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## BCJ Identities in Pure Spinor Superspace

 byElliot Nicholas Bridges

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ABSTRACT<br>FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES<br>Mathematical Sciences<br>Thesis for the degree of Doctor of Philosophy

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This thesis focuses upon work advancing the methods by which scattering amplitudes in string and field theory may be described. Through the pure spinor formalism, rapid developments have been made in this area and numerous new results have been found. Further, when amplitudes are constructed so as to satisfy BCJ identities further simplifications become apparent, and the double copy procedure relating amplitudes in super Yang-Mills and supergravity may be applied. It is this context within which this thesis falls. The work which follows can broadly be split into two categories.

The first relates to the construction of a BCJ gauge, within which amplitudes automatically satisfy BCJ identities. Such has previously been described in a small subset of cases, and we use these results to find general methods for its construction. This is done in terms of a new combinatorial map, which we describe and rigorously prove identities satisfied by it. As a consequence of this, we are then able to relate the BCJ gauge constructed using these methods to arbitrary rank with the standard Lorenz gauge by a finite gauge transformation.

The second set of work relates to the construction of amplitudes at one loop in field theory. We describe a procedure by which one may extract from genus one string correlators their corresponding results in field theory. The results are shown to satisfy BCJ identities automatically. Subtleties related to the symmetries of these results are then discussed, and an overview of why we are unable to then apply the double copy procedure is detailed. This resolves an outstanding problem related to the construction of higher point one loop amplitudes so as to satisfy BCJ identities, and raises new questions related to the application of the double copy in loop amplitudes.
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## Declaration of Authorship

I, Elliot Nicholas Bridges, declare that the thesis entitled BCJ Identities in Pure Spinor Superspace and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted form the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:
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Signed: $\qquad$
Date: $\qquad$

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In memory of my grandmother, Wendy Fox.

## CHAPTER 1

## Introduction

### 1.1 Context

At its core, the goal of physics is to describe why things behave the way they do. For three of the fundamental forces; electromagnetism and the strong and weak nuclear forces, this is done through quantum theory. The fourth, gravity, is described through general relativity. Unfortunately, gravity resists attempts to be quantised; when one attempts to introduce a particle governing it to the standard model, a "graviton", one encounters unrenormalisable results [2]. As such, there must be some other theory with which we may describe both consistently, and the most promising candidate we have for this is string theory $[3 ; 4]$.

In string theory, we replace point particles with one-dimensional objects called strings. These come in two core varieties; open strings, which may be thought of a length of rope and are so called due to their open ends, and closed strings, which resemble loops. In its limits, string theory reproduces quantum theory and general relativity as required. So for instance, by taking the string length to zero, that is looking at the low energy limit, quantum theory is reproduced [5]. A potentially useful image thus arises, with string theory corresponding with taking a point particle and "stretching" it out at higher energies. For technical reasons, the length of a string is described in terms of its square root $\alpha^{\prime}=\sqrt{l_{s}}$. There is the added complication that string theory exists in ten dimensions,
but it is well established that four dimensional results may be recovered with appropriate compactifications [4].

The properties of strings are determined by whether they are open or closed, and how they vibrate. Vibrations may move in either direction along the string, and so are described as left moving or right moving. There are then five different types of superstring theory, each with its own restrictions on which strings they contain $[4 ; 6]$. In type I, both open and closed strings exist. In type II, only closed strings exist, and this is split into type IIA and type IIB based upon having symmetry and antisymmetry between left and right movers respectively. Finally there are two classes of heterotic strings, in which there are only closed strings, and the vibrations are split up such that the left movers describe bosonic strings, while the right movers describe superstrings with symmetry group either $S O(32)$ or $E_{8} \times E_{8}$. These theories may be related to each other by symmetries known as s-duality and t-duality. Together with supergravity introduced shortly, they all represent different limits of an overarching M-theory.

One may wonder how strings correlate with specific particles, and for the purposes of this thesis we need only consider two pairs of simple cases. Gluons and gluinos correspond with the low energy limit of open strings, and gravitons and gravitinos correspond with the low energy limits of closed strings. There are of course more particles than this, in both the standard model and its extensions. These arise from different vibrational modes of these two classes of string, and in some cases interactions with D-branes. However, it is these four particles which the results of this thesis are applied to, and so we focus on them.

Gluons describe the strong nuclear force, and their motion is described with Yang-Mills (YM) theory. So far as anything can be considered such in particle physics, they are well understood; they were experimentally observed in the 1970's, and while there are still a great many calculations to be undertaken involving them, some of which will be done in this thesis, they are in a much better place than the other particles. Gravitons are the force carrying particles for gravity. As previously mentioned, the usual methods to describe such a particle fail here, and so some creativity is required.

Supersymmetry [7] adds new particles to the standard model. These are the superpartners of ordinary particles, with the superpartner of a boson being a fermion and vice versa. Though it has yet to be observed, it is a widely used tool in physics and aids calculations considerably. The superpartner of the gluon is the gluino, and of the graviton is the gravitino. The supersymmetric model of the strong force is super Yang-Mills (SYM) [8], and of gravity is supergravity [9]. Note that multiple forms of these theories exist, depending on the dimension we work in, denoted $D$, and the number of supersymmetry generators allowed, denoted $\mathcal{N}$. This thesis takes place in $D=10$ space, and in such SYM
necessarily has $\mathcal{N}=1$. With appropriate dimensional reduction any of the $\mathcal{N}=1,2,4$ options available in $D=4$ may be recovered [8].

As will be discussed, these two theories are related. The double copy states that, if things are arranged appropriately, results in supergravity may be thought of as the "square" of corresponding results in SYM $[10 ; 11 ; 12]$. We will focus upon the particle physics picture, which has a useful description in string theory. A closed string may be thought of as two open strings with their ends attached together, and so it follows from looking at the field theory limit that a graviton may be thought of as being like two gluons. While the full argument is more subtle than this, and will be detailed later, this serves as a useful image to understand the results. Though we will not discuss it further, we note that such relations follow through to larger scale problems, in what is known as the classical double copy. This has been used to find alternative simpler descriptions of a range of problems in classical relativity, and a review may be found in [12].

Theories in particle physics are tested through scattering amplitude experiments. In these, particles are fired into each other at large energies and the results, how the particles "scatter" off of each other, are measured. This is then compared with what would be expected from the theory. Quantum calculations are by their nature probabilistic, and so one cannot say with certainty that if say particles $X$ and $Y$ collide, a particle $Z$ will be produced. However, one can work out the probability of individual results, and then perform the experiment repeatedly and compare the data set produced with what would be expected. The probability of a particular event is proportional to the square of the modulus of its scattering amplitude.

The calculation of scattering amplitudes is highly non-trivial [13]. The field has been around for many decades, and while significant progress has been made it is far from a solved problem. The traditional approach relies upon a decomposition into Feynman diagrams; graphs describing the motion of particles in the scattering event, each of which corresponds with a complex integral. Suppose we wish to find the probability of two particles colliding, interacting in some way, and two particles emerging on the other side. This would be referred to as a four point amplitude, with other point amplitudes defined similarly. Suppose further that we know that this particular particle may only interact with other particles in a specific way; either two of this particle may combine into one, or one of it may split into two. Then the Feynman diagram decomposition of this interaction would correspond with a sum over all graphs with four external vertices and all other vertices of degree 3 , equivalent to summing over all possible ways two particles can go in and two come out.

There are clearly infinitely many such diagrams, and as such infinitely many calculations. Fortunately though, not all are needed. The tree level Feynman diagrams are those which
are trees in the graph theory sense. Then, all others are described as being loop level diagrams. An n-loop diagram may be thought of as having $n$ distinct circles within it; this may be more formally defined through homology. When it comes to experiments, the majority of the information needed to make predictions comes from the tree level calculations, then the next most information comes from the one-loop diagrams, then the two-loop diagrams, and so on. Similarly, the difficulty of calculation increases with loop order, with tree level calculations simpler than one-loop calculations, which are easier than two-loop calculations, and so on. As such we do not need to consider diagrams to arbitrary loop order, with the first few such being more than sufficient. In fact, nonperturbative effects mean that including diagrams beyond a certain loop order actually makes calculations less accurate, and so cutting off calculations at a certain point makes sense. Clearly though, even if we do not need to perform an infinite number of calculations, a great many must be performed.

One approach to removing this problem lies in string theory, where the number of diagrams is significantly lower. While the path of point particles describes a graph, strings are onedimensional objects and so their paths form surfaces. The sum over graphs with $n$ external vertices in field theory corresponds with the sum over topologically-distinct surfaces with $n$ points removed in string theory. The tree level part of an amplitude in field theory corresponds with surfaces of Euler characteristic 2 in string theory, and similarly the sum over $m$-loop diagrams in field theory corresponds with the sum over Euler characteristic $2-2 m$ diagrams in string theory. There are significantly fewer such surfaces than there are diagrams, and so this would suggest performing the calculation in string theory and then looking at the field theory limit should be an efficient method for amplitude calculations.

Unfortunately, the complication is that each individual calculation in string theory is harder than its equivalent in field theory. Fortunately though, recent breakthroughs have significantly reduced this complexity. In particular, the pure spinor formalism allows for the covariant quantisation of the superstring. That is, while in the Green-Schwartz formalism [14] quantisation requires introducing light cone coordinates $x^{ \pm}=x^{0} \pm x^{9}$ [15; 16], and in the RNS formalism if we attempt to work with the usual $x^{i}$ coordinates Lorenz invariance must be broken [17], in the pure spinor formalism no such issues arise. We will detail this construction in the literature review. Using this approach, string amplitudes at tree level to arbitrary points have been identified [18; 19], as well as oneloop amplitudes to seven points [20;21; 22], and results at two and three loops also [23; 24; 25; 26].

### 1.2 Outline

This thesis is split into three parts, the first of which is a review of literature relevant to this research. This begins with a discussion of the double copy. We give an overview of the general structure of amplitudes in (super) Yang-Mills, and one method by which they have been traditionally computed. We then move on to a very brief discussion of the difficulties associated with calculations in supergravity, and then detail how one may use the double copy to circumvent having to perform these calculations. This concludes with a reformulation of the relations needed for the double copy in terms of combinatorial maps, which will be relevant to the work later in this thesis.

Calculations in this thesis are done in terms of the pure spinor formalism of string theory, and in chapter 3 we detail this. This begins with an outline of a formulation of ten dimensional SYM. We then briefly discuss the origin of the pure spinor formalism, general structural properties of it, and how it may be used in general to perform calculations. This is not intended to be a complete analysis of every detail of the formalism, but rather just to describe the key features needed for this work.

Chapters 4 and 5 discuss the specifics of how these techniques have been applied to construct amplitudes. In the former, we begin by describing how string amplitude calculations may be simplified in terms of objects called multiparticle superfields, and then apply these to construct amplitudes at tree level. In chapter 5, we then discuss several generalisations of these techniques needed at loop level, and very briefly describe how one loop string amplitudes have been constructed to seven points.

We then move on to part II, which begins the original research conducted in the course of my PhD. This part has been published as [27]. We begin with chapter 6 , which details a new combinatorial map which will be used in numerous calculations. Various results relating to this map are then proved mathematically, and we use it to generalise many formulae previously found in the construction of scattering amplitudes.

In chapters 7 and 8 , we generalise procedures by which multiparticle superfields can be constructed such that the resulting amplitudes satisfy BCJ identities. As detailed in the literature review, this was previously known only for a small subset of cases, and using the methods we describe in these chapters this is now known in general. Further, we prove the validity of the arguments presented by showing that they represent a gauge transformation, and so do not affect the physics of the objects being described.

In chapter 9 , we conclude the discussion of this paper. We describe potential directions for future research on this subject, and in some cases outline how the calculations for these would begin. We then have some closing remarks on the field more broadly.

In part III we detail a scheme by which one loop correlators from string theory may be taken, and corresponding amplitudes in field theory extracted. We begin with chapter 10 , in which we first describe a clear set of rules to perform such. These methods are then illustrated with an example, and we verify a consistency relation needed to be satisfied by these rules. Then, we apply these rules, and construct amplitudes up to seven points. Expressions for these may be found at [28], as in some higher point instances they become too complex to state here.

In chapter 11, these amplitudes are then shown to satisfy BCJ identities. That such holds is the key improvement these results have over the previous results of [1], and as such showing these is important. We consider several the relations most likely to fail at six points and so them in detail, and outline how similar procedures are then performed at seven points.

Then in chapter 12, we describe how one may attempt to apply the double copy to these results in order to generate supergravity results. It is unfortunate, but as we detail in this chapter such fails at six and higher points. This failure is a result in and of itself however, and has implications for the criteria necessary for a successful application of the double copy more broadly.

In chapter 13, we then summarise the results of this part and describe a pair of possible directions for future work in the area. This concludes the discussion of one loop field theory amplitudes. We note that the work of part III has been compiled into a paper, [29].

Part IV concludes this thesis, summarising the results within and their implications. Then finally part V contains various appendices; discussing aspects of the notation in this thesis in more detail, providing several formulae which are too large for the main body, and discussing several smaller and incomplete results which have been found.

## Part I

## Literature Review

## CHAPTER 2

## The Double Copy

Super Yang-Mills (SYM) is the theory by which we describe the interactions of gluons, the particles which carry the strong nuclear force, and their supersymmetric partners gluinos. Supergravity on the other hand describes the force carrying particles for gravity, gravitons and gravitinos. Calculations in the former theory are a great deal simpler than the latter, but fortunately the two are inherently linked. Gravitational results may be reformulated as the "square" of those of gauge theories like SYM, as was first identified in specific circumstances by Kawai, Lewellen, and Tye [30], and then generalised considerably by Bern, Carrasco, Johannson [10; 31]. This allows for supergravity amplitudes to be computed far more efficiently.

In order for this link to become apparent, we must construct the amplitudes so as to satisfy BCJ relations. This correspondence is known as the double copy, and it has been applied to other gravitational phenomena also. Its origins lie in amplitudes though, and it is on this that we shall focus.

This chapter will begin with a review of scattering amplitudes in SYM; the general form amplitudes in this theory take, and some of the more prominent means by which they have been calculated historically. I will then briefly discuss amplitudes in supergravity; not in any great detail, but enough so as to illustrate the great difficulty in their calculation. Then I will proceed to the double copy, where I will outline its historic origin, its modern application. This is a large and growing field, and so I will focus only on the areas of it
most relevant to this thesis. However, a more complete review can be found in [12].

### 2.1 Amplitudes in Super Yang-Mills

### 2.1.1 An Overview

In SYM, particles have a property known as colour, which in effect means that particles should be considered to be Lie algebra valued. That is, they behave in a manner similar to that of terms from a Lie algebra. In this case that link is made explicit, with particles being represented by some fields multiplied by some function of the generators of a Lie algebra. As such, we will begin with a brief overview of various aspects of Lie algebras.

### 2.1.1.1 Lie Algebras

We first introduce the concept of a Lie bracket $[\cdot, \cdot]$, which is a function of two variables satisfying anticommutativity and the Jacobi identity,

$$
\begin{gather*}
{[a, b]=-[b, a]}  \tag{2.1.1}\\
{[[a, b], c]+[[b, c], a]+[[c, a], b]=0} \tag{2.1.2}
\end{gather*}
$$

It should be noted that the usual commutator,

$$
\begin{equation*}
[a, b]=a b-b a, \tag{2.1.3}
\end{equation*}
$$

satisfies both of these required relations, and serves as something of a trivial example.

A Lie monomial is then any function described by repeated application of the Lie bracket. This is most easily understood with a few examples,

$$
\begin{equation*}
[[1,2], 3], \quad[1,[2,3]], \quad[[1,2],[3,[4,5]]] \tag{2.1.4}
\end{equation*}
$$

For our purposes, an element of a Lie algebra is any linear combination of Lie monomials. As with groups, it is common to describe these in terms of generators, and these are denoted $T^{a}$, and satisfy the relation

$$
\begin{equation*}
\left[T^{a}, T^{b}\right]=f^{a b c} T^{c} \tag{2.1.5}
\end{equation*}
$$

where the $f^{a b c}$ are the structure constants of the Lie algebra, some set of values which
describe it. With appropriate choice of basis, the structure constants can be made antisymmetric in all of their indices [32]

$$
\begin{equation*}
f^{a b c}=f^{b c a}=f^{c a b}=-f^{a c b}=-f^{b a c}=-f^{c b a} \tag{2.1.6}
\end{equation*}
$$

### 2.1.1.2 Momentum

We then introduce our notation for denoting momentum. The momentum of a particle $a$ will be denoted $k_{a}^{m}$, with $m$ the vector index, and likewise the momentum of a set of particles $a_{1} \ldots a_{n}$ will be denoted

$$
\begin{equation*}
k_{a_{1} \ldots a_{n}}^{m}=k_{a_{1}}^{m}+\ldots+k_{a_{n}}^{m} . \tag{2.1.7}
\end{equation*}
$$

In this thesis we only consider gluons, gravitons, and their supersymmetric partners, and as all of these particles are massless their momenta will always square to zero

$$
\begin{equation*}
k^{a} \cdot k^{a}=0 \quad \forall a . \tag{2.1.8}
\end{equation*}
$$

Products of momenta are most efficiently written in terms of Mandelstam variables. We describe these with the notation

$$
\begin{equation*}
s_{12 \ldots p}=\frac{1}{2}\left(k^{12 \ldots p} \cdot k^{12 \ldots p}\right)=\sum_{i<j} k^{i} \cdot k^{j} . \tag{2.1.9}
\end{equation*}
$$

### 2.1.1.3 Amplitude Structure

We are now ready to discuss the general structure of a term from an SYM amplitude [33]. The standard description of such is done in terms of a colour piece, a kinematics piece, and a denominator. The colour piece corresponds with the Lie algebra terms; for every vertex we assign a structure constant. The indices of this are assigned by moving clockwise around the vertex, and noting the particle label for an external edge, or a dummy variable to be summed over for an internal edge.


The kinematics piece contains the majority of the information about the amplitude; it is a function of the particles momenta and polarisation vector we use to specify them. These


Figure 2.1.1: A pair of examples of diagrams which may arise in the calculation of SYM amplitudes.
two are combined into a single numerator, and the corresponding denominator is then a product of the square of the momentum passing through each internal edge. Such terms will often be referred to as the poles of the diagram. Note at tree level, this corresponds with a mandelstam variable (2.1.9) for each internal edge, with its labels corresponding to all of the external particles on one side of the edge. By momentum conservation it does not matter which side we choose. At loop level we have the added complication of loop momenta, describing the undetermined momentum of the virtual particles in the internal loops of such diagrams. As these can take any value they are integrated over.

To illustrate all of this, consider the pair of examples of diagrams in an SYM amplitude given in figure 2.1.1. Both of these will have the general form discussed. That is,

$$
\begin{equation*}
\frac{\text { Kinematics } \times \text { Colour }}{\text { Poles }} \tag{2.1.10}
\end{equation*}
$$

We first specify the colour factors. In the first case, we will have a factor $f^{12 a}$ from the leftmost vertex, with the 1 and 2 arising from the external particles, and the $a$ from the internal edge. Then working to the right, we have the factors $f^{a b 5}$ and $f^{b 34}$. Combing these, and doing likewise for the other example, we have colour factors

$$
\begin{equation*}
f^{12 a} f^{a b 5} f^{b 34}, \quad f^{f 1 a} f^{a b c} f^{b 23} f^{c 4 d} f^{d e f} f^{e 56} \tag{2.1.11}
\end{equation*}
$$

In the tree level example, we have two internal edges, and so two poles $s_{12}=s_{345}$ and $s_{125}=s_{34}$. In the one loop example, there is a loop momentum to be integrated over, and we must specify a direction and a specific value of this on one edge. Otherwise it is much the same, and we have four pole terms containing the loop momentum, and two which do not $\left(s_{23}=s_{1456}\right.$ and $\left.s_{1234}=s_{56}\right)$. Specifying the kinematics piece is a much more complex task, and is the focus of much of the work in this thesis, and so we conclude with these amplitude terms having the form

$$
\begin{equation*}
\frac{(\text { Kinematics }) \times f^{12 a} f^{a b 5} f^{b 34}}{s_{12} s_{34}} \tag{2.1.12}
\end{equation*}
$$

$$
\int d^{D} \ell \frac{(\text { Kinematics }) \times f^{f 1 a} f^{a b c} f^{b 23} f^{c 4 d} f^{d e f} f^{e 56}}{s_{23} s_{56} \ell^{2}\left(\ell-k_{1}\right)^{2}\left(\ell-k_{123}\right)^{2}\left(\ell-k_{1234}\right)^{2}}
$$

One loop diagrams are often described in terms of the nature of their loop. So for example, if the loop connects three vertices it would be called a triangle diagram, if it connects four vertices like in figure 2.1.1 it would be called a box diagram, and if it connects $n$ vertices it would be an $n$-gon diagram.

### 2.1.1.4 Colour Decomposition

It is standard convention to discuss amplitudes in terms of their colour decomposition. Including a colour factor in every term adds a significant complication to amplitude construction. Instead, we expand colour factors in terms of their constituent Lie algebra generators using

$$
\begin{equation*}
f^{a b c}=\operatorname{Tr}\left(\left[T^{a}, T^{b}\right] T^{c}\right) \tag{2.1.13}
\end{equation*}
$$

We then group terms based upon traces. At tree level, this corresponds with reexpressing the amplitude as

$$
\begin{equation*}
\mathcal{A}_{n}^{\text {tree }}=\sum_{\sigma \in \operatorname{Perm}(2,3, \ldots, n)} \operatorname{Tr}\left(T^{1} T^{\sigma(2)} \ldots T^{\sigma(n)}\right) A_{n}^{\text {tree }}(1, \sigma(2), \ldots, \sigma(n)) . \tag{2.1.14}
\end{equation*}
$$

Cyclicity of the trace allows us to always have a leading 1. By $\operatorname{Perm}(2,3, \ldots, n)$ we denote the set of permutations of $2,3, \ldots, n$, and we use $\sigma(m)$ we denote an element of $\sigma$ in the natural way. The objects $A(1, \sigma(2), \ldots, \sigma(n))$ are then referred to as being either partial or colour ordered amplitudes. They correspond with a sum over all planar diagrams in which the external particles are ordered $1 \sigma(2) \ldots \sigma(n)$ as we move clockwise around the diagram. We refer to $A(1,2,3, \ldots, n)$ as the canonical ordering. A graph is called planar if no edges pass underneath other edges, or, in mathematical language, if it may be embedded into $\mathbb{R}^{2}$. This discussion follows similarly loop level also.

### 2.1.1.5 Reducing The Number of Calculations Needed

There are a number of results which may be used to simplify amplitudes, and here we discuss just a few of particular importance. First we note that, while SYM can contain both three and four point vertices, we need not consider the four point vertices. The colour term associated with such vertices is a pair of structure constants [33], $f^{a b c} f^{c d e}$, which may be written as the product of the colour factors of a pair of three point vertices. Further, four point vertices do not introduce any extra pole structure, and so by multiplication
by factors like $s_{12} / s_{12}$ we may write them as sums over three point vertex diagrams. As such, four point diagrams can be ignored in our calculations by supposing that they form a component of the three point diagrams instead.

Then we also have that, in maximal SYM, it has been proven that there are no triangle diagrams at one loop [34;35]. Additionally at higher loops, the diagrams with a triangle sub-diagram vanish similarly. This significantly reduces the number of loop diagrams needed to be considered, meaning that for instance the three point one-loop amplitude is known to vanish without calculation, as the only diagram it would contain triangle diagrams. Likewise, at four points and one-loop, this means we need only consider box diagrams, despite the significant number of other diagrams we could construct involving triangles. There is an analogous result to this in string theory also, which is that three point amplitudes vanish at all loop orders $[36 ; 37 ; 38 ; 39]$. Consistency between string and field theory would suggest likewise is true in field theory also.

Finally, the number of linearly independent partial amplitudes may be reduced at $n$ points by a factor of $(n-1)$ using the Kleiss-Kuijf (KK) relations [40; 41]. These we express here in the form [42]

$$
\begin{equation*}
A(X 1 Y, n)=(-1)^{|X|} A(1(\bar{X} \sqcup Y), n) \tag{2.1.15}
\end{equation*}
$$

There are a number of items of notation we need to explain here. We begin by introducing the concept of a word, which is a sequence of letters. This is most easily understood with an example, so, suppose we are given the letters $a$ and $b$. With these we can form five words of length less than 2 ,

$$
\begin{equation*}
\emptyset, a, b, a b, b a \tag{2.1.16}
\end{equation*}
$$

By $\emptyset$, we denote the empty word, consisting of no letters. Longer words we can form with these words involve repeated letters, for instance $a a b b a b b a b a$. Words are generally denoted with capital letters, whereas letters are usually lower case. Given a word $X$, we use $\bar{X}$ to denote the reversal of its ordering, and $|X|$ to denote the number of letters it contains. These are both defined in the natural way, with for example

$$
\begin{equation*}
X=285437619 \quad \Rightarrow \quad \bar{X}=916734582, \quad|X|=9 \tag{2.1.17}
\end{equation*}
$$

The symbol $\amalg$ denotes the shuffle product. This takes two words $X$ and $Y$, and returns all possible words containing every letter of $X$ and $Y$ which also maintain the ordering of each word. This may be thought of as, if $X$ and $Y$ were two sets of cards, then their shuffle product would be the sum over all possible ways they could be combined in a single shuffle. We define this recursively for a pair of words $A=a_{1} \ldots a_{|A|}$ and $B=b_{1} \ldots b_{|B|}$ by [43]

$$
\begin{equation*}
\emptyset \sqcup A=A \sqcup \emptyset=A, \quad A \sqcup B=a_{1}\left(a_{2} \ldots a_{|A|} \sqcup B\right)+b_{1}\left(b_{2} \ldots b_{|B|} \amalg A\right), \tag{2.1.18}
\end{equation*}
$$

However, it is again made much clearer through examples,

$$
\begin{align*}
& 1 \sqcup 234=1234+2134+2314+2341  \tag{2.1.19}\\
& 12 \sqcup 34=1234+1324+3124+1342+3142+3412  \tag{2.1.20}\\
& 123 \sqcup 4=1234+1243+1423+4123 \tag{2.1.21}
\end{align*}
$$

Putting all of this together, we may give one example of a KK relation. Setting $X=23$, $Y=4, n=5$, and using the cyclicity of partial amplitudes, we have

$$
\begin{align*}
A(14523) & =(-1)^{2} A(1,(32 \sqcup 4), 5)  \tag{2.1.22}\\
& =A(1,3,2,4,5)+A(1,3,4,2,5)+A(1,4,3,2,5)
\end{align*}
$$

These relations may be used show that there are at most $(n-2)$ ! linearly independent partial amplitudes. Though the KK relations will not be exploited in this thesis, from a certain point of view the double copy is an extension of them, and so they are worth including here.

### 2.1.2 Traditional Methods of Computation

The focus of a significant portion of this thesis will be upon new techniques for the calculation of amplitudes in SYM. However, some background information on what has come before should be discussed, and here we briefly discuss the concept of Berends-Giele (BG) currents [44]. These were introduced in the 1980's as a recursive method by which tree level amplitudes in (non-supersymmetric) Yang-Mills could be computed. They will serve as the background theory upon which the work discussed in this thesis has generalised.

A Berends-Giele current is an object $J_{P}^{m}$, where $P$ is a word denoting particle labels, and $m$ is a vector index. These are defined recursively by [42]

$$
\begin{equation*}
J_{i}^{m}=e_{i}^{m}, \quad s_{P} J_{P}^{m}=\sum_{X Y=P}\left[J_{X}, J_{Y}\right]^{m}+\sum_{X Y Z=P}\left\{J_{X}, J_{Y}, J_{Z}\right\}^{m} . \tag{2.1.23}
\end{equation*}
$$

The notation $e_{i}^{m}$ denotes the polarisation vector of a particle $i$, and $s_{P}$ again describe mandelstams variables (2.1.9). The brackets are defined as

$$
\begin{align*}
{\left[J_{X}, J_{Y}\right]^{m} } & :=\left(k_{Y} \cdot J_{X}\right) J_{Y}^{m}+\frac{1}{2} k_{x}^{m}\left(J_{X} \cdot J_{Y}\right)-(X \leftrightarrow Y)  \tag{2.1.24}\\
\left\{J_{X}, J_{Y}, J_{Z}\right\}^{m} & :=\left(J_{X} \cdot J_{Z}\right) J_{Y}^{m}-\frac{1}{2}\left(J_{X} \cdot J_{Y}\right) J_{Z}^{m}-\frac{1}{2}\left(J_{Y} \cdot J_{Z}\right) J_{X}^{m} . \tag{2.1.25}
\end{align*}
$$

Finally we must explain the summation notation. The concatenation of two words $A$ and $B$ is the word formed by placing one word after the other. So for example, if $A=123$ and $B=45$, then their concatenation is the word $A B=12345$. A deconcatenation is the
reversal of this process, wherein we split a word into two pieces. In (2.1.23) we are summing over deconcatenations; that is, we are summing over all possible ways of deconcatenating the word. So for example, $\sum_{X Y=1234}$ means to sum over all possible words $X$ and $Y$ such that the concatenation $X Y$ is the word 1234. These words may be empty, though often it will be clear from context if deconcatenations containing empty words should be dropped. This generalises to deconcatenations into multiple words $\sum_{X_{1} \ldots X_{n}=A}$ in the natural way.

Putting all of this together, gluon amplitudes are then a simple product of BG currents $[44]^{1}$,

$$
\begin{equation*}
A_{\text {tree }}^{Y M}(1,2, \ldots, p, p+1)=s_{12 \ldots p} J_{12 \ldots p}^{m} J_{p+1}^{m} \tag{2.1.26}
\end{equation*}
$$

This is proved by induction [44]. We also note that BG currents can be found to satisfy the symmetry relations [45]

$$
\begin{equation*}
k_{P}^{m} J_{P}^{m}=0, \quad J_{A \amalg B}^{m}=0, \quad|A| \neq 0 \neq|B| \tag{2.1.27}
\end{equation*}
$$

These bear similarity with results which will be found for their generalisations going forward. Note $\amalg$ denotes the shuffle product seen in (2.1.18).

To demonstrate these methods, consider the three point amplitude $A_{\text {tree }}^{Y M}(1,2,3)$. We see from (2.1.26) that this may be expressed in terms of BG currents as

$$
\begin{equation*}
A_{\text {tree }}^{Y M}(1,2,3)=s_{12} J_{12}^{m} J_{3}^{m} \tag{2.1.28}
\end{equation*}
$$

Applying the definitions (2.1.23) this becomes

$$
\begin{align*}
A_{\text {tree }}^{Y M}(1,2,3) & =\frac{s_{12}}{s_{12}}\left(\sum_{X Y=12}\left[J_{X}, J_{Y}\right]^{m}+\sum_{X Y Z=12}\left\{J_{X}, J_{Y}, J_{Z}\right\}^{m}\right) e_{3}^{m}  \tag{2.1.29}\\
& =\left[J_{1}, J_{2}\right]^{m} e_{3}^{m} . \tag{2.1.30}
\end{align*}
$$

Note that $J_{P}$ is only defined for $P$ a non-empty word, and so we drop empty words from the summation. This eliminates all terms involving the second class of bracket. Applying the definitions, we then see that this amplitude is given by $^{2}$

$$
\begin{align*}
A_{\text {tree }}^{Y M}(1,2,3)= & \left(\left(k_{2} \cdot J_{1}\right) J_{2}^{m}+\frac{1}{2} k_{1}^{m}\left(J_{1} \cdot J_{2}\right)-(1 \leftrightarrow 2)\right) e_{3}^{m}  \tag{2.1.31}\\
= & \left(k_{2} \cdot e_{1}\right)\left(e_{2} \cdot e_{3}\right)+\frac{1}{2}\left(k_{1} \cdot e_{3}\right)\left(e_{1} \cdot e_{2}\right)  \tag{2.1.32}\\
& -\left(k_{1} \cdot e_{2}\right)\left(e_{1} \cdot e_{3}\right)-\frac{1}{2}\left(k_{2} \cdot e_{3}\right)\left(e_{1} \cdot e_{2}\right)
\end{align*}
$$

[^0]There are of course other methods of computing amplitudes in Yang-Mills theory, upon which considerable work has been performed. The approach of BCFW relations [47; 48] for instance is a particularly prominent one, and is a field of significant research. This however, is less relevant to the work of this thesis. The approach of Berends-Giele currents is considerable in its own right, and it is upon this which we will focus, so other methods will not be discussed here.

### 2.2 Supergravity Amplitudes

The intention of this brief discussion is not to provide a detailed analysis of supergravity amplitudes, but rather a short comparison of them with SYM amplitudes to illustrate the relative difficulty in their computation. The methods described here will not be used elsewhere in this thesis, and are drawn heavily from the equivalent discussion in [12].

Let us compare amplitude calculations in SYM and supergravity, using a traditional Feynman integral approach. In SYM, we have only three point vertices. In the Feynman gauge ${ }^{3}$ these correspond with terms [12]

$$
\begin{equation*}
V_{3 \mu \nu \sigma}^{a b c}\left(k_{1}, k_{2}, k_{3}\right)=g f^{a b c}\left[\left(k_{1}-k_{2}\right)_{\sigma} \eta_{\mu \nu}+\operatorname{cyclic}(1,2,3)\right] . \tag{2.2.1}
\end{equation*}
$$

where the $k_{i}$ are momenta, and $g$ the coupling constant.

We now compare this with the equivalent supergravity vertex. In the de Donder gauge, this is given by $[12 ; 49 ; 50]$

$$
\begin{align*}
G_{3 \mu \rho, \nu \lambda, \sigma \tau}\left(p_{1}, p_{2}, p_{3}\right)=i S y m & -\frac{1}{2} P_{3}\left(p_{1} \cdot p_{2} \eta_{\mu \rho} \eta_{\nu \lambda} \eta_{\sigma \tau}\right)-\frac{1}{2} P_{6}\left(p_{1 \nu} p_{1 \lambda} \eta_{\mu \rho} \eta_{\sigma \tau}\right) \\
& +\frac{1}{2} P_{3}\left(p_{1} \cdot p_{2} \eta_{\mu \nu} \eta_{\rho \lambda} \eta_{\sigma \tau}\right)+P_{6}\left(p_{1} \cdot p_{2} \eta_{\mu \rho} \eta_{\nu \sigma} \eta_{\lambda \tau}\right) \\
& +2 P_{3}\left(p_{1 \nu} p_{1 \tau} \eta_{\mu \rho} \eta_{\lambda \sigma}\right)-P_{3}\left(p_{1 \lambda} p_{2 \mu} \eta_{\rho \nu} \eta_{\sigma \tau}\right)  \tag{2.2.2}\\
& +P_{3}\left(p_{1 \sigma} p_{2 \tau} \eta_{\mu \nu} \eta_{\rho \lambda}\right)+P_{6}\left(p_{1 \sigma} p_{1 \tau} \eta_{\mu \nu} \eta_{\rho \lambda}\right) \\
& +2 P_{6}\left(p_{1 \nu} p_{2 \tau} \eta_{\lambda \mu} \eta_{\rho \sigma}\right)+2 P_{3}\left(p_{1 \nu} p_{2 \mu} \eta_{\lambda \sigma} \eta_{\tau \rho}\right) \\
& \left.-2 P_{3}\left(p_{1} \cdot p_{2} \eta_{\rho \nu} \eta_{\lambda \sigma} \eta_{\tau \mu}\right)\right]
\end{align*}
$$

The Sym denotes the symmeterisation of $\mu$ with $\rho, \nu$ with $\lambda$, and $\sigma$ with $\tau$. The $P_{i}$ are symmeterisations which generate $i$ terms. This is clearly by a wide margin more complicated than the SYM vertex.

[^1]Then, there is the further complication arising from the number of diagrams in each theory. In SYM, as discussed we may consider there to be only the 3 point vertices. In supergravity there is no such constraint, with $n$-point vertices existing for all $n \geq 3$. These higher point vertices are unsurprisingly no less complicated than the three point case, and that they must be included in amplitude calculations increases the number of diagrams considerably.

Given the significant complexity of supergravity amplitudes, it should comes as no surprise that we would like to avoid computing them wherever possible. Fortunately by virtue of the double copy, this is an option, and we now detail this.

### 2.3 The Double Copy

As has just been demonstrated, the calculation of supergravity amplitudes is no small feat. The number of diagrams we have to compute are enormous, and each individual one is an extremely complex calculation. Amplitudes in SYM however involve only a fraction of the number of diagrams, and though their calculation is still far from trivial it is a great deal simpler than the equivalent diagram in supergravity. In this section, I introduce the double copy, which relates these two by expressing supergravity amplitudes as something like the square of those in SYM. This then means that the particularly fearsome supergravity amplitudes become much simpler, arising from friendlier results in SYM. Though we shall in later sections that additional complications can arise which prevent such a link, it has been used to enormous effect in a number of papers and is an extremely useful tool.

This section will begin with a discussion of the historic origin of the double copy, and how it was first identified as a result in string theory. Its later generalisation to field theory will then be discussed, along with some discussion of its current applications.

### 2.3.1 Origins in String Theory

The origin of the double copy can be found in the work of Kawai, Lewellen, and Tye (KLT) [30]. They made the observation that tree level amplitudes for closed strings can be formulated as products of two amplitudes of open strings. For a simple intuition of why such a link may exist, one may observe that if given a pair of open strings, we may attach their respective ends together into a single closed string. By then considering the field theory counterparts of this result, relations between between tree level SYM and supergravity amplitudes were identified. These became known as the KLT relations, and
the first three of these are stated below ${ }^{4}$,

$$
\begin{align*}
\mathcal{M}^{\text {tree }}(1,2,3) & =i A^{\text {tree }}(1,2,3) \tilde{A}^{\text {tree }}(1,2,3)  \tag{2.3.1}\\
\mathcal{M}^{\text {tree }}(1,2,3,4) & =-i s_{12} A^{\text {tree }}(1,2,3,4) \tilde{A}^{\text {tree }}(1,2,4,3)  \tag{2.3.2}\\
\mathcal{M}^{\text {tree }}(1,2,3,4,5) & =i s_{12} s_{45} A^{\text {tree }}(1,2,3,4,5) \tilde{A}^{\text {tree }}(1,3,5,4,2)  \tag{2.3.3}\\
& +i s_{14} s_{25} A^{\text {tree }}(1,4,3,2,5) \tilde{A}^{\text {tree }}(1,3,5,2,4)
\end{align*}
$$

These relations have since been generalised to arbitrary points $[51 ; 52 ; 53 ; 54]$,

$$
\begin{equation*}
\mathcal{M}_{n}^{\text {tree }}=-i \sum_{\sigma, \rho \in \operatorname{Perm}(2, \ldots, m-2)} A_{m}^{\text {tree }}(1, \sigma, m-1, m) S[\sigma \mid \rho] \tilde{A}_{m}^{\text {tree }}(1, \rho, m, m-1) \tag{2.3.4}
\end{equation*}
$$

A new object has been introduced here, $S[\sigma \mid \rho]$. This is known as the KLT matrix or KLT kernel. There are numerous formulae for this, but here we will define it recursively in terms of a generalisation $[52 ; 55]$

$$
\begin{gather*}
S[A \mid B] \equiv S[A \mid B]_{1}  \tag{2.3.5}\\
S[A, j \mid B, j, C]_{i}=2 k_{i B} \cdot k_{j} S[A \mid B, C]_{i}, \quad S[j \mid j]_{i}=s_{i j} \tag{2.3.6}
\end{gather*}
$$

To illustrate, consider the KLT kernel applied to permutations of 23. One instance of such is $S[23 \mid 32]$, given by

$$
\begin{align*}
S[23 \mid 32] & =S[23 \mid 32]_{1}  \tag{2.3.7}\\
& =2\left(k_{1} \cdot k_{3}\right) S[2 \mid 2]_{1} \\
& =2 s_{12} s_{13}
\end{align*}
$$

Likewise the others will be given by

$$
\begin{equation*}
S[23 \mid 23]=2\left(k^{12} \cdot k^{3}\right) s_{12}, \quad S[32 \mid 23]=2 s_{12} s_{13}, \quad S[32 \mid 32]=2\left(k^{13} \cdot k^{2}\right) s_{13} \tag{2.3.8}
\end{equation*}
$$

These may be arranged into a matrix, which is the origin of the "KLT matrix" terminology. We do not do so here though as such will not be used elsewhere this thesis, and because complications arise in doing this when the input words have more than two letters ${ }^{5}$.

Later in the course of this literature review, and then in part III, the elements of the inverse of this matrix will be discussed. These can be computed this directly, without the need for the identification of the full matrix and then finding its inversion. This is done

[^2]in terms of Berends-Giele double currents [56], denoted $\phi_{A \mid B}$, and defined recursively by
\[

$$
\begin{equation*}
\phi_{P \mid Q}=\frac{1}{s_{P}} \sum_{X Y=P} \sum_{A B=Q}\left(\phi_{X \mid A} \phi_{Y \mid B}-(X \leftrightarrow Y)\right), \quad \phi_{i \mid j}=\delta_{i j} . \tag{2.3.9}
\end{equation*}
$$

\]

Elements of the inverse KLT matrix are then given by

$$
\begin{equation*}
S^{-1}[A \mid B]_{i}=\phi_{i A \mid i B} \tag{2.3.10}
\end{equation*}
$$

Mirroring the previous example, we then see that the elements of the inverse KLT matrix corresponding with permutations of 23 are given by

$$
\begin{align*}
& S^{-1}[23 \mid 23]=\frac{1}{s_{12} s_{123}}+\frac{1}{s_{23} s_{123}}, \quad S^{-1}[23 \mid 32]=-\frac{1}{s_{23} s_{123}},  \tag{2.3.11}\\
& S^{-1}[32 \mid 23]=-\frac{1}{s_{23} s_{123}}, \quad S^{-1}[32 \mid 32]=\frac{1}{s_{13} s_{123}}+\frac{1}{s_{23} s_{123}} .
\end{align*}
$$

If one were to assemble these into a matrix, it would be seen that it is the inverse of the KLT matrix up to an overall $1 / 2$ factor, which has been omitted from this formulation for simplicity.

The KLT relations were a significant result. They allowed a means of simplifying tree level amplitudes in supergravity. Unfortunately though, they did not proceed much further beyond this point, and it would be a number of years before loop level supergravity amplitudes could be simplified similarly, and to do so would require a new approach.

### 2.3.2 The Lie Algebraic Structure of Diagrams in Super Yang-Mills

We have already seen that the colour factors of SYM diagrams are described in terms of Lie algebras, but the graphical representation of the diagrams themselves may also be similarly represented. In this section, we shall give some details of this, and introduce some more notation and which will be of significant use in this thesis.

We begin by introducing the concept of a (planar) binary tree, which is a tree in which all vertices are three point. These are drawn with a single "root" splitting into two branches, with these either further splitting or ending in leaves. This is closely related with a similar concept in computer science [57], with an example illustrated in figure 2.3.1.

All tree level diagrams in SYM are binary trees. We may always take an external edge, usually particle $n$, and orient the diagram such that this is our root. Then as SYM amplitudes contain only multiplicity three vertices, we will be able to organise the resulting diagram to match with the form of a binary tree. Further, colour dressed $n$-point tree level amplitudes may instead be represented as the sum over all possible binary trees with


Figure 2.3.1: An example of an (unsorted) binary tree used as a data structure in computer science, and the Feynman diagram we would associate with the structure of this tree.
( $n-1$ ) leaves.

Every binary tree corresponds with a Lie monomial [58; 59]. We start from the root, and then at every vertex insert the left hand side into the left hand side of a Lie bracket, and the right hand side into the right hand side of the Lie bracket. This process we iterate until all leaves have been reached. So for example, the tree in figure 2.3.1 corresponds with

$$
\begin{equation*}
[[[1,2],[3,4]], 5] . \tag{2.3.12}
\end{equation*}
$$

Hence combining results, we see that a tree level amplitude at $n$ points may be regarded equally as a sum over all Lie monomials with $n-1$ entries. These can then be identified algorithmically.

The binary tree map, otherwise known as the b-map, generates the sum of all possible Lie brackets labelled by the letters of an input word. Further, it includes a division by Mandelstams, to describe the poles of the corresponding Feynman diagrams. It is recursively defined by [60]

$$
\begin{equation*}
b(i)=i, \quad b(P)=\frac{1}{s_{P}} \sum_{X Y=P}[b(X), b(Y)] . \tag{2.3.13}
\end{equation*}
$$

To give one example, we may generate the tree level four point Feynman diagrams and their poles with $b(123)$,

$$
\begin{align*}
b(123) & =\frac{1}{s_{123}}[b(12), b(3)]+\frac{1}{s_{123}}[b(1), b(23)] \\
& =\frac{1}{s_{12} s_{123}}[[b(1), b(2)], b(3)]+\frac{1}{s_{23} s_{123}}[b(1),[b(2), b(3)]]  \tag{2.3.14}\\
& =\frac{1}{s_{12} s_{123}}[[1,2], 3]+\frac{1}{s_{23} s_{123}}[1,[2,3]]
\end{align*}
$$

These Lie monomials may then be mapped to corresponding Feynman diagrams, as is illustrated in figure 2.3.2. Note the poles generated differ by an overall $s_{123}^{-1}$ factor, and


Figure 2.3.2: The planar binary trees corresponding with $b(123)$, equal to $1 / s_{123}$ times the partial amplitude $A(1,2,3,4)$
for $b(P)$ they will always differ by an $s_{P}^{-1}$, but this will be accounted for in calculations.

We may use this description to identify the number of diagrams in tree level amplitudes. Denoting the number of $n$-point diagrams by $C_{n}$, it follows from (2.3.13) that we have the recursion relation

$$
\begin{equation*}
C_{n}=\sum_{\substack{i+j=n \\ i, j>0}} C_{i} C_{j}, \quad C_{1}=1 . \tag{2.3.15}
\end{equation*}
$$

It is a known result that this recursion describes the Catalan numbers, the next seven terms of which are

$$
\begin{equation*}
C_{2}=1, \quad C_{3}=2, \quad C_{4}=5, \quad C_{5}=14, \quad C_{6}=42, \quad C_{7}=132, \quad C_{8}=429 \tag{2.3.16}
\end{equation*}
$$

So to conclude, at $n$ points, there are $C_{n-1}$ binary trees, and therefore $C_{n-1}$ tree level diagrams in each partial amplitude, where $C_{n}$ is the $n^{\text {th }}$ Catalan Number [61].

One loop diagrams may also be identified through the $b$-map. Each one loop diagram is based around some $m$-gon, with $m<n, n$ the number of points. At each corner of this $m$-gon, we have either an external particle, or a tree level diagram. As such, the corners of the $m$-gons may be described through the $b$-map also. This will be discussed in more detail later.

### 2.3.3 Approach of Bern, Carrasco, Johansson

The approach of KLT simplifies the calculation of tree level gravity amplitudes, but a generalisation beyond this would be desirable. Such was identified by Bern, Carrasco, and Johansson (BCJ), and we detail this here [10; 11].

First of all, we recall the form of amplitudes in SYM. Each diagram corresponds with a structure

$$
\begin{equation*}
\frac{(\text { Kinematics }) \times(\text { Colour })}{(\text { Poles })} \tag{2.3.17}
\end{equation*}
$$

The colour terms are the structure constants of some Lie algebra, and as such will satisfy


Figure 2.3.3: The colour factors in the above diagrams will form a Jacobi identity and cancel. The double copy rests on constructing the kinematic factors such that those of the above diagrams cancel also.

Jacobi identities

$$
\begin{equation*}
f^{a b c} f^{c d e}+f^{b d c} f^{c a e}+f^{d a c} f^{c b e}=0 \tag{2.3.18}
\end{equation*}
$$

This corresponds with the diagrams in figure 2.3.3. Further, as was discussed in the previous subsection, the diagrams themselves may be described in terms of Lie monomials, and generated through Lie algebraic maps. As such, one may wonder if there is a further Lie algebraic description underlying SYM amplitudes.

There is considerable freedom in the kinematics component of amplitudes. As was previously discussed, diagrams in SYM contain implicit contributions those with four point vertices, and we are always able to some degree to move these terms between certain diagrams. Then there is the usual gauge freedom, by changing the gauge we may change the components of the kinematics components. The breakthrough of BCJ was to ask what would happen if kinematics components were constructed so as to satisfy the same Jacobi identities as their corresponding colour components. How they did this specifically will not be relevant to this thesis, as more recent methods for doing so will be discussed here, but [12] may be consulted for details.

These relations between kinematic factors will be referred to in this thesis as BCJ identities or BCJ symmetries. Elsewhere they are occasionally referred to as kinematic Jacobi identities also. We may use these to identify gravity amplitudes.

Suppose we have an amplitude in Yang-Mills

$$
\begin{equation*}
\mathcal{A}_{m}=\sum_{i} \frac{N_{i} \times C_{i}}{P_{i}} \tag{2.3.19}
\end{equation*}
$$

where we use $N_{i}$ to denote the kinematic component of each term. Suppose further that we have BCJ relations between these components. Then the kinematic component, and replacing the colour component with a second copy of this, we find a corresponding gravity
amplitude,

$$
\begin{equation*}
\mathcal{M}_{m}=\sum_{i=1}^{n} \frac{N_{i} \times \tilde{N}_{i}}{P_{i}} \tag{2.3.20}
\end{equation*}
$$

This is known as the double copy.

The nature of the gravitational theory in which amplitudes are produced depends upon the specific theory to which input amplitudes belong. The majority of this thesis uses pure spinor methods, and so takes place in 10 dimensional $\mathcal{N}=1$ SYM. The double copy of this amplitude then produces corresponding results for 10 dimensional $\mathcal{N}=2$ supergravity. That is, when we compute say a five point one loop $\mathcal{N}=1 \mathrm{SYM}$ amplitude in this thesis and take its double copy, then the result is a five point one loop $\mathcal{N}=2$ supergravity amplitude.

This is far from the only choice of input theory though. For example, by choosing the kinematic terms to be from (non-supersymmetric) Yang-Mills one can obtain amplitudes in classical gravity with matter $[10 ; 62 ; 31]$. Alternatively, there is no reason why the two sets of kinematic terms have to be from the same theory. If for instance, we choose the $N$ numerators to be the kinematic components of an amplitude in $\mathcal{N}=4 \mathrm{SYM}$, and the $\tilde{N}$ numerators to be the same components from the same diagrams, but drawn from any of $\mathcal{N}=1,2,4$ SYM instead, then the resulting amplitudes belong to $\mathcal{N}=5,6,8$ supergravity accordingly [31]. These are but a few examples of how the double copy may be applied, and a far more complete list may be found in tables 4,5 and 6 of [12].

It is far from obvious why this procedure works. That is, there is not an intuitive reson why SYM amplitudes constructed with this very specific structure should be related to those of supergravity. That they are has been demonstrated through numerous examples (see [12] for many such examples). Additionally the first step, arranging numerators such that they satisfy BCJ relations, has been proven to be possible at tree level using string theory methods [63; 64].

One further consequence of the work of BCJ is that new relations emerge between partial amplitudes. We will not detail the derivation of these, but merely state that they are given by

$$
\begin{equation*}
\sum_{i=2}^{m-1} k_{1} \cdot k_{23 \ldots i} A_{m}^{\text {tree }}(2, \ldots, i, 1, i+1, \ldots, m)=0 \tag{2.3.21}
\end{equation*}
$$

These are also referred to as BCJ relations, though it should be clear from the context when we refer to BCJ relations as to whether we mean those between partial amplitudes or those between numerators. They can be viewed as an extension of the KK relations, in that where the KK relations reduced the number of linearly independent partial amplitudes to $(n-2)$ !, the KK relations plus the BCJ relations reduce the number further to $(n-3)!$.





Figure 2.3.4: There may be thought of as being two classes of BCJ relations at loop level; those involving edges of the loop as in the first diagram, and those within one of the trees branching off as in the second. The former class of these will prove substantially harder to enforce than the latter in what follows. Note that we take care to include the loop momentum $\ell$ explicitly in the diagrams.

It is believed this is the maximum reduction which can be made in the number of linearly independent partial amplitudes.

### 2.3.4 Loop Level

Unlike with the KLT relations, the approach of BCJ generalises to loop level. This generalisation is usually justified as a consequence of unitarity cuts [ $65 ; 35 ; 66 ; 67 ; 68 ; 69 ; 70]$. In this approach to loop amplitudes, one effectively "cuts" diagrams along certain edges repeatedly, to reduce loop diagrams to functions of tree level diagrams. Then as a result, since BCJ holds for tree level diagrams, it follows that it should also hold for the loop diagrams. Showing this rigorously though remains an open problem.

Enforcing the BCJ relations at loop level is similar to doing so at tree level, in that we must ensure that the kinematic numerators satisfy the same Jacobi identities as their colour partners. An extra complication which should be considered is that there is in effect two classes of BCJ relations at loop level; those which involve the loop and those which do not. An example of each is provided in figure 2.3.4.

Another difficulty, which is closely related to difficulties we will encounter in the course of this thesis, is the labelling problem. As has been discussed, one loop amplitudes are integrated over the undetermined loop momentum, $\int_{-\infty}^{\infty} d^{D} \ell$. At the level of diagrams, extra care must therefore be taken to ensure that the numerators of BCJ relations share
a loop momentum structure. That is, alike in figure 2.3.4, we must make sure we look at diagrams constructed with $\ell$ in the same place on them. Otherwise a substitution $\ell^{\prime}=\ell+k$ in certain terms may have to be performed, or in some instances something more complicated as will be detailed later.

Using the double copy approach, numerous loop level results in supergravity have been found. The four dimensional MHV amplitude ${ }^{6}$ at one loop has been has been constructed, first to seven points [72], and then for arbitrary such [73]. Using an alternative representation of the denominators of the amplitude, the double copy was also applied at one loop in [74]. Then when the number of points is limited, results up to four loop were identified in various theories $[75 ; 76 ; 77 ; 78 ; 79]$, and this was extended to five loops in $\mathcal{N}=8$ supergravity [80; 81].

### 2.4 Testing For Lie Algebraic Structure

Let us denote the kinematic component of a tree level diagram described by the Lie monomial $A$ by

$$
\begin{equation*}
N_{A} \tag{2.4.1}
\end{equation*}
$$

So for instance, the kinematic component of the four point diagrams illustrated in figure 2.3.2 would be denoted by

$$
\begin{equation*}
N_{[11,2], 3]}, \quad N_{[1,[2,3]]} . \tag{2.4.2}
\end{equation*}
$$

This will be generalised to loop level in later sections. The Lie brackets in this notation then correspond exactly with the symmetries of the factors corresponding colour term. Hence, enforcement of BCJ identities is equivalent to ensuring that these kinematic terms have the symmetries the Lie brackets notation would suggest. So in the first case for instance, this means verifying

$$
\begin{align*}
N_{[1,2], 3]} & =-N_{[[2,1], 3]} \\
& =-N_{[3,[1,2]]}  \tag{2.4.3}\\
& =+N_{[3,[1,2]]} \\
& =-N_{[[2,3], 1]}-N_{[[3,1], 2]} .
\end{align*}
$$

[^3]That is, we must have antisymmetry in both of the brackets and the Jacobi identity. Then, if these relations are satisfied, the kinematic numerator $N_{[1,2], 3]}$ can be said to satisfy BCJ identities. Clearly the number of relations we must check is going to grow rapidly. Fortunately, we may reduce this number significantly. Partially, this will is achieved implicitly by building numerators with common algorithms; if we know that $N_{[1,2], 3]}$ satisfies BCJ relations, and that $N_{[[2,1], 3]}$ say is defined in precisely the same way save for 1 and 2 being swapped, then this must also satisfy BCJ relations. The number of identities required to be verified will remain large however. In this section, we describe methods by which they may be standardised and this process made more algorithmic.

### 2.4.1 Generalised Jacobi Identities

We begin with a single class of kinematic numerators, those represented with (left-to-right) Dynkin brackets. These are denoted $\ell(P)$, for $P$ a word, and are defined recursively by [43]

$$
\begin{equation*}
\ell(P i)=[\ell(P), i], \quad \ell(i)=r(i)=i . \tag{2.4.4}
\end{equation*}
$$

However, they may be represented more clearly in the form

$$
\begin{equation*}
\ell\left(a_{1} a_{2} \ldots a_{n}\right)=\left[\left[\ldots\left[a_{1}, a_{2}\right], a_{3}\right], \ldots, a_{n}\right] \tag{2.4.5}
\end{equation*}
$$

For completeness, we note that we may also define right-to-left Dynkin brackets similarly

$$
\begin{align*}
& r(i P)=[i, r(P)], \quad r(i)=i  \tag{2.4.6}\\
& r\left(a_{1} a_{2} \ldots a_{n}\right)=\left[a_{1},\left[a_{2},\left[a_{3},\left[\ldots,\left[a_{n-1}, a_{n}\right]\right] \ldots\right]\right.\right. \tag{2.4.7}
\end{align*}
$$

However, these will play a significantly lesser role than their counterparts, and as such unless specified otherwise a Dynkin bracket should always be assumed to be left-to-right.

These will be so present in fact, that often we will omit Lie brackets entirely to denote them and simplify notation. Unless stated otherwise, when a word is used in a context in which one would expect a Lie bracket, it should be interpreted as being the Dynkin bracketing of that word. So, to give one example,

$$
\begin{align*}
N_{[[123,45], 6789]} & =N_{[[\ell(123), \ell(45)], \ell(6789)]}  \tag{2.4.8}\\
& =N_{[[[1,2], 3],[4,5]],[[[6,7], 8], 9]]} .
\end{align*}
$$

Suppose now that we have a kinematic numerator with an apparent Dynkin bracket struc-
ture,

$$
\begin{equation*}
N_{12 \ldots n} \quad\left(=N_{\ell(12 \ldots n)}\right) . \tag{2.4.9}
\end{equation*}
$$

Testing for BCJ relations in this case then corresponds with verifying that this numerator satisfies the symmetries the Lie bracket would suggest. This is described in terms of generalised Jacobi identities (GJIs) [82].

An object $K_{12 \ldots . .}$ is said to satisfy GJIs if

$$
\begin{equation*}
K_{A \ell(B) C}+K_{B \ell(A) C}=0, \quad \forall A B C=12 \ldots p, \quad A, B \neq \emptyset \tag{2.4.10}
\end{equation*}
$$

To understand the meaning of this, consider it case by case. When $C=34 \ldots p$, the above relation reduces to the constraint

$$
\begin{equation*}
K_{1234 \ldots p}+K_{2134 \ldots p}=0 . \tag{2.4.11}
\end{equation*}
$$

That is, antisymmetry in the first two indices, which would be expected if this $K$ has the structure of $\ell(12 \ldots p)$. Likewise $C=45 \ldots p$ corresponds with the Jacobi identity in the first three indices. The following cases then follow from repeated application of Jacobi identities to increasingly outer brackets. For instance, setting $A=12, B=34, C=56 \ldots p$, this becomes

$$
\begin{equation*}
K_{12345 \ldots p}-K_{12435 \ldots p}+K_{34125 \ldots p}-K_{34215 \ldots p}, \tag{2.4.12}
\end{equation*}
$$

and one may verify that Jacobi identities imply

$$
\begin{equation*}
[[[1,2], 3], 4]-[[[1,2], 4], 3]+[[[3,4], 1], 2]-[[[3,4], 2], 1]=0 . \tag{2.4.13}
\end{equation*}
$$

In general, that the GJIs are equivalent to having that $K_{12 \ldots p}$ has the structure of the left-to-right Dynkin bracket $\ell(12 \ldots p)$ can be seen by taking $C=c_{1} \ldots c_{|C|}$,

$$
\begin{align*}
\ell(A \ell(B) C) & \left.=\left[\left[\ldots\left[[\ell(A), \ell(B)], c_{1}\right], \ldots\right], c_{|C|}\right]=-\left[\left[\ldots[\ell(B), \ell(A)], c_{1}\right], \ldots\right], c_{|C|}\right]  \tag{2.4.14}\\
& =-\ell(B \ell(A) C)
\end{align*}
$$

Hence, if $K_{12 \ldots p}$ has a left-to-right Dynkin bracket structure for its indices, then the above relation and therefor GJIs will be satisfied. Note the steps in the above which are not detailed are easily proved by induction.

It will prove to be sufficient to only check one identity for each length of $C$ in (2.4.10), simplifying our discussion significantly. In accordance with [83], we define an operator $\mathcal{L}$,
and use this to generate the identities we verify

$$
\begin{align*}
\mathcal{L}_{2 n+1} \circ K_{12 \ldots m} & =K_{12 \ldots n+1 r(n+2 \ldots 2 n+1) 2 n+2 \ldots m}-K_{2 n+1 \ldots n+2 r(n+1 \ldots 1) 2 n+2 \ldots m},  \tag{2.4.15}\\
\mathcal{L}_{2 n} \circ K_{12 \ldots m} & =K_{12 \ldots n r(n+1 \ldots 2 n) 2 n+1 \ldots m}+K_{2 n \ldots n+1 r(n \ldots 1) 2 n+1 \ldots m},
\end{align*}
$$

Consistency between requiring the vanishing of the above and requiring (2.4.10) may be seen by noting that

$$
\begin{align*}
r\left(a_{1} a_{2} \ldots a_{n}\right) & =\left[a_{1},\left[\ldots,\left[a_{n-2},\left[a_{n-1}, a_{n}\right]\right]\right] \ldots\right]=(-1)^{n-1}\left[\left[\ldots\left[\left[a_{n}, a_{n-1}\right], a_{n-2}\right], \ldots\right], a_{1}\right] \\
& =(-1)^{n-1} \ell\left(a_{n} \ldots a_{2} a_{1}\right) \tag{2.4.16}
\end{align*}
$$

In the case of $\mathcal{L}_{2 n}$ we may then set $A=12 \ldots n, B=2 n(2 n-1) \ldots(n+1)$, and $\bar{A}$ and $\bar{B}$ their reversals, to see the link with (2.4.10)

$$
\begin{align*}
\mathcal{L}_{2 n} \circ K_{A \bar{B}} & =K_{A r(\bar{B})}+K_{B r(\bar{A})}  \tag{2.4.17}\\
& =(-1)^{n-1}\left(K_{A \ell(B)}+K_{B \ell(A)}\right)
\end{align*}
$$

The other case of $\mathcal{L}_{2 n+1}$ will follow similarly.

To conclude, an object $K_{12 \ldots p}$ will have the symmetries of the left-to-right Dynkin bracket $\ell(12 \ldots p)$ if it satisfies generalised Jacobi identities, defined by

$$
\begin{equation*}
\mathcal{L}_{n} \circ K_{12 \ldots p}=0 \quad \forall n \leq p \tag{2.4.18}
\end{equation*}
$$

It follows that, given a kinematic numerator $N_{12 \ldots p}$, this is satisfies BCJ relations if we have the analogous relations

$$
\begin{equation*}
\mathcal{L}_{n} \circ N_{12 \ldots p}=0 \quad \forall n \leq p \tag{2.4.19}
\end{equation*}
$$

### 2.4.2 General Lie Bracket Structures

Though Dynkin brackets serve as something of a base case for calculations in this thesis, they are by no means the only Lie bracket structures which exist, and we must discuss how BCJ relations for these are standardised. Fortunately though, this is a relatively simple process. Any Lie bracket structure may be transformed into a collection of left-to-right Dynkin brackets with repeated use of Bakers identity [43]

$$
\begin{equation*}
[\ell(P), \ell(Q)]=\ell(P \ell(Q)) \tag{2.4.20}
\end{equation*}
$$

It is a simple induction to prove this. So, to give one more complex example

$$
\begin{equation*}
[[\ell(12), \ell(34)],[5, \ell(67)]]=[\ell(12 \ell(34)), \ell(5 \ell(67))] \tag{2.4.21}
\end{equation*}
$$

$$
\begin{aligned}
= & \ell(1234 \ell(567))-\ell(1243 \ell(567))-\ell(1234 \ell(576))+\ell(1243 \ell(576)) \\
= & \ell(1234567)-\ell(1234657)-\ell(1234756)+\ell(1234765) \\
& -\ell(1243567)+\ell(1243657)+\ell(1243756)-\ell(1234765) \\
& -\ell(1234576)+\ell(1234756)+\ell(1234657)-\ell(1234675) \\
& +\ell(1243576)-\ell(1243756)-\ell(1243657)+\ell(1243675)
\end{aligned}
$$

For clarity all Dynkin bracketing functions have been made explicit in the above, and we remind the reader that for instance $\ell(1234567)=[[[[[[1,2], 3], 4], 5], 6], 7]$.

This will be the primary method by which BCJ relations are verified. The methods of the previous subsection verify BCJ relations for Dynkin brackets, and then we may use the above trick to relate the remaining numerators to those. If all such relations hold, then we may be satisfied that all BCJ relations hold

## The Pure Spinor Formalism

Calculations in string theory are notoriously difficult. In the original Green-Schwartz formalism [84], one must work in the light-cone gauge in order to quantise, and this leads to difficulties due to the lack of manifest Lorentz covariance. In the RNS formalism there are other difficulties, arising from the lack of target-space supersymmetry. Fortunately there is a third approach, the pure spinor formalism, with which considerable success has been found in amplitude construction. In this section we introduce this approach. We begin with a discussion of the description of SYM which will be used in the formalism. We then outline the key points in the origin of this formalism, and outline how amplitudes are constructed within it. The calculations needed to find tree and one-loop amplitudes will then be discussed in more detail in the following two chapters.

