Sigma^{k+1} X)$ is an isomorphism, provided $n + k < 2(r + k) - 1$. This condition rearranges to $n < 2r + k - 1$, which holds for large k, since r and n are fixed. Assuming that X is connected, we have $r \geq 0$, so the suspension map will certainly be an isomorphism for $n < k - 1$. In particular, for $k > n + 1$, the isomorphism type of the group $\pi_{n+k}(\Sigma^k X)$ is independent of k (and these groups are identified by canonical isomorphisms). We may therefore make the following definition.

Definition 1.19. Let X be a space. The *n-th stable (integral) homotopy group* of X is the group
$$\pi_n^S(X) := \pi_{n+k}(\Sigma^k X) \qquad (k > n + 1).$$

1.4 Localization

Localization provides a way to do homotopy theory 'one prime at a time'. In this subsection we record the basic theory, following [HMR75].

First, let P be a set of primes. The *integers localized at* P, denoted $\mathbb{Z}_P$, is the subring of $\mathbb{Q}$ consisting of those rationals $\frac{a}{b}$ where b is not divisible by any prime $p \in P$.

Two cases deserve particular emphasis. Firstly, if $P = \{p\}$ consists of only a single prime, then we write $\mathbb{Z}_{(p)} = \mathbb{Z}_P$. This ring is called the *integers localized at* p. If $P = \emptyset$, then the localization $\mathbb{Z}_P$ is equal to the whole of $\mathbb{Q}$.

Following [HMR75], we will write P' for the collection of primes not in P, and if some integer n is a product of primes in P', we will abuse notation to write $n \in P'$.

Definition 1.20 (P-local groups and spaces)**.** Let P be a set of primes. A group G is said to be *P-local* if the map $G \longrightarrow G$ given by $x \longmapsto x^n$ is an isomorphism for all $n \in P'$. A simply connected CW-complex X is said to be P-local if the homotopy groups of X are all P-local.

To say that a space is P-local is therefore to say that it carries only homotopy information corresponding to the set of primes P. Localizing a space at P is the process of stripping away all of the information corresponding to primes not in P to obtain a P-local object, as follows.

Definition 1.21. Let $f : X \longrightarrow Y$ be a map of simply connected CW-complexes. We say that f *P-localizes* X if

1. Y is P-local, and
2. $f^* : [Y, Z] \longrightarrow [X, Z]$ is an isomorphism for all P-local simply connected CW-complexes Z.

This definition is really saying that a P-localizing map is universal with respect to maps from X to P-local spaces. We expect objects satisfying universal properties to be unique up to an appropriate notion of isomorphism, and in this case we have the following.

Proposition 1.22. *[HMR75, Chapter II] If the maps $f_1 : X \longrightarrow Y_1$ and $f_2 : X \longrightarrow Y_2$ both P-localize X, then there is a homotopy equivalence $h : Y_1 \longrightarrow Y_2$ with $h \circ f_1 \simeq f_2$, and these properties uniquely determine the homotopy class of h.* □

Uniqueness is all very well, but is not terribly helpful without existence. The next theorem gives us existence.

Theorem 1.23. *[HMR75, Theorem II.1.1A] Every simply connected CW-complex X admits a P-localization. We write $X \longrightarrow X_P$ for a fixed choice of P-localization of X.* $\square$

In the case that $P = \{p\}$, we will write $X_{(p)} = X_P$. In the case that $P = \emptyset$, we will write $X_{\mathbb{Q}} = X_P$, and call this the *rationalization* of X.

Definition 1.24. Let X and Y be simply connected CW-complexes, and let P be a set of primes. If the P-localizations X_P and Y_P are homotopy equivalent, then we say that X and Y are P-locally equivalent, and write $X \simeq_P Y$.

The next theorem is the way that we recognise P-localizations in practice. We first need to know what it means to P-localize an abelian group. A homomorphism of abelian groups $e : A \longrightarrow A_P$ is said to be *P-localizing* if A_P is P-local (Definition 1.20) and the pullback map $e^* : \mathrm{Hom}(A_P, B) \longrightarrow \mathrm{Hom}(A, B)$ is an isomorphism for all P-local abelian groups B. For a given abelian group A, it is immediate that the natural map $A \longrightarrow A \otimes \mathbb{Z}_P$ is a P-localizing map for A, so all P-localizing maps for A are identified with this one, up to unique isomorphism.

Theorem 1.25. *[HMR75, Theorem II.1.1B] Let $f : X \longrightarrow Y$ be a map of simply connected CW-complexes. The following are equivalent.*

1. *The map f P-localizes X.*
2. *The map $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$ P-localizes for all $n \geq 1$.*
3. *The map $f_* : H_n(X) \longrightarrow H_n(Y)$ P-localizes for all $n \geq 1$.* $\square$

In short, a map of spaces is a P-localization precisely when it looks like one from the point of view of homology, or equivalently of homotopy. As a first application of this theory, we have the following proposition, which complements Proposition 1.17.

Proposition 1.26. *For primes $q \neq p$, and $r \in \mathbb{N}$, the q-localization of the mod-p^r Moore space $(P^n(p^r))_{(q)}$ is contractible.*

Proof. For $q \neq p$ prime, the homology $H_*(P^n(p^r); \mathbb{Z}_{(q)})$ is trivial. By the Universal Coefficient Theorem, the group $H_*(P^n(p^r); \mathbb{Z}) \otimes \mathbb{Z}_{(q)}$ must also be trivial for each n, and by Theorem 1.25 we have $H_*(P^n(p^r)_{(q)}; \mathbb{Z}) \cong H_*(P^n(p^r); \mathbb{Z}) \otimes \mathbb{Z}_{(q)}$. The inclusion of the basepoint of $P^n(p^r)_{(q)}$ is therefore a homology isomorphism, and hence, by Whitehead's theorem (Theorem 1.16), is actually a homotopy equivalence. $\square$

Notice that for p prime, and $r \in \mathbb{N}$, the group $\mathbb{Z}/p^r$ is p-local. By the Universal Coefficient Theorem for homology, the homology $H_*(X; \mathbb{Z}/p^r)$ depends only on the p-localization $X_{(p)}$. Likewise, the Universal Coefficient Theorem for homotopy [Nei10, Theorem 1.3.1] implies that the homotopy $\pi_*(X;\mathbb{Z}/p^r)$ depends only on the p-localization $X_{(p)}$. That is, homotopy and homology with coefficients in $\mathbb{Z}/p^r$ are properly thought of as invariants of the p-local homotopy type of X.

Lastly, localization allows for the following reformulation of a classical theorem of Serre.

Theorem 1.27. *[Ser51a] When p is odd, there is a p-local homotopy equivalence*

$$\Omega S^{2n} \simeq_p S^{2n-1} \times \Omega S^{4n-1}. \qquad \square$$

1.5 Samelson and Whitehead Products

In this subsection we introduce Samelson and Whitehead products, following the presentation in [Nei10].

Fix a space X, and let R be $\mathbb{Z}$ or $\mathbb{Z}/p^r$. Our main goal for this subsection is to define the Samelson product, which will be a bilinear operation

$$\pi_n(\Omega X;R) \times \pi_m(\Omega X;R) \longrightarrow \pi_{n+m}(\Omega X;R).$$

We will make much use of Samelson products in Papers 2 and 3. By applying the adjunction isomorphism (Proposition 1.12) to all three groups, we will obtain a related operation, the Whitehead product. This is a bilinear operation

$$\pi_n(X;R) \times \pi_m(X;R) \longrightarrow \pi_{n+m-1}(X;R),$$

which does not require that the space be a loop space.

Consider a loop space ΩX. The *H-group commutator* on ΩX is the map

$$\bar{c} : \Omega X \times \Omega X \longrightarrow \Omega X$$

$$(\gamma, \theta) \longmapsto \gamma\theta\gamma^{-1}\theta^{-1}.$$

The restriction of $\bar{c}$ to the wedge $\Omega X \vee \Omega X$ is nullhomotopic, so $\bar{c}$ descends to a map

$$c : \Omega X \wedge \Omega X \longrightarrow \Omega X.$$

Furthermore, by [Nei10, Proposition 6.6.3], the natural map

$$[\Omega X \wedge \Omega X, \Omega X] \longrightarrow [\Omega X \times \Omega X, \Omega X]$$

is an injection, so the homotopy class of c is uniquely determined.

We may now define the external Samelson product. Suppose given maps $f : A \longrightarrow \Omega X$ and $g : B \longrightarrow \Omega X$. The *external Samelson product* $c(f, g)$ is the composite

$$A \wedge B \xrightarrow{f \wedge g} \Omega X \wedge \Omega X \xrightarrow{c} \Omega X.$$

The homotopy class of the external Samelson product depends only on the homotopy classes of f and g. If $A = S^n$ and $B = S^m$ are spheres, we have a natural homeomorphism $S^{n+m} \cong S^n \wedge S^m$ (Proposition 1.2). In this case, f, g and $c(f, g)$ may all be regarded as elements of $\pi_*(\Omega X)$, in dimensions n, m, and $n + m$ respectively. With this in mind, we have the following definition.

Definition 1.28. Let X be a simply connected CW-complex, and let $f \in \pi_n(\Omega X)$, and $g \in \pi_m(\Omega X)$. The *integral Samelson product* $\langle f, g\rangle \in \pi_{n+m}(\Omega X)$ is the composite

$$\langle f, g\rangle : S^{n+m} \cong S^n \wedge S^m \xrightarrow{f \wedge g} \Omega X \wedge \Omega X \xrightarrow{c} \Omega X.$$

In particular, the Samelson product may be regarded as an operation on $\pi_*(\Omega X)$ which is additive on grading. We have the following basic properties.

Proposition 1.29. *[Whi78] The Samelson product*

$$\langle\ ,\ \rangle : \pi_n(\Omega X) \times \pi_m(\Omega X) \longrightarrow \pi_{n+m}(\Omega X)$$

satisfies the following properties, for all $f \in \pi_n(\Omega X)$, $g \in \pi_m(\Omega X)$, and $h \in \pi_\ell(\Omega X)$.

1. *It is bilinear.*
2. $\langle f, g\rangle = -(-1)^{nm}\langle g, f\rangle$.
3. $\langle f, \langle g, h\rangle\rangle = \langle\langle f, g\rangle, h\rangle + (-1)^{nm}\langle g, \langle f, h\rangle\rangle$. □

Bilinearity means that we may equivalently think of the Samelson product as a graded map $\pi_*(\Omega X) \otimes \pi_*(\Omega X) \longrightarrow \pi_*(\Omega X)$, and we will frequently do so. Recall that we write $\overline{f}$ for the image of a map f under either the isomorphism of Proposition 1.12 or the inverse of that isomorphism, and that we call this map the adjoint of f.

Definition 1.30 (Integral and external Whitehead products)**.** Let X be a simply connected CW-complex. Let $f \in \pi_n(X)$ and let $g \in \pi_m(X)$. The *integral Whitehead product* $[f, g] \in \pi_{n+m-1}(X)$ is the adjoint $\overline{\langle \overline{f}, \overline{g}\rangle}$ of the Samelson product of the adjoints $\overline{f}$ and $\overline{g}$.

In the external setting, let $c : \Omega X \wedge \Omega X \longrightarrow \Omega X$ be as usual. Given maps $f : \Sigma A \longrightarrow X$ and $g : \Sigma B \longrightarrow X$, the *external Whitehead product* $[f, g]_w : \Sigma(A \wedge B) \longrightarrow X$ is the adjoint $\overline{c(\overline{f}, \overline{g})}$ of the external Samelson product of the adjoints $\overline{f}$ and $\overline{g}$.

Setting up internal mod-p^r Samelson products is more complicated, because $P^{n+m}(p^r) \not\simeq P^n(p^r) \wedge P^m(p^r)$ (for example, the two sides have different homology with coefficients in $\mathbb{Z}/p$). We will instead find a suitably canonical map $P^{n+m}(p^r) \longrightarrow P^n(p^r) \wedge P^m(p^r)$ which we can compose with the external Samelson product to obtain an operation on $\pi_*(\Omega X; \mathbb{Z}/p^r)$. Interestingly, this operation does not behave in the same way for all prime powers p^r.

Fix a prime p and $r \in \mathbb{N}$, and consider the homotopy cofibration

$$S^{n-1} \xrightarrow{p^r} S^{n-1} \longrightarrow P^n(p^r) \longrightarrow S^n \xrightarrow{p^r} S^n$$

defining $P^n(p^r)$. Taking the smash product of a cofibration and a (locally compact) space again yields a cofibration, so $P^n(p^r) \wedge P^m(p^r)$ fits into a cofibration sequence

$$S^{n-1} \wedge P^m(p^r) \xrightarrow{p^r \wedge 1} S^{n-1} \wedge P^m(p^r) \longrightarrow P^n(p^r) \wedge P^m(p^r) \longrightarrow S^n \wedge P^m(p^r) \to \dots,$$

which, since $S^i \wedge P^n(p^r)$ is just $P^{n+i}(p^r)$, is the same as

$$P^{m+n-1}(p^r) \xrightarrow{p^r} P^{m+n-1}(p^r) \xrightarrow{\iota} P^n(p^r) \wedge P^m(p^r) \xrightarrow{\tau} P^{m+n}(p^r) \to \dots,$$

where we give the names ι and τ to the indicated maps.

Since the co-H-space structure on $P^{n+m-1}(p^r)$ comes from the fact that it is a suspension, the map that we have labelled p^r above really is p^r times the identity in the homotopy set $[P^{n+m-1}(p^r), P^{n+m-1}(p^r)]$. Now, $P^n(p^r)$ is a 'mod-p^r' object, so one might hope that p^r times its identity map was nullhomotopic. This is almost always true, as in the following proposition, in which different primes behave differently.

Proposition 1.31. *[Nei10, Proposition 6.1.7] Let $m \geq 3$, let p be prime, and let $r \in \mathbb{N}$. If $p^r \neq 2$, then $p^r : P^m(p^r) \longrightarrow P^m(p^r)$ is nullhomotopic.* □

Assume now that $p^r \neq 2$. By [Nei10, Lemma 6.2.1], the fact that p^r is nullhomotopic implies that there exists a section for τ, that is, a map s such that $\tau \circ s$ is homotopic to the identity on $P^{n+m}(p^r)$. Since $P^n(p^r) \wedge P^m(p^r)$ is a suspension, this implies the existence of a homotopy equivalence

$$\iota \vee s : P^{n+m-1}(p^r) \vee P^{n+m}(p^r) \xrightarrow{\simeq} P^n(p^r) \wedge P^m(p^r).$$

At this stage, it is convenient to reintroduce the dimensions n and m to the notation. Write $\Delta_{n,m} = s$. Since $\Delta_{n,m}$ is a choice of splitting, we might worry about whether it is unique up to homotopy. In fact, it is not, but it is unique enough for our purposes. We are assured of this by the next proposition.

Recall first that $\widetilde{H}_*(P^n(p^r);\mathbb{Z}/p^r)$ is a free $\mathbb{Z}/p^r$-module on two generators e_n and s_{n-1}, in dimensions n and $n-1$ respectively. Recall the definition of the external Whitehead product from Definition 1.30. For a space X, the *Whitehead subgroup* $\mathrm{Wh}_*(X;\mathbb{Z}/p^r)$ is defined to be the subgroup generated by all compositions

$$P^*(p^r) \xrightarrow{f} \bigvee \Sigma(A \wedge B) \xrightarrow{g} X,$$

where f is any map, and g is any bouquet of external Whitehead products.

Proposition 1.32. *[Nei10, Corollary 6.4.5] In this proposition, homology is taken with coefficients in $\mathbb{Z}/p^r$. Let p be an odd prime. The induced map*

$$(\Delta_{n,m})_* : \widetilde{H}_*(P^{n+m}(p^r)) \to \widetilde{H}_*(P^n(p^r) \wedge P^m(p^r)) \cong \widetilde{H}_*(P^n(p^r)) \otimes \widetilde{H}_*(P^m(p^r))$$

satisfies $(\Delta_{n,m})_*(e_{n+m}) = e_n \otimes e_m$, and this condition characterizes $\Delta_{n,m}$ uniquely up to addition of elements of $\mathrm{Wh}_{n+m}(P^n(p^r) \wedge P^m(p^r);\mathbb{Z}/p^r)$. □

We care about $\Delta_{n,m}$ because it allows us to define mod-p^r Samelson products, so the ambiguity in the homotopy type of $\Delta_{n,m}$ is tolerable as long as it does not produce any ambiguity in the Samelson products. The following lemma is the key.

Lemma 1.33. *Let Y and Z be simply connected CW-complexes. If $\varphi : Y \longrightarrow \Omega Z$ is any map, then the Whitehead subgroup $\mathrm{Wh}_*(Y;\mathbb{Z}/p^r)$ is contained in the kernel of $\varphi_* : \pi_*(Y;\mathbb{Z}/p^r) \longrightarrow \pi_*(\Omega Z;\mathbb{Z}/p^r)$.*

Proof. From the definition of the Whitehead subgroup, it suffices to show that the composite

$$\Sigma A \wedge B \xrightarrow{[f,g]_w} Y \xrightarrow{\varphi} \Omega Z$$

of φ with any external Whitehead product is nullhomotopic. By Proposition 1.8, it suffices to show that the adjoint of this composite is nullhomotopic.

One can show that the isomorphism of Proposition 1.8 is a categorical adjunction. This entails certain naturality properties, sometimes called triangle identities. In particular, the adjoint of the above map is $(\Omega\varphi) \circ \overline{[f,g]_w}$, which by definition of the external Whitehead product (Definition 1.30) is the composite

$$A \wedge B \xrightarrow{\overline{f} \wedge \overline{g}} \Omega Y \wedge \Omega Y \xrightarrow{c} \Omega Y \xrightarrow{\Omega\varphi} \Omega^2 Z.$$

Since $\Omega\varphi$ is a loop map, we have a commutative diagram

$$\begin{tikzcd}
\Omega Y \wedge \Omega Y \arrow[r, "\Omega\varphi\wedge\Omega\varphi"] \arrow[d, "c"] & \Omega^2 Z \wedge \Omega^2 Z \arrow[d, "c"] \\
\Omega Y \arrow[r, "\Omega\varphi"] & \Omega^2 Z.
\end{tikzcd}$$

Now, by definition, the map c is the unique map which has image $\bar{c}$ under the natural pullback $[\Omega^2 Z \wedge \Omega^2 Z, \Omega^2 Z] \longrightarrow [\Omega^2 Z \times \Omega^2 Z, \Omega^2 Z]$. But $\bar{c}$ is the honest commutator of the two projections in the group $[\Omega^2 Z \times \Omega^2 Z, \Omega^2 Z]$. By [Ark11, Proposition 2.3.8], the double loop space $\Omega^2 Z$ is a homotopy-commutative H-space, so the group $[X, \Omega^2 Z]$ is abelian for all spaces X. But this means that in particular $\bar{c} : \Omega^2 Z \times \Omega^2 Z \longrightarrow \Omega^2 Z$ is nullhomotopic, so $c : \Omega^2 Z \wedge \Omega^2 Z \longrightarrow \Omega^2 Z$ is also nullhomotopic. We have seen that $(\Omega\varphi) \circ \overline{[f,g]_w}$ factors through c, so this map is also nullhomotopic, as required. $\square$

We may now make the following definition.

Definition 1.34. Let p be an odd prime, and let $r \in \mathbb{N}$. Let X be a simply connected CW-complex, and let $f \in \pi_n(\Omega X; \mathbb{Z}/p^r)$, and $g \in \pi_m(\Omega X; \mathbb{Z}/p^r)$. The *mod-$p^r$ Samelson product* $\langle f, g \rangle \in \pi_{n+m}(\Omega X; \mathbb{Z}/p^r)$ is the composite

$$\langle f, g \rangle : P^{n+m}(p^r) \xrightarrow{\Delta_{n,m}} P^n(p^r) \wedge P^m(p^r) \xrightarrow{f \wedge g} \Omega X \wedge \Omega X \xrightarrow{c} \Omega X.$$

By Lemma 1.33 the homotopy type of $\langle f, g \rangle$ is independent of our choice of the section $\Delta_{n,m}$. The requirement that p be odd is necessary for Proposition 1.31.

The basic properties of mod-p^r Samelson products are as follows.

Proposition 1.35. *[Nei10, Proposition 6.7.2] The Samelson product*

$$\langle \ , \ \rangle : \pi_n(\Omega X; \mathbb{Z}/p^r) \times \pi_m(\Omega X; \mathbb{Z}/p^r) \longrightarrow \pi_{n+m}(\Omega X; \mathbb{Z}/p^r)$$

satisfies the following properties, for all $f \in \pi_n(\Omega X; \mathbb{Z}/p^r)$, $g \in \pi_m(\Omega X; \mathbb{Z}/p^r)$, and $h \in \pi_\ell(\Omega X; \mathbb{Z}/p^r)$.

1. *It is bilinear.*
2. $\langle f, g \rangle = -(-1)^{nm} \langle g, f \rangle$.
3. *If $p > 3$, then $\langle f, \langle g, h \rangle \rangle = \langle \langle f, g \rangle, h \rangle + (-1)^{nm} \langle g, \langle f, h \rangle \rangle$.* $\square$

We conclude this subsection by noticing that Proposition 1.31 has an important corollary for the structure of mod-p^r homotopy groups.

Corollary 1.36. *If $m \geq 3$ and $p^r \neq 2$, then the homotopy group $\pi_m(Y; \mathbb{Z}/p^r)$ is a $\mathbb{Z}/p^r$-module for all spaces Y.* $\square$

Our inability to prove Corollary 1.36 when $p^r = 2$ is a real phenomenon, rather than a sign of inadequacy in the techniques. We do, however, have the following.

Proposition 1.37. *[Nei80, Proposition 7.1] If $m \geq 3$ then $\pi_m(Y; \mathbb{Z}/2)$ is a $\mathbb{Z}/4$-module.* $\square$

1.6 Lie algebras

For the sake of completeness, we follow Neisendorfer's definition of a graded Lie algebra [Nei10], in order to avoid having to insist that 2 be a unit in the ground ring. We write $\deg(x)$ for the degree of a homogeneous element x in some graded module.

Definition 1.38. Graded Lie algebra Let R be a commutative ring. A graded Lie algebra L over R is a graded R-module together with two operations:

1. bilinear pairings, called *Lie brackets*,
$$[\ ,\ ] : L_m \otimes L_n \longrightarrow L_{m+n},$$
and
2. a 'quadratic' operation, called *squaring*, defined on odd-dimensional classes,
$$(\ )^2 : L_k \longrightarrow L_{2k},$$
such that $(ax)^2 = a^2x^2$ and $(x+y)^2 = x^2 + y^2 + [x,y]$ for all $a \in R$ and $x, y \in L$ of equal odd degree.

These operations must satisfy the following identities.

1. $[x,y] = -(-1)^{\deg(x)\deg(y)}[y,x]$ for all $x, y \in L$.
2. $[x,[y,z]] = [[x,y],z] + (-1)^{\deg(x)\deg(y)}[y,[x,z]]$, for all $x, y, z \in L$.
3. $[x,x] = 0$ for all $x \in L$ with $\deg(x)$ even.
4. $2x^2 = [x,x]$ and $[x,x^2] = 0$ for all $x \in L$ with $\deg(x)$ odd.
5. $[y,x^2] = [[y,x],x]$ for all $x, y \in L$ with $\deg(x)$ odd.

As Neisendorfer remarks, if 2 is a unit in R, then by Identity 4 we have $x^2 = \frac{1}{2}[x,x]$, so the squaring operation may be recovered from the Lie bracket. This vastly simplifies the definition when 2 is a unit, removing the need for the squaring operation altogether. Specifically, in the axioms, one may omit all reference to the squaring operation, and add the requirement that $[x,[x,x]] = 0$ for all x of odd degree. Since 2 is invertible, Identity 3 follows from Identity 1, and may be omitted. If 3 is also invertible, the new requirement that $[x,[x,x]] = 0$ follows from Identity 2, and may be omitted. These simplifications will apply in Subsection 3.2, when we work over $\mathbb{Q}$.

Our first example is as follows. It forms much of the backbone of Paper 3.

Example 1.39. *Let X be any space, let $r \in \mathbb{N}$, and let $p \geq 5$ be prime. By Corollary 1.36, the homotopy groups $\pi_*(\Omega X; \mathbb{Z}/p^r)$ are $\mathbb{Z}/p^r$-modules, and by Proposition 1.35, Identities 1 and 2 are satisfied. Since 2 is invertible we need not define the squaring operation. By the preceding remarks, since 2 and 3 are invertible, this suffices to establish that $\pi_*(\Omega X; \mathbb{Z}/p^r)$ is a graded Lie algebra.*

Our second example plays the analogous role in Paper 2.

Example 1.40. *If all of the identities of Definition 1.38 are satisfied apart from $[x, x^2] = 0$ for odd-dimensional x, then Neisendorfer calls the resulting object a* quasi-graded Lie algebra. *Localized away from 2, the integral homotopy groups of a loop space form a quasi-graded Lie algebra under the Samelson product [Nei10, Chapter 8]. Again, since 2 is invertible we need not define the squaring operation.*

Lastly, we have an important algebraic example.

Example 1.41. *A graded Lie algebra may be obtained from any graded associative algebra A by taking $L = A$ with the Lie bracket*

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx,$$

and squaring operation equal to the ordinary algebra squaring. In what follows, we take all algebras to also carry the structure of Lie algebras in this way. This is what is meant, for instance, if we say that a map $L \longrightarrow A$ is a map of Lie algebras, for a Lie algebra L and an associative algebra A.

The *Universal enveloping algebra* of a graded Lie algebra L is an algebra $U(L)$, together with a map of Lie algebras $L \longrightarrow U(L)$ such that if A is any graded associative algebra and $L \xrightarrow{\varphi} A$ is a map of Lie algebras, then there exists a unique map $\widetilde{\varphi}$ such that the following diagram commutes.

$$\begin{tikzcd} U(L) \arrow[r, "\widetilde{\varphi}"] & A \\ L. \arrow[u] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

2 Paper 1 in context

The broad goal of this paper is to bound the size of $\pi_q(S^n)$. Serre (Theorem 1.15) showed that these groups are finitely generated abelian, and that they contain a single $\mathbb{Z}$-summand when $q = n$, or when n is even and $q = 2n - 1$, and are finite otherwise. We may therefore restrict our attention to the p-torsion summand for each prime p.

When p is odd it suffices to consider n odd, because, again by work of Serre (Theorem 1.27), we have a p-local equivalence

$$\Omega S^{2n} \simeq_p S^{2n-1} \times \Omega S^{4n-1}.$$

For us, the *rank* of a finitely generated module will be the size of a minimal generating set. Selick [Sel82] proved that the rank of $\pi_q(S^n_{(p)})$, regarded as a $\mathbb{Z}_{(p)}$-module, is at most 3^{q^2}. Bödigheimer and Henn [BH83] prove that the base-p logarithm of the cardinality of the p-torsion part of $\pi_q(S^n)$ is at most $3^{(q-\frac{n}{2})}$, for n odd and all primes p. By the same method, they prove that the rank of $\pi_q(S^n_{(p)})$ satisfies the same bound. In [Hen86], Henn further improved this bound to 2^{q-n+1}. In [Iri87], Iriye states the bound $3^{\frac{q-n}{2p-3}}$, which is actually slightly better than our Theorem 2.1, but gives no proof.

For a prime p, let $s_p(n,q)$ denote the base-p logarithm of the cardinality of the p-torsion part of $\pi_q(S^n)$.

Theorem 2.1 (Paper 1 [Boy20, Theorem 1.1])**.** *For all natural numbers q, and n odd,*

$$s_p(n,q) \leq 2^{\frac{1}{p-1}(q-n+3-2p)}.$$

If $p=2$ then this bound holds also for n even. □

As a corollary we obtain the weaker but simpler bound $s_p(n,q) \leq 2^{\frac{q-n}{p-1}}$. The $3-2p$ which appears in Theorem 2.1 reflects the classical fact that the first p-torsion classes appear in the $(2p-3)$-rd stem. We can think of the bound as an exponential function of the stem $q-n$ in base $2^{\frac{1}{p-1}}$. The main advantage of this bound over its predecessors is that as p becomes large, the base of the exponential approaches 1, so our bound grows more slowly for larger primes.

As in [BH83], we also obtain a bound on the rank, but we prefer to regard it as following from Theorem 1.1, by using that the rank of a finite p-torsion group is at most $\log_p$ of its cardinality.

Corollary 2.2 (Paper 1 [Boy20, Corollary 1.2])**.** *For n odd and $q \geq 1$, the rank of the p-torsion part of $\pi_q(S^n)$ is at most $2^{\frac{q-n}{p-1}}$. If $p=2$, this bound holds also for n even.* □

Our method follows that of the preceding papers on the topic; we analyse the combinatorics of the EHP sequences of James and Toda [Jam57, Tod56]. This leads to the following lemma, which was first proven in [BH83].

Lemma 2.3. *For all q, and n odd:*

1. $s_p(n,q) \leq s_p(p(n-1)+1,q) + s_p(p(n-1)-1,q-1) + s_p(n-2,q-2)$ *if* $q \neq p(n-1)$.

2. *$s_p(n,q) \leq 1 + s_p(n-2, q-2)$ if $q = p(n-1)$.*

When $p = 2$, we no longer need to restrict to odd n, and the first inequality can be replaced by

$$s_2(n,q) \leq s_2(2n-1,q) + s_2(n-1,q-1). \qquad \square$$

All of our bounds are stable; that is, they depend only on the difference $q - n$. The stable homotopy groups of spheres have been computed up to the 90-stem with some uncertainties, the most recent paper of this sort being that of Isaksen, Wang and Xu [IWX20]. They make the following conjecture, which we give in a 'stable' version of our notation. For a prime p, let $s_p(k)$ denote the base-p logarithm of the cardinality of the p-torsion part of $\pi_{n+k}(S^n)$, for any $k \geq n+2$. This is well defined, since the isomorphism type of $\pi_{n+k}(S^n)$ is independent of n in this range (see the discussion preceding Definition 1.19).

Conjecture 2.4. *[IWX20, Conjecture 1.5] There exists a nonzero constant C such that*

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} s_2(k)}{k^2} = C.$$

This suggests that, at least stably, much better bounds than Theorem 2.1 ought to be possible. On the other hand, we have the following result, which makes precise the intuition that one cannot do better than exponential by merely improving the combinatorics; new input from topology is required.

Corollary 2.5 (Paper 1 [Boy20, Corollary 3.1])**.** *Let F_n denote the n-th Fibonacci number. Any bound on the base-p logarithm of the cardinality of the p-primary part of $\pi_{2p+j(4p+5)+n-3}(S^n)$ (or the rank of that group) which can be obtained from Lemma 2.3 is greater than or equal to F_{2j+1}. In particular, any base for an exponential bound which can be obtained from Lemma 2.3 is at least $\phi^{\frac{2}{4p+5}}$, where ϕ denotes the golden ratio.* $\square$

We add as a footnote that it is very well possible that better bounds on the size of homotopy groups of finite complexes are possible in the stable range than are possible in general. In particular, the suspension of a Whitehead product is always trivial [Whi46, Theorem 3.11], so none of the families of classes identified so far in the study of local hyperbolicity (in the papers [HW20, ZP21] or in Papers 2 and 3) survives to the stable range. This makes sense intuitively; the stable range consists of precisely those dimensions which are too low to accommodate any nontrivial Whitehead products, which are the essential scaffolding used so far to show that particular spaces are locally hyperbolic.

3 Background to Papers 2 and 3

Papers 2 and 3 concern local hyperbolicity; a new concept due to Huang and Wu [HW20]. Local hyperbolicity is one approach to the study of the global properties of homotopy groups, and there are many older results which are of interest in this broader context. In this section, we collect some of these results, concluding with a discussion of Huang and Wu's work.

3.1 Classical results

In this subsection, we record some classical results on the structure of homotopy groups. First of all, recall Corollary 1.15 from Subsection 1.3, which is due to Serre [Ser53b], and tells us that the homotopy groups of any simply connected finite CW-complex are finitely generated.

Next, we have the following theorem, which is due to Serre when $p = 2$ and Umeda when p is odd.

Theorem 3.1. *[Ser53a, Ume59] Let p be a prime, and let X be a simply connected space such that*

1. *$H_i(X;\mathbb{Z})$ is finitely generated for all $i > 0$,*
2. *$H_i(X;\mathbb{Z}/p) = 0$ for all sufficiently large i, and*
3. *$H_i(X;\mathbb{Z}/p) \neq 0$ for some $i > 0$.*

Then there exist infinitely many values of i such that $\pi_i(X)$ has a subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p$. □

Said another way, the conclusion of the theorem is that there are infinitely many values of i such that $\pi_i(X)$ contains a summand isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p^r$ for some $r \in \mathbb{N}$.

Thirty years later, this result was improved by McGibbon and Neisendorfer:

Theorem 3.2. *[MN84] Let p be a prime, and let X be a simply connected space such that*

1. *$H_i(X;\mathbb{Z}/p) = 0$ for all sufficiently large i, and*
2. *$H_i(X;\mathbb{Z}/p) \neq 0$ for some $i > 0$.*

Then there exist infinitely many values of i such that $\pi_i(X)$ has a subgroup isomorphic to $\mathbb{Z}/p$. □

Notice that if X is a finite CW-complex, then Hypothesis 1 is automatic. Consider the inclusion of the basepoint of the p-localization $* \longrightarrow X_{(p)}$. By Whitehead's second theorem (Theorem 1.16), if this map induces an isomorphism on integral homology, then it must actually be a homotopy equivalence. The integral homology of $X_{(p)}$ is isomorphic to $H_*(X;\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$ by Theorem 1.25, so, by the Universal Coefficient Theorem, the inclusion of the basepoint will be an isomorphism if and only if the reduced homology $\widetilde{H}_*(X;\mathbb{Z}/p)$ vanishes. That is, the second hypothesis of McGibbon and Neisendorfer's result is equivalent to asking that the localization $X_{(p)}$ not be contractible. We therefore have the following corollary.

Corollary 3.3. *Let p be a prime, and let X be a simply connected finite CW-complex. Then either*

1. *The localization $X_{(p)}$ is contractible (and hence $\pi_i(X)$ contains no summands isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p^r$) or*
2. *there exist infinitely many values of i such that $\pi_i(X)$ has a summand isomorphic to $\mathbb{Z}/p^r$ for some $r \in \mathbb{N}$.* □

We will see that this corollary provides an interesting contrast with the rational situation.

3.2 Rational homotopy theory

The program to which the work in this thesis belongs was initiated by Huang and Wu [HW20]. Their inspiration came principally from rational homotopy theory. Rational homotopy theory studies properties of spaces and maps that depend only on their rational homotopy type. By Theorem 1.25 and the Universal Coefficient Theorem we have

$$H_*(X_{\mathbb{Q}};\mathbb{Z}) \cong H_*(X;\mathbb{Q})$$

and

$$\pi_*(X_{\mathbb{Q}}) \cong \pi_*(X) \otimes \mathbb{Q}.$$

We will refer to the groups $\pi_*(X) \otimes \mathbb{Q}$ as the *rational homotopy groups* of X. Note in particular that the rational homotopy groups tell us where the $\mathbb{Z}$-summands were in the integral homotopy groups, since tensoring with $\mathbb{Q}$ kills torsion summands, and turns copies of $\mathbb{Z}$ into copies of $\mathbb{Q}$.

Dealing only with the rational information simplifies things in a way that is often useful. A simple example is provided by the homotopy groups of spheres. By Corollary 3.3, there are p-torsion summands in arbitrarily high dimension in the homotopy groups of S^n ($n \geq 2$) for every prime p. On the other hand, Serre [Ser53b] showed that the rational homotopy groups $\pi_q(S^n) \otimes \mathbb{Q}$ are isomorphic to $\mathbb{Q}$ when $q = n$, or when n is even and $q = 2n - 1$, and are trivial otherwise. That is, rationally, spheres 'look finite' in a way that they very much do not integrally. We can see already here that rational homotopy retains information that the homology groups do not, since $\pi_{4n-1}(S^{2n}) \otimes \mathbb{Q} \cong \mathbb{Q}$.

In fact, more can be said about the relationship between rational homotopy and rational homology, thanks to the following theorem of Henn.

Theorem 3.4. *[Hen83] Any co-H-space is rationally equivalent to a wedge of spheres.* □

Necessarily, one sphere appears in the wedge for each $\mathbb{Q}$-summand appearing in rational homology. In particular, any suspension is a co-H-space, so looks rationally and stably like a wedge of spheres. Since the Hurewicz map is an isomorphism in the stable range for wedges of spheres, we immediately obtain the following corollary.

Corollary 3.5. *The rational stable Hurewicz map*

$$\pi_*^S(X) \otimes \mathbb{Q} \longrightarrow H_*(X : \mathbb{Q})$$

is an isomorphism for all spaces X. □

That is to say, after rationalizing and stabilising, homotopy no longer knows anything that homology does not. From this point of view, one can see how rational homotopy theory and stable homotopy theory come to be so powerful.

A great deal is known about the global structure of the rational homotopy groups of a space, and this is what inspired Huang and Wu to investigate the corresponding torsion situation, as we will see in Subsection 3.4.

In order to state the theorems which follow, we need the notion of rational LS-category. Let X be a simply connected space. Recall that the ordinary Lusternik-Schnirelmann category of X is the least m such that X is the union of $m+1$ open subsets, each of which is contractible in X. We write $\mathrm{cat}(X) = m$. Let the *rational Lusternik-Schnirelmann category*, or *rational category* of X, denoted $\mathrm{cat}_0(X)$, be defined by

$$\mathrm{cat}_0(X) = \min\{\mathrm{cat}(Y) \mid Y \simeq_{\mathbb{Q}} X,\ \pi_1(Y) = 0\}.$$

That is, $\mathrm{cat}_0(X)$ is the least ordinary LS-category of a simply connected space which is rationally equivalent to X. We have [FHT15, Proposition 28.1] that $\mathrm{cat}_0(X) = \mathrm{cat}(X_{\mathbb{Q}})$.

From our point of view, the most important theorem of rational homotopy theory is the following. Recall (Definition 1.1) that a sequence α_k is said to grow exponentially if there exists $C > 1$ such that for large enough k we have $\alpha_k > C^k$, or equivalently if $\liminf_{k\to\infty} \frac{\ln(\alpha_k)}{k} > 0$.

Theorem 3.6 (The rational dichotomy [FHT15, Theorem 33.2])**.** *Let X be a simply connected space with finite rational category and rational homology of finite type. Then either*

1. *$\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, and X is said to be **rationally elliptic**, or*
2. *$\sum_{i=1}^{k} \dim(\pi_*(X) \otimes \mathbb{Q})$ grows exponentially in k, and X is said to be **rationally hyperbolic**.* $\square$

Note that a finite CW-complex automatically has finite rational homology, and has finite ordinary LS-category, hence finite rational category. Any simply connected finite CW-complex, therefore, satisfies the dichotomy.

As a first example, we have seen that the rational homotopy of S^n has finite total dimension, so S^n is rationally elliptic for each n. More generally, one checks whether a space is elliptic or hyperbolic by means of the following results. The *formal rational dimension n_X* of a space X is the largest integer n such that $H^n(X;\mathbb{Q}) \neq 0$.

Theorem 3.7. *[FHT15, Theorem 33.3] Suppose that X is a simply connected topological space with finite dimensional rational homology and formal rational dimension n_X. Then either*

1. *$\pi_i(X) \otimes \mathbb{Q} = 0$ for $i \geq 2n_X$, or else*
2. *for each $k \geq 1$, there exists i with $k < i < k + n_X$ such that $\pi_i(X) \otimes \mathbb{Q} \neq 0$.* $\square$

Of course, in Case 1, X is rationally elliptic, and in Case 2, X is rationally hyperbolic. This means that if we know the formal dimension of X, then X is elliptic if and only if $\pi_i(X) \otimes \mathbb{Q}$ vanishes for $2n_X \leq i \leq 3n_X - 2$. This is a big improvement, since we only need to check a finite dimensional range.

Various results which add detail to Theorem 3.6 have been proven, and the remainder of this section records some of them.

First, and most centrally, the relationship between homotopy and loop-homology is very well behaved rationally. Recall the definition of a graded Lie algebra L, and of its universal enveloping algebra UL, from Subsection 1.6. We do not need to know the definition of a Hopf algebra - it will suffice to know that a Hopf algebra is in particular an algebra, and that an isomorphism of Hopf algebras is an isomorphism of algebras [MM65]. For a space Y, let

$$\Delta : Y \longrightarrow Y \times Y$$

be the diagonal, defined by $\Delta(y) = (y, y)$. Recall that the Künneth theorem identifies $H_*(Y \times Y;\mathbb{Q}) \cong H_*(Y;\mathbb{Q}) \otimes H_*(Y;\mathbb{Q})$. A homology class $\alpha \in H_*(Y;\mathbb{Q})$ is called *primitive* if $\Delta_*(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$ under the Künneth identification. Recall the definition of the integral Samelson product (Definition 1.28), and of the universal enveloping algebra (Subsection 1.6). We then have the following structure theorem, which is due to Milnor and Moore.

Theorem 3.8. *[MM65] If X is a simply connected space then*

1. *Samelson products make $\pi_*(\Omega X) \otimes \mathbb{Q}$ into a graded Lie algebra, denoted by L_X.*

2. *The Hurewicz homomorphism for ΩX is an isomorphism of L_X onto the Lie algebra $P_*(\Omega X; \mathbb{Q})$ of primitive elements in $H_*(\Omega X; \mathbb{Q})$.*

3. *The Hurewicz homomorphism extends to an isomorphism of graded Hopf algebras*

$$UL_X \xrightarrow{\cong} H_*(\Omega X; \mathbb{Q}). \qquad \square$$

The good behaviour of the Hurewicz homomorphism is now plain: by point 3, the Hurewicz map for ΩX may be regarded as the natural inclusion of the Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ into its universal enveloping algebra $H_*(\Omega X; \mathbb{Q})$. This has an interesting consequence for the dimensions of the rational homotopy groups. For a space X, let P_X be the power series in z given by $\sum_{n=0}^{\infty} \dim H_n(X; \mathbb{Q})$. Then we have the following.

Theorem 3.9. *[FHT15, Formula 33.7, Proposition 33.10] Let $r_i = \dim(\pi_i(X) \otimes \mathbb{Q})$. Then*

$$P_{\Omega X} = \frac{\prod_{i=1}^{\infty}(1 + z^{2i+1})^{r_{2i+1}}}{\prod_{i=1}^{\infty}(1 + z^{2i})^{r_{2i}}},$$

and furthermore, the power series $P_{\Omega X}$ and $\sum_{i=1}^{\infty} r_i z^i$ have the same radius of convergence, R, such that

1. *$R = 1$ if X is rationally elliptic, and $R < 1$ if X is rationally hyperbolic.*

2. *If X is rationally hyperbolic and if $H^i(X; \mathbb{Q}) = 0$ for $i > n_X$ then $R < K < 1$ for some constant K depending only on n_X.* $\square$

As one might expect, this implies that loop homology suffices to determine whether a space is elliptic or hyperbolic.

Theorem 3.10. *[FHT15, Proposition 33.8] Suppose that $H^i(X; \mathbb{Q}) = 0$ for $i > n_X$. Then the integers $\dim H_i(\Omega X; \mathbb{Q})$ for $2n_X - 1 \leq i \leq 3n_X - 3$ determine whether X is rationally elliptic or rationally hyperbolic.* $\square$

Finally, we have the following result, which describes qualitatively the growth of the loop homology in the elliptic and hyperbolic cases. For a graded module M, M_{odd} denotes the direct sum of the components in odd gradings, and M_{even} denotes the direct sum of the components in even gradings.

Theorem 3.11. *[FHT15, Proposition 33.9]*

1. *If X is rationally elliptic then there exist constants $A < B \in \mathbb{R}_{>0}$ such that*

$$An^r \leq \sum_{i=0}^{n} \dim H_i(\Omega X;\mathbb{Q}) \leq Bn^r,\ n \geq 1,$$

where $r = \dim(\pi_{\mathrm{odd}}(X) \otimes \mathbb{Q})$.