### 3.1 Super Yang-Mills in Ten Dimensions

Superstring theory is necessarily a ten-dimensional theory, and as such the amplitudes we will construct will be in terms of ten-dimensional $\mathcal{N}=1 \mathrm{SYM}$. This is described through a formulation due to Witten, in which particle properties are contained within objects called superfields [85; 86]. These are functions of worldsheet vectors and spinors $x^{m}$ and $\theta_{\alpha}$ respectively, describing the position and polarisation of particles. There are four such
superfields, denoted by

$$
\begin{equation*}
\mathbb{A}_{\alpha}(x, \theta), \quad \mathbb{A}_{m}(x, \theta), \mathbb{W}^{\alpha}(x, \theta), \mathbb{F}^{m n}(x, \theta) . \tag{3.1.1}
\end{equation*}
$$

The latter pair of these correspond with the field strengths of the former pair, and the indices take the usual range of values $m, n=0, \ldots, 9, \alpha=1, \ldots, 16$. The terms these are functions of, the $x^{m}$ and $\theta^{\alpha}$, are 10 and 16 dimensional worldsheet variables. These will frequently be dropped for simplicity, and particle labels will often be added to denote which particle the superfield relates to. So, for instance,

$$
\begin{equation*}
\mathbb{A}_{\alpha}^{1} \mathbb{W}_{2}^{\beta} \tag{3.1.2}
\end{equation*}
$$

would refer to the product of the superfield $\mathbb{A}_{\alpha}\left(x_{1}, \theta_{1}\right)$ and $\mathbb{W}^{\beta}\left(x_{2}, \theta_{2}\right)$, where $x_{i}$ and $\theta_{i}$ describe particle $i$. The superfields with Roman indices are bosonic, meaning they commute with each other, while those with Greek indices are fermionic, and so they commute with bosonic superfields and anticommute with each other. Similar properties hold for the constituent $x^{m}$ and $\theta^{\alpha}$ terms.

In order to describe the equations of motion of these superfields, we introduce the supercovariant derivatives [86; 85],

$$
\begin{equation*}
\nabla_{\alpha}=D_{\alpha}-\mathbb{A}_{\alpha}, \quad \nabla_{m}=\partial_{m}-\mathbb{A}_{m} \tag{3.1.3}
\end{equation*}
$$

wherein we have introduced the superspace derivative,

$$
\begin{equation*}
D_{\alpha}=\frac{\partial}{\partial \theta^{\alpha}}+\frac{1}{2} \gamma_{\alpha \beta}^{m} \theta_{\beta} \partial_{m} . \tag{3.1.4}
\end{equation*}
$$

Note $\gamma_{\alpha \beta}^{m}$ denotes the gamma matrices, defined as being the $16 \times 16$ matrices which satisfy

$$
\begin{equation*}
\gamma_{\alpha \beta}^{(m} \gamma^{n) \beta \gamma}=2 \eta^{m n} \delta_{\alpha}^{\gamma} . \tag{3.1.5}
\end{equation*}
$$

It will frequently be necessary to deal with products of these, and so a $\gamma$ with $n$ indices will denote the $n$-form constructed out of these,

$$
\begin{equation*}
\gamma^{m_{1} \ldots m_{n}}=\gamma^{\left[m_{1}\right.} \gamma^{m_{2}} \ldots \gamma^{\left.m_{n}\right]} . \tag{3.1.6}
\end{equation*}
$$

The field strength $\mathbb{F}_{m n}$, and a similar object $\mathbb{W}_{m}^{\alpha}$, may be defined in terms of the supercovariant derivatives

$$
\begin{equation*}
\mathbb{F}_{m n}=-\left[\nabla_{m}, \nabla_{n}\right], \quad \mathbb{W}_{m}^{\alpha}=-\left[\nabla_{m}, \mathbb{W}^{\alpha}\right] . \tag{3.1.7}
\end{equation*}
$$

These derivatives satisfy a number of relations, with the superspace derivative obeying

$$
\begin{equation*}
\left\{D_{\alpha}, D_{\beta}\right\}=\gamma_{\alpha \beta}^{m} \partial_{m} \tag{3.1.8}
\end{equation*}
$$

Again we have introduced notation; the bracket $\{\cdot, \cdot\}$ denotes the anticommutator

$$
\begin{equation*}
\{A, B\}=A B+B A \tag{3.1.9}
\end{equation*}
$$

Further, the supercovariant derivatives satisfy a constraint,

$$
\begin{equation*}
\left\{\nabla_{\alpha}, \nabla_{\beta}\right\}=\gamma_{\alpha \beta}^{m} \nabla_{m} \tag{3.1.10}
\end{equation*}
$$

arising from dimensional considerations $[86 ; 87]$. These relations, plus the Bianchi identity, give the equations of motion of the superfields

$$
\begin{array}{rlr}
\left\{\nabla_{(\alpha}, \mathbb{A}_{\beta)}\right\} & =\gamma_{\alpha \beta}^{m} \mathbb{A}_{m}, & {\left[\nabla_{\alpha}, \mathbb{A}_{m}\right]=\left[\partial_{m}, \mathbb{A}_{\alpha}\right]+\left(\gamma_{m} \mathbb{W}\right)_{\alpha},} \\
\left\{\nabla_{\alpha}, \mathbb{W}^{\beta}\right\}=\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}^{\beta} \mathbb{F}_{m n}, & {\left[\nabla_{\alpha}, \mathbb{F}^{m n}\right]=\left(\mathbb{W}^{[m} \gamma^{n]}\right)_{\alpha},} \tag{3.1.11}
\end{array}
$$

It is important to note here the self contained nature of the above; as we take more derivatives, we cycle through only these four superfields, and no further objects are introduced.

These equations are invariant under the gauge transformations

$$
\begin{equation*}
\delta_{\Omega} \mathbb{A}_{\alpha}=\left[\nabla_{\alpha}, \Omega\right], \quad \delta_{\Omega} \mathbb{A}_{m}=\left[\nabla_{m}, \Omega\right] \tag{3.1.12}
\end{equation*}
$$

where $\Omega=\Omega(x, \theta)$ is some (Lie algebra valued) gauge parameter. The corresponding gauge transformations for the field strengths are then

$$
\begin{equation*}
\delta_{\Omega} \mathbb{W}^{\alpha}=\left[\Omega, \mathbb{W}^{\alpha}\right], \quad \delta_{\Omega} \mathbb{F}_{m n}=\left[\Omega, \mathbb{F}_{m n}\right], \quad \delta_{\Omega} \mathbb{W}_{m}^{\alpha}=\left[\Omega, \mathbb{W}_{m}^{\alpha}\right] \tag{3.1.13}
\end{equation*}
$$

It is common to impose the Lorenz gauge, defined by the constraint ${ }^{1}$

$$
\begin{equation*}
\left[\partial_{m}, \mathbb{A}^{m}\right]=0 \tag{3.1.14}
\end{equation*}
$$

In this gauge choice, the equations of motion can be found to be equivalent to a set of

[^4]non-linear wave equations [88],
\[

$$
\begin{align*}
\square \mathbb{A}_{\alpha} & =\left[\mathbb{A}_{m},\left[\partial^{m}, \mathbb{A}_{\alpha}\right]\right]+\left[\left(\gamma^{m} \mathbb{W}\right)_{\alpha}, \mathbb{A}_{m}\right], \\
\square \mathbb{A}_{m} & =\left[\mathbb{A}_{p},\left[\partial^{p}, \mathbb{A}^{m}\right]\right]+\left[\mathbb{F}^{m p}, \mathbb{A}_{p}\right]+\gamma_{\alpha \beta}^{m}\left\{\mathbb{W}^{\alpha}, \mathbb{W}^{\beta}\right\}, \\
\square \mathbb{W}^{\alpha} & =\left[\mathbb{A}_{m},\left[\partial^{m}, \mathbb{W}^{\alpha}\right]\right]+\left[\mathbb{A}^{m}, \mathbb{W}_{m}^{\alpha}\right]+\frac{1}{2}\left[\mathbb{F}_{m n},\left(\gamma^{m n} \mathbb{W}\right)^{\alpha}\right],  \tag{3.1.15}\\
\square \mathbb{F}^{m n} & =\left[\mathbb{A}_{p},\left[\partial^{p}, \mathbb{F}^{m n}\right]\right]+\left[\mathbb{A}_{p},\left[\nabla^{p}, \mathbb{F}^{m n}\right]\right]+2\left[\mathbb{F}_{m p}, \mathbb{F}_{p}^{n}\right]+4\left\{\left(\mathbb{W}^{[m} \gamma^{n]}\right)_{\alpha}, \mathbb{W}^{\alpha}\right\} .
\end{align*}
$$
\]

We have introduced two new items of notation here. First of all, the d'Alembertian operator $\square$, defined by

$$
\begin{equation*}
\square \mathbb{K}=\left[\partial^{m},\left[\partial_{m}, \mathbb{K}\right]\right], \tag{3.1.16}
\end{equation*}
$$

for $\mathbb{K} \in\left\{\mathbb{A}_{\alpha}, \mathbb{A}^{m}, \mathbb{W}^{\alpha}, \mathbb{F}^{m n}\right\}$. Then there is a piece of notation which will become ubiquitous in this thesis. When we have a sum over Roman indices, we denote this as a dot product between the two objects and omit the index. Likewise when we have a sum over Greek indices, we omit the index, and rely upon context to denote which indices are summed over. This is best illustrated with examples,

$$
\begin{equation*}
(\mathbb{A} \cdot \mathbb{A})=\mathbb{A}^{m} \mathbb{A}_{m}, \quad(\mathbb{W} \mathbb{A})=\mathbb{W}^{\alpha} \mathbb{A}_{\alpha}, \quad\left(\mathbb{W} \gamma^{m n p} \gamma_{q r}\right)^{\alpha}=\mathbb{W}^{\beta} \gamma_{\beta \sigma}^{m n p} \gamma_{r s}^{\sigma \alpha} . \tag{3.1.17}
\end{equation*}
$$

### 3.1.1 Linearisation

In practice, the full non-linear formulation of 10D SYM is too complex to be practical, and one uses its linearisation instead. As such, the vast majority of calculations in this thesis will invoke linearised SYM only [86]. In this, we will denote the superfields with standard capital Roman letters,

$$
\begin{equation*}
A_{\alpha}(x, \theta), \quad A^{m}(x, \theta), \quad W^{\alpha}(x, \theta), \quad F_{m n}(x, \theta) \tag{3.1.18}
\end{equation*}
$$

Dropping non-linear terms, the equations of motion (3.1.11) reduce to

$$
\begin{array}{cl}
2 D_{(\alpha} A_{\beta)}=\gamma_{\alpha \beta}^{m} A_{m}, & D_{\alpha} A_{m}=\left(\gamma_{m} W\right)_{\alpha}+k_{m} A_{\alpha}, \\
D_{\alpha} F_{m n}=2 \partial_{[m}\left(\gamma_{n]} W\right)_{\alpha}, & D_{\alpha} W^{\beta}=\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}{ }^{\beta} F_{m n} . \tag{3.1.19}
\end{array}
$$

Likewise, the Lorenz gauge condition may be restated as

$$
\begin{equation*}
k^{m} A_{m}=0, \tag{3.1.20}
\end{equation*}
$$

It follows from this the complimentary relations

$$
\begin{gather*}
k_{m}\left(\gamma^{m} W\right)_{\alpha}=0,  \tag{3.1.21}\\
k^{m} F_{m n}=0,
\end{gather*}
$$

the first arising from the equation of motion for $A_{m}$, and the second from the linearisation of the definition of $\mathbb{F}_{m n}(3.1 .7)$,

$$
\begin{equation*}
F_{m n}=k_{m} A_{n}-k_{n} A_{m} \tag{3.1.22}
\end{equation*}
$$

The form the expansions of the four superfields in terms of their components has not yet been discussed. Though the details of these are not needed for this thesis, a knowledge of their general structure is necessary to understand the calculations performed. These are expressed as series' in $\theta$, with the forms of the series identified through manipulations of the equations of motion (3.1.19). The reference [89] should be consulted for details, and to illustrate we give just one example,

$$
\begin{equation*}
A_{m}(x, \theta)=a_{m}+\left(\theta \gamma_{m} \chi\right)+\frac{1}{4}\left(\theta \gamma_{m}^{p q} \theta\right) \partial_{q} a_{p}+\frac{1}{12}\left(\theta \gamma_{m}^{q p} \theta\right)\left(\theta \gamma_{q} \partial_{p} \chi\right)+\ldots \tag{3.1.23}
\end{equation*}
$$

As will be outlined later, the nature of amplitude construction in the pure spinor formalism means that only the first few terms from these expansions will ever be needed. The $a_{m}$ and $\chi^{\alpha}$ correspond with the $\theta=0$ components of $A_{m}$ and $W^{\alpha}$ respectively.

### 3.2 The Pure Spinor Formalism

We now begin the discussion of string theory, by introducing the pure spinor formalism. This is done in terms of the superfields detailed in the previous subsection, and will serve as the primary scheme by which amplitudes are computed in this thesis, both in string and in field theory.

### 3.2.1 Origin

The pure spinor formalism had its origin in consideration of the Green-Schwarz superstring. The left-moving piece of the covariant Green-Schwarz action for the heterotic string in the conformal gauge is given by [14]

$$
\begin{equation*}
S_{h e t}=\int d^{2} z\left(\frac{1}{2} \Pi^{m} \bar{\Pi}_{m}+\frac{1}{4} \Pi_{m} \theta^{\alpha} \gamma_{\alpha \beta}^{m} \bar{\partial} \theta^{\beta}-\frac{1}{4} \bar{\Pi}_{m} \theta^{\alpha} \gamma_{\alpha \beta}^{m} \partial \theta^{\beta}\right) \tag{3.2.1}
\end{equation*}
$$

The $x^{m}$ and $\theta^{\alpha}$ are as in the previous section, and the $\Pi^{m}$ and $\bar{\Pi}$ denote supersymmetric momenta

$$
\begin{equation*}
\Pi^{m}=\partial x^{m}+\frac{1}{2} \theta^{\alpha} \gamma_{\alpha \beta}^{m} \partial \theta^{\beta}, \quad \bar{\Pi}^{m}=\bar{\partial} x^{m}+\frac{1}{2} \theta^{\alpha} \gamma_{\alpha \beta}^{m} \bar{\partial} \theta^{\beta} \tag{3.2.2}
\end{equation*}
$$

These can be checked to be invariant under the supersymmetry transformation

$$
\begin{equation*}
\delta \theta^{\alpha}=\epsilon^{\alpha}, \quad x^{m}=\frac{1}{2}\left(\theta \gamma^{m} \epsilon\right) . \tag{3.2.3}
\end{equation*}
$$

Finally, $\gamma_{\alpha \beta}^{m}$ refers to the gamma matrices of (3.1.5). From this action the canonical momentum $p_{\alpha}$ corresponding with $\theta^{\alpha}$ may be defined,

$$
\begin{equation*}
p_{\alpha}=\frac{\delta \Lambda}{\delta \partial_{0} \theta^{\alpha}}=\frac{1}{2}\left(\Pi_{m}-\frac{1}{4} \theta \gamma_{m} \partial_{1} \theta\right)\left(\gamma^{m} \theta\right)_{\alpha} . \tag{3.2.4}
\end{equation*}
$$

This then leads naturally to the Dirac constraints,

$$
\begin{equation*}
d_{\alpha}=p_{\alpha}-\frac{1}{2}\left(\Pi_{m}-\frac{1}{4} \theta \gamma_{m} \partial_{1} \theta\right)\left(\gamma^{m} \theta\right)_{\alpha} \tag{3.2.5}
\end{equation*}
$$

The pure spinor formalism uses similar objects to those in above, and introduces some new ones also. It has its origin in the 1986 proposition by Siegel, of the action [90]

$$
\begin{equation*}
S=\int d^{2} z\left(\frac{1}{2} \partial x^{m} \bar{\partial} x_{m}+p_{\alpha} \bar{\partial} \theta^{\alpha}\right) . \tag{3.2.6}
\end{equation*}
$$

The $p_{\alpha}$ is now regarded as an independent object, and correspondingly $d_{\alpha}$ no longer required to vanish. This approach was used to quantise the super-particle [91], but did not meet with the same success in quantising the superstring. This required the addition of ghost fields to the action, which will be discussed in the following section. It does serve as a basis upon which the pure spinor formalism may be built though, and as such we may use (3.2.6) to identify the operator product expansions related to the physical fields.

The operator product expansion (OPE) of a pair of operators $A(x)$ and $B(y)$, is a representation of their product as a sum of operators [3;92;93]

$$
\begin{equation*}
A(x) B(y)=\sum_{i} c_{i}(x-y)^{i} C_{i}(y) \tag{3.2.7}
\end{equation*}
$$

For the purposes of calculations in string and quantum theory, we usually neglect the non-negative powers in this expansion, as only at poles will such products contribute to physical phenomena. In the case of the action (3.2.6), the OPEs of the operators therein may be identified as

$$
\begin{equation*}
x^{m}\left(z_{1}\right) x^{n}\left(z_{2}\right) \rightarrow-\frac{\alpha^{\prime}}{2} \eta^{m n} \log \left|z_{1}-z_{2}\right|^{2}, \quad p_{\alpha}\left(z_{1}\right) \theta^{\beta}\left(z_{2}\right) \rightarrow \frac{\delta_{\alpha}^{\beta}}{z_{1}-z_{2}} \tag{3.2.8}
\end{equation*}
$$

Explaining these expansions uses techniques discussed in a number of sources, and so we direct the reader elsewhere for details $[3 ; 92 ; 93]$. The $d_{\alpha}$ and $\Pi^{m}$ operators are functions
of these operators, and their OPEs may be identified similarly

$$
\begin{array}{rlrl}
d_{\alpha}\left(z_{1}\right) d_{\beta}\left(z_{2}\right) & \rightarrow-\frac{1}{2} \frac{\gamma_{\alpha \beta}^{m} \Pi_{m}}{z_{1}-z_{2}}, & d_{\alpha}\left(z_{1}\right) \Pi^{m}\left(z_{2}\right) & \rightarrow \frac{1}{2} \frac{\left(\gamma^{m} \partial \theta\right)_{\alpha}}{z_{1}-z_{2}} \\
d_{\alpha}\left(z_{i}\right) \Pi^{m}\left(z_{j}\right) & \rightarrow \frac{\left(\gamma^{m} \partial \theta\right)_{\alpha}}{z_{i j}}, & d_{\alpha}\left(z_{i}\right) \theta^{\beta}\left(z_{j}\right) & \rightarrow \frac{\delta_{\alpha}^{\beta}}{z_{i j}} \\
d_{\alpha}\left(z_{i}\right) \partial \theta^{\beta}\left(z_{j}\right) & \rightarrow \frac{\delta_{\alpha}^{\beta}}{z_{i j}^{2}}, & \Pi^{m}\left(z_{i}\right) \Pi^{n}\left(z_{j}\right) & \rightarrow-\frac{\eta^{m n}}{z_{i j}^{2}}  \tag{3.2.9}\\
& d_{\alpha}\left(z_{i}\right) d_{\beta}\left(z_{j}\right) \rightarrow-\frac{\gamma_{\alpha \beta}^{m} \Pi_{m}}{z_{i j}}
\end{array}
$$

Note we have introduced new notation here, $z_{i j}$, defined as the difference

$$
\begin{equation*}
z_{i j}=z_{i}-z_{j} \tag{3.2.10}
\end{equation*}
$$

This will be used many times throughout this thesis.

Finally, if we then consider a superfield $K=K(x, \theta)$, OPEs between them and the $d_{\alpha}$ and $\Pi^{m}$ operators may be identified also

$$
\begin{equation*}
d_{\alpha}\left(z_{i}\right) K\left(z_{j}\right) \rightarrow \frac{D_{\alpha} K}{z_{i j}}, \quad \Pi^{m}\left(z_{i}\right) K\left(z_{j}\right) \rightarrow-\frac{k^{m} K}{z_{i j}} \tag{3.2.11}
\end{equation*}
$$

Operator product expansions, and simplifying calculations involving them, is one of the greatest difficulty in the calculation of amplitudes in the pure spinor formalism. As such, they will be discussed a great deal in this thesis, and from a certain point of view this will be the task we will be working on throughout part II.

### 3.2.2 Pure Spinors Ghosts

An object $\lambda$ is called a pure spinor if it satisfies the relation

$$
\begin{equation*}
\left(\lambda \gamma^{m} \lambda\right)=0 \tag{3.2.12}
\end{equation*}
$$

for $\gamma^{m}$ the usual gamma matrix. They were originally studied by Cartan [94]. Prior to the work of Berkovits, they had been used to describe SYM and supergravity [95; 96], but not applied to string theory. The superparticle may also be parameterised using methods similar to those discussed here [97; 98; 99], but we will not discuss this here and instead focus upon the superstring.

The pure spinor formalism is described by the action (3.2.6), plus a ghost action

$$
\begin{equation*}
S_{\lambda}=\int d^{2} z\left(\bar{\partial} t \partial s-\frac{1}{2} v^{a b} \bar{\partial} u_{a b}\right), \tag{3.2.13}
\end{equation*}
$$

where the $v_{a b}$ and $u_{a b}$ are antisymmetric in their indices. It can be shown $[100 ; 101]$ that these parameters combine to parameterise pure spinors $\lambda^{\alpha}$. The Lorentz current of this action is denoted $N^{m n}$ (that is, a conserved current associated with Lorentz symmetry) and for details of its construction [100; 99; 102] should be consulted. These ghost fields then have OPEs

$$
\begin{equation*}
t\left(z_{1}\right) s\left(z_{2}\right) \rightarrow \log \left(z_{1}-z_{2}\right), \quad v^{a b}\left(z_{1}\right) u_{c d}\left(z_{2}\right) \rightarrow \frac{\delta_{c}^{[a} \delta_{d}^{b]}}{z_{1}-z_{2}} \tag{3.2.14}
\end{equation*}
$$

and a series of calculations detailed in [102] gives

$$
\begin{equation*}
N^{m n}\left(z_{i}\right) \lambda^{\alpha}\left(z_{j}\right) \rightarrow-\frac{1}{2} \frac{\left(\lambda \gamma^{m n}\right)^{\alpha}}{z_{i j}}, \quad N^{m n}\left(z_{i}\right) N_{p q}\left(z_{j}\right) \rightarrow \frac{4}{z_{i j}} N_{[p}^{[m} \delta_{q]}^{n]}-\frac{6}{z_{i j}^{2}} \delta_{[p}^{n} \delta_{q]}^{m} \tag{3.2.15}
\end{equation*}
$$

### 3.2.3 Construction of Physical Objects

Physical states in the pure spinor formalism are identified as follows. We begin by introducing the (pure spinor) BRST operator, also known as the BRST charge, and denoted by $Q$. This is defined by ${ }^{2}$

$$
\begin{equation*}
Q=\oint d z \lambda^{\alpha}(z) d_{\alpha}(z) \tag{3.2.16}
\end{equation*}
$$

though it will frequently be abbreviated to the form

$$
\begin{equation*}
Q=\lambda^{\alpha} D_{\alpha} . \tag{3.2.17}
\end{equation*}
$$

This simplification arises as the effect of the $d_{\alpha}$ on superfields is to take their derivative and introduce a pole as in (3.2.11). That is,

$$
\begin{align*}
Q K\left(z_{j}\right) & =\oint d z_{i} \lambda^{\alpha}\left(z_{i}\right) d_{\alpha}\left(z_{i}\right) K\left(z_{j}\right) \\
& \rightarrow \oint d z_{i} \lambda^{\alpha}\left(z_{i}\right) \frac{D_{\alpha} K}{z_{i j}} . \tag{3.2.18}
\end{align*}
$$

By then performing this integral (3.2.17) follows. We note that, as such an interal has been performed, (3.2.17) represents the integral rather than the integrand, and thus is a

[^5]charge.

The BRST charge is nilpotent,

$$
\begin{equation*}
Q^{2}=0 \tag{3.2.19}
\end{equation*}
$$

which follows as a simple consequence of the pure spinor constraint (3.2.12). Physical states are then defined as being those which are in the cohomology of this operator. That is, a physical state in the pure spinor formalism is defined as being those which are annihilated by the BRST operator, but are not themselves given by the action of the BRST operator on some other state,

$$
\begin{equation*}
Q(\text { phys })=0, \quad(\text { phys }) \neq Q(\ldots) \tag{3.2.20}
\end{equation*}
$$

The action of the BRST charge upon a physical state will be referred to as the variation of that state.

Vertex operators are naturally defined using this charge ${ }^{3}$. The integrated vertex operator is the most general expression one can construct out of the worldsheet functions, with the ghost contribution constrained by the Lorenz invariance of $Q$ [104]

$$
\begin{equation*}
V=\lambda^{\alpha} A_{\alpha}(x, \theta) \tag{3.2.21}
\end{equation*}
$$

with $A_{\alpha}(x, \theta)$ for now some general function of $x^{m}$ and $\theta^{\alpha}$. The unintegrated vertex operator is then identified through its relation with the above [105]

$$
\begin{equation*}
Q U=\partial V \tag{3.2.22}
\end{equation*}
$$

The form of $U$ is

$$
\begin{equation*}
U=\partial \theta^{\alpha} A_{\alpha}+\Pi^{m} A_{m}+d_{\alpha} W^{\alpha}+\frac{1}{2} N^{p q} F_{p q} \tag{3.2.23}
\end{equation*}
$$

with $A_{\alpha}$ as in (3.2.21), and $A_{m}\left(x^{m}, \theta^{\alpha}\right), W^{\alpha}\left(x^{m}, \theta^{\alpha}\right)$, and $F_{p q}\left(x^{m}, \theta^{\alpha}\right)$ again some general expressions in $x^{m}$ and $\theta^{\alpha}$. The variation of this is then found,

$$
\begin{align*}
Q U & =\partial\left(\lambda^{\alpha} A_{\alpha}\right)+\lambda^{\alpha} \partial \theta^{\beta}\left(-D_{\alpha} A_{\beta}-D_{\beta} A_{\alpha}+\gamma_{\alpha \beta}^{m} A_{m}\right)  \tag{3.2.24}\\
& +\lambda^{\alpha} \Pi^{m}\left(D_{\alpha} A_{m}-\partial_{m} A_{\alpha}-\gamma_{m \alpha \beta} W^{\beta}\right)+\lambda^{\alpha} d_{\beta}\left(-D_{\alpha} W^{\beta}+\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}^{\beta} F_{m n}\right) \\
& +\frac{1}{2} \lambda^{\alpha} N_{m n} D_{\alpha} F^{m n}
\end{align*}
$$

Comparing this with the constraint (3.2.22), we see that (3.2.21) and (3.2.23) define a system of integrated and unintegrated vertex operators if $A_{\alpha}, A_{m}, W_{\alpha}$, and $F_{m n}$ satisfy a

[^6]number of relations. These coincide exactly with the equations of motion of the superfields describing ten dimensional SYM, (3.1.19), and so the origin of this particular choice of notation becomes clear. Vertex operators in the pure spinor formalism are given in terms of the superfield description of ten dimensional SYM.

Note that we have intentionally suppressed the plane wave factors from the vertex operators. Such an operator for a particle $m$ should be considered to contain an additional overall factor

$$
\begin{equation*}
e^{i k_{m} \cdot x\left(z_{m}\right)} \tag{3.2.25}
\end{equation*}
$$

It is standard convention to not include these terms in calculations; their contribution to an $n$-point amplitude may be summarised with the formula $[106 ; 107 ; 108]$

$$
\begin{equation*}
\mathcal{I}_{n}=\left\langle\prod_{j=1}^{n} e^{k_{j} \cdot x\left(z_{j}\right)}\right\rangle=\prod_{i<j}^{n}\left|z_{i j}\right|^{2 \alpha^{\prime} s_{i j}}, \tag{3.2.26}
\end{equation*}
$$

This is known as the Koba-Nielsen factor. It arises from the product of the plane wave factors as the result of a standard OPE calculation.

### 3.2.4 Tree Level Amplitude Formulation

Tree level amplitudes follow the standard formula, consisting of the product of $N-3$ unintegrated and 3 integrated vertex operators

$$
\begin{equation*}
A=\int d z_{4} \ldots \int d z_{N}\left\langle V_{1}\left(z_{1}\right) V_{2}\left(z_{2}\right) V_{3}\left(z_{3}\right) \prod_{r=4}^{N} U_{r}\left(z_{r}\right)\right\rangle \tag{3.2.27}
\end{equation*}
$$

Note we do not specify $z_{1}, z_{2}$ and $z_{3}$ purely for notational convenience. These are usually set to $0,1, \infty$. It will be convenient to be able to refer to the $z$ coordinate of particle $i$ as $z_{i}$ however, and so we leave them unspecified.

The angle brackets in (3.2.27) refer to the integration scheme for the ghost fields, which we have not yet specified. This was identified through the requirement that scattering amplitudes be supersymmetry and gauge invariant [97; 99], and corresponds with selecting the $\theta^{5}$ components of the vertex operators with particular structures. That is, we take (3.2.27), and perform the computations therein until we have an expression in the worldsheet functions, particle properties, and pure spinors,

$$
\begin{equation*}
A=\int d z_{4} \ldots \int d z_{N}\left\langle\lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} f_{\alpha \beta \gamma}\left(z_{r}, k_{r}, \eta_{r}, \theta\right)\right\rangle . \tag{3.2.28}
\end{equation*}
$$

The function $f_{\alpha \beta \gamma}$ is a series expansion in $\theta^{\alpha}$, and the integration over the ghost fields
corresponds with selecting the terms proportional to

$$
\begin{equation*}
\left(\gamma^{m} \theta\right)_{\alpha}\left(\gamma^{n} \theta\right)_{\beta}\left(\gamma^{p} \theta\right)_{\gamma}\left(\theta \gamma_{m n p} \theta\right), \tag{3.2.29}
\end{equation*}
$$

and setting all others to zero. Other terms of order 5 in $\theta$ may be related to this structure through gamma matrix identities, with some such relations given in [109]. Informally, all of this is to say that the integrand of $(3.2 .28)$ is given by

$$
\begin{equation*}
\left\langle\ldots+\left(\lambda \gamma^{m} \theta\right)\left(\lambda \gamma^{n} \theta\right)\left(\lambda \gamma^{p} \theta\right)\left(\theta \gamma_{m n p} \theta\right) F\left(z_{r}, k_{r}, \eta_{r}\right)+\ldots\right\rangle=F\left(z_{r}, k_{r}, \eta_{r}\right) . \tag{3.2.30}
\end{equation*}
$$

This then satisfies the necessary consistency conditions; supersymmetry and gauge invariance as discussed [99], and also agreement with known results from other formalisms, as seen in for instance [110].

When an object produces a non-zero value under the action of the pure spinor bracket it is said to be in the pure spinor superspace [111]. For convenience, we will frequently omit the pure spinor brackets $\langle\ldots\rangle$ from calculations. It should always be clear from context when they are implicit in calculations. In such instances, one then uses the procedure outlined to select the terms in the cohomology of the pure spinor BRST operator [104; 99], using identities from [109] where necessary.

### 3.2.5 Loop Amplitude Formulation

As in field theory, when one moves to looking at loop amplitudes the complexity of calculations increases significantly. Some progress has been made on their identification though, and in the course of this thesis we will discuss one loop amplitudes in particular in some detail.

We begin with the description of one loop open string amplitudes in string theory. The diagrams of these are punctured tori, with an $n$-point amplitude corresponding with $n$ punctures. Note that though there are other diagrams of the same genus, the cylinder and the Mobius strip, these may be related to the torus and so will not be discussed separately [3; 112]. We parameterise the torus based upon its fundamental group; the two distinct $S^{1}$ circles within it define what are known as the A-cycle and B-cycle. In terms of our coordinate $z$, one of these cycles is considered wholly real and adds 1 to $z$ as we complete a circuit of it, and the other is complex and adds $\tau$ similarly. It is then clear that for consistency, any function $f$ on the torus must satisfy

$$
\begin{equation*}
f(z)=f(z+1)=f(z+\tau) . \tag{3.2.31}
\end{equation*}
$$

Figure 3.2.1 should be consulted for further explanation of this discussion.


Figure 3.2.1: The torus has fundamental group $\pi_{1}\left(\mathbb{T}^{2}\right) \cong \mathbb{Z} \otimes \mathbb{Z}$. That is, there are two distinct closed loops on its surface which cannot be continuously deformed into each other or a point. These are referred to as the A and B-cycles of the torus, with the former in red and the latter in blue in the above. They serve as the basis upon which we parameterise the surface, the A-cycle corresponding with the real axis and the B-cycle an axis in some complex direction $\tau$. This $\tau$ is often called the Teichmuller parameter. This is scaled such that the addition of 1 and $\tau$ to $z$ correspond with a single circuit of the A and B cycles respectively.

In the pure spinor formalism, a general $n$-point $g$-loop scattering amplitude has the form [39; 20]

$$
\begin{equation*}
\mathcal{A}_{n}^{g-l o o p}=\int d \tau_{1} \ldots d \tau_{3 g-3}\left\langle\mathcal{Z}\left(\prod_{a=1}^{3 g-3}\left(\mu_{a}, b_{B_{a}}\right)\right) V_{1}\left(z_{1}\right)\left(\prod_{b=2}^{n} \int d z_{b} U_{b}\left(z_{b}\right)\right)\right\rangle . \tag{3.2.32}
\end{equation*}
$$

This largely corresponds with the ingredients in the equivalent formula in other formalisms [3]. The $V$ and $U$ are the vertex operators previously discussed, and the integration variables $\tau_{i}$ are the $g$-loop Teichmuller parameters. The $(\mu, b)$ bracket is defined by

$$
\begin{equation*}
\left(\mu_{a}, b_{B_{a}}\right)=\int d u_{a} \mu_{a}\left(u_{a}\right) b_{B_{a}}\left(u_{a}, z_{a}\right) \tag{3.2.33}
\end{equation*}
$$

The objects here are not present at tree level, but are in agreement with those found in other formalisms. The $\mu_{a}$ are the Beltrami differential, a function of the metric of the surface we are working on. Meanwhile the $b_{B_{a}}$ denote the b-ghosts. These appear as a consequence of BRST quantisation of the string, and are usually defined by the relation

$$
\begin{equation*}
\{Q, b(u)\}=T(u), \tag{3.2.34}
\end{equation*}
$$

where $T(u)$ is the stress-energy tensor of the string. Unfortunately by considering the distribution of ghost fields within this relation, it can be found that no such $b$ can exist in the pure spinor formalism. Instead picture changing operators must be introduced; functions $Z_{B}$ containing a ghost field $\lambda$ with which the (pure spinor) b-ghost may be defined,

$$
\begin{equation*}
\left\{Q, b_{B}(u, z)\right\}=T(u) Z_{B}(z) . \tag{3.2.35}
\end{equation*}
$$

The final element of (3.2.32) not yet discussed, the $\mathcal{Z}$, is another function of these picture changing operators, inserted to absorb zero modes when we integrate. The full depth of all of these terms is not important for this thesis, and so not discussed any further. For more details, $[39 ; 3 ; 4]$ should be consulted.

We now limit ourselves to the one loop case once again. In this situation, the amplitude formula reduces to

$$
\begin{equation*}
\mathcal{A}_{n}=\int d \tau \int d z_{2} \ldots \int d z_{n}\left\langle(\mu, b) \mathcal{Z} V_{1}\left(z_{1}\right) U_{2}\left(z_{2}\right) \ldots U_{n}\left(z_{n}\right)\right\rangle \tag{3.2.36}
\end{equation*}
$$

A skeletal expression for the form of the b-ghost is [39]

$$
\begin{align*}
b= & (\Pi d+N \partial \theta+J \partial \theta) d \delta(N)+(w \partial \lambda+J \partial N+N \partial J+N \partial N) \delta(N) \\
& +\left(N \Pi+J \Pi+\partial \Pi+d^{2}\right)\left(\Pi \delta(N)+d^{2} \delta^{\prime}(N)\right)  \tag{3.2.37}\\
& +(N d+J d)\left(\partial \theta \delta(N)+d \Pi \delta^{\prime}(N)+d^{3} \delta^{\prime \prime}(N)\right) \\
& +\left(N^{2}+J N+J^{2}\right)\left(d \partial \theta \delta^{\prime}(N)+\Pi^{2} \delta^{\prime}(N)+\Pi d^{2} \delta^{\prime \prime}(N)+d^{4} \delta^{\prime \prime \prime}(N)\right) .
\end{align*}
$$

We have not yet defined the meaning of the pure spinor bracket $\langle\ldots\rangle$ at one loop level. The role of these brackets is to ensure we have the right number of zero modes of operators in the amplitude ${ }^{4}$. At tree level, this effectively meant deleting all terms not of the form $\lambda^{3} \theta^{5}$. At genus one, this changes to become the requirement that the $b$-ghost and vertex operators only contain terms $\lambda^{3} \theta^{5} d^{5} \delta(N)$. However, we do not discuss this in detail as the specifics of this will not be used in this thesis. We direct the reader to [39] for such.

[^7]
## CHAPTER 4

## Tree Level Amplitudes From String Theory

One of the great successes of the pure spinor formalism is the improvements in the calculation of scattering amplitudes it has brought about. One in particular has been the formulation of arbitrary point tree level amplitudes in field [113;114] and string [18; 19] theory. In this chapter we discuss how these may be constructed in terms of multiparticle superfields. We will then give an overview of some properties of these amplitudes, and how the double copy might be applied to them.

### 4.1 Lorenz Gauge Construction of Multiparticle Superfields

To begin, recall the formula for tree level amplitudes in the pure spinor formalism (3.2.27),

$$
\begin{equation*}
\int_{0 \leq z_{2} \leq z_{3} \leq \ldots \leq z_{n-2}}\left\langle V_{1}\left(z_{1}\right)\left(\prod_{i=2}^{n-2} U_{i}\left(z_{i}\right)\right) V_{n-1}\left(z_{n-1}\right) V_{n}\left(z_{n}\right)\right\rangle, \tag{4.1.1}
\end{equation*}
$$

within which one makes the identifications $z_{1}=0, z_{n-1}=1, z_{n}=\infty$. Recall further the form of the vertex operators (3.2.21), (3.2.23),

$$
\begin{gather*}
V=\lambda^{\alpha} A_{\alpha}  \tag{4.1.2}\\
U=\partial \theta^{\alpha} A_{\alpha}+\Pi^{m} A_{m}+d_{\alpha} W^{\alpha}+\frac{1}{2} N^{m n} F_{m n} . \tag{4.1.3}
\end{gather*}
$$

In order to compute amplitudes using (4.1.1), we must find the product of a great many of these vertex operators. This means a considerable number of calculations of operator product expansions, and as such requires a great deal of work. Fortunately though, this process can be made algorithmic, and this is the focus of this section.

The process by which we will identify the general form of the OPEs of various vertex operators will be as follows. We begin by discussing the OPE of a pair of unintegrated vertex operators. This result we will then be able to extrapolate from, and use to find the product of an arbitrary number of unintegrated vertex operators. We then consider their OPEs of these with the integrated vertex operators, and thereby have a scheme to considerably simplify amplitude calculations.

### 4.1.0.1 Product of Two Unintegrated Vertex Operators

We begin by considering the product of two unintegrated vertex operators, $U_{1}\left(z_{1}\right)$ and $U_{2}\left(z_{2}\right)$. The calculation is lengthy, but uses nothing more complicated than the formulae of (3.2.9) (3.2.11). The conclusion of this is [115; 83]

$$
\begin{align*}
U^{1}\left(z_{1}\right) U^{2}\left(z_{2}\right) & \rightarrow z_{12}^{-k^{1} \cdot k^{2}-1}\left(\partial \theta^{\alpha}\left[\left(k^{1} \cdot A_{2}\right) A_{\alpha}^{1}-\left(k^{2} \cdot A_{1}\right) A_{\alpha}^{2}+D_{\alpha} A_{\beta}^{2} W_{1}^{\beta}-D_{\alpha} A_{\beta}^{1} W_{2}^{\beta}\right]\right. \\
& +\Pi^{m}\left[\left(k^{1} \cdot A_{2}\right) A_{m}^{1}-\left(k^{2} \cdot A_{1}\right) A_{m}^{2}+k_{m}^{2}\left(A_{2} W_{1}\right)-k_{m}^{1}\left(A_{1} W_{2}\right)-\left(W_{1} \gamma_{m} W_{2}\right)\right] \\
& +d_{\alpha}\left[\left(k^{1} \cdot A_{2}\right) W_{1}^{\alpha}-\left(k^{2} \cdot A_{1}\right) W_{2}^{\alpha}+\frac{1}{4}\left(\gamma^{m n} W_{1}\right)^{\alpha} F_{m n}^{2}-\frac{1}{4}\left(\gamma^{m n} W_{2}\right)^{\alpha} F_{m n}^{1}\right] \\
& \left.+\frac{1}{2} N^{m n}\left[\left(k^{1} \cdot A_{2}\right) F_{m n}^{1}-\left(k^{2} \cdot A_{1}\right) F_{m n}^{2}-2 k_{m}^{12}\left(W_{1} \gamma_{n} W_{2}\right)-2 F_{m a}^{2} F_{n}^{3} a\right]\right) \\
& +\left(1+k^{1} \cdot k^{2}\right) z_{12}^{-k^{1} \cdot k^{2}-2}\left[\left(A_{1} W_{2}\right)+\left(A_{2} W_{1}\right)-\left(A_{1} \cdot A_{2}\right)\right] \tag{4.1.4}
\end{align*}
$$

Note we have reintroduced the Koba-Nielsen factors (3.2.26), as they will play a role here. Namely, they allow for the poles of the second order terms to be rewritten as a partial derivative,

$$
\begin{equation*}
\left(1+k^{1} \cdot k^{2}\right) z_{12}^{-k^{1} \cdot k^{2}-2}=\partial\left(z_{12}^{-k^{1} \cdot k^{2}-1}\right) . \tag{4.1.5}
\end{equation*}
$$

We may then integrate by parts these terms within (4.1.4), and use the relation

$$
\begin{equation*}
\partial K=\partial \theta^{\alpha} D_{\alpha} K+\Pi^{m} k_{m} K . \tag{4.1.6}
\end{equation*}
$$

This results in a new expression for the $U_{1} U_{2}$ OPE [83],

$$
\begin{align*}
& U^{1}\left(z_{1}\right) U^{2}\left(z_{2}\right) \rightarrow-z_{12}^{-k^{1} \cdot k^{2}-1}\left(\partial \theta^{\alpha} A_{\alpha}^{[1,2]}+\Pi^{m} A_{m}^{[1,2]}+d_{\alpha} W_{[1,2]}^{\alpha}+\frac{1}{2} N^{m n} F_{m n}^{[1,2]}\right) \\
&+\partial_{1}\left(z_{12}^{-k^{1} \cdot k^{2}-1}\left[\frac{1}{2}\left(A_{1} \cdot A_{2}\right)-\left(A_{1} W_{2}\right)\right]\right)  \tag{4.1.7}\\
&+\partial_{2}\left(z_{12}^{-k^{1} \cdot k^{2}-1}\left[\frac{1}{2}\left(A_{1} \cdot A_{2}\right)-\left(A_{2} W_{1}\right)\right]\right)
\end{align*}
$$

The notation above has been chosen carefully to correspond with the form of (4.1.3); where at single particle level we had a term $\partial \theta^{\alpha} A_{\alpha}^{1}$, we now define the terms with coefficient $\partial \theta_{\alpha}$ as being the two particle version of the superfield $A_{\alpha}^{[1,2]}$. This is the first example of multiparticle superfields we have encountered; multiparticle versions of the superfields describing 10D SYM, identified through the calculation of OPEs between vertex operators. The two particle superfields are defined by [83]

$$
\begin{align*}
A_{\alpha}^{[1,2]} & =-\frac{1}{2}\left[A_{\alpha}^{1}\left(k^{1} \cdot A^{2}\right)+A_{m}^{1}\left(\gamma^{m} W^{2}\right)_{\alpha}-(1 \leftrightarrow 2)\right],  \tag{4.1.8}\\
A_{m}^{[1,2]} & =\frac{1}{2}\left[A_{p}^{1} F_{p m}^{2}-A_{m}^{1}\left(k^{1} \cdot A^{2}\right)+\left(W^{1} \gamma_{m} W^{2}\right)-(1 \leftrightarrow 2)\right],  \tag{4.1.9}\\
W_{[1,2]}^{\alpha} & =\frac{1}{4}\left(\gamma^{m n} W^{2}\right)^{\alpha} F_{m n}^{1}+W_{2}^{\alpha}\left(k^{2} \cdot A^{1}\right)-(1 \leftrightarrow 2),  \tag{4.1.10}\\
F_{m n}^{[1,2]} & =F_{m n}^{2}\left(k^{2} \cdot A^{1}\right)+F_{[m}^{2} F_{n] p}^{1}+k_{12}^{[m}\left(W_{1} \gamma^{n]} W_{2}\right)-(1 \leftrightarrow 2)  \tag{4.1.11}\\
& =k_{m}^{12} A_{n}^{[1,2]}-k_{n}^{12} A_{m}^{[1,2]}-\left(k^{1} \cdot k^{2}\right)\left(A_{m}^{1} A_{n}^{2}-A_{n}^{1} A_{m}^{2}\right),
\end{align*}
$$

with $k_{m}^{12}$ defined as in (2.1.7). The second expression for $F_{m n}^{12}$ is in terms of other multiparticle superfields and so arises less naturally, but it has been included because of its similarity to the single particle definition $F_{m n}=k_{m} A_{n}-k_{n} A_{m}$. Note here we do not include the Dynkin bracket explicitly in $F_{m n}^{[1,2]}$ to simplify notation, and similar will occur on many occasions going forward.

As we will eventually integrate (4.1.7) when calculating amplitudes, we may drop the total derivative terms therein. We will revisit these terms later, but until then their contribution may be ignored. Therefore, using two particle superfields the two particle unintegrated vertex operator is the natural generalisation of the one particle case,

$$
\begin{equation*}
U^{[1,2]}=\partial \theta^{\alpha} A_{\alpha}^{[1,2]}+\Pi^{m} A_{m}^{[1,2]}+d_{\alpha} W_{[1,2]}^{\alpha}+\frac{1}{2} N^{m n} F_{m n}^{[1,2]} \tag{4.1.12}
\end{equation*}
$$

It is then natural to ask what properties these two particle superfields share with their single particle equivalents. Their equations of motion, it can be found, generalise the single particle case (3.1.19) also [83],

$$
\begin{align*}
2 D_{(\alpha} A_{\beta)}^{12} & =\gamma_{\alpha \beta}^{m} A_{m}^{12}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} A_{\beta}^{2}+A_{\beta}^{1} A_{\alpha}^{2}\right)  \tag{4.1.13}\\
D_{\alpha} A_{m}^{12} & =\left(\gamma_{m} W^{12}\right)_{\alpha}+k_{m}^{12} A_{\alpha}^{12}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} A_{m}^{2}-A_{\alpha}^{2} A_{m}^{1}\right), \tag{4.1.14}
\end{align*}
$$

$$
\begin{align*}
& D_{\alpha} W_{12}^{\beta}=\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}{ }^{\beta} F_{m n}^{12}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} W_{2}^{\beta}-A_{\alpha}^{2} W_{1}^{\beta}\right),  \tag{4.1.15}\\
& \begin{aligned}
& D_{\alpha} F_{m n}^{12}=2 k_{[m}^{12}\left(\gamma_{n]} W^{12}\right)_{\alpha}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} F_{m n}^{2}-A_{\alpha}^{2} F_{m n}^{1}\right. \\
&\left.+2 A_{[n}^{1}\left(\gamma_{m]} W^{2}\right)_{\alpha}-2 A_{[n}^{2}\left(\gamma_{m]} W^{1}\right)_{\alpha}\right) .
\end{aligned} \tag{4.1.16}
\end{align*}
$$

We may ask about the gauge conditions (3.1.20), (3.1.21) also. These generalise to two particles as [83]

$$
\begin{align*}
k_{12}^{m} A_{m}^{12} & =0, \\
k_{m}^{12}\left(\gamma^{m} W^{12}\right)_{\alpha} & =\left(k^{1} \cdot k^{2}\right)\left(A_{m}^{1}\left(\gamma^{m} W^{2}\right)_{\alpha}-(1 \leftrightarrow 2)\right),  \tag{4.1.17}\\
k_{12}^{m} F_{m n}^{12} & =\left(k^{1} \cdot k^{2}\right)\left(A_{n}^{12}+A_{n}^{1}-(1 \leftrightarrow 2)\right) .
\end{align*}
$$

The only differences arising between the properties of two particle and single particle superfields are correction terms proportional to mandelstams $s_{12}$. Such is the case in general with multiparticle superfields; they share properties with their single particle equivalents, up to some correction terms proportional to mandelstam terms.

### 4.1.0.2 Product of Three Unintegrated Vertex Operators

We may now use these techniques to identify arbitrary products of unintegrated vertex operators. Due to the similarity between the single particle and two particle vertex operators, (4.1.3) and (4.1.12), we may assume that the calculation of the OPE between $U_{[1,2]}$ and $U_{3}$ repeats exactly that already discussed. That is, in the previous discussion, we found that

$$
\begin{equation*}
U_{1} U_{2} \rightarrow U_{[1,2]}=\partial \theta^{\alpha} A_{\alpha}^{[1,2]}+\Pi^{m} A_{m}^{[1,2]}+d_{\alpha} W_{[1,2]}^{\alpha}+\frac{1}{2} N^{m n} F_{m n}^{[1,2]} . \tag{4.1.18}
\end{equation*}
$$

This is identical to the expression for $U_{1}$, except for the replacement of 1 by the Lie bracket $[1,2]$. As this replacement does not alter the coefficients of the superfields, which are the source of poles in OPE calculations, we may assume that the OPE of this with a third $U_{3}$ follows an identical procedure, up to similar replacements in the result. That is, we would expect [83]

$$
\begin{equation*}
U_{[1,2]} U_{3} \rightarrow U_{[[1,2], 3]}=\partial \theta^{\alpha} A_{\alpha}^{[11,2], 3]}+\Pi^{m} A_{m}^{[1,2], 3]}+d_{\alpha} W_{[1,2], 3]}^{\alpha}+\frac{1}{2} N^{m n} F_{m n}^{[11,2], 3]} \tag{4.1.19}
\end{equation*}
$$

where the three particle superfields are defined using similar rules to those involving two particles. So for instance, where before we had

$$
\begin{equation*}
A_{\alpha}^{[1,2]}=-\frac{1}{2}\left[A_{\alpha}^{1}\left(k^{1} \cdot A^{2}\right)+A_{m}^{1}\left(\gamma^{m} W^{2}\right)_{\alpha}-(1 \leftrightarrow 2)\right], \tag{4.1.20}
\end{equation*}
$$

we now have

$$
\begin{equation*}
A_{\alpha}^{[[1,2], 3]}=-\frac{1}{2}\left[A_{\alpha}^{[1,2]}\left(k^{12} \cdot A^{3}\right)+A_{m}^{[1,2]}\left(\gamma^{m} W^{3}\right)_{\alpha}-([1,2] \leftrightarrow 3)\right] \tag{4.1.21}
\end{equation*}
$$

Analogous relations would be expected for the other superfields. However, minor modifications must be made to the $W$ and $F$ superfields in order to preserve the structure of the equations of motion. That is, following (4.1.13) the equation of motion of the $A_{\alpha}^{[[1,2], 3]}$ superfield would be expected to have the form

$$
\begin{equation*}
D_{(\alpha} A_{\beta)}^{[[1,2], 3]}=\gamma_{\alpha \beta}^{m} A_{m}^{[[1,2], 3]}+(\text { Mandelstams }) \times(\text { Correction Terms }) \tag{4.1.22}
\end{equation*}
$$

We then use this to define $A_{m}^{[[1,2], 3]}$, and repeat to similarly define the remaining superfields through further equations of motion. So, to detail this example further, one may find that [83]

$$
\begin{align*}
D_{(\alpha} A_{\beta)}^{[[1,2], 3]} & =\gamma_{\alpha \beta}^{m}\left[\frac{1}{2}\left[A_{p}^{[1,2]} F_{p m}^{3}-A_{m}^{[1,2]}\left(k^{12} \cdot A^{3}\right)+\left(W^{[1,2]} \gamma_{m} W^{3}\right)-([1,2] \leftrightarrow 3)\right]\right] \\
& +\left(k^{1} \cdot k^{2}\right)\left[A_{\alpha}^{1} A_{\beta}^{[2,3]}+A^{[1,3]_{\alpha}} A_{\beta}^{2}-(1 \leftrightarrow 2)\right] \\
& +\left(k^{12} \cdot k^{3}\right)\left[A_{\alpha}^{[1,2]} A_{\beta}^{3}-([1,2] \leftrightarrow 3)\right] \tag{4.1.23}
\end{align*}
$$

We may then read the definition of $A_{m}^{[[1,2], 3]}$ directly from this,

$$
\begin{equation*}
A_{m}^{[[1,2], 3]}=\frac{1}{2}\left(A_{p}^{[1,2]} F_{p m}^{3}-A_{m}^{[1,2]}\left(k^{12} \cdot A^{3}\right)+\left(W^{[1,2]} \gamma_{m} W^{3}\right)-([1,2] \leftrightarrow 3)\right) \tag{4.1.24}
\end{equation*}
$$

This, we then note, is very similar to the definition of $A_{m}^{[1,2]}$ (4.1.9). Similarly, by repeatedly taking derivatives we may find the other three particle superfields [83],

$$
\begin{align*}
W_{[[1,2], 3]}^{\alpha} & =\left(-\left(k^{12} \cdot A^{3}\right) W_{[1,2]}^{\alpha}+\frac{1}{4}\left(\gamma^{r s} W^{3}\right)^{\alpha} F_{r s}^{[1,2]}-([1,2] \leftrightarrow 3)\right)  \tag{4.1.25}\\
& +\frac{1}{2}\left(k^{1} \cdot k^{2}\right)\left(W_{2}^{\alpha}\left(A^{1} \cdot A^{3}\right)-(1 \leftrightarrow 2)\right) \\
F_{r s}^{[[1,2], 3]} & =\left(\left(k^{3} \cdot A^{[1,2]}\right) F_{m n}^{3}+F_{a[m}^{[1,2]} F_{n] a}^{3}+2 k_{[m}^{12}\left(W^{3} \gamma_{n]} W^{[1,2]}\right)-([1,2] \leftrightarrow 3)\right) \\
& +\left(k^{1} \cdot k^{2}\right)\left(\frac{1}{2} F_{m n}^{2}\left(A^{1} \cdot A^{3}\right)+2 A_{[m}^{1}\left(W^{3} \gamma_{n]} W^{2}\right)-(1 \leftrightarrow 2)\right) \tag{4.1.26}
\end{align*}
$$

Unlike with two particles, there is another class of superfields at three points. We have a triplet of vertex operators,

$$
\begin{equation*}
U^{1}\left(z_{1}\right) U^{2}\left(z_{2}\right) U^{3}\left(z_{3}\right) \tag{4.1.27}
\end{equation*}
$$

and are looking at terms corresponding with the poles where $z_{1}, z_{2}$, and $z_{3}$ coincide. In the previous discussion, we took it that first we approach the $z_{1}=z_{2}$ pole first, and then bring $z_{3}$ to this same value. We are limited by the integration domain of (4.1.1) $z_{1} \leq z_{2} \leq z_{3}$, but this does allow for a second way of approaching this pole. Namely, we
may take $z_{2}=z_{3}$ first, and then set $z_{1}$ equal to these. Hence, we may define another class of three particle superfields by [88]

$$
\begin{align*}
U^{1}\left(U^{2} U^{3}\right) & \rightarrow U^{1} U^{[2,3]}  \tag{4.1.28}\\
& \rightarrow U^{[1,[2,3]]}=\partial \theta^{\alpha} A_{\alpha}^{[1,[2,3]]}+\Pi^{m} A_{m}^{[1,[2,3]]}+d_{\alpha} W_{[1,[2,3]]}^{\alpha}+\frac{1}{2} N^{m n} F_{m n}^{[1,[2,3]]},
\end{align*}
$$

The resulting superfields are then defined similarly to the previous discussion, with the first case being

$$
\begin{equation*}
A_{\alpha}^{[1,[2,3]]}=-\frac{1}{2}\left[A_{\alpha}^{1}\left(k^{1} \cdot A^{[2,3]}\right)+A_{m}^{1}\left(\gamma^{m} W^{[2,3]}\right)_{\alpha}-(1 \leftrightarrow[2,3])\right] . \tag{4.1.29}
\end{equation*}
$$

To introduce some terminology, we say there are two different topologies of multiparticle superfields at multiplicity three, or equivalently of rank three. A topology in this instance refers to a Lie monomial associated with a superfield, and the multiplicity or rank refers to the number of particles associated with that superfield.

### 4.1.0.3 Product of Arbitrarily Many Unintegrated Vertex Operators

This procedure may now be applied indefinitely, to find arbitrarily complicated products of unintegrated vertex operators. Doing so leads one to conjecture the following general expressions for arbitrary multiparticle superfields [88]

$$
\begin{align*}
& A_{\alpha}^{[P, Q]}=-\frac{1}{2}\left[A_{\alpha}^{P}\left(k^{P} \cdot A^{Q}\right)+A_{m}^{P}\left(\gamma^{m} W^{Q}\right)_{\alpha}-(P \leftrightarrow Q)\right],  \tag{4.1.30}\\
& A_{m}^{[P, Q]}=\frac{1}{2}\left[A_{p}^{P} F_{p m}^{Q}-A_{m}^{P}\left(k^{P} \cdot A^{Q}\right)+\left(W^{P} \gamma_{m} W^{Q}\right)-(P \leftrightarrow Q)\right],  \tag{4.1.31}\\
& W_{[P, Q]}^{\alpha}=-\frac{1}{2}\left[W_{P}^{\alpha}\left(k_{P} \cdot A_{Q}\right)+W_{P}^{m \alpha} A_{Q}^{m}+\frac{1}{2}\left(\gamma_{r s} W_{P}\right)^{\alpha} F_{Q}^{r s}-(P \leftrightarrow Q)\right],  \tag{4.1.32}\\
& F_{\ell(A)}^{m n}=k_{m}^{A} A_{n}^{\ell(A)}-k_{n}^{A} A_{m}^{\ell(A)}+\sum_{\substack{X j Y=A \\
Y=R \amalg S}} 2\left(\left(k_{X} \cdot k_{j}\right) A_{[n}^{\ell(X R)} A_{m]}^{\ell(j S)}-(X \leftrightarrow j)\right), \tag{4.1.33}
\end{align*}
$$

where $P$ and $Q$ denote any Lie monomial. Note the definition of $F^{m n}$ given above is limited to left-to-right Dynkin brackets only. A general expression for arbitrary bracketing structures is presented for it later (see (6.3.7)), but for now we are only discussing background material and such was not known at the start of this work. Further, we note that we have introduced the multiparticle linearisation of field strength seen in (3.1.7), $W_{P}^{m \alpha}$, which is defined [88]

$$
\begin{equation*}
W_{\ell(P)}^{m \alpha}=k_{P}^{m} W_{\ell(P)}^{\alpha}+\sum_{\substack{X j Y=P \\ Y=R \uplus S}}\left(k^{X} \cdot k^{j}\right)\left(W_{\ell(X R)}^{\alpha} A_{\ell(j S)}^{m}-(X \leftrightarrow j)\right) . \tag{4.1.34}
\end{equation*}
$$

Again, this was only defined in specific cases at the start of this work, and a general form is identified later (see (6.3.8)).

### 4.1.0.4 Products of Integrated and Unintegrated Vertex Operators

The discussion of integrated vertex operators here will be considerably more brief. Poles do not arise as a result of these objects interacting with each other; we need only consider their OPEs with unintegrated vertex operators. That is, we need to find some multiparticle $V_{A}$ such that

$$
\begin{equation*}
V_{A} U_{B} \rightarrow V_{[A, B]}, \tag{4.1.35}
\end{equation*}
$$

for $A$ and $B$ some general Lie bracket structures. This turns out to be relatively simple; given that the multiparticle version of the unintegrated vertex operator is just the single particle case with the superfields replaced by their multiparticle equivalents, it may not be surprising that we do the same here,

$$
\begin{equation*}
V_{1}=\lambda^{\alpha} A_{\alpha}^{1} \quad \Rightarrow \quad V_{[P, Q]}=\lambda^{\alpha} A_{\alpha}^{[P, Q]} \tag{4.1.36}
\end{equation*}
$$

This may seem arbitrary but it is consistent, as is discussed in more detail in [83]. We note here the form of the variations of these vertex operators, which can be found to be [83]

$$
\begin{align*}
& Q V_{\ell(P)}=\sum_{\substack{X j Y=P \\
R \sqcup S=Y}}\left(k^{X} \cdot k^{j}\right) V_{\ell(X R)} V_{\ell(j S)}  \tag{4.1.37}\\
& Q U_{\ell(P)}=\partial V_{\ell(P)}+\sum_{\substack{X j Y=P \\
R \sqcup S=Y}}\left(k^{X} \cdot k^{j}\right)\left(V_{\ell(X R)} U_{\ell(j S)}-(X \leftrightarrow j)\right) \tag{4.1.38}
\end{align*}
$$

Note this may be expanded to other Lie bracket structures using methods which will be discussed in chapter 6

### 4.1.1 Berends-Giele Currents

Amplitude expressions are dominated by their unintegrated vertex operator component,

$$
\begin{equation*}
\int_{z_{1} \leq z_{2} \leq \ldots \leq z_{n}} d z_{1} \ldots d z_{n} U_{1}\left(z_{1}\right) U_{2}\left(z_{2}\right) \ldots U_{n}\left(z_{n}\right) \tag{4.1.39}
\end{equation*}
$$

By the methods outlined above, we may take objects of this form and write them as sums over multiparticle superfields with an associated pole structure. That is, if we wish to perform the above integral using the OPE methods above, we must sum over all possible ways such OPEs may be performed in order to perform the integral properly. This
corresponds with summing over all possible topologies of $U$ with rank $n$, each divided by mandelstams arising from their particular integrations. Such are known as (pure spinor) Berends-Giele currents (BG currents), and are best understood through examples.

At rank two, the BG current of a superfield $K$ is given by [83]

$$
\begin{equation*}
\mathcal{K}_{12}=\frac{K_{[1,2]}}{s_{12}} . \tag{4.1.40}
\end{equation*}
$$

This corresponds with the integral of $U_{1} U_{2}$. There is only one pole to consider the integral over, and so only one term in the above. The $s_{12}$ factor arises from the the integral over this $z_{12}^{-s_{12}-1}$ pole.

At rank three, the BG current now has two terms, [83]

$$
\begin{equation*}
\mathcal{K}_{123}=\frac{K_{[[1,2], 3]}}{s_{12} s_{123}}+\frac{K_{[1,[2,3]]}}{s_{23} s_{123}} . \tag{4.1.41}
\end{equation*}
$$

These arise from the two distinct poles in the integration domain, with the former term coming from integrating the $z_{12}$ pole first and the latter from the $z_{23}$ pole instead.

This generalises to arbitrary points, and there are two approaches to describe this. The first was through the inverse KLT matrix [83; 116],

$$
\begin{equation*}
\mathcal{K}_{123 \ldots p}:=\sum_{\rho \in S_{p-1}} S^{-1}[23 \ldots p \mid \rho]_{1} K_{1 \rho(23 \ldots p)} \tag{4.1.42}
\end{equation*}
$$

where $S_{p}$ is the symmetric group, the set of permutations of $p$ elements. This is not the form we will express them in though. Instead, we use the $b$-map (2.3.13) [60],

$$
\begin{equation*}
\mathcal{K}_{12 \ldots p}=K_{b(12 \ldots p)} . \tag{4.1.43}
\end{equation*}
$$

The examples (4.1.40), (4.1.41) then follow from the $b$-map expressions

$$
\begin{equation*}
b(12)=\frac{[1,2]}{s_{12}}, \quad b(123)=\frac{[[1,2], 3]}{s_{12} s_{123}}+\frac{[1,[2,3]]}{s_{23} s_{123}} . \tag{4.1.44}
\end{equation*}
$$

Berends-Giele currents are non-local objects, while their constituent superfields are local. As a corollary, this $b$-map definition implies that at multiplicity $n$ there are $C_{n}$ unique topologies of superfields, with $C_{n}$ the $n^{\text {th }}$ Catalan number (2.3.15). The BG current corresponding with a particular superfield is usually denoted with the calligraphic form of that superfield, with the exception being the $V$ vertex operators which have BG currents denoted $\mathcal{M}$. The etymology of BG currents lies in that they form amplitudes and that their components are defined recursively, alike their namesakes.

One particular result of importance is that it may be found that

$$
\begin{equation*}
k_{m}^{P} \mathcal{A}_{P}^{m}=0 \tag{4.1.45}
\end{equation*}
$$

for $P$ a word of any length. This satisfies the definition (3.1.14), and so we talk about multiparticle superfields constructed in this way as being in the Lorenz gauge.

### 4.2 The BCJ Gauge

The notation used in the previous subsection should be reminiscent of that of section 2.4 ; in that we are using a Lie monomial notation to denote something, but we have yet to justify that notation through the verification of Jacobi identities. This is an important step; as will be seen shortly, the presence of BCJ relations relies upon the multiparticle superfields we construct them out of satisfying Jacobi identities.

As presently defined, the superfields of rank greater than 2 do not satisfy this requirement, but this may be brought through a gauge transformation. There are two known approaches to find the BCJ gauge. The former, which we discuss now, involves an intermediate hybrid gauge. This is better understood, and the recommended approach for performing calculations. The latter, which will be in the following section, makes the gauge transformation structure of this methodology much more apparent, but is less well understood.

### 4.2.1 Hybrid Gauge Construction

We begin now with the enforcement of BCJ relations at rank 3, and build up towards a general method. We limit ourselves to what was known at the start of this research, and so we will almost exclusively focus upon left-to-right Dynkin brackets. Finally we introduce notation to denote the gauge of an arbitrary superfield $K$; a plain $K$ is in the BCJ gauge, the addition of a check $\check{K}$ denotes the hybrid gauge, and the addition of a hat $\hat{K}$ denotes the Lorenz gauge. Note the methods of this subsection are drawn from [83] entirely.

### 4.2.1.1 Finding the BCJ Gauge at Rank Three

Following the discussion of 2.4 we see there are two generalised Jacobi identities to verify at rank three,

$$
\begin{equation*}
\mathcal{L}_{2} \circ K_{[[1,2], 3]}=0 \quad \leftrightarrow \quad K_{[[1,2], 3]}+K_{[[2,1], 3]}=0 \tag{4.2.1}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{L}_{3} \circ K_{[11,2], 3]}=0 \quad \leftrightarrow \quad K_{[[1,2], 3]}+K_{[[2,3], 1]}+K_{[[3,1], 2]}=0 \tag{4.2.2}
\end{equation*}
$$

The $\mathcal{L}_{2}$ relation is already satisfied. Further, two superfields, $W_{123}^{\alpha}$ and $F_{m n}^{123}$, satisfy the $\mathcal{L}_{3}$ relation also. We then exploit this to enforce the $\mathcal{L}_{3}$ relation for the remaining superfields using their equations of motion. That of $A_{m}^{123}$ is

$$
\begin{align*}
D_{\alpha} \hat{A}_{m}^{[1,2], 3]} & =k_{m}^{123} \hat{A}_{\alpha}^{[1,2], 3]}+\left(\gamma^{m} W^{[11,2], 3]}\right)_{\alpha} \\
& +\left(k^{1} \cdot k^{2}\right)\left[A_{\alpha}^{1} A_{m}^{[2,3]}+A_{\alpha}^{[1,3]} A_{m}^{2}-(1 \leftrightarrow 2)\right]  \tag{4.2.3}\\
& +\left(k^{12} \cdot k^{3}\right)\left[A_{\alpha}^{[1,2]} A_{m}^{3}-([1,2] \leftrightarrow 3)\right] .
\end{align*}
$$

If we act on both sides of this equation with the $\mathcal{L}_{3}$ operator, the terms proportional to mandelstams can be seen to cancel, and as stated $\mathcal{L}_{3} \circ W^{123}=0$ also. Hence, we are left with

$$
\begin{equation*}
D_{\alpha}\left(\mathcal{L}_{3} \circ \hat{A}_{123}^{m}\right)=k_{m}^{123}\left(\mathcal{L}_{3} \circ \hat{A}_{\alpha}^{123}\right) . \tag{4.2.4}
\end{equation*}
$$

From this, one can deduce that $\mathcal{L}_{3} \circ \hat{A}_{m}^{123}$ should contain an overall $k_{m}^{123}$ factor. That is,

$$
\begin{equation*}
\mathcal{L}_{3} \circ \hat{A}_{123}^{m}=3 k_{123}^{m} H_{123}, \tag{4.2.5}
\end{equation*}
$$

where $H_{123}$ is some combination of superfields defined by the above, and the factor of 3 is included for convenience. Further, it follows from (4.2.4) that

$$
\begin{equation*}
\mathcal{L}_{3} \circ \hat{A}_{\alpha}^{123}=3 D_{\alpha} H_{123}, \tag{4.2.6}
\end{equation*}
$$

where this $H_{123}$ is the same as defined by (4.2.5). Finally, we note that a symmetry of $H_{123}$ can be identified,

$$
\begin{equation*}
\mathcal{L}_{3} \circ H_{123}=3 H_{123} . \tag{4.2.7}
\end{equation*}
$$

Putting all of this together, we infer that by making the redefinitions

$$
\begin{align*}
& A_{m}^{123}=\hat{A}_{m}^{123}-k_{m}^{123} H^{123},  \tag{4.2.8}\\
& A_{\alpha}^{123}=\hat{A}_{\alpha}^{123}-D_{\alpha} H^{123}, \tag{4.2.9}
\end{align*}
$$

BCJ relations may be enforced on the complete set of superfields. It remains to find the explicit form of $H_{123}$, and a simple but lengthy calculation reveals it to have the form

$$
\begin{equation*}
H_{123}=-\frac{1}{4} A_{1}^{m} A_{2}^{n} F_{3}^{m n}+\frac{1}{2}\left(W_{1} \gamma_{m} W_{2}\right) A_{3}^{m}+\operatorname{cyclic}(1,2,3) \tag{4.2.10}
\end{equation*}
$$

It is worth noting that this satisfies the relation

$$
\begin{equation*}
\mathcal{L}_{2} \circ H_{123}=0 \tag{4.2.11}
\end{equation*}
$$

and so we do not sabotage the one BCJ relation while enforcing the other. A similar situation will arise at higher ranks, and verifying relations like (4.2.11) serves as something of a consistency check in calculations.

### 4.2.1.2 Finding the BCJ Gauge at Rank Four

We then consider the case of the rank four superfields. We take advantage of the fact that we know how to construct lower rank BCJ gauge superfields, and define hybrid gauge superfields using a modification of the superfield equations (4.1.30) - (4.1.32),

$$
\begin{align*}
\check{A}_{\alpha}^{1234}= & -\frac{1}{2}\left[A_{\alpha}^{123}\left(k^{123} \cdot A^{4}\right)+A_{m}^{123}\left(\gamma^{m} W^{4}\right)_{\alpha}-(123 \leftrightarrow 4)\right]  \tag{4.2.12}\\
\check{A}_{m}^{1234}= & \frac{1}{2}\left[A_{p}^{123} F_{p m}^{4}-A_{m}^{123}\left(k^{123} \cdot A^{4}\right)+\left(W^{123} \gamma_{m} W^{4}\right)-(123 \leftrightarrow 4)\right]  \tag{4.2.13}\\
\check{W}_{1234}^{\alpha}= & \frac{1}{4}\left(\gamma^{m n} W^{4}\right)^{\alpha} F_{m n}^{123}+W_{4}^{\alpha}\left(k^{4} \cdot A^{123}\right)-(123 \leftrightarrow 4) \\
& -\frac{1}{2} \sum_{\substack{X j Y=123}}\left(k^{X} \cdot k^{j}\right)\left(W_{X R}^{\alpha}\left(A^{j S} \cdot A^{4}\right)-(X \leftrightarrow j)\right) .  \tag{4.2.14}\\
& Y=R \amalg S
\end{align*}
$$

The $F_{m n}$ superfield in the BCJ gauge will be defined separately, using a modification of (4.1.33),

$$
\begin{equation*}
F_{r s}^{1234}=k_{r}^{1234} A_{s}^{1234}-k_{s}^{1234} A_{r}^{1234}-\sum_{\substack{X j Y=1234 \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right) 2 A_{[r}^{X R} A_{s]}^{j S} \tag{4.2.15}
\end{equation*}
$$

These hybrid gauge superfields will satisfy the $\mathcal{L}_{2}$ and $\mathcal{L}_{3}$ symmetries trivially, since their constituent parts satisfy them. It remains to enforce one final symmetry, $\mathcal{L}_{4}$. To do so, we again turn to the equation of motion of $\hat{A}_{m}$ for guidance. This is again of the same structure, and will be in general,

$$
\begin{equation*}
D_{\alpha} \check{A}_{1234}^{m}=k_{m}^{1234} \check{A}_{\alpha}^{1234}+\left(\gamma_{m} \check{W}^{1234}\right)_{\alpha}+(\text { terms }) \tag{4.2.16}
\end{equation*}
$$

The difference now though is that the $\check{W}_{\alpha}^{1234}$ term does not immediately vanish under the $\mathcal{L}_{4}$ operation. Additionally the extra terms unspecified above also do not vanish under such an operation. These problems must be corrected for.

We begin with enforcing the $\mathcal{L}_{4}$ relation on $\check{W}_{\alpha}^{1234}$. This has equation of motion

$$
\begin{equation*}
D_{\alpha} \check{W}_{1234}^{\beta}=\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}^{\beta} \hat{F}_{m n}^{1234}+\sum_{\substack{X j Y=123 \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(\hat{A}_{\alpha}^{X R 4} W_{j S}^{\beta}-(X \leftrightarrow j)\right) \tag{4.2.17}
\end{equation*}
$$

The difficulties in enforcing the $\mathcal{L}_{4}$ relation can be fixed by making the redefinition

$$
\begin{equation*}
W_{1234}^{\alpha}=\hat{W}_{1234}^{\alpha}+\sum_{\substack{X j Y=123 \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(W_{X R}^{\alpha} H_{j S 4}-(X \leftrightarrow j)\right) \tag{4.2.18}
\end{equation*}
$$

We then must deal with the absence of GJIs in the correction terms in (4.2.16). These corrections have a similar structure to those of (4.2.17), and so a similar redefinition,

$$
\begin{equation*}
K_{1234}^{\prime}=K_{1234}+\sum_{\substack{X j Y=123 \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(K_{X R}^{\alpha} H_{j S 4}-(X \leftrightarrow j)\right), \quad K \in\left\{A^{m}, A_{\alpha}\right\} \tag{4.2.19}
\end{equation*}
$$

corrects for this error.

We now have the same situation as we had at rank three, and so we again apply $\mathcal{L}_{4}$ to the equation of motion of $A_{1234}^{\prime m}$. Almost all terms now cancel, and we are left with

$$
\begin{equation*}
D_{\alpha}\left(\mathcal{L}_{4} \circ \hat{A}_{1234}^{\prime m}\right)=k_{m}^{1234}\left(\mathcal{L}_{4} \circ \hat{A}_{\alpha}^{1234}\right) . \tag{4.2.20}
\end{equation*}
$$

Again, we deduce that there is some combination of superfields, which we shall call $H_{1234}$, which satisfies

$$
\begin{equation*}
\mathcal{L}_{2} \circ H_{1234}=\mathcal{L}_{3} \circ H_{1234}=0, \quad \mathcal{L}_{4} \circ H_{1234}=4 H_{1234} \tag{4.2.21}
\end{equation*}
$$

Hence, we again perform a final set of redefinitions,

$$
\begin{align*}
& A_{m}^{1234}=A_{m}^{\prime 1234}-k_{m}^{123} H^{1234}  \tag{4.2.22}\\
& A_{\alpha}^{1234}=A_{\alpha}^{\prime 1234}-D_{\alpha} H^{1234} \tag{4.2.23}
\end{align*}
$$

The resulting expressions for $A_{m}$ and $A_{\alpha}$ then satisfy all generalised Jacobi identities. The specific form of the $H_{1234}$ is more complex than at rank three, but may be conveniently represented as

$$
\begin{equation*}
H_{1234}=\frac{1}{4}\left(H_{12,3,4}^{\prime}+H_{1,2,34}^{\prime}\right), \tag{4.2.24}
\end{equation*}
$$

where the $H^{\prime}$ superfields are defined by,

$$
\begin{equation*}
H_{A, B, C}^{\prime}=H_{A, B, C}+\frac{1}{2}\left(H_{[A, B]}\left(K_{A B} \cdot A_{C}\right)+\operatorname{cyclic}(A, B, C)\right), \tag{4.2.25}
\end{equation*}
$$

and $H_{A, B, C}$ is the natural generalisation of $H_{123}$

$$
\begin{equation*}
H_{A, B, C}=-\frac{1}{4} A_{A}^{m} A_{B}^{n} F_{C}^{m n}+\frac{1}{2}\left(W_{A} \gamma_{m} W_{B}\right) A_{C}^{m}+\operatorname{cyclic}(\mathrm{A}, \mathrm{~B}, \mathrm{C}) . \tag{4.2.26}
\end{equation*}
$$

### 4.2.1.3 Finding the BCJ Gauge at General Rank

Using the machinery of rank four, we may outline the general method of enforcing GJIs upon multiparticle superfields to arbitrary order. We begin by defining a rank $n$ hybrid gauge superfield in terms of rank $(n-1)$ BCJ gauge superfields,

$$
\begin{gather*}
\check{A}_{\alpha}^{12 \ldots n}=-\frac{1}{2}\left[A_{\alpha}^{12 \ldots n-1}\left(k^{1} \cdot A^{n}\right)+A_{m}^{12 \ldots n-1}\left(\gamma^{m} W^{n}\right)_{\alpha}-(12 \ldots n-1 \leftrightarrow n)\right]  \tag{4.2.27}\\
\check{A}_{m}^{12 \ldots n}=\frac{1}{2}\left[A_{p}^{12 \ldots n-1} F_{p m}^{n}-A_{m}^{12 \ldots n-1}\left(k^{1} \cdot A^{n}\right)+\left(W^{12 \ldots n-1} \gamma_{m} W^{n}\right)\right.  \tag{4.2.28}\\
-(12 \ldots n \leftrightarrow n)] \\
\check{W}_{12 \ldots n}^{\alpha}=-\frac{1}{2}\left[W_{12 \ldots n-1}^{\alpha}\left(k_{12 \ldots n-1} \cdot A_{n}\right)+W_{12 \ldots n-1}^{m \alpha} A_{n}^{m}\right. \\
\left.\quad+\frac{1}{2}\left(\gamma_{r s} W_{12 \ldots n-1}\right)^{\alpha} F_{n}^{r s}-(12 \ldots n-1 \leftrightarrow n)\right] . \tag{4.2.29}
\end{gather*}
$$

Again the $F_{m n}$ superfields will be defined differently, as functions of superfields in the BCJ gauge,

$$
\begin{equation*}
F_{r s}^{12 \ldots n}=2\left(k_{[r}^{12 \ldots n} A_{s]}^{12 \ldots n}-\sum_{\substack{X j Y=12 \ldots n \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right) A_{[r}^{X R} A_{s]}^{j S}\right) . \tag{4.2.30}
\end{equation*}
$$

We then correct for the absence of lower rank GJIs in the correction terms of the equation of motion for $A_{m}^{12 \ldots n}$, and similar for $\check{W}_{12 \ldots . n}^{\alpha}$. These two issues can be corrected for with a single redefinition formula,

$$
\begin{equation*}
K_{12 \ldots p}^{\prime}:=\check{K}_{12 \ldots p}-\sum_{\substack{X j Y=12 \ldots p-1 \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(H_{X R p} K_{j S}-(X \leftrightarrow j)\right) \tag{4.2.31}
\end{equation*}
$$

where $K \in\left\{A_{m}, A_{\alpha}, W^{\alpha}\right\}$, and $H_{i}=H_{i j}=0$. The superfield $W_{12 \ldots n}^{\prime \alpha}$ then satisfies all required relations, and so we set

$$
\begin{equation*}
W_{12 \ldots n}^{\prime \alpha}=W_{12 \ldots n}^{\alpha} \tag{4.2.32}
\end{equation*}
$$

Taking $\mathcal{L}_{n}$ of the equation of motion for $A_{m}^{12 \ldots n}$ gives

$$
\begin{equation*}
D_{\alpha}\left(\mathcal{L}_{n} \circ A_{m}^{12 \ldots n}\right)=k_{m}^{12 \ldots n}\left(\mathcal{L}_{n} \circ A_{\alpha}^{12 \ldots n}\right) \tag{4.2.33}
\end{equation*}
$$

We therefore deduce that there exists some combination of superfields $H_{12 \ldots n}$ arising from

$$
\begin{align*}
& \mathcal{L}_{n} \circ A_{m}^{\prime 12 \ldots n}=n k_{m}^{12 \ldots n} H_{12 \ldots n}  \tag{4.2.34}\\
& \mathcal{L}_{n} \circ A_{\alpha}^{\prime 12 \ldots n}=n D_{\alpha} H_{12 \ldots n} \tag{4.2.35}
\end{align*}
$$

which satisfies

$$
\mathcal{L}_{p} \circ H_{12 \ldots n}=\left\{\begin{array}{ll}
0, & p<n  \tag{4.2.36}\\
H_{12 \ldots n}, & p=n
\end{array} .\right.
$$

A general formula for these $H$ terms was not known prior to the start of this work, but will be presented in chapter 7 . The five point case was previously known though, and is given by

$$
\begin{equation*}
H_{[1234,5]}=\frac{1}{5}\left(H_{123,4,5}^{\prime}-H_{543,2,1}^{\prime}+H_{12,3,45}^{\prime}\right) \tag{4.2.37}
\end{equation*}
$$

where the $H^{\prime}$ are defined as in (4.2.25). We therefore conclude by defining $A_{m}^{12 \ldots n}$ and $A_{\alpha}^{12 \ldots n}$,

$$
\begin{align*}
& A_{m}^{12 \ldots n}=A_{m}^{12 \ldots n}-k_{m}^{12 \ldots n} H^{12 \ldots n},  \tag{4.2.38}\\
& A_{\alpha}^{12 \ldots n}=A_{\alpha}^{\prime 12 \ldots n}-D_{\alpha} H^{12 \ldots n}, \tag{4.2.39}
\end{align*}
$$

Following this algorithm, general arbitrary rank superfields satisfying all required generalised Jacobi identities may be identified. There are of course a range of areas for generalisation within these definitions; to name two we have limited ourselves to left-toright Dynkin brackets, and we have no general formula for the $H$ terms. Save for some partial generalisations in the following section though, this was the state of the art at the beginning of my work.

### 4.2.2 Direct Transition from the Lorenz Gauge

Though the previous method will allow one to identify BCJ gauge superfields, it is far from obvious as to why the gauge description is valid. In this subsection we describe up to rank five how one may move directly from Lorenz to BCJ gauge superfields, without the need for the intermediate hybrid gauge. This description is understood for other topologies, but less so, and as such we begin again with left-to-right Dynkin brackets. Superfields in the Lorenz gauge are defined as in (4.1.30)-(4.1.33),

$$
\begin{gather*}
\hat{A}_{\alpha}^{12 \ldots n}=-\frac{1}{2}\left[\hat{A}_{\alpha}^{12 \ldots n-1}\left(k^{1} \cdot \hat{A}^{n}\right)+\hat{A}_{m}^{12 \ldots n-1}\left(\gamma^{m} \hat{W}^{n}\right)_{\alpha}-(12 \ldots n-1 \leftrightarrow n)\right],  \tag{4.2.40}\\
\begin{aligned}
& \hat{A}_{m}^{12 \ldots n}=\frac{1}{2}\left[\hat{A}_{p}^{12 \ldots n-1} \hat{F}_{p m}^{n}-\hat{A}_{m}^{12 \ldots n-1}\left(k^{1} \cdot \hat{A}^{n}\right)+\left(\hat{W}^{12 \ldots n-1} \gamma_{m} \hat{W}^{n}\right)\right. \\
&\quad-(12 \ldots n \leftrightarrow n)], \\
& \hat{W}_{12 \ldots n}^{\alpha}=-\frac{1}{2}\left[\hat{W}_{12 \ldots n-1}^{\alpha}\left(k_{12 \ldots n-1} \cdot \hat{A}_{n}\right)+\hat{W}_{12 \ldots n-1}^{m \alpha} \hat{A}_{n}^{m}\right. \\
&\left.+\frac{1}{2}\left(\gamma_{r s} \hat{W}_{12 \ldots n-1}\right)^{\alpha} \hat{F}_{n}^{r s}-(12 \ldots n-1 \leftrightarrow n)\right],
\end{aligned}
\end{gather*}
$$

$$
\begin{equation*}
\hat{F}_{r s}^{12 \ldots n}=2\left(k_{[r}^{12 \ldots n} \hat{A}_{s]}^{12 \ldots n}-\sum_{\substack{X j Y=12 \ldots n \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right) \hat{A}_{[r}^{X R} \hat{A}_{s]}^{j S}\right) . \tag{4.2.43}
\end{equation*}
$$

It has then been found that the following formula takes these superfields and outputs the BCJ gauge equivalent [88]. For $K=\left\{A_{\alpha}, A_{m}, W^{\alpha}\right\}$,

$$
\begin{align*}
K_{12 \ldots p}:=\hat{K}_{12 \ldots p}-\sum_{\substack{X j Y=12 \ldots p \\
Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(\hat{H}_{X R} \hat{K}_{j S}-\right. & (X \leftrightarrow j)) \\
& - \begin{cases}D_{\alpha} \hat{H}_{12 \ldots p} & \text { if } K=A_{\alpha} \\
k_{12 \ldots p}^{m} \hat{H}_{12 \ldots p} & \text { if } K=A^{m} \\
0 & \text { if } K=W^{\alpha}\end{cases} \tag{4.2.44}
\end{align*}
$$

The superfields $\hat{H}$ are then defined recursively in terms of the $H$ superfields [88],

$$
\begin{equation*}
\hat{H}_{12 \ldots n}=H_{12 \ldots n}-\frac{1}{2} \hat{H}_{12 \ldots n-1}\left(K_{12 \ldots n-1} \cdot \hat{A}_{n}\right) \tag{4.2.45}
\end{equation*}
$$

The origin of this formula is much more mysterious than that of the hybrid gauge approach. One may carefully examine the superfields in both and see that the two are equivalent, but why the above should arise remains unknown.

### 4.2.2.1 General Topologies

Formulae to move to the BCJ gauge for superfields of arbitrary topology up to rank five using this direct approach were identified in [88], and may be found in appendix C.1. These involve a generalisation of the $\hat{H}$ formula (4.2.45),

$$
\begin{equation*}
\hat{H}_{[A, B]}=H_{[A, B]}-\frac{1}{2}\left[\hat{H}_{A}\left(K_{A} \cdot \hat{A}_{B}\right)-(A \leftrightarrow B)\right], \tag{4.2.46}
\end{equation*}
$$

where $A$ and $B$ are Lie monomials. It is clear that when $A$ is a Dynkin bracket, and $B$ a letter, this reduces to (4.2.45). The full set of $H$ superfields needed to rank five are given by

$$
\begin{align*}
H_{[12,3]} & =\frac{1}{3} H_{1,2,3}, \\
H_{[123,4]} & =\frac{1}{4}\left(H_{12,3,4}^{\prime}+H_{34,1,2}^{\prime}\right), \\
H_{[12,34]} & =\frac{1}{4}\left(-2 H_{12,3,4}^{\prime}+2 H_{34,1,2}^{\prime}\right),  \tag{4.2.47}\\
H_{[1234,5]} & =\frac{1}{5}\left(H_{123,4,5}^{\prime}-H_{543,2,1}^{\prime}+H_{12,3,45}^{\prime}\right), \\
H_{[123,45]} & =\frac{1}{5}\left(-3 H_{123,4,5}^{\prime}-2 H_{543,2,1}^{\prime}+2 H_{12,3,45}^{\prime}\right),
\end{align*}
$$

Where the $H_{A, B, C}^{\prime}$ and $H_{A, B, C}$ are defined as in (4.2.25), (4.2.26)

$$
\begin{align*}
& H_{A, B, C}^{\prime}=H_{A, B, C}+\frac{1}{2}\left(H_{[A, B]}\left(k_{A B} \cdot A_{C}\right)+\operatorname{cyclic}(A, B, C)\right)  \tag{4.2.48}\\
& H_{A, B, C}=-\frac{1}{4} A_{A}^{m} A_{B}^{n} F_{C}^{m n}+\frac{1}{2}\left(W_{A} \gamma_{m} W_{B}\right) A_{C}^{m}+\operatorname{cyclic}(A, B, C) \tag{4.2.49}
\end{align*}
$$

### 4.2.3 Berends-Giele Currents in the BCJ Gauge

In this subsection, we present a pair of results on BG currents which will be of use in later sections. The first is that, assuming its constituent superfields are in the BCJ gauge, we may invert their definitions and find expressions for superfields in terms of their BG currents. To see this, we begin with the rank two case, which is trivial,

$$
\begin{equation*}
\mathcal{K}_{12}=\frac{K_{12}}{s_{12}} \quad \Rightarrow \quad K_{12}=\mathcal{K}_{12} s_{12} . \tag{4.2.50}
\end{equation*}
$$

At rank three though, BCJ relations are required. The BG current is given by

$$
\begin{equation*}
\mathcal{K}_{123}=\frac{K_{[[1,2], 3]}}{s_{12} s_{123}}+\frac{K_{[1,[2,3]]}}{s_{23} s_{123}} . \tag{4.2.51}
\end{equation*}
$$

We then combine this with another labelling,

$$
\begin{align*}
s_{23} \mathcal{K}_{123}-s_{13} \mathcal{K}_{213} & =\left(\frac{s_{23} K_{[11,2], 3]}}{s_{12} s_{123}}+\frac{K_{[1,[2,3]]}}{s_{123}}\right)-\left(\frac{s_{13} K_{[[2,1], 3]}}{s_{12} s_{123}}+\frac{K_{[2,[1,3]]}}{s_{123}}\right)  \tag{4.2.52}\\
& =\frac{1}{s_{12} s_{123}}\left(s_{23} K_{[[1,2], 3]}-s_{13} K_{[[2,1], 3]}\right)+\frac{1}{s_{123}}\left(K_{[1,[2,3]]}-K_{[2,[1,3]]}\right)
\end{align*}
$$

The BCJ relations may then be used to write this exclusively in terms of the $V_{123}$ vertex operator,

$$
\begin{align*}
s_{23} \mathcal{K}_{123}-s_{13} \mathcal{K}_{213} & =\frac{1}{s_{12} s_{123}}\left(s_{23} K_{123}+s_{13} K_{123}\right)+\frac{1}{s_{123}} K_{123}  \tag{4.2.53}\\
\Rightarrow V_{123} & =s_{12}\left(s_{23} M_{123}^{B C J}-s_{13} M_{213}^{B C J}\right)
\end{align*}
$$

The second result is that Berends-Giele currents are annihilated by proper shuffles. That is,

$$
\begin{equation*}
\mathcal{K}_{A Ш B}=0, \quad \forall A, B \neq \emptyset \tag{4.2.54}
\end{equation*}
$$

where we use the notation

$$
\begin{equation*}
\mathcal{K}_{A Ш B} \equiv \sum_{\sigma \in A \amalg B} \mathcal{K}_{\sigma}, \tag{4.2.55}
\end{equation*}
$$

and $\amalg$ denotes the shuffle product, defined in (2.1.18). This arises as a property of the
b-map, which is itself always annihilated by proper shuffles.

$$
\begin{equation*}
b(A \sqcup B)=0 \tag{4.2.56}
\end{equation*}
$$

To give one example,

$$
\begin{align*}
s_{123} b(12 \sqcup \sqcup 3) & =b(123)+b(132)+b(312) \\
& =s_{12}^{-1}[[1,2], 3]+s_{23}^{-1}[1,[2,3]]+s_{13}^{-1}[[1,3], 2]+s_{23}^{-1}[1,[3,2]]  \tag{4.2.57}\\
& +s_{13}^{-1}[[3,1], 2]+s_{12}^{-1}[3,[1,2]] \\
& =0,
\end{align*}
$$

with the final equality following from antisymmetry. This result arises as a consequence of a standard result in free Lie algebras, that all Lie polynomials are orthogonal to proper shuffles ${ }^{1}$ [43].

### 4.2.4 Justification for the Gauge Transformation Description

In order to justify that these methods represent a gauge transformation, consider the nonlinear gauge transformation (3.1.12). For simplicity, we limit ourselves to $\mathbb{A}_{m}$ superfields, but given the close link the transformation of these and the $\mathbb{A}_{\alpha}$ superfields, the methods will hold there also. Expanding the $\nabla_{m}$, this tells us that gauge transforms of $\mathbb{A}_{m}$ have the form

$$
\begin{equation*}
\mathbb{A}_{m}^{\prime}=\mathbb{A}_{m}+\delta_{\Omega} \mathbb{A}_{m}=\mathbb{A}+\left[\partial_{m}, \Omega\right]-\left[\mathbb{A}_{m}, \Omega\right] \tag{4.2.58}
\end{equation*}
$$

Into this, we substitute $\mathbb{H}$ for $\Omega$. We then use the approach of Selivanov to expand this, wherein one expands the superfields as a series in the Lie algebra generators [117; 118],

$$
\begin{equation*}
\mathbb{K} \equiv \sum_{P} \mathcal{K}_{P} T^{P}, \quad T^{P} \equiv T^{p_{1}} T^{p_{2}} \cdots T^{p_{|P|}} \tag{4.2.59}
\end{equation*}
$$

Extracting terms with the same Lie generator coefficients, this reduces (4.2.58) to

$$
\begin{equation*}
\mathcal{A}_{P}^{m, \mathrm{BCJ}}=\mathcal{A}_{P}^{m, \mathrm{~L}}-k_{P}^{m} \mathcal{H}_{P}+\sum_{X Y=P}\left(\mathcal{A}_{X}^{m, \mathrm{~L}} \mathcal{H}_{Y}-\mathcal{A}_{Y}^{m, \mathrm{~L}} \mathcal{H}_{X}\right) \tag{4.2.60}
\end{equation*}
$$

We take $\mathcal{H}$ to be the Berends-Giele current corresponding with the $H$ superfields. The BCJ and L have been inserted to denote which superfields are in the BCJ and Lorenz gauges respectively. This is the general form we expect gauge transformations to have, and it can be seen that this fits to rank 5 .

[^8]To demonstrate, we consider the simplest non-trivial case, that of rank three superfields,

$$
\begin{align*}
\mathcal{A}_{123}^{m, \mathrm{BCJ}} & =\frac{A_{[[1,2], 3]}^{m, \mathrm{BCJ}}}{s_{12} s_{123}}+\frac{A_{[1,[2,3]]}^{m, \mathrm{BCJ}}}{s_{23} s_{123}} \\
& =\frac{A_{[[1,2], 3]}^{m, \mathrm{~L}}-k_{123}^{m} H_{[[1,2], 3]}}{s_{12} s_{123}}+\frac{A_{[1,[2,3]]}^{m, \mathrm{~L}}-k_{123}^{m} H_{[1,[2,3]]}}{s_{23} s_{123}}  \tag{4.2.61}\\
& =\frac{A_{[[1,2], 3]}^{m, \mathrm{~L}}}{s_{12} s_{123}}+\frac{A_{[1,[2,3]]}^{m, \mathrm{~L}}}{s_{23} s_{123}}-k_{123}^{m}\left(\frac{H_{[[1,2], 3]}}{s_{12} s_{123}}+\frac{H_{[1,[2,3]]}}{s_{23} s_{123}}\right) \\
& =\mathcal{A}_{123}^{m, \mathrm{~L}}-k_{123}^{m} \mathcal{H}_{123}
\end{align*}
$$

This matches (4.2.58). There are no summation terms, as any $H$ in them will have at most two indices, and so will by definition be zero. Similar calculations at four and five points fit this same pattern [88].

### 4.3 Amplitudes in Field Theory

Recall the form of tree level amplitudes,

$$
\begin{equation*}
\int_{0 \leq z_{2} \leq z_{3} \leq \ldots \leq z_{n-2} \leq 1} d z_{2} \ldots d z_{n-2}\left\langle V_{1}(0) U_{2}\left(z_{2}\right) \ldots U_{n-2}\left(z_{n-2}\right) V_{n-1}(1) V_{n}(\infty)\right\rangle \tag{4.3.1}
\end{equation*}
$$

Using the techniques of multiparticle superfields, this may now be calculated. We begin at four points, where this reduces to

$$
\begin{equation*}
\int_{0 \leq z \leq 1} d z\left\langle V_{1}(0) U_{2}(z) V_{3}(1) V_{4}(\infty)\right\rangle . \tag{4.3.2}
\end{equation*}
$$

There are two OPE calculations we may perform here; taking the $U_{2}(z)$ into either of the $V_{1}(0)$ or $V_{3}(1)$. We sum over both, giving

$$
\begin{equation*}
\left\langle\mathcal{M}_{12}(0) V_{3}(1) V_{4}(\infty)\right\rangle+\left\langle V_{1}(0) \mathcal{M}_{23}(1) V_{4}(\infty)\right\rangle \tag{4.3.3}
\end{equation*}
$$

where $\mathcal{M}$ is the Berends-Giele current corresponding with the vertex operator $V$. Taking advantage of the fact that single particle BG currents and their corresponding superfields are identical, $\mathcal{M}_{1}=V_{1}$, we may rewrite this as a single summation

$$
\begin{equation*}
\sum_{X Y=123}\left\langle\mathcal{M}_{X}(0) \mathcal{M}_{Y}(1) \mathcal{M}_{4}(\infty)\right\rangle \tag{4.3.4}
\end{equation*}
$$

At five points, the discussion is similar. The amplitude function we are now considering is

$$
\begin{equation*}
\int_{0 \leq z_{2} \leq z_{3} \leq 1} d z\left\langle V_{1}(0) U_{2}\left(z_{2}\right) U_{3}\left(z_{3}\right) V_{4}(1) V_{5}(\infty)\right\rangle \tag{4.3.5}
\end{equation*}
$$

There are now three pole structures we must consider; when $z_{2}$ and $z_{3}$ both approach 0 , when $z_{2}$ approaches 0 and $z_{3}$ approaches 1 , and when they both approach 1 . This then results in a triplet of products of BG currents,

$$
\begin{align*}
\left\langle\mathcal{M}_{123}(0) \mathcal{M}_{4}(1) \mathcal{M}_{5}(\infty)\right\rangle & +\left\langle\mathcal{M}_{12}(0) \mathcal{M}_{34}(1) \mathcal{M}_{5}(\infty)\right\rangle+\left\langle\mathcal{M}_{1}(0) \mathcal{M}_{234}(1) \mathcal{M}_{5}(\infty)\right\rangle \\
& =\sum_{X Y=1234}\left\langle\mathcal{M}_{X}(0) \mathcal{M}_{Y}(1) \mathcal{M}_{5}(\infty)\right\rangle \tag{4.3.6}
\end{align*}
$$

The general point expression for tree level amplitudes follows from similar arguments. We have the expression for arbitrary point amplitudes

$$
\begin{equation*}
\int_{0 \leq z_{2} \leq z_{3} \leq \ldots \leq z_{n-2} \leq 1} d z_{2} \ldots d z_{n-2}\left\langle V_{1}(0) U_{2}\left(z_{2}\right) \ldots U_{n-2}\left(z_{n-2}\right) V_{n-1}(1) V_{n}\left(z_{n}\right)\right\rangle \tag{4.3.7}
\end{equation*}
$$

We split this into $n-1$ pole structures, wherein $z_{2}, \ldots, z_{i}$ approach 0 and $z_{i+1} \ldots z_{n-2}$ approach 1 for $i=1, \ldots, n-1$. Each of these then corresponds with a BG current triplet,

$$
\begin{equation*}
\left\langle\mathcal{M}_{1 \ldots i} \mathcal{M}_{i+1 \ldots . n-1} \mathcal{M}_{n}\right\rangle . \tag{4.3.8}
\end{equation*}
$$

We then sum over these, and the result is an expression for the $n$-point tree level amplitude,

$$
\begin{equation*}
A^{\text {tree }}(1,2, \ldots, n)=\sum_{X Y=12 \ldots n-1}\left\langle\mathcal{M}_{X}(0) \mathcal{M}_{Y}(1) \mathcal{M}_{n}(\infty)\right\rangle \tag{4.3.9}
\end{equation*}
$$

### 4.3.1 BRST Invariance

In order to verify the validity of the above expression, we must confirm that it lies within the cohomology of the BRST operator. That is, given

$$
\begin{equation*}
\mathcal{A}_{n}=\sum_{X Y=12 \ldots n-1}\left\langle M_{X} M_{Y} M_{n}\right\rangle, \tag{4.3.10}
\end{equation*}
$$

we require that

$$
\begin{equation*}
Q \mathcal{A}_{n}=0, \quad \mathcal{A}_{n} \neq Q(\text { something else }) . \tag{4.3.11}
\end{equation*}
$$

In order to show this, we require the form of the BRST variation of the BG currents $\mathcal{M}$.

The variation of their constituent vertex operators $V$ is as in (4.1.37),

$$
\begin{equation*}
Q V_{\ell(P)}=\sum_{\substack{X j Y=P \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right) V_{\ell(X R)} V_{\ell(j S)} . \tag{4.3.12}
\end{equation*}
$$

The sum in this relation here is over the deshuffle product of $Y$, as is described in the appendix A.1.3. From this, we find the variation of the BG currents

$$
\begin{equation*}
Q M_{P}=\sum_{X Y=P} M_{X} M_{Y} \tag{4.3.13}
\end{equation*}
$$

A corollary of a result shown in this thesis will be the rigorous proof of this result. For now though, we can demonstrate one example of this, and assure the reader that similar explicit calculations at other ranks hold similarly

$$
\begin{align*}
Q M_{123}= & Q\left(\frac{V_{[[1,2], 3]}}{s_{12} s_{123}}+\frac{V_{[3,2], 1]}}{s_{23} s_{123}}\right) \\
= & \frac{s_{12}\left(V_{13} V_{2}+V_{1} V_{23}\right)+\left(k^{12} \cdot k^{3}\right) V_{12} V_{3}}{s_{12} s_{123}} \\
& \quad+\frac{s_{23}\left(V_{31} V_{2}+V_{3} V_{21}\right)+\left(k^{23} \cdot k^{1}\right) V_{32} V_{1}}{s_{23} s_{123}}  \tag{4.3.14}\\
= & \frac{V_{1} V_{23}}{s_{123}}+\frac{\left(k^{12} \cdot k^{3}\right) V_{12} V_{3}}{s_{12} s_{123}}+\frac{V_{12} V_{3}}{s_{123}}+\frac{\left(k^{23} \cdot k^{1}\right) V_{1} V_{23}}{s_{23} s_{123}} \\
= & \frac{s_{123} V_{1} V_{23}}{s_{23} s_{123}}+\frac{s_{123} V_{12} V_{3}}{s_{12} s_{123}} \\
= & M_{1} M_{23}+M_{12} M_{3} .
\end{align*}
$$

It then follows that the amplitudes have vanishing variation,

$$
\begin{align*}
Q \sum_{X Y=12 \ldots n-1} M_{X} M_{Y} M_{n}= & \sum_{X Y=12 \ldots n-1}\left(\sum_{A B=X} M_{A} M_{B}\right) M_{Y} M_{n} \\
& \quad-\sum_{X Y=12 \ldots n-1} M_{X}\left(\sum_{A B=Y} M_{A} M_{B}\right) M_{n}  \tag{4.3.15}\\
= & \sum_{X Y Z=12 \ldots n-1} M_{X} M_{Y} M_{Z} M_{n}-M_{X} M_{Y} M_{Z} M_{n} \\
= & 0
\end{align*}
$$

Note the minus sign arises from the anticommuting of the BRST charge $Q$ and the vertex operator $V_{X}$. There is no third term in the variation in the above as $Q M_{n}=0$.

We then must show that the amplitude formula (4.3.10) is not itself the variation of some other object. To do this, we consider $M_{12 \ldots n-1} M_{n}$. By (4.3.13), this has variation

$$
\begin{equation*}
Q M_{12 \ldots n-1} M_{n}=\sum_{X Y=12 \ldots n-1} M_{X} M_{Y} M_{n} . \tag{4.3.16}
\end{equation*}
$$

This then would suggest that the amplitude formula (4.3.10) is the variation of $M_{12 \ldots n-1} M_{n}$. However, $M_{12 \ldots n-1}$ contains an overall $1 / s_{12 \ldots n-1}$ factor. By momentum conservation we then have

$$
\begin{equation*}
\frac{1}{s_{12 \ldots n-1}}=\frac{1}{k^{12 \ldots n-1} \cdot k^{12 \ldots n-1}}=\frac{1}{k^{n} \cdot k^{n}}=\frac{1}{0} \tag{4.3.17}
\end{equation*}
$$

Hence, the object with variation (4.3.10) is ill defined at $n$-points. As such, we can be assured that this amplitude formula is in the cohomology of the BRST operator, and is therefore correct [113; 119].

### 4.3.2 Deriving BCJ relations Between Amplitudes

As was discussed in section 2, BCJ discovered a set of relations between partial amplitudes (2.3.21). We may identify these relations using this formalism, by exploiting properties of the BCJ gauge. Namely, by writing vertex operators in this gauge as functions of BG currents, relations between amplitudes may be identified. So, to give one example,

$$
\begin{align*}
0 & =\left\langle Q\left(\frac{V_{123}}{s_{12}} V_{4}\right)\right\rangle \\
& =\left\langle Q\left(s_{23} M_{123}^{B C J} V_{4}-s_{13} M_{213}^{B C J} V_{4}\right)\right\rangle \\
& =s_{23} \sum_{X Y=123}\left\langle M_{X}^{B C J} M_{Y}^{B C J} M_{4}\right\rangle-s_{13} \sum_{X Y=213}\left\langle M_{X}^{B C J} M_{Y}^{B C J} M_{4}\right\rangle  \tag{4.3.18}\\
& =s_{23} A^{S Y M}(1,2,3,4)-s_{13} A^{S Y M}(2,1,3,4)
\end{align*}
$$

Hence, we have the four point BCJ relation.

Increasing the rank we can find comparable relations. The four point vertex operator may be written as a function of BG currents as

$$
\begin{array}{r}
V_{1234}=s_{12}\left(s_{23} s_{34} M_{1234}^{B C J}-s_{13} s_{34} M_{2134}^{B C J}+s_{14} s_{23} M_{3214}^{B C J}-s_{13} s_{24} M_{3124}^{B C J}\right. \\
\left.+s_{23} s_{24}\left(M_{1234}^{B C J}+M_{1243}^{B C J}\right)-s_{13} s_{14}\left(M_{2134}^{B C J}+M_{2143}^{B C J}\right)\right) \tag{4.3.19}
\end{array}
$$

By adding together combinations of this, BCJ identities may be found. For instance, we have

$$
\begin{align*}
\frac{V_{1234}}{s_{12} s_{123}}+\frac{V_{3214}}{s_{23} s_{123}} & =s_{34} M_{1234}^{B C J}+s_{12} M_{3214}^{B C J}-s_{24}\left(M_{1324}^{B C J}+M_{3124}^{B C J}\right)  \tag{4.3.20}\\
\frac{V_{1234}-V_{1243}}{s_{12} s_{34}} & =s_{23} M_{1234}^{B C J}-s_{13} M_{2134}^{B C J}-s_{24} M_{1243}^{B C J}+s_{14} M_{2143}^{B C J}, \tag{4.3.21}
\end{align*}
$$

and from these one may extract the relations

$$
\begin{align*}
& s_{34} A^{S Y M}(1,2,3,4,5)+s_{14} A^{S Y M}(3,2,1,4,5) \\
& \quad s_{24}\left(A^{S Y M}(1,3,2,4,5)+A^{S Y M}(3,1,2,4,5)\right)=0  \tag{4.3.22}\\
& s_{23} A^{S Y M}(1,2,3,4,5)-s_{13} A^{S Y M}(2,1,3,4,5) \\
&-s_{24} A^{S Y M}(1,2,4,3,5)+s_{14} A^{S Y M}(2,1,4,3,5)=0 .
\end{align*}
$$

In general, this approach may be used to find [83; 42]

$$
\begin{align*}
& 0=\sum_{i=1}^{|A|} \sum_{j=1}^{|B|}(-1)^{i-j} s_{a_{i} b_{j}}  \tag{4.3.23}\\
& A^{S Y M}\left(\left(a_{1} \ldots a_{i-1} \amalg a_{|A|} \ldots a_{i+1}\right), a_{i}, b_{j},\left(b_{j-1} \ldots b_{1} \amalg b_{j+1} \ldots b_{|B|}\right), n\right),
\end{align*}
$$

which match BCJ relations in a particular representation [10; $63 ; 64 ; 120]$. If one sets $A=1$, this matches (2.3.21) exactly [42]

$$
\begin{align*}
& s_{12} A^{S Y M}(2,1,3, \ldots, n)+\left(s_{12}+s_{13}\right) A^{S Y M}(2,3,1,4, \ldots, n)  \tag{4.3.24}\\
& \quad+\ldots+\left(s_{12}+s_{13}+\ldots+s_{1, n 1}\right) A^{S Y M}(2,3, \ldots, n-1,1, n)=0
\end{align*}
$$

The above are known as the fundamental BCJ relations [121].

### 4.4 Amplitudes in String Theory

We will not detail the calculation of amplitudes in string theory using these techniques, only highlight the key results. For the full details, $[18 ; 19]$ should be consulted.

The general $n$-point disk amplitude is given by

$$
\begin{align*}
\mathcal{A}_{n}= & \prod_{j=2}^{n-2} \int d z_{j} \prod_{i<j}\left|z_{i j}\right|^{-s_{i j}} \\
& \times \sum_{p=1}^{n-2}\left\langle\frac{V_{12 \ldots . .} V_{n-1, n-2, \ldots, p+1} V_{n}}{\left(z_{12} z_{23} \ldots z_{p-1, p}\right)\left(z_{n-1, n-2} \ldots z_{p+2, p+1}\right)}+\operatorname{Perm}(2,3, \ldots, n-2)\right\rangle . \tag{4.4.1}
\end{align*}
$$

Using the properties of the BCJ gauge, this may be reexpressed as a function of amplitudes in SYM,

$$
\begin{equation*}
\mathcal{A}_{n}=\int_{z_{i}<z_{i+1}} \prod_{i<j}\left|z_{i j}\right|^{-s_{i j}}\left[\prod_{k=2}^{n-2} \sum_{m=1}^{k-1} \frac{s_{m k}}{z_{m k}} A_{S Y M}(1,2, \ldots, n)+\operatorname{Perm}(2,3, \ldots, n-2)\right] \tag{4.4.2}
\end{equation*}
$$

Factors of $\alpha^{\prime}$ have been set to $\frac{1}{2}$, and the references should be consulted for formulae with them included.

### 4.4.1 BCJ Satisfying Numerators From String Theory

While the formula (4.3.10) does produce amplitudes correctly, they do not necessarily satisfy BCJ relations explicitly in this form. There is a generalisation of BCJ relations in $[122 ; 123]$ which they can be shown to satisfy [119], but such will not be used for this thesis. Instead, we use the string amplitudes to generate alternative field theory amplitude representations for non-canonical orderings which satisfy BCJ relations explicitly [114].

To begin, the results of [116] allow for the string amplitude integrands to be reformulated in terms of functions

$$
\begin{equation*}
Z_{\Sigma}(1,2, \ldots, n)=\int_{\Sigma} d z_{1} d z_{2} \ldots d z_{n} \frac{\prod_{i<j}^{n}\left|z_{i j}\right|^{\alpha^{\prime} s_{i j}}}{z_{12} z_{23} \ldots z_{n-1, n} z_{n 1}} \tag{4.4.3}
\end{equation*}
$$

Using such, partial amplitudes are given by [56]

$$
\begin{align*}
& A(\Sigma)=\sum_{X Y=23 \ldots n-2}\left\langle V_{1 X} V_{(n-1) \bar{Y}} V_{n}\right\rangle Z_{\Sigma}(1, X, n, Y, n-1)(-1)^{|X|}  \tag{4.4.4}\\
&+\operatorname{Perm}(2,3, \ldots, n-2)
\end{align*}
$$

In [124], the field theory limit of (4.4.3) was identified as

$$
\begin{equation*}
\lim _{\alpha^{\prime} \rightarrow 0} Z_{P}(Q)=(-1)^{|P|} m(P \mid Q) \tag{4.4.5}
\end{equation*}
$$

with the $m$ function defined in terms of the BG double currents (2.3.9)

$$
\begin{equation*}
m(P, n \mid Q, n)=s_{P} \phi_{P \mid Q} \tag{4.4.6}
\end{equation*}
$$

The arguments of the $m$ should be regarded as cyclic in order to eliminate the $n$.

SYM amplitudes satisfying BCJ relations are thus by

$$
\begin{align*}
& A(\Sigma)=\sum_{X Y=23 \ldots n-2}\left\langle V_{1 X} V_{(n-1) \bar{Y}} V_{n}\right\rangle(-1)^{|Y|+1} m(\Sigma \mid 1, X, n, Y, n-1)  \tag{4.4.7}\\
&+\operatorname{Perm}(2,3, \ldots, n-2)
\end{align*}
$$

We may then use this to generate examples. First of all we note the canonical ordering is


Figure 4.4.1: We may verify this BCJ relation using (4.4.13)
reproduced [56], as when $\Sigma=12 \ldots n$ the $m$ function separates as

$$
\begin{align*}
m(12 \ldots n \mid 1, X, n, Y, n-1) & =s_{12 \ldots n-1} \phi_{12 \ldots n-1 \mid Y(n-1) 1 X}  \tag{4.4.8}\\
& =s_{12 \ldots n-1} \phi_{1 X \mid 1 X} \phi_{Y(n-1) \mid Y(n-1)}
\end{align*}
$$

The formula (4.3.10) then follows by

$$
\begin{equation*}
M_{1 A}=\sum_{B} \phi_{1 A \mid 1 B} V_{1 B}, \tag{4.4.9}
\end{equation*}
$$

which is the definition of Berends-Giele currents in terms of the inverse momentum kernel (4.1.42), reformulated in terms of BG double currents as in (2.3.10).

We may then wish to look at another ordering, and in order to coincide with results discussed in [56] we consider $A(1,2,4,3,5)$. Applying the formula, this is

$$
\begin{align*}
A(12435)= & \sum_{X Y=23,32}\left\langle V_{1 X} V_{4 \bar{Y}} V_{5}\right\rangle(-1)^{5} m(12435 \mid 1 X 5 Y 4)  \tag{4.4.10}\\
= & \sum_{X Y=23,32}\left\langle V_{1 X} V_{4 \bar{Y}} V_{5}\right\rangle(-1)^{5} s_{1234} \phi_{1243 \mid Y 41 X}  \tag{4.4.11}\\
= & -s_{1234}\left[\left\langle V_{1} V_{432} V_{5}\right\rangle \phi_{1243 \mid 2341}+\left\langle V_{12} V_{43} V_{5}\right\rangle \phi_{1243 \mid 3412}\right. \\
& \quad+\left\langle V_{123} V_{4} V_{5}\right\rangle \phi_{1243 \mid 4123}+\left\langle V_{1} V_{423} V_{5}\right\rangle \phi_{1243 \mid 3241}  \tag{4.4.12}\\
& \left.\quad+\left\langle V_{13} V_{42} V_{5}\right\rangle \phi_{1243 \mid 2413}+\left\langle V_{132} V_{4} V_{5}\right\rangle \phi_{1243 \mid 4132}\right]
\end{align*}
$$

Applying the definition (2.3.9) to each $\phi_{A}$ in turn, one finds

$$
\begin{align*}
A(1,2,4,3,5)= & \frac{V_{12} V_{43} V_{5}+V_{123} V_{4} V_{5}}{s_{12} s_{124}}-\frac{V_{1} V_{423} V_{5}+V_{13} V_{42} V_{5}}{s_{24} s_{124}}+\frac{V_{12} V_{43} V_{5}}{s_{12} s_{34}} \\
& -\frac{V_{1} V_{432} V_{5}}{s_{34} s_{234}}-\frac{V_{1} V_{423} V_{5}}{s_{24} s_{234}} \tag{4.4.13}
\end{align*}
$$

To demonstrate that this is a BCJ representation, the identity in figure 4.4.1 may be checked. The first and second numerators come from the amplitude in the canonical
ordering,

$$
\begin{equation*}
V_{1} V_{[[2,3], 4]} V_{5}, \quad V_{1} V_{[2,[3,4]]} V_{5} \tag{4.4.14}
\end{equation*}
$$

The third is the numerator associated with the $s_{24} s_{243}$ denominator in (4.4.13),

$$
\begin{equation*}
V_{1} V_{[[4,2], 3]} V_{5} \tag{4.4.15}
\end{equation*}
$$

The identity then follows as a result of the BCJ gauge construction of the multiparticle $V$,

$$
\begin{align*}
& V_{1} V_{[[2,3], 4]} V_{5}-V_{1} V_{[2,[3,4]} V_{5}-\left(-V_{1} V_{[[4,2], 3]} V_{5}\right)=  \tag{4.4.16}\\
& V_{1} V_{[[2,3], 4]} V_{5}+V_{1} V_{[[3,4], 2]} V_{5}+V_{1} V_{[[4,2], 3]} V_{5},
\end{align*}
$$

which vanishes as it is a statement of the Jacobi identity. Note that while this was a relatively simple example and could have been shown with the naive relabelling of $A(1,2,3,4,5),(4.4 .13)$ contains more complex numerators for the $s_{124}$ poles, and identities involving these terms will not hold for such a naive approach. Another example of these methods is provided in section 10.1.

## One Loop Amplitudes From String Theory

In addition to the previous tree level results, the pure spinor formalism has also been successfully used to find amplitudes at loop level. Part III of this thesis will focus upon one loop amplitudes in such, and as such we introduce the relevant material here.

### 5.1 Multiparticle Superfield structures at One Loop

For the calculation of scattering amplitudes at one loop we may borrow some techniques from tree level, but new methods are required also. The piece of the amplitude formula we are now interested in is

$$
\begin{equation*}
\left\langle b V_{1}\left(z_{1}\right) \prod_{i=2}^{n} U_{i}\left(z_{i}\right)\right\rangle . \tag{5.1.1}
\end{equation*}
$$

While we have formulae for the interactions between the $V$ and $U$ vertex operators, there is now the $b$ ghost also which may interact with the $U$ terms also. This has an extremely complex form, as given in (3.2.37),

$$
\begin{align*}
b= & (\Pi d+N \partial \theta+J \partial \theta) d \delta(N)+(w \partial \lambda+J \partial N+N \partial J+N \partial N) \delta(N) \\
& +\left(N \Pi+J \Pi+\partial \Pi+d^{2}\right)\left(\Pi \delta(N)+d^{2} \delta^{\prime}(N)\right) \\
& +(N d+J d)\left(\partial \theta \delta(N)+d \Pi \delta^{\prime}(N)+d^{3} \delta^{\prime \prime}(N)\right)  \tag{5.1.2}\\
& +\left(N^{2}+J N+J^{2}\right)\left(d \partial \theta \delta^{\prime}(N)+\Pi^{2} \delta^{\prime}(N)+\Pi d^{2} \delta^{\prime \prime}(N)+d^{4} \delta^{\prime \prime \prime}(N)\right),
\end{align*}
$$

This presents a significant complication. Fortunately though, for lower point amplitudes the situation is far simpler than it appears. As discussed previously, the pure spinor bracket $\langle\ldots\rangle$ at loop level effectively selects only terms containing six $d_{\alpha}$ and zero $N_{m n}$ zero modes. We will be able to use this to simplify the problem considerably. Note in this section we assuming that all multiparticle superfields are constructed in the BCJ gauge using the techniques of the previous section.

### 5.1.1 Construction of Building Blocks from the $b$-ghost

To identify the new superfields describing one loop kinematics, we begin with the simplest case as at tree level and then generalise. As three point amplitudes are known to vanish in string $[36 ; 37 ; 38]$ and field [34] theory, this is four point box

$$
\begin{equation*}
\left\langle b V_{1} U_{2} U_{3} U_{4}\right\rangle . \tag{5.1.3}
\end{equation*}
$$

Within this, we need only consider terms which contain six $d_{\alpha}$ zero modes. The integrated vertex operator $V$ does not contain any of these. The unintegrated vertex operators $U$ can each provide one only, and so we may find at most three $d_{\alpha}$ terms here. Hence, we need at least three $d_{\alpha}$ terms from the $b$ ghost. This reduces the relevant terms in the $b$-ghost to

$$
\begin{equation*}
d^{4} \delta^{\prime}(N)+(N d+J d) d^{3} \delta^{\prime \prime}(N)+\left(N^{2}+J N+J^{2}\right) d^{4} \delta^{\prime \prime \prime}(N) \tag{5.1.4}
\end{equation*}
$$

We then consider the restriction to no $N_{m n}$ zero modes. Note an $n^{\text {th }}$ order derivative of $\delta(N)$ effectively contributes $-n$ zero modes of $N_{m n}$, as it must be partially integrated against $n$ such terms.

When the $b$ ghost provides three $d_{\alpha}$ zero modes, we rely upon the vertex operators for the remaining three, and so these cannot also provide a $N_{m n}$ to counteract any derivatives of $\delta(N)$ terms. Similarly, when the $b$ ghost provides four $d_{\alpha}$ terms, the vertex operators can provide at most one $N_{m n}$ term, and so the ghost cannot contain a higher derivative than $\delta^{\prime}(N)$. This then leave us with three possible contributions from the $b$-ghost at four points[125],

$$
\begin{equation*}
d^{4} \delta^{\prime}(N), \quad N d^{4} \delta^{\prime \prime}(N), \quad N^{2} d^{4} \delta^{\prime \prime \prime}(N) \tag{5.1.5}
\end{equation*}
$$

After partial integration, these combine into a single term. This relies upon a contribution of two $d_{\alpha}$ terms and a $N_{m n}$ from the unintegrated vertex operators. Hence the four point
amplitude calculation simplifies to a triplet of terms,

$$
\begin{align*}
\left\langle b V_{1} U_{2} U_{3} U_{4}\right\rangle \sim\left\langle V_{1}\left(d_{\alpha} W_{2}^{\alpha}\right)\right. & \left(d_{\beta} W_{3}^{\beta}\right)\left(\frac{1}{2} N_{m n} F_{4}^{m n}\right) \\
& +V_{1}\left(d_{\alpha} W_{2}^{\alpha}\right)\left(\frac{1}{2} N_{m n} F_{3}^{m n}\right)\left(d_{\beta} W_{4}^{\beta}\right)  \tag{5.1.6}\\
& \left.+V_{1}\left(\frac{1}{2} N_{m n} F_{2}^{m n}\right)\left(d_{\alpha} W_{3}^{\alpha}\right)\left(d_{\beta} W_{4}^{\beta}\right)\right\rangle
\end{align*}
$$

The above was only intended as a sketch of this procedure, to give an idea of where the results come from. When the calculation is more carefully performed, these terms have the form $[125 ; 20]$

$$
\begin{equation*}
b V_{1} U_{2} U_{3} U_{4} \quad \leftrightarrow \quad V_{1}\left(\lambda \gamma_{m} W_{2}\right)\left(\lambda \gamma_{m} W_{3}\right) F_{4}^{m n}+c y c(2,3,4) \tag{5.1.7}
\end{equation*}
$$

Hence, we define a (one loop) building block which captures the terms in the four point amplitude [1]

$$
\begin{equation*}
T_{A, B, C}=\left(\lambda \gamma_{m} W_{A}\right)\left(\lambda \gamma_{n} W_{B}\right) F_{C}^{m n}+\operatorname{cyc}(A, B, C) \tag{5.1.8}
\end{equation*}
$$

This has been defined with multiparticle labels, as at higher points it will arise also. For instance, the eight particle amplitude is a function of

$$
\begin{equation*}
\left\langle b V_{1} U_{2} U_{3} U_{4} U_{5} U_{6} U_{7} U_{8}\right\rangle \tag{5.1.9}
\end{equation*}
$$

Alongside contributions arising from more complex terms within the $b$-ghost, this will also contain terms arising from OPE calculations between the $V$ and $U$ superfields. For example, this will contain a contribution

$$
\begin{equation*}
b V_{[1,2]} U_{[[3,4], 5]} U_{6} U_{[7,8]} \tag{5.1.10}
\end{equation*}
$$

This then has the same structure in terms of $d_{\alpha}$ and $N_{m n}$ zero modes as the four point amplitude, and so the calculation proceeds analogously, giving a contribution

$$
\begin{equation*}
V_{[1,2]} T_{[[3,4], 5], 6,[7,8]} \tag{5.1.11}
\end{equation*}
$$

Hence the need for multiparticle indices in the $T$ superfield.

We now note some properties of the $T$ terms. The first is that, by construction, it is symmetric under permutations of its blocks of indices [1],

$$
\begin{equation*}
T_{A, B, C}=T_{A, C, B}=T_{B, C, A}=T_{B, A, C}=T_{C, A, B}=T_{C, B, A} \tag{5.1.12}
\end{equation*}
$$

The second is the form of its variation, which can be found for Dynkin brackets to conve-
niently factorise as

$$
\begin{gather*}
Q T_{\ell(A), \ell(B), \ell(C)}=\sum_{\substack{X j Y=A \\
R \amalg S=Y}}\left(k^{X} \cdot k^{j}\right)\left(V_{\ell(X R)} T_{\ell(j S), \ell(B), \ell(C)}-(X \leftrightarrow j)\right)  \tag{5.1.13}\\
+(A \leftrightarrow B, C) .
\end{gather*}
$$

This will be generalised to general Lie brackets later. The sum in this relation here is again over the deshuffle product, as defined in appendix A.1.3.

### 5.1.1.1 Vectorial Generalisation

To find the first generalisation of this, we look to the five point amplitude. There are two further classes of terms which now contribute. The first comes from selecting the same $b$-ghost term as at four points, and the same values for three of the $U$ terms, and then selecting from the fourth the $\Pi^{m} A_{m}$ term. This is the only term in $U$ without a $\theta, d_{\alpha}$, or $N_{m n}$, and as such will not affect the counting of the previous discussion. This produces terms of the form

$$
\begin{equation*}
A_{A}^{m} T_{B, C, D} . \tag{5.1.14}
\end{equation*}
$$

The second new class of terms arise from a new element of the $b$-ghost, namely [125]

$$
\begin{equation*}
\Pi d^{2} \delta(N) \tag{5.1.15}
\end{equation*}
$$

This requires four $d_{\alpha}$ modes from the $U$ vertex operators, which is now possible at five points and above. More careful manipulation of the terms involved is believed to give the contribution [126; 20]

$$
\begin{equation*}
W_{A, B, C, D}^{m} \equiv \frac{1}{12}\left(\lambda \gamma_{n} W_{A}\right)\left(\lambda \gamma_{p} W_{B}\right)\left(W_{C} \gamma^{m n p} W_{D}\right)+(A, B \mid A, B, C, D) \tag{5.1.16}
\end{equation*}
$$

Note the new summation notation introduced here is detailed in appendix A.1.

We combine (5.1.14) and (5.1.16) to create a new class of superfields, the vectorial generalisation of $T_{A, B, C}$ [1],

$$
\begin{equation*}
T_{A, B, C, D}^{m}=\left(A_{A}^{m} T_{B, C, D}+(A \leftrightarrow B, C, D)\right)+W_{A, B, C, D}^{m} . \tag{5.1.17}
\end{equation*}
$$

The coefficients of the terms in the above were determined by requiring a convenient form of the variation,
$Q T_{\ell(A), \ell(B), \ell(C), \ell(D)}^{m}=k_{A}^{m} V_{A} T_{B, C, D}+\sum_{\substack{X j Y=A \\ Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(V_{\ell(X R)} T_{\ell(j S), \ell(B), \ell(C), \ell(D)}-(X \leftrightarrow j)\right)$

$$
\begin{equation*}
+(A \leftrightarrow B, C, D) . \tag{5.1.18}
\end{equation*}
$$

Note the $T_{A, B, C, D}^{m}$ has been defined for general Lie monomial indices, as it will generalise to higher points by the same mechanism as $T_{A, B, C}$ did.

### 5.1.1.2 Tensorial Generalisation

At higher points, we may select more and more vertex operators to contribute a $\Pi^{m} A_{m}$ term. This produces the tensorial generalisation of the building blocks constructed thus far [1],

$$
\begin{align*}
T_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1}, \ldots, m_{r}} & =\left(T_{B_{1}, B_{2}, B_{3}} A_{B_{4}}^{\left(m_{1}\right.} A_{B_{5}}^{\left(m_{2}\right.} \ldots A_{B_{r+3}}^{\left(m_{r}\right.}+\left(B_{1}, B_{2}, B_{3} \mid B_{1}, B_{2}, \ldots, B_{r+3}\right)\right)  \tag{5.1.19}\\
& +\left(W_{B_{1}, B_{2}, B_{3}, B_{4}}^{\left(m_{1}\right.} A_{B_{5}}^{\left(m_{2}\right.} A_{B_{6}}^{\left(m_{3}\right.} \ldots A_{B_{r+3}}^{\left(m_{r}\right.}+\left(B_{1}, B_{2}, B_{3}, B_{4} \mid B_{1}, B_{2}, \ldots, B_{r+3}\right)\right) .
\end{align*}
$$

These have variation,

$$
\begin{align*}
Q T_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}= & \delta^{\left(m_{1} m_{2}\right.} Y_{B_{1}, B_{2}, \ldots, B_{r+3}}^{\left.m_{3} \ldots m_{r}\right)}  \tag{5.1.20}\\
+ & \left(k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} T_{B_{2}, \ldots, B_{r+3}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)\right) \\
+ & \left(\sum_{\substack{X j Y=B_{1} \\
Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(V_{X R} T_{j S, B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}-(X \leftrightarrow j)\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)\right) .
\end{align*}
$$

Here we have introduced an anomaly term. Such terms are known to arise at six points and higher $[127 ; 128 ; 129 ; 109]$. The superfield above is defined by [1]

$$
\begin{equation*}
Y_{A, B, C, D, E}=\frac{1}{2}\left(\lambda \gamma^{m} W_{A}\right)\left(\lambda \gamma^{n} W_{B}\right)\left(\lambda \gamma^{p} W_{C}\right)\left(W_{D} \gamma_{m n p} W_{E}\right) \tag{5.1.21}
\end{equation*}
$$

This then generalises to a tensorial structure by a similar procedure to that of $T_{A, B, C}$,

$$
\begin{align*}
Y_{B_{1}, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}} & \equiv Y_{B_{1}, \ldots, B_{5}} A_{B_{6}}^{\left(m_{1}\right.} A_{B_{7}}^{m_{2}} \ldots A_{B_{r+5}}^{\left.m_{r}\right)}+\left(B_{1}, \ldots, B_{5} \mid B_{1}, . ., B_{r+5}\right)  \tag{5.1.22}\\
& =A_{B_{1}}^{m_{1}} Y_{B_{2}, B_{3}, \ldots, B_{r+5}}^{m_{2} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{r+5}\right)
\end{align*}
$$

These have variation

$$
\begin{align*}
Q Y_{B_{1}, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}= & k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} Y_{B_{2}, \ldots, B_{r+5}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right)  \tag{5.1.23}\\
& +\sum_{\substack{ \\
}}\left(k_{X} \cdot k_{j}\right)\left[V_{X R} Y_{j S, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{1}}-(X \leftrightarrow j)\right]+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right) .
\end{align*}
$$

### 5.1.2 Refined Building Blocks

Starting at six points, further objects begin to appear. These were identified based upon the form their variations take, rather than via the $b$-ghost explicitly. They are known as refined building blocks, as they contain a separate refined class of indices separated by a vertical bar from the rest. The lowest rank case is $[130 ; 1]$

$$
\begin{align*}
J_{1 \mid 2,3,4,5} & \equiv A_{1}^{m} T_{2,3,4,5}^{m}-\frac{1}{2}\left[\left(A_{1} \cdot A_{2}\right) T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right]  \tag{5.1.24}\\
& =\frac{1}{2} A_{1}^{m}\left(T_{2,3,4,5}^{m}+W_{2,3,4,5}^{m}\right)
\end{align*}
$$

in which the 1 is the refined index. This has variation

$$
\begin{equation*}
Q J_{1 \mid 2,3,4,5}=k_{1}^{m} V_{1} T_{2,3,4,5}^{m}+\left[V_{12} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right]+Y_{1,2,3,4,5} . \tag{5.1.25}
\end{equation*}
$$

This then can be generalised to allow for Lie bracket indices in the usual way, though it does also require a correction to enforce the BCJ gauge correct structure in the variation [130; 1],

$$
\begin{equation*}
J_{A \mid B, C, D, E} \equiv A_{A}^{m} T_{B, C, D, E}^{m}-\left[\left(H_{[A, B]}+\frac{1}{2}\left(A_{A} \cdot A_{B}\right)\right) T_{C, D, E}+(B \leftrightarrow C, D, E)\right] \tag{5.1.26}
\end{equation*}
$$

The variation of these is then given by

$$
\begin{equation*}
Q J_{A \mid B, C, D, E}=k_{A}^{m} V_{A} T_{B, C, D, E}^{m}+\left[V_{[A, B]} T_{C, D, E}+(B \leftrightarrow C, D, E)\right]+Y_{A, B, C, D, E} \tag{5.1.27}
\end{equation*}
$$

Without the $H$ corrections in (5.1.26), the $V$ in the above would not be in the BCJ gauge.