2. *If X is rationally hyperbolic and if $H^i(X;\mathbb{Q}) = 0$ for $i > n_X$ then there exist constants $C > 1$ and $K \in \mathbb{N}$ such that*

$$\sum_{i=k+1}^{k+2(n_X-1)} \dim H_i(\Omega X;\mathbb{Q}) \geq C^k,\ k \geq K. \qquad \square$$

3.3 Moore's Conjecture and the exponent problem

In this subsection we discuss Moore's conjecture and the exponent problem; two topics which are intimately related to local hyperbolicity. We begin with a definition.

Definition 3.12. Let X be a space. The *homotopy exponent* of X at some prime p is the smallest power of p which annihilates the p-torsion in $\pi_*(X)$; if no such power exists then we say that X *has no homotopy exponent* at the prime p.

For the avoidance of doubt, by the *p-torsion* in a group G we mean the subgroup consisting of those elements whose order is a power of p.

Moore's conjecture proposes that the behaviour of the rational homotopy groups of a space is intimately related to the exponent.

Conjecture 3.13 (Moore's conjecture)**.** *Let X be a simply connected finite CW-complex. The following are equivalent.*

1. *X is rationally elliptic.*
2. *X has a finite homotopy exponent at some prime p.*
3. *X has a finite homotopy exponent at all primes p.*

Resolving Moore's conjecture even for spheres is challenging. Of course, in dimension 1, things are easy, since S^1 is a $K(\mathbb{Z}, 1)$, but higher dimensional spheres present more of a problem. We have seen that Serre showed that S^n is rationally elliptic, but does it have finite exponent? The answer is yes, as in the following theorem, which is due to James when $p = 2$, and to Toda when p is odd.

Theorem 3.14. *[Jam57, Tod58] For all primes p, the p-primary homotopy exponent of the sphere S^{2n+1} is at most p^{2n}.* $\square$

There is no loss in treating only odd-dimensional spheres: by Serre's p-local decomposition of the loops on an even-dimensional sphere (Theorem 1.27), James and Toda's results imply that the exponent of S^{2n} is at most p^{4n-2}.

Now, suppose that p is odd. Gray [Gra69] showed that the homotopy groups of S^{2n+1} contain classes of order p^n, which implies that the homotopy exponent of S^{2n+1} must be at least p^n. Selick [Sel78] showed that the homotopy exponent of S^3 is p. These results enabled Cohen, Moore, and Neisendorfer [CMN79b, CMN79a] to resolve the exponent precisely;

Theorem 3.15. *[CMN79a, Gra69] Let p be an odd prime. The p-primary homotopy exponent of S^{2n+1} is precisely p^n.* □

It now follows that the exponent of S^{2n} is precisely p^{2n-1}. Amusingly, Cohen, Moore, and Neisendorfer did not set out to compute the exponents of spheres; they originally intended to study the exponents of Moore spaces [CMN79b]. They do not resolve the exponents of Moore spaces in those papers, but later, in [Nei87], Neisendorfer shows that $\pi_*(P^n(p^r))$ has exponent p^{r+1} for $p \geq 5$. In fact, Neisendorfer claimed in [Nei87] that this result also holds when $p = 3$, but later, with Gray, discovered some mistakes in the proof (see the unpublished [Nei]). These mistakes were repaired apart from when $p = 3$. In [Nei], Neisendorfer shows that the 3-primary exponent of $P^n(3^r)$ is either 3^{r+1} or 3^{r+2}. Theriault [The08] has shown that the exponent of $P^n(2^r)$ is 2^{r+1} for $r \geq 6$, that it is at most 2^{r+2} for $3 \leq r \leq 5$, and that it is at most 32 when $r = 2$. These results are summarised in the following theorem.

Theorem 3.16. *[Nei87, Nei, The08] The homotopy exponent of the Moore space $P^n(p^r)$ is*

- *p^{r+1} when $p \geq 5$,*
- *either p^{r+1} or p^{r+2} when $p = 3$,*
- *p^{r+1} when $p = 2$ and $r \geq 6$,*
- *at most p^{r+2} when $p = 2$ and $3 \leq r \leq 5$, and*
- *at most 32 when $p = 2$ and $r = 2$.* □

Moore spaces $P^n(p^r)$ are contractible after localization at a prime different from p (Proposition 1.26) hence are rationally contractible. Moore spaces (with $p^r \neq 2$) are therefore a second class of spaces which is known to satisfy Moore's conjecture. When $p^r = 2$, no exponent is known.

Question 3.17. *Does the Moore space $P^n(2)$ have a 2-primary homotopy exponent?*

Moore's conjecture remains open, but various partial results are known. Perhaps most notably, we have the following theorem of McGibbon and Wilkerson, where by 'almost all primes' we mean 'all but perhaps finitely many primes' or equivalently 'all sufficiently large primes'.

Theorem 3.18. *[MW86] Let X be a simply connected finite CW-complex. Suppose that X is rationally elliptic. Then, for almost all primes p, there is a p-local homotopy equivalence*

$$\Omega X \simeq_p \prod_i S^{2m_i-1} \times \prod_j \Omega S^{2n_j-1}. \qquad \square$$

We have known since James and Toda's work (Theorem 3.14) that spheres have finite exponents, so this implies the following partial resolution of Moore's conjecture.

Corollary 3.19. *Let X be a simply connected finite CW-complex. If X is rationally elliptic, then X has a homotopy exponent at almost all primes.* $\square$

On the hyperbolic side, we have for example the following result of Selick, which resolves one direction of Moore's conjecture for suspensions whose homology is torsion-free.

Theorem 3.20. *[Sel83] Let p be an odd prime, and let X be a finite CW-complex with $H_*(X)$ torsion-free. If ΣX is rationally elliptic, then ΣX has no homotopy exponent at p.* $\square$

The reason for including this result in particular is that Selick's methods provide the blueprint for the methods of Paper 2.

3.4 Local hyperbolicity

The study of local hyperbolicity was initiated by Huang and Wu in [HW20]. Their goal was to investigate the torsion analogues of the theorems of Section 3.2, especially the rational dichotomy of Theorem 3.6. Recall (Definition 1.1) that a sequence α_m is said to grow exponentially if there exists $C > 1$ such that for large enough m we have $\alpha_m > C^m$, or equivalently if $\liminf_{m\to\infty} \frac{\ln(\alpha_m)}{m} > 0$.

Definition 3.21. [HW20, Definition 1.2] Let X be a space, and let p be a prime. Let T_m be the number of p-torsion summands in $\bigoplus_{i\leq m} \pi_i(X)$. We say that X is *p-hyperbolic* if T_m grows exponentially in m.

The above definition counts $\mathbb{Z}/p^r$-summands for all values of r. It is also possible to consider only a single r, and by doing so we obtain the definition of $\mathbb{Z}/p^r$-hyperbolicity.

Definition 3.22. [HW20, Definition 1.1] Let X be a space, let p be a prime, and fix $r \in \mathbb{N}$. Let t_m be the number of $\mathbb{Z}/p^r$-summands in $\bigoplus_{i \leq m} \pi_i(X)$. We say that X is *$\mathbb{Z}/p^r$-hyperbolic* if t_m grows exponentially in m.

When there is no ambiguity, in either case we may say that the space X is *locally hyperbolic*. This terminology is justified, since both above definitions depend only on the p-localization $X_{(p)}$ of X. Note that $\mathbb{Z}/p^r$-hyperbolicity for any particular r implies p-hyperbolicity.

Selick and Wu [SW00, SW06] introduced a family of functors $\widetilde{Q}_n^{\max}$ for $n \geq 2$, together with a functor $\widetilde{A}^{\min}$. These functors allow a general functorial decomposition of the loop-suspension $\Omega\Sigma X$ of any path-connected CW-complex X as

$$\Omega\Sigma X \simeq \widetilde{A}^{\min}(X) \times \Omega\Big(\bigvee_{n=2}^{\infty} \widetilde{Q}_n^{\max}(X)\Big).$$

In terms of these functors, Huang and Wu sketch a theorem statement which illustrates their approach.

Theorem 3.23. *[HW20, Theorem 1.5] Suppose X is the p-localization of a path-connected finite CW-complex. If*

1. *$\Sigma^* X$ is a homotopy retract of $\widetilde{Q}_*^{\max}(X)$,*
2. *there exists a map $\Sigma^* X \vee \Sigma^* X \longrightarrow \Sigma^* X^{\wedge *}$ which admits a left homotopy inverse, and*
3. *there is a $\mathbb{Z}/p^r$-summand in each $\pi_*(\Sigma^* X)$,*

then ΣX is $\mathbb{Z}/p^r$-hyperbolic. The symbol $*$ here refers to various arithmetic sequences which Huang and Wu define precisely later in the paper; they must satisfy some mild arithmetic conditions. □

The functorial decomposition of $\Omega\Sigma X$ allows for this general statement, but in practice one needs only to know some adequate decomposition of the space in question; it is not important to know its relation to this very sophisticated and general decomposition.

Huang and Wu establish the first example satisfying their definitions, as follows.

Theorem 3.24. *Let $n \geq 3$ and $r \geq 1$. The Moore space $P^n(p^r)$ is $\mathbb{Z}/p^r$ and $\mathbb{Z}/p^{r+1}$-hyperbolic. Additionally, $P^n(2)$ is $\mathbb{Z}/8$-hyperbolic.* □

Theorem 3.24 fits into the general pattern that Huang and Wu set out in Theorem 3.23. The main ingredient necessary for this is a loop-decomposition; and these are known for Moore spaces. Since at this stage we are only sketching the situation, for brevity we give only the even-dimensional odd-primary decomposition.

Theorem 3.25. *[CMN79b, Theorem 1.1] Let p be an odd prime, and let $n > 0$. Then*

$$\Omega P^{2n+2}(p^r) \simeq S^{2n+1}\{p^r\} \times \Omega \bigvee_{m=0}^{\infty} P^{4n+2mn+3}(p^r). \qquad \square$$

Notice that this is of the same form as Selick and Wu's general decomposition. When $p^r = 2$ no analogous decomposition is known.

The second paper dealing with local hyperbolicity is that of Zhu and Pan [ZP21]. Zhu and Pan are concerned with the category of A_n^2-*complexes*; that is, $(n-1)$-connected finite CW-complexes of dimension at most $n+2$. Any A_n^2-complex necessarily has a cell-structure which has cells only in dimensions $n, n+1$, and $n+2$. We say that $X \in A_n^2$ is *indecomposable* if whenever $X \simeq A \vee B$ for A_n^2-complexes A and B, then either A or B is contractible. The indecomposable complexes in A_n^2 were classified by Chang [Cha50], and are as follows.

- Spheres S^n, S^{n+1}, and S^{n+1}.
- Moore spaces $P^n(p^r)$ and $P^{n+1}(p^r)$, for p prime and $r \in \mathbb{N}$.
- The *elementary Chang complexes*:
  - $C_\eta^{n+2} = S^n \cup_\eta CS^{n+1}$,
  - $C^{n+2,s} = (S^n \vee S^{n+1}) \cup_{\left(\begin{smallmatrix} \eta \\ 2^s \end{smallmatrix}\right)} CS^{n+1}$,
  - $C_r^{n+2} = S^n \cup_{(2^r\ \eta)} C(S^n \vee S^{n+1})$, and
  - $C_r^{n+2,s} = (S^n \vee S^{n+1}) \cup_{\left(\begin{smallmatrix} 2^r & \eta \\ 0 & 2^s \end{smallmatrix}\right)} C(S^n \vee S^{n+1})$.

  In the above, η is an appropriate suspension of the Hopf map, and the matrix $(f_{i,j}) : \bigvee_{i=1}^n X_i \longrightarrow \bigvee_{j=1}^m Y_j$ represents the map whose composition with the inclusion of X_i and the projection onto Y_j is $f_{i,j}$. Such maps exist whenever the X_i are co-H-spaces.

Most of the complexity of Chang's classification arises at the prime 2: after localization at an odd prime, each indecomposable complex in A_n^2 is homotopy equivalent to a sphere or a Moore space. Zhu and Pan therefore begin by studying 2-primary hyperbolicity of the elementary Chang complexes, proving the following two theorems.

Theorem 3.26. *[ZP21, Theorem 1.2] The Chang complexes $C_r^{n+2,r}$ $(n \geq 4)$ are $\mathbb{Z}/2^i$-hyperbolic for $i = 1, r, r+1$.* $\square$

Theorem 3.27. *[ZP21, Theorem 1.3] Let C be either C_η^{n+2}, C_r^{n+2}, $C^{n+2,s}$, or $C_r^{n+2,s}$ with $r \neq s$. Suppose that $n \geq 4$.*

Let $u_C = \begin{cases} 1 & \text{if } C = C_\eta^{n+2}, \\ r & \text{if } C = C_r^{n+2}, \\ s & \text{if } C = C^{n+2,s}, \text{ and} \\ \min(r,s) & \text{if } C = C_r^{n+2,s}. \end{cases}$

Then C *is* $\mathbb{Z}/2$*-hyperbolic if* $u_C = 1$ *and* $\mathbb{Z}/2^i$ *hyperbolic for* $i = 1, u_C, u_C + 1$ *if* $u_C > 1$. □

With this understanding of the 2-primary behaviour, Zhu and Pan are able to prove the following theorem, which reduces the question $\mathbb{Z}/p$-hyperbolicity in A_2^n to the question of whether the spheres S^n, S^{n+1}, and S^{n+2} are $\mathbb{Z}/p$-hyperbolic.

Theorem 3.28. *[ZP21, Theorem 1.4] Let* A *be an* A_n^2 *complex, and let* p *be a prime. Then either*

1. *A is $\mathbb{Z}/p$-hyperbolic,*
2. *there is a p-local equivalence $A \simeq_p S^m$ for $m = n, n+1$, or $n+2$, or*
3. *A is p-locally contractible.* □

However, the question of whether S^n is $\mathbb{Z}/p$-hyperbolic, or even p-hyperbolic, is very hard - so hard, in fact, that Huang and Wu state it as a question.

Question 3.29. *[HW20, Question 1.7] Is* S^n $(n \geq 2)$ *p-hyperbolic?*

To answer this question in the negative, it would suffice to prove a subexponential bound for the quantity $s_p(n,q)$ of Paper 1, and as we saw in our discussion of that paper, it seems likely from computations that a subexponential bound is possible (Conjecture 2.4), but will require new techniques (Corollary 2.5). Of course, it follows from Serre and Umeda's work (Theorem 3.1) that there are infinitely many p-torsion classes in the homotopy groups of S^n, so the answer is certainly more complicated than the rational one.

3.5 Relationship of local hyperbolicity to work of Henn and Iriye

In a pair of papers in the 1980s, Henn [Hen86], and Iriye [Iri87], studied the radii of convergence of certain power series which are related to the ideas of Huang and Wu. This section summarises their results and explores the connection.

Let $R_{\pi_*(X;\mathbb{Z}/p)}$ and $R_{H_*(\Omega X;\mathbb{Z}/p)}$ be the radii of convergence of the power series in t given by

$$\sum_{n=1}^{\infty} \dim(\pi_n(X;\mathbb{Z}/p) \otimes \mathbb{Z}/p)t^n$$

and
$$\sum_{n=1}^{\infty} \dim H_n(\Omega X; \mathbb{Z}/p)t^n$$
respectively. The reason for tensoring with $\mathbb{Z}/p$ in the first case is that homotopy groups with coefficients in $\mathbb{Z}/2$ are only $\mathbb{Z}/4$-modules in general (Corollary 1.36 and Proposition 1.37). Henn and Iriye prove complementary bounds relating these quantities, as follows.

The *radius of convergence* R of a power series $\sum a_n z^n$ is the least upper bound of the nonnegative numbers r such that $\sum |a_n| r^n$ converges. The radius of convergence satisfies $R^{-1} = \limsup |a_n|^{\frac{1}{n}}$ [FHT15, Section 33].

Theorem 3.30. *[Hen86] Let X be a simply connected space of finite type, and let p be a prime. Then*
$$R_{\pi_*(X;\mathbb{Z}/p)} \geq \min(R_{H_*(\Omega X;\mathbb{Z}/p)}, c_p),$$
where c_p is a constant depending only on p and $c_p \geq \frac{1}{2}$ for all p. □

Theorem 3.31. *[Iri87] Let X be a simply connected space of finite type, and let p be a prime. Then*
$$R_{H_*(\Omega X;\mathbb{Z}/p)} \geq \min(R_{\pi_*(X;\mathbb{Z}/p)}, 1).$$ □

Firstly, it is convenient to introduce notation for the number of $\mathbb{Z}/p^t$-summands in a module. Let M be a $\mathbb{Z}$-module, let p be a prime and let $t \in \mathbb{N}$. The *$\mathbb{Z}/p^t$-dimension* of M, denoted $\dim_{\mathbb{Z}/p^t}(M)$, is the number of $\mathbb{Z}/p^t$-summands in M. The *p-dimension* of M is $\dim_p(M) = \sum_{t=1}^{\infty} \dim_{\mathbb{Z}/p^t}(M)$; this is the number of p-torsion summands in M. Similarly, let $\mathrm{rk}(M)$ denote the *rank* of M, that is, the number of $\mathbb{Z}$-summands in M, or equivalently $\dim(M \otimes \mathbb{Q})$.

For a space X, Henn and Iriye's work concerns the ordinary dimensions $a_n = \dim(\pi_n(X;\mathbb{Z}/p) \otimes \mathbb{Z}/p)$, while Huang and Wu's p-hyperbolicity (Definition 3.21) concerns the quantities $b_n = \dim_p(\pi_n(X))$. For the sake of simplicity, suppose that p is odd, so that $\pi_n(X;\mathbb{Z}/p) \otimes \mathbb{Z}/p \cong \pi_n(X;\mathbb{Z}/p)$. For each n, we have a universal coefficient sequence [Nei10, Theorem 1.3.1]
$$0 \longrightarrow \pi_n(X) \otimes \mathbb{Z}/p \longrightarrow \pi_n(X;\mathbb{Z}/p) \longrightarrow \mathrm{Tor}(\pi_{n-1}(X);\mathbb{Z}/p) \longrightarrow 0.$$

We are dealing here with a sequence of vector spaces, which is automatically split. Writing $c_n = \mathrm{rk}(\pi_n(X))$, we have
$$\begin{aligned} a_n = \dim \pi_n(X;\mathbb{Z}/p) &= \dim_p(\pi_n(X)) + \mathrm{rk}(\pi_n(X)) + \dim_p(\pi_{n-1}(X)) \\ &= b_n + c_n + b_{n-1}. \end{aligned}$$

Now, $R_{\pi_*(X;\mathbb{Z}/p)}$ is the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n t^n$. Write R_b for the radius of convergence of $\sum_{n=1}^{\infty} b_n t^n$, and write R_c for the radius of convergence of $\sum_{n=1}^{\infty} c_n t^n$. Shifting indices does not affect radius of convergence, so the radii of convergence of $\sum_{n=1}^{\infty} b_n t^n$ and $\sum_{n=1}^{\infty} b_{n-1} t^n$ are equal. Furthermore, the radius of convergence of a termwise sum of power series is the minimum of the radii of convergence of the summands, so we have

$$R_{\pi_*(X;\mathbb{Z}/p)} = \min(R_b, R_c).$$

The point is that R_c belongs to rational homotopy, while R_b is closely related to p-hyperbolicity.

Since $c_n = \dim(\pi_n(X;\mathbb{Q}))$, it follows from Theorem 3.9 that R_c is precisely the radius of convergence of the power series $P_{\Omega X} = \sum_{n=1}^{\infty} \dim(H_n(\Omega X;\mathbb{Q}))t^n$; in particular it is determined by rational loop homology. Leaving the rational part aside, we certainly have that

$$R_{\pi_*(X;\mathbb{Z}/p)} \leq R_b.$$

The following simple one-directional relationship exists between p-hyperbolicity and $R_{\pi_*(X;\mathbb{Z}/p)}$.

Lemma 3.32. *Let* $\liminf_n \frac{\ln(T_n)}{n} = d$, *where the sequence* T_n *is as in Definition 3.21. Then* $R_{\pi_*(X;\mathbb{Z}/p)} \leq \exp(-d)$.

The bound can only be one-directional, because lim inf is essentially a lower bound, while the radius of convergence is essentially an upper bound.

Proof. Suppose that $\liminf_n \frac{\ln(T_n)}{n} = d$. We have seen that it suffices to show that $R_b \leq e^{-d}$, where R_b is the radius of convergence of the power series $\sum_{n=1}^{\infty} \dim_p(\pi_n(X))t^n$.

Certainly, for any $\varepsilon < d$, once n is large we have $\frac{\ln(T_n)}{n} > \varepsilon$, so $T_n > (e^{\varepsilon})^n$. By definition, $T_n = \sum_{i=1}^{n} \dim_p(\pi_i(X))$, so there must be infinitely many n for which $\dim_p(\pi_n(X)) > \frac{1}{n}(e^{\varepsilon})^n$. If the power series $\sum_{n=1}^{\infty} \dim_p(\pi_n(X))t^n$ converges, then certainly it cannot have infinitely many terms which are greater than 1, so we must have $\frac{1}{n}(e^{\varepsilon})^n t^n < 1$ for sufficiently large values of n. Rearranging, we must have $t < (\frac{1}{n})^{\frac{1}{n}} \frac{1}{(e^{\varepsilon})} < \frac{1}{(e^{\varepsilon})}$, as required. $\square$

It would be interesting to explore the relationship between the work of Huang and Wu and that of Henn and Iriye further.

4 Papers 2 and 3 in context

Papers 2 and 3 follow the same basic pattern. Each paper consists of two main theorems; the first treating a specific example, and the second giving a general criterion for local hyperbolicity. Both results of Paper 2 concern wedges of spheres, while both results of Paper 3 concern Moore spaces. We will consider the first theorem of each paper in Subsection 4.1, and the second in Subsection 4.2. Examples are collected at the end of each subsection.

4.1 Direct calculation

As part of their paper, Zhu and Pan [ZP21] showed that the wedge of two spheres $S^n \vee S^m$ (for $n, m \geq 2$) is $\mathbb{Z}/p$-hyperbolic. Our first theorem extends that result to cover all powers of p.

Theorem 4.1 (Paper 2 [Boy21b, Theorem 1.3]). *Let $n, m \geq 2$. Then $S^n \vee S^m$ is $\mathbb{Z}/p^r$-hyperbolic for all primes p and all $r \in \mathbb{N}$.* □

Huang and Wu [HW20] showed that Moore spaces $P^n(p^r)$ are $\mathbb{Z}/p^r$ and $\mathbb{Z}/p^{r+1}$-hyperbolic, while Zhu and Pan showed that they are $\mathbb{Z}/p$-hyperbolic. Our second theorem fills in the gap.

Theorem 4.2 (Paper 3 [Boy21a, Theorem 1.3]). *Let p be a prime, and $r \in \mathbb{N}$. If $n \geq 3$, then $P^n(p^r)$ is $\mathbb{Z}/p^s$-hyperbolic for all $s \leq r$ such that $p^s \neq 2$.* □

Combined with Huang and Wu's results, we may conclude that $P^n(p^r)$ is $\mathbb{Z}/p^s$-hyperbolic for all $s \leq r+1$ such that $p^s \neq 2$. These results are close to complete, since for $p \geq 5$ the homotopy exponent of $P^n(p^r)$ is p^{r+1} (Theorem 3.16). By Propositions 1.17 and 1.26, we therefore have the following.

Corollary 4.3. *For $p \neq 2, 3$ prime, $s, \ell \in \mathbb{N}$ and $n \geq 3$, the following are equivalent:*

1. *$P^n(\ell)$ is $\mathbb{Z}/p^s$-hyperbolic.*
2. *$\pi_*(P^n(\ell))$ contains a class of order p^s.*
3. *$p^{\max(s-1,1)} | \ell$.* □

Wedges of spheres (Theorem 4.1) and Moore spaces $P^n(\ell)$ with ℓ not divisible by 2 or 3 (Corollary 4.3) therefore provide the first two families of spaces for which the questions of local hyperbolicity are totally resolved within Huang and Wu's definitions.

These examples are suggestive: both spaces are $\mathbb{Z}/p^r$-hyperbolic for each prime power p^r for which there is a $\mathbb{Z}/p^r$-summand in their homotopy groups. We might therefore ask whether we can find a finite CW-complex which is not locally hyperbolic without being contractible.

Question 4.4. *Does there exist a simply connected finite CW-complex X and a prime p, such that X is not p-locally contractible, but X is not $\mathbb{Z}/p^r$-hyperbolic for some p^r less than the homotopy exponent of X?*

One can think of this question as asking whether there exist any 'locally elliptic' spaces other than the trivial example. We have already expressed a suspicion that homotopy groups of the sphere S^n grow slower than exponentially, but it might well be easier to show that other examples have this property.

Many spaces are known to contain either a wedge of two spheres or a Moore space as a p-local retract, perhaps after looping. To conclude, we collect some examples of this sort.

- For $n, k \geq 3$, the configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$ is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Paper 2, Example 2.1).
- An $(n-1)$-connected $2n$-manifold M, where $H^n(M)$ is of rank at least 3, is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Paper 2, Example 2.2).
- An (oriented) $(n-1)$-connected $(2n+1)$-manifold M, where $H_n(M)$ is of rank at least 1, is $\mathbb{Z}/p^r$-hyperbolic whenever p^{r-1} divides the order of the torsion part of $H_n(M)$, and if $H_n(M)$ is of rank at least 2 then M is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Paper 3, Example 2.1).
- A 5-dimensional spin manifold M with $H_2(M;\mathbb{Z})$ isomorphic to a direct sum of copies of $\mathbb{Z}/p^s$ is $\mathbb{Z}/p^r$-hyperbolic for all $1 \leq r \leq s$ (Paper 3, Example 2.2). This provides an example
- A generalized moment-angle complex on a simplicial complex having two minimal missing faces which are not disjoint is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Paper 2, Example 2.3).
- Suspended complex projective space $\Sigma\mathbb{C}P^2$ is $\mathbb{Z}/p^r$-hyperbolic for all $p \neq 2$ and all r, and $\Sigma\mathbb{H}P^2$ is is $\mathbb{Z}/p^r$-hyperbolic for all $p \neq 2, 3$ and all r (Paper 2, Example 2.4).

4.2 Homological results

Generally, homological methods are more robust than homotopical ones, at least in the context of finite CW-complexes. It is therefore desirable to have some homological criteria for local hyperbolicity.

For a space X, let $\widetilde{K}^*(X)$ denote complex topological K-theory of X. In Paper 2, we show the following.

Theorem 4.5 (Paper 2 [Boy21b, Theorem 1.4])**.** *Let X be a path connected space having the homotopy type of a finite CW-complex, and let p be an odd prime. Suppose that there exists a map*

$$\mu : S^{n+1} \vee S^{m+1} \to \Sigma X$$

with $n, m \geq 1$, such that the map

$$\widetilde{K}^*(\Sigma X) \otimes \mathbb{Z}/p \xrightarrow{\mu^*} \widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

is a surjection. Then ΣX is p-hyperbolic. □

In order to state the homological result of Paper 3 in full generality, we must introduce some new definitions.

Firstly, recall from Subsection 3.5 that, for a $\mathbb{Z}$-module M, the *$\mathbb{Z}/p^t$-dimension* of M, $\dim_{\mathbb{Z}/p^t}(M)$, is the number of $\mathbb{Z}/p^t$-summands in M.

Definition 4.6. Let M be a graded $\mathbb{Z}$-module, Let p be a prime, and let $S \subset \mathbb{N}$. We say that X is *p-hyperbolic concentrated in (the set of exponents) S* if the sequence

$$a_m := \sum_{t \in S} \dim_{\mathbb{Z}/p^t}(\bigoplus_{i=1}^{m} M_i)$$

grows exponentially. For a space X we will say that X is *p-hyperbolic concentrated in S* if $\pi_*(X)$ is p-hyperbolic concentrated in S.

Definition 4.6 subsumes Definitions 3.21 and 3.22: *$\mathbb{Z}/p^s$-hyperbolicity* is precisely p-hyperbolicity concentrated in the singleton set $\{s\}$, and *p-hyperbolicity* is precisely p-hyperbolicity concentrated in $\mathbb{N}$.

Our purpose here is twofold. Firstly, the new definitions recognise that hyperbolicity is most immediately a property of graded modules, and only becomes a property of spaces by taking homotopy groups. It is convenient at various points in Paper 3 to be able to speak of various algebraic objects as being hyperbolic; for example a graded Lie algebra, or the loop homology $H_*(\Omega X)$. Secondly, these definitions reflect well what we were actually able to show, namely that the number of $\mathbb{Z}/p^t$-summands for t

in some finite range $s \leq t \leq r$ grew exponentially. This is both importantly stronger than p-hyperbolicity and importantly weaker than $\mathbb{Z}/p^t$-hyperbolicity.

These definitions allow for a slight improvement in the statement of Theorem 4.5 - one could show p-hyperbolicity concentrated in the set of exponents $\geq n$ for any $n \in \mathbb{N}$, but we do not know a means of producing an upper bound for the orders of the torsion we detect.

With these definitions in hand, we may state the second theorem of Paper 3.

Theorem 4.7 (Paper 3 [Boy21a, Theorem 1.6]). *Let X be a connected CW-complex, let $p \neq 2$ be prime, and let $s \leq r \in \mathbb{N}$. If there exists a map*

$$\mu : P^{n+1}(p^r) \longrightarrow \Sigma X$$

such that

$$\mu_* : \widetilde{H}_*(P^{n+1}(p^r); \mathbb{Z}/p^s) \longrightarrow \widetilde{H}_*(\Sigma X; \mathbb{Z}/p^s)$$

is an injection, then ΣX is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$. In particular if $s = r$ then ΣX is $\mathbb{Z}/p^r$-hyperbolic. □

Theorems 4.5 and 4.7 are in many ways analogous. The main difference is that the Hurewicz map is enough to detect p^r-torsion in the homotopy groups of the Moore space $P^n(p^r)$. In contrast, one needs more sophisticated machinery to see p^r-torsion in a wedge of spheres; we used Adams' e-invariant. This is the reason that Theorem 4.5 is stated in terms of K-theory, rather than ordinary (co)homology. There is a further apparent difference; Theorem 4.5 uses a cohomology theory, and demands a surjection, while Theorem 4.7 uses a homology theory, and demands a injection. In fact, this no difference; dualising either statement yields something 'of the same form' as the other.

We deduce Theorem 4.7 from the following theorem, which removes the requirement that the space in question be a suspension, provided that the loop homology can be shown to behave appropriately. This generalisation is not possible in Paper 2, because of the need to restrict to a finite stage of the James construction, so that the Adams operations have bounded eigenvalues.

Theorem 4.8 (Paper 3 [Boy21a, Theorem 1.5]). *Let Y be a simply connected CW-complex, let $p \neq 2$ be prime, and let $s \leq r \in \mathbb{N}$. If there exists a map*

$$\mu : P^{n+1}(p^r) \longrightarrow Y$$

such that the induced map

$$(\Omega\mu)_* : H_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s) \longrightarrow H_*(\Omega Y; \mathbb{Z}/p^s)$$

is an injection, then Y is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$. In particular if $s = r$ then Y is $\mathbb{Z}/p^r$-hyperbolic. □

Theorem 4.7 is derived from Theorem 4.8 by means of Proposition 10.12 of Paper 3, which, as one might expect, says that when $Y = \Sigma X$ is a suspension, injectivity of μ_* implies injectivity of $(\Omega\mu)_*$.

The prime 2 is excluded in both papers, but for different reasons. It is possible in both cases that with more care, analogous results can be obtained at the prime 2, although they may differ slightly from the odd-primary cases.

Since these results are homological, we automatically get a sort of stability, in the following way. If the map μ of Theorem 4.5 induces a surjection on $\widetilde{K}^*(\quad) \otimes \mathbb{Z}/p$, then so does its suspension $\Sigma\mu$. Likewise, if the map μ of Theorem 4.7 induces an injection on $\widetilde{H}^*(\quad; \mathbb{Z}/p^s)$, then so does its suspension $\Sigma\mu$. The conclusion of both theorems may therefore be strengthened in the following way.

Corollary 4.9. *With the hypothesis of Theorem 4.5, $\Sigma^n X$ is p-hyperbolic for all $n \geq 1$. With the hypothesis of Theorem 4.7, $\Sigma^n X$ is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$ for all $n \geq 1$.* □

We conclude this subsection by recording some examples of these theorems. Firstly, by Theorem 4.5, the following spaces are p-hyperbolic for $p \neq 2$.

- Suspended complex projective space $\Sigma\mathbb{CP}^n$ for $n \geq 2$ (Paper 2, Example 2.5).
- More generally, the suspended complex Grassmannian $\Sigma\mathrm{Gr}_{k,n}$ for $n \geq 3$ and $0 < k < n$ (Paper 2, Example 2.6).
- The suspended Milnor Hypersurface $\Sigma H_{m,n}$ for $m \geq 2$ and $n \geq 3$ (Paper 2, Example 2.7).
- The suspended unitary group $\Sigma U(n)$ for $n \geq 3$ (Paper 2, Example 2.8).

The easiest way to use Theorem 4.7 is by means of the following corollary.

Corollary 4.10 (Paper 3 [Boy21a, Corollary 2.4])**.** *Let n be the least natural number for which $\widetilde{H}_n(\Sigma X; \mathbb{Z})$ is nontrivial. If $\widetilde{H}_n(\Sigma X; \mathbb{Z})$ contains a $\mathbb{Z}/p^s$-summand, for p an odd prime and $s \in \mathbb{N}$, then ΣX is $\mathbb{Z}/p^s$-hyperbolic.* □

From this corollary, we obtain the following examples.

- Let p be an odd prime, and let $s \in \mathbb{N}$. The suspended Eilenberg-MacLane space $\Sigma K(\mathbb{Z}/p^s, n)$ is $\mathbb{Z}/p^s$-hyperbolic (Paper 3, Example 2.5).

- Let G be a finite group of odd order. The suspended Eilenberg-MacLane space $\Sigma K(G,1)$ is $\mathbb{Z}/p^s$-hyperbolic for some p^s dividing the order of G (Paper 3, Example 2.6).
- If G is the alternating group A_6 or A_7, or the Suzuki group Suz, then $\Sigma K(G,1)$ is $\mathbb{Z}/3$-hyperbolic (Paper 3, Example 2.6).

These examples highlight how dramatic the effect of suspending on homotopy groups can be: by definition $K(\mathbb{Z}/p^s, n)$ has only a single nontrivial homotopy group, but $\Sigma K(\mathbb{Z}/p^s, n)$ is $\mathbb{Z}/p^s$-hyperbolic. It would be interesting to obtain an example of Theorem 4.8 which is not a suspension, but we do not presently have such an example.

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Paper 1
Bounding size of homotopy groups of spheres
Guy Boyde

ABSTRACT. Let p be prime. We prove that, for n odd, the p-torsion part of $\pi_q(S^n)$ has cardinality at most $p^{2^{\frac{1}{p-1}(q-n+3-2p)}}$, and hence has rank at most $2^{\frac{1}{p-1}(q-n+3-2p)}$. For $p = 2$ these results also hold for n even. The best bounds proven in the existing literature are $p^{2^{q-n+1}}$ and 2^{q-n+1} respectively, both due to Hans-Werner Henn. The main point of our result is therefore that the bound grows more slowly for larger primes. As a corollary of work of Henn, we obtain a similar result for the homotopy groups of a broader class of spaces.

1 Introduction

Our goal is to bound the size of $\pi_q(S^n)$. Serre [9] showed that these groups are finitely generated abelian, and that they contain a single $\mathbb{Z}$-summand when $q = n$, or when n is even and $q = 2n - 1$, and are finite otherwise. We may therefore restrict our attention to the p-torsion summand for each prime p. When p is odd it suffices to consider n odd, because, again by work of Serre [8] ,

$$\pi_q(S^{2n}_{(p)}) \cong \pi_{q-1}(S^{2n-1}_{(p)}) \oplus \pi_q(S^{4n-1}_{(p)}),$$

where $X_{(p)}$ denotes the localisation of the space X at p.

For us, the *rank* of a finitely generated module will be the size of a minimal generating set. Selick [7] proved that the rank of $\pi_q(S^n_{(p)})$, regarded as a $\mathbb{Z}_{(p)}$-module, is at most 3^{q^2}. Bödigheimer and Henn [1] prove that the base-p logarithm of the cardinality of the p-torsion part of $\pi_q(S^n)$ is at most $3^{(q-\frac{n}{2})}$, for n odd and all primes p. By the same method, they prove that the rank of $\pi_q(S^n_{(p)})$ satisfies the same bound. In [3] , Henn further improved this bound to 2^{q-n+1}. In [5] , Iriye states the bound $3^{\frac{q-n}{2p-3}}$, which is similar to our Theorem 1.1, but gives no proof.

We will use the same machinery as all three of those papers, namely the EHP sequences of James [6] and Toda [10] . In particular, the new ideas in this paper are primarily combinatorial. A sub-exponential bound is not known, and Bödigheimer and Henn note that to produce such a bound one would have to introduce additional information from topology.

Our main result is as follows. Denote by $s_p(n, q)$ the base-p logarithm of the cardinality of the p-torsion part of $\pi_q(S^n)$.

Theorem 1.1. *For all natural numbers q, and n odd,*

$$s_p(n, q) \leq 2^{\frac{1}{p-1}(q-n+3-2p)}.$$

If $p = 2$ then this bound holds also for n even.

As a corollary we obtain the weaker but simpler bound $s_p(n,q) \leq 2^{\frac{q-n}{p-1}}$. The $3-2p$ which appears in the original statement reflects the classical fact that the first p-torsion classes appear in the $(2p-3)$-rd stem. We can think of the bound as an exponential function of the stem $q-n$ in base $2^{\frac{1}{p-1}}$. The main advantage of this bound over its predecessors is that as p becomes large, the base of the exponential approaches 1, so our bound grows more slowly for larger primes.

As in [1] , we also obtain a bound on the rank, but we prefer to regard it as following from Theorem 1.1, by using that the rank of a finite p-torsion group is at most $\log_p$ of its cardinality.

Corollary 1.2. *For n odd, the rank of the p-torsion part of $\pi_q(S^n)$ is at most $2^{\frac{q-n}{p-1}}$. If $p = 2$, this bound holds also for n even.*

The bound proven by Henn in [3] was a lemma used to establish results about the rank of the p-torsion part of $\pi_q(X)$ for X any simply connected space of finite type. Our improvement to the bound feeds directly into the main theorem of that paper to give that a certain constant c_p which appears there may be assumed to be at least $(\frac{1}{2})^{\frac{1}{p-1}}$ (Henn shows that it is at least $\frac{1}{2}$). This has the following corollary.

Corollary 1.3. *Let X be a simply connected space of finite type, and suppose that the radius of convergence of the power series $\sum_{q=1}^{\infty} \dim_{\mathbb{Z}/p}(H_q(\Omega X;\mathbb{Z}/p)) \cdot x^q$ is 1. Then $\sum_{q=1}^{\infty} \dim_{\mathbb{Z}/p}(\pi_q(X;\mathbb{Z}/p) \otimes \mathbb{Z}/p) \cdot x^q$ has radius of convergence at least $(\frac{1}{2})^{\frac{1}{p-1}}$. In particular, the rank of the p-torsion part of $\pi_q(X)$ is at most $2^{\frac{q}{p-1}}$ for all but perhaps finitely many q.*

The hypotheses of the above corollary are satisfied if, for example, the dimension of $H_q(\Omega X;\mathbb{Z}/p)$ is bounded above by a polynomial in q.

I would like to thank my supervisor, Stephen Theriault, for all of his help and support. I would also like to thank Hans-Werner Henn for his helpful correspondence - in particular for drawing my attention to the methods of his paper [3] , and for making me aware of the paper [2] of Flajolet and Prodinger, which lead indirectly to the idea for this paper.

2 Approach

It will be convenient for us to think of a stem in the homotopy groups of spheres as a combinatorial object, so we say that the k-th *stem* is the set

$\{(n,q) \in \mathbb{N} \times \mathbb{N} \mid q - n = k\}$. The *negative* stems, for example, are those for which $k < 0$.

Fix a prime p, and assume henceforth that all spaces are localized at p. For a space X, let $J_k(X)$ denote the k-th stage of the James Construction on X. For n odd, we have the following (p-local) fibrations, from [10] and [6] :

$$J_{p-1}(S^{n-1}) \longrightarrow \Omega S^n \longrightarrow \Omega S^{p(n-1)+1}, \text{ and}$$

$$S^{n-2} \longrightarrow \Omega J_{p-1}(S^{n-1}) \longrightarrow \Omega S^{p(n-1)-1}.$$

The long exact sequences on homotopy groups induced by these fibrations are called *EHP-sequences*. The following inequalities are proven in the first lemma of [1] . The first inequality is obtained by considering the EHP-sequences, assuming that all groups are finite, and the second inequality accounts for the possibility that one of the groups is not finite, using knowledge of the relative homotopy groups $\pi_q(\Omega^2 S^n, S^{n-2})$, as in for example Appendix 2 of [4] . When $p = 2$ the situation is simpler: the above fibrations are just the odd and even cases of

$$S^{n-1} \longrightarrow \Omega S^n \longrightarrow \Omega S^{2n-1}.$$

Lemma 2.1. *For all q, and n odd:*

1. $s_p(n,q) \leq s_p(p(n-1)+1,q) + s_p(p(n-1)-1,q-1) + s_p(n-2,q-2)$ *if* $q \neq p(n-1)$.
2. $s_p(n,q) \leq 1 + s_p(n-2,q-2)$ *if* $q = p(n-1)$.

When $p = 2$, we no longer need to restrict to odd n, and the first inequality can be replaced by

$$s_2(n,q) \leq s_2(2n-1,q) + s_2(n-1,q-1).$$

These inequalities will be used to prove Theorem 1.1. It is worth noting that the extent to which the inequalities fail to be equalities is measured by the size of the images of the boundary maps (equivalently, the kernels of the suspensions) in the EHP sequences. In some sense, therefore, the extent to which our bound fails to be sharp is measuring the aggregate size of the images of EHP boundary maps.

3 Limitations of our approach

If in Lemma 2.1, one replaces the inequalities with equalities, and regards this as an inductive definition of integers $t_p(n,q)$, then necessarily $t_p(n,q)$ is the best upper bound that can be obtained for $s_p(n,q)$ using that lemma. In [3] , Henn defines inductively integers $b_2(n,k)$. He shows that $t_2(n,q) = b_2(q-2,n)$ (note that n has switched roles). In [2] , Flajolet and Prodinger study a combinatorially defined sequence H_n. By definition, $t_2(2,q) = b_2(q-2,2) = H_{q-2}$, a fact I was made aware of by Henn. Flajolet and Prodinger obtain an asymptotic estimate $H_q \sim K \cdot \nu^q$, giving formulas for K and ν and computing both to 15 decimal places. To 3 decimal places, K is 0.255, and ν is 1.794. They remark that H_q (which is equal to $t_2(2,q+2)$) is at least F_q, where F_q is the q-th Fibonacci number. We will not do so here, but one can show by induction that $t_p(3,2p+j(4p+5)) \geq F_{2j+1}$ for $j \geq 0$. Inductively applying $t_p(n-2,q-2) \leq t_p(n,q)$ then gives that $t_p(n,2p+j(4p+5)+n-3) \geq F_{2j+1}$ for all odd $n \geq 3$. We therefore have the following.