As with the other objects discussed thus far we may generalise these to a tensorial structure[1; 130; 20],

$$
\begin{align*}
& J_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}} \equiv A_{A}^{p} T_{B_{1}, \ldots, B_{r+4}}^{p m_{1}, \ldots, m_{r}}-\left(\left(H_{\left[A, B_{1}\right]}+\frac{1}{2}\left(A_{A} \cdot A_{B_{1}}\right)\right) T_{B_{2}, \ldots, B_{r+4}}^{m_{1} \ldots, m_{r}}\right. \\
&\left.+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+4}\right)\right) \tag{5.1.28}
\end{align*}
$$

These then have BRST variation

$$
\begin{align*}
Q J_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots R m_{r}}= & k_{A}^{p} V_{A} T_{B_{1}, \ldots, B_{r+4}}^{p m_{1} \ldots m_{r}}+\delta^{\left(m_{1} m_{2}\right.} Y_{A \mid B_{1}, \ldots, B_{r+4}}^{\left.m_{3} \ldots m_{r}\right)}+Y_{A, B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}  \tag{5.1.29}\\
& +V_{\left[A, B_{1}\right]} T_{B_{2} \ldots \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} J_{A \mid B_{2}, \ldots, B_{r+4}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+4}\right) \\
& +\sum_{\substack{A=X j Y \\
Y=R \amalg S}}\left(k_{X} \cdot k_{j}\right)\left[V_{X R} J_{j S \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}-(X \leftrightarrow j)\right] \\
& +\sum\left(k_{X} \cdot k_{j}\right)\left[V_{X R} J_{A \mid j S, B_{2}, \ldots, B_{r+4}}-(X \leftrightarrow j)\right]+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+4}\right), \\
& Y=X j Y \\
& Y=R \sqcup S
\end{align*}
$$

In addition to generalising through the addition of tensor indices, we may also generalise through adding further degrees of refinement. This is defined through repeated application of the formula refining $T$ into $J$, (5.1.28), with the simplest such extension being [130; 20]

$$
\begin{equation*}
J_{1,2 \mid 3,4,5,6,7} \equiv A_{m}^{1} J_{2 \mid 3,4,5,6,7}^{m}-\frac{1}{2}\left[\left(A_{1} \cdot A_{3}\right) J_{2 \mid 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right] \tag{5.1.30}
\end{equation*}
$$

The extension of maximum generality is defined similarly,

$$
\begin{align*}
J_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}} \equiv & \frac{1}{2} A_{A_{1}}^{p}\left[J_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{p m_{1} \ldots m_{r}}+W_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r} \mid p}\right]  \tag{5.1.31}\\
& -H_{\left[A_{1}, B_{1}\right]} J_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)
\end{align*}
$$

The refined $W$ superfields likewise follow from the formula (5.1.28),

$$
\begin{align*}
W_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}} \equiv & \frac{1}{2} A_{A_{1}}^{p} W_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{p m_{1} \ldots m_{r-1} \mid m_{r}}  \tag{5.1.32}\\
& -\left[H_{\left[A_{1}, B_{1}\right]} W_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)\right] .
\end{align*}
$$

We regard the $T$ superfields as being of degree if refinement 0 , and the $J$ superfields with $n$ blocks of refined indices as being of degree $n$. The variation of these objects is then given as follows. To begin, we consider the simplest case of the doubly refined $J$,

$$
\begin{align*}
Q J_{1,2 \mid 3,4,5,6,7}= & k_{1}^{m} V_{1} J_{2 \mid 3,4,5,6,7}^{m}+k_{2}^{m} V_{2} J_{1 \mid 3,4,5,6,7}^{m}+Y_{1 \mid 2,3,4,5,6,7}+Y_{2 \mid 1,3,4,5,6,7}  \tag{5.1.33}\\
& +\left[V_{13} J_{2 \mid 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right]+\left[V_{23} J_{1 \mid 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right]
\end{align*}
$$

Then extending this to the maximum generality, we have

$$
\begin{align*}
& Q J_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}=\delta^{\left(m_{1} m_{2}\right.} Y_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{\left.m_{3} \ldots m_{r}\right)}  \tag{5.1.34}\\
& \quad+k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} J_{A_{1}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right) \\
& \quad+V_{\left[A_{1}, B_{1}\right]} J_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{m_{1},}+\left(A_{1} \leftrightarrow A_{2}, A_{3}, \ldots, A_{d} ; \quad B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right) \\
& \quad+Y_{A_{2}, \ldots, A_{d} \mid A_{1}, B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}+k_{A_{1}}^{p} V_{A_{1}} J_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{p m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right) \\
& \quad+\sum^{A_{1}=X j Y}\left(k_{X} \cdot k_{j}\right)\left[V_{X R} J_{j S, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}-(X \leftrightarrow j)\right]+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right) \\
& \quad Y=R \sqcup S \\
& \quad+\sum\left(k_{X} \cdot k_{j}\right)\left[V_{X R} J_{A_{1}, \ldots, A_{d} \mid j S, B_{2}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}-(X \leftrightarrow j)\right]+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right) . \\
& \quad B_{1}=X j Y \\
& Y=R \sqcup S
\end{align*}
$$

We note that $J$ superfields with more than seven particle labels in their indices will never be used in this report. We state their maximum generalities here for purely for completeness, and to illustrate the covariant nature of their variation.

In the above, we have included also the refined form of the anomaly building blocks. These
are defined in a similar way,

$$
\begin{equation*}
Y_{A \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}} \equiv \frac{1}{2} A_{A}^{p} Y_{B_{1}, \ldots, B_{r+6}}^{p m_{1} \ldots m_{r}}-\left[H_{\left[A, B_{1}\right]} Y_{B_{2}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+6}\right)\right], \tag{5.1.35}
\end{equation*}
$$

and have variation

$$
\begin{align*}
Q Y_{A \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}= & k_{A}^{p} V_{A} Y_{B_{1}, \ldots, B_{r+6}}^{p m_{1} \ldots m_{r}}  \tag{5.1.36}\\
& +V_{\left[A, B_{1}\right]} Y_{B_{2}, \ldots, B_{r}}^{m_{1}}+k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} Y_{A \mid B_{2}, \ldots, B_{r+6}}^{\left.m_{2}, \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+6}\right) \\
& +\sum_{X Y=B_{1}}\left(k_{X} \cdot k_{j}\right)\left[V_{X R} Y_{A|j|, B_{2}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}-(X \leftrightarrow j)\right]+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+6}\right) \\
& Y=R \amalg S \\
& +\sum_{\substack{X j Y=A}}\left(k_{X} \cdot k_{j}\right)\left[V_{X R} Y_{j S \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}-(X \leftrightarrow j)\right] .
\end{align*}
$$

A further refinement of the anomaly superfields will be needed also, denoted $\Delta$, and defined by

$$
\begin{equation*}
\Delta_{1|2| 3,4,5,6,7}=\mathcal{Y}_{2 \mid 1,3,4,5,6,7}+k_{2}^{m} \mathcal{Y}_{12,3,4,5,6,7}^{m}+\left[s_{23} \mathcal{Y}_{123,4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right], \tag{5.1.37}
\end{equation*}
$$

where $\mathcal{Y}$ denotes the Berends-Giele currents corresponding to the anomaly superfields $Y$ defined in the usual way [20]

$$
\begin{equation*}
\mathcal{Y}_{A_{1}, \ldots, A_{m} \mid B_{1}, \ldots, B_{n}}^{n-m-6}=\mathcal{Y}_{b\left(A_{1}\right), \ldots, b\left(A_{m}\right) \mid b\left(B_{1}\right), \ldots, b\left(B_{n}\right)}^{n-m-6} \tag{5.1.38}
\end{equation*}
$$

The variation of this $\Delta$ object has the form

$$
\begin{equation*}
Q \Delta_{1|2| 3,4,5,6,7}=V_{1} k_{2}^{m} Y_{2,3, \ldots, 7}^{m}-V_{12} Y_{3,4, \ldots, 7}+\left[V_{1} Y_{23,4,5,6,7}+(3 \leftrightarrow 4, \ldots, 7)\right] \tag{5.1.39}
\end{equation*}
$$

Further generalisations of $\Delta$ will not be used in this thesis, and so we do not extend its indices to general Lie monomials.

### 5.2 Amplitudes in Field Theory

We now discuss how these objects are used to construct amplitudes in field theory. Prior to the work discussed in this thesis, such was known up to six points, and the six point case was not in a BCJ representation. As such we limit ourselves to these cases here, and will extend these results considerably later. The notation discussed in appendix A. 2 will be used going forward. These results will be linked with those in string theory in section 5.4, and then in a more general class of circumstances in part III.


Figure 5.2.1: There is only one diagram at four points one loop

### 5.2.0.1 Four Points

In the canonical ordering, the only diagram is that shown in figure 5.2.1. We can read off from this diagram the form the denominator of this term must have,

$$
\begin{equation*}
I_{1,2,3,4}=\frac{1}{\ell^{2}\left(\ell-k_{1}\right)^{2}\left(\ell-k_{12}\right)^{2}\left(\ell-k_{123}\right)^{2}} . \tag{5.2.1}
\end{equation*}
$$

This uses the notation discussed in appendix A.2. There are no mandelstam terms due to the absence of tree-like structures in the diagram. As for the numerator, only one term has been identified which can arise at four points,

$$
\begin{equation*}
V_{1} T_{2,3,4} \tag{5.2.2}
\end{equation*}
$$

This is indeed the numerator and so we have the four point one loop amplitude [1],

$$
\begin{equation*}
A^{1-l o o p}(1,2,3,4)=\frac{V_{1} T_{2,3,4}}{\ell^{2}\left(\ell-k_{1}\right)^{2}\left(\ell-k_{12}\right)^{2}\left(\ell-k_{123}\right)^{2}} . \tag{5.2.3}
\end{equation*}
$$

The validity of this can be verified as it lies in the BRST cohomology. That is, its variation vanishes, as can be seen in that $Q V_{1}=0$ and $Q T_{2,3,4}=0$. As for showing that it is not the variation of another object, we note that [130]

$$
\begin{equation*}
\frac{1}{k^{1} \cdot k^{234}} Q\left(k_{m}^{1} T_{1,2,3,4}^{m}+\left[T_{12,3,4}+(2 \leftrightarrow 3,4)\right]\right)=V_{1} T_{2,3,4} . \tag{5.2.4}
\end{equation*}
$$

By momentum conservation, $k^{1} \cdot k^{234}=k^{1} \cdot k^{1}=0$. Thus, alike at tree level, the four point amplitude is the variation of something which is not valid at four points, and so it is in the cohomology. Note at higher points this part of the discussion will not be repeated, but for details one should consult [130; 1].

### 5.2.0.2 Five Points

There are six diagrams at five points; five boxes and a pentagon. The amplitude may be expressed as

$$
\begin{align*}
A(1,2,3,4,5) & =\mathcal{N}_{1 \mid 2,3,4,5}(\ell)+\sum_{A B C D E=2345} \mathcal{N}_{E 1 A \mid B, C, D}(\ell) I_{1 A, B, C, D}^{(4)}  \tag{5.2.5}\\
& =A^{\text {pent }}(1,2,3,4,5)+A^{b o x}(1,2,3,4,5)
\end{align*}
$$

where we use the Berends-Giele current notation detailed in appendix A.2.

The five point boxes are a generalisation of the four point case, namely [1]

$$
\begin{equation*}
N_{A \mid B, C, D}(\ell)=V_{A} T_{B, C, D} \tag{5.2.6}
\end{equation*}
$$

for $A, B, C$, and $D$ Lie monomials. We then move onto the pentagon. There are two possible classes of terms their numerators may be composed of; $V_{A} T_{B, C, D}$ and $V_{A} T_{B, C, D, E}^{m}$. The variation of the pentagon must be such that it cancels the variation of the boxes exactly, and so we use this to fix the precise combination of these terms needed. This occurs when we take the pentagon numerator to be [1]

$$
\begin{align*}
N_{1 \mid 2,3,4,5}(\ell)=\ell_{m} V_{1} T_{2,3,4,5}^{m} & +\frac{1}{2}\left(V_{12} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right) \\
& +\frac{1}{2}\left(V_{1} T_{23,4,5}+(2,3 \mid 2,3,4,5)\right) . \tag{5.2.7}
\end{align*}
$$

This has variation [1]

$$
\begin{align*}
Q N_{1 \mid 2,3,4,5}^{(5)}(\ell) & =\frac{1}{2} V_{1} V_{2} T_{3,4,5}\left(\left(\ell-k_{12}\right)^{2}-\left(\ell-k_{1}\right)^{2}\right) \\
& +\frac{1}{2} V_{1} V_{3} T_{2,4,5}\left(\left(\ell-k_{123}\right)^{2}-\left(\ell-k_{12}\right)^{2}\right) \\
& +\frac{1}{2} V_{1} V_{4} T_{2,3,5}\left(\left(\ell-k_{1234}\right)^{2}-\left(\ell-k_{123}\right)^{2}\right)  \tag{5.2.8}\\
& +\frac{1}{2} V_{1} V_{5} T_{2,3,4}\left(\ell^{2}-\left(\ell-k_{1234}\right)^{2}\right)
\end{align*},
$$

wherein we have reexpressed any $\left(\ell \cdot k_{i}\right)$ functions in terms of the propagators using

$$
\begin{equation*}
\left(\ell \cdot k_{i}\right)=-\frac{1}{2}\left(\ell-k_{12 \ldots i}\right)^{2}+\frac{1}{2}\left(\ell-k_{12 \ldots i-1}\right)^{2}+\left(k^{12 \ldots i-1} \cdot k^{i}\right) . \tag{5.2.9}
\end{equation*}
$$

These then cancel with the $I_{1,2,3,4,5}$ denominator of the pentagon, giving [1]

$$
\begin{align*}
Q A^{\text {pent }}(1,2,3,4,5) & =\frac{1}{2} V_{1} V_{2} T_{3,4,5}\left(I_{12,3,4,5}^{(4)}-I_{1,23,4,5}^{(4)}\right) \\
& +\frac{1}{2} V_{1} V_{3} T_{2,4,5}\left(I_{1,23,4,5}^{(4)}-I_{1,2,34,5}^{(4)}\right) \\
& +\frac{1}{2} V_{1} V_{4} T_{2,3,5}\left(I_{1,2,34,5}^{(4)}-I_{1,2,3,45}^{(4)}\right)  \tag{5.2.10}\\
& +\frac{1}{2} V_{1} V_{5} T_{2,3,4}\left(I_{1,2,3,45}^{(4)}-I_{1,2,3,4}^{(4)}\right) \\
& =-Q A^{b o x}(1,2,3,4,5)
\end{align*}
$$

Hence, the variation of the five point amplitude defined using these numerators vanishes.

### 5.2.0.3 Six Points

The number of diagrams grows rapidly as we increase the number of points. There are now a hexagon diagram, six pentagons, and twenty-one boxes. We again express the amplitude using the notation of Berends Giele currents,

$$
\begin{align*}
A(1,2,3,4,5,6)= & \mathcal{N}_{1 \mid 2,3,4,5,6}(\ell) I_{1,2,3,4,5,6}+\sum_{A B C D E F=23456} \mathcal{N}_{F 1 A \mid B, C, D, E}(\ell) I_{1 A, B, C, D, E} \\
& +\sum_{A B C D E=23456} \mathcal{N}_{E 1 A \mid B, C, D}(\ell) I_{E 1 A, B, C, D}  \tag{5.2.11}\\
= & A^{h e x}(1,2,3,4,5,6)+A^{\text {pent }}(1,2,3,4,5,6)+A^{b o x}(1,2,3,4,5,6) .
\end{align*}
$$

The boxes are again defined by (10.3.10). The pentagons are largely defined through a generalisation of (5.2.7) [1],

$$
\begin{align*}
N_{A \mid B, C, D, E}(\ell)=\ell_{m} V_{A} T_{B, C, D, E}^{m} & +\frac{1}{2}\left(V_{[A, B]} T_{C, D, E}+(B \leftrightarrow C, D, E)\right) \\
& +\frac{1}{2}\left(V_{A} T_{[B, C], D, E}+(B, C \mid B, C, D, E)\right) . \tag{5.2.12}
\end{align*}
$$

The exception to this is $N_{61 \mid 2,3,4,5}(\ell)$, which is given by [1]

$$
\begin{align*}
N_{61 \mid 2,3,4,5}(\ell)=\left(\ell_{m}+k_{m}^{6}\right) V_{61} T_{2,3,4,5}^{m} & +\frac{1}{2}\left(V_{[61,2]} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right) \\
-V_{1} J_{6 \mid 2,3,4,5} & +\frac{1}{2}\left(V_{61} T_{[2,3], 4,5}+(2,3 \mid 2,3,4,5)\right) \tag{5.2.13}
\end{align*}
$$

The reason for this exception the associated $I_{61,2,3,4,5}$ denominator. This does not contain a $\ell^{2}$ alike the other pentagons, and so the $V J$ term is included to correct for complications which arise writing the variation as a function of propagators. An alternative formulation of this numerator, and a more complete reasoning for why it differs from the rest, will be discussed later.

The exceptional term then has variation of a similar structure to all other pentagons [1],

$$
\begin{gather*}
Q \mathcal{N}_{61 \mid 2,3,4,5}^{(5)}(\ell)=V_{6} N_{1 \mid 2,3,4,5}^{(5)}(\ell)+\frac{1}{2} V_{1}\left(V_{62} T_{3,4,5}+V_{63} T_{2,4,5}+V_{64} T_{2,3,5}+V_{65} T_{2,3,4}\right)  \tag{5.2.14}\\
+M_{16} M_{2} M_{3,4,5}\left[\left(\ell-k_{1}\right)^{2}-\left(\ell-k_{12}\right)^{2}\right]+M_{16} M_{3} M_{2,4,5} \frac{1}{2}\left[\left(\ell-k_{12}\right)^{2}-\left(\ell-k_{123}\right)^{2}\right] \\
+M_{16} M_{4} M_{2,3,5} \frac{1}{2}\left[\left(\ell-k_{123}\right)^{2}-\left(\ell-k_{1234}\right)^{2}\right]+M_{16} M_{5} M_{2,3,4} \frac{1}{2}\left[\left(\ell-k_{1234}\right)^{2}-\left(\ell-k_{12345}\right)^{2}\right] \\
+V_{1} Y_{2,3,4,5,5}
\end{gather*}
$$

The only significant difference is the presence of the anomaly $Y$ term. This will be discussed more later.

The variation of the pentagons defined above cancels that of the boxes as required, and leaves some terms for the hexagon to cancel. As such, the hexagon should be defined such that [1]

$$
\begin{align*}
Q A^{\text {hex }}(1,2,3,4,5,6) & =-Q A^{p e n t}(1,2,3,4,5,6)-Q A^{b o x}(1,2,3,4,5,6)  \tag{5.2.15}\\
& +2 V_{1} Y_{2,3,4,5,6} I_{1,2,3,4,5}^{(5)}+2 V_{2} N_{1 \mid 3,4,5,6}(\ell)\left[I_{1,23,4,5,6}-I_{12,3,4,5,6}\right] \\
& +2 V_{3} N_{12,4,5,6}(\ell)\left[I_{1,2,34,5,6}-I_{1,23,4,5,6}\right]+2 V_{4} N_{1 \mid 2,3,5,6}(\ell)\left[I_{1,2,3,45,6}-I_{1,2,34,5,6}\right] \\
& +2 V_{5} N_{12,3,4,6}(\ell)\left[I_{1,2,3,4,56}-I_{1,2,3,45,6}\right]+2 V_{6} N_{1 \mid 2,3,4,5}(\ell)\left[I_{1,2,3,4,5}-I_{1,2,3,4,56}\right] \\
& +V_{1} V_{23} T_{4,5,6}\left[I_{12,3,4,5,6}-I_{1,2,34,5,6}\right]+V_{1} V_{34} T_{2,5,6}\left[I_{1,23,4,5,6}-I_{1,2,3,45,6}\right] \\
& +V_{1} V_{45} T_{2,3,6}\left[I_{1,2,34,5,6}-I_{1,2,3,4,56}\right]+V_{1} V_{56} T_{2,3,4}\left[I_{1,2,3,4,56}-I_{1,2,3,4,5}\right] \\
& +V_{1} V_{24} T_{3,5,6}\left[I_{12,3,4,5,6}-I_{1,23,4,5,6}+I_{1,2,34,5,6}-I_{1,2,3,45,6}\right] \\
& +V_{1} V_{25} T_{3,4,6}\left[I_{12,3,4,5,6}-I_{1,23,4,5,6}+I_{1,2,3,45,6}-I_{1,2,3,4,56}\right] \\
& +V_{1} V_{26} T_{3,4,5}\left[I_{12,3,4,5,6}-I_{1,23,4,5,6}+I_{1,2,3,4,56}-I_{1,2,3,4,5}\right] \\
& +V_{1} V_{35} T_{2,4,6}\left[I_{1,23,4,5,6}-I_{1,2,34,5,6}+I_{1,2,3,45,6}-I_{1,2,3,4,56}\right] \\
& +V_{1} V_{36} T_{2,4,5}\left[I_{1,23,4,5,6}-I_{1,2,34,5,6}+I_{1,2,3,4,56}-I_{1,2,3,4,5]}\right] \\
& +V_{1} V_{46} T_{2,3,5}\left[I_{1,2,34,5,6}-I_{1,2,3,45,6}+I_{1,2,3,4,56}-I_{1,2,3,4,5}\right] .
\end{align*}
$$

This is solved by assigning the hexagon numerator the value [1]

$$
\begin{align*}
N_{1 \mid 2,3,4,5,6}(\ell) & =\frac{1}{2} \ell_{m} \ell_{n} V_{1} T_{2,3,4,5,6}^{m n} \\
& +\frac{1}{2} \ell_{m}\left[V_{12} T_{3,4,5,6}^{m}+(2 \leftrightarrow 3,4,5,6)\right] \\
& +\frac{1}{2} \ell_{m} V_{1}\left[T_{23,4,5,6}^{m}+(2,3 \mid 2,3,4,5,6)\right] \\
& +\frac{1}{4}\left[V_{1} T_{23,45,6}+(2,3|4,5| 2,3,4,5,6)\right] \\
& +\frac{1}{4}\left[V_{12} T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right] \\
& +\frac{1}{6}\left[\left(V_{1} T_{234,5,6}+V_{1} T_{432,5,6}\right)+(2,3,4 \mid 2,3,4,5,6)\right]  \tag{5.2.16}\\
& -\frac{1}{12}\left[\left(k_{m}^{1}-k_{m}^{2}\right) V_{12} T_{3,4,5,6}^{m}+(2 \leftrightarrow 3,4,5,6)\right] \\
& +\frac{1}{6}\left[\left(V_{123} T_{4,5,6}+V_{321} T_{4,5,6}\right)+(2,3 \mid 2,3,4,5,6)\right] \\
& -\frac{1}{12}\left[\left(k_{m}^{2}-k_{m}^{3}\right) V_{1} T_{23,4,5,6}^{m}+(2,3 \mid 2,3,4,5,6)\right] \\
& -\frac{1}{24} V_{1} T_{2,3,4,5,6}^{m n}\left[k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4,5,6)\right]
\end{align*}
$$

Note that the terms containing loop momentum factors are essentially $\ell^{m}$ multiplied by the definition of $N_{A \mid B, C, D, E}(\ell)$ with an extra tensor index added to all terms. The same was true of $N_{A \mid B, C, D, E}(\ell)$ compared with $N_{A \mid B, C, D}(\ell)$, and would be expected to be likewise for $N_{1 \mid 2,3,4,5,6,7}(\ell)$.

This is such that the variation of the overall amplitude is $[1 ; 131]$

$$
\begin{equation*}
Q A(1,2,3,4,5,6)=\frac{1}{2} V_{1} Y_{2,3,4,5,6}\left(I_{1,2,3,4,5}-\ell^{2} I_{1,2,3,4,5,6}\right) \tag{5.2.17}
\end{equation*}
$$

which would appear to cancel, but there are complications which arise as a result of dimensional regularisation. Further analysis reveals [1; 132]

$$
\begin{equation*}
Q \int d^{D} \ell A(1,2,3,4,5,6)=-\frac{\pi^{5}}{240} V_{1} Y_{2,3,4,5,6} \tag{5.2.18}
\end{equation*}
$$

This anomalous term remaining corresponds with the known results [109; 133]. Hence, the variation of the six point one loop amplitude has the required form.

### 5.2.1 BCJ Relations

The above satisfy BCJ relations at four and five points, but at six points they fail. To see this, we begin at four points. These are trivial to check; given the vanishing of triangle diagrams, this is just a matter of checking relations of the form seen in figure 5.2.2. As


Figure 5.2.2: The vanishing of the triangle diagrams means that the four point BCJ relations essentially just verify that we have symmetry in the legs of the diagrams

 -


Figure 5.2.3: A BCJ relation at five points one loop
all four point boxes are given by $V_{1} T_{a, b, c}$, for $a b c$ some permutation of 234 , and $T$ is symmetric in its blocks of indices, this identity follows trivially. Hence, at four points the BCJ relations are satisfied.

At five points, we have non-trivial BCJ relations between pentagons and boxes. One example of such is given in figure 5.2.3. These need more work to verify, but follow without too much work. In this example, we are trying to show the identity

$$
\begin{equation*}
N_{1 \mid 2,3,4,5}(\ell)-N_{1 \mid 3,2,4,5}(\ell)-N_{1 \mid[2,3], 4,5}(\ell)=0 \tag{5.2.19}
\end{equation*}
$$

Taking the definitions of these numerators and plugging them in, this is

$$
\begin{align*}
&\left(\ell_{m} V_{1} T_{2,3,4,5}^{m}+\frac{1}{2}\left(V_{[1,2]} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right)+\frac{1}{2}\left(V_{1} T_{[2,3], 4,5}+(2,3 \mid 2,3,4,5)\right)\right) \\
&-\left(\ell_{m} V_{1} T_{3,2,4,5}^{m}+\frac{1}{2}\left(V_{[1,3]} T_{2,4,5}+(3 \leftrightarrow 2,4,5)\right)+\frac{1}{2}\left(V_{1} T_{[3,2], 4,5}+(3,2 \mid 3,2,4,5)\right)\right)  \tag{5.2.20}\\
&-V_{1} T_{23,4,5}=0 .
\end{align*}
$$

This then vanishes by symmetry in the blocks of indices the $T$ building blocks [1].

At six points, complications can arise. We consider the pair of BCJ identities given in figure 5.2.4. The first of these works as required [1];

-


 -



Figure 5.2.4: A pair of BCJ relations at 6 points

$$
\begin{align*}
N_{1 \mid 2,34,5,6}(\ell)-N_{1 \mid 2,34,6,5}(\ell)-N_{1 \mid 23,4,56}(\ell) & = \\
+\left(\ell_{m} V_{1} T_{2,34,5,6}^{m}\right. & +\frac{1}{2}\left(V_{[1,2]} T_{34,5,6}+(2 \leftrightarrow 34,5,6)\right) \\
+ & \left.\frac{1}{2}\left(V_{1} T_{[2,34], 5,6}+(2,34 \mid 2,34,5,6)\right)\right) \\
-\left(\ell_{m} V_{1} T_{2,34,6,5}^{m}\right. & +\frac{1}{2}\left(V_{[1,2]} T_{34,6,5}+(2 \leftrightarrow 34,6,5)\right)  \tag{5.2.21}\\
& \left.+\frac{1}{2}\left(V_{1} T_{[2,34], 6,5}+(2,34 \mid 2,34,6,5)\right)\right)
\end{align*}
$$

$$
\begin{aligned}
& -V_{1} T_{2,34,56} \\
& =0
\end{aligned}
$$

The second identity fails however. We have thus far been lax with the positioning of the loop momentum factors in diagrams, but now we must be precise. We must make sure that, in all diagrams in a BCJ relation, the momentum around the loop away from the branches affected by the relation are the same. This is related to the discussion of the labelling problem in section 2.3.4. The significance for this relation is that, rather than considering $N_{1 \mid 4,5,6,23}(\ell)$, we must instead look to $N_{1 \mid 4,5,6,23}\left(\ell-k_{23}\right)$,

$$
\begin{equation*}
N_{1 \mid 23,4,5,6}(\ell)-N_{1 \mid 4,5,6,23}\left(\ell-k_{23}\right)-N_{[1,23] \mid 4,5,6}(\ell)=0 \tag{5.2.22}
\end{equation*}
$$

Plugging in the numerator values, this is [1]

$$
\begin{gathered}
\left(\ell_{m} V_{1} T_{23,4,5,6}^{m}+\frac{1}{2}\left(V_{[1,23]} T_{4,5,6}+(23 \leftrightarrow 4,5,6)\right)+\frac{1}{2}\left(V_{1} T_{[23,4], 5,6}+(23,4 \mid 23,4,5,6)\right)\right) \\
-\left(\left(\ell_{m}-k_{m}^{23}\right) V_{1} T_{4,5,6,23}^{m}+\frac{1}{2}\left(V_{[1,4]} T_{5,6,23}+(4 \leftrightarrow 5,6,23)\right)+\frac{1}{2}\left(V_{1} T_{[4,5], 6,23}+(4,5 \mid 4,5,6,23)\right)\right) \\
-V_{1} T_{[2,3], 4], 5,6}
\end{gathered}
$$

$$
\begin{align*}
& =k_{m}^{23} V_{1} T_{23,4,5,6}^{m}+V_{231} T_{4,5,6}+\left(V_{1} T_{234,5,6}+(4 \leftrightarrow 5,6)\right)  \tag{5.2.23}\\
& \neq 0
\end{align*}
$$

The above is not BRST trivial, as can be seen in its non-vanishing variation

$$
\begin{align*}
& Q\left(k_{m}^{23} V_{1} T_{23,4,5,6}^{m}+V_{231} T_{4,5,6}+\left(V_{1} T_{234,5,6}+(4 \leftrightarrow 5,6)\right)\right) \\
& =\left(k^{2} \cdot k^{3}\right)\left[V_{1} V_{3} T_{2,4,5,6}^{m} k_{23}^{m}+V_{13} V_{2} T_{4,5,6}\right.  \tag{5.2.24}\\
& \left.\quad+\left(V_{1} V_{3} T_{24,5,6}+V_{1} V_{34} T_{2,5,6}+(4 \leftrightarrow 5,6)\right)-(2 \leftrightarrow 3)\right] \neq 0
\end{align*}
$$

Hence the identity fails. This result concluded the work of [1]. However, we since developed new methods which rectify this problem, and extend the results considerably.

### 5.3 Amplitudes in String Theory

This section provides only a brief summary of the computations by which one-loop string amplitudes are identified in the pure spinor formalism. For a more complete description, we refer the reader to [20; 21; 22], and the citations therein. These should be regarded as sources for the entirety of this section.

To begin, recall the string amplitude formula

$$
\begin{equation*}
\mathcal{A}_{n}=\int_{0}^{\infty} \int_{0 \leq \operatorname{Im}} d z_{i} \leq \operatorname{Im} z_{2+1} d z_{3} \ldots d z_{n}\left\langle\int \mu b \mathcal{Z} V_{1}\left(z_{1}\right) \prod_{j=2}^{n} U_{j}\left(z_{j}\right)\right\rangle, \tag{5.3.1}
\end{equation*}
$$

Through the field theory amplitude discussion, we know the form the OPEs of the vertex operator and $b$-ghost take. What remains are the features unique to string theory, namely the worldsheet factors describing the toroidal surface the amplitude is integrated over. In this section, we introduce the functions this is described in terms of, and then identify the one loop string amplitudes.

### 5.3.1 Kronecker-Eisenstein Series

The worldsheet functions of string amplitudes are a function of terms from the KroneckerEisenstein series (KE series) [134; 135; 136; 137]. This is defined by the ratio

$$
\begin{equation*}
F(z, \alpha, \tau) \equiv \frac{\theta_{1}^{\prime}(0, \tau) \theta_{1}(z+\alpha, \tau)}{\theta_{1}(\alpha, \tau) \theta_{1}(z, \tau)} \equiv \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z, \tau) \tag{5.3.2}
\end{equation*}
$$

where the $\theta_{1}$ is one of the Jacobi theta functions, defined by (for $q=e^{2 \pi i \tau}$ )

$$
\begin{equation*}
\theta_{1}(z, \tau) \equiv 2 q^{1 / 8} \sin (\pi z) \prod_{n=1}^{\infty}\left(1-q^{n}\right)\left(1-q^{n} e^{2 \pi i z}\right)\left(1-q^{n} e^{-2 \pi i z}\right) \tag{5.3.3}
\end{equation*}
$$

It is the coefficients of terms from the KE series which will appear most frequently in this section. The first two instances of these are

$$
\begin{equation*}
g^{(1)}(z, \tau)=\partial \log \theta_{1}(z, \tau), \quad g^{(2)}(z, \tau)=\frac{1}{2}\left[\left(\partial \log \theta_{1}(z, \tau)\right)^{2}-\wp(z, \tau)\right] \tag{5.3.4}
\end{equation*}
$$

where $\wp(z, \tau)=-\partial^{2} \log \tau_{1}(z, \tau)-G_{2}(\tau)$ is the Weierstrass function, and $G_{2}$ is defined in terms of Eisenstein Series

$$
\begin{equation*}
G_{2 k}(\tau)=\sum_{(m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash\{(0,0)\}} \frac{1}{(m \tau+n)^{2 k}}=-g^{(2 k)}(0, \tau) \tag{5.3.5}
\end{equation*}
$$

The poles of these functions will be critical in future calculations. The first KE term $g^{(1)}(z, \tau)$ has poles at $z=0$. All other terms $g^{(i)}(z, \tau)$ have poles only when the greater constraint of $z=m \tau+n, m, n \in \mathbb{Z}, m \neq 0$, is satisfied. Going forward, we will use the notation

$$
\begin{equation*}
g_{i j}^{(i)} \equiv g^{(i)}\left(z_{i}-z_{j}, \tau\right) \tag{5.3.6}
\end{equation*}
$$

to simplify the discussion.

Terms from the KE series satisfy Fay identities [138]. This identity is

$$
\begin{equation*}
F\left(z_{1}, \alpha_{1}, \tau\right) F\left(z_{2}, \alpha_{2}, \tau\right)=F\left(z_{1}, \alpha_{1}+\alpha_{2}, \tau\right) F\left(z_{2}-z_{1}, \alpha_{2}, \tau\right)+(1 \leftrightarrow 2) \tag{5.3.7}
\end{equation*}
$$

which upon expanding in terms of components gives [135]

$$
\begin{align*}
g_{12}^{(n)} g_{23}^{(m)} & =-g_{13}^{(m+n)}+\sum_{j=0}^{n}(-1)^{j}\binom{m-1+j}{j} g_{13}^{(n-j)} g_{23}^{(m+j)} \\
& +\sum_{j=0}^{m}(-1)^{j}\binom{n-1+j}{j} g_{13}^{(m-j)} g_{12}^{(n+j)} \tag{5.3.8}
\end{align*}
$$

Only the lowest order cases of this will be needed for this thesis, and so we state the first six cases

$$
\begin{align*}
& g_{12}^{(1)} g_{23}^{(1)}+g_{12}^{(2)}+c y c(1,2,3)=0  \tag{5.3.9}\\
& g_{12}^{(1)} g_{23}^{(2)}=g_{13}^{(1)} g_{23}^{(2)}+g_{12}^{(1)} g_{13}^{(2)}-g_{13}^{(1)} g_{12}^{(2)}+g_{12}^{(3)}-g_{13}^{(3)}-2 g_{23}^{(3)} \\
& g_{12}^{(2)} g_{23}^{(2)}=g_{12}^{(2)} g_{13}^{(2)}+g_{13}^{(2)} g_{23}^{(2)}-2 g_{13}^{(1)} g_{12}^{(3)}-2 g_{13}^{(1)} g_{23}^{(3)}+3 g_{12}^{(4)}-g_{13}^{(4)}+3 g_{23}^{(4)}, \\
& g_{12}^{(1)} g_{23}^{(3)}=-g_{12}^{(2)} g_{13}^{(2)}+g_{13}^{(1)} g_{12}^{(3)}+g_{12}^{(1)} g_{13}^{(3)}+g_{13}^{(1)} g_{23}^{(3)}-g_{12}^{(4)}-g_{13}^{(4)}-3 g_{23}^{(4)} \\
& g_{12}^{(2)} g_{23}^{(3)}=-g_{13}^{(5)}+6 g_{23}^{(5)}-4 g_{12}^{(5)}+g_{13}^{(2)} g_{23}^{(3)}-3 g_{13}^{(1)} g_{23}^{(4)}+g_{13}^{(3)} g_{12}^{(2)}-2 g_{13}^{(2)} g_{12}^{(3)}+3 g_{13}^{(1)} g_{12}^{(4)},
\end{align*}
$$

$$
g_{12}^{(1)} g_{23}^{(4)}=-g_{13}^{(5)}-4 g_{23}^{(5)}+g_{12}^{(5)}+g_{13}^{(1)} g_{23}^{(4)}+g_{13}^{(4)} g_{12}^{(1)}-g_{13}^{(3)} g_{12}^{(2)}+g_{13}^{(2)} g_{12}^{(3)}-g_{13}^{(1)} g_{12}^{(4)}
$$

The coefficients of the Kronecker-Eisenstein series have all properties required to describe one-loop string amplitudes. We will not go into any great detail on this point, but to give a flavor of these results, it follows from a known result of Jacobi theta functions [136],

$$
\begin{equation*}
\theta_{1}(z+1, \tau)=-\theta(z, \tau), \tag{5.3.10}
\end{equation*}
$$

that the KE series is invariant under A-cycles,

$$
\begin{equation*}
F(z+1, \alpha, \tau)=F(z, \alpha, \tau) . \tag{5.3.11}
\end{equation*}
$$

A more complex result holds for B-cycles, but we will not detail this here as it is not used in this work. Further, it can be seen that

$$
\begin{equation*}
g_{i j}^{(1)} \sim 1 / z_{i j}+O\left(z_{i j}\right) \tag{5.3.12}
\end{equation*}
$$

which serves as something of a generalisation of the $1 / z_{i j}$ terms associated with tree level string amplitudes.

### 5.3.2 One Loop String Correlators

We describe one loop amplitudes in string theory in terms of string correlators, which when integrated on an appropriate domain give amplitudes. These are split into two sectors; the anomalous sector, containing only anomalous contributions and denoted $\mathcal{K}_{n}^{Y}(\ell)$, and the Lie polynomial sector, containing the rest of the information and denoted by $\mathcal{K}_{n}^{L i e}(\ell)$. The name of the latter of these two will be explained shortly. The correlator is the sum of the two sectors [22],

$$
\begin{equation*}
\mathcal{K}_{n}(\ell)=\mathcal{K}_{n}^{L i e}(\ell)+\mathcal{K}_{n}^{Y}(\ell) . \tag{5.3.13}
\end{equation*}
$$

Explicit formulae for these are known to seven points. These are given in terms of worldsheet functions, $\mathcal{Z}_{A}$. The anomaly sector is vanishing up to six points, and so the only non-vanishing value it takes is

$$
\begin{equation*}
\mathcal{K}_{7}^{Y}(\ell)=-\Delta_{1|2| 3,4,5,6,7} \mathcal{Z}_{12 \mid 3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7) . \tag{5.3.14}
\end{equation*}
$$

The Lie polynomial sector is in general non-vanishing, and is given by the sum

$$
\begin{equation*}
\mathcal{K}_{n}^{L i e}(\ell) \equiv \sum_{d=0}^{\left\lfloor\frac{n-4}{2}\right\rfloor}(-1)^{d} \mathcal{K}_{n}^{(d)}(\ell) \tag{5.3.15}
\end{equation*}
$$

where the $\mathcal{K}_{n}^{(d)}(\ell)$ refer to the part of the Lie polynomial sector composed of building blocks with degree of refinement $d$, defined by

$$
\begin{align*}
\mathcal{K}_{n}^{(d)}(\ell)=\sum_{r=0}^{n-4-2 d} \frac{1}{r!}( & \left(V_{A_{1}} J_{A_{2}, \ldots, A_{d+1} \mid A_{d+2}, \ldots, A_{r+4+2 d}}^{m_{1} \ldots m_{r}} \mathcal{Z}_{A_{2}, \ldots, A_{d+1} \mid A_{1}, A_{d+2}, \ldots, A_{r+4+2 d}}^{m_{1} \ldots m_{r}}\right.  \tag{5.3.16}\\
& \left.\left.+\left(A_{2}, \ldots, A_{d+1} \mid A_{2}, \ldots, A_{r+4+2 d}\right)\right)+\left[12 \ldots n \mid A_{1}, \ldots, A_{r+4+2 d}\right]\right) .
\end{align*}
$$

Note the second sum is over Stirling cycles, as is defined in appendix A.1. So, for instance, $\mathcal{K}^{(0)}$ is a function of $V T$ terms, $\mathcal{K}^{(1)}$ a function of $V J_{A \mid B, C, \ldots}$ terms, $\mathcal{K}^{(2)}$ a function of $V J_{A, B \mid C, D, \ldots}$ terms, and so on. Bringing this all together, we find expressions for the correlators, stated here in their entirety [22]

$$
\begin{align*}
\mathcal{K}_{4}(\ell) & =V_{1} T_{2,3,4} \mathcal{Z}_{1,2,3,4},  \tag{5.3.17}\\
\mathcal{K}_{5}(\ell) & =V_{1} T_{2,3,4,5}^{m} \mathcal{Z}_{1,2,3,4,5}^{m}  \tag{5.3.18}\\
& +V_{A} T_{B, C, D}^{m} \mathcal{Z}_{A, B, C, D}+[12345 \mid A, B, C, D], \\
\mathcal{K}_{6}(\ell) & =\frac{1}{2} V_{1} T_{2,3,4,5,6}^{m n} \mathcal{Z}_{1,2,3,4,5,6}^{m n} \\
& +V_{A} T_{B, C, D, E}^{m} \mathcal{Z}_{A, B, C, D, E}^{m}+[123456 \mid A, B, C, D, E]  \tag{5.3.19}\\
& +V_{A} T_{B, C, D} \mathcal{Z}_{A, B, C, D}+[123456 \mid A, B, C, D], \\
\mathcal{K}_{7}(\ell) & =\frac{1}{6} V_{1} T_{2,3,4,5,6,7}^{m n p} \mathcal{Z}_{1,2,3,4,5,6,7}^{m n p} \\
& +\frac{1}{2} V_{A} T_{B, C, D, E, F}^{m n} \mathcal{Z}_{A, B, C, D, E, F}^{m n}+[1234567 \mid A, B, C, D, E, F] \\
& +V_{A} T_{B, C, D, E}^{m} \mathcal{Z}_{A, B, C, D, E}^{m}+[123456 \mid A, B, C, D, E]  \tag{5.3.20}\\
& +V_{A} T_{B, C, D} \mathcal{Z}_{A, B, C, D}+[123456 \mid A, B, C, D] \\
& -\left(V_{1} J_{2 \mid 3,4,5,6,7}^{m} \mathcal{Z}_{2 \mid 1,3,4,5,6,7}^{m}+(2 \leftrightarrow 3,4,5,6,7)\right) \\
& -\left(V_{A} J_{B \mid C, D, E, F}^{m} \mathcal{Z}_{B \mid A, C, D, E, F}^{m}+(B \leftrightarrow C, D, E, F)\right)+[1234567 \mid A, B, C, D, E, F] \\
& -\Delta_{1|2| 3,4,5,6,7} \mathcal{Z}_{12 \mid 3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7) .
\end{align*}
$$

We have not yet stated the form of the $\mathcal{Z}$ functions. These are symmetric in their blocks
of indices, and all cases up to six points are given by [21]

$$
\begin{align*}
\mathcal{Z}_{1,2,3,4} & =1, \quad \mathcal{Z}_{1,2,3,4,5}=\ell^{m}, \quad \mathcal{Z}_{12,3,4,5}=g_{12}^{(1)} \\
\mathcal{Z}_{123,4,5,6} & =g_{12}^{(1)} g_{23}^{(1)}+g_{12}^{(2)}+g_{23}^{(2)}-g_{13}^{(2)}, \\
\mathcal{Z}_{12,34,5,6} & =g_{12}^{(1)} g_{34}^{(1)}+g_{13}^{(2)}+g_{24}^{(2)}-g_{14}^{(2)}-g_{23}^{(2)}, \\
\mathcal{Z}_{12,3,4,5,6}^{m} & =\ell^{m} g_{12}^{(1)}+\left(k_{2}^{m}-k_{1}^{m}\right) g_{12}^{(2)}+\left[k_{3}^{m}\left(g_{13}^{(2)}-g_{23}^{(2)}\right)+(3 \leftrightarrow 4,5,6)\right],  \tag{5.3.21}\\
\mathcal{Z}_{1,2,3,4,5,6}^{m n} & =\ell^{m} \ell^{n}+\left[\left(k_{1}^{m} k_{2}^{n}+k_{1}^{n} k_{2}^{m}\right) g_{12}^{(2)}+(1,2 \mid 1,2,3,4,5,6)\right], \\
\mathcal{Z}_{2 \mid 1,3,4,5,6} & =0 .
\end{align*}
$$

The rank seven cases can be found in [21].

### 5.3.2.1 Lie Polynomial Sector

Here, we briefly state two properties of Lie polynomials, and justify the name of the Lie polynomial sector. As previously alluded to, for the purposes of this thesis a collection of words $P$ is a Lie polynomial if it is a linear combination of words which can be expressed as a linear combination of nested commutators. So for example, $P=123-213-312+321$ would be a Lie polynomial, since it can be written as $P=[[1,2], 3]$. However, no similar such relation exists for say $P=123$, and so this would not be a Lie polynomial.

It is often not immediately obvious whether a collection of words is a Lie polynomial. For example, it is far from obvious that

$$
\begin{align*}
1324+1423 & -1432-2134+2341-3124+3214-3241 \\
& -4123+4213-4231+4312=[[[1,2], 3], 4]+[[[2,3], 4], 1] . \tag{5.3.22}
\end{align*}
$$

As such, we need a theorem by Dynkin-Specht-Wever [139; 43], which states that a sum of words $P$ is a Lie polynomial if and only if $\ell(P)=|P| P$. So, one simple example would be that

$$
\begin{align*}
\ell(123-213-312+321) & =(123-213-312+321)-(213-123-321+312) \\
& -(312-132-231+213)+(321-231-132+123) \\
& =3 \cdot(123-213-312+321) . \tag{5.3.23}
\end{align*}
$$

Hence, it follows that $123-213-312+321$ is a Lie polynomial (it is $[[1,2], 3]$ ).
A theorem by Ree [140] states that if an object $M_{A}$ satisfies shuffle symmetries,

$$
\begin{equation*}
M_{A \amalg B}=0 \quad \forall A, B \neq \emptyset, \tag{5.3.24}
\end{equation*}
$$

and if $t^{a_{i}}$ are the letters describing our Lie algebra (1, 2, and 3 in (5.3.23)), $t^{A}=$ $t^{a_{1}} t^{a_{2}} \ldots t^{a_{|A|}}$, then the sum

$$
\begin{equation*}
P=\sum_{A} M_{A} t^{A} \tag{5.3.25}
\end{equation*}
$$

is a Lie polynomial. This result is not used any further in this thesis, so we will not cover it in any further detail. However, one should note the shuffle symmetric object times lie symmetric object structure of the above, as this is shared by the Lie polynomial sector of the string correlator. This shared structure is the origin of the sector name.

### 5.4 Relating Amplitudes in String and Field Theory

In this section, we briefly review previous work relating one loop field theory and string theory amplitudes at four and five points. This largely follows the appendix of [1].

The integral structures of string and field theory amplitudes are considerably different in appearance. Fortunately though, they may be related in the field theory limit $[34 ; 141$; $142 ; 143 ; 144 ; 145 ; 146]$. At four points, the relation is

$$
\begin{equation*}
\int d^{D} \ell I_{1,2,3,4}=\pi^{4} \int_{0}^{\infty} \frac{d \tau}{\tau} \tau^{4-D / 2} \int_{0 \leq z_{i} \leq z_{i+1} \leq 1} d z_{2} d z_{3} d z_{4} e^{-\pi \tau Q_{4}\left[k_{1}, k_{2}, k_{3}, k_{4}\right]} \tag{5.4.1}
\end{equation*}
$$

where the $Q$ denotes the field theory limit of the Koba-Nielsen factor,

$$
\begin{equation*}
Q_{n}\left[k_{A_{1}}, k_{A_{2}}, \ldots, k_{A_{n}}\right]=\sum_{i<j}^{n}\left(k_{A_{i}} \cdot k_{A_{j}}\right)\left(z_{i j}^{2}-\left|z_{i j}\right|\right) . \tag{5.4.2}
\end{equation*}
$$

In this case, equality between the string and field theory amplitudes is immediate. We take the string correlator integrated in the relevant terms as it should be, and by the above the field theory amplitude follows immediately,

$$
\begin{equation*}
\pi^{4} \int_{0}^{\infty} \frac{d \tau}{\tau} \tau^{4-D / 2} \int_{0 \leq z_{i} \leq z_{i+1} \leq 1} d z_{2} d z_{3} d z_{4} e^{-\pi \tau Q_{4}\left[k_{1}, k_{2}, k_{3}, k_{4}\right]} V_{1} T_{2,3,4}=\int d^{D} \ell I_{1,2,3,4}^{(4)} V_{1} T_{2,3,4} \tag{5.4.3}
\end{equation*}
$$

Hence at four points, we have equivalence between the string and field theory amplitudes.

At five points, the relation (5.4.1) generalises in the natural way,

$$
\begin{equation*}
\int d^{D} \ell I_{1,2,3,4,5}=\pi^{5} \int_{0}^{\infty} \frac{d \tau}{\tau} \tau^{5-D / 2} \int_{0 \leq z_{i} \leq z_{i+1} \leq 1} d z_{2} d z_{3} d z_{4} d z_{5} e^{-\pi \tau Q_{5}\left[k_{1}, k_{2}, k_{3}, k_{4}, k_{5}\right]} \tag{5.4.4}
\end{equation*}
$$

In fact, under certain constraints on the words $A_{i}$ such is the case in general, ${ }^{1}$

$$
\begin{equation*}
\int d^{D} \ell I_{A_{1}, A_{2}, \ldots, A_{n}}=\pi^{n} \int_{0}^{\infty} \frac{d \tau}{\tau} \tau^{n-D / 2} \int_{0 \leq z_{i} \leq z_{i+1} \leq 1} d z_{2} d z_{3} \ldots d z_{n} e^{-\pi \tau Q_{n}\left[k_{A_{1}}, k_{A_{2}}, \ldots, k_{A_{n}}\right]} \tag{5.4.5}
\end{equation*}
$$

We do have a significant further complication at five points and higher however; terms from the Kronecker-Eisenstein series with unclear limits. Consider the five point string amplitude,

$$
\begin{aligned}
& \pi^{5} \int_{0}^{\infty} \frac{d \tau}{\tau} \tau^{5-D / 2} \int_{0 \leq z_{i} \leq z_{i+1} \leq 1} d z_{2} d z_{3} d z_{4} d z_{5}\left(V_{1} T_{2,3,4,5}^{m} \ell^{m}+\left(V_{12} T_{3,4,5}^{m} g_{12}^{(1)}+(2 \leftrightarrow 3,4,5)\right)\right. \\
&+\left.\left(V_{1} T_{23,4,5}^{m} g_{23}^{(1)}+(2,3 \mid 2,3,4,5)\right) e^{-\pi \tau Q_{5}\left[k_{1}, k_{2}, k_{3}, k_{4}, k_{5}\right]}\right)
\end{aligned}
$$

As previously referenced, $g_{i j}^{(1)}$ has the behaviour of $z_{i j}^{-1}$ as $z_{i} \rightarrow z_{j}$. At such a pole, the $k_{i}$ and $k_{j}$ in the Koba-Nielsen factor coincide. So, for example

$$
\begin{equation*}
Q_{5}\left[k_{1}, k_{2}, k_{3}, k_{4}, k_{5}\right] \underset{z_{1} \rightarrow z_{2}}{ } Q_{5}\left[k_{12}, k_{3}, k_{4}, k_{5}\right] . \tag{5.4.7}
\end{equation*}
$$

As such, poles in the string integral correspond with lower order $n$-gons in the field theory amplitude. Only a select few terms in the five point string amplitude have poles in the integration domain, namely

$$
\begin{equation*}
g_{12}^{(1)}, g_{23}^{(1)}, g_{34}^{(1)}, g_{45}^{(1)}, g_{15}^{(1)} \tag{5.4.8}
\end{equation*}
$$

The poles of the first term then corresponds with the 12 -box, the second the 23 -box, etc. By taking the $g_{i j}^{(1)}$ functions to have value $\frac{1}{2}$ at and away from their poles, we then recover the field theory amplitude ${ }^{2}$. That is, we take for instance

$$
\begin{array}{r}
\pi^{5} \int_{0}^{\infty} \frac{d \tau}{\tau} \tau^{5-D / 2} \int_{0 \leq z_{i} \leq z_{i+1} \leq 1} d z_{2} d z_{3} d z_{4} d z_{5} g_{35}^{(1)} V_{12} T_{3,4,5}^{m} e^{-\pi \tau Q_{5}\left[k_{1}, k_{2}, k_{3}, k_{4}, k_{5}\right]} \\
\rightarrow \int d^{D} \ell V_{12} T_{3,4,5} \frac{1}{2} I_{1,2,3,4,5} \\
\pi^{5} \int_{0}^{\infty} \frac{d \tau}{\tau} \tau^{5-D / 2} \int_{0 \leq z_{i} \leq z_{i+1} \leq 1} d z_{2} d z_{3} d z_{4} d z_{5} g_{12}^{(1)} V_{12} T_{3,4,5}^{m} e^{-\pi \tau Q_{5}\left[k_{1}, k_{2}, k_{3}, k_{4}, k_{5}\right]}  \tag{5.4.10}\\
\rightarrow \int d^{D} \ell V_{12} T_{3,4,5}\left(\frac{1}{2} I_{1,2,3,4,5}+\frac{1}{2} I_{12,3,4,5}\right)
\end{array}
$$

Performing this procedure for all terms in the string amplitude gives that of field theory.

[^9]
## Part II

## Generalising the Construction of the BCJ Gauge

This part focuses on the construction of multiparticle superfields, and in particular the construction of those in the BCJ gauge. Currently, we have a scheme with which most Lorenz gauge superfields can be constructed with arbitrary topology to arbitrary multiplicity. Furthermore, we have a pair of approaches with which we can produce multiparticle superfields with a Dynkin bracket structure in the BCJ gauge. One of these is based upon an intermediate hybrid gauge and has a clear origin, and the other represents a direct transition from the Lorenz gauge and is more mysterious. That this represents a gauge transformation has been shown to rank five. Finally, up to rank five, more complex topology superfields in the BCJ gauge have been constructed also.

There are a number of areas in which this situation can be improved. Some Lorenz gauge superfields are currently undescribed for topologies beyond Dynkin brackets. Then those in the BCJ gauge are restricted even more so to this bracket structure. We would like to extend the current results to arbitrary topologies, rather than just this heavily constrained subset. Another issue is the absence of a general formula for the redefinition $H$ terms; we know the explicit form of these to rank five, but these were only found and simplified through long calculations and a general structure for these is not yet known. The calculations needed each time to find these grows exponentially with multiplicity, and finding the general form of these terms would save a great deal of work in computations.

In this part, we will make considerable steps towards such generalisations. This will take significant advantage of a new map, the so called "contact term map", introduced shortly. With this, several results will be extended, and in many cases general results will be found. We will then conclude with some avenues for future development. A large amount of the research which led to this part has been published in [27].

## The Contact Term Map

### 6.1 Definition and Examples

We begin with the introduction of the contact term map, which we denote $C$. This acts on Lie monomials recursively, and is defined by

$$
\begin{gather*}
C \circ i=0, \\
C \circ[P, Q]=(C \circ P) \wedge Q+P \wedge(C \circ Q)+\left(k^{P} \cdot k^{Q}\right)(P \otimes Q-Q \otimes P), \tag{6.1.1}
\end{gather*}
$$

for $i$ a letter, $P$ and $Q$ Lie monomials, and the wedge $\wedge$ defined by

$$
\begin{align*}
& (P \otimes Q) \wedge R \equiv[P, R] \otimes Q+P \otimes[Q, R]  \tag{6.1.2}\\
& P \wedge(Q \otimes R) \equiv[P, Q] \otimes R+Q \otimes[P, R],
\end{align*}
$$

On occasion, we may act upon a sum of Lie monomials. The contact term map should be taken to be linear in such situations; that is,

$$
\begin{equation*}
C \circ \sum_{m=1}^{n} a_{i}\left[P_{i}, Q_{i}\right]=\sum_{m=1}^{n} a_{i} C \circ\left[P_{i}, Q_{i}\right], \tag{6.1.3}
\end{equation*}
$$

for $a_{i}$ some constants and $P_{i}$ and $Q_{i}$ Lie monomials.
This will be made much clearer through examples. First of all, we consider a simple case,
$C \circ[1,2]$. Applying (6.1.1) once, we see we have

$$
\begin{equation*}
C \circ[1,2]=(C \circ 1) \wedge 2+1 \wedge(C \circ 2)+\left(k^{1} \cdot k^{2}\right)(1 \otimes 2-2 \otimes 1) . \tag{6.1.4}
\end{equation*}
$$

The result then follows as a consequence of the vanishing of $C \circ 1$ and $C \circ 2$,

$$
\begin{equation*}
C \circ[1,2]=\left(k^{1} \cdot k^{2}\right)(1 \otimes 2-2 \otimes 1) . \tag{6.1.5}
\end{equation*}
$$

For this example, the iterative nature of the definition (6.1.1) was not needed, and the definition of the (6.1.2) was not used either. As such, we consider the more complex example of $C \circ[1,[[2,3], 4]]$. Repeatedly applying (6.1.1) until all appearances of the contact term map vanish as a result of $C \circ i=0$, this is

$$
\begin{align*}
& C \circ[1,[[2,3], 4]]=(C \circ 1) \wedge[[2,3], 4]+1 \wedge(C \circ[[2,3], 4]) \\
&+\left(k^{1} \cdot k^{234}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1) \\
&=1 \wedge((C \circ[2,3]) \wedge 4)+1 \wedge([2,3] \wedge(C \circ 4)) \\
&+\left(k^{23} \cdot k^{4}\right) 1 \wedge([2,3] \otimes 4) \\
&+\left(k^{1} \cdot k^{234}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1) \\
&=1 \wedge(((C \circ 2) \wedge 3) \wedge 4)+1 \wedge((2 \wedge(C \circ 3)) \wedge 4)  \tag{6.1.6}\\
&+\left(k^{2} \cdot k^{3}\right) 1 \wedge((2 \otimes 3-3 \otimes 2) \wedge 4) \\
&+\left(k^{23} \cdot k^{4}\right) 1 \wedge([2,3] \otimes 4) \\
&+\left(k^{1} \cdot k^{234}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1) \\
&=\left(k^{2} \cdot k^{3}\right) 1 \wedge((2 \otimes 3-3 \otimes 2) \wedge 4)+\left(k^{23} \cdot k^{4}\right) 1 \wedge([2,3] \otimes 4) \\
&+\left(k^{1} \cdot k^{234}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1) .
\end{align*}
$$

Then the definition of the wedge operation (6.1.2) must be applied repeatedly, until all have been removed

$$
\begin{align*}
& C \circ[1,[[2,3], 4]]=\left(k^{2} \cdot k^{3}\right) 1 \wedge((2 \otimes 3-3 \otimes 2) \wedge 4)+\left(k^{23} \cdot k^{4}\right) 1 \wedge([2,3] \otimes 4) \\
&+\left(k^{1} \cdot k^{234}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1) \\
&\left.=\left(k^{2} \cdot k^{3}\right)\right) 1 \wedge([2,4] \otimes3+2 \otimes[3,4]) \\
&+\left(k^{23} \cdot k^{4}\right)([1,[2,3]] \otimes 4+[2,3] \otimes[1,4]) \\
&+\left(k^{234} \cdot k^{4}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1)  \tag{6.1.7}\\
&\left.=\left(k^{2} \cdot k^{3}\right)\right)([1,[2,4]] \otimes 3+[2,4] \otimes[1,3]+[1,2] \otimes[3,4] \\
&+2 \otimes[1,[3,4]]) \\
&+\left(k^{23} \cdot k^{4}\right)([1,[2,3]] \otimes 4+[2,3] \otimes[1,4]) \\
&+\left(k^{234} \cdot k^{4}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1) .
\end{align*}
$$

A significant number of further examples of this map are given in appendix E.

One detail which we should make a point to acknowledge is that the $\wedge$ operations should be removed in the reverse order to that which they are introduced. Without such a criterion, ambiguities can arise with objects of the form $A \wedge(B \otimes C) \wedge D$;

$$
\begin{align*}
(P \wedge(Q \otimes R)) \wedge S & =([P, Q] \otimes R) \wedge S+(Q \otimes[P, R]) \wedge S  \tag{6.1.8}\\
& =[[P, Q], S] \otimes R+[P, Q] \otimes[R, S]+[Q, S] \otimes[P, R]+Q \otimes[[P, R], S] \\
P \wedge((Q \otimes R) \wedge S) & =P \wedge([Q, S] \otimes R)+P \wedge(Q \otimes[R, S])  \tag{6.1.9}\\
& =[P,[Q, S]] \otimes R+[Q, S] \otimes[P, R]+[P, Q] \otimes[R, S]+Q \otimes[P,[R, S]] \\
& \neq(P \wedge(Q \otimes R)) \wedge S
\end{align*}
$$

Mistaking these two functions as being equal can lead to significant errors.