Corollary 3.1. *Any bound on the base-p logarithm of the cardinality of the p-primary part of $\pi_{2p+j(4p+5)+n-3}(S^n)$ (or the rank of that group) which can be obtained from Lemma 2.1 is greater than or equal to F_{2j+1}. In particular, any base for an exponential bound which can be obtained from Lemma 2.1 is at least $\phi^{\frac{2}{4p+5}}$, where ϕ denotes the golden ratio.*

4 Proof of Theorem 1.1

Proof of Theorem 1.1. We will actually prove the slightly stronger result that $s_p(n,q) \leq 2^{\lfloor \frac{1}{p-1}(q-n+3-2p) \rfloor}$. The floor function forces the exponent to be an integer, which will be useful in the proof. We will use a (slightly modified) strong double induction over stems. More precisely, the proof of the result for (n,q) will use the result for all (m,r) with $r-m < q-n$ (that is, on lower stems) and for $(1,q-n+1)$ (that is, the entry at the base of the stem on which (n,q) lies). A proof using the other lower entries on the same stem in the induction is possible, but results in a more unwieldy inductive hypothesis. The case $p=2$, n even will be treated at the end.

Suppose first that $q \leq n$. In this case, $\pi_q(S^n)$ is torsion free (indeed, it is zero for $q < n$) so $s_p(n,q) = 0$. This proves the result for all non-positive stems. The higher homotopy groups of S^1 are trivial (since it has a contractible universal cover) so $s_p(1,q) = 0$ for all q. This proves the base case of each stem.

It remains only to treat an inductive step on a positive stem. Thus, let $(n,q) \in \mathbb{N} \times \mathbb{N}$ with n odd, and suppose that the result is proven for all (m,r) with $r-m < q-n$. Consider the two inequalities of Lemma 2.1. We wish to apply the first inequality inductively down the stem to bound $s_p(n,q)$ by a sum of terms on lower stems and $s_p(1,q-n+1)$, which is zero by the discussion above. The only complicating factor is the second inequality, which may require us to add one to our bound at certain steps. However, the second case of Lemma 2.1 can occur at most once per stem, so at worst we will have to add one to the bound that we would obtain if the first case of the

Lemma held everywhere. More precisely, we obtain

$$s_p(n,q) \leq 1 + \sum_{i=0}^{\frac{1}{2}(n-3)} (s_p(p(n-2i-1)+1, q-2i) + s_p(p(n-2i-1)-1, q-2i-1)).$$

Each $s_p(m,r)$ is an integer. Therefore, for those (m,r) for which we inductively have $s_p(m,r) \leq 2^{\lfloor \frac{1}{p-1}(r-m+3-2p) \rfloor}$, we actually have the sightly stronger statement that $s_p(m,r) \leq \lfloor 2^{\lfloor \frac{1}{p-1}(r-m+3-2p) \rfloor} \rfloor$. Including this fact into the above inequality, we find that

$$\begin{aligned} s_p(n,q) \leq 1 + \sum_{i=0}^{\frac{1}{2}(n-3)} (&\lfloor 2^{\lfloor \frac{1}{p-1}(q-2i-(p(n-2i-1)+1)+3-2p) \rfloor} \rfloor \\ &+ \lfloor 2^{\lfloor \frac{1}{p-1}(q-2i-1-(p(n-2i-1)-1)+3-2p) \rfloor} \rfloor). \end{aligned} \tag{$*$}$$

Notice that the value of the floor function on an integer power of 2 is given by

$$\lfloor 2^i \rfloor = \begin{cases} 2^i & i \geq 0 \\ 0 & i < 0. \end{cases}$$

We now bound this summation by another where the nonzero exponents are distinct integers. More precisely, adding $1 - \frac{1}{p-1}$ to the exponent of the second term in $(*)$ (inside the floor function) gives that

$$\begin{aligned} s_p(n,q) &\leq 1 + \sum_{i=0}^{\frac{1}{2}(n-3)} (\lfloor 2^{\lfloor \frac{1}{p-1}(q-2i-(p(n-2i-1)+1)+3-2p) \rfloor} \rfloor \\ &\qquad + \lfloor 2^{\lfloor \frac{1}{p-1}(q-(2i+1)-(p(n-(2i+1)-1)+1)+3-2p) \rfloor} \rfloor) \\ &= 1 + \sum_{i=0}^{n-2} (\lfloor 2^{\lfloor \frac{1}{p-1}(q-i-(p(n-i-1)+1)+3-2p) \rfloor} \rfloor. \end{aligned}$$

In particular, $s_p(n,q)$ is at most one greater than a sum of powers of 2. It suffices to show that those powers of 2 that are not killed off by the outer floor function are all distinct and strictly smaller than $2^{\lfloor \frac{1}{p-1}(q-n+3-2p) \rfloor}$, because $\sum_{i=0}^{k-1} 2^i = 2^k - 1$.

To see that they are distinct, notice that changing i by 1 changes the exponent by 1. To see that they are strictly smaller than $2^{\lfloor \frac{1}{p-1}(q-n+3-2p) \rfloor}$, consider the largest power occurring in the summation, which is the $i = n-2$ term. Its exponent rearranges to $\lfloor \frac{1}{p-1}(q-n+3-2p) - 1 \rfloor = \lfloor \frac{1}{p-1}(q-n+3-2p) \rfloor - 1$, as required. This completes the proof for n odd.

It remains to treat the case $p = 2$, n even. Since the simplification at $p = 2$ in Lemma 2.1 holds for all n, the above proof may be repeated without restricting to n odd, and doing so gives the result for all n. $\square$

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Paper 2

p-hyperbolicity of homotopy groups via K-theory

Guy Boyde

ABSTRACT. We show that $S^n \vee S^m$ is $\mathbb{Z}/p^r$-hyperbolic for all primes p and all $r \in \mathbb{N}$, provided $n, m \geq 2$, and consequently that various spaces containing $S^n \vee S^m$ as a p-local retract are $\mathbb{Z}/p^r$-hyperbolic. We then give a K-theory criterion for a suspension ΣX to be p-hyperbolic, and use it to deduce that the suspension of a complex Grassmannian $\Sigma Gr_{k,n}$ is p-hyperbolic for all odd primes p when $n \geq 3$ and $0 < k < n$. We obtain similar results for some related spaces.

1 Introduction

A space X is called *rationally elliptic* if $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, and *rationally hyperbolic* if the dimension of $\bigoplus_{i \leq m} \pi_i(X) \otimes \mathbb{Q}$ grows exponentially in m. It was proved in [13, Chapter 33] that simply connected CW-complexes with rational homology of finite type and finite rational category are either rationally elliptic or rationally hyperbolic. In order to study the p-torsion analogue of this dichotomy, Huang and Wu [21] introduced the definitions of $\mathbb{Z}/p^r$- and p-hyperbolicity.

For p prime, by a *p-torsion summand* in an abelian group A, we mean a direct summand isomorphic to $\mathbb{Z}/p^r$ for some $r \geq 1$.

Definition 1.1. Let X be a space, and let p be a prime. We say that X is *p-hyperbolic* if the number of p-torsion summands in $\pi_*(X)$ grows exponentially, in the sense that

$$\liminf_m \frac{\ln(T_m)}{m} > 0,$$

where T_m is the number of p-torsion summands in $\bigoplus_{i \leq m} \pi_i(X)$.

The above definition counts $\mathbb{Z}/p^r$-summands for all values of r. It is also possible to consider only a single r, and by doing so we obtain the definition of $\mathbb{Z}/p^r$-hyperbolicity.

Definition 1.2. Let X be a space, let p be a prime, and fix $r \in \mathbb{N}$. We say that X is *$\mathbb{Z}/p^r$-hyperbolic* if the number of $\mathbb{Z}/p^r$-summands in $\pi_*(X)$ grows exponentially, in the sense that

$$\liminf_m \frac{\ln(t_m)}{m} > 0,$$

where t_m is the number of $\mathbb{Z}/p^r$-summands in $\bigoplus_{i \leq m} \pi_i(X)$.

Note that $\mathbb{Z}/p^r$-hyperbolicity for any r implies p-hyperbolicity. It follows immediately from a result of Henn [19, Corollary of Theorem 1] that the lim infs appearing in the above definitions must be finite if X is a simply connected finite CW-complex.

Huang and Wu show that for $n \geq 3$, $r \geq 1$ and p any prime, the Moore space $P^n(p^r)$ is $\mathbb{Z}/p^r$-hyperbolic and $\mathbb{Z}/p^{r+1}$-hyperbolic, and that $P^n(2)$ is also $\mathbb{Z}/8$-hyperbolic [21, Theorem 1.6] . More generally, they give criteria in terms of a functorial loop space decomposition due to Selick and Wu [29, 30] for a suspension ΣX to be $\mathbb{Z}/p^r$-hyperbolic.

More recently, Zhu and Pan [37] use a classification of $(n-1)$-connected CW-complexes of dimension at most $n+2$, due to Chang [11] , to show that, for $n \geq 4$, such a complex is $\mathbb{Z}/p$-hyperbolic, provided that it is not contractible or a sphere after p-localization. They also prove hyperbolicity results for several complexes that have become known as elementary Chang complexes.

This paper studies p- and $\mathbb{Z}/p^r$-hyperbolicity of certain suspensions. Our first result is as follows.

Theorem 1.3. *Let $q_1, q_2 \geq 1$. Then $S^{q_1+1} \vee S^{q_2+1}$ is $\mathbb{Z}/p^r$-hyperbolic for all primes p and all $r \in \mathbb{N}$.*

Let p be a prime. If a space X contains a wedge of two spheres as a p-local retract, then Theorem 1.3 implies that X is $\mathbb{Z}/p^r$-hyperbolic for all r. Various spaces have been shown to contain such a wedge - examples of this sort are given in Section 2.1. A summary is as follows:

- for $n, k \geq 3$, the configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$ is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Example 2.1);
- an $(n-1)$-connected $2n$-dimensional manifold M, where $H^n(M)$ is of rank at least 3, is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Example 2.2);
- a generalized moment-angle complex on a simplicial complex having two minimal missing faces which are not disjoint is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Example 2.3);
- $\Sigma\mathbb{C}P^2$ is $\mathbb{Z}/p^r$-hyperbolic for all $p \neq 2$ and all r, and $\Sigma\mathbb{H}P^2$ is is $\mathbb{Z}/p^r$-hyperbolic for all $p \neq 2, 3$ and all r (Example 2.4).

Our other result is as follows.

Theorem 1.4. *Let X be a path connected space having the homotopy type of a finite CW-complex, and let p be an odd prime. Suppose that there exists a map*

$$\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$$

with $q_i \geq 1$, such that the map

$$\widetilde{K}^*(\Sigma X) \otimes \mathbb{Z}/p \xrightarrow{(\mu_1 \vee \mu_2)^*} \widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

is a surjection. Then ΣX is p-hyperbolic.

This criterion is quite different to that given by Huang and Wu [21, Theorem 1.5] . Their criterion is homotopical, using hypotheses on X to produce retracts of $\Omega\Sigma X$, wheras ours is cohomological, which makes it easier to check. On the other hand, their criterion is stronger, since it gives $\mathbb{Z}/p^r$-hyperbolicity, rather than just p-hyperbolicity. The examples they give, primarily various Moore spaces, differ from those we obtain, which are the suspensions of spaces related to complex projective space. More precisely, in Section 2.2, we show that the following spaces are p-hyperbolic for all $p \neq 2$:

- suspended complex projective space $\Sigma\mathbb{CP}^n$ for $n \geq 2$ (Example 2.5), and more generally;
- the suspended complex Grassmannian $\Sigma\mathrm{Gr}_{k,n}$ for $n \geq 3$ and $0 < k < n$ (Example 2.6);
- the suspended Milnor Hypersurface $\Sigma H_{m,n}$ for $m \geq 2$ and $n \geq 3$, (Example 2.7);
- the suspended unitary group $\Sigma U(n)$ for $n \geq 3$ (Example 2.8).

Both Theorem 1.3 and Theorem 1.4 will be proven by constructing an exponentially growing family of classes which generate summands in the relevant homotopy groups. We think of this family as 'witnessing' the hyperbolicity. For Theorem 1.3, one can proceed directly from the Hilton-Milnor decomposition of $S^n \vee S^m$ [20] . For Theorem 1.4, we employ K-theoretic methods originally used by Selick [28] to prove one direction of Moore's conjecture for suspensions having torsion-free homology.

If the map $\mu_1 \vee \mu_2$ of Theorem 1.4 induces a surjection on $\widetilde{K}^*(\ \ ) \otimes \mathbb{Z}/p$, then so does its suspension $\Sigma\mu_1 \vee \Sigma\mu_2$. The conclusion of Theorem 1.4 may therefore be strengthened in the following way.

Corollary 1.5. *With the hypothesis of Theorem 1.4, $\Sigma^n X$ is p-hyperbolic for all $n \geq 1$.* □

One might be motivated by this observation to ask whether, in the circumstances of Theorem 1.4, the stable homotopy groups of X satisfy the growth conditions of Definition 1.1 or 1.2. In the proofs of both Theorem 1.3 and 1.4, the classes that witness the hyperbolicity are composites involving Whitehead products. The suspension of a Whitehead product is always trivial [34, Theorem 3.11] , so the classes we detect cannot be stable. Therefore, Corollary 1.5 does not suggest that the stable homotopy of ΣX should be p- or $\mathbb{Z}/p^r$-hyperbolic. On the other hand, it follows from our methods that, under the hypotheses of Theorem 1.4, $\Omega\Sigma X$ is stably p-hyperbolic.

By a result of Henn [18] , any co-H space, and in particular any suspension, decomposes rationally as a wedge of spheres. It then follows from the Hilton-Milnor theorem [20] and the computation of the rational homotopy groups of spheres [31] that such a suspension is rationally hyperbolic precisely when there are at least two spheres (of dimension ≥ 2) in this decomposition.

We will see in Corollary 7.12 that if ΣX satisfies the hypotheses of Theorem 1.4 for any prime (including 2), then ΣX is rationally hyperbolic, hence is rationally a wedge of at least two spheres by the preceding discussion. This rational equivalence is a local equivalence at all but perhaps finitely many primes, so by Theorem 1.3, ΣX is $\mathbb{Z}/p^r$ hyperbolic for all r at all but finitely many primes p. One might therefore conjecture that the conclusion of Theorem 1.4 can be strengthened to give $\mathbb{Z}/p^r$-hyperbolicity for all r rather than p-hyperbolicity, but we do not know whether this is possible.

We now discuss situations in which it is adequate to consider ordinary cohomology, rather than K-theory. If ΣX has torsion-free integral (co)homology, or if its cohomology is concentrated in even degrees, then the Atiyah-Hirzebruch spectral sequence for $K^*(\Sigma X)$ collapses on the E^2 page [22] . It follows by naturality that the image of the map induced by $\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$ on K-theory is identified with the image of the induced map on cohomology. We may therefore replace K-theory with cohomology in Theorem 1.4, as follows.

Corollary 1.6. *Let X be a path connected space having the homotopy type of a finite CW-complex, such that the Atiyah-Hirzebruch spectral sequence for $K^*(\Sigma X)$ collapses on the E^2 page. Let p be an odd prime. Suppose that there exists a map $\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$ with $q_i \geq 1$, such that the map induced by $\mu_1 \vee \mu_2$ on $\widetilde{H}^*(\ \ ) \otimes \mathbb{Z}/p$ is a surjection. Then ΣX is p-hyperbolic.* □

One advantage of ordinary cohomology is that it is connected to the homotopy groups integrally, via the universal coefficient theorem and Hurewicz map. We can exploit this as follows.

Example 1.7. *Suppose that the Atiyah-Hirzebruch spectral sequence for $K^*(\Sigma X)$ collapses (for example, if ΣX has torsion-free homology) and that there exists $q \in \mathbb{N}$ so that $\widetilde{H}_i(\Sigma X) = 0$ for $i \leq q$, and $\dim_{\mathbb{Q}}(\widetilde{H}_{q+1}(\Sigma X) \otimes \mathbb{Q}) \geq 2$. The Hurewicz map $\pi_{q+1}(\Sigma X) \to \widetilde{H}_{q+1}(\Sigma X)$ is an isomorphism, so there exists a map $\mu_1 \vee \mu_2 : S^{q+1} \vee S^{q+1} \to \Sigma X$ inducing the inclusion of a $\mathbb{Z}^2$-summand in $\widetilde{H}_{q+1}(\Sigma X)$. By the universal coefficient theorem relating ordinary homology and cohomology, $\mu_1 \vee \mu_2$ induces a surjection on integral cohomology, so by Corollary 1.6, ΣX is p-hyperbolic for all odd primes p.*

I would like to thank my PhD supervisor, Stephen Theriault, for suggesting the problems that this paper tries to address, and for many helpful conversations along the way. From a technical point of view, much is owed to papers of Huang and Wu

[21] , and of Selick [28] . Neil Strickland's 'Bestiary' [33] was extremely helpful in providing examples of Theorem 1.4. Conversations with Sam Hughes were very useful in formulating Corollary 1.6.

2 Applications

2.1 Spaces having a wedge of two spheres as a retract

Theorem 1.3 immediately implies that any space X which has $S^{q_1+1} \vee S^{q_2+1}$ as a retract after p-localization is $\mathbb{Z}/p^r$-hyperbolic for that p and all r. This implies that for all $n \geq 1$, $\Sigma^n X$ contains $S^{q_1+n+1} \vee S^{q_2+n+1}$ as a p-local retract, and so is $\mathbb{Z}/p^r$-hyperbolic for all r. We first consider examples of this form.

Example 2.1. *It is known [24, Section 3.1] that* $\mathrm{Conf}_k(\mathbb{R}^n)$*, the ordered configuration space of k points in $\mathbb{R}^n$, contains $\bigvee_{k-1} S^{n-1}$ as a retract. It follows that, when $n, k \geq 3$,* $\mathrm{Conf}_k(\mathbb{R}^n)$ *is $\mathbb{Z}/p^r$-hyperbolic for all p and r.*

Example 2.2. *Let M be an $(n-1)$-connected $2n$-dimensional manifold. By the universal coefficient theorem, there can be no torsion in $H^n(M)$. Suppose that the rank of $H^n(M)$ is at least 3. By work of Beben and Theriault [10, Theorem 1.4] , ΩM contains a wedge of two spheres as a retract after looping. Thus, M is again $\mathbb{Z}/p^r$-hyperbolic for all p and r.*

Example 2.3. *Let K be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$, and let $(\underline{X}, \underline{A})$ be any sequence of pairs (D^{n_i}, S^{n_i-1}) with $n_i \geq 2$ for $1 \leq i \leq m$. If there exist two distinct minimal missing faces of K which are not disjoint, then by work of Hao, Sun and Theriault [17, Theorem 4.2] the polyhedral product $(\underline{X}, \underline{A})^K$ contains a wedge of two spheres as a retract after looping, and hence is $\mathbb{Z}/p^r$-hyperbolic for all p and all r.*

Example 2.4. *Localized away from 2, $\Sigma\mathbb{C}P^2 \simeq S^3 \vee S^5$. To see this, note that $\Sigma\mathbb{C}P^2$ has a CW-structure consisting of one 3-cell and one 5-cell, and that $\pi_4(S^3) \cong \mathbb{Z}/2$ [14] . This implies that the attaching map for the 5-cell is nullhomotopic after localization at an odd prime. Thus, $\Sigma\mathbb{C}P^2$ is $\mathbb{Z}/p^r$-hyperbolic for all r when $p \neq 2$.*

Similarly, $\Sigma\mathbb{H}P^2$ admits a cell structure with one 5-cell and one 9-cell, and $\pi_8(S^5) \cong \mathbb{Z}/24$. Thus, $\Sigma\mathbb{H}P^2$ is $\mathbb{Z}/p^r$-hyperbolic for all r when $p \neq 2, 3$.

2.2 Suspensions of spaces related to $\mathbb{C}P^n$

Suppose that one has verified the hypotheses of Theorem 1.4 for a given space X and odd prime p, using a map $\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$. If another space Y admits a map $\sigma : \Sigma X \to \Sigma Y$ which induces a surjection on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$, then it is immediate that $\sigma \circ (\mu_1 \vee \mu_2)$ satisfies the hypotheses of Theorem 1.4, and hence that ΣY is

p-hyperbolic. The slogan is that K-theory surjections allow us to generate new examples from old ones.

In this section, we will apply this idea. We will first show that $\Sigma\mathbb{C}P^2$ satisfies the hypotheses of Theorem 1.4 at all odd primes p. We will then consider spaces X which are known to admit maps $\mathbb{C}P^2 \to X$ which induce surjections on integral K-theory, and hence on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$ for all odd p. It follows in each case that ΣX is p-hyperbolic, and further by Corollary 1.5, that $\Sigma^n X$ is p-hyperbolic for all $n \geq 1$.

To start, let η be the Hopf invariant one class which is the attaching map for the top cell of $\mathbb{C}P^2$. Since $\Sigma\eta$ lies in $\pi_4(S^3) \cong \mathbb{Z}/2$, we have that $2\Sigma\eta = 0$. This gives the following map of cofibre sequences.

$$\begin{tikzcd}
S^4 \arrow[r, "\Sigma\eta"] & S^3 \arrow[r] & \Sigma\mathbb{C}P^2 \arrow[r] & S^5 \arrow[r] & S^4 \\
S^4 \arrow[r, "*"] \arrow[u, "2"] & S^3 \arrow[r] \arrow[u, equal] & S^3 \vee S^5 \arrow[r] \arrow[u, "\mu"] & S^5 \arrow[r, "*"] \arrow[u, "2"] & S^4. \arrow[u, equal]
\end{tikzcd}$$

Let the restrictions of μ to S^3 and S^5 be μ_1 and μ_2 respectively, so that $\mu = \mu_1 \vee \mu_2$. The map $\Sigma\eta$ induces zero on reduced integral K-theory for degree reasons, so we obtain a diagram of short exact sequences:

$$\begin{tikzcd}
0 & \widetilde{K}^*(S^3) \arrow[l] & \widetilde{K}^*(\Sigma\mathbb{C}P^2) \arrow[l] \arrow[d, "(\mu_1\vee\mu_2)^*"] & \widetilde{K}^*(S^5) \arrow[l] \arrow[d, "2"] & 0 \arrow[l] \\
0 & \widetilde{K}^*(S^3) \arrow[l] \arrow[u, equal] & \widetilde{K}^*(S^3) \oplus \widetilde{K}^*(S^5) \arrow[l] & \widetilde{K}^*(S^5) \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

We have obtained a map $\mu_1 \vee \mu_2 : S^3 \vee S^5 \to \Sigma\mathbb{C}P^2$ which induces a surjection on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$ for all odd primes p. We now seek maps of spaces which allow us to extend to $\Sigma\mathbb{C}P^n$.

The inclusion of $\mathbb{C}P^n$ into $\mathbb{C}P^{n+1}$ induces a surjection on K-theory, so it must still induce a surjection after suspending. Composing these inclusions with $\mu_1 \vee \mu_2$ gives, for each $n \geq 2$, a map $S^3 \vee S^5 \to \Sigma\mathbb{C}P^n$ which still induces a surjection on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$ for all odd primes p. Applying Theorem 1.4 to this map gives the following.

Example 2.5. *For $n \geq 2$, $\Sigma\mathbb{C}P^n$ is p-hyperbolic for all $p \neq 2$.*

Now let $\mathrm{Gr}_{k,n}$ be the Grassmannian of k-dimensional complex subspaces of $\mathbb{C}^n$. First note that orthogonal complement gives a homeomorphism $\mathrm{Gr}_{k,n} \cong \mathrm{Gr}_{n-k,n}$. In particular $\mathrm{Gr}_{n-1,n} \cong \mathrm{Gr}_{1,n} \cong \mathbb{C}P^{n-1}$, so $\Sigma\mathrm{Gr}_{n-1,n}$ is p-hyperbolic. Other Grassmannians can be treated more uniformly, as follows.

Let $\gamma_{k,n}$ denote the tautological bundle over $\mathrm{Gr}_{k,n}$. Consider the inclusion

$$\iota_n : \mathbb{C}^n \to \mathbb{C}^{n+1}$$

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n, 0).$$

This inclusion induces a map $i_{k,n} : \mathrm{Gr}_{k,n} \to \mathrm{Gr}_{k,n+1}$, defined on $V \in \mathrm{Gr}_{k,n}$ by $V \mapsto \iota_n(V)$. It follows from this definition that the pullback bundle $i^*_{k,n}(\gamma_{k,n+1})$ is isomorphic to $\gamma_{k,n}$. Letting e_i denote the i-th standard basis vector in $\mathbb{C}^n$, we also have a map $j_{k,n} : \mathrm{Gr}_{k,n} \to \mathrm{Gr}_{k+1,n+1}$, defined on $V = \mathrm{Span}(v_1, v_2, \ldots, v_k) \in \mathrm{Gr}_{k,n}$ by $V \mapsto \mathrm{Span}(\iota(v_1), \iota(v_2), \ldots, \iota(v_k), e_{n+1})$. It follows from this definition that $j^*_{k,n}(\gamma_{k+1,n+1}) = \gamma_{k,n} \oplus \underline{\mathbb{C}^1}$, where $\underline{\mathbb{C}^1}$ is the 1-dimensional trivial bundle.

Since $K^*(\mathbb{C}P^n)$ is generated by the class of the tautological bundle, composing the maps $i_{k,n}$ and $j_{k,n}$ for different values of k and n will give maps $\mathbb{C}P^2 = \mathrm{Gr}_{1,3} \to \mathrm{Gr}_{k,n}$ for all $1 \leq k \leq n-2$ and $n \geq 3$ which induce surjections in integral K-theory. As in Example 2.5, this implies the following (the case $k = n-1$ is $\mathrm{Gr}_{n-1,n}$, which was treated first).

Example 2.6. *For $n \geq 3$ and $0 < k < n$, the suspended complex Grassmannian $\Sigma\mathrm{Gr}_{k,n}$ is p-hyperbolic for all $p \neq 2$.*

For $m \leq n$, the *Milnor Hypersurface* $H_{m,n}$ is defined by

$$H_{m,n} = \{([z], [w]) \in \mathbb{C}P^m \times \mathbb{C}P^n \mid \sum_{i=0}^{m} z_i w_i = 0\}.$$

Suppose that $m \geq 2$ and $n \geq 3$. Then there is an inclusion $\iota : \mathbb{C}P^2 \to H_{m,n}$, defined by

$$\iota([z_0 : z_1 : z_2]) = ([z_0 : z_1 : z_2 : 0 : \cdots : 0], [0 : \cdots : 0 : 1]).$$

Write π_1 for the projection $H_{m,n} \to \mathbb{C}P^m$. Then the inclusion $\mathbb{C}P^2 \to \mathbb{C}P^m$ factors as

$$\begin{tikzcd} \mathbb{C}P^2 \arrow[r, "\iota"] \arrow[dr] & H_{m,n} \arrow[d, "\pi_1"] \\ & \mathbb{C}P^m. \end{tikzcd}$$

This implies that ι induces a surjection on integral K-theory, so we obtain the following.

Example 2.7. *For $m \geq 2$ and $n \geq 3$, the suspended Milnor Hypersurface $\Sigma H_{m,n}$ is p-hyperbolic for all $p \neq 2$.*

Let $U(n)$ denote the unitary group. There is a well-known map $r : \Sigma\mathbb{C}P^{n-1} \to U(n)$ (see for example [35]) which induces a surjection on K-theory. From this we obtain

Example 2.8. *For $n \geq 3$, the suspended unitary group $\Sigma U(n)$ is p-hyperbolic for all $p \neq 2$.*

2.3 Quantitative lower bounds on growth

In Section 4, we will derive the following simple lower bound for the lim inf in the definition of $\mathbb{Z}/p^r$-hyperbolicity, for the space $S^{q_1+1} \vee S^{q_2+1}$.

Corollary 2.9. *Let p be a prime and $r \in \mathbb{N}$. Let t_m be the constants of Definition 1.2 for $X = S^{q_1+1} \vee S^{q_1+1}$. Then*

$$\liminf_m \frac{\ln(t_m)}{m} \geq \frac{\ln(2)}{\max(q_1, q_2)}.$$

This implies that the t_m eventually exceed $((1-\varepsilon)2)^{\frac{m}{\max(q_1,q_2)}}$ for any $\varepsilon > 0$. The constant 2 reflects the number of wedge summands. Note that this bound is independent of p and r.

Example 2.10. *Taking $\varepsilon = \frac{1}{4}$, we find that for all $r \in \mathbb{N}$ and all primes p the number of $\mathbb{Z}/p^r$-summands in $\bigoplus_{i \leq m} \pi_i(S^2 \vee S^2)$ eventually exceeds $(\frac{3}{2})^m$.*

One can produce an analogous quantitative bound on the lim inf in the case of Theorem 1.4, but this bound is very weak. In particular, it depends on knowledge of the Adams operations on $K^*(X)$, and is at best $\frac{\ln(2)}{2(p-1)}$.

3 Preliminary results

Both Theorem 1.3 and Theorem 1.4 will be proven by means of Lemma 3.3. Our first goal is to establish this lemma.

Let L be the free Lie algebra over $\mathbb{Q}$ on basis elements $x_1, \ldots, x_n$. Write $\mathscr{L}_k$ for the subset of L consisting of the *basic products* of the x_i of *weight* k, in the sense of [20] , where the basic products of weight 1 are taken to be the x_i, ordered by $x_1 < x_2 < \cdots < x_n$. The union $\mathscr{L} = \bigcup_{k=1}^{\infty} \mathscr{L}_k$ is a vector space basis for L (see for example [32, Theorem 5.3] , but note that what we call basic products, Serre calls a *Hall basis*).

Let $\mu : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ be the Möbius inversion function, defined by

$$\mu(s) = \begin{cases} 1 & s = 1 \\ 0 & s > 1 \text{ is not square free} \\ (-1)^\ell & s > 1 \text{ is a product of } \ell \text{ distinct primes.} \end{cases}$$

The Witt Formula $W_n(k)$ is then defined by

$$W_n(k) = \frac{1}{k}\sum_{d|k}\mu(d)n^{\frac{k}{d}}.$$

Theorem 3.1. *[20, Theorem 3.3] Let L be the free Lie algebra over $\mathbb{Q}$ on basis elements $x_1, \ldots, x_n$. Then $|\mathscr{L}_k| = W_n(k)$.* □

Lemma 3.2. *[8, Introduction] The ratio*

$$\frac{W_n(k)}{\frac{1}{k}n^k}$$

tends to 1 as k tends to ∞. □

In particular, this implies that for $n \geq 2$, the Witt function $W_n(k)$ grows exponentially in k. It should follow that if the number of p-torsion summands in $\bigoplus_{i\leq k}\pi_i(Y)$ exceeds $W_2(k)$, then Y is p-hyperbolic. The following lemma makes a slightly generalised form of this idea precise.

Lemma 3.3. *Let Y be a space. Suppose that there exist $a, b \in \mathbb{N}$ such that the number of p-torsion summands (respectively, $\mathbb{Z}/p^r$-summands) in $\bigoplus_{i\leq ak+b}\pi_i(Y)$ exceeds $W_2(k)$, for all k large enough. Then Y is p-hyperbolic (respectively, $\mathbb{Z}/p^r$-hyperbolic).*

Proof. The proofs for p- and $\mathbb{Z}/p^r$-hyperbolicity are identical, so we give only the former. Reframing the hypothesis in terms of the sequence T_m of Definition 1.1, we are assuming precisely that $T_{ak+b} > W_2(k)$ for sufficiently large k. We then have that

$$\liminf_m \frac{\ln(T_m)}{m} = \liminf_k \frac{\ln(T_{ak+b})}{ak+b} \geq \liminf_k \frac{\ln(W_2(k))}{ak+b}.$$

It then follows from Lemma 3.2 that if $1 > \varepsilon > 0$, once k is large enough, we have

$$W_2(k) > (1-\varepsilon)\frac{1}{k}2^k.$$

This implies that

$$\liminf_k \frac{\ln(W_2(k))}{ak+b} \geq \liminf_k \frac{\ln((1-\varepsilon)\frac{1}{k}2^k)}{ak+b},$$

and since this holds for all $\varepsilon > 0$,

$$\liminf_m \frac{\ln(T_m)}{m} \geq \liminf_k \frac{\ln(W_2(k))}{ak+b} \geq \liminf_k \frac{\ln(\frac{1}{k}2^k)}{ak+b} = \liminf_k \frac{\ln(\frac{1}{k}) + k\ln(2)}{ak+b} = \frac{\ln(2)}{a},$$

which is greater than zero, as required. □

3.1 Existence of summands in the stable stems

We write π_j^S for the j-th stable stem in the homotopy groups of spheres, that is

$$\pi_j^S := \lim_{n\to\infty} \pi_{n+j}(S^n).$$

The proof of Theorem 1.3, depends on having, for each p and r, some j such that π_j^S contains a $\mathbb{Z}/p^r$-summand. The purpose of this subsection is to show that the existence of such a j follows from existing work of Adams and others.

Lemma 3.4. *For any prime p and any $r \in \mathbb{N}$, there exists j such that $\mathbb{Z}/p^r$ is a direct summand in π_j^S. That is, for a fixed choice of such a j, $\mathbb{Z}/p^r$ is a direct summand in $\pi_{n+j}(S^n)$ for all $n \geq j+2$.*

Proof. We write $\nu_p(s)$ for the largest power of p dividing the integer s.

CASE 1 (p odd): Set $t := p^{r-1}(p-1)$, and notice that, since $(p-1)$ is even, $j := 4t-1$ is congruent to 7 mod 8. Theorem 1.6 of [2] , and the discussion immediately following it, then tells us that π_j^S contains a direct summand isomorphic to $\mathbb{Z}/m(2t)$, for a function m which Adams defines. By decomposing this subgroup into direct summands of prime power order, it suffices to show that $\nu_p(m(2t)) = r$.

The discussion after Theorem 2.5 in [1] gives that since $t \equiv 0 \bmod (p-1)$,

$$\nu_p(m(2t)) = 1 + \nu_p(2t).$$

Now, $\nu_p(2t)$ is equal to $(r-1)$, by definition of t, so $\nu_p(m(2t)) = r$, as required.

CASE 2 ($p = 2$, $r \geq 3$): Set $t := 2^{r-3}$, and set $j := 4t-1$. From Theorem 1.5, and the discussion following Theorem 1.6 in [2] , π_j^S has a direct summand isomorphic to $\mathbb{Z}/m(2t)$, regardless of whether j is congruent to 3 or 7 mod 8. Again, referring to the discussion after Theorem 2.5 of [1] , we see that

$$\nu_2(m(2t)) = 2 + \nu_2(2t) = 3 + \nu_2(t) = r,$$

as required.

CASE 3 ($p^r = 2$ and $p^r = 4$): It is known from [14] that $\pi_1^S \cong \mathbb{Z}/2$, and from [9] that $\pi_{34}^S \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3$. $\square$

4 Proof of Theorem 1.3

In this section we prove Theorem 1.3, which says that the wedge of two spheres is $\mathbb{Z}/p^r$-hyperbolic for all p and r. We also prove Corollary 2.9, which extracts some simple quantitative information from the proof of Theorem 1.3. We first record the following simple observation.

Remark 4.1. Let $k_1, \ldots, k_n$ and $q_1, \ldots q_n$ be non-negative integers. Suppose that $q_1 \leq q_2 \leq \cdots \leq q_n$, and let $k = \sum_{i=1}^{n} k_i$. Then

$$kq_1 \leq \sum_{i=1}^{n} k_i q_i \leq kq_n.$$

Proof of Theorem 1.3. Assume without loss of generality that $q_1 \leq q_2$. By Lemma 3.3 It suffices to show that there exist constants a and b such that the number of $\mathbb{Z}/p^r$-summands in

$$\bigoplus_{i \leq ak+b} \pi_i(S^{q_1+1} \vee S^{q_2+1})$$

exceeds $W_2(k)$, for k large enough.

We first apply the Hilton-Milnor Theorem. Since we are dealing with spheres, we need only the original form, due to Hilton in [20] :

$$\Omega(S^{q_1+1} \vee S^{q_2+1}) \simeq \Omega\Sigma(S^{q_1} \vee S^{q_2}) \simeq \Omega \prod_{B \in \mathscr{L}} S^{k_1 q_1 + k_2 q_2 + 1},$$

where, as in Section 3, $\mathscr{L} = \bigcup_{k=1}^{\infty} \mathscr{L}_k$ is Hilton's 'basic product' basis for L, the free Lie Algebra over $\mathbb{Q}$ on two generators x_1 and x_2, and k_i is the number of occurrences of the generator x_i in the bracket B. Recall also from Section 3 that the weight k of a bracket B is equal to $k_1 + k_2$, and that the cardinality of $\mathscr{L}_k$ is given by the Witt formula $W_2(k)$ by Theorem 3.1.

For fixed $k \in \mathbb{N}$, consider the factor in the above product corresponding to $\mathscr{L}_k \subset \mathscr{L}$:

$$F_k := \Omega \prod_{B \in \mathscr{L}_k} S^{k_1 q_1 + k_2 q_2 + 1}.$$

The associated subgroup of $\pi_*(S^{q_1+1} \vee S^{q_2+1})$,

$$\bigoplus_{B \in \mathscr{L}_k} \pi_*(S^{k_1 q_1 + k_2 q_2 + 1}),$$

is a direct summand.

We will first find a $\mathbb{Z}/p^r$-summand in the homotopy groups of each of the spheres appearing in F_k. Since $q_1 \leq q_2$, Remark 4.1 applies, and we may lower bound the dimensions of spheres appearing in F_k by $k_1q_1 + k_2q_2 + 1 \geq kq_1 + 1$. By Lemma 3.4, there exists $j \in \mathbb{N}$ such that $\pi_{j+\ell}(S^\ell)$ has a direct summand $\mathbb{Z}/p^r$ for $\ell \geq j+2$. Therefore, if k is large enough that $kq_1 \geq j+1$, then $k_1q_1 + k_2q_2 + 1 \geq j+2$ - that is, the j-th stem is stable on all of the spheres occurring in F_k. Thus, for k large enough, there is a $\mathbb{Z}/p^r$ summand in $\pi_{j+k_1q_1+k_2q_2+1}(S^{k_1q_1+k_2q_2+1})$ whenever $k_1 + k_2 = k$.

We now upper bound the dimension of the homotopy groups in which these summands appear. Since $q_1 \leq q_2$ we have by Remark 4.1 that $j + k_1q_1 + k_2q_2 + 1 \leq kq_2 + 1 + j$, so each of the $\mathbb{Z}/p^r$-summands we have identified is a distinct direct summand in

$$\bigoplus_{i \leq kq_2+1+j} \bigoplus_{B \in \mathscr{L}_k} \pi_i(S^{k_1q_1+k_2q_2+1}),$$

hence in

$$\bigoplus_{i \leq kq_2+1+j} \pi_i(S^{q_1+1} \vee S^{q_2+1}).$$

We have identified one such summand for each $B \in \mathscr{L}_k$, so the number of $\mathbb{Z}/p^r$-summands in $\bigoplus_{i \leq kq_2+1+j} \pi_i(S^{q_1+1} \vee S^{q_2+1})$ is at least $|\mathscr{L}_k| = W_2(k)$. Thus, taking $a = q_2$ and $b = 1+j$ in Lemma 3.3 suffices. □

Proof of Corollary 2.9. The last line of the proof of Lemma 3.3 shows that $\liminf_m \frac{\ln t_m}{m} > \frac{\ln 2}{a}$. The last line of the proof of Theorem 1.3 implies that a may be taken to be q_2, under the assumption that $q_1 \leq q_2$, which implies the result. □

5 K-theory and K-homology of $\Omega\Sigma X$

The remainder of this paper proves Theorem 1.4. Sections 5 and 6 assemble necessary background, which we will use in Section 7 to prove the result.

When studying the homotopy groups of a suspension ΣX, as in Theorem 1.4, the following approach is natural. Since $\pi_*(\Sigma X) \cong \pi_{*-1}(\Omega\Sigma X)$, we may instead study $\Omega\Sigma X$. This is useful because $\Omega\Sigma X$ is well understood homologically via the Bott-Samelson theorem, which decomposes its homology as the tensor algebra on $\widetilde{H}_*(X)$. Because we will need to use Adams' e-invariant, which is defined in terms of K-theory, we wish to replace ordinary homology with K-homology.

The purpose of Section 5 is to record the version of the Bott-Samelson theorem which applies to (torsion-free) K-homology, along with a universal coefficient theorem for passing between K-theory and K-homology. All of the material here is already known (in particular much of it is in [28]) so its summary here is for convenience and clarity.

Our conventions on definition of $\widetilde{K}^*(X)$ are those of [7] . In particular, we define $\widetilde{K}^{-1}(X) := \widetilde{K}^0(\Sigma X)$, and set $\widetilde{K}^*(X) := \widetilde{K}^0(X) \oplus \widetilde{K}^{-1}(X)$. We regard $\widetilde{K}^*(X)$ and $\widetilde{K}_*(X)$ as being $\mathbb{Z}/2$-graded. It is shown in [7] that $\widetilde{K}^*(X)$ is a $\mathbb{Z}/2$-graded ring.

We will wish to work with K-theory and K-homology modulo the torsion subgroup. For a space X, write $\widetilde{K}_*^{\mathrm{TF}}(X)$ and $\widetilde{K}^*_{\mathrm{TF}}(X)$ for the quotients of the reduced K-homology and K-theory of X by their torsion subgroups. The same convention applies in the unreduced case.

5.1 Künneth and universal coefficient theorems

The universal coefficient theorem for K-theory first appears in some unpublished lecture notes of Anderson [4] , and is first published by Yosimura [36] .

Theorem 5.1 (Universal coefficient theorem)**.** *For any CW-complex X and each integer n there is a short exact sequence*

$$0 \to \mathrm{Ext}(K_{n-1}(X), \mathbb{Z}) \to K^n(X) \to \mathrm{Hom}(K_n(X), \mathbb{Z}) \to 0. \qquad \square$$

In the torsion-free case, the universal coefficient theorem simplifies as follows.

Corollary 5.2. *Let Y be a finite CW-complex. Then we have a natural isomorphism which descends from the second map in the theorem above,*

$$K^n_{\mathrm{TF}}(Y) \xrightarrow{\cong} \mathrm{Hom}(K_n^{\mathrm{TF}}(Y), \mathbb{Z}).$$

There is an analogous isomorphism for the reduced theories:

$$\widetilde{K}^n_{\mathrm{TF}}(Y) \xrightarrow{\cong} \mathrm{Hom}(\widetilde{K}_n^{\mathrm{TF}}(Y), \mathbb{Z}).$$

Proof. Firstly, since Y is assumed to be a finite complex, the group $K_{n-1}(Y)$ is finitely generated abelian, so $\mathrm{Ext}(K_{n-1}(Y), \mathbb{Z})$ is torsion. Secondly, for any group G, $\mathrm{Hom}(G, \mathbb{Z})$ is torsion-free. Together, these observations show that the exact sequence of Theorem 5.1 has first term torsion and last term torsion-free. That means that it induces an isomorphism $K^n_{\mathrm{TF}}(Y) \xrightarrow{\cong} \mathrm{Hom}(K_n(Y), \mathbb{Z})$. Any homomorphism $K_n(Y) \to \mathbb{Z}$ is zero on the torsion subgroup of $K_n(Y)$, so the injection $\mathrm{Hom}(K_n^{\mathrm{TF}}(Y), \mathbb{Z}) \hookrightarrow \mathrm{Hom}(K_n(Y), \mathbb{Z})$ is an isomorphism, and the unreduced result follows. The reduced statement follows immediately from the unreduced one. $\square$

Selick [28] deduces the following from work of Atiyah [6] , Mislin [25] and Adams [3] .

Theorem 5.3 (Künneth theorem for K-homology). *Let X and Y be of the homotopy type of finite complexes. Then there is an isomorphism of $\mathbb{Z}/2$-graded $\mathbb{Z}$-modules:*

$$\widetilde{K}_*^{\mathrm{TF}}(X \wedge Y) \cong \widetilde{K}_*^{\mathrm{TF}}(X) \otimes \widetilde{K}_*^{\mathrm{TF}}(Y). \qquad \square$$

Remark 5.4. It follows immediately from Corollary 5.2 (and knowledge of $\widetilde{K}^*(S^q)$) that $\widetilde{K}_*^{\mathrm{TF}}(S^q) \cong \mathbb{Z}$. We write ξ_q for the generator of $\widetilde{K}_*^{\mathrm{TF}}(S^q)$. By the Künneth Theorem 5.3, we may choose the ξ_q so that $\xi_n \otimes \xi_m$ is identified with ξ_{n+m} under the homeomorphism $S^n \wedge S^m \cong S^{n+m}$.

In the case of K-theory, the analogous result follows directly from [3] .

Theorem 5.5 (Künneth theorem for K-theory). *Let X and Y be of the homotopy type of finite complexes. Then the external product on K-theory defines an isomorphism of $\mathbb{Z}/2$-graded rings:*

$$\widetilde{K}^*_{\mathrm{TF}}(X) \otimes \widetilde{K}^*_{\mathrm{TF}}(Y) \xrightarrow{\cong} \widetilde{K}^*_{\mathrm{TF}}(X \wedge Y).$$

5.2 The James construction

For a space X, let X^s denote the product of s copies of X. Let $\sim$ be the relation on X^s defined by

$$(x_1, \dots, x_{i-1}, *, x_{i+1}, x_{i+2}, \dots x_s) \sim (x_1, \dots, x_{i-1}, x_{i+1}, *, x_{i+2}, \dots x_s).$$

Let $J_s(X)$ be the space ${X^s}/{\sim}$. There is a natural inclusion

$$J_s(X) \hookrightarrow J_{s+1}(X)$$

$$(x_1, \dots, x_s) \mapsto (x_1, \dots, x_s, *).$$

The *James construction* JX is defined to be the colimit of the diagram consisting of the spaces $J_s(X)$ and the above inclusions. Write $i_s : J_s(X) \to JX$ for the map associated to the colimit. Notice that JX carries a product given by concatenation, which makes it into the free topological monoid on X, and that a topological monoid is in particular an H-space.

Let $X^{\wedge i}$ denote the smash product of i copies of X, and let $\eta : X \to \Omega\Sigma X$ be the unit of the adjunction $\Sigma \dashv \Omega$. Explicitly, $\eta(x) = (t \mapsto \langle x, t \rangle \in \Sigma X)$.

Theorem 5.6. *[23]*

1. *There is a homotopy equivalence $JX \xrightarrow{\simeq} \Omega\Sigma X$ which respects the H-space structures and identifies i_1 with η.*

2. *There is a homotopy equivalence* $\bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \xrightarrow{\simeq} \Sigma JX$ *which restricts to a homotopy equivalence* $\bigvee_{i=1}^{s} \Sigma X^{\wedge i} \xrightarrow{\simeq} \Sigma J_s(X)$ *for each* $s \in \mathbb{N}$. □

Lemma 5.7. *[28, Lemma 7] Let X have the homotopy type of an $(r-1)$-connected CW-complex.*

1. $(i_s)_* : \pi_N(J_s(X)) \to \pi_N(JX)$ *is an isomorphism for* $N < r(s+1) - 1$.

2. *Let* $x \in \pi_N(J_s(X))$ *for any* N. *If* Σx *is nontrivial then* $(i_s)_*(x)$ *is also nontrivial.*

Proof. The first part follows by cellular approximation from the observation that $J_s(X)$ contains the $(r(s+1)-1)$-skeleton of JX. The second part follows from the observation that Σi_s has a retraction by Theorem 5.6. □

For spaces X and Y, let $X * Y$ denote the *join*, which we define to be the homotopy pushout of the projections $X \times Y \to X$ and $X \times Y \to Y$. The join is naturally a quotient of $X \times I \times Y$, where I denotes the unit interval. Following the treatment in [5] , let C_1 denote the subspace of $X * Y$ consisting of points of the form $(x, t, *)$, for $t \in I$ and $x \in X$, and let C_2 be the subspace consisting of points of the form $(*, t, y)$. The subspace $C_1 \cup C_2 \cong CX \cup CY$ is contractible, so the quotient map $q : X * Y \to {X * Y}/{C_1 \cup C_2}$ is a homotopy equivalence. The quotient ${X * Y}/{C_1 \cup C_2}$ is homeomorphic to $\Sigma X \wedge Y$. The suspended product $\Sigma(X \times Y)$ is also a quotient of $X \times I \times Y$, and this quotient lies between $X * Y$ and ${X * Y}/{C_1 \cup C_2}$.

This gives a factorization of q as $X * Y \to \Sigma(X \times Y) \to \Sigma(X \wedge Y)$. Let q^{-1} denote any choice of homotopy inverse to q; all possible choices are homotopic. We may form a new map $\delta_{X,Y}$ as the composite $\Sigma(X \wedge Y) \xrightarrow{q^{-1}} X * Y \to \Sigma(X \times Y)$. It is automatic that $\delta_{X,Y}$ splits the quotient map $\pi : \Sigma(X \times Y) \to \Sigma X \wedge Y$. The homotopy class of $\delta_{X,Y}$ is well-defined, and we will call $\delta_{X,Y}$ the *canonical splitting* of π. Note that $\delta_{X,Y}$ is natural in maps of spaces in the sense that given $f : A \to X$ and $g : B \to Y$ we obtain a commutative diagram

$$\begin{tikzcd} \Sigma A \wedge B \arrow[r, "\delta_{A,B}"] \arrow[d, "\Sigma(f\wedge g)"] & \Sigma(A \times B) \arrow[d, "\Sigma(f\times g)"] \\ \Sigma X \wedge Y \arrow[r, "\delta_{X,Y}"] & \Sigma(X \times Y). \end{tikzcd}$$

For $s \geq 3$, consider the quotient map $\Sigma X^s \to \Sigma X^{\wedge s}$. We define the *canonical splitting* of this quotient to be the composite of canonical splittings

$$\Sigma X^{\wedge s} \to \Sigma(X \times X) \wedge X^{\wedge(s-2)} \to \Sigma((X \times X) \times X) \wedge X^{\wedge(s-3)} \to \cdots \to \Sigma X^s.$$

Of course, we chose an order of multiplication here. This canonical splitting is natural as before.

Definition 5.8. For a $\mathbb{Z}$-graded (respectively $\mathbb{Z}/2$-graded) module M, let $T(M) = \bigoplus_{k=1}^{\infty} M^{\otimes k}$ denote the *tensor algebra* on M. The product is given by concatenation. We refer to $M^{\otimes k}$ as the *weight k component* of the tensor algebra $T(M)$. We define a $\mathbb{Z}$-grading (respectively $\mathbb{Z}/2$-grading) on $T(M)$ by setting $|x_1 \otimes x_2 \otimes \cdots \otimes x_k| = \sum_{i=1}^{k} |x_i|$.

Definition 5.9. For a space Y, let $\sigma : \widetilde{K}_*^{\mathrm{TF}}(Y) \xrightarrow{\cong} \widetilde{K}_{*+1}^{\mathrm{TF}}(\Sigma Y)$ be the suspension isomorphism. Let $\varphi : \widetilde{K}_*^{\mathrm{TF}}(\Sigma Y) \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma Y)$ be a homomorphism of graded groups, not necessarily induced by a map of spaces. We call the composite $\sigma^{-1} \circ \varphi \circ \sigma$ the *desuspension* of θ, denoting it by $S^{-1}\varphi$.

Write $m_s : (\Omega\Sigma X)^s \to \Omega\Sigma X$ for the map given by iteratively performing the standard loop multiplication on $\Omega\Sigma X$ in any choice of order. Up to homotopy, m_s is independent of this choice of order, since $\Omega\Sigma X$ is homotopy associative.

Theorem 5.6 gives the existence of a homotopy equivalence $\Gamma : \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \to \Sigma\Omega\Sigma X$. There are many choices of Γ, up to homotopy. The next lemma asserts that Γ can be chosen in a way which suits our purpose. Selick [28] describes the composite $\bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \xrightarrow{\Gamma} \Sigma\Omega\Sigma X \xrightarrow{\simeq} \Sigma JX$ of Γ with the homotopy equivalence of Theorem 5.6 (1). This immediately implies the following description of Γ.

Lemma 5.10. *[28] Let X be a finite CW-complex. The homotopy equivalence* $\Gamma : \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \to \Sigma\Omega\Sigma X$ *may be chosen such that:*

1. $S^{-1}(\Gamma_*) : T(\widetilde{K}_*^{\mathrm{TF}}(X)) \xrightarrow{\cong} K_*^{\mathrm{TF}}(\Omega\Sigma X)$ *is an isomorphism of algebras;*
2. *the restriction of* Γ *to* $\Sigma X^{\wedge s}$ *is homotopic to the composite*

$$\Sigma X^{\wedge s} \to \Sigma X^s \xrightarrow{\Sigma(\eta)^s} \Sigma(\Omega\Sigma X)^s \xrightarrow{\Sigma m_s} \Sigma\Omega\Sigma X,$$

where the unlabelled arrow is the canonical splitting.

The description of the map Γ in Lemma 5.10 has the following consequence. For a space Y, let $\mathrm{ev} : \Sigma\Omega Y \to Y$ be the evaluation map, which may be described explicitly by $\mathrm{ev}(\langle \gamma, t\rangle) = \gamma(t)$ for $\gamma \in \Omega Y$.

Lemma 5.11. *Let* $\Gamma : \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \to \Sigma\Omega\Sigma X$ *be the homotopy equivalence of Lemma 5.10. The composite* $\mathrm{ev} \circ \Gamma$ *is homotopic to the projection onto the first wedge summand.*

Proof. Let $\iota_s : \Sigma X^{\wedge s} \to \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i}$ be the inclusion of the s-th wedge summand. We must show that

$$\mathrm{ev} \circ \Gamma \circ \iota_s \simeq \begin{cases} 1_{\Sigma X} & \text{if } s = 1, \text{ and} \\ * & \text{otherwise.} \end{cases}$$

The following diagram commutes up to homotopy

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma\eta"] \arrow[dr, "1_{\Sigma X}"'] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}"] \\
& \Sigma X.
\end{tikzcd}$$

By Lemma 5.10, $\Sigma\eta = \Gamma \circ \iota_1$, which implies the $s = 1$ statement.

Now let $s \geq 2$. Ganea [15, Theorems 1.1 and 1.4] shows that the homotopy fibre of ev is given by

$$\Sigma(\Omega\Sigma X \wedge \Omega\Sigma X) \xrightarrow{v} \Sigma\Omega\Sigma X \xrightarrow{\mathrm{ev}} \Sigma X,$$

where the map v is equal to the composite

$$\Sigma(\Omega\Sigma X \wedge \Omega\Sigma X) \to \Sigma(\Omega\Sigma X \times \Omega\Sigma X) \xrightarrow{\Sigma m_2} \Sigma\Omega\Sigma X$$

of Σm_2 with the canonical splitting. We will show that $\Gamma \circ \iota_s$ factors through v, and hence composes trivially with ev. Consider the following diagram, where the unlabelled arrows are all canonical splittings:

$$\begin{tikzcd}
& \Sigma(\Omega\Sigma X \wedge \Omega\Sigma X) \arrow[r] \arrow[rr, bend left, "v"] & \Sigma(\Omega\Sigma X \times \Omega\Sigma X) \arrow[r, "\Sigma m_2"'] & \Sigma\Omega\Sigma X \arrow[d, equal] \\
\Sigma(\Omega\Sigma X)^{\wedge s} \arrow[r] & \Sigma((\Omega\Sigma X)^{(s-1)} \wedge \Omega\Sigma X) \arrow[r] \arrow[u, "\Sigma(m_{(s-1)}\wedge 1)"] & \Sigma(\Omega\Sigma X)^s \arrow[r, "\Sigma m_s"'] \arrow[u, "\Sigma(m_{(s-1)}\times 1)"] & \Sigma\Omega\Sigma X \\
\Sigma X^{\wedge s} \arrow[r] \arrow[u, "\Sigma\eta^{\wedge s}"] & \Sigma(X^{(s-1)} \wedge X) \arrow[r] \arrow[u, "\Sigma(\eta^{(s-1)}\wedge\eta)"] & \Sigma X^s. \arrow[u, "\Sigma\eta^s"] &
\end{tikzcd}$$

The composite along the bottom of the diagram is $\Gamma \circ \iota_s$, so to obtain the desired factorization of $\Gamma \circ \iota_s$ through v, it suffices to show that the diagram commutes up to homotopy.

The top right square commutes because $m_2 \circ m_{(s-1)} \simeq m_s$, by homotopy associativity of the H-space $\Omega\Sigma X$. The remaining three squares commute by naturality of our canonical splitting. This completes the proof. $\square$

Let ρ_k be the projection $T(\widetilde{K}^{\mathrm{TF}}_*(X)) \to \widetilde{K}^{\mathrm{TF}}_*(X)^{\otimes k}$. The next corollary is immediate from Lemma 5.11.

Corollary 5.12. $S^{-1}(\mathrm{ev}_* \circ \Gamma_*) = \rho_1 : T(\widetilde{K}^{\mathrm{TF}}_*(X)) \to \widetilde{K}^{\mathrm{TF}}_*(X)$. $\square$

5.3 Primitives and commutators

It follows from the Künneth Theorem (Theorem 5.3), and the fact that $\Sigma(Y \times Y) \simeq \Sigma Y \vee \Sigma Y \vee \Sigma(Y \wedge Y)$, that $K^{\mathrm{TF}}_*(Y \times Y) \cong K^{\mathrm{TF}}_*(Y) \otimes K^{\mathrm{TF}}_*(Y)$. We may therefore make the following definition. A class $y \in \widetilde{K}_*^{\mathrm{TF}}(Y)$ is called *primitive* if $\Delta_*(y) = y \otimes 1 + 1 \otimes y$, where $\Delta : Y \to Y \times Y$ is the diagonal, defined by $\Delta(y) = (y, y)$.

The comultiplication $Y \to Y \vee Y$ on a co-H-space Y is a factorization of Δ via the inclusion $Y \vee Y \hookrightarrow Y \times Y$. From this point of view, the following lemma is immediate.

Lemma 5.13. *If Y is a co-H-space, then all elements in $\widetilde{K}_*^{\mathrm{TF}}(Y)$ are primitive.* □

If Y is an H-group, then the multiplication $m : Y \times Y \to Y$ induces a map $\widetilde{K}_*^{\mathrm{TF}}(Y) \otimes \widetilde{K}_*^{\mathrm{TF}}(Y) \to \widetilde{K}_*^{\mathrm{TF}}(Y)$. We will denote this map by juxtaposition, so that $m_*(y_1 \otimes y_2) = y_1 y_2$. Furthermore, the commutator $Y \times Y \to Y$ descends to a map $c : Y \wedge Y \to Y$. Expanding the definition of the commutator in terms of the K-homology Künneth Theorem (Theorem 5.3) gives the following lemma.

Lemma 5.14. *Let Y be an H-group, and let $c : Y \wedge Y \to Y$ be the commutator. If y_1 and $y_2 \in \widetilde{K}_*^{\mathrm{TF}}(Y)$ are primitive, then $c_*(y_1 \otimes y_2) = y_1 y_2 - (-1)^{|y_1||y_2|} y_2 y_1$.* □

6 The category of ψ-modules

In [2] , Adams defines an abelian category which we will follow Selick [28] in calling ψ-modules. The e-invariant, which is our central tool, is defined by Adams in terms of ψ-modules. The purpose of this section is to record results about ψ-modules for later use.

A ψ-module consists of an abelian group M, with homomorphisms

$$\psi^\ell : M \to M \ (\ell \in \mathbb{Z})$$

satisfying the axioms of [2, Section 6] . If X is a space then the group $\widetilde{K}^0(X)$, together with its Adams operations, is a ψ-module. Since we defined $\widetilde{K}^{-1}(X)$ by setting $\widetilde{K}^{-1}(X) = \widetilde{K}^0(\Sigma X)$, it too has the structure of a ψ-module. Maps of spaces induce maps of ψ-modules. The Adams operation ψ^ℓ on $\widetilde{K}^0(S^{2n})$ is multiplication by ℓ^n, so in particular Adams operations do not commute with the Bott isomorphism.

For graded ψ-modules M and N we will write $\mathrm{Hom}_{\psi-\mathrm{Mod}}(M, N)$ for the abelian group consisting of graded ψ-module homomorphisms. The unadorned notation $\mathrm{Hom}(M, N)$ will mean homomorphisms of the underlying graded abelian groups.

Lemma 6.1. *Let M and N be ψ-modules, with N torsion-free. The inclusion of $\mathbb{Z}$-modules $\mathrm{Hom}_{\psi-\mathrm{Mod}}(M, N) \hookrightarrow \mathrm{Hom}(M, N)$ is an injection onto a summand.*

Proof. Let $\varphi : M \to N$ be a homomorphism of underlying $\mathbb{Z}$-modules. If, for some $k \in \mathbb{Z} \setminus \{0\}$, $k \cdot \varphi$ is a ψ-module homomorphism, then, since N is torsion-free, φ is also

a ψ-module homomorphism. This implies that
$\operatorname{coker}(\operatorname{Hom}_{\psi-\mathrm{Mod}}(M,N) \hookrightarrow \operatorname{Hom}(M,N))$ is torsion-free, which implies the result. □

For the avoidance of doubt, by the e-invariant we will always mean the map that Adams calls the complex e-invariant e_C [1, 2] .

Definition 6.2 (Adams' e-invariant)**.** Suppose that $f : X \to Y$ induces the trivial map on $\widetilde{K}^*$. Then the cofibre sequence of f gives a short exact sequence of ψ-modules

$$0 \leftarrow \widetilde{K}^0(Y) \leftarrow \widetilde{K}^0(C_f) \leftarrow \widetilde{K}^0(\Sigma X) \leftarrow 0.$$

The *e-invariant* of f is the element of $\operatorname{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Y), \widetilde{K}^0(\Sigma X))$ represented by this exact sequence.

The e-invariant does not commute with the Bott isomorphism, but the interaction between the Bott isomorphism and the Adams operations is easy to describe, as follows. Let ψ^ℓ_Y be the homomorphism $\psi^\ell : \widetilde{K}^0(Y) \to \widetilde{K}^0(Y)$. Then, modulo the Bott isomorphism, we have $\psi^\ell_{\Sigma^2 X} = \ell \cdot \psi^\ell_X$. That is 'upon double suspending, the Adams operations gain a factor ℓ'. In terms of the e-invariant, all we need to know is the following.

Lemma 6.3. *[2, Proposition 3.4b)] There is a homomorphism*
$T : \operatorname{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Y), \widetilde{K}^0(\Sigma X)) \to \operatorname{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(\Sigma^2 Y), \widetilde{K}^0(\Sigma^3 X))$*, such that*
$T(e(f)) = e(\Sigma^2 f)$.

We will be concerned only with the e-invariants of maps whose domain is a sphere. One of the two K-groups of a sphere vanishes, in the dimension matching the parity of the sphere, but the e-invariant, as defined above, lives only in K^0. In order to detect maps regardless of the parity of the sphere on which they are defined, we will need to keep track of the e-invariants of f and Σf, so we will use the following modified e-invariant.

Definition 6.4 (Double e-invariant)**.** Let

$$\begin{aligned}\operatorname{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Y), \widetilde{K}^*(\Sigma X)) := {} & \operatorname{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Y), \widetilde{K}^0(\Sigma X)) \\ & \oplus \operatorname{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^{-1}(Y), \widetilde{K}^{-1}(\Sigma X)).\end{aligned}$$

Suppose that $f : X \to Y$ induces the trivial map on $\widetilde{K}^*$. Then the *double e-invariant* of f is $\bar{e}(f) = (e(f), e(\Sigma f)) \in \operatorname{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Y), \widetilde{K}^*(\Sigma X))$.

Pullback of an extension along a homomorphism defines a map

$$\operatorname{Hom}_{\psi-\mathrm{Mod}}(M,B) \otimes \operatorname{Ext}_{\psi-\mathrm{Mod}}(B,A) \to \operatorname{Ext}_{\psi-\mathrm{Mod}}(M,A).$$

If $g : Y \to Z$ then $e(g \circ f)$ is represented by the pullback of $e(f)$ and $g^* : \widetilde{K}^0(Z) \to \widetilde{K}^0(Y)$ [2, Proposition 3.2 b)] . To describe $\bar{e}(g \circ f)$ we need only apply this result degree-wise, as follows. For convenience, we write $g^* \cdot e(f)$ for the pullback of g^* and $e(f)$. Define the map

$$\theta_0(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Z), \widetilde{K}^0(Y)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Z), \widetilde{K}^0(\Sigma X))$$

$$\theta_0(f)(x) = x \cdot e(f).$$

Likewise, define

$$\theta_{-1}(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^{-1}(Z), \widetilde{K}^{-1}(Y)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^{-1}(Z), \widetilde{K}^{-1}(\Sigma X))$$

$$\theta_{-1}(f)(x) = x \cdot e(\Sigma f).$$

Combining these, let

$$\theta(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z), \widetilde{K}^*(Y)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z), \widetilde{K}^*(\Sigma X))$$

be the direct sum $\theta_0(f) \oplus \theta_{-1}(f)$. These definitions, together with Adams' above result, give the following lemma.

Lemma 6.5. *For a map $f : X \to Y$, the following diagram commutes:*

$$\begin{tikzcd}
{[Y,Z]} \arrow[r, "f^*"] \arrow[d, "\deg"'] & {[X,Z]} \arrow[d, "\bar{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z), \widetilde{K}^*(Y)) \arrow[r, "\theta(f)"] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z), \widetilde{K}^*(\Sigma X)).
\end{tikzcd}$$

Following [28] , write $\mathbb{Z}(n)$ for the ψ-module $\widetilde{K}^0(S^{2n})$. Explicitly, $\mathbb{Z}(n)$ has underlying abelian group $\mathbb{Z}$, and ψ^ℓ acts by multiplication by ℓ^n. It follows that $\widetilde{K}^{-1}(S^{2n+1}) := \widetilde{K}^0(S^{2n+2}) \cong \mathbb{Z}(n+1)$.

Lemma 6.6. *[2, Proposition 7.8, 7.9] If $n < m$ then $\mathrm{Ext}_{\psi-\mathrm{Mod}}(\mathbb{Z}(n), \mathbb{Z}(m))$ injects into $\mathbb{Q}/\mathbb{Z}$. The e-invariant of a map $f : S^{2m-1} \to S^{2n}$ may therefore be regarded as an element of $\mathbb{Q}/\mathbb{Z}$. Furthermore, the value $e(f)$ in $\mathbb{Q}/\mathbb{Z}$ satisfies $e(\Sigma^2 f) = e(f)$, so in particular, when f is a map between spheres, $e(f)$ depends only on the stable homotopy class of f.* □

The following theorem is the main technical component of Selick's paper [28] .

Theorem 6.7. *[28, Theorem 6] Let $f' : S^{2m-1} \to S^{2n}$ be such that $p^{t-1}e(f') \neq 0$ in $\mathbb{Q}/\mathbb{Z}$, for p prime and some $t \in \mathbb{N}$. Let Y have the homotopy type of a finite CW-complex and let $g : S^{2n} \to Y$ be such that $\mathrm{Im}(g^* : \widetilde{K}^0(Y) \to \widetilde{K}^0(S^{2n}))$ contains $up^s\widetilde{K}^0(S^{2n})$, for $s \in \mathbb{N}$ and u prime to p. If $s < t$, and there exists some $\ell \in \mathbb{N}$ for which*

$$\psi^\ell \otimes \mathbb{Q} : \widetilde{K}^0(Y) \otimes \mathbb{Q} \to \widetilde{K}^0(Y) \otimes \mathbb{Q}$$

does not have ℓ^m as an eigenvalue, then $e(g \circ f') \neq 0$. □

The following theorem of Gray [16] will provide the map f' for Theorem 6.7. Specifically, this theorem provides a linearly spaced family of stems, each of which has a stable p-torsion class which is born on S^3 and detected by the e-invariant.

Theorem 6.8. *[16, Corollary of Theorem 6.2] Let p be an odd prime and let $j \in \mathbb{N}$. Then there exists a class $f_{p,j} \in \pi_{2j(p-1)+2}(S^3)$ with $e(f_{p,j}) = \frac{-1}{p} \in \mathbb{Q}/\mathbb{Z}$.* □

The corresponding 2-primary result is as follows. Adams [2, Theorem 1.5 and Proposition 7.14] shows that, for $j > 0$, the $(8j+3)$-rd stem contains a direct summand whose 2-primary component has order 8, and that on this component the e-invariant is a surjection onto $\mathbb{Z}/4$. The sphere of origin of the classes in this component was deduced by Curtis in [12] .

Theorem 6.9. *[2, Cur69] Let $j \in \mathbb{N}$. Then there exists a class $f_{2,j} \in \pi_{8j+6}(S^3)$ of order 4, with $e(f_{2,j}) = \frac{-1}{2} \in \mathbb{Q}/\mathbb{Z}$.* □

7 Main construction

Having assembled preliminaries in Sections 5 and 6, we can begin to work towards the proof of Theorem 1.4. Our approach is as follows. From the data of Theorem 1.4, we will construct a commutative diagram of (roughly) the following form, where $\mathscr{B}$ is a set and the other objects are $\mathbb{Z}$-modules.

$$\begin{tikzcd}
\mathscr{B}^k \arrow[r] \arrow[d] & \pi_*(\Omega\Sigma X) \arrow[d] \\
I^k \arrow[r] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(\Omega\Sigma X), \widetilde{K}^*_{\mathrm{TF}}(S^*)).
\end{tikzcd}$$

We will argue that

- The image of the top map consists of classes of order dividing p.
- The image of the left vertical map generates a submodule isomorphic to the weight k component of the free graded Lie algebra over $\mathbb{Z}/p$ on two generators.
- The bottom map is injective.

Together, these facts imply that there is a submodule of $\pi_*(\Omega\Sigma X) \cong \pi_{*+1}(\Sigma X)$, consisting of classes of order dividing p, and surjecting onto a module isomorphic to the weight k component of the free graded Lie algebra over $\mathbb{Z}/p$ on two generators.

This submodule (which is necessarily a $\mathbb{Z}/p$-vector space) must therefore have dimension at least $W_2(k)$ (Theorem 7.5), which will imply that ΣX is p-hyperbolic (Lemma 3.3).

The diagram will be obtained by juxtaposing three squares. Subsections 7.1, 7.2, and 7.3 each construct one of these squares. In Subsection 7.4 we put them together and prove Theorem 1.4. Roughly speaking, the top map of the diagram should be thought of as first taking a family of Samelson products and then pulling them back along some suitable map f coming from Gray's work (Theorem 6.8). The vertical maps should be thought of as passing from maps of spaces to K-theoretic invariants, and the bottom map (therefore) should be thought of as tracking the effect of the top map on these invariants.

Because of the need to work with a finite CW-complex in Selick's Theorem (Theorem 6.7) we will restrict the right hand side of the diagram to instead refer to some finite skeleton $J_s(X)$ of the James construction.

7.1 Samelson products and their Hurewicz images in K-homology

Let R be a commutative ring with unit. We take a *graded Lie algebra* over R to be defined as in [26] . For a non-negatively graded R-module V, let $L(V)$ denote the *free graded Lie algebra* [26, Section 8.5] . Write $L^k(V)$ for the submodule of $L(V)$ generated by the brackets of length k in the elements of V. We will call $L^k(V)$ the *weight k component* of $L(V)$. Note that this convention differs from Neisendorfer's - he writes $L(V)_k$ for the weight k component.

Definition 7.1. Let Y be an H-group, and let $c : Y \wedge Y \to Y$ be the commutator of Lemma 5.14. Let $\alpha \in \pi_N(Y)$, and let $\beta \in \pi_M(Y)$. The *Samelson product* of α and β, written $\langle \alpha, \beta \rangle \in \pi_{N+M}(Y)$, is the composite

$$\langle \alpha, \beta \rangle : S^{N+M} \cong S^N \wedge S^M \xrightarrow{\alpha\wedge\beta} Y \wedge Y \xrightarrow{c} Y.$$

Samelson products fail to make $\pi_*(Y)$ into a graded Lie algebra over $\mathbb{Z}$ [27, Section 7] , but they do define the structure of a graded magma. In fact, they define a sort of 'pseudo-Lie algebra' structure, since they are graded anticommutative and satisfy the graded Jacobi identity. One could define an appropriate notion of 'free graded pseudo-Lie algebra', and proceed as follows with that in place of the magma we will use, but we prefer the more lightweight approach.

For a graded R-module V, let $U(V)$ denote the graded set of homogeneous elements in V. Let $\mathscr{B}(V)$ be the free magma on $U(V)$, where we write the product as a bracket $[x, y]$. We think of $\mathscr{B}$ as the 'set of brackets of homogeneous elements in V'. Elements of $\mathscr{B}(V)$ are nonassociative words in the elements of $U(V)$, so we may define a grading on $\mathscr{B}(V)$ which extends the grading on $U(V)$ via $|[x,y]| = |x| + |y|$. The *weight* of an element of $\mathscr{B}(V)$ is its word length. Write $\mathscr{B}_N(V)$ for the subset of elements in degree N, $\mathscr{B}^k(V)$ for the subset of elements of weight k, and set $\mathscr{B}^k_N(V) = \mathscr{B}^k(V) \cap \mathscr{B}_N(V)$.

Let $\nu : A \to \Omega\Sigma X$ be a map. By the universal property of the free magma $\mathscr{B}(\pi_*(A))$, there exists a map

$$\Phi^\pi_\nu : \mathscr{B}(\pi_*(A)) \to \pi_*(\Omega\Sigma X)$$

which extends ν_* and satisfies $\Phi^\pi_\nu([x,y]) = \langle \Phi^\pi_\nu(x), \Phi^\pi_\nu(y)\rangle$ for all $x, y \in \mathscr{B}(\pi_*(A))$.

For a $\mathbb{Z}/2$-graded $\mathbb{Z}$-module V, we define a non-negatively graded $\mathbb{Z}$-module $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), V)$, by setting

$$\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), V)_N = \begin{cases} \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), V) & \text{if } N > 0, \text{ and} \\ 0 & \text{if } N \leq 0, \end{cases}$$

where the homomorphisms are understood to respect the $\mathbb{Z}/2$-grading on $\widetilde{K}_*$ and V.

In the case that $V = L$ is a $\mathbb{Z}/2$-graded Lie algebra over $\mathbb{Z}$, $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L)$ inherits a non-negatively graded Lie algebra structure as follows. Let the generators ξ_N of $\widetilde{K}^{\mathrm{TF}}_*(S^N)$ be as in Remark 5.4. Then the bracket $[f, g]$ of $f \in \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), L)$ and $g \in \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^M), L)$ is the homomorphism $\widetilde{K}^{\mathrm{TF}}_*(S^M) \to L$ carrying ξ_{N+M} to $[f(\xi_N), g(\xi_M)] \in L$. The squaring operation is defined in the same way. Likewise, if V is a $\mathbb{Z}/2$-graded associative algebra over $\mathbb{Z}$, then $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), V)$ inherits the structure of a non-negatively graded associative algebra.

Let $\nu : A \to \Omega\Sigma X$ be a map. There is a composition

$$L(\widetilde{K}^{\mathrm{TF}}_*(A)) \to T(\widetilde{K}^{\mathrm{TF}}_*(A)) \to \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X),$$

where the first map is the natural map which is the identity on $\widetilde{K}^{\mathrm{TF}}_*(A)$ and satisfies $[x, y] \mapsto xy - (-1)^{|x||y|}yx$, and the second map is obtained by applying the universal property of the tensor algebra to ν_*. Let

$$\Phi^K_\nu : \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A))) \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X))$$

be the pushforward along the above composite. It is then automatic that Φ^K_ν is a map of non-negatively graded Lie algebras over $\mathbb{Z}$, where the structures are defined as above.

We write $\deg : \pi_N(Y) \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(Y))$ for the map $f \mapsto f_*$. Let $\deg' : \mathscr{B}(\pi_*(A)) \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A)))$ be the unique map which restricts to

$\deg : \pi_*(A) \to \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(A)) \subset \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), L(\widetilde{K}_*^{\mathrm{TF}}(A)))$ and carries brackets to brackets. The above maps are related as follows.

Lemma 7.2. *Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes. The following diagram commutes:*

$$\begin{tikzcd}
\mathscr{B}(\pi_*(A)) \arrow[r, "\Phi_\nu^\pi"] \arrow[d, "\deg'"] & \pi_*(\Omega\Sigma X) \arrow[d, "\deg"] \\
\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), L(\widetilde{K}_*^{\mathrm{TF}}(A))) \arrow[r, "\Phi_\nu^K"] & \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(\Omega\Sigma X)).
\end{tikzcd}$$

Proof. By the universal property of the free magma $\mathscr{B}(\pi_*(A))$, it suffices to show that the restriction of the diagram to the weight 1 component $\mathscr{B}^1(\pi_*(A)) = \pi_*(A)$ commutes, and that all maps respect the bracket operations. By definition, $L^1(\widetilde{K}_*^{\mathrm{TF}}(A)) = \widetilde{K}_*^{\mathrm{TF}}(A)$. It then follows immediately from the definitions of Φ_ν^π and Φ_ν^K that restricting the left hand side of the diagram to weight 1 components gives the diagram

$$\begin{tikzcd}
\pi_*(A) \arrow[r, "\nu_*"] \arrow[d, "\deg"] & \pi_*(\Omega\Sigma X) \arrow[d, "\deg"] \\
\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(A)) \arrow[r, "\nu_*"] & \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(\Omega\Sigma X)),
\end{tikzcd}$$

which commutes, since it just expresses naturality of deg.

It remains to show that all maps respect bracket operations. The maps Φ_ν^π and $\deg'$ respect the bracket operations by definition, and Φ_ν^K respects bracket operations by construction. We therefore only need show that deg respects brackets. Let $f \in \pi_N(\Omega\Sigma X)$, and let $g \in \pi_M(\Omega\Sigma X)$. We must show that $\deg(\langle f, g\rangle)$ is the commutator $\deg(f)\deg(g) - (-1)^{NM}\deg(g)\deg(f)$ with respect to the algebra operation on $\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(\Omega\Sigma X))$.

Since $\widetilde{K}_*^{\mathrm{TF}}(S^{N+M}) \cong \mathbb{Z}$, it suffices to show that the two homomorphisms agree on the generator ξ_{N+M} (Remark 5.4). By Definition 7.1 and the Künneth Theorem (Theorem 5.3),

$$\deg(\langle f, g\rangle)(\xi_{N+M}) = c_* \circ (f_* \otimes g_*)(\xi_N \otimes \xi_M) = c_* \circ (f_*(\xi_N) \otimes g_*(\xi_M)).$$

Spheres of dimension at least 1 are co-H spaces, so by Lemma 5.13, ξ_N and ξ_M are primitive. By naturality of the diagonal $f_*(\xi_N)$ and $g_*(\xi_M)$ are still primitive, so by Lemma 5.14,

$$c_* \circ (f_*(\xi_N) \otimes g_*(\xi_M)) = f_*(\xi_N)g_*(\xi_M) - (-1)^{NM}g_*(\xi_M)f_*(\xi_N),$$

which by definition of the multiplication on $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X))$ is the result of evaluating $\deg(f)\deg(g) - (-1)^{NM}\deg(g)\deg(f)$ on ξ_{N+M}, as required. $\square$

We now lift the previous result to $J_s(X)$, thereby producing the first square of the diagram promised at the start of this section. Recall that we write $i_s : J_s(X) \to JX$ for the inclusion, and that by Theorem 5.6 we have a homotopy equivalence $JX \xrightarrow{\simeq} \Omega\Sigma X$. We will abuse notation and also write i_s for the composite $J_s(X) \to JX \xrightarrow{\simeq} \Omega\Sigma X$.

Corollary 7.3. *Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. If $N, s \in \mathbb{N}$ satisfy $N < r(s+1) - 1$, then $(i_s)_* : \pi_N(J_s X) \to \pi_N(\Omega\Sigma X)$ is an isomorphism and for each $k \leq s$ there exists a commutative diagram:*

$$\begin{tikzcd}
\mathscr{B}^k_N(\pi_*(A)) \arrow[r, "\widetilde{\Phi^\pi_\nu}"] \arrow[d, "\deg'"] & \pi_N(J_s X) \arrow[d, "\deg"] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), L^k(\widetilde{K}^{\mathrm{TF}}_*(A))) \arrow[r, "\widetilde{\Phi^K_\nu}"] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(J_s X)),
\end{tikzcd}$$

with $(i_s)_* \circ \widetilde{\Phi^\pi_\nu} = \Phi^\pi_\nu$ and $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), (i_s)_*) \circ \widetilde{\Phi^K_\nu} = \Phi^K_\nu$.

Proof. Consider the diagram of Lemma 7.2. Lemma 5.7 shows that $(i_s)_*$ is an isomorphism on π_N, so let $\widetilde{\Phi^\pi_\nu}$ be the unique map such that the condition $(i_s)_* \circ \widetilde{\Phi^\pi_\nu} = \Phi^\pi_\nu$ holds. By Theorem 5.6 (2) and Lemma 5.10, the map $(i_s)_* : \widetilde{K}^{\mathrm{TF}}_N(J_s(X)) \to \widetilde{K}^{\mathrm{TF}}_N(\Omega\Sigma X)$ is the inclusion of the tensors of length at most s. Since $k \leq s$, we may therefore define $\widetilde{\Phi^K_\nu}$ to be the unique map such that the condition $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), (i_s)_*) \circ \widetilde{\Phi^K_\nu} = \Phi^K_\nu$ holds. Commutativity then follows from Lemma 7.2 by naturality of deg, since $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), (i_s)_*)$ is injective. $\square$

Lemma 7.4. *Let V be a non-negatively- or $\mathbb{Z}/2$-graded $\mathbb{Z}$-module which is free and finitely generated in each dimension. Then*

- *$L(V)$ and $T(V)$ are free $\mathbb{Z}$-modules in every dimension.*
- *The natural map $L(V) \to T(V)$, $[x, y] \mapsto xy - (-1)^{|x||y|}yx$ is an injection onto a summand.*

Proof. The non-negatively graded case is immediate from [26] , Proposition 8.3.1 and p282. For the $\mathbb{Z}/2$-graded case, first observe that there is a forgetful functor U from $\mathbb{Z}$-graded modules to $\mathbb{Z}/2$-modules which carries $\mathbb{Z}$-graded (Lie) algebras to $\mathbb{Z}/2$-graded (Lie) algebras, and a functor C from $\mathbb{Z}/2$-graded modules to $\mathbb{Z}$-modules which puts V_0 in any even dimension and V_1 in any odd dimension. Both C and U respect freeness and split injections, and there are natural isomorphisms $UT(CV) \cong T(V)$ and $UL(CV) \cong L(V)$. This implies the $\mathbb{Z}/2$-graded result. $\square$

The graded version of Theorem 3.1 now follows immediately from Hilton's paper:

Theorem 7.5. *[20, Theorem 3.2, 3.3] Let V be a torsion-free $\mathbb{Z}$- or $\mathbb{Z}/2$-graded $\mathbb{Z}$-module of total dimension n. Then the total dimension of $L^k(V)$ is $W_n(k)$.* $\square$

Let R be a commutative ring with unit. Let M be an R-module, and as usual let $T(M)$ denote the tensor algebra on M. Let $\iota_k : M^{\otimes k} \to T(M)$ be the inclusion, and let $\rho_k : T(M) \to M^{\otimes k}$ be the projection. Let $\tau : T(M) \to T(M)$ be the composite $\iota_1 \circ \rho_1$. Given an R-algebra A, and a map $\varphi : M \to A$, we write $\widetilde{\varphi}$ for the induced map $T(M) \to A$, that is, the unique map of algebras such that $\widetilde{\varphi} \circ \iota_1 = \varphi$.

Now, let M and N be R-modules, and let $\varphi : M \to T(N)$ be a map. In the proof of Theorem 7.7, we will wish to make a 'leading terms' style argument. This is made precise in the next Lemma, which compares $\widetilde{\varphi}$ with $\widetilde{\tau \circ \varphi}$. Informally, we think of $\widetilde{\tau \circ \varphi}$ as the 'leading terms part' of $\widetilde{\varphi}$.

Lemma 7.6. *Let R be a commutative ring with unit. Let M and N be $\mathbb{Z}$- or $\mathbb{Z}/2$-graded R-modules. Let $\iota_k : M^{\otimes k} \to T(M)$ be the inclusion, let $\rho_k : T(N) \to N^{\otimes k}$ be the projection, and let $\tau : T(N) \to T(N)$ be as above. Let $\varphi : M \to T(N)$ be a map. Then $\rho_k \circ \widetilde{\varphi} \circ \iota_k = \rho_k \circ \widetilde{\tau \circ \varphi} \circ \iota_k$.*

Proof. It suffices to check equality on basic tensors. Let $v \in M^{\otimes k}$ be a basic tensor, so that $v = v_1 \otimes v_2 \otimes \cdots \otimes v_k$, for $v_i \in M$. Then

$$\widetilde{\varphi} \circ \iota_k(v) = \widetilde{\varphi}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \varphi(v_1) \otimes \varphi(v_2) \otimes \cdots \otimes \varphi(v_k)$$

$$= \tau(\varphi(v_1)) \otimes \tau(\varphi(v_2)) \otimes \cdots \otimes \tau(\varphi(v_k)) + \text{terms of weight} > k.$$

Applying ρ_k to both sides yields the result. $\square$

Theorem 7.7. *Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}/p$ for p prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes. Let $\overline{\nu} : \Sigma A \to \Sigma X$ be the adjoint of ν. If*

$$\overline{\nu}_* \otimes \mathbb{F} : \widetilde{K}^{\mathrm{TF}}_*(\Sigma A) \otimes \mathbb{F} \to \widetilde{K}^{\mathrm{TF}}_*(\Sigma X) \otimes \mathbb{F}$$

is an injection, then

$$\Phi^K_\nu \otimes \mathbb{F} : \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A))) \otimes \mathbb{F} \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)) \otimes \mathbb{F}$$

is also an injection.

Remark 7.8. In the case where $\overline{\nu}$ is a suspension $\Sigma\zeta$, we have a diagram

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "\Omega\overline{\nu}"] & \Omega\Sigma X \\ A \arrow[u, "\eta"] \arrow[ur, "\nu"] \arrow[r, "\zeta"] & X, \arrow[u, "\eta"'] \end{tikzcd}$$

so in particular ν_* factors through the weight 1 component $\widetilde{K}^{\mathrm{TF}}_*(X)$ of the tensor algebra decomposition of $\widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)$. This dramatically simplifies the proof, removing the need for Lemma 7.6. In practice this is not a reasonable assumption - for example, the map $\mu : S^3 \vee S^5 \to \Sigma\mathbb{C}P^2$ of Example 2.5 (which plays the role of $\overline{\nu}$) does not desuspend.

Proof. In this proof, for a space Y, we will identify the algebras $T(\widetilde{K}^{\mathrm{TF}}_*(Y))$ and $\widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma Y)$, omitting the isomorphism $S^{-1}\Gamma_*$ of Lemma 5.10. We defined Φ^K_ν to be the pushforward along a certain map $L(\widetilde{K}^{\mathrm{TF}}_*(A)) \to \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)$. Call this map $\Phi^{K'}_\nu$. It suffices to prove that $\Phi^{K'}_\nu \otimes \mathbb{F}$ is an injection.

The triangle identities for the adjunction $\Sigma \dashv \Omega$ give a commutative diagram

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "\Omega\overline{\nu}"] & \Omega\Sigma X \\ A. \arrow[u, "\eta"] \arrow[ur, "\nu"'] & \end{tikzcd}$$

Since $\Phi^{K'}_\nu$is the unique map of Lie algebras extending ν, we have a commuting diagram

$$\begin{tikzcd} T(\widetilde{K}^{\mathrm{TF}}_*(A)) \cong \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma A) \arrow[r, "(\Omega\overline{\nu})_*"] & \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X) \\ L(\widetilde{K}^{\mathrm{TF}}_*(A)), \arrow[u, hook] \arrow[ur, "\Phi^{K'}_\nu"'] & \end{tikzcd}$$

where we note that that by Lemma 7.4, the natural map $L(\widetilde{K}^{\mathrm{TF}}_*(A)) \to T(\widetilde{K}^{\mathrm{TF}}_*(A))$ is an injection onto a summand. It therefore suffices to show that $(\Omega\overline{\nu})_* \otimes \mathbb{F}$ is an injection.