### 6.1.1 Related Maps

Further maps related to the contact term map (6.1.1) will be used in this thesis. The most important of these is denoted $\tilde{C}$, and referred to as the modified contact term map. It is defined by

$$
\begin{equation*}
\tilde{C} \circ i \equiv 0, \quad \tilde{C} \circ[P, Q] \equiv(C \circ P) \tilde{\wedge} Q+P \tilde{\wedge}(C \circ Q) . \tag{6.1.10}
\end{equation*}
$$

Note the standard contact term $C$ map (6.1.1) on the right-hand side. The modified wedge $\tilde{\Lambda}$ is a restricted form of $\wedge$,

$$
\begin{equation*}
(P \otimes Q) \tilde{\wedge} R \equiv[P, R] \otimes Q, \quad P \tilde{\wedge}(Q \otimes R) \equiv[P, Q] \otimes R \tag{6.1.11}
\end{equation*}
$$

Again, we give examples. The first case found for the $C$ map now vanishes,

$$
\begin{equation*}
\tilde{C} \circ[1,2]=(C \circ 1) \tilde{\wedge} 2+1 \tilde{\wedge}(C \circ 2)=0 \tag{6.1.12}
\end{equation*}
$$

As such, the trivial case we now consider is $\tilde{C} \circ[[1,2], 3]$. Taking advantage of (6.1.5), this is given by

$$
\begin{align*}
\tilde{C} \circ[[1,2], 3] & =(C \circ[1,2]) \tilde{\wedge} 3+[1,2] \tilde{\wedge}(C \circ 3) \\
& =\left(k^{1} \cdot k^{2}\right)(1 \otimes 2-2 \otimes 1) \tilde{\wedge} 3 \\
& =\left(k^{1} \cdot k^{2}\right)([1,3] \otimes 2-[2,3] \otimes 1) . \tag{6.1.13}
\end{align*}
$$

The more complex example of the previous discussion, that of applying the map to
$[1,[[2,3], 4]]$, is given by

$$
\begin{align*}
\tilde{C} \circ[1,[[2,3], 4]= & (C \circ 1) \tilde{\wedge}[[2,3], 4]+1 \tilde{\wedge}(C \circ[[2,3], 4]) \\
= & 1 \tilde{\wedge}((C \circ[2,3]) \wedge 4+[2,3] \wedge(C \circ 4) \\
& \left.\quad+\left(k^{23} \cdot k^{4}\right)([2,3] \otimes 4-4 \otimes[2,3])\right) \\
= & 1 \tilde{\wedge}(((C \circ 2) \wedge 3) \wedge 4+(2 \wedge C \circ 3) \wedge 4 \\
& \quad+\left(k^{2} \cdot k^{3}\right)(2 \otimes 3-3 \otimes 2) \wedge 4  \tag{6.1.14}\\
& \left.\quad+\left(k^{23} \cdot k^{4}\right)([2,3] \otimes 4-4 \otimes[2,3])\right) \\
= & 1 \tilde{\wedge}\left(\left(k^{2} \cdot k^{3}\right)([2,4] \otimes 3+2 \otimes[3,4]-(2 \leftrightarrow 3))\right. \\
& \left.\quad+\left(k^{23} \cdot k^{4}\right)([2,3] \otimes 4-4 \otimes[2,3])\right) \\
= & \left(k^{2} \cdot k^{3}\right)([1,[2,4]] \otimes 3+[1,2] \otimes[3,4]-(2 \leftrightarrow 3)) \\
& \quad+\left(k^{23} \cdot k^{4}\right)([1,[2,3]] \otimes 4-[1,4] \otimes[2,3])
\end{align*}
$$

This we note bears more resemblance to the application of the $C$ map to the bracket $[[2,3], 4]$ than $[1,[[2,3], 4]]$. Such similarities will be generalised later. For further examples of the $\tilde{C}$ map, appendix E should be consulted.

One final related algorithm $\tilde{C}^{\prime}$ was presented in the paper [27]. It is not strictly necessary, as it only appears in one instance and represents only a small modification of the $\tilde{C}$ map, and so results concerning it can be alternatively described using the $\tilde{C}$ map. However, we will be following the results of [27], and so we present it here. This is defined by

$$
\begin{equation*}
\tilde{C}^{\prime} \circ i=0, \quad \tilde{C}^{\prime} \circ[P, Q]=\tilde{C} \circ[P, Q]-\frac{1}{2}\left(k^{P} \cdot k^{Q}\right)(P \otimes Q-Q \otimes P) \tag{6.1.15}
\end{equation*}
$$

Alike with the $\tilde{C}$ map, we should note that this is defined in terms of another map on its right hand side, in this case $\tilde{C}$. Examples of this follow naturally from those previously discussed, with the one such being

$$
\begin{align*}
\tilde{C}^{\prime} \circ[[1,2], 3] & =\tilde{C} \circ[[1,2], 3]-\frac{1}{2}\left(k^{12} \cdot k^{3}\right)([1,2] \otimes 3-3 \otimes[1,2])  \tag{6.1.16}\\
& =\left(k^{1} \cdot k^{2}\right)([1,3] \otimes 2-[2,3] \otimes 1)-\frac{1}{2}\left(k^{12} \cdot k^{3}\right)([1,2] \otimes 3-3 \otimes[1,2])
\end{align*}
$$

Due to the close relationship between this map and $\tilde{C}$, we will not detail any further examples.

### 6.2 Various Relations Satisfied by the Map

The contact term map has certain key properties which we may prove rigorously. These will then be important in describing multiparticle superfields in the BCJ gauge. In this
section, we will state and prove these identities.

### 6.2.1 Proposition Regarding Application to the $b$-map

A key result regarding the contact term map is the following.

Proposition: The $C$ map satisfies

$$
\begin{equation*}
C \circ b(P)=\sum_{X Y=P}(b(X) \otimes b(Y)-(X \leftrightarrow Y)) . \tag{6.2.1}
\end{equation*}
$$

Proof: We may prove this using induction. First the base case; when the word $P$ has length two the statement is

$$
\begin{align*}
C \circ b(12) & =\sum_{X Y=12}(b(X) \otimes b(Y)-(X \leftrightarrow Y))  \tag{6.2.2}\\
& =b(1) \otimes b(2)-b(2) \otimes b(1)
\end{align*}
$$

Verifying that this is satisfied is not terribly complex,

$$
\begin{align*}
C \circ b(12) & =\frac{1}{s_{12}} C \circ[1,2] \\
& =\frac{1}{s_{12}}\left((C \circ 1) \wedge 2+1 \wedge(C \circ 2)+s_{12}(1 \otimes 2-2 \otimes 1)\right)  \tag{6.2.3}\\
& =\frac{1}{s_{12}} s_{12}(1 \otimes 2-2 \otimes 1) \\
& =b(1) \otimes b(2)-b(2) \otimes b(1) .
\end{align*}
$$

Now we assume that the relation (6.2.1) is satisfied for any word of length less than $n$, and let $Q$ be a length $n$ word. Applying the iterative definition of the $b$-map once and then taking the contact term map within the resulting sum, this becomes

$$
\begin{align*}
s_{Q} C \circ b(Q)= & C \circ \sum_{X Y=Q}[b(X), b(Y)] \\
= & \sum_{X Y=Q}[(C \circ b(X)) \wedge b(Y)+b(X) \wedge(C \circ b(Y))  \tag{6.2.4}\\
& \left.+\left(k^{X} \cdot k^{Y}\right)(b(X) \otimes b(Y)-b(Y) \otimes b(X))\right]
\end{align*}
$$

We separate this into the three possible cases; both of $|X|$ and $|Y|$ being greater than 1, $|X|=1$, and $|Y|=1$. We then use that $C \circ b(i)=0$ for $i$ a letter, and that the induction hypothesis (6.2.1) holds for all $C \circ b(P)$ such that $|P|<|Q|$, to remove every explicit
application of the map $C$ from this equation. We are left with

$$
\begin{align*}
& s_{Q} C \circ b(Q)= \sum_{X Y=Q}\left(k^{X} \cdot k^{Y}\right)(b(X) \otimes b(Y)-b(Y) \otimes b(X))  \tag{6.2.5}\\
&+\sum_{\substack{X Y=Q \\
|X|>1,|Y|>1}} \sum_{A B=X}(b(A) \otimes b(B)-b(B) \otimes b(A)) \wedge b(Y) \\
&+\sum_{\substack{X Y=Q \\
|Y|=1}} \sum_{C D=X}(b(C) \otimes b(D)-b(D) \otimes b(C)) \wedge b(Y) \\
&+\sum_{\substack{X Y=Q \\
|X|>1,|Y|>1}} b(X) \wedge \sum_{C D=Y}(b(C) \otimes b(D)-b(D) \otimes b(C)) \\
&+\sum_{\substack{X Y=Q \\
|X|=1}} b(X) \wedge \sum_{A B=Y}(b(A) \otimes b(B)-b(B) \otimes b(A))
\end{align*}
$$

Absorbing the $|X|=1$ and $|Y|=1$ summations into the $|X|>1,|Y|>1$ cases, this reduces to

$$
\begin{align*}
s_{Q} C \circ b(Q) & =\sum_{X Y=Q}\left(k^{X} \cdot k^{Y}\right)(b(X) \otimes b(Y)-b(Y) \otimes b(X)) \\
& +\sum_{\substack{X Y=Q A B=X \\
|X|>1}} \sum_{\substack{X Y=Q \\
|Y|>1}}(b(A) \otimes b(B)-b(B) \otimes b(A)) \wedge b(Y)  \tag{6.2.6}\\
& +\sum_{\substack{X=Y}} b(X) \wedge(b(C) \otimes b(D)-b(D) \otimes b(C))
\end{align*}
$$

The two double sums may be reduces to a single sum using relations of the form

$$
\begin{equation*}
\sum_{X Y=Q,|X|>1} \sum_{A B=X}=\sum_{A B Y=Q} \tag{6.2.7}
\end{equation*}
$$

Applying this to the double sums of (6.2.6), these become

$$
\begin{aligned}
& \sum_{A B Y=Q}(b(A) \otimes b(B)-b(B) \otimes b(A)) \wedge b(Y) \\
& \quad \quad+\sum_{X C D=Q} b(X) \wedge(b(C) \otimes b(D)-b(D) \otimes b(C)) \\
& =\sum_{A B Y=Q}([b(A), b(Y)] \otimes b(B)+b(A) \otimes[b(B), b(Y)]-[b(B), b(Y)] \otimes b(A)-b(B) \otimes[b(A), b(Y)]) \\
& +\quad \sum_{X C D=Q}([b(X), b(C)] \otimes b(D)+b(C) \otimes[b(X), b(D)]-[b(X), b(D)] \otimes b(C)-b(D) \otimes[b(X), b(C)])
\end{aligned}
$$

Where we have used the definition (6.1.2) to remove the wedges $\wedge$. We now group this
into two sets of four terms in a convenient way

$$
\begin{align*}
& =\left(\sum_{A B Y=Q}([b(A), b(Y)] \otimes b(B)-b(B) \otimes[b(A), b(Y)])\right.  \tag{6.2.9}\\
& \left.\quad+\sum_{X C D=Q}(b(C) \otimes[b(X), b(D)]-[b(X), b(D)] \otimes b(C))\right) \\
& +\left(\sum_{A B Y=Q}(b(A) \otimes[b(B), b(Y)]-[b(B), b(Y)] \otimes b(A))\right. \\
& \left.\quad+\sum_{X C D=Q}([b(X), b(C)] \otimes b(D)-b(D) \otimes[b(X), b(C)])\right)
\end{align*}
$$

The first set of terms is identically zero, which follows from relabelling the second sum

$$
\begin{align*}
\sum_{A B Y=Q}([b(A), b(Y)] \otimes b(B)- & b(B) \otimes[b(A), b(Y)]  \tag{6.2.10}\\
& +b(B) \otimes[b(A), b(Y)]-[b(A), b(Y)] \otimes b(B)) .
\end{align*}
$$

The second set of terms in (6.2.9) can be simplified by using the definition of the $b$-map (2.3.13),

$$
\begin{align*}
\sum_{A B Y=Q}\left(b(A) \otimes b(B Y) s_{B Y}\right. & \left.-s_{B Y} b(B Y) \otimes b(A)\right)  \tag{6.2.11}\\
& +\sum_{X C D=Q}\left(s_{X C} b(X C) \otimes b(D)-b(D) \otimes b(X C) s_{X C}\right)
\end{align*}
$$

Then, as the words $B$ and $Y$ only appear consecutively in the first sum, we can condense them into a single word, and likewise for $X$ and $C$ in the second sum. This reduces the above to

$$
\begin{align*}
\sum_{X Y=Q} s_{Y}( & (b(X) \otimes b(Y)-b(Y) \otimes b(X))  \tag{6.2.12}\\
& +\sum_{X Y=Q} s_{X}(b(X) \otimes b(Y)-b(Y) \otimes b(X)) .
\end{align*}
$$

We now return to (6.2.4). Using the methods discussed, the double sum terms reduce to (6.2.12). Hence this becomes

$$
\begin{align*}
C \circ b(Q) & =\frac{1}{s_{Q}} \sum_{X Y=Q}\left[\left(s_{X}+s_{Y}+\left(k^{X} \cdot k^{Y}\right)\right)(b(X) \otimes b(Y)-b(Y) \otimes b(X))\right]  \tag{6.2.13}\\
& =\sum_{X Y=Q}(b(X) \otimes b(Y)-b(Y) \otimes b(X)),
\end{align*}
$$

with the second line following by the relation

$$
\begin{equation*}
s_{X}+s_{Y}+\left(k^{X} \cdot k^{Y}\right)=s_{X Y} . \tag{6.2.14}
\end{equation*}
$$

Hence the result is proved.

A similar result holds for the $\tilde{C}$ map,

$$
\begin{equation*}
\tilde{C} \circ P=\frac{1}{s_{P}} \sum_{X Y=P} s_{X} b(X) \otimes b(Y)-s_{Y} b(Y) \otimes b(X) . \tag{6.2.15}
\end{equation*}
$$

The proof of this follows similar lines to the above. This result is not applied in this thesis however, and so we note it only for completeness.

We conclude this discussion by noting that an alternative proof of this result has since been identified in [147], as well as several further results regarding the contact term map. The methods therein are not relevant to our discussion here however, and so we will not detail them further.

### 6.2.2 General Form when Applied to Dynkin Brackets

The general form of the contact term map $C$ and its modification $\tilde{C}$ when applied to Dynkin brackets are known, and here we state and prove them as lemmata. These will prove to be useful for showing consistency between formulae produced here and those given in the literature review.

Lemma: The application of the contact term map to any left-to-right Dynkin bracket $\left.P=\left[\left[\ldots\left[p_{1}, p_{2}\right], p_{3}\right], \ldots\right], p_{|P|}\right]$ is given by

$$
\begin{equation*}
C \circ P=\sum_{\substack{X j Y=P \\ Y=R \sqcup S}}\left(k^{X} \cdot k^{j}\right)[X R \otimes j S-(X \leftrightarrow j)] . \tag{6.2.16}
\end{equation*}
$$

where the deshuffle product is as defined in appendix A.1.3.

Proof: This again follows from induction. First we show the base case, which is a simple application of (6.1.1),

$$
\begin{equation*}
C \circ[1,2]=\left(k^{1} \cdot k^{2}\right)(1 \otimes 2-2 \otimes 1) . \tag{6.2.17}
\end{equation*}
$$

We then suppose that the relation (6.2.16) is satisfied for the Dynkin bracket $\ell(P)$, and consider $C \circ[P, q]$, for $q$ a letter,

$$
\begin{align*}
& C \circ[P, q]=(C \circ P) \wedge q+P \wedge(C \circ q)+\left(k^{P} \cdot k^{q}\right)(P \otimes q-q \otimes P)  \tag{6.2.18}\\
&= \sum_{\substack{X j Y=P \\
Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)(X R \otimes j S-(X \leftrightarrow j)) \wedge q+\left(k^{P} \cdot k^{q}\right)(P \otimes q-q \otimes P) \\
&= \sum_{\substack{X j Y=P \\
Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)(X R q \otimes j S+X R \otimes j S q-(X \leftrightarrow j))+\left(k^{P} \cdot k^{q}\right)(P \otimes q-q \otimes P) \\
&= \sum_{\substack{X j Y=P}}\left(k^{X} \cdot k^{j}\right)(X R \otimes j S-(X \leftrightarrow j))+\left(k^{P} \cdot k^{q}\right)(P \otimes q-q \otimes P) \\
&=\sum^{Y}=R \amalg S\left(k^{X} \cdot k^{j}\right)(X R \otimes j S-(X \leftrightarrow j)) . \\
& Y=R=P q \\
& Y=R \amalg S
\end{align*}
$$

Hence by induction the result (6.2.16) is proved.

Moving onto the modified contact term map $\tilde{C}$, the general form of this applied to a pair of Dynkin brackets is given by the following lemma.

Lemma: For $P$ and $Q$ left-to-right Dynkin brackets, the modified contact term map $\tilde{C}$ satisfies

$$
\begin{equation*}
\tilde{C} \circ[P, Q]=\sum_{\substack{X j Y=P \\ \delta(Y)=R \otimes S}}\left(k^{X} \cdot k^{j}\right)([X R, Q] \otimes j S-(X \leftrightarrow j))-(P \leftrightarrow Q), \tag{6.2.19}
\end{equation*}
$$

Proof: This follows as a corollary of the identity (6.2.16),

$$
\begin{align*}
& \tilde{C} \circ[P, Q]=(C \circ P) \tilde{\wedge} Q+P \tilde{\wedge}(C \circ Q)  \tag{6.2.20}\\
&= \sum_{\substack{X j Y=P \\
\delta(Y)=R \otimes S}}\left(k^{X} \cdot k^{j}\right)(X R \otimes j S-(X \leftrightarrow j)) \tilde{\wedge} Q \\
&+P \tilde{\wedge} \sum_{\substack{X j Y=Q \\
\delta(Y)=R \otimes S}}\left(k^{X} \cdot k^{j}\right)(X R \otimes j S-(X \leftrightarrow j)) \\
&=\sum_{\substack{X j Y=P \\
\delta(Y)=R \otimes S}}\left(k^{X} \cdot k^{j}\right)([X R, Q] \otimes j S-(X \leftrightarrow j)) \\
&+\sum_{\substack{X j Y=Q \\
\delta(Y)=R \otimes S}}\left(k^{X} \cdot k^{j}\right)([P, X R] \otimes j S-(X \leftrightarrow j))
\end{align*}
$$

where the second equality follows from the definition of the modified wedge (6.1.11). The result follows after using the antisymmetry $[P, X R]=-[X R, P]$ in the final line.

Though we will not use such, for the sake of completeness we note that the results of this subsection may be generalised to right-to-left Dynkin brackets using the relation

$$
\begin{equation*}
\ell(P)=(-1)^{|P|+1} r(\bar{P}) \tag{6.2.21}
\end{equation*}
$$

with $\bar{P}$ the reversal of the word $P$.

### 6.3 Constructing Superfields With Arbitrary Indices

One of the most immediate consequences of the identification of the contact term map, is that it allows us to generalise previous formulae for Dynkin brackets to arbitrary Lie monomials. In essence, this amounts to making a series of substitutions. Where before we had a left-to-right Dynkin bracket $P$ and a sum

$$
\begin{equation*}
\sum_{\substack{X j Y=P \\ R \sqcup S=Y}} \tag{6.3.1}
\end{equation*}
$$

we may generalise this to an arbitrary Lie monomial $Q$ using the contact map or its modification, with consistency following from the results of subsection 6.2.2. We now detail this on a case by case basis.

We begin by introducing some notation. Given some $\left(k^{X} \cdot k^{Y}\right) A \otimes B$ arising as a result of the contact term map, we assign the Lie monomials $A$ and $B$ as the indices of superfields $K$ and $S$ by acting on them with the double bracket $\llbracket K, S \rrbracket$. That is,

$$
\begin{equation*}
\llbracket K, S \rrbracket \circ\left(k^{X} \cdot k^{Y}\right) A \otimes B=\left(k^{X} \cdot k^{Y}\right) K_{A} S_{B} \tag{6.3.2}
\end{equation*}
$$

We then generalise this to all terms arising from applying the contact term map to a Lie monomial $[P, Q]$ using

$$
\begin{equation*}
C \llbracket K, S \rrbracket \circ[P, Q]=\llbracket K, S \rrbracket \circ(C \circ[P, Q]) \tag{6.3.3}
\end{equation*}
$$

This notation generalises to the $\tilde{C}$ map in the natural way.

To illustrate, let us give some examples. The contact term map applied to $[1,2]$ is

$$
\begin{equation*}
C \circ[1,2]=\left(k^{1} \cdot k^{2}\right)(1 \otimes 2-2 \otimes 1) \tag{6.3.4}
\end{equation*}
$$

We then denote the sum over superfields $V$ and $A^{m}$ with this sum as its indices by

$$
\begin{align*}
C \llbracket V, A^{m} \rrbracket \circ[1,2] & =\llbracket V, A^{m} \rrbracket \circ(C \circ[1,2])  \tag{6.3.5}\\
& =\left(k^{1} \cdot k^{2}\right)\left(V_{1} A_{2}^{m}-V_{2} A_{1}^{m}\right) .
\end{align*}
$$

If we encounter an object with several slots of indices, we use a $\cdot$ to denote the slot to show where these maps should insert indices. For example,

$$
\begin{equation*}
\llbracket V, T_{\cdot, 3,4} \rrbracket \circ[1,2]=\left(k^{1} \cdot k^{2}\right)\left(V_{1} T_{2,3,4}-V_{2} T_{1,3,4}\right) . \tag{6.3.6}
\end{equation*}
$$

### 6.3.1 Lorenz Gauge Superfields

In relations (4.1.30) - (4.1.34), the multiparticle superfields in the Lorenz gauge $\hat{A}_{\alpha}$ and $\hat{A}_{m}$ were defined for arbitrary Lie monomial indices. However, the multiparticle $\hat{W}^{\alpha}$ and $\hat{F}_{m n}$ were only defined in specific circumstances; the former for indices $[P, Q]$ and the latter for indices $P$, where $P$ and $Q$ denote left-to-right Dynkin brackets. We may generalise this, by virtue of the contact term algorithm. The full set of Lorenz gauge multiparticle superfields are defined with the recursion

$$
\begin{align*}
\hat{A}_{\alpha}^{[P, Q]} & =-\frac{1}{2}\left[\hat{A}_{\alpha}^{P}\left(k^{P} \cdot \hat{A}^{Q}\right)+\hat{A}_{m}^{P}\left(\gamma^{m} \hat{W}^{Q}\right)_{\alpha}-(P \leftrightarrow Q)\right]  \tag{6.3.7}\\
\hat{A}_{m}^{[P, Q]} & =-\frac{1}{2}\left[\hat{A}_{m}^{P}\left(k^{P} \cdot \hat{A}^{Q}\right)+\hat{A}_{n}^{P} \hat{F}_{m n}^{Q}-\left(\hat{W}^{P} \gamma_{m} \hat{W}^{Q}\right)-(P \leftrightarrow Q)\right] \\
\hat{W}_{[P, Q]}^{\alpha} & =\frac{1}{4} \hat{F}_{r s}^{P}\left(\gamma^{r s} \hat{W}^{Q}\right)^{\alpha}-\frac{1}{2}\left(k^{P} \cdot \hat{A}^{Q}\right) \hat{W}_{P}^{\alpha}-\frac{1}{2} \hat{W}_{P}^{m \alpha} A_{Q}^{m}-(P \leftrightarrow Q), \\
\hat{F}_{m n}^{[P, Q]} & =-\frac{1}{2}\left[\hat{F}_{P}^{m n}\left(k_{P} \cdot \hat{A}_{Q}\right)+\hat{F}_{P}^{p \mid m n} \hat{A}_{p}^{Q}+2 \hat{F}_{P}^{m p} \hat{F}_{Q p}^{n}+4 \gamma_{\alpha \beta}^{[m} \hat{W}_{P}^{n] \alpha} \hat{W}_{Q}^{\beta}-(P \leftrightarrow Q)\right] \\
& =k_{P Q}^{m} \hat{A}_{[P, Q]}^{n}-k_{P Q}^{m} \hat{A}_{[P, Q]}^{m}-C \llbracket \hat{A}^{m}, \hat{A}^{n} \rrbracket \circ[P, Q] .
\end{align*}
$$

where the higher weight superfields are defined using the contact term map,

$$
\begin{align*}
\hat{W}_{[P, Q]}^{m \alpha} & =k_{P Q}^{m} \hat{W}_{[P, Q]}^{\alpha}-C \llbracket \hat{A}^{m}, \hat{W}^{\alpha} \rrbracket \circ[P, Q]  \tag{6.3.8}\\
\hat{F}_{[P, Q]}^{m \mid p q} & =k_{P Q}^{m} \hat{F}_{[P, Q]}^{p q}-C \llbracket \hat{A}^{m}, \hat{F}^{p q} \rrbracket \circ[P, Q] .
\end{align*}
$$

The two definitions of the $F_{m n}$ have differing appearances, and in different circumstances one will prove more useful than the other, but they are believed to be identical. Note the formulae for superfields in the hybrid gauge may be generalised using the contact term map also, with the result being analogous with the above.

To give examples of these formulae, the Lorenz gauge superfield $\hat{A}_{[[1,2],[[3,4], 5]}^{m}$ is given by

$$
\begin{align*}
\hat{A}_{[[1,2],[[3,4], 5]]}^{m}=-\frac{1}{2}\left[\hat{A}_{m}^{[1,2]}\left(k^{12} \cdot \hat{A}^{[3,4], 5]}\right)\right. & +\hat{A}_{n}^{[1,2]} \hat{F}_{m n}^{[33,4], 5]}  \tag{6.3.9}\\
& \left.-\left(\hat{W}^{[1,2]} \gamma_{m} \hat{W}^{[3,4], 5]}\right)-([1,2] \leftrightarrow[[3,4], 5])\right] .
\end{align*}
$$

The superfield $\hat{F}_{[11,2],[3,4]]}^{m n}$ may now be expressed as

$$
\begin{align*}
\hat{F}_{[[1,2],[3,4]]}^{m n} & =k_{1234}^{m} \hat{A}_{[[1,2],[3,4]]}^{n}-k_{1234}^{n} \hat{A}_{[[1,2],[3,4]]}^{m} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{A}_{[1,[3,4]}^{m} \hat{A}_{2}^{n}+\hat{A}_{1}^{m} \hat{A}_{[2,[3,4]]}^{n}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{3} \cdot k^{4}\right)\left(\hat{A}_{[[1,2], 3]}^{m} \hat{A}_{4}^{n}+\hat{A}_{3}^{m} \hat{A}_{[1,2], 4]}-(3 \leftrightarrow 4)\right)  \tag{6.3.10}\\
& -\left(k^{12} \cdot k^{34}\right)\left(\hat{A}_{[1,2]}^{m} \hat{A}_{[3,4]}^{n}-\hat{A}_{[3,4]}^{m} \hat{A}_{[1,2]}^{n}\right) .
\end{align*}
$$

This uses the contact term map applied to $[[1,2],[3,4]]$, which is given in appendix E.

### 6.3.2 Equations of Motion of Local Superfields

Previously, equations of motion were defined only for left-to-right Dynkin brackets, and we may now use this technology to generalise these to arbitrary Lie monomials. This is done in terms of an object $\nabla_{\alpha}^{(L)}$, the local counterpart of $\nabla_{\alpha} \equiv D_{\alpha}-\mathbb{A}_{\alpha}$, defined in terms of the contact term map

$$
\begin{equation*}
\nabla_{\alpha}^{(L)} \equiv D_{\alpha}-C \llbracket \hat{A}_{\alpha}, \cdot \rrbracket, \quad C \llbracket \hat{A}_{\alpha}, \cdot \rrbracket K_{[P, Q]} \equiv C \llbracket \hat{A}_{\alpha}, K \rrbracket \circ[P, Q] . \tag{6.3.11}
\end{equation*}
$$

With this, the equations of motion take the form

$$
\begin{align*}
\nabla_{(\alpha}^{(L)} \hat{A}_{\beta)}^{[P, Q]} & =\gamma_{\alpha \beta}^{m} \hat{A}_{[P, Q]}^{m}, \\
\nabla_{\alpha}^{(L)} \hat{A}_{[P, Q]}^{m} & =\left(\gamma^{m} \hat{W}_{[P, Q]}\right)_{\alpha}+k_{P Q}^{m} \hat{A}_{\alpha[P, Q]}, \\
\nabla_{\alpha}^{(L)} \hat{W}_{[P, Q]}^{\beta} & =\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}{ }^{\beta} \hat{F}_{m n}^{[P, Q]},  \tag{6.3.12}\\
\nabla_{\alpha}^{(L)} \hat{F}_{[P, Q]}^{m n} & =\left(\hat{W}_{[P, Q]}^{[m} \gamma^{n]}\right)_{\alpha} .
\end{align*}
$$

In essence, these relations say that a multiparticle superfield $K_{B}$ has the same equation of motion as its single particle equivalent, plus some correction terms $\hat{A}_{\alpha}^{P} K_{Q}$ with their indices generated by the contact term map.

To illustrate, consider the equation of motion of $\hat{A}_{[1,2]}^{m}$. This is given by

$$
\begin{equation*}
\nabla_{\alpha}^{(L)} \hat{A}_{[1,2]}^{m}=\left(\gamma^{m} \hat{W}_{[1,2]}\right)_{\alpha}+k_{12}^{m} \hat{A}_{\alpha[1,2]} . \tag{6.3.13}
\end{equation*}
$$

We may then expand the $\nabla^{(L)}$ function,

$$
\begin{align*}
\nabla_{\alpha}^{(L)} \hat{A}_{[1,2]}^{m} & =D_{\alpha} \hat{A}_{[1,2]}^{m}-C \llbracket \hat{A}_{\alpha}, \hat{A}^{m} \rrbracket \circ[1,2]  \tag{6.3.14}\\
& =D_{\alpha} \hat{A}_{[1,2]}^{m}-\left(k_{1} \cdot k_{2}\right)\left(\hat{A}_{\alpha}^{1} \hat{A}_{2}^{m}-\hat{A}_{\alpha}^{2} \hat{A}_{1}^{m}\right), \tag{6.3.15}
\end{align*}
$$

and so we arrive at the more familiar form of the equation of motion

$$
\begin{equation*}
D_{\alpha} \hat{A}_{[1,2]}^{m}=\left(\gamma^{m} \hat{W}_{[1,2]}\right)_{\alpha}+k_{12}^{m} \hat{A}_{\alpha[1,2]}+\left(k_{1} \cdot k_{2}\right)\left(\hat{A}_{\alpha}^{1} \hat{A}_{2}^{m}-\hat{A}_{\alpha}^{2} \hat{A}_{1}^{m}\right) . \tag{6.3.16}
\end{equation*}
$$

Similar also holds for the $T_{A_{1}, \ldots, A_{m+3}}^{n_{1} \ldots n_{m}}$ superfields arising at one loop, and for their refinements also. These generalise from their Dynkin bracket expressions in the natural way. So, for instance, take the equation of motion (5.1.18)

$$
\begin{gather*}
Q T_{\ell(A), \ell(B), \ell(C), \ell(D)}^{m}=k_{A}^{m} V_{A} T_{B, C, D}+\sum_{\substack{X j Y=A \\
Y=R \sqcup S}}\left(k^{X} \cdot k^{j}\right)\left(V_{\ell(X R)} T_{\ell(j S), \ell(B), \ell(C), \ell(D)}-(X \leftrightarrow j)\right) \\
+(A \leftrightarrow B, C, D) \tag{6.3.17}
\end{gather*}
$$

For simplicity we will not use the $\nabla^{(L)}$ operator. This generalises to arbitrary Dynkin brackets by replacing the sum with an application of the contact term map,

$$
\begin{equation*}
Q T_{A, B, C, D}^{m}=k_{A}^{m} V_{A} T_{B, C, D}+C \llbracket V, T_{\cdot, B, C, D} \rrbracket \circ A+(A \leftrightarrow B, C, D) \tag{6.3.18}
\end{equation*}
$$

So, to give just one example,

$$
\begin{align*}
Q T_{[1,2],[3,[4,5]], 6,7}^{m} & =k_{12}^{m} V_{[1,2]} T_{[3,[4,5]], 6,7}+k_{345}^{m} V_{[3,[4,5]]} T_{[1,2], 6,7} \\
& +k_{6}^{m} V_{6} T_{[1,2],[3,[4,5]], 7}+k_{7}^{m} V_{7} T_{[1,2],[3,[4,5]], 6} \\
& +\left(k^{1} \cdot k^{2}\right)\left(V_{1} T_{2,[3,[4,5]], 6,7}-V_{2} T_{1,[3,[4,5]], 6,7}\right) \\
& +\left(k^{4} \cdot k^{5}\right)\left(V_{4} T_{[1,2],[3,5], 6,7}+V_{[3,4]} T_{[1,2], 5,6,7}-V_{5} T_{[1,2],[3,4]], 6,7}-V_{[3,5]} T_{[1,2], 4,6,7}\right) \\
& +\left(k^{3} \cdot k^{45}\right)\left(V_{3} T_{[1,2],[4,5]], 6,7}-V_{[4,5]} T_{[1,2], 3,6,7}\right) \tag{6.3.19}
\end{align*}
$$

Likewise will hold for higher weight $T$ superfields, with the variation of arbitrary tensorial $T$ (5.1.20) becoming

$$
\begin{align*}
Q T_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}} & =\delta^{\left(m_{1} m_{2}\right.} Y_{B_{1}, B_{2}, \ldots, B_{r+3}}^{\left.m_{3} \ldots m_{r}\right)}  \tag{6.3.20}\\
& +\left(k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} T_{B_{2}, \ldots, B_{r+3}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)\right) \\
& +\left(C \llbracket V, T_{\cdot, B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}} \rrbracket \circ B_{1}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)\right) .
\end{align*}
$$

For completeness, we also give the general form of the variation of the refined $J$ superfields

$$
Q J_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}=\delta^{\left(m_{1} m_{2}\right.} Y_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{\left.m_{3} \ldots m_{r}\right)}
$$

$$
\begin{align*}
& +k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} J_{A_{1}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)  \tag{6.3.21}\\
& +V_{\left[A_{1}, B_{1}\right]} J_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r}}^{m_{1}}+\left(A_{1} \leftrightarrow A_{2}, A_{3}, \ldots, A_{d} ; \quad B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right) \\
& +Y_{A_{2}, \ldots A_{r}\left|A_{d}\right| A_{1}, B_{1}, \ldots, B_{d+r+3}}^{m_{1}}+k_{A_{1}}^{p} V_{A_{1}} J_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{p m_{1}, m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right) \\
& +C \llbracket V, J_{\cdot, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1}} \rrbracket \circ A_{1}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right) \\
& +C \llbracket V, J_{A_{1}, \ldots, A_{d} \mid, B_{2}, \ldots, B_{d+r+3}}^{m_{1}} \rrbracket \circ B_{1}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right),
\end{align*}
$$

and of the anomalous $Y$ superfields,

$$
\begin{align*}
Q Y_{B_{1}, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}= & k_{B_{1}}^{\left(m_{1}\right.} V_{B_{1}} Y_{B_{2}, \ldots, B_{r+5}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right)  \tag{6.3.22}\\
& +C \llbracket V, Y_{, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots r_{r}} \rrbracket \circ B_{1}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right) .
\end{align*}
$$

### 6.3.3 Deconcatenation of Berends-Giele Currents

We may combine these results with those of section 6.2.1, in order to find that the variation of Berends-Giele currents have a very specific and useful form; namely the terms generated by the contact term map combine into a deconcatenation. To illustrate, let us consider the simple case of the Berends-Giele current $\mathcal{A}_{12}^{m}$. This is described by a single local superfield, $A_{[1,2]}^{m}$, which has equation of motion

$$
\begin{equation*}
D_{\alpha} \hat{A}_{[1,2]}^{m}=\left(\gamma^{m} \hat{W}_{[1,2]}\right)_{\alpha}+k_{12}^{m} \hat{A}_{\alpha[1,2]}+\left(k_{1} \cdot k_{2}\right)\left(\hat{A}_{\alpha}^{1} \hat{A}_{2}^{m}-\hat{A}_{\alpha}^{2} \hat{A}_{1}^{m}\right) . \tag{6.3.23}
\end{equation*}
$$

Dividing by $s_{12}$, we may convert all superfields to BG currents,

$$
\begin{equation*}
D_{\alpha} \hat{\mathcal{A}}_{12}^{m}=\left(\gamma^{m} \hat{\mathcal{W}}_{12}\right)_{\alpha}+k_{12}^{m} \hat{\mathcal{A}}_{\alpha}^{12}+\sum_{X Y=12}\left(\hat{\mathcal{A}}_{\alpha}^{X} \hat{\mathcal{A}}_{Y}^{m}-(X \leftrightarrow Y)\right) . \tag{6.3.24}
\end{equation*}
$$

That is, the equation of motion of the rank two BG current is the rank one equation of motion, plus a sum over a deconcatenation. This result, and that it appears true for higher ranks also, had been observed prior. Now though, we may prove it rigorously using the contact term map.

We first note that, for $P$ some Lie monomial of rank $p$, the superfield $\hat{A}_{P}^{m}$ has variation

$$
\begin{equation*}
D_{\alpha} \hat{A}_{P}^{m}=\left(\gamma^{m} \hat{W}_{P}\right)_{\alpha}+k_{P}^{m} \hat{A}_{\alpha}^{P}+C \llbracket \hat{A}_{\alpha}, \hat{A}^{m} \rrbracket \circ P . \tag{6.3.25}
\end{equation*}
$$

We now consider the variation of the BG current $\mathcal{A}_{12 \ldots p}^{m}$. Generating it using the $b$-map, this has variation

$$
\begin{equation*}
D_{\alpha} \hat{\mathcal{A}}_{P}^{m}=\left(\gamma^{m} \hat{\mathcal{W}}_{P}\right)_{\alpha}+k_{P}^{m} \hat{\mathcal{A}}_{\alpha}^{P}+C \llbracket \hat{A}_{\alpha}, \hat{A}^{m} \rrbracket \circ b(P) . \tag{6.3.26}
\end{equation*}
$$

We then apply the relation (6.2.1) to simplify the application of the contact term map on
the $b$-map,

$$
\begin{align*}
D_{\alpha} \hat{\mathcal{A}}_{P}^{m} & =\left(\gamma^{m} \hat{\mathcal{W}}_{P}\right)_{\alpha}+k_{P}^{m} \hat{\mathcal{A}}_{\alpha}^{P}+\sum_{X Y=12 \ldots p} \llbracket \hat{A}_{\alpha}, \hat{A}^{m} \rrbracket \circ(b(X) \otimes b(Y)-b(Y) \otimes b(X)) \\
& =\left(\gamma^{m} \hat{\mathcal{W}}_{P}\right)_{\alpha}+k_{P}^{m} \hat{\mathcal{A}}_{\alpha}^{P}+\sum_{X Y=12 \ldots p}\left(\hat{\mathcal{A}}_{\alpha}^{X} \hat{\mathcal{A}}_{Y}^{m}-\hat{\mathcal{A}}_{\alpha}^{Y} \hat{\mathcal{A}}_{X}^{m}\right) \tag{6.3.27}
\end{align*}
$$

That is, the variation of the BG current is the same as the single particle superfield, plus a deconcatenation. Analogous procedures show similar results for the other superfields.

One particular case of note is that of the vertex operator $V_{P}$, as its variation has previously been found to have an exclusively contact term map-like structure. Namely,

$$
\begin{equation*}
Q V_{P}=\sum_{\substack{P=X j Y \\ Y=R \amalg S}}\left(k_{X} \cdot k_{j}\right) V_{X R} V_{j S} . \tag{6.3.28}
\end{equation*}
$$

This generalises to arbitrary Lie monomials in the usual way,

$$
\begin{equation*}
Q V_{[P, Q]}=\frac{1}{2} C \llbracket V, V \rrbracket \circ[P, Q] \tag{6.3.29}
\end{equation*}
$$

The variation of the BG current is therefore a sum over deconcatenations,

$$
\begin{equation*}
Q M_{P}=\sum_{X Y=P} M_{X} M_{Y} \tag{6.3.30}
\end{equation*}
$$

This approach proves a relation which was crucial in a number of places, but particularly in demonstrating the validity of the tree level amplitude formulae.

## CHAPTER 7

## Redefinitions Using the Hybrid Gauge

In order to move superfields to the BCJ gauge, we have two approaches. One based upon a construction with an intermediate hybrid gauge, and the other based upon taking superfields in the Lorenz gauge and moving directly to the BCJ gauge. The former of these is much easier to develop formulae for and much better understood, and we detail this here. We begin by generalising the results up to rank five. We then proceed to higher ranks, first by detailing problems at rank six and solving them, and then moving onto higher and then general ranks. We note here that we will limit ourselves to the case of superfields with topology $[P, Q]$, for $P$ and $Q$ left-to-right Dynkin brackets. Other BCJ gauge superfields may be then found by using BCJ relations to express them in terms of these.

### 7.1 Further Topologies Up To Rank Five

In previous discussions, superfields in the BCJ gauge using the hybrid gauge approach were only constructed for left-to-right Dynkin bracket topologies. Using direct gauge transformations however, this mapping was found exactly, with one example being

$$
\begin{align*}
A_{[123,45]}^{m}= & \hat{A}_{[123,45]}^{m}-k_{12345}^{m} \hat{H}_{[123,45]}  \tag{7.1.1}\\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[13,45]} A_{2}^{m}+\hat{H}_{[45,2]} A_{13}^{m}-(1 \leftrightarrow 2)\right)
\end{align*}
$$

$$
\begin{aligned}
& -\left(k^{12} \cdot k^{3}\right)\left(\hat{H}_{[12,45]} A_{3}^{m}+\hat{H}_{[45,3]} A_{12}^{m}\right) \\
& -\left(k^{123} \cdot k^{45}\right) \hat{H}_{[12,3]} A_{45}^{m} \\
& -\left(k^{4} \cdot k^{5}\right)\left(\hat{H}_{[123,4]} A_{5}^{m}-\hat{H}_{[123,5]} A_{4}^{m}\right)
\end{aligned}
$$

Appendix C. 1 should be consulted for further examples. We would expect to be able to take these results, and rearrange the right hand side in terms of hybrid gauge $\check{A}^{m}$ superfields and unhatted $H$ redefinition terms, as these are the only objects which appear in the usual hybrid gauge redefinition formula (4.2.31). To do so, we begin by first removing the hats from $\hat{H}$ terms

$$
\begin{align*}
A_{[123,45]}^{m}= & \hat{A}_{[123,45]}^{m}-k_{12345}^{m}\left(H_{[123,45]}-\frac{1}{2} H_{[12,3]}\left(k^{123} \cdot A_{45}\right)\right)  \tag{7.1.2}\\
& -\left(k^{1} \cdot k^{2}\right)\left(H_{[13,45]} A_{2}^{m}+H_{[45,2]} A_{13}^{m}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(H_{[12,45]} A_{3}^{m}+H_{[45,3]} A_{12}^{m}\right) \\
& -\left(k^{123} \cdot k^{45}\right) H_{[12,3]} A_{45}^{m} \\
& -\left(k^{4} \cdot k^{5}\right)\left(\left(H_{[123,4]}-\frac{1}{2} H_{[12,3]}\left(k_{123} \cdot A_{4}\right)\right) A_{5}^{m}-\left(H_{[123,5]}-\frac{1}{2} H_{[12,3]}\left(k_{123} \cdot A_{5}\right)\right) A_{4}^{m}\right) .
\end{align*}
$$

Rank one and two $A^{m}$ superfields are identical in the Lorenz, hybrid, and BCJ gauges, so we may leave these as they are. The difficulty comes with the $\hat{A}_{[123,45]}^{m}$ superfield, which we must expand and manipulate in order to find its relation with the hybrid gauge $\check{A}_{[123,45]}^{m}$,

$$
\begin{aligned}
2 \hat{A}_{[123,45]}^{m}= & \hat{A}_{p}^{123} \hat{F}_{p m}^{45}-\hat{A}_{m}^{123}\left(k^{123} \cdot \hat{A}^{45}\right)+2\left(\hat{W}^{123} \gamma_{m} \hat{W}^{45}\right) \\
& \quad-\hat{A}_{p}^{45} \hat{F}_{p m}^{123}+\hat{A}_{m}^{45}\left(k^{45} \cdot \hat{A}^{123}\right) \\
= & \left(A_{p}^{123}+k_{p}^{123} H_{[12,3]}\right) F_{p m}^{45}-\left(A_{m}^{123}+k_{m}^{123} H_{[12,3]}\right)\left(k^{123} \cdot A^{45}\right)+2\left(W^{123} \gamma_{m} W^{45}\right) \\
& -A_{p}^{45} F_{p m}^{123}+A_{m}^{45}\left(k^{45} \cdot\left(A^{123}+k^{123} H_{[12,3]}\right)\right) \\
= & 2 \check{A}_{[123,45]}^{m}+k_{p}^{123} H_{[12,3]} F_{p m}^{45}-k_{m}^{123} H_{[12,3]}\left(k^{123} \cdot A^{45}\right)+A_{m}^{45} H_{[12,3]}\left(k^{45} \cdot k^{123}\right) \\
= & 2 \check{A}_{[123,45]}^{m}-k_{m}^{12345} H_{[12,3]}\left(k^{123} \cdot A^{45}\right)+2 A_{m}^{45} H_{[12,3]}\left(k^{45} \cdot k^{123}\right) \\
& \quad-k_{p}^{123} H_{[12,3]}\left(k^{4} \cdot k^{5}\right)\left(A_{p}^{4} A_{m}^{5}-A_{m}^{4} A_{p}^{5}\right) .
\end{aligned}
$$

We should be reassured by the appearance of the negative of the more complex terms in (7.1.2). Substituting this in, we have

$$
\begin{align*}
A_{[123,45]}^{m}= & \check{A}_{[123,45]}^{m}-k_{12345}^{m} H_{[123,45]}  \tag{7.1.4}\\
& -\left(k^{1} \cdot k^{2}\right)\left(H_{[13,45]} A_{2}^{m}+H_{[45,2]} A_{13}^{m}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(H_{[12,45]} A_{3}^{m}+H_{[45,3]} A_{12}^{m}\right) \\
& -\left(k^{4} \cdot k^{5}\right)\left(H_{[123,4]} A_{5}^{m}-H_{[123,5]} A_{4}^{m}\right) .
\end{align*}
$$

By studying this, and the similar formula which can be found for $A_{[12,34]}^{m}$, we arrive at a proposed generalisation of (4.2.31),

$$
\begin{align*}
& K^{[P, Q]}=\hat{K}^{[P, Q]}-\sum_{\substack{P=X j Y \\
Y=R \sqcup S}}\left(k_{X} \cdot k_{j}\right)\left[H_{[X R, q]} K_{j S}-(X \leftrightarrow j)\right]  \tag{7.1.5}\\
&+\sum_{\substack{Q=X j Y \\
Y=R \amalg S}}\left(k_{X} \cdot k_{j}\right)\left[H_{[X R, p]} K_{j S}-(X \leftrightarrow j)\right]- \begin{cases}D_{\alpha} H_{[P, Q]} & : K=A_{\alpha} \\
k_{P Q}^{m} H_{[P, Q]} & : K=A^{m} \\
0 & : K=W^{\alpha}\end{cases}
\end{align*}
$$

for $P$ and $Q$ left-to-right Dynkin brackets, and $p$ and $q$ the letterifications of $P$ and $Q$, which is to say $p$ is the word $P$ treated as a letter, and likewise for $q$.

This form of the redefinitions was the primary tool used to find BCJ gauge superfields using the hybrid gauge in the course of the work in [27]. However, towards the end of this project we realized that it may be generalised to arbitrary bracketing structures using the contact term map, and as such we present it in this form here

$$
K^{[P, Q]}=\check{K}^{[P, Q]}-\tilde{C} \llbracket H, K \rrbracket \circ[P, Q]-\left\{\begin{array}{ll}
D_{\alpha} H_{[P, Q]} & : K=A_{\alpha}  \tag{7.1.6}\\
k_{P Q}^{m} H_{[P, Q]} & : K=A^{m} \\
0 & : K=W^{\alpha}
\end{array} .\right.
$$

Consistency between the formulae follows from the lemma (6.2.19). We will now see that these formulae allow us to find higher rank redefinitions also.

### 7.2 Redefinitions at Rank Six

Having identified the general redefinition formulae, our goals now will be to verify these, and to find formulae for the thus far undefined $H$ superfields which arise higher ranks. The verification will be implicit, in that all calculations will proceed as expected and so we do not need an extra step confirming this. Our goal therefore, will be finding formulae for the $H$ superfields. We begin this task by gathering more data, and start with the three topologies at rank six, $[12345,6],[1234,56]$, and $[123,456]$. These calculations were performed with the aid of FORM [148; 149; 150].

### 7.2.0.1 The $[12345,6]$ topology

The redefinition of the superfield $A_{m}^{123456}$ follows from either (7.1.5) or (7.1.6),

$$
\begin{align*}
& A_{[12345,6]}^{m}=\check{A}_{[12345,6]}^{m}-k_{123456}^{m} H_{[12345,6]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(H_{13456} A_{2}^{m}+H_{1346} A_{25}^{m}+H_{1356} A_{24}^{m}+H_{1456} A_{23}^{m}\right. \\
& \\
& \left.\quad+H_{136} A_{245}^{m}+H_{146} A_{235}^{m}+H_{156} A_{234}^{m}-(1 \leftrightarrow 2)\right)  \tag{7.2.1}\\
& -\left(k^{12} \cdot k^{3}\right)\left(H_{12456} A_{3}^{m}+H_{1246} A_{35}^{m}+H_{1256} A_{34}^{m}+H_{126} A_{345}^{m}\right. \\
& \\
& \left.\quad-H_{3456} A_{12}^{m}-H_{346} A_{125}^{m}-H_{356} A_{124}^{m}\right) \\
& \\
& -\left(k^{123} \cdot k^{4}\right)\left(H_{12356} A_{4}^{m}+H_{1236} A_{45}^{m}-H_{456} A_{123}^{m}\right) \\
& \\
& -\left(k^{1234} \cdot k^{5}\right) H_{12346} A_{5}^{m}
\end{align*}
$$

To identify $H_{[12345,6]}$, we proceed as in [83] and act on the above with $\mathcal{L}_{6}$. This then sends the left hand side to zero by assumption, and by taking ${ }^{1} \mathcal{L}_{6} \circ H_{[12345,6]}=6 H_{[12345,6]}$ we can rearrange to find $H_{[12345,6]}$. While it is possible to some degree to express this purely in terms of the $H^{\prime}$ defined in $(4.2 .25)$ as at lower ranks, this is not completely the case. Instead, one finds

$$
\begin{align*}
H_{[12345,6]}= & \frac{1}{6}\left(H_{123,4,56}^{\prime}+H_{1234,5,6}^{\prime}+H_{654,3,21}^{\prime}+H_{6543,2,1}^{\prime}\right.  \tag{7.2.2}\\
& -\left(k^{1} \cdot k^{2}\right)\left(H_{134} H_{652}-H_{135} H_{642}+H_{136} H_{542}-H_{145} H_{632}+H_{146} H_{532}-H_{156} H_{432}\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(H_{124} H_{653}-H_{125} H_{643}+H_{126} H_{543}\right) \\
& -\left(k^{6} \cdot k^{5}\right)\left(H_{643} H_{125}-H_{642} H_{135}+H_{641} H_{235}-H_{632} H_{145}+H_{631} H_{245}-H_{621} H_{345}\right) \\
& -\left(k^{65} \cdot k^{4}\right)\left(H_{653} H_{124}-H_{652} H_{134}+H_{651} H_{234}\right)
\end{align*}
$$

At lower ranks, terms of second order in $H_{[P, Q]}$ could not appear as each $H$ must have at least three particle labels to be non-vanishing. We now look to other rank six topologies to see if similar terms appear there, and if a pattern in such may be identified.

[^10]
### 7.2.0.2 The $[1234,56]$ topology

We would now like to see the form of $H_{[1234,56]}$, and see if this also has $H^{2}$ terms within it also. Again using (7.1.5) or (7.1.6) we generate the redefinition formula

$$
\begin{align*}
A_{m}^{[1234,56]}=\check{A}_{m}^{[1234,56]} & -\left(k^{1} \cdot k^{2}\right)\left(H_{[1,56]} A_{m}^{[23,4]}+H_{[13,56]} A_{m}^{[2,4]}+H_{[14,56]} A_{m}^{[2,3]}\right. \\
& \left.+H_{[134,56]} A_{m}^{2}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(H_{[12,56]} A_{m}^{[3,4]}+H_{[124,56]} A_{m}^{3}-(12 \leftrightarrow 3)\right)  \tag{7.2.3}\\
& -\left(k^{123} \cdot k^{4}\right)\left(H_{[123,56]} A_{m}^{4}-(123 \leftrightarrow 4)\right) \\
& +\left(k^{5} \cdot k^{6}\right)\left(H_{[5,1234]} A_{m}^{6}-(5 \leftrightarrow 6)\right) \\
& -k_{m}^{123456} H_{[1234,56]} .
\end{align*}
$$

We identify the form of $H_{[1234,56]}$ through the relation of this superfield with known Dynkin bracket structures,

$$
\begin{equation*}
A_{m}^{[1234,56]}=A_{m}^{1234 \ell(56)}=A_{m}^{123456}-A_{m}^{123465} \tag{7.2.4}
\end{equation*}
$$

Enforcing this leads to the form of $H_{[1234,56]}$,

$$
\begin{align*}
H_{[1234,56]}= & \frac{1}{3}\left(H_{123,4,56}^{\prime}-2 H_{1234,5,6}^{\prime}+H_{654,3,21}^{\prime}+H_{6543,2,1}^{\prime}\right. \\
& +\left(k^{1} \cdot k^{2}\right)\left(2 H_{145} H_{236}+2 H_{135} H_{246}+H_{134} H_{256}\right. \\
& \left.-2 H_{146} H_{235}-2 H_{136} H_{245}-H_{156} H_{234}\right) \\
& +\left(k^{12} \cdot k^{3}\right)\left(2 H_{125} H_{346}+H_{124} H_{356}-2 H_{126} H_{345}\right)  \tag{7.2.5}\\
& +\left(k^{5} \cdot k^{6}\right)\left(H_{146} H_{235}+H_{125} H_{346}+H_{136} H_{245}\right. \\
& \left.-H_{145} H_{236}-H_{135} H_{246}-H_{126} H_{345}\right) \\
& \left.+\left(k^{4} \cdot k^{56}\right)\left(H_{156} H_{234}-H_{134} H_{256}+H_{124} H_{356}\right)\right)
\end{align*}
$$

We note the similarities with $H_{[12345,6]}$; this is again four $H^{\prime}$ terms plus a collection of $H^{2}$ terms. This provides some clue towards a general structure. That is for example, based upon what has been seen thus far it seems reasonable to assume that a term $H_{[P, Q]}$ will contain $(|P|+|Q|-2) H^{\prime}$ terms, and several $H^{2}$ terms potentially linked to the $H^{\prime}$. An explicit general formula for $H_{[P, Q]}$ will be identified in (7.3.4).

### 7.2.0.3 The $[123,456]$ topology

The redefinition for $A_{[123,456]}^{m}$ can be found with the usual formulae, and is given by

$$
\begin{align*}
A_{m}^{[123,456]}=\hat{A}_{m}^{[123,456]} & -k_{m}^{123456} H_{[123,456]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(H_{[13,456]} A_{m}^{2}+H_{[1,456]} A_{m}^{23}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(H_{[12,456]} A_{m}^{3}-(12 \leftrightarrow 3)\right)  \tag{7.2.6}\\
& +\left(k^{4} \cdot k^{5}\right)\left(H_{[46,123]} A_{m}^{5}+H_{[4,123]} A_{m}^{56}-(4 \leftrightarrow 5)\right) \\
& +\left(k^{45} \cdot k^{6}\right)\left(H_{[45,123]} A_{m}^{6}-(45 \leftrightarrow 6)\right)
\end{align*}
$$

We again relate this to known superfields,

$$
\begin{equation*}
A_{m}^{[123,456]}=A_{m}^{123 \ell(456)}=A_{m}^{123654}-A_{m}^{123645}-A_{m}^{123546}+A_{m}^{123456} \tag{7.2.7}
\end{equation*}
$$

By enforcing this we determine the value of $H_{[123,456]}$,

$$
\begin{align*}
H_{[123,456]}= & \frac{1}{6}\left(-3 H_{123,45,6}^{\prime}-3 H_{1236,5,4}^{\prime}+3 H_{12,3,456}^{\prime}+3 H_{4563,2,1}^{\prime}\right. \\
& +\left(k^{1} \cdot k^{2}\right)\left(H_{[1,45]} H_{[23,6]}+H_{[13,5]} H_{[26,4]}+H_{[16,5]} H_{[23,4]}\right. \\
& \left.\quad-H_{[13,6]} H_{[2,45]}-H_{[13,4} H_{[26,5]}-H_{[16,4]} H_{[23,5]}\right) \\
& +\left(k^{12} \cdot k^{3}\right)\left(H_{[12,5]} H_{[36,4]}-H_{[3,45]} H_{[12,6]}-H_{[12,4]} H_{[36,4]}\right)  \tag{7.2.8}\\
& -\left(k^{4} \cdot k^{5}\right)\left(H_{[4,12]} H_{[56,3]}+H_{[46,2]} H_{[53,1]}+H_{[43,2]} H_{[56,1]}\right. \\
& \left.\quad-H_{[46,3]} H_{[5,12]}-H_{[46,1]} H_{[53,2]}-H_{[43,1]} H_{[56,2]}\right) \\
& \left.-\left(k^{45} \cdot k^{6}\right)\left(H_{[45,2]} H_{[63,1]}-H_{[6,12]} H_{[45,3]}-H_{[45,1]} H_{[63,1]}\right)\right)
\end{align*}
$$

### 7.2.1 Simplifying the Redefinition Terms

We now identify a pattern within the $H^{2}$ contributions to the rank six $H_{[P, Q]}$ terms in order to simplify them. The method that was used was to create a new $\tilde{H}_{A, B, C}$ object to be used in the definition of $H$ terms, such that it contains the corresponding $H^{\prime}$, and several $H^{2}$ terms. That is,

$$
\begin{equation*}
\tilde{H}_{A, B, C}=H_{A, B, C}^{\prime}+\text { some of the } H^{2} \mathrm{~S} \tag{7.2.9}
\end{equation*}
$$

There are a number of methods by which the pattern within the $H^{2}$ terms could then be identified and, on reflection, the most obvious would have been to begin with (7.2.5). In this equation, there is one $H^{\prime}$ with a coefficient of 2 , and so we might expect all $H^{2}$ terms with the same coefficient to be associated with this $H^{\prime}$. We could then identify the relation between the two sets of terms, and validate it by looking at the other $H^{\prime}$ terms.

The result is that one identifies the definition of $\tilde{H}_{A, B, C}$,

$$
\begin{align*}
& \tilde{H}_{A, B, C}=H_{A, B, C}^{\prime}-\left[\sum_{\substack{X j Y=A \\
Y=R \sqcup S}}\left(k^{X} \cdot k^{j}\right)\left(H_{[X R, B]} H_{[j S, C]}-(B \leftrightarrow C)\right)\right.  \tag{7.2.10}\\
&+\operatorname{cyclic}(A, B, C)]
\end{align*}
$$

With this, the rank six $H$ definitions are reduced to a sum of four terms each,

$$
\begin{align*}
H_{[12345,6]} & =\frac{1}{6}\left(\tilde{H}_{123,4,56}+\tilde{H}_{1234,5,6}+\tilde{H}_{654,3,21}+\tilde{H}_{6543,2,1}\right)  \tag{7.2.11}\\
H_{[1234,56]} & =\frac{1}{3}\left(\tilde{H}_{123,4,56}-2 \tilde{H}_{1234,5,6}+\tilde{H}_{654,3,21}+\tilde{H}_{6543,2,1}\right)  \tag{7.2.12}\\
H_{[123,456]} & =\frac{1}{2}\left(-\tilde{H}_{123,45,6}-\tilde{H}_{1236,5,4}+\tilde{H}_{12,3,456}+\tilde{H}_{4563,2,1}\right) \tag{7.2.13}
\end{align*}
$$

Further, the formula (7.2.10) is such that up to rank 5 it reduces to $\tilde{H}_{A, B, C}=H_{A, B, C}^{\prime}$. Hence, we may replace all $H^{\prime}$ terms in (4.2.47) with their corresponding $\tilde{H}$, and the definitions will be unchanged. Thus we will make the $\tilde{H}$ the primary tool with which the $H$ terms are defined.

To give one example of $(7.2 .10)$, we consider $\tilde{H}_{1234,5,6}$. This is given by

$$
\begin{align*}
& \tilde{H}_{1234,5,6}=H_{1234,5,6}^{\prime}-\left(k^{1} \cdot k^{2}\right)\left(H_{[134,5]} H_{[2,6]}\right. \\
&+H_{[13,5]} H_{[24,6]}+H_{[14,5]} H_{[23,6]} \\
&\left.+H_{[1,5]} H_{[234,6]}-(5 \leftrightarrow 6)\right) \\
&-\left(k^{12} \cdot k^{3}\right)\left(H_{[124,5]} H_{[3,6]}+H_{[12,5]} H_{[34,6]}-(5 \leftrightarrow 6)\right)  \tag{7.2.14}\\
&-\left(k^{123} \cdot k^{4}\right)\left(H_{[123,5]} H_{[4,6]}-(5 \leftrightarrow 6)\right)
\end{align*}
$$

Note the cyclic sum plays no role in this example, as in these cases we sum over $X j Y=5$ or 6 , and both $X$ and $j$ are required to be non-empty. Using the vanishing of $H_{i}=H_{i j}=0$, this simplifies further to

$$
\begin{align*}
\tilde{H}_{1234,5,6}=H_{1234,5,6}^{\prime} & -\left(k^{1} \cdot k^{2}\right)\left(H_{[13,5]} H_{[24,6]}+H_{[14,5]} H_{[23,6]}-(5 \leftrightarrow 6)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(H_{[12,5]} H_{[34,6]}-(5 \leftrightarrow 6)\right) . \tag{7.2.15}
\end{align*}
$$

This it can then be seen links the terms with coefficient 2 in (7.2.12) and (7.2.5), as required.

It should be noted that (7.2.10) is not the only form $\tilde{H}$ could have taken; for example we could replace the $H^{2}$ terms summed by

$$
\begin{equation*}
9 H_{X, R, B}^{\prime} H_{j, S, C}^{\prime}-(B \leftrightarrow C) \tag{7.2.16}
\end{equation*}
$$

or even with something much uglier, say

$$
\begin{equation*}
H_{X, R, B}^{\prime}\left(\hat{H}_{j, S, C}\right)^{\frac{1}{2}} \frac{H_{j, S, C}}{\sqrt{3}\left(H_{j, S, C}^{\prime}\right)^{\frac{1}{2}}}-(B \leftrightarrow C), \tag{7.2.17}
\end{equation*}
$$

and this would still give the same factorisation at rank six, but something much different at higher ranks. (7.2.10) is the correct formula though, as is proved by its working at higher ranks. We just note that at this point it was far from clear that this was the form it must take.

The $\tilde{H}_{A, B, C}$ preserve the symmetries of $H_{A, B, C}^{\prime}$; they are symmetric in their blocks of indices and they satisfy generalised Jacobi identities in each of $A, B$, and $C$ individually. More discussion of the symmetries of the $\tilde{H}$ terms can be found in appendix D .

### 7.3 Higher Ranks

We may express the rank seven $H$ superfields using the technology found at rank six. Once we have these, we will have enough data to identify the arbitrary rank case. In this section we do just this $[148 ; 149 ; 150]$, and then perform a strong check on the formula we find.

### 7.3.1 Rank Seven

The rank seven redefinitions follow from the usual formulae. We then identify the $H$ superfields as at six points; starting by finding $H_{[1234567,8]}$, and then using this and the associated $A^{m}$ superfield to fix the remaining $H$ terms. These calculations all proceed as required, and the $\tilde{H}$ formulae of the previous section continue to capture all of the $H^{2}$ terms involved. The redefinition terms are found to be

$$
\begin{align*}
& H_{[123456,7]}=\frac{1}{7}\left(\tilde{H}_{12345,6,7}-\tilde{H}_{1234,5,76}-\tilde{H}_{76543,2,1}+\tilde{H}_{7654,3,12}-\tilde{H}_{765,4,123}\right),  \tag{7.3.1}\\
& H_{[12345,67]}=\frac{1}{7}\left(-5 \tilde{H}_{12345,6,7}-2 \tilde{H}_{1234,5,76}-2 \tilde{H}_{76543,2,1}+2 \tilde{H}_{7654,3,12}-2 \tilde{H}_{765,4,123}\right), \\
& H_{[1234,567]}=\frac{1}{7}\left(-4 \tilde{H}_{1234,7,65}-4 \tilde{H}_{12347,6,5}+3 \tilde{H}_{123,4,567}-3 \tilde{H}_{56743,2,1}-3 \tilde{H}_{5674,3,21}\right) .
\end{align*}
$$

These have similar aspects to those previously found, and these similarities will be exploited to find a general formula. For completeness we state all redefinition terms found so far, presented with their indices in a particularly useful form

$$
\begin{equation*}
H_{[12,3]}=\frac{1}{3}\left(\tilde{H}_{1,2,3}\right) \tag{7.3.2}
\end{equation*}
$$

$$
\begin{aligned}
H_{[123,4]} & =\frac{1}{4}\left(\tilde{H}_{12,3,4}-\tilde{H}_{1,2,43}\right) \\
H_{[12,34]} & =\frac{1}{4}\left(2 \tilde{H}_{1,2,34}-2 \tilde{H}_{3,4,12}\right) \\
H_{[1234,5]} & =\frac{1}{5}\left(\tilde{H}_{123,4,5}-\tilde{H}_{12,3,54}+\tilde{H}_{1,2,543}\right) \\
H_{[123,45]} & =\frac{1}{5}\left(2 \tilde{H}_{12,3,45}-2 \tilde{H}_{1,2,453}-3 \tilde{H}_{4,5,123}\right) \\
H_{[12345,6]} & =\frac{1}{6}\left(\tilde{H}_{1234,5,6}-\tilde{H}_{123,4,65}+\tilde{H}_{12,3,654}-\tilde{H}_{1,2,6543}\right) \\
H_{[1234,56]} & =\frac{1}{6}\left(2 \tilde{H}_{123,4,56}-2 \tilde{H}_{12,3,564}+2 \tilde{H}_{1,2,5643}-4 \tilde{H}_{5,6,1234}\right) \\
H_{[123,456]} & =\frac{1}{6}\left(3 \tilde{H}_{12,3,456}-3 \tilde{H}_{1,2,4563}-3 \tilde{H}_{45,6,123}+3 \tilde{H}_{4,5,1236}\right) \\
H_{[123456,7]} & =\frac{1}{7}\left(\tilde{H}_{12345,6,7}-\tilde{H}_{1234,5,76}+\tilde{H}_{123,4,765}-\tilde{H}_{12,3,7654}+\tilde{H}_{1,2,76543}\right) \\
H_{[12345,67]} & =\frac{1}{7}\left(2 \tilde{H}_{1234,5,67}-2 \tilde{H}_{123,4,675}+2 \tilde{H}_{12,3,6754}-2 \tilde{H}_{1,2,67543}-5 \tilde{H}_{6,7,12345}\right) \\
H_{[1234,567]} & =\frac{1}{7}\left(3 \tilde{H}_{123,4,567}-3 \tilde{H}_{12,3,5674}+3 \tilde{H}_{1,2,56743}-4 \tilde{H}_{56,7,1234}+4 \tilde{H}_{5,6,12347}\right),
\end{aligned}
$$

These must be studied carefully in order to identify an underlying pattern. However, certain aspects of this are clear already; for instance that the rank $n H$ is associated with an overall $1 / n$ factor and $n-2 \tilde{H}$ terms. Further, for a term $H_{[P, Q]},|P|-1$ of these are multiplied by $|Q|$, and $|Q|-1$ are multiplied by $|P|$. By making observations of this sort we may begin to find the underlying structure.