Let $\widetilde{(\nu_*)}$ denote the extension of ν_* to $T(\widetilde{K}^{\mathrm{TF}}_*(A))$, so that $\widetilde{(\nu_*)} = (\Omega\overline{\nu})_*$ (modulo the isomorphism $S^{-1}\Gamma_*$, as above). Since $(\rho_i \circ \widetilde{(\nu_*)} \circ \iota_k) = 0$ for $i < k$, it further suffices to show that $(\rho_k \circ \widetilde{(\nu_*)} \circ \iota_k) \otimes \mathbb{F}$ is an injection for each k. By Lemma 7.6, with $M = \widetilde{K}^{\mathrm{TF}}_*(A)$ and $N' = \widetilde{K}^{\mathrm{TF}}_*(X)$, we have that $\rho_k \circ \widetilde{(\nu_*)} \circ \iota_k = \rho_k \circ \widetilde{(\tau \circ \nu_*)} \circ \iota_k$.

As previously, let $\mathrm{ev} : \Sigma\Omega Y \to Y$ denote the evaluation map. The following diagram commutes:

$$\begin{tikzcd} \Sigma A \arrow[r, "\Sigma\nu"] \arrow[dr, "\overline{\nu}"'] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}"] \\ & \Sigma X. \end{tikzcd}$$

The hypothesis therefore implies that the composite $(\mathrm{ev}_* \circ \Sigma\nu_*) \otimes \mathbb{F}$ is an injection. Desuspending and applying Lemma 5.12 gives that

$$(\rho_1 \circ \nu_*) \otimes \mathbb{F} : \widetilde{K}_*^{\mathrm{TF}}(A) \otimes \mathbb{F} \to \widetilde{K}_*^{\mathrm{TF}}(X) \otimes \mathbb{F}$$

is an injection of $\mathbb{F}$-vector spaces. Thus, the image $(\rho_1 \circ \nu_*)(\widetilde{K}_*^{\mathrm{TF}}(A)) \otimes \mathbb{F} \subset \widetilde{K}_*^{\mathrm{TF}}(X) \otimes \mathbb{F}$ is a direct summand. Thus, the extension $\widetilde{(\tau \circ \nu_*)} \otimes \mathbb{F}$ is an injection, and $\widetilde{(\tau \circ \nu_*)}\widetilde{K}_*^{\mathrm{TF}}(A)^{\otimes k} \subset \widetilde{K}_*^{\mathrm{TF}}(X)^{\otimes k}$ for each k. This implies that $\rho_k \circ \widetilde{(\tau \circ \nu_*)} \circ \iota_k$ is an injection for each k, as required. □

The following corollary, which lifts the injectivity back to $J_s(X)$, is immediate from Theorem 7.7 and Lemma 7.3.

Corollary 7.9. *Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}/p$ for p prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. Suppose that $N, s, k \in \mathbb{N}$ satisfy $k \leq s$, so that the map $\widetilde{\Phi_\nu^K}$ is as in Corollary 7.3. If*

$$\overline{\nu}_* \otimes \mathbb{F} : \widetilde{K}_*^{\mathrm{TF}}(\Sigma A) \otimes \mathbb{F} \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma X) \otimes \mathbb{F}$$

is an injection, then

$$\widetilde{\Phi_\nu^K} \otimes \mathbb{F} : \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), L^k(\widetilde{K}_*^{\mathrm{TF}}(A))) \otimes \mathbb{F} \to \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(J_s X)) \otimes \mathbb{F}$$

is also an injection. □

We have now established all that we will need to know about this 'first square'. Before we move on, we will prove that a space satisfying the hypotheses of Theorem 1.4 at any prime must be rationally hyperbolic. We will do so as Corollary 7.12, but we first require two lemmas. The first is needed because the hypotheses of Theorem 1.4 and Corollary 7.12 are given in terms of surjections on K-theory, but the machinery we have built so far deals with injections on K-homology.

Lemma 7.10. *Let X be a space, and let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}/p$ for p prime. Let $\mu : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$ be a map with $q_i \geq 1$, such that the map*

$$\widetilde{K}^*(\Sigma X) \otimes \mathbb{F} \xrightarrow{\mu^* \otimes \mathbb{F}} \widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{F} \cong \mathbb{F} \oplus \mathbb{F}$$

is a surjection. Then $\mu_* \otimes \mathbb{F} : \widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{F} \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma X) \otimes \mathbb{F}$ is an injection.

Proof. Because $\widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1})$ is torsion-free, the hypothesis implies that the map

$$\widetilde{K}^*_{\mathrm{TF}}(\Sigma X) \otimes \mathbb{F} \xrightarrow{\mu^* \otimes \mathbb{F}} \widetilde{K}^*_{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{F} \cong \mathbb{F} \oplus \mathbb{F}$$

is also a surjection. Thus, there exist elements x and y in $\widetilde{K}^*_{\mathrm{TF}}(\Sigma X) \otimes \mathbb{F}$ with $(\mu^* \otimes \mathbb{F})(x)$ and $(\mu^* \otimes \mathbb{F})(y)$ linearly independent in $\widetilde{K}^*_{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{F}$. Via the Universal Coefficient Theorem (Corollary 5.2) we may regard x and y as elements of $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(\Sigma X), \mathbb{Z}) \otimes \mathbb{F} \cong \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(\Sigma X) \otimes \mathbb{F}, \mathbb{F})$, with $x \circ (\mu_* \otimes \mathbb{F})$ and $y \circ (\mu_* \otimes \mathbb{F})$ linearly independent. This implies that $\mathrm{Im}(\mu_* \otimes \mathbb{F})$ has dimension at least 2. Since $\widetilde{K}^{\mathrm{TF}}_*(S^{q_1+1} \vee S^{q_2+1})$ (the domain of μ_*) has dimension 2, it follows that $\mu_* \otimes \mathbb{F}$ is injective, as required. $\square$

Some preamble to the second lemma is necessary. Let $h : \pi_*(A) \to \widetilde{K}^{\mathrm{TF}}_*(A)$ be the K-homological Hurewicz map, which sends $f \in \pi_N(A)$ to $f_*(\xi_N) \in \widetilde{K}^{\mathrm{TF}}_*(A)$. As with $\deg'$, let $h' : \mathscr{B}(\pi_*(A)) \to L(\widetilde{K}^{\mathrm{TF}}_*(A))$ be the unique map which restricts to $h : \pi_*(A) \to \widetilde{K}^{\mathrm{TF}}_*(A) \subset L(\widetilde{K}^{\mathrm{TF}}_*(A))$ and respects brackets.

Let M be a $\mathbb{Z}/2$-graded $\mathbb{Z}$-module. Let $\chi : \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), M) \to M$ be the map which carries $\varphi \in \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), M)$ to $\varphi(\xi_N) \in M$ (Remark 5.4). If $M = L$ is a $\mathbb{Z}/2$-graded Lie algebra, then it follows immediately from the definition of the bracket in $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L)$ that χ is a map of Lie algebras.

Lemma 7.11. *For any space A, there is a commuting diagram*

$$\begin{tikzcd} \mathscr{B}(\pi_*(A)) \arrow[r, "\deg'"] \arrow[d, "h'"'] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A))) \arrow[dl, "\chi"] \\ L(\widetilde{K}^{\mathrm{TF}}_*(A)). & \end{tikzcd}$$

Proof. Commutativity of the diagram

$$\begin{tikzcd} \pi_*(A) \arrow[r, "\deg"] \arrow[d, "h"'] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(A)) \arrow[dl, "\chi"] \\ \widetilde{K}^{\mathrm{TF}}_*(A). & \end{tikzcd}$$

follows from the definitions. Commutativity of the diagram from the lemma statement then follows from the universal property of $\mathscr{B}(\pi_*(A))$, since χ respects brackets. $\square$

We are now ready to prove that a space satisfying the hypotheses of Theorem 1.4 at any prime must be rationally hyperbolic.

Corollary 7.12. *Let X be a path connected space having the homotopy type of a finite CW-complex. Suppose that there exists a map*

$$\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$$

with $q_i \geq 1$, such that the map

$$\widetilde{K}^*(\Sigma X) \otimes \mathbb{Z}/p \xrightarrow{(\mu_1 \vee \mu_2)^* \otimes \mathbb{Z}/p} \widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

is a surjection for some prime p (not necessarily odd). Then ΣX is rationally hyperbolic.

Proof. Set $\mu = \mu_1 \vee \mu_2$. By Lemma 7.10 we have that $\mu_* \otimes \mathbb{Z}/p$ is injective. The codomain of μ_* is torsion free, so $\mu_* \otimes \mathbb{Q}$ is also injective. Theorem 7.7 (with $A = S^{q_1} \vee S^{q_2}$ and $\nu = \overline{\mu}$) then implies that $\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), L(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2}))) \otimes \mathbb{Q}$ injects into $\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(\Omega\Sigma X)) \otimes \mathbb{Q}$ via the map $\Phi_\nu^K \otimes \mathbb{Q}$.

The Hurewicz map h is a surjection $\pi_*(S^{q_1} \vee S^{q_2}) \to \widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2})$, so the submodule generated by the image of $h' : \mathscr{B}(\pi_*(S^{q_1} \vee S^{q_2})) \to L(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2}))$ is precisely the submodule generated by the image of h under the bracket operation. By Theorem 7.5, this submodule is certainly infinite dimensional, so by Lemma 7.11 the image of $\deg'$ is also infinite dimensional. Thus, the image of $(\Phi_\nu^K \circ \deg') \otimes \mathbb{Q}$ is also infinite dimensional.

By Lemma 7.2, the image of $\deg : \pi_*(\Omega\Sigma X) \to \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(\Omega\Sigma X))$ contains the image of $\Phi_\nu^K \circ \deg'$. Thus, $\pi_*(\Omega\Sigma X) \otimes \mathbb{Q} \cong \pi_{*+1}(\Sigma X) \otimes \mathbb{Q}$ surjects onto an infinite dimensional rational vector space, and hence also has infinite rational dimension, and thus ΣX is rationally hyperbolic, as required. □

7.2 Maps derived from the universal coefficient isomorphism

In this subsection we will build the second square of our diagram. This square is really just the Universal Coefficient theorem (Corollary 5.2) in a different form. We will write deg for both K-homological and K-theoretic degree.

Lemma 7.13. *Let Y be a space having the homotopy type of a finite CW-complex. There exists an isomorphism $\mathscr{U}$ making the following diagram commute.*

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, equal] \arrow[d, "\deg"] & \pi_N(Y) \arrow[d, "\deg"] \\
\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(Y)) \arrow[r, "\mathscr{U}"] & \mathrm{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)).
\end{tikzcd}$$

Proof. For $\beta : \widetilde{K}_*^{\mathrm{TF}}(S^N) \to \widetilde{K}_*^{\mathrm{TF}}(Y)$, let $\mathscr{U}(\beta)$ be the unique map making the following diagram commute

$$\begin{tikzcd}
\widetilde{K}^*_{\mathrm{TF}}(Y) \arrow[r, "\cong"] \arrow[d, "\mathscr{U}(\beta)"] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(Y), \mathbb{Z}) \arrow[d, "\mathrm{Hom}(\beta, \mathbb{Z})"] \\
\widetilde{K}^*_{\mathrm{TF}}(S^N) \arrow[r, "\cong"] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \mathbb{Z})
\end{tikzcd}$$

where the isomorphisms are those of Corollary 5.2. Since $\widetilde{K}^{\mathrm{TF}}_*(Y)$ is a finitely generated free $\mathbb{Z}$-module, $\beta \mapsto \mathrm{Hom}(\beta, \mathbb{Z})$ is an isomorphism, so $\mathscr{U}$ is also an isomorphism. Commutativity of the diagram from the statement of this lemma is by naturality of Lemma 5.2. □

Corollary 7.14. *Let Y be a space having the homotopy type of a finite CW-complex. For a $\mathbb{Z}$-module M, let $\tau_p : M \to M \otimes \mathbb{Z}/p$ be the natural map. There exists an injection $\mathscr{U}'$ making the following diagram commute.*

$$\begin{tikzcd}
\pi_N(Y) \arrow[rr, equal] \arrow[d, "\mathrm{deg}"] \arrow[ddr, bend left] & & \pi_N(Y) \arrow[d, "\mathrm{deg}"] \arrow[ddr, bend left] & \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(Y)) \arrow[dd, "\tau_p"] & & \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[dd, "\tau_p"] & \\
& \mathrm{Im}(\tau_p \circ \mathrm{deg}) \arrow[rr, "\mathscr{U}'"] \arrow[dl, hook] & & \mathrm{Im}(\tau_p \circ \mathrm{deg}) \arrow[dl, hook] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(Y)) \otimes \mathbb{Z}/p & & \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. &
\end{tikzcd}$$

Proof. By Lemma 7.13, we have a commutative diagram

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, equal] \arrow[d, "\mathrm{deg}"] & \pi_N(Y) \arrow[d, "\mathrm{deg}"] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(Y)) \arrow[r, "\mathscr{U}"] & \mathrm{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)).
\end{tikzcd}$$

with $\mathscr{U}$ an isomorphism, so $\mathscr{U} \otimes \mathbb{Z}/p$ is also an isomorphism. By Lemma 6.1, the map

$$\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p \to \mathrm{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p$$

is an injection. Maps of spaces induce maps of ψ-modules on K-theory, so the image of $\mathscr{U} \circ \mathrm{deg}$ is contained in $\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N))$, and hence there exists a map $\mathscr{U}'$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Im}(\tau_p \circ \mathrm{deg}) \arrow[r, dotted, "\mathscr{U}'"] \arrow[d, hook] & \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p \arrow[d, hook] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(Y)) \otimes \mathbb{Z}/p \arrow[r, "\mathscr{U} \otimes \mathbb{Z}/p \cong"] & \mathrm{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p.
\end{tikzcd}$$

Both vertical maps are injections, so $\mathscr{U}'$ has the required properties. $\square$

7.3 Pulling back along classes defined on S^3

Let $f \in \pi_j(S^3)$, and let $N \geq 3$. Then, for $\omega \in \pi_N(Y)$, the composite

$$S^{N+j-3} \xrightarrow{\Sigma^{N-3}f} S^N \xrightarrow{\omega} Y$$

is defined. The class $\omega \circ \Sigma^{N-3}f$ lies in $\pi_{M-1}(Y)$, where $M-1 = N+j-3$.

Thus motivated, we define the map $f_\Sigma^* : \pi_*(Y) \to \pi_*(Y)$ on $\omega \in \pi_N(Y)$ by setting $f_\Sigma^*(\omega) = (\Sigma^{N-3}f)^*\omega = \omega \circ \Sigma^{N-3}f$. In words, f_Σ^* pulls classes back along the appropriate suspension of f. Strictly speaking, f_Σ^* is only a partial map, because it is undefined on π_N for $N \leq 2$, but this will be unimportant.

Recall the definition of the double e-invariant $\bar{e}$ (Definition 6.4). On $\pi_N(Y)$, we have by definition that $f_\Sigma^* = (\Sigma^{N-3}f)^*$. By Lemma 6.5 we have a commuting square:

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, "f_\Sigma^*"] \arrow[d, "\deg"'] & \pi_{N+j-3}(Y) \arrow[d, "\bar{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta(\Sigma^{N-3}f)"] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^{N+j-2})).
\end{tikzcd}$$

Mimicking the convention for f_Σ^*, let

$$\theta_\Sigma(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^*)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^{*+j-2}))$$

be the map which is defined to be equal to $\theta(\Sigma^{N-3}f)$ on the degree N component $\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N))$ of $\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^*))$.

Lemma 7.15. *Let p be a prime, and let $f \in \pi_j(S^3)$ with $e(f)$ defined. If $pf = 0$, then there exists a map $\theta_\Sigma^p(f)$ making the following diagram commute for all N:*

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, "f_\Sigma^*"] \arrow[d, "\deg"'] & \pi_{N+j-3}(Y) \arrow[d, "\bar{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta_\Sigma(f)"] \arrow[d, "\tau_p"'] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^{N+j-2})) \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. \arrow[ur, dotted, "\theta_\Sigma^p(f)"'] &
\end{tikzcd}$$

Proof. Since $pf = 0$, we have that $p\bar{e}(\Sigma^{N-3}f) = 0$ for all N, which implies that $\theta_\Sigma(f)$ vanishes on p-divisible elements, so there exists a unique map $\theta_\Sigma^p(f)$ making the diagram commute, as required. $\square$

Lemma 7.16. *Let X be a finite CW-complex. Let λ_ℓ^X be the largest eigenvalue of the rational Adams operation*

$$\psi^\ell \otimes \mathbb{Q} : \widetilde{K}^0(X) \otimes \mathbb{Q} \to \widetilde{K}^0(X) \otimes \mathbb{Q}.$$

Then, for $i \geq 0$

- *the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma^{2i} J_s(X)) \otimes \mathbb{Q}$ is $\ell^i (\lambda_\ell^X)^s$, and*
- *the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma^{2i+1} J_s(X)) \otimes \mathbb{Q}$ is $\ell^i \lambda_\ell^{\Sigma X} (\lambda_\ell^X)^{s-1}$.*

Proof. When $i \geq 1$, Theorem 5.6 gives that $\Sigma J_s(X) \simeq \Sigma \bigvee_{t=1}^s X^{\wedge t}$, so $\Sigma^{2i} J_s(X) \simeq S^{2i} \wedge \bigvee_{t=1}^s X^{\wedge t}$, and $\Sigma^{2i+1} J_s(X) \simeq S^{2i} \wedge \Sigma X \wedge \bigvee_{t=1}^{s-1} X^{\wedge t}$. By the Künneth theorem (Theorem 5.5), this implies isomorphisms of rings

$$\widetilde{K}^0_{\mathrm{TF}}(\Sigma^{2i} J_s(X)) \cong \bigoplus_{t=1}^{s} \widetilde{K}^0_{\mathrm{TF}}(S^{2i}) \otimes \widetilde{K}^0_{\mathrm{TF}}(X)^{\otimes t}, \text{ for } i \geq 1,$$

and

$$\widetilde{K}^0_{\mathrm{TF}}(\Sigma^{2i+1} J_s(X)) \cong \bigoplus_{t=0}^{s-1} \widetilde{K}^0_{\mathrm{TF}}(S^{2i}) \otimes \widetilde{K}^0_{\mathrm{TF}}(\Sigma X) \otimes \widetilde{K}^0_{\mathrm{TF}}(X)^{\otimes t} \text{ for } i \geq 0.$$

The Künneth isomorphism of Theorem 5.5 is given by the external product on K-theory. Since the Adams operations are ring homomorphisms, the above isomorphisms are also isomorphisms of ψ-modules. In particular, the Adams operations on the left are the tensor product of the corresponding operations on the right.

These decompositions hold for $\widetilde{K}^0_{\mathrm{TF}}$, so they also hold for $\mathbb{Q} \otimes \widetilde{K}^0$, and the remaining problem is to determine the largest eigenvalue of the relevant tensor products of Adams operations. The eigenvalues of a tensor product of linear endomorphisms are precisely the products of the eigenvalues. The operation ψ^ℓ acts on S^{2i} by multiplication by ℓ^i. Together, these observations imply the result. □

Lemma 7.17. *Let p be an odd prime. Let X be an $(r-1)$-connected finite CW-complex. Let $N, s \in \mathbb{N}$. Consider the diagram of Lemma 7.15 for $Y = J_s(X)$ and $f = f_{p,j} \in \pi_{2j(p-1)+2}(S^3)$, the map of Theorem 6.8:*

$$\begin{tikzcd}
\pi_N(J_s(X)) \arrow[r, "f_\Sigma^*"] \arrow[d, "\deg"'] & \pi_{N+2j(p-1)-1}(J_s(X)) \arrow[d, "\bar{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta_\Sigma(f)"] \arrow[d, "\tau_p"'] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+2j(p-1)})) \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. \arrow[ur, "\theta_\Sigma^p(f)"'] &
\end{tikzcd}$$

For $\ell \in \mathbb{N}$, let λ_ℓ^Y be the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(Y) \otimes \mathbb{Q}$, and let $\lambda_\ell = \max(\lambda_\ell^X, \lambda_\ell^{\Sigma X})$. If there exists $\ell \in \mathbb{N}$ such that $\ell^{j(p-1)+\frac{N-1}{2}} > \lambda_\ell^s$ then $\mathrm{Ker}(\bar{e} \circ f_\Sigma^*) \subset \mathrm{Ker}(\tau_p \circ \deg)$, and hence the restriction of $\theta_\Sigma^p(f)$ to $\mathrm{Im}(\tau_p \circ \deg)$ is an injection.

Proof. First note that $pf = 0$ by Theorem 6.8, so $\theta_\Sigma^p(f)$ is well-defined by Lemma 7.15. Let $\omega \in \pi_N(J_s(X))$. Suppose that $\omega \in \mathrm{Ker}(\bar{e} \circ f_\Sigma^*)$, that is, that the $\bar{e}$-invariant of the composite

$$S^{N+2j(p-1)-1} \xrightarrow{\Sigma^{N-3}f} S^N \xrightarrow{\omega} J_s(X)$$

is trivial. By Lemma 6.3, this implies that $\Sigma^i(\omega \circ \Sigma^{N-3}f)$ has trivial e-invariant for all i. In particular, $e(\Sigma\omega \circ \Sigma^{N-2}f)$ and $e(\omega \circ \Sigma^{N-3}f)$ are both 0.

By Lemma 7.16, the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(J_s(X)) \otimes \mathbb{Q}$ is at most λ_ℓ^s, and the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma J_s(X)) \otimes \mathbb{Q}$ is also at most λ_ℓ^s. We now divide into cases, based on the parity of N.

CASE 1 (N even): Write $N = 2n$. Let $f' = \Sigma^{N-3}f$ and $g = \omega$ in Theorem 6.7. The domain of $\omega \circ \Sigma^{N-3}f$ is S^{M-1}, where $M - 1 = N + 2j(p-1) - 1$, so M is even, as is required. To check the eigenvalue hypothesis of Theorem 6.7, write $M = 2m$. By Lemma 7.16, the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(J_s(X)) \otimes \mathbb{Q}$ is at most λ_ℓ^s, and $\ell^m = \ell^{j(p-1)+n} > \ell^{j(p-1)+\frac{N-1}{2}}$, which we assumed was greater than λ_ℓ^s. This means that ℓ^m cannot be an eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(J_s(X)) \otimes \mathbb{Q}$. Now, $e(f) \neq 0$ by construction (Theorem 6.8), so $e(\Sigma^{N-3}f) \neq 0$ by stability (Lemma 6.6). Since $e(\omega \circ \Sigma^{N-3}f) = 0$, the contrapositive of Theorem 6.7 gives that ω^* has p-divisible image in $\widetilde{K}^0(S^N)$. Since N is even, this implies that $\tau_p \circ \deg(\omega) = 0$, as required.

CASE 2 (N odd): Write $n = 2n + 1$. Let $f' = \Sigma^{N-2}f$ and $g = \Sigma\omega$ in Theorem 6.7, and proceed similarly to case 1. The domain of $\Sigma\omega \circ \Sigma^{N-2}f$ is S^{M-1}, where $M - 1 = N + 2j(p-1)$, so M is even, as is required. To check the eigenvalue hypothesis of Theorem 6.7, write $M = 2m$. By Lemma 7.16, the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma J_s(X)) \otimes \mathbb{Q}$ is at most λ_ℓ^s, and $\ell^m = \ell^{j(p-1)+n} = \ell^{j(p-1)+\frac{N-1}{2}}$, which we assumed was greater than λ_ℓ^s. This means that ℓ^m cannot be an eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma J_s(X)) \otimes \mathbb{Q}$. As in the previous case, $e(\Sigma^{N-2}f) \neq 0$. Since $e(\Sigma\omega \circ \Sigma^{N-2}f) = 0$, the contrapositive of Theorem 6.7 gives that $(\Sigma\omega)^*$ has p-divisible image in $\widetilde{K}^0(S^{N+1})$. Since N is odd, this implies that $\tau_p \circ \deg(\omega) = 0$, as required. This completes the case, and hence the proof. $\square$

7.4 Proof of Theorem 1.4

Construction 7.18. *Let p be an odd prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. Let*

$f \in \pi_i(S^3)$ with $\bar{e}(f)$ defined. Suppose that $N, k, s \in \mathbb{N}$ satisfy $N < r(s+1) - 1$ and $k \leq s$. The diagrams of the preceding subsections may be combined as follows.

Recall the definition of $\deg'$ from the preamble to Lemma 7.2. Let $I(A)$ be the submodule of $\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), L(\widetilde{K}_*^{\mathrm{TF}}(A))) \otimes \mathbb{Z}/p$ generated by $\mathrm{Im}(\tau_p \circ \deg')$. The same grading conventions as usual apply: we write $I^k(A)$ for the weight k part, we write $I_N(A)$ for the degree N part, and let $I_N^k(A) = I^k(A) \cap I_N(A)$.

From Corollary 7.3, using the assumptions that $N < r(s+1) - 1$ and $k \leq s$ (which make $\widetilde{\Phi_v^\pi}$ and $\widetilde{\Phi_v^K}$ well-defined) we obtain the following diagram, where the images of the vertical maps have been 'popped out' to their right.

$$\begin{tikzcd}
\mathscr{B}_N^k(\pi_*(A)) \arrow[rr, "\widetilde{\Phi_v^\pi}"] \arrow[dd, "\tau_p \circ \deg'"] \arrow[dr] & & \pi_N(J_s(X)) \arrow[dd, "\tau_p \circ \deg"] \arrow[dr] & \\
& I_N^k(A) \arrow[rr] \arrow[dl] & & \mathrm{Im}(\tau_p \circ \deg) \arrow[dl] \\
\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), L^k(\widetilde{K}_*^{\mathrm{TF}}(A))) \otimes \mathbb{Z}/p \arrow[rr, "\widetilde{\Phi_v^K} \otimes \mathbb{Z}/p"] & & \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(J_s(X))) \otimes \mathbb{Z}/p. &
\end{tikzcd}$$

Next, from Corollary 7.14 (with $Y = J_s(X)$) we have a diagram

$$\begin{tikzcd}
\pi_N(J_s(X)) \arrow[rr, equal] \arrow[dd, "\tau_p \circ \deg"] \arrow[dr] & & \pi_N(J_s(X)) \arrow[dd] \arrow[dr] \arrow[dd, "\tau_p \circ \deg"] & \\
& \mathrm{Im}(\tau_p \circ \deg) \arrow[rr, "\mathscr{U}'"] \arrow[dl] & & \mathrm{Im}(\tau_p \circ \deg) \arrow[dl] \\
\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(J_s(X))) \otimes \mathbb{Z}/p & & \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. &
\end{tikzcd}$$

Lastly, we obtain the following diagram from Lemma 7.15:

$$\begin{tikzcd}
\pi_N(J_s(X)) \arrow[r, "f_\Sigma^*"] \arrow[d, "\deg"'] & \pi_{N+i-3}(J_s(X)) \arrow[d, "\bar{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta_\Sigma(f)"] \arrow[d, "\tau_p"'] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+i-2})) \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. \arrow[ur, "\theta_\Sigma^p(f)"] &
\end{tikzcd}$$

Concatenating these diagrams gives a diagram as follows:

$$\begin{tikzcd}
\mathscr{B}^k_N(\pi_*(A)) \arrow[r, "f^*_\Sigma\circ\widetilde{\Phi^\pi_\nu}"] \arrow[d, "\tau_p\circ\deg'"'] & \pi_{N+i-3}(J_s(X)) \arrow[d, "\bar{e}"] \\
I^k_N(A) \arrow[r, "\theta^p_\Sigma(f)\circ\mathscr{U}'\circ(\widetilde{\Phi^K_\nu}\otimes\mathbb{Z}/p)"] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+i-2})).
\end{tikzcd}$$

In this subsection, we combine the results of the previous subsections to produce results about this diagram.

Theorem 7.19. *Let p be an odd prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. Let $N, k, s \in \mathbb{N}$ with $N < r(s+1) - 1$ and $k \leq s$. Let $f = f_{p,j} \in \pi_{2j(p-1)+2}(S^3)$, the map of Theorem 6.8.*

For $\ell \in \mathbb{N}$, let λ^Y_ℓ be the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(Y) \otimes \mathbb{Q}$, and let $\lambda_\ell = \max(\lambda^X_\ell, \lambda^{\Sigma X}_\ell)$. If

- *$\overline{\nu}_* \otimes \mathbb{Z}/p : \widetilde{K}^{\mathrm{TF}}_*(\Sigma A) \otimes \mathbb{Z}/p \to \widetilde{K}^{\mathrm{TF}}_*(\Sigma X) \otimes \mathbb{Z}/p$ is an injection, and*
- *there exists $\ell \in \mathbb{N}$ such that $\ell^{j(p-1)+\frac{N-1}{2}} > \lambda^s_\ell$,*

then $\theta^p_\Sigma(f) \circ \mathscr{U}' \circ (\widetilde{\Phi^K_\nu} \otimes \mathbb{Z}/p) : I^k_N(A) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+2j(p-1)}))$ is an injection.

Proof. By Corollary 7.9, since $\overline{\nu}_* \otimes \mathbb{Z}/p$ is an injection, $\widetilde{\Phi^K_\nu} \otimes \mathbb{Z}/p$ is also an injection. By Corollary 7.14 $\mathscr{U}'$ is an injection. By Lemma 7.17 the hypothesis on ℓ implies that the restriction of $\theta^p_\Sigma(f)$ to $\mathrm{Im}(\tau_p \circ \deg)$ is an injection. The map $\theta^p_\Sigma(f) \circ \mathscr{U}' \circ (\widetilde{\Phi^K_\nu} \otimes \mathbb{Z}/p)$ is thus a composite of injections, hence an injection, as required. □

In the proof of Theorem 1.4, we will wish to restrict attention to those elements of $\mathscr{B}(\pi_*(A))$ who are brackets of classes in $\pi_*(A)$ in some dimensional range $q_{\min} \leq n \leq q_{\max}$. All such classes lie in dimensions $kq_{\min} \leq N \leq kq_{\max}$. Said more precisely, we have an inclusion $\mathscr{B}^k(\bigoplus_{n=q_{\min}}^{q_{\max}} \pi_n(A)) \subset \bigcup_{N=kq_{\min}}^{kq_{\max}} \mathscr{B}^k_N(\pi_*(A))$. We will now study the diagram of Construction 7.18 in this dimensional range.

Construction 7.20. *Let p be an odd prime, $\nu : A \to \Omega\Sigma X$ for finite CW-complexes A and X with X $(r-1)$-connected for $r \geq 1$, and $f \in \pi_i(S^3)$ with $\bar{e}(f)$ defined. Let $q_{\max} > q_{\min}$ be natural numbers. Fix $k \in \mathbb{N}$, and let $s = kq_{\max} + 1$. For $N \in \mathbb{N}$ with $kq_{\min} \leq N \leq kq_{\max}$, we have that $N < r(s+1) - 1$ and $k \leq s$. Combining the diagrams obtained from Construction 7.18 for this range of values of N gives the following diagram:*

$$\begin{tikzcd}
\bigcup_{N=kq_{\min}}^{kq_{\max}} \mathscr{B}^k_N(\pi_*(A)) \arrow[r, "f^*_\Sigma\circ\widetilde{\Phi^\pi_\nu}"] \arrow[d, "\tau_p\circ\deg'"'] & \bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+i-3}(J_s(X)) \arrow[d, "\bar{e}"] \\
\bigoplus_{N=kq_{\min}}^{kq_{\max}} I^k_N(A) \arrow[r, "\theta^p_\Sigma(f)\circ\mathscr{U}'\circ(\widetilde{\Phi^K_\nu}\otimes\mathbb{Z}/p)"] & \bigoplus_{N=kq_{\min}}^{kq_{\max}} \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+i-2})).
\end{tikzcd}$$

We now show that by choosing a large enough $c \in \mathbb{N}$, and setting $f = f_{p,ck}$, the eigenvalue hypothesis of Theorem 7.19 may be satisfied across the dimensional range of Construction 7.20 for all sufficiently large k.

Corollary 7.21. *Let p be an odd prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X path-connected. Let $q_{\max} > q_{\min}$ be natural numbers. Let $c, k \in \mathbb{N}$. Let $f = f_{p,ck} \in \pi_{2ck(p-1)+2}(S^3)$ be the map of Theorem 6.8. If*

$$\overline{\nu}_* \otimes \mathbb{Z}/p : \widetilde{K}_*^{\mathrm{TF}}(\Sigma A) \otimes \mathbb{Z}/p \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma X) \otimes \mathbb{Z}/p$$

is an injection then there exists $c \in \mathbb{N}$ such that for large enough $k \in \mathbb{N}$,

$$\theta^p_\Sigma(f) \circ \mathscr{U}' \circ (\widetilde{\Phi^K_\nu} \otimes \mathbb{Z}/p) : I^k_N(A) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+2ck(p-1)}))$$

is an injection for $kq_{\min} \le N \le kq_{\max}$.

Proof. By Theorem 7.19, it suffices to show that for each N with $kq_{\min} \le N \le kq_{\max}$ there exists $\ell \in \mathbb{N}$ such that $\ell^{ck(p-1)+\frac{N-1}{2}} > \lambda^s_\ell = \lambda_\ell^{kq_{\max}+1}$. Take any $\ell \ge 2$. Since $N \ge kq_{\min}$, it suffices to find c such that for large enough k we have $\ell^{ck(p-1)+\frac{kq_{\min}-1}{2}} > \lambda_\ell^{kq_{\max}}$. Taking logs on both sides, this is equivalent to

$$(ck(p-1) + \frac{kq_{\min}-1}{2}) \log(\ell) > kq_{\max} \log(\lambda_\ell).$$

It is now clear that we may choose c large enough that this equation holds for large enough k, in particular, any $c \ge \frac{1}{p-1}(q_{\max}\frac{\log(\lambda_\ell)}{\log(\ell)} - \frac{q_{\min}}{2})$ will do. $\square$

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\mu = \mu_1 \vee \mu_2$, with adjoint $\overline{\mu} : S^{q_1} \vee S^{q_2} \to \Omega\Sigma X$. Let $f = f_{p,ck} \in \pi_{2ck(p-1)+2}(S^3)$. Consider the diagram of Construction 7.20, with $A = S^{q_1} \vee S^{q_2}$, $q_{\max} = \max(q_1, q_2)$, $q_{\min} = \min(q_1, q_2)$, and $\nu = \overline{\mu}$. We have such a diagram for each $k \in \mathbb{N}$:

$$\begin{tikzcd}
\bigcup_{N=kq_{\min}}^{kq_{\max}} \mathscr{B}^k_N(\pi_*(S^{q_1} \vee S^{q_2})) \arrow[r, "f^*_\Sigma \circ \widetilde{\widetilde{\Phi}^\pi_{\overline{\mu}}}"] \arrow[d, "\tau_p \circ \deg'"'] & \bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(J_s(X)) \arrow[d, "\overline{e}"] \\
\bigoplus_{N=kq_{\min}}^{kq_{\max}} I^k_N(S^{q_1} \vee S^{q_2}) \arrow[r] & \bigoplus_{N=kq_{\min}}^{kq_{\max}} \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+2ck(p-1)})).
\end{tikzcd}$$

By assumption, $\mu^* \otimes \mathbb{Z}/p : \widetilde{K}^*(\Sigma X) \otimes \mathbb{Z}/p \to \widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p$ is a surjection. By Lemma 7.10 this implies that

$$\mu_* \otimes \mathbb{Z}/p : \widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma X) \otimes \mathbb{Z}/p$$

is an injection. Thus, by Corollary 7.21, we may fix c such that for large enough k, $\theta_p(f) \circ \mathscr{U}' \circ (\widetilde{\Phi^K_{\mu}} \otimes \mathbb{Z}/p)$ is an injection.

The Hurewicz map h is a surjection $\pi_*(S^{q_1} \vee S^{q_2}) \to \widetilde{K}^{\mathrm{TF}}_*(S^{q_1} \vee S^{q_2})$, so the submodule generated by the image of the map $h' : \mathscr{B}(\pi_*(S^{q_1} \vee S^{q_2})) \to L(\widetilde{K}^{\mathrm{TF}}_*(S^{q_1} \vee S^{q_2}))$ of Lemma 7.11 contains the submodule generated by $\widetilde{K}^{\mathrm{TF}}_*(S^{q_1} \vee S^{q_2})$ under the bracket operation. In particular, it contains the weight k component $L^k(\widetilde{K}^{\mathrm{TF}}_*(S^{q_1} \vee S^{q_2}))$ for each k. By Theorem 3.1, $\dim_{\mathbb{Z}}(L^k(\widetilde{K}^{\mathrm{TF}}_*(S^{q_1} \vee S^{q_2}))) = W_2(k)$. Note that $L^k(\widetilde{K}^{\mathrm{TF}}_*(S^{q_1} \vee S^{q_2})) = \bigoplus_{N=kq_{\min}}^{kq_{\max}} L^k_N(\widetilde{K}^{\mathrm{TF}}_*(S^{q_1} \vee S^{q_2}))$.

It then follows from Lemma 7.11 that $\dim_{\mathbb{Z}/p}(\bigoplus_{N=kq_{\min}}^{kq_{\max}} I^k_N(S^{q_1} \vee S^{q_2})) \geq W_2(k)$. Since $\theta^p_{\Sigma}(f) \circ \mathscr{U}' \circ (\widetilde{\Phi^K_{\mu}} \otimes \mathbb{Z}/p)$ is an injection for large enough k, it follows that the dimension of $\bar{e}(\bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(J_s(X)))$ is at least $W_2(k)$. By Corollary 7.3 $(i_s)_*$ is an injection, so the dimension of $(i_s)_*(\bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(J_s(X))) \subset \bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(\Omega\Sigma X)$ is also at least $W_2(k)$. Thus, ΣX satisfies the hypotheses of Lemma 3.3 with $a = 2c(p-1) + q_{\max} = 2c(p-1) + \max(q_1, q_2)$ and $b = 0$, and hence is p-hyperbolic.

□

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Paper 3
$\mathbb{Z}/p^r$-hyperbolicity via homology

Guy Boyde

ABSTRACT. We show that the homotopy groups of a Moore space $P^n(p^r)$ are $\mathbb{Z}/p^s$-hyperbolic for $s \leq r$ and $p^s \neq 2$. Combined with work of Huang-Wu and Neisendorfer, this completely resolves the question of when such a Moore space is $\mathbb{Z}/p^s$-hyperbolic for $p \geq 5$. We also give a homological criterion for a space to be $\mathbb{Z}/p^r$-hyperbolic, and deduce some examples.

1 Introduction

Given a space X, one can ask about the behaviour of the partial sum of homotopy groups

$$\bigoplus_{i=1}^{m} \pi_i(X) \text{ as } m \to \infty.$$

Rationally, deep results have been obtained, notably the famous dichotomy of Félix, Halperin and Thomas [14, Chapter 33] . Interpreted integrally, this dichotomy says that if X is a simply connected finite CW-complex with finite rational category then either

- the rank of $\bigoplus_{i=1}^{\infty} \pi_i(X)$ is finite, and X is called *rationally elliptic*, or
- the rank of $\bigoplus_{i=1}^{m} \pi_i(X)$ grows exponentially with m, and X is called *rationally hyperbolic*.

Study of the corresponding behaviour for the torsion parts of these groups, which is the subject of this paper, was initiated by Huang and Wu in [19] .

Let M be a $\mathbb{Z}$-module, let p be a prime and let $t \in \mathbb{N}$. The $\mathbb{Z}/p^t$*-dimension* or $\mathbb{Z}/p^t$*-rank* of M, denoted $\dim_{\mathbb{Z}/p^t}(M)$, is the greatest $d \in \mathbb{N} \cup \{0, \infty\}$ such that there is an isomorphism $M \cong (\mathbb{Z}/p^t)^d \oplus C$ for some complementary module C. Said another way, $\dim_{\mathbb{Z}/p^t}(M)$ is the number of $\mathbb{Z}/p^t$-summands in M.

Definition 1.1. Let M be a graded $\mathbb{Z}$-module, Let p be a prime, and let $S \subset \mathbb{N}$. We say that X is *p-hyperbolic concentrated in (the set of exponents) S* if

$$a_m := \sum_{t \in S} \dim_{\mathbb{Z}/p^t}(\bigoplus_{i=1}^{m} M_i)$$

grows exponentially, in the sense that

$$\liminf_m \frac{\ln(a_m)}{m} > 0.$$

For a space X we will say that X is *p-hyperbolic concentrated in S* if $\pi_*(X)$ is p-hyperbolic concentrated in S. If X is p-hyperbolic concentrated in $\mathbb{N}$ then we will say simply that X is p-hyperbolic.

This definition generalises and interpolates between two definitions due to Huang and Wu [19] . Namely, their *$\mathbb{Z}/p^s$-hyperbolicity* is precisely our p-hyperbolicity concentrated in the singleton set $\{s\}$, and their *p-hyperbolicity* is precisely our p-hyperbolicity concentrated in $\mathbb{N}$, as defined above.

Definition 1.2. Let $P^n(\ell)$ denote the *mod-ℓ Moore space*, which we take to be the cofibre

$$S^{n-1} \xrightarrow{\ell} S^{n-1} \longrightarrow P^n(\ell)$$

of the degree ℓ map.

Huang and Wu show that for p prime, $n \geq 3$, and $r \geq 1$ the Moore space $P^n(p^r)$ is $\mathbb{Z}/p^r$ and $\mathbb{Z}/p^{r+1}$-hyperbolic, and additionally that $P^n(2)$ is $\mathbb{Z}/8$-hyperbolic. In [32] , Zhu and Pan show that $P^n(p^r)$ is also $\mathbb{Z}/p$-hyperbolic. Our first main result fills in the gap between these exponents:

Theorem 1.3. *Let p be a prime, and $r \in \mathbb{N}$. If $n \geq 3$, then $P^n(p^r)$ is $\mathbb{Z}/p^s$-hyperbolic for all $s \leq r$ such that $p^s \neq 2$.*

The key is to show that the stable homotopy of $P^n(p^r)$ contains a $\mathbb{Z}/p^s$-summand for each $s \leq r$. This follows from work of Adams on the J-homomorphism [1, 2] , which allows us to find such summands in the stable homotopy of spheres, and classical work of Barratt [7] allows us to transplant these summands to Moore spaces. Once this is done, the proof follows the same lines as those in [19] and [32] .

For $p > 3$ Huang and Wu's results and Theorem 1.3 together are best possible, in the following sense. In [25] , Neisendorfer shows that $\pi_*(P^n(p^r))$ contains no element of order p^s for $s > r+1$. In fact, Neisendorfer claimed in [25] that this result also holds when $p = 3$, but later, with Brayton Gray, discovered some mistakes in the proof (see the unpublished [24]). These mistakes were repaired apart from when $p = 3$. In [24] , Neisendorfer shows that the 3-primary exponent of $P^n(3^r)$ is either 3^{r+1} or 3^{r+2}.

Neisendorfer's result allows us to combine Huang and Wu's result with Theorem 1.3 to obtain the following (using Proposition 3.1):

Corollary 1.4. *For $p \neq 2, 3$ prime, $s, \ell \in \mathbb{N}$ and $n \geq 3$, the following are equivalent:*

1. *$P^n(\ell)$ is $\mathbb{Z}/p^s$-hyperbolic.*
2. *$\pi_*(P^n(\ell))$ contains a class of order p^s.*

3. $p^{\max(s-1,1)} | \ell$. □

Our second main result is a homological criterion for hyperbolicity:

Theorem 1.5. *Let Y be a simply connected CW-complex, let $p \neq 2$ be prime, and let $s \leq r \in \mathbb{N}$. If there exists a map*

$$\mu : P^{n+1}(p^r) \longrightarrow Y$$

such that the induced map

$$(\Omega\mu)_* : H_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s) \longrightarrow H_*(\Omega Y; \mathbb{Z}/p^s)$$

is an injection, then Y is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$. In particular if $s = r$ then Y is $\mathbb{Z}/p^r$-hyperbolic.

The point is that $\mathbb{Z}/p^r$-hyperbolicity of $P^{n+1}(p^r)$ can be detected in $H_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^r)$, and under the hypotheses of the theorem, the exponential family of classes which gives the hyperbolicity can be composed with μ to give classes in the homotopy groups $\pi_*(\Omega Y) \cong \pi_{*+1}(Y)$ with orders in the set $\{p^s, p^{s+1}, \ldots p^r\}$.

We will see (using Proposition 10.12) that the hypotheses of Theorem 1.5 simplify in the case that $Y = \Sigma X$ is a suspension, as follows:

Theorem 1.6. *Let X be a connected CW-complex, let $p \neq 2$ be prime, and let $s \leq r \in \mathbb{N}$. If there exists a map*

$$\mu : P^{n+1}(p^r) \longrightarrow \Sigma X$$

such that

$$\mu_* : \widetilde{H}_*(P^{n+1}(p^r); \mathbb{Z}/p^s) \longrightarrow \widetilde{H}_*(\Sigma X; \mathbb{Z}/p^s)$$

is an injection, then ΣX is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$. In particular if $s = r$ then ΣX is $\mathbb{Z}/p^r$-hyperbolic.

Together, Theorems 1.3 and 1.6 may be thought of as doing for Moore spaces what [10] did for wedges of spheres. The main difference between the homological results of that paper and this is that the Hurewicz map is enough to detect p^r-torsion in the homotopy groups of the Moore space $P^n(p^r)$. In contrast, one needs more sophisticated machinery to see p^r-torsion in a wedge of spheres; [10] used Adams' e-invariant. This meant that the theorems of that paper had to be stated in terms of K-theory, rather than ordinary homology.

This document is organized as follows. The proof of Theorem 1.3 may be read independently of the proof of Theorems 1.5 and 1.6, and vice versa. Section 2 contains applications of our results. Section 3 contains definitions needed throughout. Sections

4, 5 and 6 prove Theorem 1.3, while Sections 7, 8, 9 and 11 prove Theorem 1.5, and Section 10 shows that Theorem 1.5 implies Theorem 1.6.

I would like to thank my PhD supervisor, Stephen Theriault, for many helpful conversations and much encouragement. Changes to the proof of Theorem 1.3 which make the result work for powers of 2 are due to him.

2 Applications

2.1 Spaces containing a Moore space as a retract

Various spaces have been shown to contain wedges of Moore spaces and spheres as p-local retracts after looping. This section collects some examples of this form.

Example 2.1. *Let M be an (oriented) $(n-1)$-connected $(2n+1)$-manifold for $n \geq 2$. By Poincaré duality, the homology of M is determined entirely by*

$$H_n(M) \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{\ell} \mathbb{Z}/p_i^{r_i}.$$

When $r \geq 1$, Basu [8, Theorem 5.4] gives a decomposition of ΩM, which shows in particular that ΩM contains a retract $\Omega(\bigvee_{r-1} S^n \vee \bigvee_{r-1} S^{n+1} \vee \bigvee_{i=1}^{\ell} P^n(p_i^{r_i}))$. By Theorem 1.3 and the work of Huang-Wu [19] and Zhu-Pan [32], it follows that M is $\mathbb{Z}/p^s$-hyperbolic whenever p^{s-1} divides the order of the torsion part of $H_n(M)$. In fact, if $r \geq 2$ then ΩM contains $\Omega(S^n \vee S^m)$ as a retract, so is $\mathbb{Z}/p^s$-hyperbolic for all p and s by [10]. Conversely, if M is not $\mathbb{Z}/p^s$ hyperbolic for any p and s (and is not the sphere S^{2n+1}) then we must have $H_n(M) \cong \mathbb{Z}$. An example of such a manifold is $S^{n-1} \times S^n$, whose homotopy groups satisfy $\pi_i(S^{n-1} \times S^n) \cong \pi_i(S^{n-1}) \times \pi_i(S^n)$. Determining hyperbolicity for these examples is therefore as difficult as determining hyperbolicity of S^n.

In order to use Basu's result, we require that there be a $\mathbb{Z}$-summand in $H_n(M)$. In contrast, our next example has $H_n(M)$ a torsion group.

Example 2.2. *Let p be an odd prime, let $r \in \mathbb{N}$, and let M be a 5-dimensional spin manifold with $H_2(M;\mathbb{Z})$ isomorphic to a direct sum of copies of $\mathbb{Z}/p^r$. In [30] Theriault notes that his Theorem 1.3, together with a classification of simply connected 5-dimensional Poincaré duality complexes by Stöcker [29], gives a decomposition of ΩM. This decomposition shows that ΩM contains $\Omega P^3(p^r)$ as a retract. In particular, by Theorem 1.3, M is $\mathbb{Z}/p^s$-hyperbolic for all $1 \leq s \leq r$.*

2.2 Suspensions

This section deduces some examples of Theorem 1.6. As a first example, note that the identity map on the Moore space $P^n(p^r)$ satisfies the hypotheses of that theorem, and so we recover the $s = r$ case of Theorem 1.3.

Let $h : \pi_n(Y) \longrightarrow H_n(Y;\mathbb{Z})$ be the Hurewicz map, which sends a homotopy class $f : S^n \longrightarrow Y$ to the image $f_*(\xi_n)$ of a generator ξ_n of $H_n(S^n;\mathbb{Z})$ under the map induced on homology by f.

Corollary 2.3 (of Theorem 1.6)**.** *Let p be an odd prime and let $s \in \mathbb{N}$. Suppose that $H_{n-1}(\Sigma X;\mathbb{Z})$ contains a $\mathbb{Z}/p^s$-summand, generated by a class $z \in \mathrm{Im}(h)$. Let $\nu : S^{n-1} \longrightarrow \Sigma X$ be a map with $h(\nu) = z$, and let $r \in \mathbb{N}$ be such that the order of ν is equal to $p^r c$, for c prime to p. Then ΣX is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$.*

Before proving this Corollary, we note that by the Hurewicz Theorem it immediately implies the following.

Corollary 2.4. *Let n be the least natural number for which $\widetilde{H}_n(\Sigma X;\mathbb{Z})$ is nontrivial. If $\widetilde{H}_n(\Sigma X;\mathbb{Z})$ contains a $\mathbb{Z}/p^s$-summand, for p an odd prime and $s \in \mathbb{N}$, then ΣX is $\mathbb{Z}/p^s$-hyperbolic.* □

Proof of Corollary 2.3. By replacing ν with $c\nu$ (and z with cz) we may assume without loss of generality that $c = 1$. Since ν has order p^r, it extends to a map $\mu : P^n(p^r) \longrightarrow \Sigma X$.

Let x generate $H_n(P^n(p^r);\mathbb{Z}/p^s)$, and let y generate $H_{n-1}(P^n(p^r);\mathbb{Z}/p^s)$. The Bockstein β satisfies $\beta(x) = y$. We have $\mu_*(y) = h(\nu) = z$, and $\beta(\mu_*(x)) = \mu_*(\beta(x)) = \mu_*(y) = z$. This implies that $\mu_*(x)$ and $\mu_*(y)$ must both have order p^s, hence that

$$\mu_* : H_*(P^n(p^r);\mathbb{Z}/p^s) \longrightarrow H_*(\Sigma X;\mathbb{Z}/p^s)$$

is an injection. Thus, by Theorem 1.6, ΣX is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$, as required. □

A first example of this sort highlights how much bigger the homotopy of an Eilenberg-MacLane space becomes upon suspending.

Example 2.5. *The least-dimensional homology of $\Sigma K(\mathbb{Z}/p^s, n)$ is isomorphic to $\mathbb{Z}/p^s$, so Corollary 2.4 implies that $\Sigma K(\mathbb{Z}/p^s, n)$ is $\mathbb{Z}/p^s$-hyperbolic for p odd.*

More generally, we have:

Example 2.6. *Let G be a finite group. Atiyah [5, Theorem 13.1] has shown that the cohomology of G (which is the cohomology of $K(G,1)$) is nonvanishing in infinitely many degrees. Since the cohomology of G is annihilated by multiplication by $|G|$ [3, Corollary II.5.4] the lowest-dimensional nontrivial cohomology $H^n(K(G,1);\mathbb{Z})$ must contain a $\mathbb{Z}/p^s$-summand for some p^s dividing $|G|$. By the universal coefficient theorem, the least nontrivial homology is $H_{n-1}(K(G,1);\mathbb{Z})$, which must also contain such a summand. By the suspension isomorphism and Corollary 2.4, $\Sigma K(G,1)$ is $\mathbb{Z}/p^s$-hyperbolic, provided that $p \neq 2$. In particular, this means that if $|G|$ is odd, then $\Sigma K(G,1)$ is $\mathbb{Z}/p^s$-hyperbolic for some p^s dividing the order of G.*

If the (co)homology of G is known in least nontrivial dimension, then we can be more precise. Algebraic interpretations exist for the first few nontrivial homology groups: $H_1(K(G,1),\mathbb{Z})$ is the abelianization G_{ab}, and $H_2(K(G,1),\mathbb{Z})$ is known as the Schur multiplier. Consider the Alternating groups A_n. These are simple, hence have trivial abelianization, and the Schur multiplier is $\mathbb{Z}/2$ unless $n = 6,7$, in which case it is $\mathbb{Z}/6$ [26] . In particular, Corollary 2.4 implies that the suspended Eilenberg-MacLane spaces of A_6 and A_7 are $\mathbb{Z}/3$-hyperbolic. Another example is the Suzuki group Suz, *which is one of the sporadic simple groups, and has Schur Multiplier $\mathbb{Z}/6$ [15] , so again $\Sigma K(\mathrm{Suz},1)$ is $\mathbb{Z}/3$-hyperbolic.*

3 Common preamble

This section collects some foundational material which will be used in the proofs of both main results. First, we have the following well-known proposition, which we use to deduce Corollary 1.4 from Theorem 1.3.

Proposition 3.1. *Let $n \geq 3$. If $\ell \in \mathbb{N}$ has a prime power factorization $\ell = p_1^{r_1} p_2^{r_2} \dots p_m^{r_m}$ then*

$$P^n(\ell) \simeq P^n(p_1^{r_1}) \vee P^n(p_2^{r_2}) \vee \cdots \vee P^n(p_m^{r_m}),$$

and furthermore $P^n(p^r)$ is q-locally contractible for any prime $q \neq p$.

Proof. Define a map $f : P^n(p_1^{r_1}) \vee P^n(p_2^{r_2}) \vee \cdots \vee P^n(p_m^{r_m}) \longrightarrow P^n(\ell)$ which is given on the wedge summand $P^n(p_i^{r_i})$ as degree 1 on the top cell and degree $\frac{\ell}{p_i^{r_i}}$ on the bottom cell; that is, according to the following diagram of defining cofibrations.

$$\begin{tikzcd}
S^{n-1} \arrow[r, "p_i^{r_i}"] \arrow[d, equal] & S^{n-1} \arrow[r] \arrow[d, "\frac{\ell}{p_i^{r_i}}"] & P^n(p_i^{r_i}) \arrow[d, "f|_{P^n(p_i^{r_i})}"] \\
S^{n-1} \arrow[r, "\ell"] & S^{n-1} \arrow[r] & P^n(\ell)
\end{tikzcd}$$

By the Chinese Remainder Theorem, f induces an isomorphism on integral homology. Thus, by Whitehead's theorem [31] , f is in fact a homotopy equivalence.

To see that $P^n(p^r)$ is contractible after localization at $q \neq p$, note that the homology with coefficients in the integers localized at q, $H_*(P^n(p^r); \mathbb{Z}_{(q)})$, is trivial, and thus by Whitehead's theorem, the inclusion of the basepoint is a homotopy equivalence. □

3.1 The Witt Formula and the Hilton-Milnor Theorem

We will be interested in counting the dimension of various 'weighted components' of free Lie algebras. These Lie algebras will be ungraded in the proof of Theorem 1.3 and will be graded for the proof of Theorems 1.5 and 1.6. In both cases, the quantities we wish to count are determined by the Witt formula, which we now define.

Let $\mu : \mathbb{N} \longrightarrow \{-1, 0, 1\}$ be the *Möbius inversion function*, defined by

$$\mu(s) = \begin{cases} 1 & s = 1 \\ 0 & s > 1 \text{ is not square free} \\ (-1)^{\ell} & s > 1 \text{ is a product of } \ell \text{ distinct primes.} \end{cases}$$

The *Witt Formula* $W_n(k)$ is then defined by

$$W_n(k) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}.$$

The Witt formula feeds into the proof of Theorem 1.3 via Theorem 3.3, and into the proof of Theorems 1.5 and 1.6 via Theorem 8.3. The asymptotics of the Witt formula are as follows:

Lemma 3.2. *[6, Introduction] The ratio*

$$\frac{W_n(k)}{\frac{1}{k} n^k}$$

tends to 1 as k tends to ∞. □

We now introduce the Hilton-Milnor Theorem. Let L be the free (ungraded) Lie algebra over $\mathbb{Z}$ on basis elements $x_1, \ldots, x_n$. For an iterated bracket B of the elements x_i, let $k_i(B) \in \mathbb{N} \cup \{0\}$ be the number of instances of the generator x_i occurring in B. The sum $k(B) = \sum_{i=1}^{n} k_i(B)$ is called the *weight* of B, following Hilton [18] . By induction on k, Hilton defines a subset $\mathscr{L}_k$ of the brackets of weight k, which he calls the set of *basic products* of weight k. The basic products of weight 1 are precisely the x_i. The union $\mathscr{L} = \bigcup_{k=1}^{\infty} \mathscr{L}_k$ is a free basis for L (see for example [28, Theorem 5.3] , but note that what we call basic products, Serre calls a *Hall basis*).

Theorem 3.3. *[18, Theorems 3.2, 3.3] Let L be the free Lie algebra over $\mathbb{Z}$ on basis elements $x_1, \ldots, x_n$. Then the cardinality $|\mathscr{L}_k|$ of the set of basic products of weight k is equal to $W_n(k)$.* □

We are now ready to state the Hilton-Milnor Theorem. Write $X^{\wedge k}$ for the smash product of k copies of the space X.

Theorem 3.4. *[18, 21] Let $X_1, X_2, \ldots, X_n$ be connected CW-complexes. There is a homotopy equivalence*

$$\Omega\Sigma(X_1 \vee \cdots \vee X_n) \simeq \prod_{B \in \mathscr{L}} \Omega\Sigma(X_1^{\wedge k_1(B)} \wedge \cdots \wedge X_n^{\wedge k_n(B)}),$$

where the right hand side is the weak infinite product. □

4 Decompositions of Moore spaces

In this section we make the first step in the proof of Theorem 1.3. Namely, we will see that it follows from work of Cohen, Moore, and Neisendorfer that a Moore space $P^n(p^r)$ with $p^r \neq 2$ contains $P^{n_1}(p^r) \vee P^{n_2}(p^r)$ as a retract after looping, and so it suffices to prove that $P^{n_1}(p^r) \vee P^{n_2}(p^r)$ is $\mathbb{Z}/p^s$-hyperbolic. We will also record Corollary 4.6, which describes the behaviour of Moore spaces under iterated smash products.

When p is odd, the loop-decomposition of $P^n(p^r)$ depends on the parity of n. We have the following three theorems, which give the three cases of the decomposition.

Theorem 4.1. *[12, Theorem 1.1] Let p be an odd prime, and let $n > 0$. Then*

$$\Omega P^{2n+2}(p^r) \simeq S^{2n+1}\{p^r\} \times \Omega \bigvee_{m=0}^{\infty} P^{4n+2mn+3}(p^r).$$ □

Theorem 4.2. *[13] Let p be an odd prime, and let $n > 0$. Then there is a space $T^{2n+1}\{p^r\}$ so that*

$$\Omega P^{2n+1}(p^r) \simeq T^{2n+1}\{p^r\} \times \Omega\Sigma \bigvee_{\alpha} P^{n_\alpha}(p^r),$$

where $\bigvee_\alpha P^{n_\alpha}(p^r)$ is an infinite bouquet of mod-p^r Moore spaces, and each n_α satisfies $n_\alpha \geq 4n - 1$. □

Lemma 4.3. *[11, Lemma 2.6] Let $n \geq 3$ and $r \geq 2$. Then there exist spaces $T^n\{2^r\}$ such that*

$$\Omega P^n(2^r) \simeq T^n\{2^r\} \times \Omega \bigvee_{\alpha} P^{m_\alpha}(2^r),$$

where $\bigvee_\alpha P^{m_\alpha}(2^r)$ is an infinite bouquet of mod-2^r Moore spaces, and each m_α satisfies $m_\alpha \geq n$. □

Theorems 4.1 and 4.2, together with Lemma 4.3 immediately imply the following corollary.

Corollary 4.4. *Let p be prime and let $r \in \mathbb{N}$. Suppose that $p^r \neq 2$, and let $n \geq 3$. Then $\Omega P^n(p^r)$ has $\Omega(P^{n_1}(p^r) \vee P^{n_2}(p^r))$ as a retract for some $n_1, n_2 \geq n$.* □

Smash powers of Moore spaces are well-understood, by means of the following Lemma.

Lemma 4.5. *[22] Let p be prime, and let $r \in \mathbb{N}$, with $p^r \neq 2$. For $n, m \geq 2$,*

$$P^n(p^r) \wedge P^m(p^r) \simeq P^{m+n}(p^r) \vee P^{m+n-1}(p^r).$$ □

For a space X, write $X^{\vee i}$ for the wedge sum of i copies of X. Applying Lemma 4.5 repeatedly gives the following binomial-type formula.

Corollary 4.6. *Let p be prime, and let $r \in \mathbb{N}$, with $p^r \neq 2$. For $n, m \geq 2$, and $k_1, k_2 \in \mathbb{N}$. Letting $k = k_1 + k_2$, we have*

$$P^n(p^r)^{\wedge k_1} \wedge P^m(p^r)^{\wedge k_2} \simeq \bigvee_{i=0}^{k-1} (P^{k_1 n + k_2 m - i}(p^r))^{\vee \binom{k-1}{i}}.$$ □

5 Classes in the homotopy groups of $P^n(p^r)$

In this section, we identify some stable classes in the homotopy groups of $P^n(p^r)$. The identification of these classes is the way in which we go beyond Huang and Wu's work. We will transfer known classes from the stable homotopy groups of spheres (Lemma 5.4) into the stable homotopy groups of Moore Spaces by means of the stable homotopy exact sequence of the cofibration defining the Moore space. To show that the resulting classes have the correct order, we need assurances about the maximum order of the torsion in the stable homotopy groups of Moore spaces, and these assurances are provided by Corollary 5.2.

Cohen, Moore, and Neisendorfer have shown that the homotopy groups of $P^n(p^r)$ contain classes of order p^{r+1} [12] . However, these classes are all outside the stable range; the stable homotopy groups of $P^n(p^r)$ were already known to be annihilated by multiplication by p^r. The proof of this fact is due to Barratt.

Lemma 5.1. *[7] Let A be $(n-1)$-connected, and let p be a prime. Suppose that we have $p^s \mathrm{id}_{\Sigma A} \simeq *$ in the group $[\Sigma A, \Sigma A]$, for some $s \in \mathbb{N}$. Then $p^s \pi_{n+j}(\Sigma A) = 0$ for $j \leq (p-1)n$.* □

Corollary 5.2. *Let p be prime, and let $s \in \mathbb{N}$ such that $p^s \neq 2$. Then we have $p^s \pi_{n+j}(P^n(p^s)) = 0$ for $j \leq (p-1)(n-2) - 2$.*

Proof. By definition, $P^n(p^s) \simeq \Sigma P^{n-1}(p^s)$, and $P^{n-1}(p^s)$ is $(n-3)$-connected. By Lemma 5.1 the result therefore follows from the fact that the identity map on $P^n(p^s)$ has order p^s [23, Proposition 6.1.7] . □

We continue in a similar vein. In general, the degree ℓ map on S^n does not induce multiplication by ℓ on homotopy groups. However, it follows from the Hilton-Milnor Theorem (Theorem 3.4) that it must do so in the stable range, as in the next lemma.

Lemma 5.3. *The degree ℓ map $S^n \xrightarrow{\ell} S^n$ induces multiplication by ℓ on $\pi_j(S^n)$ for $j \leq 2n-2$.*

Proof. Write $n = m+1$ and $i = j-1$. By the adjoint isomorphism, it suffices to show that $\Omega\ell$ induces multiplication by ℓ on $\pi_i(\Omega S^{m+1})$ for $i < 2m$. The map ℓ is the composition

$$S^{m+1} \xrightarrow{c} \bigvee_{i=1}^{\ell} S^{m+1} \xrightarrow{\nabla} S^{m+1}$$

of the ℓ-fold suspension comultiplication c on S^{m+1} with the fold map ∇. Let $\mathscr{L}$ be the free Lie algebra on ℓ generators, as in Subsection 3.1. The Hilton-Milnor Theorem (Theorem 3.4) gives a decomposition

$$\Omega \bigvee_{i=1}^{\ell} S^{m+1} \simeq \Omega \prod_{B \in \mathscr{L}} S^{km+1},$$

where k is the weight of $B \in \mathscr{L}$, so in particular is implicitly a function of B.

Let $f \in \pi_i(\Omega S^{m+1})$. Applying the above decomposition to $(\Omega\ell)_*(f) = (\Omega\ell) \circ f$ gives factorizations φ and θ as in the following diagram

$$\begin{tikzcd}
\Omega S^{m+1} \arrow[r, "\Omega c"] & \Omega \bigvee_{i=1}^{\ell} S^{m+1} \arrow[r, "\Omega\nabla"] & \Omega S^{m+1} \\
S^i \arrow[u, "f"] \arrow[r, "\varphi"] & \Omega \prod_{B \in \mathscr{L}} S^{km+1}. \arrow[u, "\simeq"] \arrow[ur, "\theta"'] &
\end{tikzcd}$$

We must show that $\theta \circ \varphi \simeq \ell f$. Since $i < 2m$, cellular approximation tells us that φ factors through the sub-product $\Omega \prod_{i=1}^{\ell} S^{m+1}$ consisting of those terms where $k = 1$. Hilton [18] tells us that the restriction of the Hilton-Milnor map to these summands is given by the product under the loop multiplication of the looped wedge factor inclusions $\Omega S^{m+1} \longrightarrow \Omega \bigvee_{i=1}^{\ell} S^{m+1}$. Thus, the restriction of θ to these summands is the ℓ-fold loop multiplication map

$$m : \Omega \prod_{i=1}^{\ell} S^{m+1} \longrightarrow \Omega S^{m+1}.$$

This map is a left homotopy inverse to the looped inclusion
$\Omega\iota : \Omega \bigvee_{i=1}^{\ell} S^{m+1} \longrightarrow \Omega \prod_{i=1}^{\ell} S^{m+1}$ of the wedge into the product, so φ is homotopic to $\Omega\iota \circ \Omega c \circ f$.

We may now modify our diagram to obtain

$$\begin{tikzcd}
\Omega S^{m+1} \arrow[r, "\Omega c"] & \Omega \bigvee_{i=1}^{\ell} S^{m+1} \arrow[r, "\Omega\nabla"] \arrow[d, bend right, "\Omega\iota"'] & \Omega S^{m+1} \\
S^i \arrow[u, "f"] \arrow[r, "\varphi"] & \Omega \prod_{i=1}^{\ell} S^{m+1}. \arrow[u] \arrow[ur, "m"] &
\end{tikzcd}$$

To finish, we note that by the axiomatic definition of a comultiplication [4] we have that $\Omega\iota \circ \Omega c = \Delta$, the diagonal map into the ℓ-fold product, and the composition $m \circ \Delta$ is by definition the map inducing multiplication by ℓ in the group structure on $[S^i, \Omega S^{m+1}] = \pi_i(\Omega S^{m+1})$ coming from the fact that ΩS^{m+1} is an H-group. But this group structure coincides with that of the homotopy group [4] , and so we are done. □

Let π_j^S denote the j-th stable homotopy group of spheres. Work of Adams on the J-homomorphism [1, 2] implies that any cyclic group of prime power order occurs as a summand in some π_j^S:

Lemma 5.4. *[10, Lemma 3.4] For any prime p and any $s \in \mathbb{N}$, there exists j such that $\mathbb{Z}/p^s$ is a direct summand in π_j^S. That is, for a fixed choice of such a j, $\mathbb{Z}/p^s$ is a direct summand in $\pi_{n+j}(S^n)$ for all $n \geq j+2$.* □

These summands can be transplanted to $P^n(p^r)$ as follows.

Corollary 5.5. *Let p be prime, and let $r \geq s \in \mathbb{N}$. If $p^s \neq 2$, then there exists j such that $\mathbb{Z}/p^s$ is a direct summand in $\pi_{n+j}(P^n(p^r))$ for all $n > j+3$.*

Proof. The cofibration $P^n(p^r) \longrightarrow S^n \xrightarrow{p^r} S^n$ gives a truncated long exact sequence on homotopy groups [17] :

$$\pi_{2n-3}(P^n(p^r)) \longrightarrow \pi_{2n-3}(S^n) \longrightarrow \pi_{2n-3}(S^n) \longrightarrow \pi_{2n-4}(P^n(p^r)) \longrightarrow \dots$$

$$\cdots \longrightarrow \pi_n(P^n(p^r)) \longrightarrow \pi_n(S^n) \longrightarrow \pi_n(S^n) \longrightarrow \pi_{n-1}(P^n(p^r)) \longrightarrow 0.$$

By Lemma 5.4, there exists j such that $\mathbb{Z}/p^s$ is a direct summand in $\pi_{n+j}(S^n)$ for all $n \geq j+2$. Fix $n \geq j+4$, and let $f : S^{n+j} \longrightarrow S^n$ generate a $\mathbb{Z}/p^s$-summand. By Lemma 5.3, since we are in the stable range, the composite $p^s \circ f$ is homotopic to $p^s f$, and by assumption f has order p^s. Thus, since $n \geq j+3$, the exact sequence applies, and taking $r = s$ we obtain a lift $\widetilde{f} \in \pi_{n+j}(P^n(p^s))$ making the following diagram commute.

$$
\begin{tikzcd}
P^n(p^s) \arrow[r] & S^n \arrow[r, "p^s"] & S^n \\
& S^{n+j}. \arrow[ul, dotted, "\tilde{f}"] \arrow[u, "f"'] \arrow[ur, "\simeq *"']
\end{tikzcd}
$$

We also have, for each $r \geq s$, a diagram

$$
\begin{tikzcd}
S^{n-1} \arrow[r, "p^r"] & S^{n-1} \\
S^{n-1} \arrow[u, equal] \arrow[r, "p^s"] & S^{n-1}. \arrow[u, "p^{r-s}"']
\end{tikzcd}
$$

Extending the rows of this diagram to cofibre sequences and combining with the previous one gives a diagram

$$
\begin{tikzcd}
S^{n-1} \arrow[r] & P^n(p^r) \arrow[r, "\underline{\varrho}"] & S^n \arrow[r, "p^r"] & S^n \\
S^{n-1} \arrow[u, "p^{r-s}"'] \arrow[r] & P^n(p^s) \arrow[u, "\varphi"'] \arrow[r, "\underline{\varrho}"] & S^n \arrow[u, equal] \arrow[r, "p^s"] & S^n \arrow[u, "p^{r-s}"'] \\
& & S^{n+j}, \arrow[ul, dotted, "\tilde{f}"] \arrow[u, "f"'] \arrow[ur, "\simeq *"']
\end{tikzcd}
$$

We have that $\underline{\varrho}_*(\varphi \circ \widetilde{f}) = f$, so the image of $\underline{\varrho}_* : \pi_{n+j}(P^n(p^r)) \longrightarrow \pi_{n+j}(S^n)$ contains f. Since f generates a $\mathbb{Z}/p^s$-summand, this gives a surjection $\pi_{n+j}(P^n(p^r)) \longrightarrow \mathbb{Z}/p^s$, and it suffices to argue that this surjection is split. From the diagram, it further suffices to do so in the case $r = s$.

By Corollary 5.2, since $n \geq j+4$ we have $p^s \pi_{n+j}(P^n(p^s)) = 0$. This means that the above surjection $\pi_{n+j}(P^n(p^s)) \longrightarrow \mathbb{Z}/p^s$ is a map of $\mathbb{Z}/p^s$-modules with free codomain, so is split, as required. □

6 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. In Section 4, we reduced the problem to showing $\mathbb{Z}/p^s$-hyperbolicity of the wedge $P^n(p^r) \vee P^m(p^r)$. By the Hilton-Milnor Theorem (Theorem 3.4) and Corollary 4.6, we will see that each of the stable classes identified in Section 5 will give exponentially many summands in the homotopy groups of $P^n(p^r) \vee P^m(p^r)$, which will suffice.

Proof of Theorem 1.3. By Corollary 4.4, it suffices to prove that if $n, m \geq 2$ then $\Omega(P^{n+1}(p^r) \vee P^{m+1}(p^r))$ is $\mathbb{Z}/p^s$-hyperbolic for all $s \leq r$. Let $\mathscr{L}$ be the free ungraded

Lie algebra over $\mathbb{Z}$ on two generators. The Hilton-Milnor theorem (Theorem 3.4) gives

$$\Omega(P^{n+1}(p^r) \vee P^{m+1}(p^r)) \simeq \Omega\Sigma(P^n(p^r) \vee P^m(p^r)) \simeq \prod_{B \in \mathscr{L}} \Omega\Sigma P^n(p^r)^{\wedge k_1} \wedge P^m(p^r)^{\wedge k_2},$$

where we have written $k_i = k_i(B)$, leaving the fact that k_i is a function of B implicit. Applying Lemma 4.6 factor-wise, this last is homotopy equivalent to

$$\Omega \prod_{B \in \mathscr{L}} \Sigma \bigvee_{i=0}^{k-1} (P^{k_1 n + k_2 m - i}(p^r))^{\vee \binom{k-1}{i}} \simeq \Omega \prod_{B \in \mathscr{L}} \bigvee_{i=0}^{k-1} (P^{k_1 n + k_2 m + 1 - i}(p^r))^{\vee \binom{k-1}{i}},$$

where $k = k_1 + k_2$ is also implicitly a function of B.

By Corollary 5.5, let j be such that $\pi_{N+j}(P^N(p^r))$ contains a $\mathbb{Z}/p^s$-summand for all $N > j + 3$. For each $B \in \mathscr{L}$, the associated factor of the above decomposition contains 2^{k-1} Moore spaces. Supposing without loss of generality that $n \leq m$, the dimensions of these Moore spaces are at least $k(n-1)+2$. Thus, for $k > \frac{j+1}{n-1}$, the homotopy groups of each factor

$$\bigvee_{i=0}^{k-1} (P^{k_1 n + k_2 m + 1 - i}(p^r))^{\vee \binom{k-1}{i}}$$

contain 2^{k-1} summands isomorphic to $\mathbb{Z}/p^s$ in dimensions at most $km + 1 + j$.

The number of factors for which the weight of B is k is equal to $W_2(k)$ (Theorem 3.3), so we may conclude that

$$\bigoplus_{i=1}^{km+1+j} \pi_i(P^{n+1}(p^r) \vee P^{m+1}(p^r))$$

contains at least $2^{k-1}W_2(k)$ summands isomorphic to $\mathbb{Z}/p^s$. The sequence $2^{k-1}W_2(k)$ certainly grows exponentially in k (in fact, by Lemma 3.2, it grows like $\frac{1}{2k}4^k$) and this completes the proof. $\square$

7 Modules over $\mathbb{Z}/p^s$

The purpose of this section is to prove various elementary facts about modules over $\mathbb{Z}/p^s$ which we will use later. These facts are mostly intuitively clear, so we recommend that the reader skip this section on first reading, referring back only as necessary.

7.1 Injections

The main point of this subsection is to develop the 'linear algebra' to prove Lemma 7.4, which says that injections from free $\mathbb{Z}/p^s$-modules are split, and that therefore the 'dimension' of the codomain must be at least the 'dimension' of the domain.

Let p be prime and let $s \in \mathbb{N}$. Let M be a finitely generated module over $\mathbb{Z}/p^s$. By the structure theorem for finitely generated $\mathbb{Z}$-modules (for example as in [20, Theorem 7.5]) M decomposes as a direct sum

$$M \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{s_i},$$

where each s_i satisfies $1 \leq s_i \leq s$. Further, if we order the summands so that $s_{i+1} \geq s_i$, then the sequence $(s_i \mid 1 \leq i \leq n)$ is uniquely determined. In particular, if we fix $t \in \mathbb{N}$, then the number of values of i for which $s_i = t$ is uniquely determined. This number is then precisely the $\mathbb{Z}/p^t$*-dimension* $\dim_{\mathbb{Z}/p^t}(M)$ of Definition 1.1. We will often use without comment the fact that a $\mathbb{Z}/p^s$-module is equivalently a $\mathbb{Z}$-module M satisfying $p^s M = 0$.

We will wish to mimic the approach of ordinary linear algebra as far as possible. We will wish to be able to 'change basis', and to do so we need a notion of basis, which must generalize the idea of a free basis in that our elements may have variable order.

Definition 7.1. Let M be a $\mathbb{Z}/p^s$-module. A *basis* of M is a list

$$((e_i, s_i) \in M \times \mathbb{N} \mid 1 \leq i \leq n),$$

such that the following conditions are satisfied:

- Each $x \in M$ is expressible as $x = \sum_{i=1}^{n} \lambda_i e_i$ for $\lambda_i \in \mathbb{Z}/p^s$ (*spanning*).
- $\sum_{i=1}^{n} \lambda_i e_i = 0$ if and only if $p^{s_i} | \lambda_i$ for each i (*linear independence*).

Lemma 7.2. *Any finitely generated $\mathbb{Z}/p^s$-module has a basis. Conversely, if $((e_i, s_i) \mid 1 \leq i \leq n)$ is a basis of M, then the map*

$$\bigoplus_{i=1}^{n} \mathbb{Z}/p^{s_i} \longrightarrow M$$

defined by sending the generator of the i-th summand to e_i is an isomorphism.

Proof. To see that M has a basis write $M \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p^{t_i}$, taking e_i to be a generator of the i-th summand, and taking $s_i = t_i$. It follows immediately that this is a basis.

Conversely, let $\varphi : \bigoplus_{i=1}^n \mathbb{Z}/p^{s_i} \longrightarrow M$ be as in the theorem statement. By linear independence of the basis, $p^{s_i}e_i = 0$ for each i, so φ is well-defined. Surjectivity of φ follows immediately from the spanning condition, while injectivity follows immediately from linear independence. Thus, φ is an isomorphism, as required. □

Lemma 7.3. *Let $((e_i, s_i) \mid 1 \le i \le n)$ be a basis of M.*

- *If λ is a unit in $\mathbb{Z}/p^s$, then replacing the basis element (e_k, s_k) with $(\lambda e_k, s_k)$ again yields a basis.*
- *If $j \neq k$ and $s_j \le s_k$, then replacing the basis element (e_k, s_k) with $(e_k + \mu e_j, s_k)$ for any $\mu \in \mathbb{Z}/p^s$ again yields a basis.*

Proof. We will show only that the basis obtained by the second replacement is linearly independent; the other parts are similar.

Write (e'_i, s_i) for the new basis, and suppose that $\sum_{i=1}^n \lambda_i e'_i = 0$. We must show that p^{s_i} divides λ_i for each i. Substituting in, we have $(\sum_{i\neq j,k} \lambda_i e_i) + \lambda_j e_j + \lambda_k(e_k + \mu e_j) = 0$. Since the original basis was linearly independent, we have that $p^{s_i}|\lambda_i$ for $i \neq j$. In particular, $p^{s_k}|\lambda_k$. We also have $p^{s_j}|(\lambda_j + \mu\lambda_k)$. Since $s_j \le s_k$ we have $p^{s_j}|\lambda_k$, so $p^{s_j}|\lambda_j$. Thus, $p^{s_i}|\lambda_i$ for all i, and thus the (e'_i, s_i) form a basis, as required. □

It is always true that a surjection onto a free module splits; over $\mathbb{Z}/p^s$, it is additionally true that an injection from a free module splits.

Lemma 7.4. *Let M and N be finitely-generated $\mathbb{Z}/p^s$-modules, with M free. The image of any injection of $\mathbb{Z}/p^s$-modules $\varphi : M \longrightarrow N$ is a summand, and $\dim_{\mathbb{Z}/p^s}(N) \ge \dim_{\mathbb{Z}/p^s}(M)$.*

Proof. Let $(x_1, t_1), \ldots, (x_m, t_m)$ be a basis of M, and let

$$(e_1, s_1), \ldots, (e_n, s_n), (e'_1, s'_1), \ldots (e'_{n'}, s'_{n'})$$

be a basis of N, such that each $s_i = s$ and each $s'_i < s$.

Thus we have $f(x_1) = \sum_{i=1}^n \lambda_i e_i + \sum_{i=1}^{n'} \lambda'_i e'_i$ for some coefficients λ_i and λ'_i. In particular, since $f(x_1)$ has order p^s, there must be some λ_i which is not divisible by p.By repeated use of Lemma 7.3 we may therefore change basis in M by replacing e_i by $\sum_{i=1}^n \lambda_i e_i + \sum_{i=1}^{n'} \lambda'_i e'_i$. After this change we have $f(x_1) = e_i$, and by renumbering we may assume that $i = 1$.

We repeat this procedure inductively: at the j-th stage we have $f(x_i) = e_i$ for all $i < j$ and we wish to arrange that $f(x_j) = e_j$. We have that $f(x_j) = \sum_{i=1}^n \lambda_i e_i + \sum_{i=1}^{n'} \lambda'_i e'_i$ for some coefficients λ_i and λ'_i, and the set $f(x_1), \ldots, f(x_{j-1})$ spans the submodule

$\langle e_1, \ldots, e_{j-1} \rangle \subset M$. By changing basis according to Lemma 7.3, we may arrange that $\lambda_i = 0$ for $i < j$, and this does not change the fact that $f(x_i) = e_i$ for these values of i. Again, $f(x_j)$ has order p^s, so there must be $i \geq j$ with λ_i not divisible by p, and by renumbering we may assume that $i = j$. By changing basis we may arrange that $f(x_j) = e_j$. This completes the inductive step, hence the proof that $\mathrm{Im}(f)$ is a summand. Since after this procedure we have $f(x_i) = e_i$ for $i = 1, \ldots, m$ we must have $n \geq m$, which is the other part of the theorem statement. $\square$

We also have the following technical lemma, which will be used in the proof of Proposition 10.12.

Lemma 7.5. *Let X, A, B, and Y be $\mathbb{Z}/p^s$-modules, with X free and $p^{s-1}B = 0$. Let $f : X \longrightarrow A \oplus B$ and $g : A \oplus B \longrightarrow Y$ be homomorphisms. Let i_A be the inclusion of A in $A \oplus B$, and let π_A be the projection $A \oplus B \longrightarrow A$. If $g \circ f$ is injective, then the composite $g \circ i_A \circ \pi_A \circ f$ is also injective.*

Proof. Since X is free, a map defined on X is an injection if and only if its restriction to $p^{s-1}X$ is an injection. It therefore suffices to show that if $g \circ i_A \circ \pi_A \circ f(p^{s-1}x) = 0$ then $p^{s-1}x = 0$.

Thus, suppose that $g \circ i_A \circ \pi_A \circ f(p^{s-1}x) = 0$. Write $f(x) = a + b \in A \oplus B$, for $a \in A$ and $b \in B$. Then $f(p^{s-1}x) = p^{s-1}a$, since $p^{s-1}B = 0$. In particular, $f(p^{s-1}x) = i_A \circ \pi_A \circ f(p^{s-1}x)$. Thus, $g \circ f(p^{s-1}x) = 0$, and $g \circ f$ is an injection, so $p^{s-1}x = 0$, as required. $\square$

7.2 Surjections

The main result of this subsection is Lemma 7.9, which is the basic algebraic scaffolding for the proof of Theorem 1.5.

Lemma 7.6. *Let $\varphi : M \longrightarrow N$ be a surjection of $\mathbb{Z}/p^s$-modules. Then*

$$\dim_{\mathbb{Z}/p^s}(M) \geq \dim_{\mathbb{Z}/p^s}(N).$$

Proof. Write $N = F \oplus C$, where F is free over $\mathbb{Z}/p^s$, and the complementary module C satisfies $p^{s-1}C = 0$. Let $\pi : N \longrightarrow F$ be the projection. The map $\pi \circ \varphi$ is a composite of surjections, hence a surjection, so is split by freeness of F. Thus, we have an isomorphism $M \cong F \oplus D$ for some complementary module D, so

$$\dim_{\mathbb{Z}/p^s}(M) \geq \dim_{\mathbb{Z}/p^s}(F) = \dim_{\mathbb{Z}/p^s}(N),$$

as required. $\square$

Lemma 7.7. *Let A be a submodule of a $\mathbb{Z}/p^s$-module N, such that $A + pN = N$. Then $A = N$.*

Proof. Because N is a $\mathbb{Z}/p^s$-module, we have $p^s N = 0$, so certainly $A \supset p^s N$. We will now show that if $A \supset p^k N$ then $A \supset p^{k-1}N$. By induction, this implies that $A \supset p^0 N = N$, which suffices.

Assume that $A \supset p^k N$, and let $z \in N$. We have by assumption that $z = x + py$ for $x \in A$ and $y \in N$. Thus, $p^{k-1}z = p^{k-1}x + p^k y$. But now, $p^k y \in p^k N$, which by induction is a subset of A, so $p^{k-1}z \in A$, and since z is an arbitrary element of N, this implies that $p^{k-1}N \subset A$. This completes the inductive step, hence the proof. $\square$

Lemma 7.8. *Let M, M', N be $\mathbb{Z}$-modules. Let p be prime and let $s \leq r \in \mathbb{N}$. Suppose that $p^r M = 0$, so M may be regarded as a module over $\mathbb{Z}/p^r$, and that $p^s N = 0$. Let $\varphi : M \longrightarrow N$ be a surjection which admits a factorization*

$$\begin{tikzcd} M' \arrow[dr, "\widetilde{\varphi}"] & \\ M \arrow[u, "\iota"] \arrow[r, "\varphi"] & N. \end{tikzcd}$$

Then $\sum_{t=s}^{r} \dim_{\mathbb{Z}/p^t}(M') \geq \dim_{\mathbb{Z}/p^s}(N)$.

Proof. We will first argue that we may assume $p^r M' = 0$ without loss of generality. Write $M' = A \oplus B$, where $p^r A = 0$, and B is a direct sum of copies of $\mathbb{Z}$, $\mathbb{Z}/q^t$ for various $q \neq p$ and $t \in \mathbb{N}$, and $\mathbb{Z}/p^t$ for $t > r$. This gives a decomposition $M = \iota^{-1}(A) \oplus \iota^{-1}(B)$. The restriction of ι to $\iota^{-1}(B)$ must have image contained in pB, so the same restriction of $\widetilde{\varphi} \circ \iota$ has image contained in pN. Furthermore, since $\widetilde{\varphi}$ is a surjection, we have that $\mathrm{Im}(\widetilde{\varphi} \circ \iota|_{\iota^{-1}(A)}) + \mathrm{Im}(\widetilde{\varphi} \circ \iota|_{\iota^{-1}(B)}) = N$, so in particular $\mathrm{Im}(\widetilde{\varphi} \circ \iota|_{\iota^{-1}(A)}) + pN = N$. By Lemma 7.7 we then have $\mathrm{Im}(\widetilde{\varphi} \circ \iota|_{\iota^{-1}(A)}) = N$. We may therefore restrict M' to A and M to $\iota^{-1}(A)$ in the diagram without affecting the hypotheses. In particular, since $p^r A = 0$ it suffices to prove the lemma in the case that $p^r M' = 0$.