### 7.3.2 Arbitrary Rank

The general rank $n$ redefinition superfield $H_{[A, B]}$, for $A$ and $B$ Dynkin brackets, is proposed to be given by

$$
\begin{equation*}
H_{[A, B]}=\frac{1}{|A|+|B|}\left(|A| \sum_{X j Y=a(\bar{B})}(-1)^{|X|} \tilde{H}_{\bar{Y}, j, X}-|B| \sum_{X j Y=b(\bar{A})}(-1)^{|X|} \tilde{H}_{\bar{Y}, j, X}\right), \tag{7.3.3}
\end{equation*}
$$

where $a$ and $b$ are the letterifications of $A$ and $B$ respectively. We note that letterifications contribute only 1 of the length of a word involving such, so for instance if $\mathrm{X}=123(456)$, where the 456 has been letterified, then $|X|=4$.

This we may reexpress as

$$
\begin{align*}
H_{[A, B]}= & \frac{1}{|A|+|B|}\left(|A| \sum_{X j Y=a \tilde{B}}(-1)^{|Y|+|B|} \tilde{H}_{\tilde{Y}, j, X}\right.  \tag{7.3.4}\\
& \left.-|B| \sum_{X j Y=b \tilde{A}}(-1)^{|Y|+|A|} \tilde{H}_{\tilde{Y}, j, X}\right),
\end{align*}
$$

which follows from $|X|+|Y|+1=|B|+1$ in the first sum, $|X|+|Y|+1=|A|+1$ in the second. In this representation, all powers of -1 correspond with words we take the transpose of. This can make more sense if there is some implicit Lie algebraic structure, as one may apply relations of the form

$$
\begin{equation*}
\ell(X)=(-1)^{|X|+1} r(\bar{X}) . \tag{7.3.5}
\end{equation*}
$$

We will explore this further in subsection 9.2.2.
As an example, consider $H_{[1234,567]}$. Applying the above formula, we see that this is given by

$$
\begin{align*}
H_{[1234,567]}= & \frac{1}{4+3}\left(4 \sum_{X j Y=\langle 1234\rangle 765}(-1)^{|Y|+3} \tilde{H}_{\tilde{Y}, j, X}-3 \sum_{X j Y=\langle 567\rangle 4321}(-1)^{|Y|+4} \tilde{H}_{\tilde{Y}, j, X}\right) \\
= & \frac{1}{7}\left(4(-1)^{4} \tilde{H}_{(\tilde{5}), 6,12347}+4(-1)^{5} \tilde{H}_{(6 \tilde{5}), 7,1234}\right.  \tag{7.3.6}\\
& \left.\quad-3(-1)^{5} \tilde{H}_{(\tilde{1}), 2,56743}-3(-1)^{6} \tilde{H}_{(\tilde{2} 1), 3,5674}-3(-1)^{7} \tilde{H}_{(3 \tilde{2} 1), 4,567}\right) \\
= & \frac{1}{7}\left(4 \tilde{H}_{5,6,12347}-4 \tilde{H}_{56,7,1234}+3 \tilde{H}_{1,2,56743}-3 \tilde{H}_{12,3,5674}+3 \tilde{H}_{123,4,567}\right) \\
= & \frac{1}{7}\left(-4 \tilde{H}_{1234,7,65}-4 \tilde{H}_{12347,6,5}+3 \tilde{H}_{123,4,567}-3 \tilde{H}_{56743,2,1}-3 \tilde{H}_{5674,3,21}\right)
\end{align*}
$$

where in the last line we used the symmetries of the $\tilde{H}$ terms to present it as was given in (7.3.1). Note I have used the convention $\langle P\rangle$ to denote the letterification of a word $P$, and I have bracketed words which are to be reversed. This will not be standard notation in this thesis.

As it is only a conjecture that (7.3.4) is the form of the general redefinition terms $H_{[P, Q]}$, we should test this. As such these were used to generate the next most complex $H$, $H_{[1234567,8]}$, and check it. The formula suggests this should be

$$
\begin{align*}
H_{[1234567,8]}=\frac{1}{8}\left(\tilde{H}_{123456,7,8}-\tilde{H}_{12345,6,87}\right. & +\tilde{H}_{1234,5,876}-\tilde{H}_{123,4,8765} \\
& \left.+\tilde{H}_{12,3,87654}-\tilde{H}_{1,2,876543}\right) . \tag{7.3.7}
\end{align*}
$$

Following the usual procedure of defining a the superfield $A_{m}^{12345678}$ using (7.1.5), wherein we now define $H_{[1234567,8]}$ as being the above, we calculate $\mathcal{L}_{8} \circ A_{m}^{12345678}$. We are then reassured to find that the answer is zero, meaning that we (7.3.4) has correctly predicted the value of $H_{[1234567,8]}$. This is a good sign for the validity of our formulas.

### 7.3.2.1 Rank Nine Test

The formula (7.3.4) was tested to rank nine. As there was a possibility of as yet unidentified terms of third order in $H_{[P, Q]}$ at such a rank, we felt that this was a strong test. Equation (7.3.4) would suggest that $H_{[12345678,9]}$ is given by,

$$
\begin{align*}
H_{[12345678,9]}= & \frac{1}{9}\left(\tilde{H}_{1234567,8,9}-\tilde{H}_{123456,7,98}+\tilde{H}_{12345,6,987}\right.  \tag{7.3.8}\\
& \left.\quad-\tilde{H}_{1234,5,9876}+\tilde{H}_{123,4,98765}-\tilde{H}_{12,3,987654}+\tilde{H}_{1,2,9876543}\right) .
\end{align*}
$$

This passes the test as required; that is, defining the corresponding $A_{m}^{123456789}$ superfield using this, the corresponding generalised Jacobi identity is satisfied [148; 149; 150],

$$
\begin{equation*}
\mathcal{L}_{9} \circ A_{m}^{123456789}=0 \tag{7.3.9}
\end{equation*}
$$

This is strong evidence that the formula (7.3.4) is likely correct in general. Note that in testing this, we have implicitly verified the formula gives the correct values of for all unchecked rank eight topologies, as these are components of the above. For example, $H_{[12345678,9]}$ contains a term $\tilde{H}_{123456,7,89}$, which contains a term $H_{123456,7,89}^{\prime}$, which then contains a term $\frac{1}{2} H_{[123456,89]}\left(k^{12345689} \cdot A_{7}\right)$. Similar is true for all other rank eight topologies.

It should be noted that implicitly, (7.3.4) does actually contain some $H^{3}$ terms, as each $\tilde{H}$ is defined containing $H^{2}$ terms, and one of these may itself contain another $H^{2}$. For example, $H_{[12345678,9]}$ should contain a term $\tilde{H}_{1234567,8,9}$. From (7.2.10) this then contains a term $\left(k^{1} \cdot k^{2}\right) H_{138} H_{245679}$, which contains a term $\left(k^{1} \cdot k^{2}\right) H_{138} \tilde{H}_{2456,7,9}$, and then finally this contains a term $\left(k^{1} \cdot k^{2}\right)\left(k^{2} \cdot k^{4}\right) H_{138} H_{257} H_{469}$. It would have been bold to assume that these terms were going to be all of the $H^{3}$ terms at rank nine though, and so the test was necessary.

## CHAPTER 8

## Gauge Transformation Construction

In the literature review we detailed how, up to five points, the construction discussed thus far may be formulated as a gauge transformation. This required finding relations between local superfields in the Lorenz and BCJ gauges, and showing that they combined in the non-local scheme into the form of a non-linear gauge transformation. In this section, we generalise this discussion. We begin by exploring extra complications which arise at six points, and detail how these may be remedied. We then use these methods to describe relations between local superfields in the two gauges which we expect to hold to arbitrary rank. Finally we show that, when the BCJ gauge is described as such, it represents a finite gauge transformation from the Lorenz gauge.

### 8.1 Initial Attempts at Generalisation

### 8.1.1 Generalising the Redefinition Formula

To rank five, we have a general local gauge transformation formula for Dynkin bracket structures, (4.2.44). Further, we have explicit formula for other topologies, as detailed in appendix C.1. By studying these, we discover we may extend the Dynkin bracket formula
to all superfields $K^{[\ell(P), \ell(Q)]}$ using

$$
\begin{align*}
& K^{[P, Q]}=\hat{K}^{[P, Q]}-\sum_{\substack{P q=X j_{j} Y \\
Y=R \amalg S}}\left(k_{X} \cdot k_{j}\right)\left[\hat{H}_{X R} \hat{K}_{j S}-(X \leftrightarrow j)\right]  \tag{8.1.1}\\
&+\sum_{\substack{Q p=X X Y Y \\
Y=R \amalg S}}\left(k_{X} \cdot k_{j}\right)\left[\hat{H}_{X R} \hat{K}_{j S}-(X \leftrightarrow j)\right]- \begin{cases}D_{\alpha} \hat{H}_{[P, Q]} & : K=A_{\alpha} \\
k_{P Q}^{m} \hat{H}_{[P, Q]} & : K=A^{m} \\
0 & : K=W^{\alpha}\end{cases}
\end{align*}
$$

Note that when a letterification $p$ is inserted into a Dynkin bracket as $\ell(Q p)$, it should be regarded as a single letter in the Dynkin bracketing, and then reexpanded as a Dynkin bracket itself. That is,

$$
\begin{equation*}
\ell(Q p)=[\ell(Q), p]=[\ell(Q), \ell(P)] . \tag{8.1.2}
\end{equation*}
$$

So for example,

$$
\begin{equation*}
\ell(123(456))=[[[1,2], 3],[[4,5], 6]], \tag{8.1.3}
\end{equation*}
$$

where we use (456) to denote the letterification of 456 .
It should not come as a surprise that (8.1.1) may be generalised to arbitrary bracket structures using the contact term map,

$$
K^{[P, Q]}=\hat{K}^{[P, Q]}-C \llbracket \hat{H}, \hat{K} \rrbracket \circ[P, Q]-\left\{\begin{array}{ll}
D_{\alpha} \hat{H}_{[P, Q]} & : K=A_{\alpha}  \tag{8.1.4}\\
k_{P Q}^{m} \hat{H}_{[P, Q]} & : K=A^{m} \\
0 & : K=W^{\alpha}
\end{array} .\right.
$$

### 8.1.2 Difficulties at Rank Six

This works for superfields up to rank 5 , but when we attempted to use it at rank six it failed in all cases. Fortunately though, the hybrid gauge approach allows for the construction of the BCJ gauge superfields we are struggling to construct here. Hence, we may fix this problem by comparing the explicit terms in the two objects, and seeing how they differ. Then, we can try and spot the pattern in these terms and find some way to add them to our redefinition formula (8.1.4)

As expected, the two methods do produce different definitions for the rank six superfields in the BCJ gauge. Lengthy calculations revealed that the $A^{m}$ superfields differ by the
following terms for each rank six topology,

$$
\begin{align*}
{[12345,6]: } & -\left(k^{1} \cdot k^{2}\right)\left(k_{m}^{134} H_{134} H_{256}+k_{m}^{135} H_{135} H_{246}+k_{m}^{145} H_{145} H_{236}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(k_{m}^{124} H_{124} H_{356}+k_{m}^{125} H_{125} H_{346}-k_{m}^{345} H_{345} H_{126}\right) \\
& -\left(k^{123} \cdot k^{4}\right) k_{m}^{123} H_{123} H_{456}, \\
{[1234,56]: } & +\left(k^{1} \cdot k^{2}\right)\left(k_{m}^{134} H_{134} H_{256}-k_{m}^{234} H_{234} H_{156}\right)  \tag{8.1.5}\\
& +\left(k^{12} \cdot k^{3}\right) k_{m}^{124} H_{124} H_{356}+\left(k^{123} \cdot k^{4}\right) k_{m}^{123} H_{123} H_{456}, \\
{[123,456]: } & +\frac{1}{2}\left(k^{123} \cdot k^{456}\right)\left(k_{m}^{123}-k_{m}^{456}\right) H_{123} H_{456},  \tag{8.1.6}\\
{[[12,34], 56]: } & +\left(k^{1} \cdot k^{2}\right)\left(H_{156} H_{234} k_{234}^{m}-H_{256} H_{134} k_{134}^{m}\right)  \tag{8.1.7}\\
& +\left(k^{3} \cdot k^{4}\right)\left(H_{123} H_{456} k_{123}^{m}-H_{124} H_{356} k_{124}^{m}\right), \\
{[[123,45], 6]: } & +\left(k^{1} \cdot k^{2}\right)\left(H_{145} H_{236} k_{145}^{m}-H_{245} H_{136} k_{245}^{m}\right)  \tag{8.1.8}\\
& +\left(k^{12} \cdot k^{3}\right)\left(-H_{126} H_{345} k_{345}^{m}\right) \\
& +\left(k^{123} \cdot k^{45}\right)\left(-H_{123} H_{456} k_{123}^{m}\right), \\
&  \tag{8.1.9}\\
& +\left(k^{1} \cdot k^{2}\right)\left(-H_{156} H_{234} k_{234}^{m}+H_{134} H_{256} k_{134}^{m}\right) \\
& +\left(k^{3} \cdot k^{4}\right)\left(H_{356} H_{412} k_{412}^{m}-H_{312} H_{456} k_{123}^{m}\right) \\
& +\left(k^{12} \cdot k^{34}\right)\left(H_{126} H_{345} k_{345}^{m}-H_{346} H_{125} k_{125}^{m}\right) . \tag{8.1.10}
\end{align*}
$$

We need to find an algorithm by which we may add such terms to the redefinition formula (8.1.4), and this is detailed in the following section.

### 8.2 Local Gauge Transformations to Arbitrary Order

The above extensions to the redefinitions may be combined with (8.1.4) in the recursive formula

$$
\begin{equation*}
K^{[P, Q]}=L_{1} \circ \hat{K}^{[P, Q]}, \tag{8.2.1}
\end{equation*}
$$

where the operator $L_{j}$ is defined by

$$
L_{j} \circ \hat{K}^{[P, Q]} \equiv \hat{K}^{[P, Q]}-\frac{1}{j} C \llbracket \hat{H}, L_{(j+1)} \circ \hat{K} \rrbracket \circ[P, Q]-\frac{1}{j} \begin{cases}D_{\alpha} \hat{H}_{[P, Q]} & : K=A_{\alpha}  \tag{8.2.2}\\ k_{P Q}^{m} \hat{H}_{[P, Q]} & : K=A^{m} \\ 0 & : K=W^{\alpha}\end{cases}
$$

Note that $L_{j} \circ \hat{K}^{[P, Q]}$ gives rise to the operator $L_{(j+1)} \circ \hat{K}^{[A, B]}$ on the right-hand side, with $|A|+|B|<|P|+|Q|$. Therefore this iteration over the index $j$ will eventually stop.

The following are some examples of redefinitions computed using this method, keeping all the nested Lie brackets explicit

$$
\begin{align*}
A_{[1,2]}^{m}= & \hat{A}_{[1,2]}^{m},  \tag{8.2.3}\\
A_{[[1,2], 3]}^{m}= & \hat{A}_{[[1,2], 3]}^{m}-k_{123}^{m} \hat{H}_{[[1,2], 3]}, \\
A_{[[1,2],[3,4]]}^{m}= & \hat{A}_{[[1,2],[3,4]]}^{m}-\left(k_{1} \cdot k_{2}\right)\left(\hat{H}_{[1,[3,4]]} \hat{A}_{2}^{m}-\hat{H}_{[2,[3,4]} \hat{A}_{1}^{m}\right) \\
& +\left(k_{3} \cdot k_{4}\right)\left(\hat{H}_{[[1,2], 4]} \hat{A}_{3}^{m}-\hat{H}_{[[1,2], 3]} \hat{A}_{4}^{m}\right)-k_{1234}^{m} \hat{H}_{[11,2],[3,4]]}, \\
A_{[[11,2], 3], 4]}^{m}= & \hat{A}_{[[1,2], 3], 4]}^{m}-\left(k_{1} \cdot k_{2}\right)\left(\hat{H}_{[[1,3], 4]} \hat{A}_{2}^{m}-\hat{H}_{[[2,3], 4]} \hat{A}_{1}^{m}\right) \\
& -\left(k_{12} \cdot k_{3}\right)\left(\hat{H}_{[[1,2], 4]} \hat{A}_{3}^{m}\right)-\left(k_{123} \cdot k_{4}\right)\left(\hat{H}_{[[1,2], 3]} \hat{A}_{4}^{m}\right)-k_{1234}^{m} \hat{H}_{[[1,2], 3], 4]}, \\
A_{[[[1,2], 3], 4], 5]}^{m}= & \hat{A}_{[[[1,2], 3], 4], 5]}^{m}-\left(k_{1} \cdot k_{2}\right)\left(\hat{H}_{[[1,3], 4]} \hat{A}_{[2,5]}^{m}+\hat{H}_{[[1,3], 5]} \hat{A}_{[2,4]}^{m}+\hat{H}_{[[1,4], 5]} \hat{A}_{[2,3]}^{m}\right. \\
& \left.+\hat{H}_{[[1,3], 4], 5]} \hat{A}_{2}^{m}-(1 \leftrightarrow 2)\right) \\
& -\left(k_{12} \cdot k_{3}\right)\left(\hat{H}_{[[1,2], 4]} \hat{A}_{[3,5]}^{m}+\hat{H}_{[[1,2], 5]} \hat{A}_{[3,4]}^{m}+\hat{H}_{[[[1,2], 4], 5]} \hat{A}_{3}^{m}-([1,2] \leftrightarrow 3)\right) \\
& -\left(k_{123} \cdot k_{4}\right)\left(\hat{H}_{[[1,2], 3]} \hat{A}_{[4,5]}^{m}+\hat{H}_{[[11,2], 3,5] 5} \hat{A}_{4}^{m}\right) \\
- & \left(k_{1234} \cdot k_{5}\right)\left(\hat{H}_{[[[1,2], 3], 4]} \hat{A}_{5}^{m}\right)-\hat{H}_{[[[1,2], 3], 4], 5]} k_{12345}^{m}, \\
A_{[[1,2], 3], 4,5]]]}^{m}= & \hat{A}_{[[1,2], 3],[4,5]]}^{m}-\left(k_{1} \cdot k_{2}\right)\left(\hat{H}_{[1,[4,5]]} \hat{A}_{[2,3]}^{m}+\hat{H}_{[[1,3],[4,5]]} \hat{A}_{2}^{m}-(1 \leftrightarrow 2)\right) \\
- & \left(k_{12} \cdot k_{3}\right)\left(\hat{H}_{[[1,2],[4,5]]} \hat{A}_{3}^{m}-\hat{H}_{[3,[4,5]]} \hat{A}_{[1,2]}^{m}\right) \\
& -\left(k_{123} \cdot k_{45}\right)\left(\hat{H}_{[[1,2], 3]} \hat{A}_{[4,5]}^{m}\right) \\
& +\left(k_{4} \cdot k_{5}\right)\left(\hat{H}_{[[[1,2], 3], 5]} \hat{A}_{4}^{m}-\hat{H}_{[[[1,2], 3], 4]} \hat{A}_{5}^{m}\right)-k_{12345}^{m} \hat{H}_{[[11,2], 3],[4,5]]} .
\end{align*}
$$

These are identical to those described previously, since the redefinition formula (8.2.1) reduces to (8.1.4) for superfields of rank five and below.

To illustrate (8.2.1) when there is more than one iteration, consider the redefinition of the superfield $\hat{A}_{m}^{[12,34], 56]}$ to the BCJ gauge. It starts as

$$
\begin{equation*}
A_{m}^{[[12,34], 56]}=L_{1} \circ \hat{A}_{m}^{[12,34], 56]} \tag{8.2.4}
\end{equation*}
$$

$$
\begin{equation*}
=\hat{A}_{m}^{[[12,34], 56]}-k_{m}^{123456} \hat{H}_{[[12,34], 56]}-C \llbracket \hat{H}, L_{2} \circ \hat{A}^{m} \rrbracket \circ[[12,34], 56] \tag{8.2.5}
\end{equation*}
$$

Using the definition of the contact term map leads to

$$
\begin{align*}
A_{m}^{[[12,34], 56]}= & \hat{A}_{m}^{[[12,34], 56]}-k_{m}^{123456} \hat{H}_{[[12,34], 56]}  \tag{8.2.6}\\
- & \left(k^{1} \cdot k^{2}\right)\left(\left(L_{2} \circ \hat{A}_{m}^{2}\right) \hat{H}_{[[1,34], 56]}+\left(L_{2} \circ \hat{A}_{m}^{[2,34]}\right) \hat{H}_{[1,56]}\right. \\
& \left.+\left(L_{2} \circ \hat{A}_{m}^{[2,56]}\right) \hat{H}_{[1,34]}-(1 \leftrightarrow 2)\right) \\
- & \left(k^{12} \cdot k^{34}\right)\left(\left(L_{2} \circ \hat{A}_{m}^{34}\right) \hat{H}_{[12,56]}-(12 \leftrightarrow 34)\right) \\
- & \left(k^{1234} \cdot k^{56}\right)\left(L_{2} \circ \hat{A}_{m}^{56}\right) \hat{H}_{[12,34]} \\
- & \left(k^{3} \cdot k^{4}\right)\left(\left(L_{2} \circ \hat{A}_{m}^{4}\right) \hat{H}_{[123,56]}+\left(L_{2} \circ \hat{A}_{m}^{[12,4]}\right) \hat{H}_{[3,56]}\right. \\
+ & \left.\left(L_{2} \circ \hat{A}_{m}^{[4,56]}\right) \hat{H}_{[12,3]}-(3 \leftrightarrow 4)\right) \\
- & \left(k^{5} \cdot k^{6}\right)\left(\left(L_{2} \circ \hat{A}_{m}^{6}\right) \hat{H}_{[[12,34], 5]}-(5 \leftrightarrow 6)\right) .
\end{align*}
$$

Note that on most of the terms the iteration stops since $L_{2} \circ \hat{A}_{m}^{i}=\hat{A}_{m}^{i}$ and $L_{2} \circ \hat{A}_{m}^{i j}=\hat{A}_{m}^{i j}$. The only remaining non-trivial action $L_{2} \circ \hat{A}_{m}^{P}$ are on terms are of multiplicity three. From (8.2.1) we obtain,

$$
\begin{equation*}
L_{2} \circ \hat{A}_{m}^{[12,3]}=\hat{A}_{m}^{[12,3]}-\frac{1}{2} k_{123}^{m} \hat{H}_{[12,3]}, \quad L_{2} \circ \hat{A}_{m}^{[1,23]}=\hat{A}_{m}^{[1,23]}-\frac{1}{2} k_{123}^{m} \hat{H}_{[1,23]} \tag{8.2.7}
\end{equation*}
$$

Plugging all of this into (8.2.6) yields

$$
\begin{align*}
A_{m}^{[[12,34], 56]}= & \hat{A}_{m}^{[[12,34], 56]}-k_{m}^{123456} \hat{H}_{[[12,34], 56]}  \tag{8.2.8}\\
- & \left(k^{1} \cdot k^{2}\right)\left(\hat{A}_{m}^{2} \hat{H}_{[[1,34], 56]}+\hat{A}_{m}^{[2,34]} \hat{H}_{[1,56]}+\hat{A}_{m}^{[2,56]} \hat{H}_{[1,34]}\right. \\
& \left.\quad-\frac{1}{2} k_{m}^{234} \hat{H}_{[2,34]} \hat{H}_{[1,56]}-\frac{1}{2} k_{m}^{256} \hat{H}_{[2,56]} \hat{H}_{[1,34]}-(1 \leftrightarrow 2)\right) \\
- & \left(k^{12} \cdot k^{34}\right)\left(\hat{A}_{m}^{34} \hat{H}_{[12,56]}-(12 \leftrightarrow 34)\right) \\
- & \left(k^{1234} \cdot k^{56}\right) \hat{A}_{m}^{56} \hat{H}_{[12,34]} \\
- & \left(k^{3} \cdot k^{4}\right)\left(\hat{A}_{m}^{4} \hat{H}_{[123,56]}+\hat{A}_{m}^{[12,4]} \hat{H}_{[3,56]}+\hat{A}_{m}^{[4,56]} \hat{H}_{[12,3]}\right. \\
& \left.\quad-\frac{1}{2} k_{m}^{124} \hat{H}_{[12,4]} \hat{H}_{[3,56]}-\frac{1}{2} k_{m}^{456} \hat{H}_{[4,56]} \hat{H}_{[12,3]}-(3 \leftrightarrow 4)\right) \\
& -\left(k^{5} \cdot k^{6}\right)\left(\hat{A}_{m}^{6} \hat{H}_{[[12,34], 5]}-(5 \leftrightarrow 6)\right) .
\end{align*}
$$

Higher-rank examples can be similarly generated from the recursion (8.2.2). However, as they grow rapidly in complexity we shall not detail them here. We tested this redefinition formula in instances up to rank eight, and it worked in all tested cases. It was computational limitations rather than doubts over the formula which impeded our testing of further ranks.

### 8.2.1 Form of the $\hat{H}_{[P, Q]}$ Redefinition Terms at Higher Orders

Unfortunately, a significant difficulty with this method was in finding a general expression for the $\hat{H}_{[P, Q]}$ terms. These become much more complicated as we move past the five particle case. Defining them through the enforcement of Jacobi identities on the corresponding superfields, it has been found that up to multiplicity eight that these can be simplified as [148; 149; 150]

$$
\begin{aligned}
\hat{H}_{[A, B]} & =\hat{H}_{[A, B]}^{\prime}-\frac{1}{2} \tilde{C} \llbracket \hat{H}, \hat{H} \rrbracket \circ[A, B], \\
\hat{H}_{\left[\left[A_{1}, A_{2}\right],\left[B_{1}, B_{2}\right]\right]}^{\prime} & =H_{\left[\left[A_{1}, A_{2}\right],\left[B_{1}, B_{2}\right]\right]}-\frac{1}{2}\left[\left(\hat{H}_{\left[A_{1}, A_{2}\right]}^{\prime} k_{A_{1} A_{2}}^{m}+\frac{1}{2}\left(k^{A_{1}} \cdot k^{A_{2}}\right)\left(\hat{H}_{A_{1}}^{m} \hat{H}_{A_{2}}-\hat{H}_{A_{2}}^{m} \hat{H}_{A_{1}}\right)\right.\right. \\
& \left.\left.-\tilde{C} \llbracket \hat{H}^{m}, \hat{H} \rrbracket \circ\left[A_{1}, A_{2}\right]\right) A_{m}^{\left[B_{1}, B_{2}\right]}-(A \leftrightarrow B)\right], \\
\hat{H}_{i}^{\prime} & =\hat{H}_{[i, j]}^{\prime}=0,
\end{aligned}
$$

where the $H_{[A, B]}$ are defined as they were previously, and $\hat{H}_{A}^{m} \equiv k_{A}^{m} \hat{H}_{A}$.
To demonstrate, we now provide several examples. First of all note that the $\tilde{C}$ maps in (8.2.9) are associated with pairs of $\hat{H}$ superfields. As each of these requires three indices, these terms will vanish identically when $|A|+|B|<6$. Thus at lower multiplicities these relations reduce to (4.2.46), as the $\tilde{C}$ terms only start contributing at multiplicity $6+$. An example of the relations in this case is as follows:

$$
\begin{align*}
\hat{H}_{[[[1,2], 3],[4,5]]} & =\hat{H}_{[[1,2], 3],[4,5]]}^{\prime}  \tag{8.2.10}\\
& =H_{[[[1,2], 3],[4,5]]}-\frac{1}{2} k_{123}^{m} \hat{H}_{[[1,2], 3]}^{\prime} A_{m}^{[4,5]} \\
& =H_{[[[1,2], 3],[4,5]]}-\frac{1}{2} H_{[[1,2], 3]}\left(k_{123} \cdot A^{[4,5]}\right) .
\end{align*}
$$

We will now outline an example of (8.2.9) for the multiplicity six redefinition term $\hat{H}_{[[[1,2], 3],[4,5]], 6]}$, which should demonstrate the formulae more clearly.

$$
\begin{equation*}
\hat{H}_{[[[1,2], 3],[4,5]], 6]}=\hat{H}_{[[[11,2], 3],[4,5]], 6]}^{\prime}-\frac{1}{2} \tilde{C}[\hat{H}, \hat{H} \rrbracket \circ[[[[1,2], 3],[4,5]], 6] . \tag{8.2.11}
\end{equation*}
$$

The expansion of the $\tilde{C}$ term above is given as the example (E.2.2) in appendix E, and from it we see that

$$
\begin{align*}
\tilde{C} \llbracket \hat{H}, \hat{H} \rrbracket \circ[[[[1,2], 3],[4,5]], 6] & =\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[[1,3], 6]} \hat{H}_{[2,[4,5]]}-\hat{H}_{[[2,3], 66} \hat{H}_{[1,[4,5]]}\right)  \tag{8.2.12}\\
& +\left(k^{12} \cdot k^{3}\right)\left(\hat{H}_{[[1,2], 6]} \hat{H}_{[3,[4,5]}\right)+\left(k^{123} \cdot k^{45}\right)\left(\hat{H}_{[[4,5], 6]} \hat{H}_{[[1,2], 3]}\right) \\
& =\left(k^{1} \cdot k^{2}\right)\left(H_{[[1,3], 6]} H_{[2,[4,5]]}-H_{[[2,3], 6]} H_{[1,[4,5]]]}\right) \\
& +\left(k^{12} \cdot k^{3}\right)\left(H_{[[1,2], 6]} H_{[3,[4,5]]}\right)+\left(k^{123} \cdot k^{45}\right)\left(H_{[[4,5], 6]} H_{[[1,2], 3]}\right)
\end{align*}
$$

As for the $\hat{H}_{[[[[1,2], 3],[4,5]], 6]}^{\prime}$ term, this piece is given by

$$
\begin{align*}
\hat{H}_{[[[[1,2], 3],[4,5]], 6]}^{\prime}= & H_{[[[[1,2], 3],[4,5]], 6]}-\frac{1}{2}\left[\left(\hat{H}_{[[[1,2], 3],[4,5]]}^{\prime} k_{12345}^{m}-\tilde{C}^{\prime} \llbracket \hat{H}, \hat{H} \rrbracket \circ[[[1,2], 3],[4,5]]\right) A_{m}^{6}\right] \\
= & H_{[[[[1,2], 3],[4,5]], 6]}-\frac{1}{2} H_{[[[1,2], 3],[4,5]]}\left(k_{12345}^{m} \cdot A^{6}\right)  \tag{8.2.13}\\
& +\frac{1}{4} H_{[[1,2], 3]}\left(k_{123} \cdot A^{45}\right)\left(k^{12345} \cdot A^{6}\right)
\end{align*}
$$

where we have used (8.2.10) and that the action of $\tilde{C}^{\prime} \llbracket \hat{H}, \hat{H} \rrbracket$ on any Lie polynomial with less than six letters is zero. Putting this all together we thus have that

$$
\begin{align*}
\hat{H}_{[[[[1,2], 3],[4,5]], 6]} & =H_{[[[[1,2], 3],[4,5]], 6]}  \tag{8.2.14}\\
& -\frac{1}{2} H_{[[[1,2], 3],[4,5]]}\left(k_{12345}^{m} \cdot A^{6}\right)+\frac{1}{4} H_{[[1,2], 3]}\left(k_{123} \cdot A^{45}\right)\left(k^{12345} \cdot A^{6}\right) \\
& -\frac{1}{2}\left(k_{1} \cdot k_{2}\right)\left(H_{[[1,3], 6]} H_{[2,[4,5]]}-H_{[[2,3], 6]} H_{[1,[4,5]]}\right) \\
& -\frac{1}{2}\left(k_{12} \cdot k_{3}\right)\left(H_{[[1,2], 6]} H_{[3,[4,5]]}\right)-\frac{1}{2}\left(k_{123} \cdot k_{45}\right)\left(H_{[[4,5], 6]} H_{[[1,2], 3]}\right) .
\end{align*}
$$

Unfortunately to see an example where the separation of $[A, B]$ into $\left[\left[A_{1}, A_{2}\right],\left[B_{1}, B_{2}\right]\right]$ in the definition of $\hat{H}^{\prime}$ comes into affect requires going to multiplicity seven, which considerably increases the number of terms involved and makes any such example less easy to follow. Such terms are no different from the one just outlined though, so we will not detail them.

It might raise some concerns that various equations seen here are in some places defined in terms of BCJ gauge superfields, and so this might not represent a true gauge transformation. This is however not an issue, as a purely Lorenz gauge version of (8.2.9) can be found by just replacing the BCJ superfields with their Lorenz gauge expansions (8.2.1). Some difficulty may arise doing this for $H_{A, B, C}$ due to the presence of $F_{P}^{m n}$ terms. However, we do the same thing, and plug the Lorenz gauge expansions into the definition of $F$ in the BCJ gauge to get

$$
\begin{equation*}
F_{[P, Q]}^{m n}=k_{m}^{P Q}\left(L_{1} \circ \hat{A}_{n}^{[P, Q]}\right)-k_{m}^{P Q}\left(L_{1} \circ \hat{A}_{m}^{[P, Q]}\right)-C \llbracket\left(L_{1} \circ \hat{A}_{m}\right),\left(L_{1} \circ \hat{A}_{n}\right) \rrbracket \circ[P, Q] . \tag{8.2.15}
\end{equation*}
$$

The notation of (8.2.9) has just been chosen for its compactness and clarity.

### 8.3 Non-Local Transformation

We would now like to use the form of the local redefinitions (8.2.1) to show that at the nonlocal level the mapping we are performing is a gauge transformation. There are additional complications compared to lower ranks in the form of products of $H$ terms, and as we
shall see these shall combine to make an overall finite gauge transformation [151],

$$
\begin{equation*}
\mathbb{A}_{m}^{B C J}=U \mathbb{A}_{m}^{L} U^{-1}-\partial U U^{-1}, \quad U=e^{-\mathbb{H}} \tag{8.3.1}
\end{equation*}
$$

To begin though, we recall the form of gauge transformations up to five points (4.2.60),

$$
\begin{equation*}
\mathcal{A}_{P}^{m, \mathrm{BCJ}}=\mathcal{A}_{P}^{m, \mathrm{~L}}-k_{P}^{m} \mathcal{H}_{P}+\sum_{X Y=P}\left(\mathcal{A}_{X}^{m, \mathrm{~L}} \mathcal{H}_{Y}-\mathcal{A}_{Y}^{m, \mathrm{~L}} \mathcal{H}_{X}\right) \tag{8.3.2}
\end{equation*}
$$

This will serve as a starting point for our calculations.

### 8.3.1 Six Points

At six points, we now have some $H^{2}$ terms which do not fit into the formula above, and some experimentation is needed to see how these affect the gauge transformation. The six point Berends-Giele current expansion in terms of local superfields given by,

$$
\begin{align*}
& \mathcal{K}_{123456}=\frac{K_{[\mid[[1,2], 3], 4], 5], 6]}}{s_{12} s_{123} s_{1234} s_{12345} s_{123456}}+\frac{K_{[[\mid 11,[2,3], 4,4], 5], 6]}}{s_{123} s_{1234} s_{12345} s_{123456} s_{23}}+\frac{K_{[[[11,2]][3,4]], 5], 6]}}{s_{12} s_{1234} s_{12345}} \\
& +\frac{K_{[[[[1,2], 3],[4,5]], 6]}}{s_{12} s_{123} s_{12345} s_{123456} s_{45}}+\frac{K_{[[[[1,2], 3], 4],[5,6]]}}{s_{12} s_{123} s_{1234} s_{123456} s_{56}}+\frac{K_{[[[1,[[2,3], 4]], 5], 6]}}{s_{1234} s_{12345} s_{123456} s_{23} s_{234}} \\
& +\frac{K_{[[[1,[2,[3,4]]], 5], 6]}}{s_{1234} s_{12345} s_{123456} s_{234} s_{34}}+\frac{K_{[[[1,[2,3]],[4,5]], 6]}}{s_{123} s_{12345} s_{123456} s_{23} s_{45}}+\frac{K_{[[[1,[2,3]], 4],[5,6]]}}{s_{123} s_{1234} s_{123456} s_{23} s_{56}} \\
& +\frac{K_{[[[1,2],[[3,4], 5]], 6]}}{s_{12} s_{12345} s_{123456} s_{34} s_{345}}+\frac{K_{[[[1,2],[3,[4,5]]], 6]}}{s_{12} s_{12345} s_{123456} s_{345} s_{45}}+\frac{K_{[[[1,2],[3,4]],[5,6]]}}{s_{12} s_{1234} s_{123456} s_{34} s_{56}} \\
& +\frac{K_{[[[1,2], 3],[[4,5], 6]]}}{s_{12} s_{123} s_{123456} s_{45} s_{456}}+\frac{K_{[[[1,2], 3],[4,[5,6]]]}}{s_{12} s_{123} s_{123456} s_{456} s_{56}}+\frac{K_{[[1,[[[2,3], 4], 5]], 6]}}{s_{12345} s_{123456} s_{23} s_{234} s_{2345}} \\
& +\frac{K_{[[1,[[2,[3,4]], 5]], 6]}}{s_{12345} s_{123456} s_{234} s_{2345} s_{34}}+\frac{K_{[[1,[[2,3],[4,5]]], 6]}}{s_{12345} s_{123456} s_{23} s_{2345} s_{45}}+\frac{K_{[[1,[[2,3], 4]],[5,6]]}}{s_{1234} s_{123456} s_{23} s_{234} s_{56}}  \tag{8.3.3}\\
& +\frac{K_{[[1,[2,[[3,4], 5]]], 6]}}{s_{12345} s_{123456} s_{2345} s_{34} s_{345}}+\frac{K_{[[1,[2,[3,[4,5]]]], 6]}}{s_{12345} s_{123456} s_{2345} s_{345} s_{45}}+\frac{K_{[[1,[2,[3,4]]],[5,6]]}}{s_{1234} s_{123456} s_{234} s_{34} s_{56}} \\
& +\frac{K_{[[1,[2,3]],[[4,5], 6]]}}{s_{123} s_{123456} s_{23} s_{45} s_{456}}+\frac{K_{[[1,[2,3]],[4,[5,6]]]}}{s_{123} s_{123456} s_{23} s_{456} s_{56}}+\frac{K_{[[1,2],[[[3,4], 5], 6]]}}{s_{12} s_{123456} s_{34} s_{345} s_{3456}} \\
& +\frac{K_{[[1,2],[[3,[4,5]], 6]]}}{s_{12} s_{123456} s_{345} s_{3456} s_{45}}+\frac{K_{[[1,2],[[3,4],[5,6]]]}}{s_{12} s_{123456} s_{34} s_{3456} s_{56}}+\frac{K_{[[1,2],[3,[[4,5], 6]]]}}{s_{12} s_{123456} s_{3456} s_{45} s_{456}}
\end{align*}
$$

$$
\begin{aligned}
& +\frac{K_{[1,[2[,[3,[4,5]]], 6]]}}{s_{123456} s_{2345} s_{23456} s_{345} s_{45}}+\frac{K_{[1,[2,[3,4]][5,6,6]]}}{s_{123456} s_{234} s_{23456} 3_{34} s_{56}}+\frac{K_{[1,[[2,3],[4,5]], 6]]]}}{s_{123456} s_{23} s_{23456} s_{45} s_{456}}
\end{aligned}
$$

Setting $K=A^{m}$, we equate this in the BCJ gauge with each superfields expansion in terms of the Lorenz gauge and redefinition terms generated by (8.2.1). We then rearrange the output of this, forming Berends-Giele currents with the resulting superfields. This is no small task due to the volume of terms involved in the calculation, but eventually one
finds that it can be simplified as $[148 ; 149 ; 150]$

$$
\begin{align*}
\mathcal{A}_{123456}^{m, B C J} & =\mathcal{A}_{123456}^{m, L}-k_{123456}^{m} \mathcal{H}_{123456} \\
& +\mathcal{A}_{1}^{m} \mathcal{H}_{23456}+\mathcal{A}_{12}^{m} \mathcal{H}_{3456}+\mathcal{A}_{123}^{m} \mathcal{H}_{456}  \tag{8.3.4}\\
& -\mathcal{A}_{6}^{m} \mathcal{H}_{12345}-\mathcal{A}_{56}^{m} \mathcal{H}_{1234}-\mathcal{A}_{456}^{m} \mathcal{H}_{123} \\
& +k_{m}^{456} \mathcal{H}_{123} \mathcal{H}_{456}-k_{m}^{123} \mathcal{H}_{123} \mathcal{H}_{456}
\end{align*}
$$

This calculation can be performed much more elegantly with the use of relations found for the contact term map. We take the general form of the gauge transformation (8.2.1), and consider its application to Berends-Giele currents defined through the $b$-map,

$$
\begin{equation*}
\mathcal{A}_{123456}^{m, B C J}=\llbracket \hat{A}^{m} \rrbracket \circ b(123456)-C \llbracket \hat{H}, L_{2} \circ \hat{A}^{m} \rrbracket \circ b(123456)-\llbracket \hat{H}^{m} \rrbracket \circ b(123456) . \tag{8.3.5}
\end{equation*}
$$

Applying the proposition (6.2.1), the middle $b$-map deconcatenates

$$
\begin{align*}
\mathcal{A}_{123456}^{m, B C J}= & \mathcal{A}_{123456}^{m, L}-k_{m}^{123456} \mathcal{H}_{123456}-\llbracket \hat{H}, L_{2} \circ \hat{A}^{m} \rrbracket \circ \sum_{X Y=12 \ldots 6}(b(X) \otimes b(Y)-b(Y) \otimes b(X)) \\
= & \mathcal{A}_{123456}^{m, L}-k_{m}^{123456} \mathcal{H}_{123456}  \tag{8.3.6}\\
& \quad-\sum_{X Y=12 \ldots 6}\left(\mathcal{H}_{X} \llbracket L_{2} \circ \hat{A}^{m} \rrbracket \circ b(Y)-\mathcal{H}_{Y} \llbracket L_{2} \circ \hat{A}^{m} \rrbracket \circ b(X)\right) .
\end{align*}
$$

Completing another round of the same sort of calculation on the $\llbracket L_{2} \circ \hat{A}^{m} \rrbracket$ terms yields ${ }^{1}$

$$
\begin{align*}
\mathcal{A}_{123456}^{m, B C J}=\mathcal{A}_{123456}^{m, L} & -k_{m}^{123456} \mathcal{H}_{123456}-\sum_{X Y=12 \ldots 6}\left(\mathcal{H}_{X} \mathcal{A}_{Y}^{m, L}-\mathcal{H}_{Y} \mathcal{A}_{X}^{m, L}\right)  \tag{8.3.7}\\
& +\frac{1}{2} \sum_{X Y=12 \ldots 6}\left(\mathcal{H}_{X} \mathcal{H}_{Y} k_{Y}^{m}-\mathcal{H}_{Y} \mathcal{H}_{X} k_{X}^{m}\right),
\end{align*}
$$

This is then (8.3.4), produced without the need for complex rearrangements of superfields into BG currents.

We now transition to the non-linear picture using the methods of section 4.2.4, and see that this six point gauge transformation above follows from the perturbiner expansion of ${ }^{2}$

$$
\begin{equation*}
\mathbb{A}_{m}^{B C J}=\mathbb{A}_{m}^{L}-\left[\partial_{m}, \mathbb{H}\right]+\left[\mathbb{A}_{m}^{L}, \mathbb{H}\right]-\frac{1}{2}\left[\left[\partial_{m}, \mathbb{H}\right], \mathbb{H}\right] \tag{8.3.8}
\end{equation*}
$$

Which suggests a non-linear gauge transformation [88]. We will now show that this, and all higher point superfields generated using (8.2.1), amount to an infinite series of non-

[^11]linear corrections to the gauge transformation at five points. This gives an overall finite gauge transformation [151].

### 8.3.2 General Points

We may reformulate (8.2.1) as a perturbiner series using

$$
\begin{equation*}
\mathbb{L}_{j} \circ \mathbb{A}_{m}=\mathbb{A}_{m}-\frac{1}{j}\left[\partial_{m}, \mathbb{H}\right]-\frac{1}{j}\left[\mathbb{H}, \mathbb{L}_{j+1} \circ \mathbb{A}_{m}\right], \tag{8.3.9}
\end{equation*}
$$

This allows us to state the form of the gauge transformation as

$$
\begin{equation*}
\mathbb{A}_{m}^{B C J}=\mathbb{L}_{1} \circ \mathbb{A}_{m}^{L} \tag{8.3.10}
\end{equation*}
$$

which upon repeated application of the $\mathbb{L}$ map gives

$$
\begin{align*}
\mathbb{A}_{m}^{B C J}=\mathbb{A}_{m}^{L}+\left[\mathbb{H}, \partial_{m}\right]-\left[\mathbb{H}, \mathbb{A}_{m}^{L}\right] & -\frac{1}{2}\left[\mathbb{H},\left[\mathbb{H}, \partial_{m}\right]\right]+\frac{1}{2}\left[\mathbb{H},\left[\mathbb{H}, \mathbb{A}_{m}^{L}\right]\right] \\
& +\frac{1}{3!}\left[\mathbb{H},\left[\mathbb{H},\left[\mathbb{H}, \partial_{m}\right]\right]\right]+\ldots \tag{8.3.11}
\end{align*}
$$

We may simplify this in terms of the supercovariant derivative (3.1.3) in the Lorenz gauge, $\nabla_{m}^{L} \equiv \partial_{m}-\mathbb{A}_{m}^{L}$. The transformation thus becomes

$$
\begin{equation*}
\mathbb{A}_{m}^{B C J}=\mathbb{A}_{m}^{L}+\left[\mathbb{H}, \nabla_{m}^{L}\right]-\frac{1}{2}\left[\mathbb{H},\left[\mathbb{H}, \nabla_{m}^{L}\right]\right]+\frac{1}{3!}\left[\mathbb{H},\left[\mathbb{H},\left[\mathbb{H}, \nabla_{m}^{L}\right]\right]\right]+\ldots \tag{8.3.12}
\end{equation*}
$$

This is then the series expansion of a finite gauge transformation [151],

$$
\begin{equation*}
\mathbb{A}_{m}^{B C J}=U \mathbb{A}_{m}^{L} U^{-1}-\partial U U^{-1}, \quad U=e^{-\mathbb{H}} \tag{8.3.13}
\end{equation*}
$$

Alternatively, this can be rewritten as $\nabla_{m}^{B C J}=e^{-a d_{\mathbb{H}}}\left(\nabla_{m}^{L}\right)$, where $a d_{\mathbb{H}}(X) \equiv[\mathbb{H}, X]$. We thus have that, to arbitrary multiplicity, the transformations defined in this section are gauge transformations, and thus performing them does not affect results. As doing so gives BCJ symmetries and introduces no extra complications, we therefore for the remainder of this thesis always construct superfields in the BCJ gauge.

## Summary and Outlook

### 9.1 Summary of Results

Using the methods described in this part of this thesis, the speed at which multiparticle superfields (and therefore amplitudes) in the BCJ gauge may be computed has improved significantly. To summarise this effectively, we will now outline how a tree level amplitude in $D=10$ SYM satisfying BCJ relations may be efficiently computed, and will clearly state when results developed in the course of this research are used. In the following section, we will then discuss some potential directions for future research in this area.

Suppose one wishes to compute the amplitude $A^{\text {tree }}(1,2, \ldots, n)$. By (4.3.10), this is given by

$$
\begin{equation*}
\mathcal{A}_{n}=\sum_{X Y=12 \ldots n-1}\left\langle M_{X} M_{Y} M_{n}\right\rangle . \tag{9.1.1}
\end{equation*}
$$

A result which was speculated to hold in general previously, but in this research has been proved, is that the BG currents $M_{A}$ have variation

$$
\begin{equation*}
Q M_{A}=\sum_{X Y=A} M_{X} M_{Y} \tag{9.1.2}
\end{equation*}
$$

As detailed in the literature review, that the amplitude (9.1.1) is in the cohomology of the

BRST operator follows from this result.
We then expand (9.1.1), and this is done using the $b$-map (2.3.13)

$$
\begin{equation*}
M_{A}=V_{b(A)}=\lambda^{\alpha} A_{\alpha}^{b(A)} \tag{9.1.3}
\end{equation*}
$$

That is, we expand it as a series of fractions, with numerator $\lambda^{\alpha} A_{\alpha}^{C}$ for $C$ some Lie monomial, and the denominator some combination of mandelstams. Using the hybrid gauge approach for simplicity, this superfield $A_{\alpha}^{C}$ is defined by recursion. First one expands it in terms of hybrid gauge and redefinition superfields using,

$$
K^{[P, Q]}=\check{K}^{[P, Q]}-\tilde{C} \llbracket H, K \rrbracket \circ[P, Q]-\left\{\begin{array}{ll}
D_{\alpha} H_{[P, Q]} & : K=A_{\alpha}  \tag{9.1.4}\\
k_{P Q}^{m} H_{[P, Q]} & : K=A^{m} \\
0 & : K=W^{\alpha}
\end{array} .\right.
$$

This general Lie monomial formula was a new result of this research. Then one expands the hybrid gauge superfields in terms of those of lower rank in the BCJ gauge,

$$
\begin{align*}
\check{A}_{\alpha}^{[P, Q]} & =-\frac{1}{2}\left[A_{\alpha}^{P}\left(k^{P} \cdot A^{Q}\right)+A_{m}^{P}\left(\gamma^{m} W^{Q}\right)_{\alpha}-(P \leftrightarrow Q)\right],  \tag{9.1.5}\\
\check{A}_{m}^{[P, Q]} & =-\frac{1}{2}\left[A_{m}^{P}\left(k^{P} \cdot A^{Q}\right)+A_{n}^{P} F_{m n}^{Q}-\left(W^{P} \gamma_{m} W^{Q}\right)-(P \leftrightarrow Q)\right], \\
\check{W}_{[P, Q]}^{\alpha} & =\frac{1}{4} F_{r s}^{P}\left(\gamma^{r s} W^{Q}\right)^{\alpha}-\frac{1}{2}\left(k^{P} \cdot A^{Q}\right) W_{P}^{\alpha}-\frac{1}{2} W_{P}^{m \alpha} A_{Q}^{m}-(P \leftrightarrow Q), \\
F_{m n}^{[P, Q]} & =k_{P Q}^{m} A_{[P, Q]}^{n}-k_{P Q}^{m} A_{[P, Q]}^{m}-C \llbracket A^{m}, A^{n} \rrbracket \circ[P, Q] .,
\end{align*}
$$

with the higher weight superfields defined by

$$
\begin{align*}
W_{[P, Q]}^{m \alpha} & =k_{P Q}^{m} W_{[P, Q]}^{\alpha}-C \llbracket \hat{A}^{m}, W^{\alpha} \rrbracket \circ[P, Q]  \tag{9.1.6}\\
F_{[P, Q]}^{m p q} & =k_{P Q}^{m} F_{[P, Q]}^{p q}-C \llbracket A^{m}, F^{p q} \rrbracket \circ[P, Q] .
\end{align*}
$$

The first three of these formulae were known prior to this research, but the latter three were only known for the case of left-to-right Dynkin brackets. Then one also needs to expand the $H$ superfields. For a general $H_{[P, Q]}$ this is done by using Jacobi identities within the $P$ and $Q$ to relate it to some $H_{[R, S]}$, for $R$ and $S$ Dynkin brackets. Then one applies the following sequence of relations

$$
\begin{align*}
& H_{[A, B]}= \frac{1}{|A|+|B|}\left(|A| \sum_{X j Y=a \tilde{B}}(-1)^{|Y|+|B|} \tilde{H}_{\tilde{Y}, j, X}\right.  \tag{9.1.7}\\
&\left.-|B| \sum_{X j Y=b \tilde{A}}(-1)^{|Y|+|A|} \tilde{H}_{\tilde{Y}, j, X}\right), \\
& \tilde{H}_{A, B, C}=H_{A, B, C}^{\prime}-\left[\sum_{\substack{X j Y=A \\
Y=R \amalg S}}\left(k^{X} \cdot k^{j}\right)\left(H_{[X R, B]} H_{[j S, C]}-(B \leftrightarrow C)\right)\right. \tag{9.1.8}
\end{align*}
$$

$$
\left.\begin{array}{rl}
\quad+\operatorname{cyclic}(A, B, C)
\end{array}\right], ~ 子 \begin{aligned}
& H_{A, B, C}^{\prime}= H_{A, B, C}+\frac{1}{2}\left(H_{[A, B]}\left(k_{A B} \cdot A_{C}\right)+\operatorname{cyclic}(A, B, C)\right), \\
& H_{A, B, C}=-\frac{1}{4} A_{A}^{m} A_{B}^{n} F_{C}^{m n}+\frac{1}{2}\left(W_{A} \gamma_{m} W_{B}\right) A_{C}^{m}+\operatorname{cyclic}(A, B, C) .
\end{aligned}
$$

The first two of these were key new results of this research. By repeated use of these relations, one finds an expression for the amplitude in terms of single particle superfields ${ }^{1}$,

$$
\begin{equation*}
\mathcal{A}_{n}=\left\langle f\left(\lambda^{\alpha}, A_{\alpha}^{i}, A_{i}^{m}, W_{i}^{\alpha}\right)\right\rangle \tag{9.1.11}
\end{equation*}
$$

These are then expanded using the results of [89], and the $\theta^{5}$ components are selected as required by the pure spinor bracket [104].

### 9.2 Future Research

There are numerous directions for possible future research in this area, which have been explored to varying degrees. In this section we discuss a few of these, and what progress may have been made upon them. We should also draw the readers attention to [147] which generalised upon the contact term map in particular; by reformulating it and by relating it to the $S$-map of [60].

### 9.2.1 Simplifying Amplitudes In Terms Of Redefinition Terms

In the course of this work, an interesting feature of the redefinition terms was noticed. Namely, that SYM amplitudes could be expressed in terms of the objects $\tilde{H}$ directly, skipping several steps of calculation. Where we have the amplitude expression

$$
\begin{equation*}
A^{S Y M}(1,2, \ldots, p)=\sum_{X Y=12 \ldots p-1}\left\langle M_{X} M_{Y} M_{p}\right\rangle \tag{9.2.1}
\end{equation*}
$$

and we would ordinarily expand the BG currents using the procedure outlined in the previous section in terms of single particle superfields, and then expand these in terms of their components and select the relevant terms within the pure spinor bracket, we may

[^12]alternatively make the identification
\[

$$
\begin{equation*}
\left\langle V_{X} V_{Y} V_{Z}\right\rangle=\left.\tilde{H}_{X, Y, Z}\right|_{\theta=0} \tag{9.2.2}
\end{equation*}
$$

\]

That is, rather than expanding the $V_{A} V_{B} V_{C}$, we instead identify their pure spinor bracket with the $\theta=0$ component of the $\tilde{H}_{A, B, C}$. These components are

$$
\begin{equation*}
A_{m}^{i} \mapsto e_{m}^{i}, \quad W_{i}^{\alpha} \mapsto \chi_{i}^{\alpha}, \quad F_{m n}^{i} \mapsto f_{m n}^{i} \tag{9.2.3}
\end{equation*}
$$

This then skips a significant amount of calculations which would otherwise be needed.

To give one example, we consider the three point amplitude,

$$
\begin{equation*}
A(1,2,3)=\left\langle M_{1} M_{2} M_{3}\right\rangle=\left\langle V_{1} V_{2} V_{3}\right\rangle . \tag{9.2.4}
\end{equation*}
$$

We identify this with the $\theta=0$ component of

$$
\begin{align*}
\tilde{H}_{1,2,3} & =H_{A, B, C}^{\prime}=H_{A, B, C}  \tag{9.2.5}\\
& =-\frac{1}{4} A_{1}^{m} A_{2}^{n} F_{3}^{m n}+\frac{1}{2}\left(W_{1} \gamma_{m} W_{2}\right) A_{3}^{m}+\operatorname{cyclic}(1,2,3) .
\end{align*}
$$

Making the identifications (9.2.3) thus yields the three point amplitude

$$
\begin{equation*}
-\frac{1}{4} e_{1}^{m} e_{2}^{n} f_{3}^{m n}+\frac{1}{2}\left(\chi_{1} \gamma_{m} \chi_{2}\right) e_{3}^{m}+\operatorname{cyclic}(1,2,3) \tag{9.2.6}
\end{equation*}
$$

By traditional methods however, we would have to expand the $V_{i}=\lambda^{\alpha} A_{\alpha}^{i}$ in terms of its components,

$$
\begin{align*}
& \left\langle\lambda ^ { \alpha } \lambda ^ { \beta } \lambda ^ { \sigma } \left(\frac{1}{2}\left(\theta \gamma^{m}\right)_{\alpha} e_{m}^{1}+\frac{1}{3}\left(\theta \gamma^{m}\right)_{\alpha}\left(\theta \gamma_{m} \chi^{1}\right)\right.\right.  \tag{9.2.7}\\
& \left.\left.\quad+\frac{1}{16}\left(\theta \gamma^{m}\right)_{\alpha}\left(\theta \gamma_{m}^{p q} \theta\right) k_{q}^{1} e_{p}^{1}\right)\left(\frac{1}{2}\left(\theta \gamma^{n}\right)_{\beta} e_{n}^{2}+\ldots\right)\left(\frac{1}{2}\left(\theta \gamma^{r}\right)_{\sigma} e_{r}^{3}+\ldots\right)\right\rangle
\end{align*}
$$

and then select the $\theta^{5}$ components of this. Clearly this is significantly more complicated, and so this trick is worthy of further exploration.

It should be noted that in previous papers, similar connections to this have been made [42]. Therein, the analogous result with the replacement

$$
\begin{equation*}
\left\langle M_{X} M_{Y} M_{Z}\right\rangle=\left.H_{X, Y, Z}\right|_{\theta=0} \tag{9.2.8}
\end{equation*}
$$

was identified implicitly. That these two possible replacement rules agree with each other (once momentum conservation is used) is an interesting result in and of itself, and would be interesting to study.

### 9.2.1.1 Origin Of This Simplification

This result comes seemingly out of the ether, and here we discuss possible routes by which it may be explained. Recall the construction of multiparticle superfields in the literature review, and in particular equation (4.1.7). When we were defining the two particle superfields we had a relation of the form

$$
\begin{equation*}
U^{1} U^{2} \rightarrow U^{12}+(\text { total derivatives }) \tag{9.2.9}
\end{equation*}
$$

and as we knew we would integrate this we dropped the total derivative terms. Higher rank superfields are constructed by the same method, and so similar terms will have been implicitly dropped. However, is may be the case that simplification is flawed. The total derivatives come from $1 / z_{12}^{2}$ terms in the OPE, and it may be the case that at rank three and higher some non-trivial relation between such terms occurs and results in extra terms which contribute to the amplitude.

In recent work [153], the $n$-point tree level amplitude was reformulated as

$$
\begin{equation*}
A=\left\langle z_{12} z_{23} z_{31} U^{1}\left(z_{1}\right) U^{2}\left(z_{2}\right) U^{3}\left(z_{3}\right) \int d z_{4} U^{4} \ldots \int d z_{n} U^{n}\right\rangle \tag{9.2.10}
\end{equation*}
$$

with this bracket defined by $\langle 1\rangle=1$. That is, in this formulation the pure spinor bracket effectively selects the $\theta=0$ component of its constituents. This aligns with that we select the $\theta=0$ component of the $\tilde{H}_{A, B, C}$, and so seems like a promising line of inquiry. However, why the amplitude constructed in this way should correlate with the redefinition terms only remains a mystery, and requires further study.

### 9.2.2 $H$ Superfields With General Structure

The $H_{[P, Q]}$ of the previous discussion were defined only for $P$ and $Q$ Dynkin brackets, and containing elements bearing some similarity with the contact term map applied to such. This might lead one to wonder if the redefinition terms can be extended to arbitrary Lie bracket structures. To do so, the definitions of $H_{A, B, C}$ and $H_{A, B, C}^{\prime}$ would generalise naturally. The $\tilde{H}_{A, B, C}$ superfields meanwhile may be reformulated in terms of the contact term map

$$
\begin{equation*}
\tilde{H}_{A, B, C}=H_{A, B, C}^{\prime}-\left(C \llbracket H_{[\cdot, B]} H_{[\cdot, C]} \rrbracket \circ A+\operatorname{cyclic}(A, B, C)\right) \tag{9.2.11}
\end{equation*}
$$

The difficulty lies in finding a general expression for $H_{[P, Q]}$ though. However, some clues to this may be found by reformulating the current expression. To begin, we have

$$
\begin{equation*}
H_{[\ell(A), \ell(B)]}=\frac{1}{|A|+|B|}\left(|A| \sum_{X j Y=a \tilde{B}}(-1)^{|Y|+|B|} \tilde{H}_{\ell(\bar{Y}), j, \ell(X)}-(A \leftrightarrow B)\right) \tag{9.2.12}
\end{equation*}
$$

We may remove the $(-1)^{|Y|}$ and the reversal of the word $Y$ using

$$
\begin{equation*}
\ell(A)=(-1)^{|A|-1} r(\bar{A}) \tag{9.2.13}
\end{equation*}
$$

Doing so yields

$$
\begin{equation*}
H_{[\ell(A), \ell(B)]}=\frac{1}{|A|+|B|}\left(|A| \sum_{X j Y=a \tilde{B}}(-1)^{|B|-1} \tilde{H}_{r(Y), j, \ell(X)}-(A \leftrightarrow B)\right) \tag{9.2.14}
\end{equation*}
$$

Then, doing likewise for the $\ell(B)$ on the left hand side gives

$$
\begin{equation*}
H_{[\ell(A), r(B)]}=\frac{1}{|A|+|B|}\left(|A| \sum_{X j Y=a B} \tilde{H}_{\ell(X), j, r(Y)}-(A \leftrightarrow B)\right) \tag{9.2.15}
\end{equation*}
$$

where we use the symmetry of $\tilde{H}$ under permutations also.

It is not difficult to imagine that a map should exist which generates this. However, time constraints and commitments to other projects have led to this not been explored as fully as would be liked. Some progress was made, and this is discussed in appendix F. However for reasons described therein these results should be treated with considerable caution, and are likely incorrect.

### 9.2.3 Further Unexplored Research Directions

While the previous two subsections detailed areas where some work has began, there are many other directions where work could be performed. One such is finding higher point $\hat{H}_{[A, B]}$ superfields. While we have found an expression for these which holds in the cases we checked, it was not a particularly appealing one in the same way that that of $H_{[A, B]}$ was. This suggests the possibility of irregularities in the formula, which would present themselves at higher points. This we have not checked for though, due to computational limitations. Alternatively, it may be that the formula we have is correct and just not presented in its best form. This would also be worthy of investigation. We should note that though we have doubts about the specific form of $\hat{H}_{[A, B]}$, those do not extend in the same way to the local gauge transformation rules they apply to, and so that the BCJ
gauge is achieved by a gauge transformation from the Lorenz gauge is not in doubt.

One further possible direction would be to attempt to find a defining equation of the BCJ gauge. That is, the Lorenz gauge is defined by the relation $k_{P}^{m} \mathcal{A}_{P}^{m}=0$. Similar holds for other gauges, with for instance the Harnad-Shnider gauge defined by $\theta^{\alpha} \mathcal{A}_{\alpha}^{P}=0[88 ; 152]$. However, no such relation is known for the BCJ gauge, and we instead use an indirect method based around local superfields satisfying numerous relations to find it. It would be interesting to see if a simple relation alike these could be found for the BCJ gauge also.

By looking at the lower energy regime of (open bosonic) string theory, that is if one takes $\alpha^{\prime}$ to be small but not simply zero, one finds a SYM theory with deformations proportional to $\alpha^{\prime} F^{3}$ and $\alpha^{\prime 2} F^{4}$ [3]. A Berends-Giele approach to amplitudes in such was discussed in [154], and this involved the identification of $\alpha^{\prime}$ corrections to the $H_{[P, Q]}$ fields. No general expression was found for these, though descriptions for such were found in terms of $\alpha^{\prime}$ corrected $H_{A, B, C}$ fields. As such, it may be possible to take the results discussed here, and use them to find a general expression for such.

Efforts have been made to describe a kinematic algebra. That is, to describe the specific structure of the Lie algebra which describes the kinematic numerators in accordance with the BCJ relations $[155 ; 156]$. It may be possible to gain some insights into this through work in this part.

Finally, the Berends-Giele recursions of Yang-Mills theory have been derived in terms of $L_{\infty}$-algebras [157]. As such, it may be possible to use some similar approach in order to derive the $H_{[A, B]}$ superfields.

## Part III

## One Loop Field Theory Amplitudes From String Theory

As detailed in the literature review, amplitudes in 10D SYM have been identified to six points, and have been shown to satisfy BCJ relations up to five points [1]. However, amplitudes in string theory have more recently been identified up to seven points, and there has been some work progress towards higher point extensions of this [20; 21; 22]. Clearly, as field theory should follow as a limit of sting theory [5], there is something of a disconnect here. We should be able to extend the results of field theory using the results developed in string theory, and in this section we do just this.

This begins with the following chapter, wherein we detail a procedure to efficiently take field theory limits of results in string theory. This begins with a justification for the methods used based upon comparable results at tree level, and the underlying symmetry between the two different loop levels in such calculations. We then state the rules we have developed, whereby one takes a term from the one-loop string correlator and outputs a contribution to the corresponding field theory amplitude based upon the Kronecker-Eisenstein coefficient associated with it. The amplitudes produced we will see unfortunately depend in part upon the choice of loop momentum parameterisation of the amplitude, and so a necessary consistency condition of the field theory limits will be needed and will be proven. We then construct amplitudes up to seven points, and discuss how we expect these methods to generalise to higher points.

This is then followed by a chapter discussing BCJ relations. A key motivation behind this work was the absence of BCJ relations in the work of [1], and using the scheme described in this part we will see that such relations are restored. We will give several examples of such at six points, and discuss aspects of the equivalent seven point calculations. The expressions for seven point numerators are so complex as for it to be unfeasible to discuss them in any detail in this part, but we are able to assure the reader of the truth of any properties described, and refer the reader to the full amplitude expressions at [28] if they wish to check them for themselves.

Given an SYM amplitude satisfying BCJ relations, it is natural to ask if we may then use this to generate supergravity amplitudes. In the next chapter we discuss this, detailing the success in doing so at five points and the unfortunate failure of the calculation at six points. Significant work has been performed in an attempt to rectify this problem, and we detail this, but unfortunately the correct solution has not yet been found. We then conclude with a brief discussion of the results of this part, and some potential directions for further investigation.

The work discussed in this part largely follows the paper [29].

## CHAPTER 10

## SYM Integrands From String Correlators

In this chapter, we detail the rules which have been identified in order to find the field theory limits of string correlators. The most obvious approach to identifying these would be an extremely complex mathematical analysis; expanding the Kronecker-Eisenstein coefficient functions $g^{(n)}(z, \tau)$, turning the strings into point particles by looking at the $\alpha^{\prime} \rightarrow 0$ limit, and sending $\operatorname{Im}(\tau) \rightarrow \infty$ to turn the genus-one surfaces into point particle diagrams [5]. Approaches to doing this may be found in [158] or in the string-based formalism $[146 ; 144 ; 143 ; 142 ; 141 ; 34]$. However, this would be no small feat, given the complexity of the KE terms particularly when $n \geq 2$.

Instead, an approach taking better advantage of the methods of the pure spinor formalism was used. We know that in such, amplitudes are in the cohomology of the BRST operator, and this may be used to more efficiently construct them. By using our knowledge of where their poles of KE coefficients should lie, we are able to identify which terms from the string correlators may contribute to which amplitudes in the field theory limit. By then enforcing the vanishing of the resulting amplitude expression, we are able to fix the specific coefficients which arise in the limit. This approach, as well as some understanding arising from tree level amplitude calculations, led to the identification of the rules which follow shortly.


Figure 10.1.1: We wish to verify this BCJ relation and we must use (4.4.8) in order to find the third numerator, as the naive relabelling of the equivalent term from $A(1,2,3,4,5)$ will fail.

### 10.1 Insights From Tree Level Considerations

One of the key steps which we make in order to rectify the problems of previous work is to no longer assume that numerators in different colour orderings follow from a simple substitution of one set of particle labels by another. This is similar to the discussion of subsection 4.4.1, in which partial amplitudes in non-canonical orderings were identified at tree level, and it was found that in order for them to satisfy BCJ relations they could not be a simple relabelling of the canonical ordering. To illustrate this further, we will discuss one more tree level example, which was also detailed in [29].

Suppose we wish to check the BCJ identity in figure 10.1.1. The tree level five-point SYM amplitude in the canonical colour ordering is given by (4.3.9),

$$
\begin{align*}
A^{\mathrm{SYM}}(1,2,3,4,5)= & \sum_{X Y=1234}\left\langle\mathcal{M}_{X} \mathcal{M}_{Y} \mathcal{M}_{5}\right\rangle \\
= & \frac{V_{[12,3]} V_{4} V_{5}}{s_{12} s_{123}}+\frac{V_{[1,23]} V_{4} V_{5}}{s_{23} s_{123}}+\frac{V_{[1,2]} V_{[3,4]} V_{5}}{s_{12} s_{34}}  \tag{10.1.1}\\
& +\frac{V_{1} V_{[23,4]} V_{5}}{s_{23} s_{234}}+\frac{V_{1} V_{[2,34]} V_{5}}{s_{34} s_{234}} .
\end{align*}
$$

From this two of the numerators may be identified, but the third may not. If we suppose that the missing $[[1,[4,3]], 2]$ numerator follows from taking the numerator of the $[[1,[2,3]], 4]$ diagram, and swapping each 2 for a 4 and vice versa, then the BCJ relation would be

$$
\begin{equation*}
V_{[1,2]} V_{[3,4]} V_{5}-V_{1} V_{[2,[3,4]]} V_{5}+V_{[1,[2,3]]} V_{4} V_{5} \neq 0 . \tag{10.1.2}
\end{equation*}
$$

Using (4.4.8) however, yields the BCJ representation of the noncanonical ordering,

$$
\begin{equation*}
A(1,4,3,2,5)=\sum_{X Y=23,32} V_{1 X} V_{4 \bar{Y}} V_{5} m(14325 \mid 1, X, 5, Y, 4)(-1)^{|Y|+1} \tag{10.1.3}
\end{equation*}
$$

$$
\begin{align*}
& =\frac{1}{s_{14} s_{25}}\left(V_{1} V_{432}+V_{12} V_{43}+V_{13} V_{42}+V_{132} V_{4}\right) V_{5} \\
& +\frac{1}{s_{15} s_{34}} V_{1} V_{432} V_{5}-\frac{1}{s_{23} s_{14}}\left(V_{1} V_{[4,23]}+V_{[1,23]} V_{4}\right) V_{5}  \tag{10.1.4}\\
& +\frac{1}{s_{25} s_{34}}\left(V_{1} V_{432}+V_{12} V_{43}\right) V_{5}-\frac{1}{s_{15} s_{23}} V_{1} V_{[4,23]} V_{5}
\end{align*}
$$

We may then read off the relevant numerator from this, $\left(V_{1} V_{432}+V_{12} V_{43}\right) V_{5}$, and using such the BCJ relation is satisfied by the symmetries of $V_{A}$ in the BCJ gauge,

$$
\begin{equation*}
V_{[1,2]} V_{[3,4]} V_{5}-V_{1} V_{[2,[3,4]]} V_{5}+\left(V_{1} V_{432}+V_{12} V_{43}\right) V_{5}=0 . \tag{10.1.5}
\end{equation*}
$$

At one loop, as we have discussed, different field theory diagrams correspond with different poles in the string correlator. These occur at small distances, as $z_{i} \rightarrow z_{j}$ on the Riemann surface, and as such the behavior of the terms involved becomes independent of the genus of the surface. Therefore, one would expect properties of the one loop field theory amplitudes to be similar to those of tree level. This includes having different orderings not necessarily related by relabelling to each other, and also being able to describe amplitude pole structures in terms of the Berends-Giele double currents $\phi_{P \mid Q}$ as in (2.3.9). Extra complications arise, with for instance the addition Feynman loop momentum integrands now defined using the notation of (A.2.16), but these guiding principals from tree level aid our understanding of how to proceed.

### 10.2 Field-theory limit of Kronecker-Eisenstein coefficients

The nature of the limit we take will depend upon how the loop momentum is chosen to be represented in the field theory amplitude. As such, we introduce the notation

$$
\begin{equation*}
A\left(1,2, \ldots, n ; \ell+a_{1} k_{1}+\cdots+a_{n} k_{n}\right) \tag{10.2.1}
\end{equation*}
$$

meaning the amplitude with colour ordering $1,2, \ldots, n$, constructed such that the momentum going from the $n^{t h}$ leg to the $1^{s t} \operatorname{leg}$ is $\ell+a_{1} k_{1}+\ldots+a_{n} k_{n}$. For example, the field-theory limit of the five-point correlator with insertion points ordered according to $z_{1} \leq z_{3} \leq z_{5} \leq z_{2} \leq z_{4}$ and loop momentum $\ell$ running between legs 4 and 1 is represented by the SYM integrand ${ }^{1} A(1,3,5,2,4 ; \ell)$. A more complex example is illustrated in figure 10.2.1

At one loop, as at tree level, the colour ordering of the resulting SYM integrand from the field-theory limit of the string correlator corresponds with the relative ordering of the

[^13]

Figure 10.2.1: The first hexagon above is left as general as possible, and this would belong to the amplitude $A\left(\sigma_{1}, \ldots, \sigma_{6} ; \ell+\sum_{i} a_{i} k_{i}\right)$. The second is a specific case, and is the hexagon of the amplitude $A\left(3,5,4,1,6,2 ; \ell+6 k_{1}-5 k_{3}\right)$. Note this latter case could equally be represented in five other ways, for instance $A\left(4,1,6,2,3,5 ; \ell+6 k_{1}-6 k_{3}-k_{5}\right)$. However, when an internal edge has its momentum made explicit, we choose choose to write the amplitude correspondingly.
$z_{i}$ variables on the boundary of the surface in the string amplitude. For example, the ordering $z_{1} \leq z_{3} \leq z_{5} \leq z_{4} \leq z_{2}$ yields an integrand with colour ordering $\sigma=13542$. The ordering of such $z_{i}$ determines which poles are present in an amplitude, and we encode such in terms of a map $\operatorname{Ord}_{A}(B)$. This acts on two words $A$ and $B$, and gives the maximum cropping of the word $A$ which maintains every letter it shares with $B$. That is, we take the word $B$ and return the smallest sequence of consecutive letters in the cyclic-symmetric object $A$ containing every letter in $B$. This is most clearly understood through examples,

$$
\begin{gather*}
\operatorname{Ord}_{123456}(32)=23, \quad \operatorname{Ord}_{123456}(13)=123, \quad \operatorname{Ord}_{123456}(15)=561,  \tag{10.2.2}\\
\operatorname{Ord}_{24856317}(58)=85, \quad \operatorname{Ord}_{24856317}(465)=4856, \quad \operatorname{Ord}_{24856317}(78)=7248 .
\end{gather*}
$$

This map may be defined algebraically by

$$
\operatorname{Ord}_{A}(B)= \begin{cases}A_{i} A_{i+1} \ldots A_{j-1} A_{j} & : \text { if } A_{i}, A_{j} \in B, \quad B \subseteq A_{i} \ldots A_{j}, \quad j-i \leq \frac{|A|}{2}  \tag{10.2.3}\\ A_{j} A_{j+1} \ldots A_{|A|} A_{1} A_{2} \ldots A_{i} & : \text { if } A_{i}, A_{j} \in B, B \subseteq A_{i} \ldots A_{j}, \quad j-i>\frac{|A|}{2} \\ 0 & : \text { else }\end{cases}
$$

This will be used in conjunction with BG double currents to generate kinematic poles for each ordering $\sigma$. This will be more conveniently summarised with the notation

$$
\begin{equation*}
\hat{\phi}(\sigma \mid A) \equiv \phi_{O r d_{\sigma}(A) \mid A}, \tag{10.2.4}
\end{equation*}
$$

for an amplitude with colour ordering $\sigma$.
We are now ready to give the limits. The field theory limits of terms from the KroneckerEisenstein series are

$$
\begin{equation*}
g_{i j}^{(p)} \rightarrow b_{i j}^{(p)} P+c_{i j}^{(p)} P(i j), \tag{10.2.5}
\end{equation*}
$$

$$
\begin{align*}
& g_{i j}^{(p)} g_{k l}^{(q)} \rightarrow b_{i j}^{(p)} b_{k l}^{(q)} P+b_{i j}^{(p)} c_{k l}^{(q)} P(k l)+c_{i j}^{(p)} b_{k l}^{(p)} P(i j)+c_{i j}^{(p)} c_{k l}^{(q)} P(i j, k l),  \tag{10.2.6}\\
& \left.g_{i_{1} j_{1}}^{\left(p_{1}\right)} g_{i_{2} j_{2}}^{\left(p_{2}\right)} g_{i_{3} j_{3}}^{\left(p_{3}\right)} \rightarrow b_{i_{1} j_{1}}^{\left(p_{1}\right)}\right)_{i_{2} j_{2}}^{\left(p_{2}\right)} b_{i_{3} j_{3}}^{\left(p_{3}\right)} P+b_{i_{1} j_{1}}^{\left(p_{1}\right)} b_{i_{2} j_{2}}^{\left(p_{2}\right)} c_{i_{3} j_{3}}^{\left(p_{3}\right)} P\left(i_{3} j_{3}\right) \\
& +b_{i_{1} j_{1}}^{\left(p_{1}\right)} c_{i_{2} j_{2}}^{\left(p_{2}\right)} b_{i_{3} j_{3}}^{\left(p_{3}\right)} P\left(i_{2} j_{2}\right)+c_{i_{1} j_{1}}^{\left(p_{1}\right)} b_{i_{2} j_{2}} p_{2} b_{i_{3} j_{3}} b^{\left(p_{3}\right)} P\left(i_{1} j_{1}\right) \\
& +b_{i_{1} j_{1}}^{\left(p_{1}\right)} c_{i_{2} j_{2}}^{\left(p_{2}\right)} c_{i_{3} j_{3}}^{\left(p_{3}\right)} P\left(i_{2} j_{2}, i_{3} j_{3}\right)+c_{i_{1} j_{1}}^{\left(p_{1}\right)} b_{i_{2} j_{2}}^{\left(p_{2}\right)} c_{i_{3} j_{3}}^{\left(p_{3}\right)} P\left(i_{1} j_{1}, i_{3} j_{3}\right)  \tag{10.2.7}\\
& +c_{i_{1} j_{1}}^{\left(p_{1}\right)} c_{i_{2} j_{2}}^{\left(p_{2}\right)} b_{i_{3} j_{3}}^{\left(p_{3}\right)} P\left(i_{1} j_{1}, i_{2} j_{2}\right)+c_{i_{1} j_{1}}^{\left(p_{1}\right)} c_{i_{2} j_{2}}^{\left(p_{2}\right)} c_{i_{3} j_{3}}^{\left(p_{3}\right)} P\left(i_{1} j_{1}, i_{2} j_{2}, i_{3} j_{3}\right) .
\end{align*}
$$

These are all of the limits necessary to calculate amplitudes using the known string correlators. They always have the same form; we take the subscripts of the $g_{i j}^{(p)}$, and sum over the possible ways to assign these to either a $b^{(p)}$ or a $c^{(p)}$ (to be defined below), and whenever we assign them to a $c^{(p)}$ they are also entered into the $P$ function. In turn these are defined by

$$
\begin{align*}
P & =I  \tag{10.2.8}\\
P(i j) & =\hat{\phi}(\sigma \mid i j) I_{i j}  \tag{10.2.9}\\
P(i j, k l) & = \begin{cases}\hat{\phi}(\sigma \mid i j l) I_{i j l} & \text { if } j=k \\
\hat{\phi}(\sigma \mid i j) \hat{\phi}(\sigma \mid k l) I_{i j, k l} & \text { if all } i \text { unique }\end{cases}  \tag{10.2.10}\\
P(i j, k l, m n) & = \begin{cases}\hat{\phi}(\sigma \mid i j l n) I_{i j l n} & \text { if } j=k, l=m \\
\hat{\phi}(\sigma \mid i j l) \hat{\phi}(\sigma \mid m n) I_{i j l, m n} & \text { if } j=k, \quad m, n \notin\{i, j, k, l\} \\
\hat{\phi}(\sigma \mid i j n) \hat{\phi}(\sigma \mid k l) I_{i j n, k l} & \text { if } j=m, \quad k, l \notin\{i, j, m, n\} \\
\hat{\phi}(\sigma \mid i j) \hat{\phi}(\sigma \mid k l n) I_{i j, k l n} & \text { if } l=m, \quad i, j \notin\{k, l, m, n\} \\
\hat{\phi}(\sigma \mid i j) \hat{\phi}(\sigma \mid k l) \hat{\phi}(\sigma \mid m n) I_{i j, k l, m n} & \text { if all } i \text { unique }\end{cases} \tag{10.2.11}
\end{align*}
$$

where we used the notation (10.2.4).
Finally, the coefficients $b^{(p)}$ and $c^{(p)}$ depend upon the loop momentum structure of the amplitude we aim to produce. For an amplitude $A\left(\sigma ; \ell+\sum_{i=1}^{n} a_{i} k_{i}\right)$, they are given by

$$
\begin{align*}
& b_{i j}^{(p)}=\sum_{m=0}^{p}\left(\operatorname{sgn}_{i j}^{\sigma}\right)^{m} \frac{B_{m}\left(a_{j}-a_{i}\right)^{p-m}}{m!(p-m)!},  \tag{10.2.12}\\
& c_{i j}^{(p)}=\frac{1}{2(p-1)!}\left(\left(a_{j}-a_{i}\right)+\operatorname{sgn}_{i j}^{\sigma} \operatorname{dist}_{4}^{\sigma}(i, j)\right)^{p-1}, \tag{10.2.13}
\end{align*}
$$

where $B_{n}$ denotes the $n^{\text {th }}$ Bernoulli number. Only the first three values of this are required to produce amplitudes up to seven points, and these are $B_{0}=1, B_{1}=\frac{1}{2}, B_{2}=\frac{1}{6}, B_{3}=0$. The $\operatorname{sgn} n_{i j}^{\sigma}$ in the above is the sign of $i$ and $j$ with respect to the ordering $\sigma$, and is defined by[159]

$$
\operatorname{sgn}_{i j}^{B}= \begin{cases}+1 & : i \text { is left of } j \text { in } B  \tag{10.2.14}\\ -1 & : i \text { is right of } j \text { in } B\end{cases}
$$

Finally, the function $\operatorname{dist}_{a}^{B}(i, j)$ measures the distance between $i$ and $j$ in the word $B$, and returns +1 if it is larger than $a$ and 0 otherwise,

$$
\operatorname{dist}_{a}^{B}(i, j)= \begin{cases}+1 & : \text { if } i \text { is a or more letters to the left or right of } j \text { in } B  \tag{10.2.15}\\ 0 & : \text { if } i \text { is fewer than } a \text { letters to the left or right of } j \text { in } B\end{cases}
$$

Note that when $a_{i}=0 \forall i$, we must take $0^{0}=1$ in this. We may justify this in terms of continuity. To give a few brief examples of these two maps,

$$
\begin{array}{cl}
d i s t_{3}^{1234567}(4,6) \operatorname{sgn}_{46}^{1234567}=0, & \operatorname{dist}_{5}^{1234567}(1,6) \operatorname{sgn}_{16}^{1234567}=1 \\
\operatorname{dist}_{5}^{1234567}(7,1) \operatorname{sgn}_{71}^{1234567}=-1, & \operatorname{dist}_{3}^{536214}(4,6) \operatorname{sgn}_{46}^{536214}=-1  \tag{10.2.16}\\
\text { dist }_{5}^{1234576}(1,6) \operatorname{sgn}_{16}^{1234576}=1, & \operatorname{dist}_{4}^{918364527}(2,3) \operatorname{sgn}_{23}^{918364527}=-1
\end{array}
$$

Using these tools, amplitudes have been constructed up to seven points which have been verified to vanish under the action of the BRST operator. These rules are expected to generalise naturally at higher points, and speculations about such will be discussed shortly. We now illustrate these methods with an example of a limit at seven points.

### 10.2.1 A Seven-Point Example

Here we detail how one takes the field-theory limit of the term $g_{25}^{(1)} g_{57}^{(1)} g_{76}^{(1)} V_{1} T_{2576,3,4}$ in the seven-point string correlator (5.3.20), in order to find the SYM integrand with colour ordering $A\left(1,2,3,4,5,6,7 ; \ell+4 k_{4}-6 k_{5}\right)$. We first apply (10.2.7),

$$
\begin{align*}
g_{25}^{(1)} g_{57}^{(1)} g_{76}^{(1)} & \rightarrow b_{25}^{(1)} b_{57}^{(1)} b_{76}^{(1)} P+b_{25}^{(1)} b_{57}^{(1)} c_{76}^{(1)} P(76) \\
& +b_{25}^{(1)} c_{57}^{(1)} b_{76}^{(1)} P(57)+c_{25}^{(1)} b_{57}^{(1)} b_{76}^{(1)} P(25) \\
& +b_{25}^{(1)} c_{57}^{(1)} c_{76}^{(1)} P(57,76)+c_{25}^{(1)} b_{57}^{(1)} c_{76}^{(1)} P(25,76)  \tag{10.2.17}\\
& +c_{25}^{(1)} c_{57}^{(1)} b_{76}^{(1)} P(25,57)+c_{25}^{(1)} c_{57}^{(1)} c_{76}^{(1)} P(25,57,76) .
\end{align*}
$$

Many of these terms vanish. For example using (10.2.8) the factor $P(57)$ is proportional to $\hat{\phi}(1234567 \mid 57)=\phi_{57 \mid \operatorname{Ord}_{1234567}(57)}=\phi_{57 \mid 567}=0$. Similarly, we find

$$
\begin{equation*}
P(25)=P(25,76)=P(25,57)=P(25,57,76)=0 . \tag{10.2.18}
\end{equation*}
$$

The non-zero terms are then given by ${ }^{2}$

$$
\begin{align*}
P & =I=I_{1,2,3,4,5,7}^{a_{4}, a_{5}}  \tag{10.2.19}\\
P(76) & =\hat{\phi}(1234567 \mid 76) I_{76}=\phi_{76 \mid 67} I_{76}=-\frac{1}{s_{67}} I_{1,2,3,4,5,76}^{a_{4}, a_{5}} \tag{10.2.20}
\end{align*}
$$

[^14]\[

$$
\begin{equation*}
P(57,76)=\hat{\phi}(1234567 \mid 576) I_{576}=\phi_{576 \mid 567} I_{1,2,3,4,576}=-\frac{1}{s_{67} s_{567}} I_{1,2,3,4,576}^{a_{4}, a_{5}} \tag{10.2.21}
\end{equation*}
$$

\]

Note we include the $a_{4}$ and $a_{6}$ labels only as these are the only non-zero shifts in this particular amplitude, with $a_{4}=4$ and $a_{5}=-6$.

To find the various $b_{i j}^{(1)}$ and $c_{i j}^{(1)}$ terms we apply (10.2.12) and (10.2.13). If we first look at the case of $g_{25}^{(1)}$, setting $a_{4}=4, a_{5}=-6$ we see that these are given by

$$
\begin{align*}
& b_{25}^{(1)}=\frac{B_{0}(-6)^{1}}{0!1!}+\frac{B_{1}(-6)^{0}}{1!0!}=-6+\frac{1}{2}=-\frac{11}{2}  \tag{10.2.22}\\
& c_{25}^{(1)}=\frac{\left(a_{5}-a_{2}+\operatorname{sgn}_{25}^{1234567} \operatorname{dist}_{4}^{1234567}(2,5)\right)^{1-1}}{2(1-1)!}=\frac{\left(-6+(-1)^{0} \times 0\right)^{0}}{2}=\frac{1}{2} . \tag{10.2.23}
\end{align*}
$$

The other $b$ and $c$ terms may be found similarly, and are given by

$$
\begin{equation*}
b_{57}^{(1)}=\frac{13}{2}, \quad c_{57}^{(1)}=\frac{1}{2}, \quad b_{76}^{(1)}=-\frac{1}{2}, \quad c_{76}^{(1)}=\frac{1}{2} . \tag{10.2.24}
\end{equation*}
$$

Putting everything together, we see that the limit is given by

$$
\begin{equation*}
g_{25}^{(1)} g_{57}^{(1)} g_{76}^{(1)} \rightarrow \frac{143}{8} I_{1,2,3,4,5,6,7}^{a_{4}, a_{5}}+\frac{143}{8} \frac{1}{s_{67}} I_{1,2,3,4,5,67}^{a_{4}, a_{5}}+\frac{11}{8} \frac{1}{s_{67} s_{567}} I_{1,2,3,4,567}^{a_{4}, a_{5}} \tag{10.2.25}
\end{equation*}
$$

Doing this analysis for the full seven point correlator leads to a BRST closed expression up to anomalous terms. The explicit expression for this is available to download from [28]. We discuss this example further in appendix H , with the BRST variation of a numerator in this amplitude described in full and shown to have the desired property of canceling propagators.

### 10.2.2 Consistency Between Amplitude Representations

Our representations of amplitudes should be cyclic symmetric. That is, it should be possible to describe any amplitude $A\left(1,2, \ldots, n ; \ell+\sum_{i} a_{i} k_{i}\right)$ with any other particle label leading and make appropriate shifts in the loop momentum, and get the same result. Hence, our field theory limit rules should be invariant under relations of the form

$$
\begin{equation*}
A\left(1,2, \ldots, n ; \ell+a_{1} k_{1}+\ldots+a_{n} k_{n}\right)=A\left(2,3, \ldots, n, 1 ; \ell+\left(a_{1}-1\right) k_{1}+a_{2} k_{2} \ldots+a_{n} k_{n}\right), . \tag{10.2.26}
\end{equation*}
$$

In this subsection we prove that they are, and it therefore follows naturally that we can always choose to fix the colour ordering of the SYM integrand to start with a leading 1.

It is only the $b$ and $c$ terms which may differ between these two representations. We refer to those relating to $A\left(1,2, \ldots, n ; \ell+\Sigma_{i} a_{i} k_{i}\right)$ with a (I), and those relating to $A(2,3, \ldots, n, 1 ; \ell-$ $k_{1}+\Sigma_{i} a_{i} k_{i}$ ) with a (II). To begin, we compare their $b_{i j}^{(p)}$ terms. Note we restrict this
discussion to the limit of a single Kronecker-Eisenstein coefficient function $g_{i j}^{(p)}$, as the limits of products of such are the natural generalization of this and will follow accordingly. Referring to (10.2.12), and using the notation $a_{j i}:=a_{j}-a_{i}$, we see that they differ by

$$
\begin{align*}
b_{i j}^{I(p)}-b_{i j}^{I I(p)}= & \sum_{m=0}^{p}\left(\left(\operatorname{sgn}_{i j}^{12 \ldots n}\right)^{m} \frac{B_{m} a_{j i}^{p-m}}{m!(p-m)!}\right. \\
& \left.-\left(\operatorname{sgn}_{i j}^{23 \ldots n 1}\right)^{m} \frac{B_{m}\left(a_{j i}+\delta_{j 1}-\delta_{i 1}\right)^{p-m}}{m!(p-m)!}\right)  \tag{10.2.27}\\
= & \sum_{m=0}^{p} \frac{B_{m}}{m!(p-m)!}\left(\left(\operatorname{sgn}_{i j}^{12 \ldots n}\right)^{m} a_{j i}^{p-m}-\left(\operatorname{sgn}_{i j}^{23 \ldots n 1}\right)^{m}\left(a_{j i}-\delta_{j 1}+\delta_{i 1}\right)^{p-m}\right)
\end{align*}
$$

Clearly in all cases where neither of $i$ or $j$ is 1 this vanishes. If we suppose $i=1$, the first $s g n$ function is 1 , and the second is -1 . Hence this difference becomes

$$
\begin{equation*}
b_{i j}^{I(p)}-b_{i j}^{I I}{ }^{(p)}=\sum_{m=0}^{p} \frac{B_{m}}{m!(p-m)!}\left(a_{j 1}^{p-m}-(-1)^{m}\left(a_{j 1}+1\right)^{p-m}\right) \tag{10.2.28}
\end{equation*}
$$

This can be verified to vanish on a case by case basis with relative ease. Taking for instance the $p=3$ case, we have

$$
\begin{align*}
b_{i j}^{I(3)}-b_{i j}^{I I(3)} & =\frac{B_{0}}{6}\left(a_{j 1}^{3}-(-1)^{0}\left(a_{j 1}+1\right)^{3}\right)+\frac{B_{1}}{2}\left(a_{j 1}^{2}-(-1)^{1}\left(a_{j 1}+1\right)^{2}\right)  \tag{10.2.29}\\
& +\frac{B_{2}}{2}\left(a_{j 1}^{1}-(-1)^{2}\left(a_{j 1}+1\right)^{1}\right)+\frac{B_{3}}{6}\left(a_{j 1}^{0}-(-1)^{3}\left(a_{j 1}+1\right)^{0}\right) \\
& =\frac{1}{6}\left(a_{j 1}^{3}-a_{j 1}^{3}-3 a_{j 1}^{2}-3 a_{j 1}-1\right)+\frac{1}{4}\left(a_{j 1}^{2}+a_{j 1}^{2}+2 a_{j 1}+1\right)  \tag{10.2.30}\\
& +\frac{1}{12}\left(a_{j 1}-a_{j 1}-1\right)+0=0
\end{align*}
$$

This was then verified to vanish with the aid of FORM [148; 149] in at least the first 700 cases. We now may prove it in general. Taking (10.2.28), and expanding the internal bracket one finds

$$
\begin{align*}
b_{i j}^{I(p)}-b_{i j}^{I I(p)} & =\sum_{m=0}^{p} \frac{B_{m}}{m!(p-m)!}\left(a_{j 1}^{p-m}-(-1)^{m} \sum_{n=0}^{p-m}\binom{p-m}{n} a_{j 1}^{n}\right) \\
& =\sum_{m=0}^{p} \sum_{n=0}^{p-m-1} \frac{-(-1)^{m} B_{m}}{m!n!(p-m-n)!} a_{j 1}^{n}+\sum_{m=0}^{p} \frac{B_{m}\left(1-(-1)^{m}\right) a_{j 1}^{p-m}}{m!(p-m)!}, \tag{10.2.31}
\end{align*}
$$

where we have separated out the $a_{j 1}^{p-m}$ terms in the second line. Consider these terms. When $m$ is even $\left(1-(-1)^{m}\right)$ vanishes, and when $m$ is odd and not $1, B_{m}$ vanishes. Hence, all terms in this sum vanish but one,

$$
\begin{equation*}
\frac{a_{j 1}^{p-1}}{(p-1)!} \tag{10.2.32}
\end{equation*}
$$

where that $B_{1}=\frac{1}{2}$ has been used. As for the double summation in (10.2.31), we reorder these sums to give

$$
\begin{equation*}
\sum_{n=0}^{p} \sum_{m=0}^{p-n-1} \frac{-(-1)^{m} B_{m}}{m!n!(p-m-n)!} a_{j 1}^{n} \tag{10.2.33}
\end{equation*}
$$

It is a known result that Bernoulli numbers satisfy the relation $[160]^{3}$

$$
\begin{equation*}
\sum_{k=0}^{n-1}\binom{n}{k}(-1)^{k} B_{k}=0, \quad n>1 \tag{10.2.34}
\end{equation*}
$$

and from this it follows that

$$
\begin{equation*}
\sum_{m=0}^{p-n-1} \frac{(-1)^{m} B_{m}}{m!(p-n-m)!}=-\frac{\delta_{p-n-1,0}}{(p-n-1)!} \tag{10.2.35}
\end{equation*}
$$

Plugging this into (10.2.36) gives

$$
\begin{equation*}
\sum_{n=0}^{p} \frac{-a_{j 1}^{n} \delta_{p-n-1}}{n!(p-n-1)!}=\frac{-a_{j 1}^{p-1}}{(p-1)!} \tag{10.2.36}
\end{equation*}
$$

This then perfectly cancels (10.2.32), and so (10.2.31) vanishes. Hence the difference between the two representations of the $b_{1 j}^{(p)}$ terms (10.2.28) vanishes in general. Similar will hold if we instead take $j=1$, and hence the $b$ part of the field theory limits matches in both representations.

Then, we move onto the $c$ piece. This difference is given by

$$
\begin{align*}
c_{i j}^{I(p)}-c_{i j}^{I I(p)}=\frac{1}{2(p-1)!} & \left(\left(a_{j i}+\operatorname{sgn}_{i j}^{12 \ldots n} \operatorname{dist}_{4}^{12 \ldots n}(i, j)\right)^{p-1}\right.  \tag{10.2.37}\\
& \left.-\left(a_{j i}-\delta_{j 1}+\delta_{i 1}+\operatorname{sgn}_{i j}^{23 \ldots n 1} \operatorname{dist}_{4}^{23 \ldots n 1}(i, j)\right)^{p-1}\right)
\end{align*}
$$

Again, we need only consider the cases where one of $i$ and $j$ is 1 . If we take $i=1$ we get

$$
\begin{equation*}
c_{i j}^{I(p)}-c_{i j}^{I I(p)}=\frac{\left(a_{j 1}+\operatorname{dist}_{4}^{12 \ldots n}(1, j)\right)^{p-1}-\left(a_{j 1}+1-\operatorname{dist}_{4}^{23 \ldots n 1}(1, j)\right)^{p-1}}{2(p-1)!} \tag{10.2.38}
\end{equation*}
$$

We now consider the two pieces of the numerator, and see that these are given by

$$
\begin{align*}
\left(a_{j 1}+\operatorname{dist}_{4}^{12 \ldots n}(1, j)\right)^{p-1} & =\left\{\begin{array}{ll}
a_{j 1}^{p-1} & j \leq 4 \\
\left(a_{j 1}+1\right)^{p-1} & j>4
\end{array},\right.  \tag{10.2.39}\\
\left(a_{j 1}+1-\operatorname{dist}_{4}^{23 \ldots n 1}(1, j)\right)^{p-1} & = \begin{cases}a_{j 1}^{p-1} & j \leq n-2 \\
\left(a_{j 1}+1\right)^{p-1} & j>n-2\end{cases} \tag{10.2.40}
\end{align*} .
$$

[^15]When $n=4,5$, the only Kronecker-Eisenstein functions in amplitudes is $g_{i j}^{(1)}$, and we see that setting $p=1$ in the above gives equivalence. When $n=6$, these coincide in that $n-2=4$. When $n=7$ and $p>1$, they differ when $j=5$. However, this disagreement will not matter. At 7 points a term $g_{15}^{(2+)}$ is multiplied by at most one other $g_{i j}^{(q)}$ function, but we need at least two KE terms in order to make the corresponding $P$ function non-zero. That is, for example,

$$
\begin{align*}
g_{15}^{(2)} g_{56}^{(1)} \Rightarrow P(15,56) & =\phi_{156 \mid 5671} I_{156}=0  \tag{10.2.41}\\
g_{15}^{(2)} g_{56}^{(1)} g_{67}^{(1)} \Rightarrow P(15,56,67) & =\phi_{1567 \mid 5671} I_{5671} \neq 0 \tag{10.2.42}
\end{align*}
$$

At 8 points, this will of course become an issue. However, the description of the dist function was chosen purely for simplicity. If we instead think of this function as asking whether the pole being approached crosses the boundary between particles $n$ and 1 , then consistency should be maintained to higher points.

### 10.3 One Loop SYM Field Theory Integrands

We now construct one loop amplitudes in field theory using these rules and the string correlators. We represent an $n$-point one loop field theory amplitude by

$$
\begin{equation*}
A\left(i_{1} i_{2} \ldots i_{n} ; \ell+a^{j} k_{j}\right)=\sum_{p=4}^{n} \sum_{A_{1} \ldots A_{p+1}=i_{2} \ldots i_{n}} \mathcal{N}_{A_{p+1} i_{1} A_{1} \mid A_{2}, \ldots, A_{p}}^{a_{1}, a_{2}, \ldots, a_{n}}(\ell) I_{i_{1} A_{1}, A_{2}, \ldots, A_{p}}^{a_{1}, a_{2}, \ldots, a_{n}} \tag{10.3.1}
\end{equation*}
$$

where $\mathcal{N}_{A_{1} \mid A_{2}, \ldots, A_{p}}^{a_{1}, a_{p}, \ldots, a_{n}}(\ell)$ denotes the kinematic Berends-Giele numerator of a $p$-gon. Note for such, and in this part of this thesis, it will be convenient to redefine the $b$-map (2.3.13) with an additional $\frac{1}{2}$ factor. That is,

$$
\begin{equation*}
b(i)=i, \quad b(P)=\frac{1}{2 s_{P}} \sum_{X Y=P}[b(X), b(Y)] . \tag{10.3.2}
\end{equation*}
$$

So to give a few examples of this notation,

$$
\begin{gather*}
\mathcal{N}_{1 \mid 2,3,4,5}(\ell)=N_{1 \mid 2,3,4,5}(\ell), \quad \mathcal{N}_{12 \mid 3,4,5}(\ell)=\frac{1}{2 s_{12}} N_{[1,2] \mid 3,4,5}(\ell), \\
\mathcal{N}_{12 \mid 34,56,7,8,9}(\ell)=\frac{1}{8 s_{12} s_{34} s_{56}} N_{[1,2][[3,4],[5,6], 7,8,9}(\ell),  \tag{10.3.3}\\
\mathcal{N}_{123 \mid 4,56,7}(\ell)=\frac{1}{8 s_{12} s_{123} s_{56}} N_{[[1,2], 3] \mid 4,[5,6], 7,8,9}(\ell)+\frac{1}{8 s_{23} s_{123} s_{56}} N_{[1,[2,3]] \mid 4,[5,6], 7,8,9}(\ell) .
\end{gather*}
$$

The $I_{A_{1}, A_{2}, \ldots, A_{p}}^{a_{1}, a_{2}, \ldots, a_{n}}$ in (10.3.1) represents the $p$-gon integrand, as described in the appendix A.2.1. We now discuss amplitudes generated using the field theory limit rules we have outlined.

### 10.3.1 Four Points

At four points, there are no terms for which the field theory limit rules are needed [39]. There is only the Koba-Nielsen factor, which gives us the $I_{1,2,3,4}$ factor in the limit,

$$
\begin{equation*}
A\left(\sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4} \mid \ell+a_{1} k_{\sigma_{1}}+\ldots+a_{4} k_{\sigma_{4}}\right)=V_{1} T_{2,3,4} I_{1,2,3,4} . \tag{10.3.4}
\end{equation*}
$$

### 10.3.2 Five Points

The five-point genus-one superstring correlator is given by [22]

$$
\begin{align*}
\mathcal{K}_{5}(\ell) & =V_{1} T_{2,3,4,5}^{m} \mathcal{Z}_{1,2,3,4,5}^{m}+\left[V_{12} T_{3,4,5} \mathcal{Z}_{12,3,4,5}+(2 \leftrightarrow 3,4,5)\right]  \tag{10.3.5}\\
& +\left[V_{1} T_{23,4,5} \mathcal{Z}_{23,1,4,5}+(2,3 \mid 2,3,4,5)\right],
\end{align*}
$$

with the worldsheet functions [21]

$$
\begin{equation*}
\mathcal{Z}_{1,2,3,4,5}^{m}=\ell^{m}, \quad \mathcal{Z}_{12,3,4,5}=g_{12}^{(1)} . \tag{10.3.6}
\end{equation*}
$$

This correlator contains five terms with non-vanishing poles in the canonical colour ordering, namely $g_{12}^{(1)}, g_{23}^{(1)}, g_{34}^{(1)}, g_{45}^{(1)}$, and $g_{51}^{(1)}$. These have limits of the form

$$
\begin{equation*}
g_{12}^{(1)} \rightarrow \frac{1}{2} I_{1,2,3,4,5}+\frac{1}{2} I_{12,3,4,5}, \tag{10.3.7}
\end{equation*}
$$

while the other KE terms will only contribute to the pentagon due to their lack of poles in the integration domain, with say

$$
\begin{equation*}
g_{14}^{(1)} \rightarrow \frac{1}{2} I_{1,2,3,4,5} . \tag{10.3.8}
\end{equation*}
$$

We represent the overall integrand of $A\left(1,2,3,4,5 ; \ell+a^{i} k_{i}\right)$ with

$$
\begin{align*}
A\left(1,2,3,4,5 ; \ell+a^{i} k_{i}\right) & =N_{1 \mid 2,3,4,5}(\ell) I_{1,2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell) \\
& +\frac{1}{2 s_{12}} N_{12 \mid 3,4,5}(\ell) I_{12,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell)+\frac{1}{2 s_{23}} N_{1 \mid 23,4,5}(\ell) I_{1,23,4,5}^{a_{1}, \ldots, a_{5}}(\ell) \\
& +\frac{1}{2 s_{34}} N_{1 \mid 2,34,5}(\ell) I_{1,2,34,5}^{a_{1}, \ldots, a_{5}}(\ell)+\frac{1}{2 s_{45}} N_{1 \mid 2,3,45}(\ell) I_{1,2,3,45}^{a_{1}, \ldots, a_{5}}(\ell)  \tag{10.3.9}\\
& +\frac{1}{2 s_{51}} N_{51 \mid 2,3,4}^{\prime}(\ell) I_{1,2,3,4}^{a_{1}, \ldots, a_{5}}(\ell) .
\end{align*}
$$

Note the 51-box numerator is denoted $N_{51 \mid 2,3,4}^{\prime}(\ell)$, different to the other pentagons. The reason for this is not clear at five points, but at higher points we shall see that terms of this form receive extra contributions when the field theory limits are taken and so it will be convenient to notate them differently.

Applying the field-theory limit rules (10.2.5) to the correlator and comparing the outcome with (10.3.9), we can read off the box numerators. They are independent of the loop momentum and are uniformly described by

$$
\begin{equation*}
N_{A \mid B, C, D}=V_{A} T_{B, C, D} \tag{10.3.10}
\end{equation*}
$$

So for example,

$$
\begin{equation*}
N_{1 \mid 23,4,5}^{\prime}=V_{1} T_{23,4,5}, \quad N_{51 \mid 2,3,4}^{\prime}=V_{51} T_{2,3,4} \tag{10.3.11}
\end{equation*}
$$

This result agrees with the work in [1].

The pentagon $I_{1,2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell)$ arising from the limit rules (10.2.5) is

$$
\begin{align*}
N_{1 \mid 2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell)= & V_{1} T_{2,3,4,5}^{m} \ell^{m}+\left[V_{12} T_{3,4,5}\left(a_{2}-a_{1}+\frac{1}{2}\right)+(2 \leftrightarrow 3,4,5)\right]  \tag{10.3.12}\\
& +\left[V_{1} T_{23,4,5}\left(a_{3}-a_{2}+\frac{1}{2}\right)+(2,3 \mid 2,3,4,5)\right]
\end{align*}
$$

Note that in the $a_{i}=0 \forall i$ case this reduces to the results of [1] also. A straightforward but tedious calculation shows that [148; 149; 150]

$$
\begin{align*}
Q N_{1 \mid 2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell) & =\frac{1}{2} V_{1} V_{2} T_{3,4,5}\left(\left(\ell+f_{a_{1} \ldots a_{5}}-k_{12}\right)^{2}-\left(\ell+f_{a_{1} \ldots a_{5}}-k_{1}\right)^{2}\right) \\
& +\frac{1}{2} V_{1} V_{3} T_{2,4,5}\left(\left(\ell+f_{a_{1} \ldots a_{5}}-k_{123}\right)^{2}-\left(\ell+f_{a_{1} \ldots a_{5}}-k_{12}\right)^{2}\right) \\
& +\frac{1}{2} V_{1} V_{4} T_{2,3,5}\left(\left(\ell+f_{a_{1} \ldots a_{5}}-k_{1234}\right)^{2}-\left(\ell+f_{a_{1} \ldots a_{5}}-k_{123}\right)^{2}\right)  \tag{10.3.13}\\
& +\frac{1}{2} V_{1} V_{5} T_{2,3,4}\left(\left(\ell+f_{a_{1} \ldots a_{5}}-k_{12345}\right)^{2}-\left(\ell+f_{a_{1} \ldots a_{5}}-k_{1234}\right)^{2}\right)
\end{align*}
$$

with the $f_{a_{1} \ldots a_{5}}$ defined as in (A.2.13). It is then not hard to check that the above cancels the BRST variation of the box terms. For example, the terms proportional to $\left(\ell+f_{a_{1} \ldots a_{5}}-k_{123}\right)^{2}$ are given by

$$
\begin{equation*}
\frac{1}{2}\left(V_{1} V_{3} T_{2,4,5}-V_{1} V_{4} T_{2,3,5}\right)=-\frac{1}{2 s_{34}} Q V_{1} T_{2,34,5} \tag{10.3.14}
\end{equation*}
$$

and cancel the BRST variation of the 34 -box in (10.3.9) since

$$
\begin{equation*}
\left(\ell+f_{a_{1} \ldots a_{5}}-k_{123}\right)^{2} I_{1,2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell)=I_{1,2,34,5}^{a_{1}, \ldots, a_{5}}(\ell) \tag{10.3.15}
\end{equation*}
$$

Similar calculations show that $Q N_{1 \mid 2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell) I_{1,2,3,4,5}^{a_{1}, \ldots, a_{5}}=-Q A_{\text {box }}(1,2,3,4,5)$ and therefore the five-point SYM integrand (10.3.9) is BRST invariant.

A result which will prove important later is that if we take the five point pentagon and shift the loop momentum, the result is equivalent within pure spinor superspace to the pentagon found using the field theory limits with that loop momentum assignment. That
is,

$$
\begin{equation*}
\left\langle N_{1 \mid 2,3,4,5}\left(\ell+a^{i} k_{i}\right)\right\rangle=\left\langle N_{1 \mid 2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell)\right\rangle \tag{10.3.16}
\end{equation*}
$$

where $N_{1 \mid 2,3,4,5}(\ell)$ is given by (10.3.12) and $I_{1,2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell)=I_{1,2,3,4,5}\left(\ell+a^{i} k_{i}\right)$. Showing such relies upon the BRST cohomology identities [130]

$$
\begin{align*}
\left\langle V_{1} k_{m}^{1} T_{2,3,4,5}^{m}\right\rangle & =\left\langle-V_{12} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right\rangle  \tag{10.3.17}\\
\left\langle V_{1} k_{m}^{2} T_{2,3,4,5}^{m}\right\rangle & =\left\langle V_{12} T_{3,4,5}+\left[-V_{1} T_{23,4,5}+(3 \leftrightarrow 4,5)\right]\right\rangle \tag{10.3.18}
\end{align*}
$$

A file containing explicit formulae for all colour ordered permutations of the five-point SYM integrand is available to download from [28].

### 10.3.3 Six points

The six-point genus-one superstring correlator is given by [22]

$$
\begin{align*}
\mathcal{K}_{6}(\ell)= & \frac{1}{2} V_{A_{1}} T_{A_{2}, \ldots, A_{6}}^{m n} \mathcal{Z}_{A_{1}, \ldots, A_{6}}^{m n}+\left[123456 \mid A_{1}, \ldots, A_{6}\right] \\
& +V_{A_{1}} T_{A_{2}, \ldots, A_{5}}^{m} \mathcal{Z}_{A_{1}, \ldots, A_{5}}^{m}+\left[123456 \mid A_{1}, \ldots, A_{5}\right]  \tag{10.3.19}\\
& +V_{A_{1}} T_{A_{2}, \ldots, A_{4}} \mathcal{Z}_{A_{1}, \ldots, A_{4}}+\left[123456 \mid A_{1}, \ldots, A_{4}\right]
\end{align*}
$$

The worldsheet functions have grown in complexity slightly, and are now given by [21],

$$
\begin{align*}
& \mathcal{Z}_{123,4,5,6}=g_{12}^{(1)} g_{23}^{(1)}+g_{12}^{(2)}+g_{23}^{(2)}-g_{13}^{(2)}  \tag{10.3.20}\\
& \mathcal{Z}_{12,34,5,6}=g_{12}^{(1)} g_{34}^{(1)}+g_{13}^{(2)}+g_{24}^{(2)}-g_{14}^{(2)}-g_{23}^{(2)}  \tag{10.3.21}\\
& \mathcal{Z}_{12,3,4,5,6}^{m}=\ell^{m} g_{12}^{(1)}+\left(k_{2}^{m}-k_{1}^{m}\right) g_{12}^{(2)}+\left[k_{3}^{m}\left(g_{13}^{(2)}-g_{23}^{(2)}\right)+(3 \leftrightarrow 4,5,6)\right]  \tag{10.3.22}\\
& \mathcal{Z}_{1,2,3,4,5,6}^{m n}=\ell^{m} \ell^{n}+\left[\left(k_{1}^{m} k_{2}^{n}+k_{1}^{n} k_{2}^{m}\right) g_{12}^{(2)}+(1,2 \mid 1,2,3,4,5,6)\right] \tag{10.3.23}
\end{align*}
$$

A fully general expression for the amplitude at six points is significantly more complex than that at five points. As such, here we detail the construction of a single amplitude instead, $A(2,3,4,5,6,1 ; \ell)=A\left(1,2,3,4,5,6 ; \ell+k_{1}\right)$. We then reserve the fully general amplitude for the appendix $\mathrm{G}^{4}$.

The field theory limit rules we require are (10.2.5) and (10.2.6). Applied to find this amplitude, they become

$$
\begin{align*}
g_{i j}^{(1)} & \rightarrow \frac{1}{2} \operatorname{sgn}_{i j}^{234561} I^{234561}+\frac{1}{2} \phi_{i j \mid \operatorname{Ord}_{234561}(i j)} I_{i j}^{234561},  \tag{10.3.24}\\
g_{i j}^{(2)} & \rightarrow \frac{1}{12} I^{234561}+\frac{1}{2 s_{12}}\left(-\delta_{1 i} \delta_{2 j}+\delta_{1 j} \delta_{2 i}\right) I_{12}^{234561}, \tag{10.3.25}
\end{align*}
$$

[^16]\[

$$
\begin{align*}
g_{i j}^{(1)} g_{k l}^{(1)} & \rightarrow \frac{1}{4} \operatorname{sgn}_{i j}^{234561} \operatorname{sgn}_{k l}^{234561} I^{234561}+\frac{1}{4} \operatorname{sgn}_{k l}^{234561} \phi_{i j \mid \operatorname{Ord}_{234561}(i j)} I_{i j}^{234561} \\
& +\frac{1}{4} \operatorname{sgn}_{i j}^{234561} \phi_{k l \mid \operatorname{Ord}_{234561}(k l)} I_{k l}^{234561}+\frac{1}{4} P(i j, k l), \tag{10.3.26}
\end{align*}
$$
\]

where the double pole function is given by

$$
P(i j, k l)= \begin{cases}\phi_{i j l \mid O r d_{234561}(i j l)} I_{i j l}^{234561} & \text { if } j=k  \tag{10.3.27}\\ -\phi_{i j k \mid \operatorname{Ord}_{234561}(i j k)} I_{i j k}^{234561} & \text { if } j=l \\ -\phi_{j i l \mid O r d_{234561}(i j l)} I_{j i l}^{234561} & \text { if } i=k \\ \phi_{k i j \mid O r d_{234561}(k i j)} I_{k i j}^{234561} & \text { if } i=l \\ \phi_{i j \mid O r d_{234561}(i j)} \phi_{k l \mid O r d_{234561}(k l)} I_{i j, k l}^{234561} & \text { else }\end{cases}
$$

So for instance, the term $V_{1} T_{24,3,5,6}^{m} \ell^{m} g_{24}^{(1)}$ would have field theory limit

$$
\begin{align*}
V_{1} T_{24,3,5,6}^{m} \ell^{m} g_{24}^{(1)} & \rightarrow V_{1} T_{24,3,5,6}^{m} \ell^{m} \frac{1}{2}\left(\operatorname{sgn}_{24}^{234561} I^{234561}+\frac{1}{2} \phi_{24 \mid \operatorname{Ord}_{234561}(24)} I_{24}^{234561}\right) \\
& =V_{1} T_{24,3,5,6}^{m} \ell^{m} \frac{1}{2}\left((+1) I^{234561}+\frac{1}{2} \phi_{24 \mid 234} I_{24}^{234561}\right)  \tag{10.3.28}\\
& =\frac{1}{2} V_{1} T_{24,3,5,6}^{m} \ell^{m} I^{234561}
\end{align*}
$$

with the last line following due to the vanishing of $\phi_{23 \mid 234}$. Another more complex example would be $V_{1} T_{243,5,6} g_{24}^{(1)} g_{43}^{(1)}$, which contains many terms including double poles. The field theory limit here would be given by

$$
\begin{align*}
V_{1} T_{243,5,6} g_{24}^{(1)} g_{43}^{(1)} \rightarrow & V_{1} T_{243,5,6}\left(\frac{1}{4} \operatorname{sgn}_{24}^{234561} \operatorname{sgn}_{43}^{234561} I^{234561}+\frac{1}{4} \operatorname{sgn}_{43}^{234561} \phi_{24 \mid \operatorname{Ord}_{234561}(24)} I_{24}^{234561}\right. \\
& \left.+\frac{1}{4} \operatorname{sgn}_{24}^{234561} \phi_{43 \mid \operatorname{Ord}_{234561}(43)} I_{43}^{234561}+\frac{1}{4} P(24,43)\right) \quad(10.3 .29)  \tag{10.3.29}\\
= & \frac{1}{4} V_{1} T_{243,5,6}\left(-I^{234561}-\frac{1}{4} \phi_{24 \mid 234} I_{24}^{234561}+\phi_{43 \mid 34} I_{34}^{234561}+\phi_{243 \mid \operatorname{Ord}_{234561}(243)} I_{243}^{234561}\right) \\
= & \frac{1}{4} V_{1} T_{243,5,6}\left(-I^{234561}-\frac{1}{s_{34}} I_{34}^{234561}-\frac{1}{s_{34} s_{234}} I_{234}^{234561}\right)
\end{align*}
$$

Performing similar calculations for all terms in the correlator, and then extracting the
terms proportional to $I^{234561}=I_{2,3,4,5,6,1}$, we find the hexagon numerator

$$
\begin{align*}
N_{2 \mid 3,4,5,6,1}(\ell) & =\frac{1}{2} V_{1} T_{2,3,4,5,6}^{m n}\left(\ell^{m} \ell^{n}-\frac{1}{12}\left[k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4,5,6)\right]\right) \\
& +\frac{1}{2}\left(V_{1} T_{23,4,5,6}^{m}\left(\ell^{m}-\frac{1}{6} k_{2}^{m}+\frac{1}{6} k_{3}^{m}\right)+(2,3 \mid 2,3,4,5,6)\right) \\
& -\frac{1}{2}\left(V_{12} T_{3,4,5,6}^{m}\left(\ell^{m}+\frac{1}{6} k_{1}^{m}-\frac{1}{6} k_{2}^{m}\right)+(2 \leftrightarrow 3,4,5,6)\right) \\
& +\frac{1}{6} V_{1}\left(T_{[[2,3], 4], 5,6}+T_{[2,[3,4], 5,6}+(2,3,4 \mid 2,3,4,5,6)\right)  \tag{10.3.30}\\
& +\frac{1}{4}\left(V_{1} T_{23,45,6}+(2,3|4,5| 2,3,4,5,6)\right) \\
& -\frac{1}{4}\left(V_{12} T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right) \\
& -\frac{1}{6}\left(\left(V_{123}-2 V_{132}\right) T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right)
\end{align*}
$$

So for example, the $V_{1} T_{25,46,3}$ term has coefficient $\frac{1}{4}$ as this is in the correlator associated with a term

$$
\begin{equation*}
\mathcal{Z}_{25,46,1,3}=g_{25}^{(1)} g_{46}^{(1)}+g_{24}^{(2)}+g_{56}^{(2)}-g_{26}^{(2)}-g_{54}^{(2)} \tag{10.3.31}
\end{equation*}
$$

and these each contribute $I^{234561}$ terms as

$$
\begin{equation*}
\frac{1}{4}+\frac{1}{12}+\frac{1}{12}-\frac{1}{12}-\frac{1}{12}=\frac{1}{4} \tag{10.3.32}
\end{equation*}
$$

Similarly the $V_{1} T_{243,5,6}$ terms ${ }^{5}$ have coefficient $-\frac{1}{6}$. This arises as a result of the $I^{234561}$ terms arising from

$$
\begin{equation*}
\mathcal{Z}_{243,1,5,6}=g_{24}^{(1)} g_{43}^{(1)}+g_{24}^{(2)}+g_{43}^{(2)}-g_{23}^{(2)} \tag{10.3.33}
\end{equation*}
$$

which are given by

$$
\begin{equation*}
\frac{1}{2} \cdot \frac{-1}{2}+\frac{1}{12}+\frac{1}{12}-\frac{1}{12}=-\frac{1}{6} \tag{10.3.34}
\end{equation*}
$$

Doing likewise for all terms in the correlator will reveal the hexagon numerator.

We then identify the pentagon numerators. In all but one case these are given by a generalization of the formula from [1] ,

$$
\begin{align*}
N_{A \mid B, C, D, E 1} & =V_{E 1} T_{A, B, C, D}^{m} \ell^{m}+\frac{1}{2}\left(V_{[A, E 1]} T_{B, C, D}+(A \leftrightarrow B, C, D)\right)  \tag{10.3.35}\\
& +\frac{1}{2}\left(V_{E 1} T_{[A, B], C, D}+(A, B \mid A, B, C, D)\right)
\end{align*}
$$

[^17]So for instance, the 34-pentagon is given by

$$
\begin{align*}
N_{2 \mid 34,5,6,1} & =V_{1} T_{2,34,5,6}^{m} \ell^{m}+\frac{1}{2} V_{[2,1]} T_{34,5,6}+\frac{1}{2} V_{[34,1]} T_{2,5,6}+\frac{1}{2} V_{[5,1]} T_{2,34,6} \\
& +\frac{1}{2} V_{[6,1]} T_{2,34,5}+\frac{1}{2} V_{1} T_{[2,34], 5,6}+\frac{1}{2} V_{1} T_{[2,5], 34,6}+\frac{1}{2} V_{1} T_{[2,6], 34,5}  \tag{10.3.36}\\
& +\frac{1}{2} V_{1} T_{[34,5], 2,6}+\frac{1}{2} V_{1} T_{[34,6], 2,5}+\frac{1}{2} V_{1} T_{[5,6], 23,4}
\end{align*}
$$

The exception to the above rule is the 12-pentagon, which differs as it has a contribution from the $g^{(2)}$ terms due to the colour ordering 234561. So for example, the worldsheet function $\mathcal{Z}_{1,2,3,4,5,6}^{m n}$ is given by

$$
\begin{equation*}
\mathcal{Z}_{1,2,3,4,5,6}^{m n}=\ell^{m} \ell^{n}+\left[\left(k_{1}^{m} k_{2}^{n}+k_{1}^{n} k_{2}^{m}\right) g_{12}^{(2)}+(1,2 \mid 1,2,3,4,5,6)\right] \tag{10.3.37}
\end{equation*}
$$

This can only contain pentagon terms through the $g_{i j}^{(2)}$ terms. These however have vanishing pole contributions in all instances where $i$ and $j$ are not 1 and 2 . Hence, only the 12-pentagon contains a $V_{1} T_{2,3,4,5,6}^{m n}$ term.

Collating all terms with coefficient $1 / 2 s_{12} I_{2,3,4,5,6}^{a_{1}=1}$ we find the 12-pentagon,

$$
\begin{align*}
N_{21 \mid 3,4,5,6}^{\prime}(\ell)= & -V_{1} T_{2,3,4,5,6}^{m n} k_{2}^{m} k_{1}^{n} \\
& -\left(V_{1} T_{23,4,5,6}^{m} k_{1}^{m}+(3 \leftrightarrow 4,5,6)\right) \\
& +V_{12} T_{3,4,5,6}^{m}\left(\ell^{m}+k_{1}^{m}-k_{m}^{2}\right) \\
& -\left(V_{13} T_{2,4,5,6}^{m} k_{2}^{m}+(3 \leftrightarrow 4,5,6)\right)  \tag{10.3.38}\\
& +\frac{1}{2}\left(V_{12} T_{34,5,6}+(3,4 \mid 3,4,5,6)\right) \\
& -\left(V_{13} T_{24,5,6}+(3|4| 3,4,5,6)\right) \\
& +\frac{1}{2}\left(\left(2 V_{132}-V_{123}\right) T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right)
\end{align*}
$$

The box numerators have the standard form, with the word containing the label 1 assigned to the $V$ superfield, and the other blocks of indices assigned to the $T$

$$
\begin{equation*}
N_{A \mid B, C, D 1 E}=V_{D 1 E} T_{A, B, C}, \quad N_{E 1 A \mid B, C, D}=V_{E 1 A} T_{B, C, D} \tag{10.3.39}
\end{equation*}
$$

A long calculation shows that the BRST variation of the above integrand is purely anomalous and given by ${ }^{6}$ [148; 149; 150]

$$
\begin{equation*}
Q A^{a_{1}=1}(1,2,3,4,5,6)=\frac{1}{2} V_{1} Y_{2,3,4,5,6}\left(I_{2,3,4,5,6}-\ell^{2} I_{2,3,4,5,6,1}\right) \tag{10.3.40}
\end{equation*}
$$

This is then of a similar form to the $a_{1}=\ldots=a_{6}=0$ result found in [1], and by an

[^18]analogous argument to the one presented there one finds the same result for the integrated anomaly
\[

$$
\begin{equation*}
\int d^{10} \ell Q A^{a_{1}=1}(1,2,3,4,5,6)=-\frac{\pi^{5}}{240} V_{1} Y_{2,3,4,5,6} . \tag{10.3.41}
\end{equation*}
$$

\]

It is a known result that type I superstring theory with gauge group $S O(32)$ does not contain gauge anomalies [128; 133]. This property does not survive the field-theory limit of its planar sector however, and the six-point one-loop SYM amplitude in ten dimensions is anomalous $[161 ; 162]$. The result (10.3.41), written in terms of the anomalous building block $Y_{2,3,4,5,6}$ [130], is the pure spinor superspace encoding of the field-theory anomaly [109; 131].

### 10.3.4 Seven Points

At seven points, the majority of the numerators become far too complex to state here. One example can be found in the appendix H. However, we may demonstrate the methods in specific instances, and here discuss how the box terms in the canonical amplitude $A(1,2,3,4,5,6,7 ; \ell)$ proportional to $I_{1234}$ are found.

Following similar results at lower points, we would expect these to be given by the BerendsGiele current

$$
\begin{equation*}
\mathcal{M}_{1234} T_{3,4,5} I_{1234}, \tag{10.3.42}
\end{equation*}
$$

that is,

$$
\begin{align*}
\left(\frac{V_{[[1,2], 3], 4]}}{8 s_{12} s_{123} s_{1234}}+\frac{V_{[11,[2,3], 4]}}{8 s_{23} s_{123} s_{1234}}\right. & +\frac{V_{[1,2],[3,4]]}}{8 s_{12} s_{34} s_{1234}} \\
& \left.+\frac{V_{[1,[2,3,4], 4]}}{8 s_{23} s_{234} s_{1234}}+\frac{V_{[1,,[2,[3,4]]]}}{8 s_{34} s_{234} s_{1234}}\right) T_{4,5,6} I_{1234} \tag{10.3.43}
\end{align*}
$$

These boxes will be the terms proportional to $\frac{1}{8} I_{1234}$ in the field theory limit of the correlator. Such a propagator structure arises only in the terms

$$
\begin{equation*}
V_{1234} T_{5,6,7} \mathcal{Z}_{1234,5,6,7}+\operatorname{Perm}(2,3,4) . \tag{10.3.44}
\end{equation*}
$$

There is only one term in each of these worldsheet functions which can contain triple poles, $g_{12}^{(1)} g_{23}^{(1)} g_{34}^{(1)}$ and its permutations in 2,3,4. By (10.2.7), the relevant piece of the field theory limits of these are

$$
\begin{align*}
c_{12}^{(1)} c_{23}^{(1)} c_{34}^{(1)} P(12,23,34) & =\frac{1}{8} \hat{\phi}(1234567 \mid 1234) I_{1234}  \tag{10.3.45}\\
& =\frac{1}{8} \phi_{1234 \mid 1234} I_{1234}
\end{align*}
$$

and its permutations in 2, 3, 4 in the right hand side of the $\phi$ and $\hat{\phi}$ functions. Note we
applied (10.2.13), (10.2.11), and (10.2.4) in the above. The relevant $\phi_{A \mid B}$ functions may then be found using (2.3.9), and are given by

$$
\begin{gather*}
s_{1234} \phi_{1234 \mid 1234}=\frac{1}{s_{12} s_{123}}+\frac{1}{s_{23} s_{123}}+\frac{1}{s_{23} s_{234}}+\frac{1}{s_{34} s_{234}}+\frac{1}{s_{12} s_{34}}, \\
s_{1234} \phi_{1234 \mid 1243}=-\frac{1}{s_{34} s_{234}}-\frac{1}{s_{12} s_{34}}, \quad s_{1234} \phi_{1234 \mid 1342}=-\frac{1}{s_{34} s_{234}},  \tag{10.3.46}\\
s_{1234} \phi_{1234 \mid 1324}=-\frac{1}{s_{23} s_{123}}-\frac{1}{s_{23} s_{234}}, \quad s_{1234} \phi_{1234 \mid 1423}=-\frac{1}{s_{23} s_{234}}, \\
s_{1234} \phi_{1234 \mid 1432}=\frac{1}{s_{23} s_{234}}+\frac{1}{s_{34} s_{234}} .
\end{gather*}
$$

We then collate the terms with the same poles. For instance, those with an $s_{34} s_{234} s_{1234}$ pole term

$$
\begin{equation*}
V_{1234} T_{5,6,7}-V_{1243} T_{5,6,7}-V_{1342} T_{5,6,7}+V_{1432} T_{5,6,7}=V_{[1,[2,[3,4]]]} T_{5,6,7}, \tag{10.3.47}
\end{equation*}
$$

Doing similar for the other pole structures in (10.3.46) thus reveals the expected form of these box terms, equation (10.3.43).