We now tensor the diagram with $\mathbb{Z}/p^s$; since $p^s N = 0$, we have $N \otimes \mathbb{Z}/p^s \cong N$. Since $p^r M' = 0$, we have $\dim_{\mathbb{Z}/p^s}(M' \otimes \mathbb{Z}/p^s) = \sum_{t=s}^{r} \dim_{\mathbb{Z}/p^t}(M')$. By Lemma 7.6, since $\widetilde{\varphi} \otimes \mathbb{Z}/p^s$ is a surjection we have $\dim_{\mathbb{Z}/p^s}(M' \otimes \mathbb{Z}/p^s) \geq \dim_{\mathbb{Z}/p^s}(N \otimes \mathbb{Z}/p^s)$, which completes the proof. $\square$

By applying Lemma 7.8 in each degree we immediately obtain the following.

Corollary 7.9 (The 'Sandwich' Lemma)**.** *Let M, M', N be graded $\mathbb{Z}$-modules. Let p be prime and let $r \geq s \in \mathbb{N}$. Suppose that $p^r M = 0$ and that $p^s N = 0$. Let $\varphi : M \longrightarrow N$ be a surjection which admits a factorization*

$$\begin{tikzcd} M' \arrow[dr] & \\ M \arrow[u] \arrow[r, "\varphi"] & N. \end{tikzcd}$$

If N is $\mathbb{Z}/p^s$-hyperbolic then M' is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$. □

Lemma 7.10. *Let $\varphi : M \longrightarrow N$ be a map of $\mathbb{Z}/p^s$-modules, with N free. Then* $\dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi)) = \dim_{\mathbb{Z}/p}(\mathrm{Im}(\varphi \otimes \mathbb{Z}/p))$.

Proof. Let $(e_1, s_1), \ldots, (e_m, s_m), (e'_1, s'_1), \ldots, (e'_{m'}, s'_{m'})$ be a basis of M, where $s_i = s$ and $s'_i < s$. Let S be a maximal subset of the e_i such that the restriction of φ to the submodule of M generated by S is an injection. Denote this submodule by $\langle S \rangle$. By renumbering we may assume that $S = \{e_1, \ldots e_k\}$ for some $k \leq n$. We clearly have $\mathrm{Im}(\varphi|_{\langle S \rangle}) \subset \mathrm{Im}(\varphi)$, and we will now show that $\mathrm{Im}(\varphi) \subset \mathrm{Im}(\varphi|_{\langle S \rangle}) + pN$.

Since N is assumed free, and the elements e'_i have order p^{s_i} for $s_i < s$, we must have $\varphi(e'_i) \in pN$. Now consider e_j, for $k+1 \leq j \leq m$. By construction of S, the restriction of φ to $\langle S \cup \{e_j\} \rangle$ is not injective, so there exist $\lambda_1, \ldots \lambda_k, \lambda \in \mathbb{Z}/p^s$ with $\lambda \neq 0$ such that $\varphi(\sum_{i=1}^k \lambda_i e_i + \lambda e_j) = 0$. This implies that $\lambda\varphi(e_j) \in \mathrm{Im}(\varphi|_{\langle S \rangle})$. Thus, $p^t\varphi(e_j) \in \mathrm{Im}(\varphi|_{\langle S \rangle})$ for some $t < s$. By Lemma 7.4 we may write $N = \mathrm{Im}(\varphi|_{\langle S \rangle}) \oplus C$ for some complementary module C, and under this correspondence we have $\varphi(e_j) = (\beta, \gamma)$ for $\gamma \in C$ and $\beta \in \mathrm{Im}(\varphi|_{\langle S \rangle})$. Since $p^t\varphi(e_j) \in \mathrm{Im}(\varphi|_{\langle S \rangle})$, we have $p^t\gamma = 0$, so by freeness of N, $t < s$ implies that $\gamma \in pN$, so $\varphi(e_j) \in \mathrm{Im}(\varphi|_{\langle S \rangle}) + pN$. We have now shown that all elements of the basis of M are carried under φ to $\mathrm{Im}(\varphi|_{\langle S \rangle}) + pN$, so $\mathrm{Im}(\varphi) \subset \mathrm{Im}(\varphi|_{\langle S \rangle}) + pN$, as claimed.

Now, $\varphi|_{\langle S \rangle}$ is split by Lemma 7.4, so $\dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi|_{\langle S \rangle})) = k$. Furthermore, by taking the inclusion on each summand there is a surjection $\mathrm{Im}(\varphi|_{\langle S \rangle}) \oplus pN \longrightarrow \mathrm{Im}(\varphi|_{\langle S \rangle}) + pN \subset N$, and pN is annihilated by multiplication by p^{s-1}, so by Lemma 7.6 $\dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi|_{\langle S \rangle}) + pN) \leq k$. Since $\dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi|_{\langle S \rangle}) + pN) \geq \dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi|_{\langle S \rangle}))$, this implies that the former is equal to k. Thus, since

$$\mathrm{Im}(\varphi|_{\langle S \rangle}) \subset \mathrm{Im}(\varphi) \subset \mathrm{Im}(\varphi|_{\langle S \rangle}) + pN$$

applying Lemma 7.4 to the inclusions gives

$$k = \dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi|_{\langle S \rangle})) \leq \dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi)) \leq \dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi|_{\langle S \rangle}) + pN) = k,$$

so $\dim_{\mathbb{Z}/p^s}(\mathrm{Im}(\varphi)) = k$.

To finish the proof we must show that $\dim_{\mathbb{Z}/p}(\mathrm{Im}(\varphi \otimes \mathbb{Z}/p)) = k$. Since the images of φ and $\varphi|_{\langle S \rangle}$ differ only by at most pN, we have $\mathrm{Im}(\varphi \otimes \mathbb{Z}/p) = \mathrm{Im}(\varphi|_{\langle S \rangle} \otimes \mathbb{Z}/p)$. Since $\varphi|_{\langle S \rangle}$ is split injective, $\varphi|_{\langle S \rangle} \otimes \mathbb{Z}/p$ is injective, so $\dim_{\mathbb{Z}/p}(\varphi|_{\langle S \rangle} \otimes \mathbb{Z}/p) = k$, which completes the proof. □

7.3 Tor and the Universal Coefficient Theorem

The purpose of this section is to prove that for $t < s$ a map inducing an injection on homology with $\mathbb{Z}/p^s$-coefficients also induces an injection on homology with $\mathbb{Z}/p^t$-coefficients (Lemma 7.13) provided that the domain is free. This follows straightforwardly from the Universal Coefficient Theorem for homology, where we regard $\mathbb{Z}/p^t$ as a module over $\mathbb{Z}/p^s$. The inclusion of the bottom cell of a Moore space provides an easy counterexample to the converse; the algebraic point being that the converse of Lemma 7.12 is false.

Lemma 7.11. *For any finitely generated $\mathbb{Z}/p^s$-modules M, N we have*

$$p^{s-1}\mathrm{Tor}_{\mathbb{Z}/p^s}(M,N) = 0,$$

and furthermore if M or N is free then $\mathrm{Tor}_{\mathbb{Z}/p^s}(M,N) = 0$.

Proof. For any ring R and R-module M we have $\mathrm{Tor}_R(R,M) = 0$, since R is free as an R-module. If $1 \leq t < s$, then a free resolution of $\mathbb{Z}/p^t$ over $\mathbb{Z}/p^s$ is given by

$$0 \longrightarrow \mathbb{Z}/p^s \xrightarrow{\cdot p^t} \mathbb{Z}/p^s \longrightarrow 0,$$

so, for any $\mathbb{Z}/p^s$-module M, $\mathrm{Tor}_{\mathbb{Z}/p^s}(\mathbb{Z}/p^t, M) = \mathrm{Ker}(M \xrightarrow{\cdot p^s} M)$, which is annihilated by multiplication by p^t, hence in particular is annihilated by multiplication by p^{s-1}. Since any $\mathbb{Z}/p^s$-module decomposes as a direct sum of modules isomorphic to $\mathbb{Z}/p^t$ for $1 \leq t \leq s$, both parts of the Lemma now follow by additivity of Tor. □

Lemma 7.12. *Let $\varphi : M \longrightarrow N$ be a map of $\mathbb{Z}/p^s$-modules, with M free. Let $t < s$. If φ is injective then $\varphi \otimes \mathbb{Z}/p^t : M \otimes \mathbb{Z}/p^t \longrightarrow N \otimes \mathbb{Z}/p^t$ is injective.*

Proof. Note that $M \otimes \mathbb{Z}/p^t$ is a free $\mathbb{Z}/p^t$-module. Suppose that $\varphi \otimes \mathbb{Z}/p^t$ is not injective. Then there exists $x \in M$ which is not divisible by p^t such that $\varphi(x)$ is divisible by p^t. By freeness of M, $p^{s-t}x$ is not divisible by p^s, hence is nonzero, but $\varphi(p^{s-t}x) = p^{s-t}\varphi(x)$ is divisible by p^s, hence is zero. That is, φ is not injective. □

Lemma 7.13. *Let $t < s \in \mathbb{N}$. Let $f : X \longrightarrow Y$ be a map of spaces, and suppose that $H_*(X;\mathbb{Z}/p^s)$ is a free $\mathbb{Z}/p^s$-module. If*

$$f_* : H_*(X;\mathbb{Z}/p^s) \longrightarrow H_*(Y;\mathbb{Z}/p^s)$$

is injective then

$$f_* : H_*(X;\mathbb{Z}/p^t) \longrightarrow H_*(Y;\mathbb{Z}/p^t)$$

is injective.

Proof. Write f_*^t for the induced map on homology with $\mathbb{Z}/p^t$-coefficients, and likewise f_*^s. Applying the universal coefficient theorem for the module $\mathbb{Z}/p^t$ over the ring $\mathbb{Z}/p^s$ we get a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H_n(X;\mathbb{Z}/p^s)\otimes\mathbb{Z}/p^t \arrow[r] \arrow[d, "f_*^s\otimes\mathbb{Z}/p^t"] & H_n(X;\mathbb{Z}/p^t) \arrow[r] \arrow[d, "f_*^t"] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_n(Y;\mathbb{Z}/p^s)\otimes\mathbb{Z}/p^t \arrow[r] & H_n(Y;\mathbb{Z}/p^t) \arrow[r] & \mathrm{Tor}(H_{n-1}(Y;\mathbb{Z}/p^s),\mathbb{Z}/p^t) \arrow[r] & 0. \end{tikzcd}$$

The Tor term in the top row vanishes by the freeness hypothesis on $H_*(X;\mathbb{Z}/p^s)$. Since the first map in each exact sequence is an injection, f_*^t is injective if and only if $f_*^s\otimes\mathbb{Z}/p^t$ is injective. By Lemma 7.12, if f_*^s is injective, then $f_*^s\otimes\mathbb{Z}/p^t$ is injective, so f_*^t is injective, as required. $\square$

8 Free differential Lie algebras

In this section we will show that the module of boundaries $BL(x,dx)$ in the free differential Lie algebra $L(x,dx)$ over $\mathbb{Z}/p^r$ is $\mathbb{Z}/p^r$-hyperbolic. In the situation of Theorem 1.5 we will obtain a factorization of the tensor map

$$BL(x,dx) \longrightarrow \pi_*(\Omega Y) \longrightarrow BL(x,dx)\otimes\mathbb{Z}/p^s,$$

which will imply by Corollary 7.9 (The 'Sandwich' Lemma) that ΩY must be p-hyperbolic concentrated in exponents $s, s+1,\ldots,r$. The desired $\mathbb{Z}/p^r$-hyperbolicity of $BL(x,dx)$ will follow from Cohen, Moore, and Neisendorfer's description of the homology of $L(x,dx)$, which is Proposition 8.4.

Throughout this section we work over a ground ring $R=\mathbb{Z}/p^r$ for $p\neq 2$. The next definitions are as in [12] .

Definition 8.1. A *graded Lie algebra* is a graded $\mathbb{Z}/p^r$-module L, together with a $\mathbb{Z}/p^r$-bilinear pairing

$$[\ ,\ ] : L_n \times L_m \longrightarrow L_{n+m},$$

called a *Lie bracket* which satisfies the relations of

- (antisymmetry): $[x,y] = -(-1)^{\deg(x)\deg(y)}[y,x]$ for all x and y in L.
- (the Jacobi identity): $[x,[y,z]] = [[x,y],z] + (-1)^{\deg(x)\deg(y)}[y,[x,z]]$ for all x, y, and z in L.
- $[x,[x,x]] = 0$ for all x of odd degree.

Let V be a graded $\mathbb{Z}/p^r$-module. Denote by $L(V)$ the *free Lie algebra* on V. There is a linear map $j : V \longrightarrow L(V)$ and $L(V)$ is characterized up to canonical isomorphism as follows. For any map $f : V \longrightarrow L$ where L is a graded Lie algebra, there is a unique map $g : L(V) \longrightarrow L$ so that $g \circ j = f$. The Lie algebra $L(V)$ may be constructed as follows.

Let $L'(V)$ be the free nonassociative graded algebra on V, where we think of the operation as a bracket. Precisely, let B_k be the set of bracketings of a string of k symbols. Concatenation of bracketings gives an operation $B_{k_1} \times B_{k_2} \longrightarrow B_{k_1+k_2}$, which makes $B = \bigcup_{i=1}^{\infty} B_i$ into a magma. As a module,

$$L'(V) = \bigoplus_{k=1}^{\infty}(\bigoplus_{b\in B_k} V^{\otimes k}),$$

where we think of each copy of $V^{\otimes k}$ as being bracketed according to b. The bracket operation on $L'(V)$ is obtained by extending the operation on B bilinearly.

The free Lie algebra $L(V)$ is obtained as the quotient of $L'(V)$ by the relations of Definition 8.1, and automatically has the desired universal property. Denote by θ the quotient map $L'(V) \longrightarrow L(V)$. It follows that for $s < r$, we have $L(V \otimes \mathbb{Z}/p^s) = L(V) \otimes \mathbb{Z}/p^s$.

Note also that any map from V into a graded $\mathbb{Z}/p^r$-module A with a bilinear operation (that is to say, a nonassociative $\mathbb{Z}/p^r$-algebra) extends uniquely to a map of graded nonassociative algebras $L'(V) \longrightarrow A$. The map θ is a map of nonassociative algebras, hence is uniquely determined by its effect on V, and we will call it the *natural quotient*.

Definition 8.2. A *differential Lie algebra* is a graded Lie algebra together with an $\mathbb{Z}/p^r$-linear map $d : L \longrightarrow L$ of degree -1, which

- is a differential: $d^2(x) = 0$ for all x in L.
- is a derivation: $d[x, y] = [dx, y] + (-1)^{\deg(x)}[x, dy]$ for all x and y in L.

If V carries a differential d, then we may define a differential on $L'(V)$ which is the unique derivation extending d. This differential can be seen to satisfy the relations of Definition 8.1, and therefore descends to give a differential on $L(V)$, which makes $L(V)$ into a differential Lie algebra.

When $p = 3$, Samelson products in $\pi_*(\Omega X; \mathbb{Z}/3^r)$ fail to satisfy the Jacobi identity, so $L'(V)$ will also serve as a version of $L(V)$ which does not satisfy the Jacobi identity. For $p \neq 3$, $L'(V)$ may be replaced with $L(V)$ everywhere in this paper, which slightly simplifies things [12, Remark 6.3] .

Write $L(V)^k$ for the weight-k component of $L(V)$, that is, the submodule generated by brackets of length k in the elements of V. It follows from our construction of the free Lie algebra $L(V)$ that $L(V) \cong \bigoplus_{k=1}^{\infty} L(V)^k$, so weight gives a second grading on $L(V)$, and we shall write $\mathrm{wt}(x) = k$ whenever $x \in L(V)^k$. We will use subscripts (as in $L(V)_i$) for ordinary grading, and superscripts (as in $L(V)^k$) for weight. The dimension of the weighted components is given by the Witt formula, which we defined in Section 3.

Theorem 8.3. *[18, Theorems 3.2, 3.3] Let V be a free graded $\mathbb{Z}$- or $\mathbb{Z}/p^s$-module of total dimension n. Then the total dimension of $L(V)^k$ is $W_n(k)$.* □

8.1 Homology and boundaries

Let x be an even-dimensional class in a graded Lie algebra L over $\mathbb{Z}/p^r$ for $p \neq 2$. Let

$$\tau_k(x) = \mathrm{ad}^{p^k-1}(x)(dx),$$

$$\text{so } \deg(\tau_k(x)) = p^k \deg(x) - 1$$

and let

$$\sigma_k(x) = \frac{1}{2}\sum_{j=1}^{p^k-1} \frac{1}{p}\binom{p^k}{j}[\mathrm{ad}^{j-1}(x)(dx), \mathrm{ad}^{p^k-1-j}(x)(dx)],$$

$$\text{so } \deg(\sigma_k(x)) = p^k \deg(x) - 2,$$

where we understand the coefficients $\frac{1}{p}\binom{p^k}{j}$ to be computed in the integers and then reduced mod p.

Proposition 8.4. *[12, Proposition 4.9] Let V be an acyclic differential $\mathbb{Z}/p$-vector space. Write $L(V) \cong HL(V) \oplus K$, for an acyclic module K. If K has an acyclic basis, that is, a basis*

$$\{x_\alpha, y_\alpha, z_\beta, w_\beta\},$$

where α and β range over index sets $\mathscr{I}$ and $\mathscr{J}$ respectively, and we have

$$d(x_\alpha) = y_\alpha,\ \deg(x_\alpha) \text{ even},$$

$$d(z_\beta) = w_\beta,\ \deg(z_\beta) \text{ odd},$$

then $HL(V)$ has a basis

$$\{\tau_k(x_\alpha), \sigma_k(x_\alpha)\}_{\alpha \in \mathscr{I}, k \geq 1}.$$

Remark 8.5. An acyclic basis for K may always be chosen, by the following inductive procedure. Write K_i for the i-th graded component of K. Then $d : K_{i+1} \longrightarrow K_i$, and since K is acyclic we have $\mathrm{Im}(d) = \mathrm{Ker}(d)$ in each K_i. Assume that we have a basis of $\mathrm{Ker}(d) \subset K_i$. Because $\mathrm{Ker}(d) = \mathrm{Im}(d)$, d induces an isomorphism $K_{i+1}/\mathrm{Ker}(d) \longrightarrow \mathrm{Im}(d)$. Choose representatives of this basis in K_{i+1}, and choose a basis of $\mathrm{Ker}(d) \subset K_{i+1}$. Combining these two sets gives a basis of K_{i+1}, and the subset which forms a basis of $\mathrm{Ker}(d)$ is precisely what we need to continue the induction. The induction can be started using the fact that $K_{-1} = 0$.

Recall that we write $L(V)^k$ for the weight-k component of $L(V)$, that is, the submodule generated by brackets of length k in the elements of V, and recall also that weight defines a grading. Note that the differential d preserves weight. The operations τ_k and σ_k satisfy

$$\mathrm{wt}(\tau_k(x)) = p^k \mathrm{wt}(x),$$
$$\mathrm{wt}(\sigma_k(x)) = p^k \mathrm{wt}(x).$$

We will use weight to produce a modified dimension function which makes precise the idea that 'most' of the decomposition of $L(V)$ in Proposition 8.4 consists of the summand K; the summand $HL(V)$ is comparatively small.

Definition 8.6. Let M be a $\mathbb{Z}/p^r$-module, together with a grading wt, which we think of as a weight, such that each weight-component M^i is free and finitely generated. Define $\dim^k(M) \in \mathbb{R}$ by setting

$$\dim^k(M) = \sum_{i=1}^{k} \frac{\dim(M^i)}{i}.$$

It follows immediately from the definition that

$$\dim^k(A \oplus B) = \dim^k(A) + \dim^k(B).$$

We will be concerned with evaluating the functions $\dim^k$ on submodules of the free Lie algebra $L(V)$. We write BM for the module $\mathrm{Im}(d)$ of boundaries in a differential module (M, d).

Lemma 8.7. *Let V be an acyclic differential $\mathbb{Z}/p$-vector space. For all $k \in \mathbb{N}$ we have:*

- $\dim^k(HL(V)) < \frac{1}{p} \dim^k(L(V))$*, and*
- $\dim^k(BL(V)) > \frac{p-1}{2p} \dim^k(L(V))$.

Proof. Decompose $L(V) \cong HL(V) \oplus K$ as in Proposition 8.4, and choose a basis $\{x_\alpha, y_\alpha, z_\beta, w_\beta\}$ of K as in Remark 8.5, where α and β run over indexing sets $\mathscr{I}$ and $\mathscr{J}$ respectively. The differential preserves weight, so by choosing such a basis in each weighted component separately, we may assume that the basis vectors are homogenous in weight. Let S_k be the set of those $\alpha \in \mathscr{I}$ with $\mathrm{wt}(x_\alpha) \leq k$. Proposition

8.4 gives that

$$\dim^k(HL(V)) < \sum_{\alpha\in S_k}\sum_{j=1}^{\infty}\frac{1}{\mathrm{wt}(\tau_j(x_\alpha))}+\frac{1}{\mathrm{wt}(\sigma_j(x_\alpha))} = \sum_{\alpha\in S_k}\sum_{j=1}^{\infty}\frac{1}{p^j\mathrm{wt}(x_\alpha)}+\frac{1}{p^j\mathrm{wt}(x_\alpha)}$$

$$= \sum_{\alpha\in S_k}\frac{2}{\mathrm{wt}(x_\alpha)}\sum_{j=1}^{\infty}\frac{1}{p^j} = \frac{1}{p-1}\sum_{\alpha\in S_k}\frac{2}{\mathrm{wt}(x_\alpha)}.$$

On the other hand, the contribution of the x_α and y_α to the dimension of K gives that

$$\dim^k(K) \geq \sum_{\alpha\in S_k}\frac{2}{\mathrm{wt}(x_\alpha)},$$

so

$$\dim^k(K) > (p-1)\dim^k(HL(V)).$$

Since $L(V) \cong HL(V) \oplus K$, we have that $\dim^k(L(V)) = \dim^k(K) + \dim^k(HL(V))$, so

$$\dim^k(L(V)) > p\dim^k(HL(V)),$$

which proves the first inequality. This also implies that $\dim^k(K) > \frac{p-1}{p}\dim^k(L(V))$, and since K is acyclic, we must have

$$\dim^k(BL(V)) \geq \frac{1}{2}\dim^k(K).$$

Combining these proves the second inequality and completes the proof. □

All we will require for our application is the case when V is the free $\mathbb{Z}/p^r$-module on two generators x and y satisfying $d(x) = y$. In this case we will write $L(x,dx) = L(V)$ and $L'(x,dx) = L'(V)$. Note that $L(x,dx)\otimes\mathbb{Z}/p^s$ is the free Lie algebra on $V\otimes\mathbb{Z}/p^s$.

Lemma 8.8. *Let V be a graded acyclic $\mathbb{Z}/p^r$-module, free and finitely generated in each dimension, of total dimension at least 2. Then the module of boundaries $BL(V)$ is $\mathbb{Z}/p^r$-hyperbolic. In particular, the module of boundaries $BL(x,dx)$ in the free differential Lie algebra $L(x,dx)$ is $\mathbb{Z}/p^r$-hyperbolic.*

Proof. Since $L(V)\otimes\mathbb{Z}/p$ is the free Lie algebra over $\mathbb{Z}/p$ on $V\otimes\mathbb{Z}/p$, by Lemma 7.10 applied to the differential d it suffices to prove the $r=1$ case, for which we can use Proposition 8.4, in the guise of Lemma 8.7.

By Lemma 8.7, we know that

$$\dim^k(BL(V)) > \frac{p-1}{2p}\dim^k(L(V)).$$

Thus,

$$\sum_{i=1}^{k} \dim(BL(V)^i) \geq \sum_{i=1}^{k} \frac{\dim(BL(V)^i)}{i} > \frac{p-1}{2p} \sum_{i=1}^{k} \frac{\dim(L(V)^i)}{i} \geq \frac{p-1}{2p} \sum_{i=1}^{k} \frac{\dim(L(V)^i)}{k}.$$

Let n be the maximum i for which $V_i \neq 0$. The leftmost term is equal to $\dim(\bigoplus_{i=1}^{k} BL(V)^i)$, and $BL(V)^i \subset L(V)^i \subset L(V)_{ni}$, so we have

$$\dim(\bigoplus_{j=1}^{nk} BL(V)_j) > \frac{p-1}{2pk} \sum_{i=1}^{k} \dim(L(V)^i) \geq \frac{p-1}{2pk} \dim(L(V)^k) = \frac{p-1}{2pk} W_\ell(k),$$

by Theorem 8.3, where we let $\ell = \dim(V)$, so

$$\dim(\bigoplus_{i=1}^{k} BL(V)_i) > \frac{p-1}{2p\lfloor \frac{k}{n} \rfloor} W_\ell(\lfloor \frac{k}{n} \rfloor) \sim \frac{p-1}{2p\lfloor \frac{k}{n} \rfloor^2} \ell^{\lfloor \frac{k}{n} \rfloor}$$

by Lemma 3.2. Now, ℓ is assumed greater than 1, so

$$\frac{p-1}{2p\lfloor \frac{k}{n} \rfloor^2} \ell^{\lfloor \frac{k}{n} \rfloor} \geq \frac{p-1}{2p(\frac{k}{n})^2} \ell^{\frac{k}{n}-1},$$

so for any $\varepsilon > 0$, once k is large enough we have $\dim(\bigoplus_{i=1}^{k} BL(V)_i) > (\ell^{\frac{1}{n}} - \varepsilon)^k$. That is, $\dim(\bigoplus_{i=1}^{k} BL(V)_i)$ grows faster than an exponential in any base smaller than $\ell^{\frac{1}{n}}$. In particular, if $\dim(V) = \ell \geq 2$, then $BL(V)$ is $\mathbb{Z}/p$-hyperbolic, as required. □

Since $\theta : L'(V) \longrightarrow L(V)$ is surjective and commutes with d, we immediately obtain the following corollary.

Corollary 8.9. *The submodule* $\mathrm{Im}(\theta \circ d)$ *in the free differential Lie algebra* $L(x, dx)$ *is* $\mathbb{Z}/p^r$*-hyperbolic.* □

9 Loop-homology of Moore spaces

In this section we will study the question 'what part of $H_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^r)$ can be shown to consist of Hurewicz images?' The answer is 'the module of boundaries in a differential sub-Lie algebra isomorphic to $L(x, dx)$'. In Section 8 we have seen that such a module is $\mathbb{Z}/p^r$-hyperbolic. The hypotheses of Theorem 1.5 are really conditions under which the image of this module under the map $(\Omega\mu)_*$ remains $\mathbb{Z}/p^s$-hyperbolic, and we thus obtain a $\mathbb{Z}/p^s$-hyperbolic submodule of the image of the Hurewicz map.

We follow the notation from Neisendorfer's book [23] . Let p be a prime and let $s \leq r \in \mathbb{N}$. For a space Y, recall that the *homotopy groups of Y with coefficients in $\mathbb{Z}/p^s$*, denoted $\pi_n(Y;\mathbb{Z}/p^s)$ are the based homotopy sets $[P^n(p^s), Y]$, which are groups for $n \geq 3$. There are a number of useful operations relating the integral and mod-p^s homotopy groups, which we introduce next.

Let $\underline{\beta}^s : S^{n-1} \longrightarrow P^n(p^s)$ be the inclusion from the cofibration sequence of Definition 1.2. This defines a map of degree -1

$$\beta^s : \pi_n(Y;\mathbb{Z}/p^s) \longrightarrow \pi_{n-1}(Y)$$

$$f \longmapsto f \circ \underline{\beta}^s.$$

Similarly, let $\underline{\rho}^s : P^n(p^s) \longrightarrow S^n$ be the pinch map, which is obtained by extending the cofibration sequence of Definition 1.2 to the right. This defines a map of degree 0

$$\rho^s : \pi_n(Y) \longrightarrow \pi_n(Y;\mathbb{Z}/p^s)$$

$$f \longmapsto f \circ \underline{\rho}^s.$$

Lastly, let $\underline{\mathrm{red}}^{r,s} : P^n(p^s) \longrightarrow P^n(p^r)$ be the map defined by the diagram of cofibrations

$$\begin{tikzcd}
S^{n-1} \arrow[r, "p^s"] \arrow[d, equal] & S^{n-1} \arrow[r] \arrow[d, "p^{r-s}"] & P^n(p^s) \arrow[d, "\underline{\mathrm{red}}^{r,s}"] \\
S^{n-1} \arrow[r, "p^r"] & S^{n-1} \arrow[r] & P^n(p^r),
\end{tikzcd}$$

and let

$$\mathrm{red}^{r,s} : \pi_n(Y;\mathbb{Z}/p^r) \longrightarrow \pi_n(Y;\mathbb{Z}/p^s)$$

$$f \longmapsto f \circ \underline{\mathrm{red}}^{r,s}.$$

It follows from the definitions that β^s, ρ^s and $\mathrm{red}^{r,s}$ are all natural in Y.

We will now use these operations to produce elements u and v of $\pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s)$. The Hurewicz images of v and u will play the roles of the elements x and dx of Section 8. Although these elements are easily described in terms of things we already have, we will give them new names for clarity.

Let

$$v' : P^n(p^s) \longrightarrow P^n(p^r)$$

be equal to $\underline{\mathrm{red}}^{r,s}$.

Let

$$u' : P^{n-1}(p^s) \longrightarrow P^n(p^r)$$

be the composite

$$P^{n-1}(p^s) \xrightarrow{\rho^s} S^{n-1} \xrightarrow{\beta^r} P^n(p^r).$$

Recall that for any space X there is a natural map $\eta : X \longrightarrow \Omega\Sigma X$, which is the unit of the adjunction $\Sigma \dashv \Omega$ and sends $x \in X$ to the loop $\gamma_x = (t \longmapsto \langle t, x\rangle)$ on ΣX. Let $v = \eta \circ v' : P^n(p^s) \longrightarrow \Omega P^{n+1}(p^r)$, and let $u = \eta \circ u' : P^{n-1}(p^s) \longrightarrow \Omega P^{n+1}(p^r)$.

Now let G be an H-group, and suppose that the prime p is odd. As in the integral setting, the homotopy groups with coefficients $\pi_*(G; \mathbb{Z}/p^s)$ carry a *Samelson product*; a bilinear operation which resembles a Lie bracket [12] . Loop spaces are in particular H-groups, so we have Samelson products in $\pi_*(\Omega X; \mathbb{Z}/p^s)$ for any X.

Lemma 9.1. *Let p be an odd prime. The map*

$$\pi_*(\Omega X; \mathbb{Z}/p^s) \xrightarrow{\beta^s} \pi_*(\Omega X) \xrightarrow{\rho^s} \pi_*(\Omega X; \mathbb{Z}/p^s)$$

is a differential (that is, $(\rho^s \circ \beta^s)^2 = 0$) of degree -1, which satisfies the Leibniz identity relative to Samelson products.

Proof. By [12, Section 7] , we have the Leibniz identity. To see that it is a differential, note that $\beta^s \circ \rho^s = 0$, so $(\rho^s \circ \beta^s)^2 = \rho^s \circ (\beta^s \circ \rho^s) \circ \beta^s = 0$. $\square$

By construction of u and v we have $(\rho^s \circ \beta^s)(v) = p^{r-s}u$ in $\pi_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s)$. Let $L'(x, dx)$ and $L(x, dx)$ be as in Section 8, where we let $\deg(x) = n$ and $\deg(y) = n - 1$. Let $\langle x, dx\rangle$ be the free graded $\mathbb{Z}/p^r$-module of dimension 2 on basis $\{x, dx\}$, so that $L(x, dx) = L(\langle x, dx\rangle)$, and with this notation note that $L'(x, dx) \otimes \mathbb{Z}/p^s = L'(\langle x, dx\rangle \otimes \mathbb{Z}/p^s)$, the analogous construction over $\mathbb{Z}/p^s$.

We define a map of $\mathbb{Z}/p^s$-modules $\phi_\pi^{r,s} : \langle x, dx\rangle \otimes \mathbb{Z}/p^s \longrightarrow \pi_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s)$ by sending $x \longmapsto v$ and $dx \longmapsto u$. Samelson products in $\pi_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s)$ are bilinear, so by the universal property of $L'(x, dx) \otimes \mathbb{Z}/p^s$, $\phi_\pi^{r,s}$ extends to a map

$$\Phi_\pi^{r,s} : L'(x, dx) \otimes \mathbb{Z}/p^s \longrightarrow \pi_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s)$$

of graded (nonassociative) $\mathbb{Z}/p^s$-algebras.

The following lemma relates $\Phi_\pi^{r,s}$ to $\Phi_\pi^{r,r}$.

Lemma 9.2. *If $s \leq r$ then $p^{r-s}\Phi_\pi^{r,s} \circ d = \rho^s \circ \beta^s \circ \Phi_\pi^{r,s}$. In particular, if $s = r$, then $\Phi_\pi^{r,s} = \Phi_\pi^{r,r}$ is a map of differential Lie algebras.*

Proof. It suffices to show that the composites $p^{r-s}\Phi_\pi^{r,s} \circ d$ and $\rho^s \circ \beta^s \circ \Phi_\pi^{r,s}$ agree on brackets of length k in $L'(x, dx) \otimes \mathbb{Z}/p^s$ for each $k \in \mathbb{N}$. We will do this by induction.

In the case $k = 1$, the restriction of $\Phi^{r,s}_\pi$ to brackets of length 1 is $\phi^{r,s}_\pi$. By construction of u and v we have $(\rho^s \circ \beta^s)(v) = p^{r-s}u$ in $\pi_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s)$, so $\phi^{r,s}_\pi$ satisfies $p^{r-s}\phi^{r,s}_\pi \circ d = \rho^s \circ \beta^s \circ \phi^{r,s}_\pi$, as required.

Now let $a \in L'(x, dx) \otimes \mathbb{Z}/p^s$ be a bracket of length $k > 1$. We have $a = [b, c]$ for brackets b, c of lengths i and j respectively with $i + j = k$, $i < k$, $j < k$. Thus

$$\rho^s \circ \beta^s \circ \Phi^{r,s}_\pi(a) = \rho^s \circ \beta^s \circ \Phi^{r,s}_\pi([b,c]) = \rho^s \circ \beta^s([\Phi^{r,s}_\pi(b), \Phi^{r,s}_\pi(c)])$$

$$= [\rho^s \circ \beta^s \circ \Phi^{r,s}_\pi(b), \Phi^{r,s}_\pi(c)] + (-1)^{\deg b}[\Phi^{r,s}_\pi(b), \rho^s \circ \beta^s \circ \Phi^{r,s}_\pi(c)],$$

where the last equality is by Lemma 9.1. By induction we have $\rho^s \circ \beta^s \circ \Phi^{r,s}_\pi(b) = p^{r-s}\Phi^{r,s}_\pi \circ d(b)$ and $\rho^s \circ \beta^s \circ \Phi^{r,s}_\pi(c) = p^{r-s}\Phi^{r,s}_\pi \circ d(c)$, so the above is equal to

$$[p^{r-s}\Phi^{r,s}_\pi \circ d(b), \Phi^{r,s}_\pi(c)] + (-1)^{\deg b}[\Phi^{r,s}_\pi(b), p^{r-s}\Phi^{r,s}_\pi \circ d(c)]$$

$$= p^{r-s}\Phi^{r,s}_\pi([d(b), c] + (-1)^{\deg b}[b, d(c)]) = p^{r-s}\Phi^{r,s}_\pi \circ d([b,c]).$$

This completes the induction, and hence the proof. □

Lemma 9.2 identifies a factor of p^{r-s}. The next lemma makes precise the idea that this factor comes from the map β^s, rather than the map ρ^s, by relating each $\Phi^{r,s}_\pi$ to $\Phi^{r,r}_\pi$.

Lemma 9.3. *The following diagram commutes:*

$$\begin{tikzcd}
L'(x,dx) \arrow[r, "\Phi^{r,r}_\pi"] \arrow[dd, "d"'] & \pi_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^r) \arrow[d, "\beta^r"] \\
 & \pi_*(\Omega P^{n+1}(p^r)) \arrow[d, "\rho^s"] \\
L'(x,dx) \otimes \mathbb{Z}/p^s \arrow[r, "\Phi^{r,s}_\pi"] & \pi_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s).
\end{tikzcd}$$

In particular, $\mathrm{Im}(\rho^s) \supset \mathrm{Im}(\Phi^{r,s}_\pi \circ d)$.

Proof. By Lemma 9.2, the top face of the following diagram commutes, and the bottom face commutes up to a factor of p^{r-s}, in the sense that $p^{r-s}\Phi^{r,s}_\pi \circ d = \rho^s \circ \beta^s \circ \Phi^{r,s}_\pi$:

$$\begin{tikzcd}
 & & L'(x,dx) \arrow[dll, "d"'] \arrow[d, "\text{quotient}"] \arrow[drr, "\Phi^{r,r}_\pi"] & & \\
L'(x,dx) \arrow[dd, "\text{quotient}"] \arrow[ddrr, "\Phi^{r,r}_\pi"' near start] & & L'(x,dx)\otimes\mathbb{Z}/p^s \arrow[ddll, "d" near start] \arrow[dr, "\Phi^{r,s}_\pi"] & & \pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^r) \arrow[dl, "\beta^r"] \arrow[dd, "\mathrm{red}^{r,s}"] \\
 & & & \pi_*(\Omega P^{n+1}(p^r)) \arrow[dl, "\rho^r"] \arrow[dd, equal] & \\
L'(x,dx)\otimes\mathbb{Z}/p^s \arrow[ddrr, "\Phi^{r,s}_\pi"] & & \pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^r) \arrow[dd, "\mathrm{red}^{r,s}"] & & \pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s) \arrow[dl, "\beta^s"] \\
 & & & \pi_*(\Omega P^{n+1}(p^r)) \arrow[dl, "\rho^s"] & \\
 & & \pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s). & &
\end{tikzcd}$$

Commutativity of the back left face is clear. We now check commutativity of the front left and back right faces, which are identical. Since the reduction map red is a map of Lie algebras, both composites are maps of nonassociative algebras, and by the uniqueness part of the universal property of $L'(x,dx)$, it suffices to show that the restrictions to $\langle x, dx\rangle$ agree, and this is easily seen by direct calculation.

We now turn to the front right face. The square involving ρ^s commutes, since the composite

$$P^m(p^s) \xrightarrow{\mathrm{red}^{r,s}} P^m(p^r) \xrightarrow{\underline{\rho}^r} S^m$$

is equal to $\underline{\rho}^s : P^m(p^s) \longrightarrow S^m$. For the square involving β^s, we have that the composite

$$S^{m-1} \xrightarrow{\underline{\beta}^s} P^m(p^s) \xrightarrow{\mathrm{red}^{r,s}} P^m(p^s)$$

is equal to $p^{r-s}\underline{\beta}^r : S^{m-1} \longrightarrow P^m(p^r)$.

Putting all of this together, we have that

$$\Phi^{r,s}_\pi \circ d \circ \text{quotient} = \mathrm{red}^{r,s} \circ \Phi^{r,r}_\pi \circ d = \mathrm{red}^{r,s} \circ \rho^r \circ \beta^r \circ \Phi^{r,r}_\pi = \rho^s \circ \beta^s \circ \Phi^{r,r}_\pi,$$

as required. $\square$

Let $s \leq r$. The homology $\widetilde{H}_*(P^m(p^r);\mathbb{Z}/p^s)$ is free over $\mathbb{Z}/p^s$; in particular we have

$$\widetilde{H}_i(P^m(p^r);\mathbb{Z}/p^s) = \begin{cases} \mathbb{Z}/p^s & i = m, m-1, \\ 0 & \text{otherwise.} \end{cases}$$

Write e_m for a choice of generator of $H_m(P^m(p^r);\mathbb{Z}/p^s)$, and $s_{m-1} = \beta(e_m)$, where β is the homology Bockstein. The group $H_{m-1}(P^m(p^r);\mathbb{Z}/p^s)$ is generated by s_{m-1}.

The Pontrjagin product makes $\widetilde{H}_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s)$ into a $\mathbb{Z}/p^s$-algebra. Any graded associative algebra carries a Lie bracket, defined by setting $[x,y] = xy - (-1)^{\deg(x)\deg(y)}yx$, and this is what will be meant by 'the bracket on $H_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s)$'.

Recall that an element of $\pi_m(Y;\mathbb{Z}/p^r)$ is a homotopy class of maps $P^m(p^r) \longrightarrow Y$. Let $h : \pi_*(Y;\mathbb{Z}/p^s) \longrightarrow H_*(Y;\mathbb{Z}/p^s)$ be the *Hurewicz map*, which sends $f \in \pi_*(Y;\mathbb{Z}/p^s)$ to $f_*(e_m) \in H_*(Y;\mathbb{Z}/p^s)$. By [12, Proposition 6.4] , the generators e_m may be chosen so that h carries Samelson products to commutators; that is, so that $h([f,g]) = [h(f), h(g)] \in H_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s)$.

Thus, the composition $h \circ \Phi_\pi^{r,s}$ respects brackets, and the codomain, $H_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s)$ carries a Lie algebra structure. We therefore obtain a factorization of $h \circ \Phi_\pi^{r,s}$ through θ to give a map of Lie algebras $\Phi_H^{r,s}$ which satisfies the following lemma:

Lemma 9.4. *The following diagram commutes:*

$$\begin{array}{ccc} L'(x,dx) \otimes \mathbb{Z}/p^s & \xrightarrow{\Phi_\pi^{r,s}} & \pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s) \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle h} \\ L(x,dx) \otimes \mathbb{Z}/p^s & \xrightarrow{\Phi_H^{r,s}} & H_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s). \end{array}$$

□

9.1 Tensor algebras and the Bott-Samelson Theorem

The purpose of this section is to introduce some notation for dealing with tensor algebras, and to recall the Bott-Samelson Theorem (Theorem 9.5). We define the *tensor algebra* on a graded R-module V to be $T(V) = \bigoplus_{k=1}^{\infty} V^{\otimes k}$, where $V^{\otimes k}$ is the tensor product of k copies of V. Note in particular that this definition is 'reduced' since we do not insert a copy of R in degree 0. The multiplication is given by concatenation of tensors, and makes $T(V)$ into the free graded associative algebra on V. Let A be an algebra and let $\varphi : V \longrightarrow A$ be a homomorphism. We write $\widetilde{\varphi} : T(V) \longrightarrow A$ for the map of algebras induced by φ. Let

$$\iota_i : V^{\otimes i} \longrightarrow T(V)$$

be the inclusion, and let

$$\zeta_i : T(V) \longrightarrow V^{\otimes i}$$

be the projection.

Bott and Samelson first proved their theorem in [9] ; we give the formulation from Selick's book [27] .

Theorem 9.5 (Bott-Samelson)**.** *Let R be a PID, and let X be a connected space with $\widetilde{H}_*(X;R)$ free over R. Then $\widetilde{H}_*(\Omega\Sigma X;R) \cong T(\widetilde{H}_*(X;R))$ and $\eta : X \longrightarrow \Omega\Sigma X$ induces the canonical map $\widetilde{H}_*(X;R) \longrightarrow T(\widetilde{H}_*(X;R))$.*

The Bott-Samelson Theorem immediately allows us to find a free Lie algebra in the loop-homology of a Moore space.

Lemma 9.6. *The map $\Phi_H^{r,s} : L(x,dx) \otimes \mathbb{Z}/p^s \longrightarrow H_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s)$ is an injection.*

Proof. Since $r \geq s$, the module $H_*(P^n(p^r);\mathbb{Z}/p^s)$ is free over $\mathbb{Z}/p^s$. By the Bott-Samelson Theorem 9.5, $\widetilde{H}_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s) \cong T(x,dx) \otimes \mathbb{Z}/p^s$, and this isomorphism identifies $\Phi_H^{r,s}$ with the natural map $L(x,dx) \otimes \mathbb{Z}/p^s \longrightarrow T(x,dx) \otimes \mathbb{Z}/p^s$. But this latter map is an injection by Proposition 2.9 and Corollary 2.7 of [12] . □

We have the following corollary, which will be the main ingredient in the proof of Theorem 1.5.

Corollary 9.7. *Let Y be a simply connected CW-complex, let p be an odd prime, and let $r \in \mathbb{N}$. Let $\mu : P^{n+1}(p^r) \longrightarrow Y$ be a continuous map. If the induced map*

$$(\Omega\mu)_* : H_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s) \longrightarrow H_*(\Omega Y;\mathbb{Z}/p^s)$$

is an injection, then the module $\mathrm{Im}((\Omega\mu)_* \circ \Phi_H^{r,s} \circ \theta \circ d)$ is $\mathbb{Z}/p^s$-hyperbolic. □

Proof. By Lemma 9.6, $\Phi_H^{r,s}$ is an injection, and by Corollary 8.9, the module $\mathrm{Im}(\theta \circ d)$ is $\mathbb{Z}/p^s$-hyperbolic. It follows that $(\Omega\mu)_* \circ \Phi_H^{r,s}(\mathrm{Im}(\theta \circ d)) = \mathrm{Im}((\Omega\mu)_* \circ \Phi_H^{r,s} \circ \theta \circ d)$ is also $\mathbb{Z}/p^s$-hyperbolic. □

10 The suspension case

The purpose of this section is to show that Theorem 1.5 implies Theorem 1.6. This will be accomplished by means of Proposition 10.12, whose proof is the goal of this section. The main point is that even if $\widetilde{H}_*(X;\mathbb{Z}/p^s)$ is not free over $\mathbb{Z}/p^s$, the canonical map of the Bott-Samelson Theorem (Theorem 9.5) is still an injection. That is, the homology $\widetilde{H}_*(\Omega\Sigma X;\mathbb{Z}/p^s)$ always contains the tensor algebra on $\widetilde{H}_*(X;\mathbb{Z}/p^s)$, but if $\widetilde{H}_*(X;\mathbb{Z}/p^s)$ is not free then it will contain other things too.

In Subsection 10.1, we recall the James splitting $\Sigma\Omega\Sigma X \simeq \bigvee_{k=1}^{\infty}\Sigma X^{\wedge k}$. This gives us Proposition 10.5, which describes the structure of the Pontrjagin algebra $\widetilde{H}_*(\Omega\Sigma X;\mathbb{Z}/p^s)$, in particular identifying the tensor algebra $T(\widetilde{H}_*(X;\mathbb{Z}/p^s))$ as a subalgebra. Subsection 10.2 proves Lemma 10.9, which describes the effect of a certain evaluation map on $H_*(\Omega\Sigma X;\mathbb{Z}/p^s)$. Subsection 10.3 draws these ingredients together to prove Proposition 10.12.

Let $\sigma : \widetilde{H}_*(Y) \longrightarrow \widetilde{H}_{*+1}(\Sigma Y)$ denote the suspension isomorphism. For a space X, let X^k denote the product of k copies of X, and let $X^{\wedge k}$ denote the smash product. Let $\sim$ be the relation on X^k defined by

$$(x_1,\ldots,x_{i-1},*,x_{i+1},x_{i+2},\ldots x_k) \sim (x_1,\ldots,x_{i-1},x_{i+1},*,x_{i+2},\ldots x_k).$$

Let $J_k(X)$ be the space ${X^k}/{\sim}$. There is a natural inclusion

$$J_k(X) \longrightarrow J_{k+1}(X)$$

$$(x_1,\ldots,x_k) \mapsto (x_1,\ldots,x_k,*).$$

The *James construction* JX is defined to be the colimit of the diagram consisting of the spaces $J_k(X)$ and the above inclusions. Notice that JX carries a product given by concatenation, which makes it into the free topological monoid on X, and that a topological monoid is in particular an H-space.

The adjunction isomorphism $[\Sigma X, Y] \cong [X, \Omega Y]$ will be written in both directions as $f \longmapsto \overline{f}$. We write η for the unit of the adjunction, which is the map $X \longrightarrow \Omega\Sigma X$ sending $x \in X$ to $(t \mapsto \langle t, x\rangle) \in \Omega\Sigma X$. We will write ev for the *evaluation map*; the counit $\Sigma\Omega Y \longrightarrow Y$, which sends $\langle t,\gamma\rangle \in \Sigma\Omega Y$ to $\gamma(t) \in Y$.

10.1 The tensor algebra inside $H_*(\Omega\Sigma X)$

In this section we will generalise the Bott-Samelson theorem to suit our purpose. Specifically, the map $\eta : X \longrightarrow \Omega\Sigma X$ induces a map $\eta_* : \widetilde{H}_*(X) \longrightarrow \widetilde{H}_*(\Omega\Sigma X)$ on

homology. By the universal property of the tensor algebra, η_* extends to a map of algebras
$$\widetilde{\eta}_* : T(\widetilde{H}_*(X)) \longrightarrow \widetilde{H}_*(\Omega\Sigma X).$$

The Bott-Samelson Theorem (Theorem 9.5) says that if the homology $H_*(X;\mathbb{Z}/p^s)$ is free then $\widetilde{\eta}_*$ is an isomorphism. We will show that even if $H_*(X;\mathbb{Z}/p^s)$ is not free, the map $\widetilde{\eta}_*$ is still an injection. This is by no means new, but follows reasonably easily from better-known results, so we shall derive it in this way. In this section homology is taken with $\mathbb{Z}/p^s$-coefficients (unless otherwise stated).

Lemma 10.1. *The cross product map $\widetilde{H}_*(X)^{\otimes k} \xrightarrow{\times} \widetilde{H}_*(X^{\wedge k})$ is injective, split (although not naturally) and its cokernel C satisfies $p^{s-1}C = 0$.*

Proof. For spaces A and B The Künneth Theorem gives an exact sequence

$$0 \longrightarrow H_*(A) \otimes H_*(B) \xrightarrow{\times} H_*(A \times B) \longrightarrow \operatorname{Tor}(H_*(A), H_{*-1}(B)) \longrightarrow 0,$$

where the Tor is taken over $\mathbb{Z}/p^s$, and this sequence is (unnaturally) split. By Lemma 7.11 we have $p^{s-1}\operatorname{Tor}(H_*(A), H_{*-1}(B)) = 0$.

Let $a_0 : pt \longrightarrow A$ denote the inclusion of the basepoint of A and let b_0 denote the inclusion of the basepoint of B. Let $j : H_*(A) \oplus H_*(B) \longrightarrow H_*(A) \otimes H_*(B)$ be the composite

$$\begin{tikzcd}
H_*(A) \oplus H_*(B) \arrow[d, "\cong"] \arrow[dr, "j"] & \\
H_*(A) \otimes H_*(pt) \oplus H_*(pt) \otimes H_*(B) \arrow[r, "(id_A)_*\otimes(b_0)_*\oplus(a_0)_*\otimes(id_B)_*"'] & H_*(A) \otimes H_*(B).
\end{tikzcd}$$

To relate the reduced and unreduced situations we have the following diagram (which we take to define the reduced cross product) where i, i_1 and i_2 are the inclusions and p is the quotient.

$$\begin{tikzcd}
0 \arrow[r] & H_*(A) \oplus H_*(B) \arrow[r, "(i_1)_*\oplus(i_2)_*"] \arrow[d, hook, "j"] & H_*(A \vee B) \arrow[r] \arrow[d, hook, "i_*"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_*(A) \otimes H_*(B) \arrow[r, "\times"] \arrow[d] & H_*(A \times B) \arrow[r] \arrow[d, "p_*"] & \operatorname{Tor}(H_*(A), H_{*-1}(B)) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{H}_*(A) \otimes \widetilde{H}_*(B) \arrow[r, "\times"] & \widetilde{H}_*(A \wedge B) \arrow[r] & \operatorname{Tor}(H_*(A), H_{*-1}(B)) \arrow[r] & 0.
\end{tikzcd}$$

The top map is an isomorphism, so the bottom row is exact, and it therefore suffices to check that the top two squares commute. We first check that the top left square commutes. It suffices to check commutativity on each summand of the domain individually. We will do so for $H_*(A)$; the case of $H_*(B)$ is analogous. Identifying $H_*(A)$ with $H_*(A) \otimes H_*(pt)$, the restriction of j becomes $(id_A)_* \otimes (b_0)_*$. The composite with the cross product is written $(id_A)_* \times (b_0)_*$, and by bilinearity of cross product this is the same as $(id_A \times b_0)_*$, where now the product is taken in spaces. But under the identification $A \cong A \times \{pt\}$, this is just the inclusion $A \longrightarrow A \times B$, which is the map obtained by going the other way round the square, as required.

The top right square commutes because the map $(i_1)_* \oplus (i_2)_*$ is an isomorphism, so, by commutativity of the top left square, the composite of i_* with the map into the Tor term factors through two terms of an exact sequence, hence is zero, as required.

To finish, we note that since the middle row is split, the bottom row is also split. □

The understanding of the cross product from Lemma 10.1 allows us to understand part of the homology of JX, by constructing a map φ as in the following lemma.

Lemma 10.2. *The maps*

$$\widetilde{H}_*(X)^{\otimes k} \longrightarrow H_*(X)^{\otimes k} \xrightarrow{\times} H_*(X^k) \longrightarrow H_*(J_k(X)) \longrightarrow H_*(J(X))$$

define an injection of algebras $T(\widetilde{H}_*(X)) \xrightarrow{\varphi} \widetilde{H}_*(J(X))$. *Furthermore,* $\mathrm{Im}(\varphi)$ *is a direct summand, and we may write* $\widetilde{H}_*(J(X)) \cong T(\widetilde{H}_*(X)) \oplus C$ *such that the complementary module* C *satisfies* $p^{s-1}C = 0$.

Proof. We use a modified version of the argument in [16, Proposition 3C.8] . First, φ is a ring homomorphism, because the product in $J(X)$ descends from the natural map $X^i \times X^j \longrightarrow X^{i+j}$. To see that we have an injection, we consider the following diagram, where we follow Hatcher's notation and set $T_k(M) = \bigoplus_{i=1}^k M^{\otimes i}$:

$$\begin{tikzcd}
0 \arrow[r] & T_{k-1}(\widetilde{H}_*(X)) \arrow[r] \arrow[d, "\varphi"] & T_k(\widetilde{H}_*(X)) \arrow[r] \arrow[d, "\varphi"] & (\widetilde{H}_*(X))^{\otimes k} \arrow[r] \arrow[d, "\times"] & 0 \\
0 \arrow[r] & \widetilde{H}_*(J_{k-1}(X)) \arrow[r] & \widetilde{H}_*(J_k(X)) \arrow[r] & \widetilde{H}_*(X^{\wedge k}) \arrow[r] & 0.
\end{tikzcd}$$

Commutativity of the diagram follows from the definition of φ. Exactness of the top row is clear. The bottom row is obtained from the long exact sequence of the pair $(J_k(X), J_{k-1}(X))$, applying excision to pass to the quotient $J_k(X)/J_{k-1}(X) \simeq X^{\wedge k}$. This sequence is split because the quotient $X^k \longrightarrow X^{\wedge k}$ factors through the map $J_k(X) \longrightarrow X^{\wedge k}$, and the former map is split after suspending. Thus we get that $\widetilde{H}_*(J_k(X)) \cong \widetilde{H}_*(J_{k-1}(X)) \oplus \widetilde{H}_*(X^{\wedge k})$. Lemma 10.1 tells us that $\widetilde{H}_*(X^{\wedge k}) \cong (\widetilde{H}_*(X))^{\otimes k} \oplus C$ with $p^{s-1}C = 0$, so the result follows immediately by inducting over k. □

Our next job is to translate this understanding of JX into an understanding of $\Omega\Sigma X$. It is well-known that the two are homotopy equivalent, but we wish to be precise about the maps. For a based space Y, let $\Omega' Y$ denote the space of loops of any length in Y, so that ΩY is the subspace of $\Omega' Y$ consisting of loops of length 1. We will write $\gamma_1 \# \gamma_2$ for the concatenation of loops γ_1 and γ_2. For $\gamma \in \Omega' Y$ and $\ell \in \mathbb{R}_{>0}$, let γ^ℓ denote the linear reparameterization of γ which has length ℓ. Note that $\gamma \longmapsto \gamma^1$ is a continuous map $\Omega' Y \longrightarrow \Omega Y$, which is a retraction for the inclusion $\Omega Y \subset \Omega' Y$. For $x \in X$, let $\gamma_x \in \Omega\Sigma X$ be the loop defined by $\gamma_x(t) = \langle t, x\rangle$, which is equal to $\eta(x)$.

Now let X be a connected CW-complex, which we take without loss of generality to have a single 0-cell, which is the basepoint. Let $d : X \longrightarrow [0,1]$ be any continuous map such that $d^{-1}(0) = \{*\}$. Define a map

$$\lambda : J(X) \longrightarrow \Omega\Sigma X$$

$$(x_1, \dots x_k) \longmapsto (\gamma_{x_1}^{d(x_1)} \# \gamma_{x_2}^{d(x_2)} \# \dots \# \gamma_{x_k}^{d(x_k)})^1.$$

The reparameterization is necessary so that λ is well-defined when some $x_i = *$.

Hatcher proves the following as [16, Theorem 4J.1] .

Lemma 10.3. *The map λ is a weak homotopy equivalence for any connected CW-complex X. Furthermore, λ is an H-map, so it induces a map of algebras on homology.* □

The following lemma is immediate from the definition of λ.

Lemma 10.4. *The composite*

$$X^k \to J_k(X) \to J(X) \xrightarrow{\lambda} \Omega\Sigma X$$

is homotopy equivalent to $m \circ \eta^k$, where m is any choice of k-fold loop multiplication on $\Omega\Sigma X$. □

Recall from Subsection 9.1 that $\iota_k : V^{\otimes k} \longrightarrow T(V)$ is the inclusion.

We are now ready to prove the main result of this subsection, which is what we will use later.

Proposition 10.5. *The map*

$$\widetilde{\eta}_* : T(\widetilde{H}_*(X)) \longrightarrow H_*(\Omega\Sigma X)$$

is an injection onto a summand, each restriction $\widetilde{\eta}_* \circ \iota_k$ is equal to

$$\widetilde{H}_*(X)^{\otimes k} \xrightarrow{\times} \widetilde{H}_*(X^k) \xrightarrow{(\eta^k)_*} \widetilde{H}_*((\Omega\Sigma X)^k) \xrightarrow{m_*} \widetilde{H}_*(\Omega\Sigma X),$$

and we may write

$$H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X)) \oplus C$$

such that the complementary module C satisfies $p^{s-1}C = 0$.

Proof. By Lemmas 10.2, 10.3 and 10.4, it suffices to show that $\lambda_* \circ \varphi = \widetilde{\eta}_*$. Since both maps are algebra maps, by the universal property of the tensor algebra it further suffices to show that the composite

$$\widetilde{H}_*(X) \xrightarrow{\iota_1} T(\widetilde{H}_*(X)) \xrightarrow{\varphi} \widetilde{H}_*(JX) \xrightarrow{\lambda_*} \widetilde{H}_*(\Omega\Sigma X)$$

is equal to η_*.

To see this, first note that the composite $\widetilde{H}_*(X) \xrightarrow{\iota_1} T(\widetilde{H}_*(X)) \xrightarrow{\varphi} \widetilde{H}_*(JX)$ is equal to the map induced by the inclusion $X \longrightarrow J_1(X) \subset J(X)$ which carries $x \in X$ to the equivalence class of x in $J(X)$. By definition of λ we then have $\lambda(x) = \gamma_x$, which by definition is $\eta(x)$, as required. $\square$

10.2 The effect of the evaluation map

The goal of this section is to prove Lemma 10.9, which says that up to suspension isomorphisms, the evaluation map $\mathrm{ev} : \Sigma\Omega\Sigma X \longrightarrow \Sigma X$ induces the projection onto the tensors of length 1. Our strategy is to first prove Lemma 10.6, the point of which is that when one evaluates a concatenation of k loops at some time t, the result only depends on one of the loops - this is the i appearing in the proof of Lemma 10.6. We will then see that this, together with simple formal properties of the cross product, is enough to prove Lemma 10.9.

In this section, for a co-H-space Y, $c : Y \longrightarrow Y \vee Y$ denotes the comultiplication, and for a product $\prod_{i=1}^k X_i$, the map π_i is the projection onto the i-th factor.

Lemma 10.6. *The following diagram commutes up to homotopy.*

$$\begin{tikzcd}
\Sigma X^k \arrow[r, "\Sigma\eta^k"] \arrow[d, "c"] & \Sigma(\Omega\Sigma X)^k \arrow[r, "\Sigma m"] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}"] \\
(\Sigma X^k)^{\vee k} \arrow[d, "\bigvee_{i=1}^k \Sigma\pi_i"] & & \Sigma X \\
(\Sigma X)^{\vee k} \arrow[r, "\mathrm{fold}"] & \Sigma X \arrow[r, "\Sigma\eta"] & \Sigma\Omega\Sigma X. \arrow[u, "\mathrm{ev}"']
\end{tikzcd}$$

Proof. It suffices to show that the diagram commutes strictly if we take c and m to both be parameterized so as to spend equal time on each component. We will do so by evaluating both composites explicitly.

A point of ΣX^k may be written in suspension coordinates as $\langle t, x_1, x_2, \ldots, x_k\rangle$, for $t \in I$ and $x_i \in X$. There exists some integer i with $1 \leq i \leq k$ so that $\frac{i-1}{k} \leq t \leq \frac{i}{k}$.

For the top right composite,

$$\begin{aligned}\mathrm{ev} \circ \Sigma m \circ \Sigma \eta^k \langle t, x_1, \ldots, x_k\rangle &= \mathrm{ev}\langle t, m(\gamma_{x_1}, \ldots, \gamma_{x_k})\rangle \\ &= (\gamma_{x_1} \# \ldots \# \gamma_{x_k})(t) \\ &= \gamma_{x_i}(kt - (i-1)).\end{aligned}$$

For the bottom left composite, we first introduce some notation. For a point y of a space Y, we write $(y)_i$ for the image of y under the inclusion of the i-th wedge summand in $Y \longrightarrow Y^{\vee k}$. With this notation, taking $Y = \Sigma X^k$, we have $c\langle t, x_1, \ldots, x_k\rangle = (\langle kt - (i-1), x_1, \ldots, x_k\rangle)_i$. Therefore,

$$\mathrm{ev} \circ \Sigma\eta \circ \mathrm{fold} \circ (\bigvee_{i=1}^{k} \Sigma\pi_i) \circ c\langle t, x_1, \ldots, x_k\rangle = \mathrm{ev} \circ \Sigma\eta \circ \mathrm{fold}((\langle kt - (i-1), x_i\rangle)_i$$

$$= \mathrm{ev} \circ \Sigma\eta\langle kt - (i-1), x_i\rangle = \gamma_{x_i}(kt - (i-1)),$$

as required. □

Lemma 10.7. *Let X be a space. The composite*

$$\widetilde{H}_*(X)^{\otimes k} \xrightarrow{\times} \widetilde{H}_*(X^k) \xrightarrow{(\pi_i)_*} \widetilde{H}_*(X)$$

of the cross product with any projection is trivial for $k \geq 2$.

Proof. Up to homeomorphism, X may be regarded as the space $\prod_{j=1}^k Y_j$, where $Y_j = *$ for $j \neq i$ and $Y_i = X$. Under this identification, π_i is identified with the map $\prod_{j=1}^k f_j : X^k \longrightarrow \prod_{j=1}^k Y_j$, where f_j is the identity on X when $j = i$, and is the trivial map otherwise.

The composite of maps $(\prod_{j=1}^k f_j)_* \circ \times$ is the cross product of homomorphisms $(f_1)_* \times (f_2)_* \times \cdots \times (f_k)_*$. Cross product of homomorphisms is k-multilinear, and since $k \geq 2$ there is at least one j with f_j equal to the constant map, hence $(f_j)_* = 0$. This means that $(\prod_{j=1}^k f_j)_* \circ \times$ is trivial for $k \geq 2$, as required. □

Corollary 10.8. *Let X be a space. The composite*

$$\widetilde{H}_*(X)^{\otimes k} \xrightarrow{\times} \widetilde{H}_*(X^k) \xrightarrow{\sigma} \widetilde{H}_*(\Sigma X^k) \xrightarrow{c_*} \widetilde{H}_*((\Sigma X^k)^{\vee k}) \xrightarrow{(\bigvee_{i=1}^k \Sigma\pi_i)_*} \widetilde{H}_*((\Sigma X)^{\vee k})$$

is trivial for $k \geq 2$.

Proof. For a space Y, let $p_i : Y^{\vee k} \longrightarrow Y$ be the projection onto the i-th wedge summand. The comultiplication c satisfies $p_i \circ c \simeq id_{\Sigma X^k}$ for each i, so on homology we have

$$c_* : \widetilde{H}_*(\Sigma X^k) \longrightarrow \widetilde{H}_*((\Sigma X^k)^{\vee k}) \cong \bigoplus_{i=1}^{k} \widetilde{H}_*(\Sigma X^k)$$

$$x \longmapsto (x, x, \dots, x).$$

That is, c_* may be identified with the diagonal map $\Delta : \widetilde{H}_*(\Sigma X^k) \longrightarrow \bigoplus_{i=1}^{k} \widetilde{H}_*(\Sigma X^k)$.

Thus,

$$(\bigvee_{i=1}^{k} \Sigma\pi_i)_* \circ c_* \circ \sigma \circ \times(x_1 \otimes \cdots \otimes x_k) = (\bigvee_{i=1}^{k} \Sigma\pi_i)_* \circ c_*(\sigma(x_1 \times \cdots \times x_k))$$

$$= \bigoplus_{i=1}^{k} (\Sigma\pi_i)_* \circ \Delta(\sigma(x_1 \times \cdots \times x_k)) = 0,$$

since by Lemma 10.7 we have

$$(\Sigma\pi_i)_*(\sigma(x_1 \times \cdots \times x_k)) = \sigma \circ (\pi_i)_*(x_1 \times \cdots \times x_k) = 0.$$

This completes the proof. □

Lemma 10.9. *The composite*

$$T(\widetilde{H}_*(X)) \xrightarrow{\widetilde{\eta}_*} \widetilde{H}_*(\Omega\Sigma X) \xrightarrow{\sigma} \widetilde{H}_*(\Sigma\Omega\Sigma X) \xrightarrow{\mathrm{ev}_*} \widetilde{H}_*(\Sigma X) \xrightarrow{\sigma^{-1}} \widetilde{H}_*(X)$$

is equal to the projection ζ_1.

Proof. Write Γ for the above composite. We must show that $\Gamma \circ \iota_k$ is the identity map on $H_*(X)$ when $k = 1$, and is 0 otherwise.

For the $k = 1$ statement, note that $\widetilde{\eta}_* \circ \iota_1 = \eta_*$ (this is the definition of $\widetilde{\eta}_*$). We may therefore write

$$\Gamma \circ \iota_1 = \sigma^{-1} \circ \mathrm{ev}_* \circ \sigma \circ \widetilde{\eta}_* \circ \iota_1 = \sigma^{-1} \circ \mathrm{ev}_* \circ \sigma \circ \eta_* = \sigma^{-1} \circ \mathrm{ev}_* \circ (\Sigma\eta)_* \circ \sigma,$$

and by the triangle identities for the adjunction $\Sigma \dashv \Omega$ we have a commuting diagram

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma\eta"] \arrow[dr, "id_{\Sigma X}"'] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}"] \\ & \Sigma X. \end{tikzcd}$$

Thus, $\Gamma \circ \iota_1 = \sigma^{-1} \circ \sigma = id_{H_*(X)}$, as we required.

Now let $k > 1$. Juxtaposing the diagram of Lemma 10.6 (after taking homology) with the result of Corollary 10.8 gives a commuting diagram

$$\begin{tikzcd}
\widetilde{H}_*(X)^{\otimes k} \arrow[r, "\times"] \arrow[ddrr, "0"] & \widetilde{H}_*(X^k) \arrow[r, "\sigma"] & \widetilde{H}_*(\Sigma X^k) \arrow[r, "(\Sigma\eta^k)_*"] \arrow[d, "c_*"] & \widetilde{H}_*(\Sigma(\Omega\Sigma X)^k) \arrow[r, "(\Sigma m)_*"] & \widetilde{H}_*(\Sigma\Omega\Sigma X) \arrow[d, "\mathrm{ev}_*"] \\
& & \widetilde{H}_*((\Sigma X^k)^{\vee k}) \arrow[d, "(\bigvee_{i=1}^k \Sigma\pi_i)_*"] & & \widetilde{H}_*(\Sigma X) \\
& & \widetilde{H}_*((\Sigma X)^{\vee k}) \arrow[r, "\mathrm{fold}_*"] & \widetilde{H}_*(\Sigma X) \arrow[r, "(\Sigma\eta)_*"] & \widetilde{H}_*(\Sigma\Omega\Sigma X). \arrow[u, "\mathrm{ev}_*"']
\end{tikzcd}$$

The description of $\widetilde{\eta}_* \circ \iota_k$ of Proposition 10.5 implies that the top-right route round the diagram is equal to $\sigma \circ \Gamma \circ \iota_k$. The diagram shows that this factors through the zero map, so $\sigma \circ \Gamma \circ \iota_k = 0$, and since σ is an isomorphism, this implies that $\Gamma \circ \iota_k$ is itself zero, which completes the $k > 1$ case and hence the proof. $\square$

10.3 Loops on homology injections

The goal of this section is to prove Proposition 10.12. We first prove two lemmas. Recall that $\iota_i : V^{\otimes i} \longrightarrow T(V)$ is the inclusion, and that $\zeta_i : T(V) \longrightarrow V^{\otimes i}$ is the projection. Similarly, let $\iota_{\leq k}$ and $\zeta_{\leq k}$ be respectively the inclusion and projection associated to the submodule $\bigoplus_{i=1}^k V^{\otimes i}$ of $T(V)$.

Lemma 10.10. *Let $a_1, a_2, \ldots a_k$ be elements of a tensor algebra $T(V)$. We have that*

$$\zeta_i(a_1 \otimes \cdots \otimes a_k) = \begin{cases} \zeta_1(a_1) \otimes \cdots \otimes \zeta_1(a_k) & i = k \\ 0 & i < k. \end{cases}$$

Proof. The multiplication in $T(V)$ repects weight, so we have the formula $\zeta_k(a \otimes b) = \sum_{i=1}^{k-1} \zeta_i(a) \otimes \zeta_{k-i}(b)$, which we will use to induct.

When $k = 1$ the result is automatic. Assuming the result for $k-1$, we have

$$\zeta_j(a_1 \otimes \cdots \otimes a_k) = \sum_{i=1}^{j-1} \zeta_i(a_1 \otimes \cdots \otimes a_{k-1}) \otimes \zeta_{j-i}(a_k).$$

By induction $\zeta_i(a_1 \otimes \cdots \otimes a_{k-1}) = 0$ for $i < k-1$, so the above is 0 when $j < k$ and when $j = k$ it becomes

$$\zeta_{k-1}(a_1 \otimes \cdots \otimes a_{k-1}) \otimes \zeta_1(a_k) = \zeta_1(a_1) \otimes \cdots \otimes \zeta_1(a_{k-1}) \otimes \zeta_1(a_k),$$

by induction, as required. $\square$

The following lemma does not depend on the algebra structure in the tensor algebras; only on the fact that tensor algebras are graded by weight. Nonetheless, we will state it only for tensor algebras because we already have the necessary notation. It formalizes the sort of 'leading terms' argument that we wish to make in proving Proposition 10.12.

Lemma 10.11. *Let $f : T(A) \longrightarrow T(B)$ be a homomorphism of $\mathbb{Z}/p^s$-modules (not necessarily of algebras) with A free. Suppose that $p^{s-1}\zeta_j \circ f \circ \iota_k = 0$ whenever $j < k$ and that for each $k \in \mathbb{N}$, the map $\zeta_k \circ f \circ \iota_k$ is an injection. Then f is also an injection.*

Proof. Firstly, since $T(A)$ is a free $\mathbb{Z}/p^s$-module, it suffices to show that if $f(p^{s-1}x) = 0$, for $x \in T(A)$, then $p^{s-1}x = 0$. This is precisely showing injectivity of the restriction of f to $p^{s-1}T(A)$. The module $T(A)$ is filtered by the submodules $\bigoplus_{i=1}^k A^{\otimes i}$ for $k \in \mathbb{N}$, so it further suffices to show that each map

$$\zeta_{\leq k} \circ f \circ \iota_{\leq k} : p^{s-1} \bigoplus_{i=1}^{k} A^{\otimes i} \longrightarrow p^{s-1} \bigoplus_{i=1}^{k} B^{\otimes i}$$

is injective.

We proceed by induction. The case $k = 1$ is immediate, so assume that the result is known for $k - 1$. Write $\bigoplus_{i=1}^k A^{\otimes i} \cong \bigoplus_{i=1}^{k-1} A^{\otimes i} \oplus A^{\otimes k}$, so that $\iota_{\leq k}$ is identified with $\iota_{\leq (k-1)} \oplus \iota_k$. Suppose that $f(y) = 0$ for $y \in p^{s-1} \bigoplus_{i=1}^k A^{\otimes i}$, so that there exists $x \in \bigoplus_{i=1}^k A^{\otimes i}$ with $y = p^{s-1}x$. We must show that $y = 0$. Write $x = x' + x_k$, for $x' \in \bigoplus_{i=1}^{k-1} A^{\otimes i}$ and $x_k \in A^{\otimes k}$. Now,

$$\zeta_{\leq (k-1)} \circ f \circ \iota_{\leq k}(y) = p^{s-1}\zeta_{\leq (k-1)} \circ f \circ \iota_{\leq k}(x)$$

$$= p^{s-1}\zeta_{\leq (k-1)} \circ f(\iota_{\leq (k-1)}x' + \iota_k(x_k)) = \zeta_{\leq (k-1)} \circ f(p^{s-1}\iota_{\leq (k-1)}x'),$$

since $p^{s-1}\zeta_j \circ f \circ \iota_k = 0$ for $j < k$. By inductive hypothesis, this implies that $p^{s-1}x' = 0$, so $y = x_k$, and

$$\zeta_k \circ f \circ \iota_{\leq k}(y) = \zeta_k \circ f \circ \iota_k(p^{s-1}x_k).$$

By assumption, $\zeta_k \circ f \circ \iota_k$ is an injection, so $p^{s-1}x_k = 0$, and therefore $y = 0$, as required. □

Proposition 10.12. *Let X be a connected CW-complex, let p be an odd prime, and let $s \leq r \in \mathbb{N}$. Let $\mu : P^{n+1}(p^r) \longrightarrow \Sigma X$ be a continuous map. If the induced map*

$$\mu_* : H_*(P^{n+1}(p^r); \mathbb{Z}/p^s) \longrightarrow H_*(\Sigma X; \mathbb{Z}/p^s)$$

is an injection, then

$$(\Omega\mu)_* : H_*(\Omega P^{n+1}(p^r); \mathbb{Z}/p^s) \longrightarrow H_*(\Omega\Sigma X; \mathbb{Z}/p^s)$$

is also an injection.

The principal difficulty in the proof is that $\mathrm{Im}(\mu_*)$ might not be contained in the tensor algebra $T(\widetilde{H}_*(X;\mathbb{Z}/p^s))$ inside $\widetilde{H}_*(\Omega\Sigma X;\mathbb{Z}/p^s)$. We deal with this using the condition $p^{s-1}C = 0$ of Proposition 10.5, which prevents the complementary part C from interfering too much. This proposition is much simpler to prove if one assumes that the map μ is a suspension, but this assumption is not necessary.

Proof. Homology is taken with $\mathbb{Z}/p^s$-coefficients throughout. By the Bott-Samelson theorem (Theorem 9.5), we have an isomorphism

$$\widetilde{\eta}_* : T(\widetilde{H}_*(P^n(p^r))) \longrightarrow H_*(\Omega P^{n+1}(p^r)),$$

so it suffices to show that $(\Omega\mu)_* \circ \widetilde{\eta}_*$ is an injection. By definition, $(\Omega\mu)_* \circ \widetilde{\eta}_*$ is the unique map of algebras extending

$$(\Omega\mu)_* \circ \eta_* : \widetilde{H}_*(P^n(p^r)) \longrightarrow H_*(\Omega\Sigma X),$$

and by the triangle identities for the adjunction $\Sigma \dashv \Omega$, we have that $(\Omega\mu) \circ \eta = \overline{\mu}$. Thus, $(\Omega\mu)_* \circ \widetilde{\eta}_*$ is the unique map of algebras extending $\overline{\mu}_*$.

The other triangle identity tells us that we have a commuting diagram

$$\begin{tikzcd}
P^{n+1}(p^r) \arrow[r, "\Sigma\overline{\mu}"] \arrow[dr, "\mu"'] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}"] \\
& \Sigma X.
\end{tikzcd}$$

By assumption, μ induces an injection on homology, so $\mathrm{ev} \circ (\Sigma\overline{\mu})$ must also induce an injection on homology.

The next step is to turn the problem into one about tensor algebras. By Proposition 10.5 we may choose a module decomposition of $\widetilde{H}_*(\Omega\Sigma X)$ as a direct sum $T(\widetilde{H}_*(X)) \oplus C$ with $p^{s-1}C = 0$. Under this decomposition, the inclusion associated to the factor $T(\widetilde{H}_*(X))$ is $\widetilde{\eta}_*$. Write τ for the projection. Consider the diagram

$$\begin{tikzcd}
& T(\widetilde{H}_*(X)) \arrow[d, bend left, "\sigma\circ\widetilde{\eta}_*"] \\
\widetilde{H}_*(P^{n+1}(p^r)) \arrow[r, "(\Sigma\overline{\mu})_*"] \arrow[dr, "\mu_*"'] & \widetilde{H}_*(\Sigma\Omega\Sigma X) \arrow[u, bend left, "\tau\circ\sigma^{-1}"] \arrow[d, "\mathrm{ev}_*"] \\
& \widetilde{H}_*(\Sigma X).
\end{tikzcd}$$

The maps $\sigma \circ \widetilde{\eta}_*$ and $\tau \circ \sigma^{-1}$ differ from $\widetilde{\eta}_*$ and τ only up to suspension isomorphisms, so they are the inclusion and projection associated to the decomposition of $\widetilde{H}_*(\Sigma\Omega\Sigma X)$ obtained by suspending that of Proposition 10.5. Lemma 7.5 (with $g = \mathrm{ev}_*$, $f = (\Sigma\overline{\mu})_*$, $i_A = \sigma \circ \widetilde{\eta}_*$, and $\pi_A = \tau \circ \sigma^{-1}$) then tells us that the whole composite $\mathrm{ev}_* \circ (\sigma \circ \widetilde{\eta}_*) \circ (\tau \circ \sigma^{-1}) \circ (\Sigma\overline{\mu})_*$ is an injection. Furthermore, by Lemma 10.9, the composite $\mathrm{ev}_* \circ (\sigma \circ \widetilde{\eta}_*)$ is identified via suspension isomorphisms with the projection $\zeta_1 : T(\widetilde{H}_*(X)) \longrightarrow \widetilde{H}_*(X)$, so the composite $\zeta_1 \circ \tau \circ \overline{\mu}_*$ is an injection.

Let a and b form a basis of the free $\mathbb{Z}/p^s$-module $\widetilde{H}_*(P^n(p^r))$. By Lemma 7.4, the images of a and b under $\zeta_1 \circ \tau \circ \overline{\mu}_*$ generate a summand isomorphic to $(\mathbb{Z}/p^s)^2$ inside $\widetilde{H}_*(X)$.

Since $\widetilde{H}_*(P^n(p^r))$ is free on a and b, a basis of $T(\widetilde{H}_*(P^n(p^r)))$ consists of the elements $x_1 \otimes \cdots \otimes x_k$, for $k \in \mathbb{N}$, where each x_i is equal to a or b. We will show that the image of this basis under $(\Omega\mu)_* \circ \widetilde{\eta}_*$ is the basis of a free $\mathbb{Z}/p^s$-submodule of $H_*(\Omega\Sigma X)$, which will imply the result. Firstly, since $(\Omega\mu)_* \circ \widetilde{\eta}_*$ is the unique map of algebras extending $\overline{\mu}_*$, we have

$$\begin{aligned} p^{s-1}\zeta_j \circ \tau \circ (\Omega\mu)_* \circ \widetilde{\eta}_*(x_1 \otimes \cdots \otimes x_k) &= p^{s-1}\zeta_j \circ \tau(\overline{\mu}_*(x_1) \otimes \cdots \otimes \overline{\mu}_*(x_k)) \\ &= \zeta_j \circ \tau(p^{s-1}(\overline{\mu}_*(x_1) \otimes \cdots \otimes \overline{\mu}_*(x_k))). \end{aligned}$$

By Proposition 10.5, we may write each $\overline{\mu}_*(x_i)$ as $\widetilde{\eta}_*(t_i) + c_i$, for $t_i = \tau(\overline{\mu}_*(x_i)) \in T(\widetilde{H}_*(X))$ and some c_i with $p^{s-1}c_i = 0$. The above is therefore equal to

$$\begin{aligned} \zeta_j \circ \tau(p^{s-1}((\widetilde{\eta}_*(t_1) + c_1) \otimes \cdots \otimes (\widetilde{\eta}_*(t_k) + c_k)) &= \zeta_j \circ \tau(p^{s-1}(\widetilde{\eta}_*(t_1) \otimes \cdots \otimes \widetilde{\eta}_*(t_k))) \\ = \zeta_j(p^{s-1}t_1 \otimes \cdots \otimes t_k) &= \begin{cases} p^{s-1}\zeta_1(t_1) \otimes \cdots \otimes \zeta_1(t_k) & j = k \\ 0 & j < k \end{cases} \end{aligned}$$

by Lemma 10.10. Since $t_i = \tau(\overline{\mu}_*(x_i))$, we have

$$p^{s-1}\zeta_1(t_1) \otimes \cdots \otimes \zeta_1(t_k) = p^{s-1}(\zeta_1 \circ \tau(\overline{\mu}_*(x_1))) \otimes \cdots \otimes (\zeta_1 \circ \tau(\overline{\mu}_*(x_k))).$$

Now, each x_i is equal to a or b, and we have seen that the images of a and b under $\zeta_1 \circ \tau \circ \overline{\mu}_*$ generate a $(\mathbb{Z}/p^s)^2$-summand inside $\widetilde{H}_*(X)$. It follows that the elements $\zeta_1 \circ \tau(\overline{\mu}_*(x_1)) \otimes \cdots \otimes \zeta_1 \circ \tau(\overline{\mu}_*(x_k))$ generate a copy of $T((\mathbb{Z}/p^s)^2)$ inside $T(\widetilde{H}_*(X))$.

The above calculation therefore tells us that the map $\zeta_k \circ \tau \circ (\Omega\mu)_* \circ \widetilde{\eta}_* \circ \iota_k$ carries p^{s-1} times a basis of $\widetilde{H}_*(P^n(p^r))^{\otimes k} \subset T(\widetilde{H}_*(P^n(p^r)))$ to p^{s-1} times a basis of $((\mathbb{Z}/p^s)^2)^{\otimes k} \subset T((\mathbb{Z}/p^s)^2)$ inside $T(\widetilde{H}_*(X))$. This implies that the restriction of $\zeta_k \circ \tau \circ (\Omega\mu)_* \circ \widetilde{\eta}_* \circ \iota_k$ to $p^{s-1}\widetilde{H}_*(P^n(p^r))^{\otimes k}$ is an injection, so $\zeta_k \circ \tau \circ (\Omega\mu)_* \circ \widetilde{\eta}_* \circ \iota_k$

must itself be an injection and we have also seen that $p^{s-1}\zeta_j \circ \tau \circ (\Omega\mu)_* \circ \widetilde{\eta}_* \circ \iota_k = 0$ for $j < k$

Thus, by Lemma 10.11, $\tau \circ (\Omega\mu)_* \circ \widetilde{\eta}_*$ is an injection, so $(\Omega\mu)_* \circ \widetilde{\eta}_*$ is an injection, as required. □

11 Proof of Theorems 1.5 and 1.6

In this section we will prove Theorem 1.5, and then from that, together with Proposition 10.12, deduce Theorem 1.6.

Proof of Theorem 1.5. By Lemma 7.13 it suffices to prove the theorem when $t = s$. Combining Lemmas 9.3, 9.4, and naturality of the maps β^r, ρ^s, and h with respect to the map of spaces $\Omega\mu$, we obtain the following commuting diagram:

$$\begin{tikzcd}
L'(x,dx) \arrow[r, "\Phi_\pi^{r,r}"] \arrow[dd, "d"] & \pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^r) \arrow[r, "(\Omega\mu)_*"] \arrow[d, "\beta^r"] & \pi_*(\Omega Y;\mathbb{Z}/p^r) \arrow[d, "\beta^r"] \\
 & \pi_*(\Omega P^{n+1}(p^r)) \arrow[r, "(\Omega\mu)_*"] \arrow[d, "\rho^s"] & \pi_*(\Omega Y) \arrow[d, "\rho^s"] \\
L'(x,dx)\otimes\mathbb{Z}/p^s \arrow[r, "\Phi_\pi^{r,s}"] \arrow[d, "\theta"] & \pi_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s) \arrow[r, "(\Omega\mu)_*"] \arrow[d, "h"] & \pi_*(\Omega Y;\mathbb{Z}/p^s) \arrow[d, "h"] \\
L(x,dx)\otimes\mathbb{Z}/p^s \arrow[r, "\Phi_H^{r,s}"] & H_*(\Omega P^{n+1}(p^r);\mathbb{Z}/p^s) \arrow[r, "(\Omega\mu)_*"] & H_*(\Omega Y;\mathbb{Z}/p^s).
\end{tikzcd}$$

By Corollary 9.7, $\mathrm{Im}((\Omega\mu)_* \circ \Phi_H^{r,s} \circ \theta \circ d)$ is $\mathbb{Z}/p^s$-hyperbolic. By commutativity of the diagram, $(\Omega\mu)_* \circ \Phi_H^{r,s} \circ \theta \circ d = h \circ \rho^s \circ (\Omega\mu)_* \circ \beta^r \circ \Phi_\pi^{r,r}$, so the image of the latter map is also $\mathbb{Z}/p^s$-hyperbolic.

We thus obtain a diagram

$$\begin{tikzcd}
\pi_*(\Omega Y) \arrow[dr, "h\circ\rho^s"] & \\
\mathrm{Im}((\Omega\mu)_* \circ \beta^r \circ \Phi_\pi^{r,r}) \arrow[u, hook] \arrow[r, "h\circ\rho^s"] & \mathrm{Im}((h\circ\rho^s)\circ((\Omega\mu)_* \circ \beta^r \circ \Phi_\pi^{r,r})).
\end{tikzcd}$$

The bottom map is a surjection by choice of codomain, and we have shown above that this codomain is $\mathbb{Z}/p^s$-hyperbolic. The domain of $(\Omega\mu)_* \circ \beta^r \circ \Phi_\pi^{r,r}$ is $L'(x,dx)$, which is a $\mathbb{Z}/p^r$-module, hence is automatically annihilated by multiplication by p^r. Therefore, the group in the bottom left, $\mathrm{Im}((\Omega\mu)_* \circ \beta^r \circ \Phi_\pi)$, is also annihilated by multiplication by p^r. The group in the bottom right, $\mathrm{Im}((\Omega\mu)_* \circ \beta^r \circ \Phi_\pi)$, is contained in $H_*(\Omega Y;\mathbb{Z}/p^s)$, hence is annihilated by multiplication by p^s. This means that we can apply Corollary 7.9 (The 'Sandwich' Lemma) to see that $\pi_*(\Omega Y) \cong \pi_{*+1}(Y)$ is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$, so by definition Y is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$, which completes the proof. □

Theorem 1.6 now follows.

Proof of Theorem 1.6. By Proposition 10.12, $(\Omega\mu)_*$ is an injection, so by Theorem 1.5, ΣX is p-hyperbolic concentrated in exponents $s, s+1, \ldots, r$, as required. □

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