There is one additional complication with higher order $n$-gon diagrams at seven points, regarding the refined superfields and worldsheet functions. As was discussed in [22] and in the review, the refined worldsheet functions are given by

$$
\begin{equation*}
\mathcal{Z}_{12 \mid 3,4,5,6,7}=\partial g_{12}^{(2)}+s_{12} g_{12}^{(1)} g_{12}^{(2)}-3 s_{12} g_{12}^{(3)} \tag{10.3.48}
\end{equation*}
$$

The derivative and the double pole are then removed by using partial integration with the Koba-Nielsen factor, $\mathcal{I}_{7}(\ell)$ as defined in (3.2.26)

$$
\begin{align*}
\left(\partial_{1} g_{12}^{(2)}\right) \mathcal{I}_{7}(\ell) & =\partial_{1}\left(g_{12}^{(2)} \mathcal{I}_{7}(\ell)\right)+g_{12}^{(2)} \partial_{2} \mathcal{I}_{7}(\ell) \\
& =\partial_{1}\left(g_{12}^{(2)} \mathcal{I}_{7}(\ell)\right)+g_{12}^{(2)}\left(\left(\ell \cdot k_{2}\right)+s_{21} g_{21}^{(1)}+s_{23} g_{23}^{(1)}+\ldots+s_{27} g_{27}^{(1)}\right) \mathcal{I}_{7}(\ell) . \tag{10.3.49}
\end{align*}
$$

Note the subscript on the partial derivative changes to account for the missing minus sign. This we then insert into (10.3.48), and identify its alternative formulation

$$
\begin{equation*}
\mathcal{Z}_{12 \mid 3,4,5,6,7}=-3 s_{12} g_{12}^{(3)}+g_{12}^{(2)}\left(\ell \cdot k_{2}+s_{23} g_{23}^{(1)}+s_{24} g_{24}^{(1)}+\ldots+s_{27} g_{27}^{(1)}\right) \tag{10.3.50}
\end{equation*}
$$

This is the form of the refined worldsheet function we use to take field theory limits and extract numerators. The resulting expressions have been verified to have vanishing BRST variation, and so we can be assured of the validity of this method. However, additional complications arise as a result of this regarding BCJ relations, which will be detailed in the following chapter.

### 10.3.5 Higher Points

We anticipate that the field theory limit rules for an arbitrary product of $g_{i j}^{(n)}$ functions should generalize in the natural way,

$$
\begin{equation*}
\left.\prod_{a=1}^{n} g_{i_{a} j_{a}}^{\left(p_{a}\right)} \rightarrow \sum_{A \in \mathcal{P}(12 \ldots n)}\left(\left(\prod_{a \in A} b_{i_{a} j_{a}}^{\left(p_{a}\right)}\right)\left(\prod_{b \in A^{c}} c_{i_{b} j_{b}}^{\left(p_{b}\right)}\right) P\left(i_{B_{1}} j_{B_{1}}, \ldots, i_{B_{|B|}} j_{B_{|B|}}\right)\right)\right) \tag{10.3.51}
\end{equation*}
$$

where $\mathcal{P}(12 \ldots n)$ denotes the power set of $12 \ldots n$, $A$ is an element of this, and $A^{c}$ its complement. We stress that the indices of the $c^{(p)}$ and those in the $P$ function are identical.

The general $P$ functions will be as in (10.2.8), with $P\left(i_{1} j_{1}, \ldots, i_{n} j_{n}\right)$ chaining together $i_{m} j_{m}$ pairs as much as possible, and then using these as indices for $\phi$ and $I$ functions. So for instance, for an amplitude $A(1,2, \ldots, n)$ we would expect

$$
\begin{align*}
P(12,23,34,45,56,67) & \leftrightarrow \hat{\phi}(\sigma \mid 1234567) I_{1234567},  \tag{10.3.52}\\
P(15,32,56,24) & \leftrightarrow \hat{\phi}(\sigma \mid 156) \hat{\phi}(\sigma \mid 324) I_{156,324} . \tag{10.3.53}
\end{align*}
$$

As for the limits of $b^{(p)}$ and $c^{(p)}$ at higher points, we expect these will generalise from (10.2.5) in the natural way.

We may provide strong evidence in favour of this with the use of Fay identities [138], discussed in section 5.3.1. One such relation is

$$
\begin{equation*}
g_{12}^{(n)} g_{23}^{(1)}=-g_{13}^{(n+1)}+g_{13}^{(1)} g_{12}^{(n)}-n g_{12}^{(n+1)}+\sum_{j=0}^{n}(-1)^{j} g_{13}^{(n-j)} g_{23}^{(1+j)} \tag{10.3.54}
\end{equation*}
$$

We begin by looking at $b^{(n)}$, and restrict ourselves to the case $a_{i}=0 \forall i$. In these circumstances we know that $b_{i j}^{(1)}=\frac{1}{2} \operatorname{sgn} n_{i j}^{12 \ldots n}$, and we would expect that for any $n, b_{i j}^{(n)}$ should be a function only of the relative ordering of $i$ and $j$ with respect to the colour ordering. Hence, we substitute into (10.3.54) the values

$$
\begin{equation*}
g_{13}^{(1)}, g_{23}^{(1)} \rightarrow \frac{1}{2}, \quad g_{12}^{(n)}, g_{13}^{(n)}, g_{23}^{(n)} \rightarrow b^{(n)} \tag{10.3.55}
\end{equation*}
$$

We then rearrange, and arrive at the recursion relation

$$
\begin{equation*}
b^{(n+1)}=-\frac{1}{n+1-(-1)^{n}} \sum_{j=1}^{n}(-1)^{j} b^{(n-j+1)} b^{(j)} \tag{10.3.56}
\end{equation*}
$$

This vanishes for $n$ even, $n>0$, due to the symmetry in the $g g$ terms and the antisymmetry
of the $(-1)^{j}$. For $n$ odd, it simplifies to

$$
\begin{equation*}
b^{(2 n)}=-\frac{1}{2 n+1} \sum_{j=1}^{2 n-1}(-1)^{j} b^{(2 n-j)} b^{(j)}=-\frac{1}{2 n+1} \sum_{j=1}^{n-1} b^{(2 n-2 j)} b^{(2 j)} \tag{10.3.57}
\end{equation*}
$$

where the second equality follows from the vanishing of the $b$ with odd indices. It may then be proved by induction that this is solved by

$$
\begin{equation*}
b^{(n)}=\frac{B_{n}}{n!} \tag{10.3.58}
\end{equation*}
$$

where $B_{n}$ is the $n^{t h}$ Bernoulli number. Showing this requires an identity due to Euler [163],

$$
\begin{equation*}
\sum_{k=1}^{n-1}\binom{2 n}{2 k} B_{2 k} B_{2 n-2 k}=-(2 n+1) B_{2 n}, \quad n \geq 2 \tag{10.3.59}
\end{equation*}
$$

Hence, we speculate that when $a_{i}=0 \forall i$, the field theory limit of a general term from the Kronecker-Eisenstein series away from poles is given by (10.3.58). The first few (non-zero) values are

$$
\begin{align*}
b^{(0)} & =1, \quad b^{(1)}=\frac{1}{2}, \quad b^{(2)}=\frac{1}{12}, \quad b^{(4)}=-\frac{1}{720}, \quad b^{(6)}=\frac{1}{30240} \\
b^{(8)} & =-\frac{1}{1209600}, \quad b^{(10)}=\frac{1}{47900160}, \quad b^{(12)}=-\frac{691}{1307674368000} \tag{10.3.60}
\end{align*}
$$

We can then extend this to the general $a_{i}$ case, though with less elegance. If we substitute the general $a_{i}$ expressions for the $b^{(1)}$ terms into (10.3.54), rather than the simplified values in (10.3.55), we find the relation

$$
\begin{align*}
\left(\frac{1}{2}+a_{3}-a_{2}\right) b_{12}^{(n)}=-b_{13}^{(n+1)} & +\left(\frac{1}{2}+a_{3}-a_{1}\right) b_{12}^{(n)}-n b_{12}^{(n+1)} \\
& +\left(\frac{1}{2}+a_{3}-a_{2}\right) b_{13}^{(n)}+\sum_{j=1}^{n}(-1)^{j} b_{13}^{(n-j)} b_{23}^{(1+j)} \tag{10.3.61}
\end{align*}
$$

This cannot be as easily rearranged into a recursion relation unfortunately. However, we may assume that $b_{i j}^{(n)}$ is an order $n$ polynomial in $a_{j}-a_{i}$ with unknown coefficients, and use the above to identify these unknowns. This then reveals the value of $b_{i j}^{(4)}$ as would be expected from (10.2.5) is the unique solution. We have then also verified that, if we assume (10.2.5) is the general value of $b_{i j}^{(n)}$, the relation (10.3.61) is satisfied in many further instances.

We can perform a similar exercise for the $c_{i j}^{(n)}$ pole terms. Rather than using (10.3.54), we consider an alternative Fay identity in order to have non-zero dist functions. Supposing
the amplitude we are looking at is $A(1,2, \ldots, m)$ for convenience, consider

$$
\begin{equation*}
g_{1 m}^{(n)} g_{m(m-1)}^{(1)}=-g_{1(m-1)}^{(n+1)}+g_{1(m-1)}^{(1)} g_{1 m}^{(n)}-n g_{1 m}^{(n+1)}+\sum_{j=0}^{n}(-1)^{j} g_{1(m-1)}^{(n-j)} g_{m(m-1)}^{(1+j)} \tag{10.3.62}
\end{equation*}
$$

We need not restrict ourselves to the $a_{i}=0 \forall i$ case here, as the computation is simpler. We look to the $s_{1 m}$ single poles in this relation and their associated $c_{1 m}^{(n)}$ factors, and find the relation

$$
\begin{align*}
& c_{1 m}^{(n)}\left(-\frac{1}{2}+a_{m-1}-a_{m}\right)=\left(\frac{1}{2}+a_{m-1}-a_{1}\right) c_{1 m}^{(n)}-n c_{1 m}^{(n+1)}  \tag{10.3.63}\\
\Rightarrow & c_{1 m}^{(n+1)}=\frac{1}{n} c_{1 m}^{(n)}\left(1+a_{m}-a_{1}\right) \tag{10.3.64}
\end{align*}
$$

This has the form of a geometric progression, and thus as we know $c_{1 m}^{(1)}=\frac{1}{2}$ we find the general expression

$$
\begin{equation*}
c_{1 m}^{(n)}=\frac{1}{2(n-1)!}\left(1+a_{m}-a_{1}\right)^{n-1} \tag{10.3.65}
\end{equation*}
$$

This agrees with the known values of $c_{17}^{(2)}$ and $c_{17}^{(3)}$. We can repeat this calculation for poles of $g_{12}^{(n)}$ to find what happens when the dist function is zero, and find the similar relation

$$
\begin{equation*}
c_{12}^{(n)}=\frac{1}{2(n-1)!}\left(a_{2}-a_{1}\right)^{n-1} \tag{10.3.66}
\end{equation*}
$$

Hence the definition (10.2.13) of $c_{i j}^{(n)}$ appears to generalise naturally at higher points.

## BCJ Identities at One Loop

In this section we demonstrate that the numerators obtained by the procedure outlined in the previous chapter satisfy BCJ relations. This resolves an open problem from [1], regarding why the numerators from such did not appear to satisfy BCJ relations at six and higher points. Our solution is to follow a similar logic to that of tree level, and not suppose that amplitudes with different color orderings have precisely the same representation. This is what results from the field theory limit procedure we have outlined, and we detail this here.

Note that for the four and five point amplitudes identified using the field theory limit procedure, the BCJ relations are trivial. At four points, due to the vanishing of triangle diagrams all such relations reduce to four-point boxes being equal to each other. However all box numerators are the same at four points; they are given by $V_{1} T_{2,3,4}$ due to the symmetry of $T$ in its indices. Hence the BCJ relations follow immediately.

At five points, there are more complex BCJ relations between pentagons and boxes. A pair of these are illustrated in figure 11.0.1, in which we keep the loop momentum as general as possible. Applying the formula (10.3.12) for the pentagons and the usual box
formula gives that these relations are satisfied. In the first case, this is

$$
\begin{align*}
& N_{1 \mid 2,3,4,5}^{a_{1}, \ldots, a_{5}}(\ell)-N_{1 \mid 3,2,4,5}^{a_{1}, \ldots, a_{5}}(\ell)-N_{1 \mid 23,4,5}^{a_{1}, \ldots, a_{5}}(\ell)= \\
& \quad+\left(V_{1} T_{2,3,4,5}^{m} \ell^{m}+\left[V_{12} T_{3,4,5}\left(a_{2}-a_{1}+\frac{1}{2}\right)+(2 \leftrightarrow 3,4,5)\right]\right. \\
& \left.\quad+\left[V_{1} T_{23,4,5}\left(a_{3}-a_{2}+\frac{1}{2}\right)+(2,3 \mid 2,3,4,5)\right]\right)  \tag{11.0.1}\\
& \quad-\left(V_{1} T_{3,2,4,5}^{m} \ell^{m}+\left[V_{13} T_{2,4,5}\left(a_{3}-a_{1}+\frac{1}{2}\right)+(3 \leftrightarrow 2,4,5)\right]\right. \\
& \left.\quad+\left[V_{1} T_{32,4,5}\left(a_{2}-a_{3}+\frac{1}{2}\right)+(3,2 \mid 3,2,4,5)\right]\right) \\
& \quad-V_{1} T_{23,4,5}=0
\end{align*}
$$

The vanishing of the above relies only upon the symmetry of $T$ in its blocks of indices, and the usual Jacobi identities satisfied by indices when we work in the BCJ gauge.

The second example in figure 11.0 .1 requires a little more work to show, as one must be careful to track the loop momentum and ensure that in the edges not involved in the BCJ relation, the momentum along them is identical across diagrams. In effect, this means that the second diagram comes from an amplitude with a different loop momentum structure. That is, while two of the diagrams in this relation come from the amplitude

$$
\begin{equation*}
A\left(1,2,3,4,5 ; \ell+\sum_{i} a_{i} k^{i}\right) \tag{11.0.2}
\end{equation*}
$$

the second does not come from the naive relabelling $A\left(1,5,2,3,4 ; \ell+\sum_{i} a_{i} k^{i}\right)$, but rather

$$
\begin{equation*}
A\left(1,5,2,3,4 ; \ell+k^{5}+\sum_{i} a_{i} k^{i}\right) \tag{11.0.3}
\end{equation*}
$$

The BCJ relation we wish to verify is thus

$$
\begin{gather*}
N_{1 \mid 2,3,4,5}^{a_{1}, a_{2}, a_{3}, a_{4}, a_{5}}(\ell)-N_{1 \mid 5,2,3,4}^{a_{1}, a_{2}, a_{3}, a_{4}, a_{5}+1}(\ell)-N_{51 \mid 2,3,4}^{a_{1}, a_{2}, a_{3}, a_{4}, a_{5}}(\ell)=  \tag{11.0.4}\\
+\left(V_{1} T_{2,3,4,5}^{m} \ell^{m}+\left[V_{12} T_{3,4,5}\left(a_{2}-a_{1}+\frac{1}{2}\right)+(2 \leftrightarrow 3,4,5)\right]\right. \\
\left.+\left[V_{1} T_{23,4,5}\left(a_{3}-a_{2}+\frac{1}{2}\right)+(2,3 \mid 2,3,4,5)\right]\right) \\
-\left(V_{1} T_{5,2,3,4}^{m} \ell^{m}+V_{15} T_{2,3,4}\left(a_{5}+1-a_{1}+\frac{1}{2}\right)\right. \\
+\left[V_{12} T_{3,4,5}\left(a_{2}-a_{1}+\frac{1}{2}\right)+(2 \leftrightarrow 3,4)\right] \\
+\left[V_{1} T_{52,3,4}\left(a_{2}-\left(a_{5}+1\right)+\frac{1}{2}\right)+(2 \leftrightarrow 3,4)\right] \\
\left.+\left[V_{1} T_{23,4,5}\left(a_{3}-a_{2}+\frac{1}{2}\right)+(2,3 \mid 2,3,4)\right]\right) \\
-V_{51} T_{2,3,4}=0 .
\end{gather*}
$$








Figure 11.0.1: A pair of BCJ relations at five points. Note that one must ensure that the loop momentum is the same on all parts of the diagrams not involved in the BCJ relation. It is simple to show that these identities are satisfied, using either the methods of this part or those of [1].

Again, this follows by the symmetries of the $V T$ superfields, and does not require any identities to show. We again note that having a BCJ representation at five points is not a new result, as one had been found previously using similar machinery in [1]. However, the property of not requiring cohomology identities to verify the BCJ relations is new to these methods, and we include the above for completeness.

### 11.1 Six points

We now demonstrate that the BCJ relations at six points are satisfied, when the relevant amplitudes are constructed using the procedures of the previous chapter. Note we intentionally focus on only those relations which are likely to fail. That is, BCJ relations within external trees in the Feynman diagrams will be satisfied already, by properties of the BCJ gauge [88; 27]. Additionally, relations in which the loop momentum is unchanged between diagrams (that is, relations of the form of the first line in 11.0.1) were already satisfied by the representation found in [1], and by similar methods it is not complex to show they are satisfied here. As such, we focus only upon those relations in which the BCJ relation relates two $n$-gons to an $(n-1)$-gon, and in which the loop momentum structure changes between diagrams (that is, relations of the form of the second line in 11.0.1).


Figure 11.1.1: This was the BCJ identity which could not be satisfied using the representation identified in [1]. As a demonstration of the usefulness of the methods developed in the previous chapter, we show that this relation is now satisfied.

### 11.1.1 Relation Between Two Pentagons and a Box

We begin by showing that the numerators constructed using the methods outlined satisfy the BCJ relation in figure 11.1.1. This example was chosen as it is the non-satisfying of this which was identified in [1]. That the numerators described here satisfy this therefore represents an improvement on the previous situation. Two graphs in this relation are drawn from amplitudes in the canonical colour ordering $A(1,2,3,4,5,6 ; \ell)$. The middle graph must be constructed so that the momentum along its edges not involved in the BCJ relation have equal momentum to the other diagrams in the relation. Therefore the middle graph must have momentum $\ell$ going from leg 6 to the 23 branch. This diagram is therefore the 23-pentagon $N_{23 \mid 1,4,5,6}(\ell)$ drawn from the amplitude

$$
\begin{equation*}
A(2,3,1,4,5,6 ; \ell) \tag{11.1.1}
\end{equation*}
$$

Note we choose this representation, rather than the 1-leading equivalent $A(1,4,5,6,2,3 ; \ell-$ $k^{23}$ ), as this simplifies the notation. However, by the results of section 10.2 .2 we could have equally chosen this instead.

This step was not used in the discussion in the review of the methods of [1], and instead it was assumed that this pentagon could be obtained instead by shifting the momentum of a relabelling of the canonical ordering, $N_{1 \mid 4,5,6,23}\left(\ell-k_{23}\right)$. Though this makes an intuitive sense, it leads to the BCJ relation being violated [1]

$$
\begin{gather*}
N_{1 \mid 23,4,5,6}(\ell)-N_{1 \mid 4,5,6,23}\left(\ell-k_{23}\right)-N_{[1,23] \mid 4,5,6}(\ell)=  \tag{11.1.2}\\
k_{m}^{23} V_{1} T_{23,4,5,6}^{m}+V_{231} T_{4,5,6}+\left[V_{1} T_{234,5,6}+(4 \leftrightarrow 5,6)\right] .
\end{gather*}
$$

The variation of this is non-vanishing, and so it is not in the cohomology of the BRST operator and is therefore non-vanishing.

Using the field-theory limit rules of the previous chapter however, the BCJ relation is
satisfied. The relation is now,

$$
\begin{equation*}
N_{1 \mid 23,4,5,6}(\ell)-N_{23 \mid 1,4,5,6}(\ell)-N_{[1,23] \mid 4,5,6}(\ell)=0 . \tag{11.1.3}
\end{equation*}
$$

To show this, we begin with the box numerator $N_{[1,23] \mid 4,5,6}(\ell)$, the coefficient of $\frac{1}{4 s_{23} s_{123}} I_{123,4,5,6}$ in the integrand $A(1,2,3,4,5,6 ; \ell)$. Following the rules (10.2.6) and (10.2.8), there are only two functions in the string correlator which can generate such;

$$
\begin{equation*}
g_{12}^{(1)} g_{23}^{(1)} \quad \text { and } \quad g_{13}^{(1)} g_{23}^{(1)}, \tag{11.1.4}
\end{equation*}
$$

owing to their constituent factors of

$$
\begin{equation*}
P(12,23)=\frac{1}{s_{12} s_{123}} I_{123}+\frac{1}{s_{23} s_{123}} I_{123} \quad \text { and } \quad P(13,23)=-\frac{1}{s_{23} s_{123}} I_{123} \tag{11.1.5}
\end{equation*}
$$

respectively. There are only two terms featuring these functions in the six-point string correlator (5.3.19),

$$
\begin{equation*}
V_{123} T_{4,5,6} g_{12}^{(1)} g_{23}^{(1)}+V_{132} T_{4,5,6} g_{13}^{(1)} g_{32}^{(1)} . \tag{11.1.6}
\end{equation*}
$$

Taking the field-theory limit we therefore find

$$
\begin{align*}
& \frac{1}{4} V_{123} T_{4,5,6} P(12,23)+\frac{1}{4} V_{132} T_{4,5,6} P(13,32)  \tag{11.1.7}\\
& \quad=\frac{1}{4} V_{123} T_{4,5,6}\left(\frac{1}{s_{12} s_{123}}+\frac{1}{s_{23} s_{123}}\right) I_{123}+\frac{1}{4} V_{132} T_{4,5,6}\left(-\frac{1}{s_{23} s_{123}}\right) I_{123} .
\end{align*}
$$

The box numerator $N_{[1,23] \mid 4,5,6}(\ell)$ is given by the coefficient of $\frac{1}{4} \frac{1}{s_{23} s_{123}} I_{123}$,

$$
\begin{equation*}
N_{[1,23] \mid 4,5,6}=V_{123} T_{4,5,6}-V_{132} T_{4,5,6}=V_{[1,23]} T_{4,5,6} \tag{11.1.8}
\end{equation*}
$$

We then find the other numerator drawn from the canonical ordering, the pentagon $N_{1 \mid 23,4,5,6}(\ell)$. This is given by the coefficient of $\frac{1}{2 s_{23}} I_{23}$ in the field theory limit of the canonically-ordered correlator $\mathcal{K}_{6}(\ell)$. These factors will arise from any appearance of $g_{23}^{(1)}$ in (5.3.19), of which there are many

$$
\begin{align*}
& V_{1} T_{23,4,5,6}^{m} \ell^{m} g_{23}^{(1)}+\left[V_{123} T_{4,5,6} g_{12}^{(1)} g_{23}^{(1)}+(2 \leftrightarrow 3)\right]+\left[V_{1} T_{234,5,6} g_{23}^{(1)} g_{34}^{(1)}+(4 \leftrightarrow 5,6)\right] \\
& +\left[V_{14} T_{23,5,6} g_{14}^{(1)} g_{23}^{(1)}+(4 \leftrightarrow 5,6)\right]+\left[V_{1} T_{23,45,6} g_{23}^{(1)} g_{45}^{(1)}+(4,5 \mid 4,5,6)\right] . \tag{11.1.9}
\end{align*}
$$

We take the limits of these, and collect terms proportional to $\frac{1}{2 s_{23}} I_{23}$. We thus arrive at the numerator

$$
\begin{align*}
N_{1 \mid 23,4,5,6}(\ell)=V_{1} T_{23,4,5,6}^{m} \ell^{m} & +\frac{1}{2}\left[V_{[1,23]} T_{4,5,6}+(23 \leftrightarrow 4,5,6)\right] \\
& +\frac{1}{2}\left[V_{1} T_{[23,4], 5,6}+(23,4 \mid 23,4,5,6)\right] . \tag{11.1.10}
\end{align*}
$$

We note that the expressions (11.1.8) and (11.1.10) agree with the numerators obtained in [1], it is only the numerators with other loop momentum structures which will differ.

The middle pentagon in figure 11.1 .1 is the 23 -pentagon in the integrand of $A(2,3,1,4,5,6 ; \ell)$, as the internal edge between leg 6 and 2 has momentum $\ell$. The calculation proceeds similarly to the above. The relevant terms are again those containing $g_{23}^{(1)}$, which are now ${ }^{1}$

$$
\begin{align*}
& V_{1} T_{4,5,6,23}^{m} \ell^{m} g_{23}^{(1)}+\left[V_{123} T_{4,5,6} g_{12}^{(1)} g_{23}^{(1)}+(2 \leftrightarrow 3)\right] \\
& \quad+\left[V_{1} T_{423,5,6} g_{42}^{(1)} g_{23}^{(1)}+V_{1} T_{432,5,6} g_{43}^{(1)} g_{32}^{(1)}+(4 \leftrightarrow 5,6)\right]  \tag{11.1.12}\\
& \quad+\left[V_{14} T_{5,6,23} g_{14}^{(1)} g_{23}^{(1)}+(4 \leftrightarrow 5,6)\right]+\frac{1}{2}\left[V_{1} T_{45,6,23} g_{45}^{(1)} g_{23}^{(1)}+(4,5 \mid 4,5,6)\right]
\end{align*}
$$

Taking the field theory limits and extracting terms proportional to $\frac{1}{2 s_{23}}$, we see that the numerator is given by

$$
\begin{align*}
N_{23 \mid 1,4,5,6}(\ell) & =V_{1} T_{4,5,6,23}^{m} \ell^{m}-\frac{1}{2} V_{[1,23]} T_{4,5,6}+\frac{1}{2}\left(V_{[1,4]} T_{5,6,23}+(4 \leftrightarrow 5,6)\right)  \tag{11.1.13}\\
& +\frac{1}{2}\left(V_{1} T_{[23,4], 5,6}+(23,4 \mid 23,4,5,6)\right)
\end{align*}
$$

This differs considerably from the parameterisation of this graph used in [1],

$$
\begin{align*}
N_{1 \mid 4,5,6,23}\left(\ell-k_{23}\right)=V_{1} T_{4,5,6,23}^{m}\left(\ell^{m}-k_{23}^{m}\right) & +\frac{1}{2}\left(V_{[1,4]} T_{5,6,23}+(4 \leftrightarrow 5,6,23)\right)  \tag{11.1.14}\\
& +\frac{1}{2}\left(V_{1} T_{[4,5], 6,23}+(4,5 \mid 4,5,6,23)\right)
\end{align*}
$$

The new representation derived here can then be seen to obey the colour-kinematics duality. To see this we plug the superfield expressions of the new field-theory representations of the box (11.1.8) and pentagons (11.1.10), (11.1.13) into the kinematic Jacobi relation (11.1.3), and obtain

$$
\begin{equation*}
N_{1 \mid 23,4,5,6}(\ell)-N_{23 \mid 1,4,5,6}(\ell)-N_{[1,23] \mid 4,5,6}(\ell)=0 . \tag{11.1.15}
\end{equation*}
$$

Verifying this is similar to that of five points, in that no BRST cohomology identities are needed to show it. This trivial vanishing for the BCJ triplet at one loop parallels the vanishing of BCJ triplets of tree-level numerators found from the field-theory of the string

[^19]Likewise is true for lower and higher points, when computing similar orderings.




Figure 11.1.2: In this subsection, we verify this BCJ relation. Note the 61 pentagon in the above is the exceptional pentagon in the amplitude; the equivalent of (10.3.38) in the example of the previous chapter.
correlators, as described in section 4.4.1.

### 11.1.2 Relation Between Two Hexagons and a Pentagon

In a given colour ordering, all of the pentagons have a similar structure apart from the $i j$-pentagon whose labels are cyclically split at the extremities $A(i, \ldots, j ; \ell)$. In this subsection we will demonstrate the validity of these expressions by verifying a BCJ relation involving such a numerator; that illustrated in figure 11.1.2. In our numerator notation, this corresponds with

$$
\begin{equation*}
N_{1 \mid 2,3,4,5,6}(\ell)-N_{1 \mid 6,2,3,4,5}^{a_{6}=1}(\ell)-N_{61 \mid 2,3,4,5}(\ell)=0 . \tag{11.1.16}
\end{equation*}
$$

To find the hexagon numerators, we look at the piece of the field theory limits proportional to $P=I$. In the first case, this means making the substitution

$$
\begin{equation*}
g_{i j}^{(1)} \rightarrow \frac{1}{2} \operatorname{sgn}_{i j}^{123456} I, \quad g_{i j}^{(1)} g_{k l}^{(1)} \rightarrow \frac{1}{4} \operatorname{sgn}_{i j}^{123456} \operatorname{sgn}_{k l}^{123456} I, \quad g_{i j}^{(2)} \rightarrow \frac{1}{12} I . \tag{11.1.17}
\end{equation*}
$$

This then gives the value of the first hexagon numerator as

$$
\begin{align*}
N_{1 \mid 2,3,4,5,6}(\ell)= & +\frac{1}{6}\left(\left(V_{[[1,2], 3]}+V_{[1,[2,3]]}\right) T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right) \\
& +\frac{1}{6} V_{1}\left(T_{[2,3], 4], 5,6}+T_{[2,[3,4]], 5,6}+(2,3,4 \mid 2,3,4,5,6)\right) \\
& \left.+\frac{1}{4} V_{[1,2]} T_{[3,4], 5,6}+(2|3,4| 2,3,4,5,6)\right) \\
& \left.+\frac{1}{4} V_{1} T_{[2,3],[4,5], 6}+(2,3|4,5| 2,3,4,5,6)\right)  \tag{11.1.18}\\
& +\frac{1}{2}\left(V_{[1,2]} T_{3,4,5,6}^{m}\left(\ell^{m}-\frac{1}{6} k_{1}^{m}+\frac{1}{6} k_{2}^{m}\right)+(2 \leftrightarrow 3,4,5,6)\right) \\
& +\frac{1}{2}\left(V_{1} T_{[2,3], 4,5,6}^{m}\left(\ell^{m}-\frac{1}{6} k_{2}^{m}+\frac{1}{6} k_{3}^{m}\right)+(2,3 \mid 2,3,4,5,6)\right) \\
& +\frac{1}{2} V_{1} T_{2,3,4,5,6}^{m n}\left(\ell^{m} \ell^{n}-\frac{1}{12} k_{1}^{m} k_{1}^{n}-\frac{1}{12} k_{2}^{m} k_{2}^{n}-\cdots-\frac{1}{12} k_{6}^{m} k_{6}^{n}\right) .
\end{align*}
$$

For the second hexagon, we consider the field-theory limit of the correlator with the colour ordering $A\left(1,6,2,3,4,5 ; \ell+k_{1}\right)$. The limits needed now have the form

$$
\begin{align*}
g_{i j}^{(1)} & \rightarrow \frac{1}{2} \operatorname{sgn}_{i j}^{162345}+\delta_{j 6}-\delta_{i 6}, \\
g_{i j}^{(1)} g_{k l}^{(1)} & \rightarrow\left(\frac{1}{2} \operatorname{sgn}_{i j}^{162345}+\delta_{j 6}-\delta_{i 6}\right)\left(\frac{1}{2} \operatorname{sgn}_{k l}^{162345}+\delta_{l 6}-\delta_{k 6}\right),  \tag{11.1.19}\\
g_{i j}^{(2)} & \rightarrow \frac{1}{12}+\delta_{i 6} \delta_{j 1}+\delta_{j 6} \delta_{i 1} .
\end{align*}
$$

Using these, the numerator is identified as

$$
\begin{align*}
N_{1 \mid 6,2,3,4,5}^{a_{6}=1}(\ell)= & +\frac{1}{2} V_{1} T_{2,3,4,5,6}^{m n}\left(\ell^{m} \ell^{n}+2 k_{1}^{m} k_{6}^{n}-\frac{1}{12}\left(k_{m}^{1} k_{n}^{1}+k_{m}^{2} k_{n}^{2}+\cdots k_{m}^{6} k_{n}^{6}\right)\right) \\
& +\frac{1}{2}\left(V_{1} T_{[2,3], 4,5,6}^{m}\left(\ell^{m}-\frac{1}{6} k_{2}^{m}+\frac{1}{6} k_{3}^{m}\right)+(2,3 \mid 2,3,4,5,6)\right) \\
& -\left(V_{1} T_{[2,6], 3,4,5}^{m} k_{1}^{m}+(2 \leftrightarrow 3,4,5)\right)  \tag{11.1.20}\\
& +\frac{1}{2}\left(V_{[1,2]} T_{3,4,5,6}^{m}\left(\ell^{m}-\frac{1}{6} k_{1}^{m}+\frac{1}{6} k_{2}^{m}+2 k_{6}^{m}\right)+(2 \leftrightarrow 3,4,5)\right) \\
& +V_{[1,6]} T_{2,3,4,5}^{m}\left(\frac{3}{2} \ell^{m}-\frac{13}{12} k_{1}^{m}+\frac{13}{12} k_{6}^{m}\right) \\
& +\frac{1}{6} V_{1}\left(T_{[[2,3], 4], 5,6}+T_{[2,[3,4]], 5,6}+(2,3,4 \mid 2,3,4,5,6)\right) \\
& +\frac{1}{6}\left(\left(V_{[[1,2], 3]}+V_{[1,[2,3]]}\right) T_{4,5,6}+(2,3 \mid 2,3,4,5)\right) \\
& -\frac{1}{3}\left(\left(V_{[[1,2], 6]}+V_{[1,[2,6]]}\right) T_{4,5,6}+(2 \leftrightarrow 3,4,5)\right) \\
& +\frac{1}{4}\left(V_{1} T_{[2,3],[4,5], 6}+(2,3|4,5| 2,3,4,5,6)\right) \\
& +\frac{1}{4}\left(V_{[1,2]} T_{[3,4], 5,6}+(2|3,4| 2,3,4,5)\right) \\
& -\frac{3}{4}\left(V_{[1,2]} T_{[3,6], 4,5}+(2|3| 2,3,4,5)\right) \\
& +\frac{3}{4}\left(V_{[1,6]} T_{[2,3], 4,5}+(2,3 \mid 2,3,4,5)\right) .
\end{align*}
$$

For example, the factor associated with $V_{16} T_{2,3,4,5}^{m}$ in the above follows as the worldsheet function is

$$
\begin{equation*}
\mathcal{Z}_{16,2,3,4,5}^{m}=\ell^{m} g_{16}^{(1)}+\left(k_{6}^{m}-k_{1}^{m}\right) g_{16}^{(2)}+\left[k_{2}^{m}\left(g_{12}^{(2)}-g_{62}^{(2)}\right)+(2 \leftrightarrow 3,4,5)\right] \tag{11.1.21}
\end{equation*}
$$

Within the square brackets, none of the $g^{(2)}$ terms are $g_{16}^{(2)}$ or $g_{61}^{(2)}$, and so by (11.1.19) these all contribute $\frac{1}{12}$. Hence this bracket vanishes,

$$
\begin{equation*}
\left[k_{2}^{m}\left(g_{12}^{(2)}-g_{62}^{(2)}\right)+(2 \leftrightarrow 3,4,5)\right] \rightarrow\left[k_{2}^{m}\left(\frac{1}{12}-\frac{1}{12}\right)+(2 \leftrightarrow 3,4,5)\right] I=0 \tag{11.1.22}
\end{equation*}
$$

As for the $g_{16}^{(1)}$ and $g_{16}^{(2)}$, these have limits

$$
\begin{align*}
g_{16}^{(1)} & \rightarrow \frac{1}{2} \operatorname{sgn}_{16}^{162345}+\delta_{66}-\delta_{16}=\frac{3}{2}  \tag{11.1.23}\\
g_{16}^{(2)} & \rightarrow \frac{1}{12}+\delta_{16} \delta_{61}+\delta_{66} \delta_{11}=\frac{13}{12} \tag{11.1.24}
\end{align*}
$$

Plugging these values into (11.1.21) reproduces the coefficient of the $V_{16} T_{2,3,4,5}^{m}$ term of (11.1.20).

Finally we have the pentagon term of the BCJ relation to find. This is the coefficient of $\frac{1}{2 s_{16}} I_{61,2,3,4,5}$ in the integrand $A(1,2,3,4,5,6 ; \ell)$. This can be found to be

$$
\begin{align*}
N_{61 \mid 2,3,4,5}^{\prime}(\ell)= & +\frac{1}{2}\left[\left(V_{[[1,2], 6]}+V_{[1,[2,6]]}\right) T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right] \\
& +\left[V_{[1,2]} T_{[3,6], 4,5}+(2|3| 2,3,4,5)\right] \\
& -\frac{1}{2}\left[V_{[1,6]} T_{[2,3], 4,5}+(2,3 \mid 2,3,4,5)\right] \\
& -\left[V_{[1,2]} T_{3,4,5,6}^{m} k_{6}^{m}+(2 \leftrightarrow 3,4,5)\right]  \tag{11.1.25}\\
& +\left[V_{1} T_{[2,6], 3,4,5}^{m} k_{1}^{m}+(2 \leftrightarrow 3,4,5)\right] \\
& -V_{[1,6]} T_{2,3,4,5}^{m}\left(\ell^{m}+k_{6}^{m}-k_{1}^{m}\right) \\
& -V_{1} T_{2,3,4,5,6}^{m n} k_{1}^{m} k_{6}^{n}
\end{align*}
$$

The identity (11.1.16) may then be verified by plugging in these numerator values. Again, no BRST cohomology identities are needed for this.

### 11.1.3 Antisymmetry of the $i j$-Pentagon From $A(i, P, j ; \ell)$ in $i$ and $j$

As mentioned above, the BCJ relations within external tree diagrams are satisfied due to the properties of superfields in the BCJ gauge. For example, all the boxes and all but one of the pentagons for an amplitude $A(P ; \ell)$ can be described by

$$
\begin{align*}
N_{A \mid B, C, D}(\ell) & =V_{A} T_{B, C, D}(\ell)+(A \leftrightarrow B, C, D)  \tag{11.1.26}\\
N_{A \mid B, C, D, E}(\ell) & =\left[V_{A} T_{B, C, D, E}^{m} \ell_{m}+(A \leftrightarrow B, C, D, E)\right] \\
& +\frac{1}{2}\left[V_{A} T_{[B, C], D, E}+(A|B, C| A, B, C, D, E)\right]  \tag{11.1.27}\\
& +\frac{1}{2}\left[V_{[A, B]} T_{C, D, E}+(A, B \mid A, B, C, D, E)\right]
\end{align*}
$$

with the additional constraint that $T_{\ldots, A 1 B, \ldots}=0$ (i.e., setting to zero all terms in which the label 1 is not assigned to a multiparticle vertex $V_{P}$ ). For example, using (11.1.27) we



Figure 11.1.3: The relation we need to be satisfied in order to demonstrate the antisymmetry of the 61 -pentagon in $A(1,2,3,4,5,6 ; \ell)$. The momentum running into the 61 external tree in the right hand graph is $\ell+k_{6}$, as in the amplitude $A(1,2,3,4,5,6 ; \ell)$ all diagrams are constructed as such. The left hand graph should have the same momentum assignment to the edges shared with the right hand graph, and so has momentum $\ell+k^{6}$ running into the fork also. Hence, the pentagon on the left belongs to the amplitude $A\left(1,6,2,3,4,5 ; \ell+k_{6}\right)$, following the convention (10.2.1). Therefore to extract this pentagon, we must use the general field-theory rules in this ordering, and set $a_{6}=1$.
recover the 23-pentagon (11.1.13)

$$
\begin{align*}
N_{23 \mid 1,4,5,6}(\ell) & =V_{1} T_{4,5,6,23}^{m} \ell^{m}-\frac{1}{2} V_{[1,23]} T_{4,5,6}+\frac{1}{2}\left(V_{[1,4]} T_{5,6,23}+(4 \leftrightarrow 5,6)\right) \\
& +\frac{1}{2}\left(V_{1} T_{[23,4], 5,6}+(23,4 \mid 23,4,5,6)\right) . \tag{11.1.28}
\end{align*}
$$

Likewise, the 46 -pentagon in the amplitude $A(5,1,2,4,6,3 ; \ell)$ say may be identified as

$$
\begin{align*}
N_{5 \mid 1,2,46,3} & =V_{1} T_{2,3,46,5}^{m}+\frac{1}{2}\left[V_{1} T_{[5,2], 46,3}+(5,2 \mid 5,2,46,3)\right]  \tag{11.1.29}\\
& +\frac{1}{2} V_{51} T_{2,46,3}+\frac{1}{2}\left[V_{12} T_{5,46,3}+(2 \leftrightarrow 46,3)\right] .
\end{align*}
$$

Since in the BCJ gauge [88; 27] the blocks of indices in (11.1.26) and (11.1.27) satisfy generalized Jacobi identities, the external tree BCJ relations are manifest. That is, in the above example, we have antisymmetry in 4 and 6 in

$$
\begin{equation*}
V_{[1,46]}, \quad T_{\ldots, 46, \ldots,}, \quad T_{\ldots, \ldots, i, 46], \ldots,}, \tag{11.1.30}
\end{equation*}
$$

and so as one of these is present in every part of the numerator, the overall numerator is antisymmetric in 4 and 6,

$$
\begin{equation*}
N_{5 \mid 1,2,46,3}=-N_{5 \mid 1,2,64,3} . \tag{11.1.31}
\end{equation*}
$$

Hence the antisymmetry in the external tree is a direct consequence of the BCJ gauge.
There is however, one notable exception, for the $i j$-pentagon in an amplitude $A(j, \ldots, i)$. So for instance, the 61-pentagon in $A(1,2,3,4,5,6 ; \ell)$ or the 12-pentagon in $A(2,3,4,5,6,1 ; \ell)$ do not follow the general formula (11.1.27), as can be seen for example in (10.3.38). These
will require more work to show they satisfy antisymmetry, as with the former example say one needs to compare it to the 16 -pentagon from $A\left(1,6,2,3,4,5 ; \ell+k_{6}\right)$. See the figure 11.1.3 for further details.

We thus construct the 61-pentagon from the amplitude $A\left(1,6,2,3,4,5 ; \ell+k^{6}\right)$, using the field-theory rules section 10.2 with $a_{6}=1$, and $a_{i}=0$ for all other $i$. The resulting numerator is

$$
\begin{align*}
N_{16 \mid 2,3,4,5}^{a_{6}=1}(\ell) & =-\frac{1}{2}\left[\left(V_{[11,2], 6]}+V_{[1,[2,6]]}\right) T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right] \\
& -\left[V_{[1,2]} T_{[3,6], 4,5}+(2|3| 2,3,4,5)\right] \\
& +\frac{1}{2}\left[V_{[1,6]} T_{[2,3], 4,5}+(2,3 \mid 2,3,4,5)\right]  \tag{11.1.32}\\
& +\left[V_{[1,2]} T_{3,4,5,6}^{m} k_{6}^{m}-V_{1} T_{[2,6], 3,4,5}^{m} k_{1}^{m}+(2 \leftrightarrow 3,4,5)\right] \\
& +V_{[1,6]} T_{2,3,4,5}^{m}\left(\ell^{m}+k_{6}^{m}-k_{1}^{m}\right) \\
& +V_{1} T_{2,3,4,5,6}^{m n} k_{1}^{m} k_{6}^{n} .
\end{align*}
$$

We then compare the two numerators, (11.1.32) and (11.1.25), and the colour-kinematics identity depicted in 11.1.3 follows immediately,

$$
\begin{equation*}
N_{16 \mid 2,3,4,5}^{a_{6}=1}(\ell)+N_{61 \mid 2,3,4,5}(\ell)=0 \tag{11.1.33}
\end{equation*}
$$

Before we conclude this example, we should note that the field-theory limit rules yield a very different expression for the 16 -pentagon in the same colour ordering without a shift in the loop momentum. That is, the 16 -pentagon of $A(1,6,2,3,4,5 ; \ell)$ is

$$
\begin{align*}
N_{16 \mid 2,3,4,5}(\ell)=V_{16} T_{2,3,4,5}^{m} \ell_{m} & +\frac{1}{2}\left[V_{16} T_{23,4,5}+(2,3 \mid 2,3,4,5)\right]  \tag{11.1.34}\\
& \left.+\frac{1}{2} V_{162} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right]
\end{align*}
$$

The above with a shift $\ell \rightarrow \ell+k_{6}$ applied to is not BRST equivalent to the 16-pentagon from the shifted amplitude $A\left(1,6,2,3,4,5 ; \ell+k_{6}\right)$. They differ as

$$
\begin{equation*}
Q\left(N_{16 \mid 2,3,4,5}^{a_{6}=1}(\ell)-N_{16 \mid 2,3,4,5}\left(\ell+k_{6}\right)\right)=Q\left(s_{16} V_{1} J_{6 \mid 2,3,4,5}\right) . \tag{11.1.35}
\end{equation*}
$$

This shows that the field-theory rules we have described capture the shifts in the loop momentum parameterisation in a non trivial way, as the limit of $A\left(1,6,2,3,4,5 ; \ell+k_{6}\right)$ does not follow from simply shifting $\ell \rightarrow \ell+k_{6}$ in $A(1,6,2,3,4,5 ; \ell)$.

### 11.1.4 Remaining BCJ triplets

The above is intended as a representative set of examples of the sorts of calculations one can perform to verify BCJ relations using these field theory methods. There are of course several further BCJ relations between pentagons and boxes left to show in order to confirm for certain that we have a BCJ representation of the amplitude. These we illustrate in figure 11.1.4. For each of these in turn we act completely analogously to the cases discussed; following the rules (10.2.5) to extract the two canonical ordering numerators, and the non-canonical ordered amplitudes needed for the third numerators are given below, along with the relevant assignments of values for the $a_{i}$

$$
\begin{align*}
A\left(1,2,6,3,4,5 ; \ell+k_{6}\right), & a_{1}=a_{2}=a_{3}=a_{4}=a_{5}=0, a_{6}=1  \tag{11.1.36}\\
A\left(1,6,5,2,3,4 ; \ell+k_{56}\right), & a_{1}=a_{2}=a_{3}=a_{4}=0, a_{5}=a_{6}=1  \tag{11.1.37}\\
A\left(1,3,4,5,2,6 ; \ell-k_{2}\right), & a_{1}=a_{3}=a_{4}=a_{5}=a_{6}=0, a_{2}=-1  \tag{11.1.38}\\
A\left(1,5,2,3,4,6 ; \ell+k_{5}\right), & a_{1}=a_{2}=a_{3}=a_{4}=a_{6}=0, a_{5}=1 \tag{11.1.39}
\end{align*}
$$

We have verified that each of the amplitudes in the above is BRST invariant ${ }^{2}$, and gives numerators which satisfy the BCJ relations in the figure. We will not detail their construction any further, as they can be obtained by analogous methods as discussed above.

As all of these relations are satisfied, as well as those we have not detailed which are easier to show, we conclude that we have a BCJ representation of the six point one-loop ten dimensional SYM amplitude.

### 11.1.5 Other Parameterisations of Graphs

The choice of loop momentum assignment to the graphs plays an important role, with the nature of the numerators produced differing significantly between different representations. The BCJ identities considered in the previous discussion are those which maximize the chances of failure, but we should note that if different choices of loop momentum assignments are made they are simple. For example, the first BCJ relation in figure 11.1.4 would have been simple to show if we parameterised such that the momentum between legs 3 and 4 was $\ell$. This is depicted in figure 11.1.5. In this parameterisation, the BCJ relation is

$$
\begin{equation*}
N_{4 \mid 5,6,12,3}(\ell)-N_{4 \mid 5,12,6,3}(\ell)-N_{4 \mid 5,[6,12], 3}(\ell)=0 \tag{11.1.40}
\end{equation*}
$$

[^20]













Figure 11.1.4: The other special cases at six points, for which amplitudes with shifted loop momentum have to be computed to extract BCJ numerators.




Figure 11.1.5: We may parameterise our graphs in any number of ways, there is no reason why the momentum going into the 1 leg has to be $\ell$. By choosing a different parameterisation, more complex BCJ relations may be made simple. The above for instance is the same BCJ identity as the first example in figure 11.1.4, but if the loop momentum were assigned as it is here showing the BCJ relation becomes significantly easier. As this is showing something different, the previous discussion is still needed however.
which follows immediately by using the formulae for pentagons and boxes of this form, equations (11.1.27) and (11.1.26)

$$
\begin{aligned}
& N_{4 \mid 5,6,12,3}(\ell)-N_{4 \mid 5,12,6,3}(\ell)-N_{4 \mid 5,[6,12], 3}(\ell)= \\
& \quad\left(V_{12} T_{3,4,5,6}^{m} \ell_{m}-\frac{1}{2} V_{12} T_{34,5,6}-\frac{1}{2} V_{12} T_{35,4,6}-\frac{1}{2} V_{12} T_{36,4,5}+\frac{1}{2} V_{12} T_{45,3,6}+\frac{1}{2} V_{12} T_{46,3,5}\right. \\
& \left.\quad+\frac{1}{2} V_{12} T_{56,3,4}+\frac{1}{2} V_{123} T_{4,5,6}-\frac{1}{2} V_{124} T_{3,5,6}-\frac{1}{2} V_{125} T_{3,4,6}-\frac{1}{2} V_{126} T_{3,4,5}\right) \\
& -\left(V_{12} T_{3,4,5,6}^{m} \ell_{m}-\frac{1}{2} V_{12} T_{34,5,6}-\frac{1}{2} V_{12} T_{35,4,6}-\frac{1}{2} V_{12} T_{36,4,5}+\frac{1}{2} V_{12} T_{45,3,6}+\frac{1}{2} V_{12} T_{46,3,5}\right. \\
& \left.\quad+\frac{1}{2} V_{12} T_{56,3,4}+\frac{1}{2} V_{123} T_{4,5,6}-\frac{1}{2} V_{124} T_{3,5,6}-\frac{1}{2} V_{125} T_{3,4,6}+\frac{1}{2} V_{126} T_{3,4,5}\right) \\
& - \\
& \quad V_{[6,12]} T_{3,4,5}=0 .
\end{aligned}
$$

The vanishing can be seen by the $V_{[6,12]}=-V_{126}$ property of the BCJ gauge.

### 11.2 Seven points

At seven points, BCJ relations are analogously satisfied. Given their significantly more complex structure, we will not give examples of these in the same detail. For more complex checks, full expressions for numerators of the amplitude $A(1,2,3,4,5,6,7 ; \ell)$ may be found in appendix I, and for $A\left(1,2,3,4,5,6,7 ; \ell+\sum_{i} a_{i} k^{i}\right)$ at [28]. We here only verify that individual terms in numerators satisfy relations.

We may take for example the BCJ identity

$$
\begin{equation*}
N_{1 \mid 2,3,4,5,6,7}(\ell)-N_{1 \mid 2,4,3,5,6,7}(\ell)-N_{1 \mid 2,34,5,6,7}(\ell)=0 \tag{11.2.1}
\end{equation*}
$$

and consider the $V_{12} T_{34,56,7}^{m}$ terms within this. The first and third numerators are from the canonically ordered amplitude, with their terms of this form corresponding with the worldsheet function [21]

$$
\begin{align*}
\mathcal{Z}_{12,34,56,7} & =g_{12}^{(1)} g_{34}^{(1)} g_{56}^{(1)}+g_{12}^{(1)}\left(g_{35}^{(2)}-g_{36}^{(2)}-g_{45}^{(2)}+g_{46}^{(2)}\right) \\
& +g_{34}^{(1)}\left(g_{15}^{(2)}-g_{16}^{(2)}-g_{25}^{(2)}+g_{26}^{(2)}\right)+g_{56}^{(1)}\left(g_{13}^{(2)}-g_{14}^{(2)}-g_{23}^{(2)}+g_{24}^{(2)}\right) \\
& +g_{15}^{(1)}\left(g_{13}^{(2)}-g_{14}^{(2)}-g_{35}^{(2)}+g_{45}^{(2)}\right)+g_{16}^{(1)}\left(g_{14}^{(2)}-g_{13}^{(2)}-g_{36}^{(2)}+g_{46}^{(2)}\right)  \tag{11.2.2}\\
& +g_{25}^{(1)}\left(g_{24}^{(2)}-g_{23}^{(2)}-g_{45}^{(2)}+g_{35}^{(2)}\right)+g_{26}^{(1)}\left(g_{23}^{(2)}-g_{24}^{(2)}-g_{36}^{(2)}+g_{46}^{(2)}\right) .
\end{align*}
$$

The heptagon term arises from the terms proportional to $I$, which in effect means setting

$$
g_{i j}^{(p)} \rightarrow \frac{\left(\operatorname{sgn}_{i j}^{1234567}\right)^{p} B_{p}}{p!}= \begin{cases}\frac{\operatorname{sgn}_{i j}^{1234567}}{2} & : p=1  \tag{11.2.3}\\ \frac{1}{12} & : p=2 \\ 0 & : p=3\end{cases}
$$

Plugging these values into (11.2.2), all of the quartets of $g^{(2)}$ terms vanish and we are left with a term

$$
\begin{equation*}
\frac{1}{8} V_{12} T_{34,56,7} \tag{11.2.4}
\end{equation*}
$$

in the numerator $N_{1 \mid 2,3,4,5,6,7}(\ell)$.

The hexagon $N_{1 \mid 2,34,5,6,7}(\ell)$ corresponds with the terms proportional to $\frac{1}{2 s_{34}} I_{34}$ in the field theory limit of (11.2.2). Such poles only arise in this in the $g_{12}^{(1)}$ terms, and so we effectively set

$$
\begin{equation*}
g_{12}^{(1)} \rightarrow \frac{1}{2 s_{12}} I_{12}, \tag{11.2.5}
\end{equation*}
$$

and take the limits (11.2.3) again for all other terms. Again, all of the quartets of $g^{(2)}$ terms vanish and we are left with a contribution to the hexagon of

$$
\begin{equation*}
\frac{1}{4} V_{12} T_{34,56,7} \tag{11.2.6}
\end{equation*}
$$

Finally, we must find the middle term of (11.2.1). In this instance, the inherent symmetry of the string correlator in 234567 is used to construct it based upon Stirling sums in 1243567 (See the discussion around (11.1.11) for more on this). As such, the correlator contains a term

$$
\begin{equation*}
V_{12} T_{43,56,7} \mathcal{Z}_{12,43,56,7}=-V_{12} T_{34,56,7} \mathcal{Z}_{12,43,56,7} \tag{11.2.7}
\end{equation*}
$$

This worldsheet function is (11.2.2) with 3 and 4 swapped, and the rules needed to find
the heptagon are the modification of (11.2.3),

$$
g_{i j}^{(p)} \rightarrow \frac{\left(\operatorname{sgn}_{i j}^{1243567}\right)^{p} B_{p}}{p!}=\left\{\begin{array}{ll}
\frac{\operatorname{sn}_{i j}^{1243567}}{2} & : p=1  \tag{11.2.8}\\
\frac{1}{12} & : p=2 . \\
0 & : p=3
\end{array} .\right.
$$

This thus produces the heptagon term

$$
\begin{equation*}
-\frac{1}{8} V_{12} T_{34,56,7} \tag{11.2.9}
\end{equation*}
$$

We then see that the relation (11.2.1) is satisfied by these terms,

$$
\begin{equation*}
\frac{1}{8} V_{12} T_{34,56,7}-\left(-\frac{1}{8} V_{12} T_{34,56,7}\right)-\frac{1}{4} V_{12} T_{34,56,7}=0 . \tag{11.2.10}
\end{equation*}
$$

As briefly discussed previously, at seven points there is an extra complication in the presence of refined superfields which must be dealt with. To find the field theory limits of these refined terms, we have to partially integrate the worldsheet functions against the Koba-Nielsen factor. As a consequence of this less direct method, when we wish to verify BCJ relations we must rearrange the refined terms to counteract this manipulation. For relations in which the loop momentum structure is unchanged between terms (that is, BCJ relations in which there is always momentum $\ell$ going into leg 1 ), this amounts to canceling all $(\ell \cdot k)$ terms against propagators. So for example, consider the relation

$$
\begin{equation*}
N_{122,3,4,5,6,7}(\ell)-N_{1 \mid 2,4,3,5,6,7}(\ell)-N_{1 \mid 2,34,5,6,7}(\ell)=0, \tag{11.2.11}
\end{equation*}
$$

within which we focus upon the refined terms $V_{1} J_{34 \mid 2,5,6,7}$. For the two terms from the canonical amplitude, this is associated with the worldsheet function

$$
\begin{equation*}
\mathcal{Z}_{34 \mid 1,2,5,6,7}=-3 s_{34} g_{34}^{(3)}+g_{34}^{(2)}\left(\ell \cdot k_{4}+\left(s_{41} g_{41}^{(1)}+(1 \leftrightarrow 2,5,6,7)\right) .\right. \tag{11.2.12}
\end{equation*}
$$

If we naively plug in the field theory limit values, we would expect the heptagon numerator $N_{1 \mid 2,3,4,5,6,7}(\ell)$ to contain the terms

$$
\begin{equation*}
-\frac{1}{12} V_{1} J_{34 \mid 2,5,6,7}\left(\ell \cdot k^{4}-\frac{1}{2} k^{12} \cdot k^{4}+\frac{1}{2} k^{4} \cdot k^{567}\right) . \tag{11.2.13}
\end{equation*}
$$

Likewise, the other numerators we would expect to contain the terms

$$
\begin{align*}
N_{1 \mid 2,4,3,5,6,7}(\ell) & \leftrightarrow-\frac{1}{12} V_{1} J_{43 \mid 2,5,6,7}\left(\ell \cdot k^{3}-\frac{1}{2} k^{12} \cdot k^{3}+\frac{1}{2} k^{3} \cdot k^{567}\right)  \tag{11.2.14}\\
N_{1 \mid 2,34,5,6,7}(\ell) & \leftrightarrow 0 . \tag{11.2.15}
\end{align*}
$$

The relation (11.2.11) is clearly not satisfied with these values.

Instead, we should cancel the $\ell \cdot k$ terms against the propagators in the denominators. So for example, we may reformulate (11.2.13) as

$$
\begin{align*}
& -\frac{1}{12} V_{1} J_{34 \mid 2,5,6,7}\left(\frac{1}{2}\left(\ell-k^{123}\right)^{2}-\frac{1}{2}\left(\ell-k^{1234}\right)^{2}+k^{123} \cdot k^{4}-\frac{1}{2} k^{12} \cdot k^{4}+\frac{1}{2} k^{4} \cdot k^{567}\right) \\
= & -\frac{1}{12} V_{1} J_{34 \mid 2,5,6,7}\left(\frac{1}{2}\left(\ell-k^{123}\right)^{2}-\frac{1}{2}\left(\ell-k^{1234}\right)^{2}+\frac{1}{2} k^{3} \cdot k^{4}\right) \tag{11.2.16}
\end{align*}
$$

We then cancel the $(\ell-k)^{2}$ terms with the corresponding piece of the Feynman loop integrand associated with this term,

$$
\begin{equation*}
I_{1,2,3,4,5,6,7}(\ell)=\frac{1}{\left(\ell-k^{1}\right)^{2}\left(\ell-k^{12}\right)^{2} \ldots\left(\ell-k^{1234567}\right)^{2}} \tag{11.2.17}
\end{equation*}
$$

Thus, these terms contribute to hexagons instead. Hence there is only one term of this form associated with the heptagon,

$$
\begin{equation*}
N_{1 \mid 2,3,4,5,6,7}(\ell) \leftrightarrow-\frac{1}{24} s_{34} V_{1} J_{34 \mid 2,5,6,7} \tag{11.2.18}
\end{equation*}
$$

A similar calculation should be performed on the other heptagon (11.2.14). This is reexpressed as

$$
\begin{equation*}
\frac{1}{12} V_{1} J_{34 \mid 2,5,6,7}\left(\frac{1}{2}\left(\ell-k^{124}\right)^{2}-\frac{1}{2}\left(\ell-k^{1243}\right)^{2}+\frac{1}{2} k^{3} \cdot k^{4}\right) \tag{11.2.19}
\end{equation*}
$$

Again, we cancel against the Feynman loop integrand, which in this instance is $I_{1,2,4,3,5,6,7}(\ell)$, and are left with a single contribution to the heptagon

$$
\begin{equation*}
N_{1 \mid 2,4,3,5,6,7}(\ell) \leftrightarrow \frac{1}{24} s_{34} V_{1} J_{34 \mid 2,5,6,7} \tag{11.2.20}
\end{equation*}
$$

The hexagons then inherit extra terms from the canceled portion of the heptagons. The 34-hexagon we are interested in inherits a term from the cancellation (11.2.16), and so we now have

$$
\begin{equation*}
N_{1 \mid 2,34,5,6,7}(\ell) \leftrightarrow-\frac{1}{12} s_{34} V_{1} J_{34 \mid 2,5,6,7} \tag{11.2.21}
\end{equation*}
$$

Note this differs from what may be naively expected from (11.2.16) due to the hexagon containing an extra $2 s_{34}$ in its denominator compared with the heptagon. Now plugging $(11.2 .18),(11.2 .20),(11.2 .21)$ into the relation (11.2.11) we see it is now satisfied

$$
\begin{equation*}
-\frac{1}{24} s_{34} V_{1} J_{34 \mid 2,5,6,7}-\frac{1}{24} s_{34} V_{1} J_{34 \mid 2,5,6,7}-\left(-\frac{1}{12} s_{34} V_{1} J_{34 \mid 2,5,6,7}\right)=0 \tag{11.2.22}
\end{equation*}
$$

Similar manipulations hold for other BCJ relations of this sort. We do have additional complications however when the BCJ relation we wish to verify involves terms of different loop momentum structure. We have yet to identify a general algorithm for such cases. However, by explicitly rearranging amplitudes term by term, we have been able to arrange
them such that they satisfy every BCJ relation we have tested. In particular, we have been able to simultaneously satisfy the following

$$
\begin{align*}
N_{1 \mid 2,3,4,5,6,7}(\ell)-N_{1 \mid 7,2,3,5,6}^{a_{7}=1}(\ell)-N_{[7,1] \mid 2,3,4,5,6}(\ell) & =0,  \tag{11.2.23}\\
N_{1 \mid 2,3,4,5,7,6}(\ell)-N_{1 \mid 6,2,3,4,5,7}^{a_{6}=1}(\ell)-N_{[1,6] \mid 2,3,4,5,7}(\ell) & =0,  \tag{11.2.24}\\
N_{[7,1] \mid 2,3,4,5,6}(\ell)-N_{[7,1] \mid 6,2,3,4,5}(\ell)-N_{[6,[7,1]] \mid 2,3,4,5} & =0,  \tag{11.2.25}\\
N_{[6,[7,1] \mid] \mid 2,3,4,5}(\ell)-N_{[6,[7,1]| | 5,2,3,4}^{a_{5}=1}(\ell)-N_{[5,[6,[7,1]]] \mid 2,3,4}(\ell) & =0,  \tag{11.2.26}\\
N_{[1,6] \mid 2,3,4,5,7}^{a_{0}=1}(\ell)+N_{[6,1] \mid 2,3,4,5,7} & =0,  \tag{11.2.27}\\
N_{[7,1] \mid 2,3,4,5,6}(\ell)+N_{[1,7] \mid 2,3,4,5,6}^{a_{7}=1} & =0 . \tag{11.2.28}
\end{align*}
$$

Though this is not an exhaustive test, we hope that it is sufficient to serve as a proof of concept that it should always be possible to rearrange the refined terms to satisfy BCJ identities.

## CHAPTER 12

## One-Loop Supergravity Amplitudes

Given that we have found a BCJ representation of one loop amplitudes, it natural to ask if we may then use these results to find corresponding amplitudes in supergravity with the double-copy construction [12]. For five points, this has previously been carried out in four dimensions in [164], while in ten dimensions it was computed using pure spinor superspace in [1]. Unfortunately as we shall see, the methods described here are not sufficient to expand these results to six points, owing to the absence of dihedral symmetries between numerators ${ }^{1}$.

In pure spinor superspace, we can test whether results obtained by the double copy are correct based upon if they are BRST invariant [104; 99]. We now repeat the five point supergravity construction of [1], to highlight that it is BRST invariant in part because the numerators satisfy dihedral symmetries. While at five points our numerators satisfy these symmetries in addition to the Jacobi identities, the corresponding symmetries at six points fail with our BCJ-satisfying six-point numerators. This will prevent the doublecopy construction of a BRST-closed supergravity integrand. We are therefore forced to

[^21]leave applying the double-copy procedure at six points to future work.

### 12.1 Five Points

At five points, the calculation proceeds as one would hope. We begin by constructing the colour dressed amplitude (that is, the SYM amplitude with its colour factors included, corresponding with a sum over diagrams in all possible orderings, each multiplied by a colour factor), and verifying it has vanishing variation. The colour factors are then replaced by the corresponding kinematic factors, and the variation of this is once again confirmed to vanish. Both of these results depend in part upon the dihedral symmetries of the graphs being present, which they are in this case.

### 12.1.1 The five-point colour-dressed integrand

We express the five-point colour-dressed one-loop integrand as

$$
\begin{align*}
M_{5}(\ell) & =\left(\frac{1}{2} \mathcal{N}_{1 \mid 2,3,45} I_{1,2,3,45} B_{1,2,3,45}+\frac{1}{2} \mathcal{N}_{1 \mid 2,34,5} I_{1,2,34,5} B_{1,2,34,5}\right.  \tag{12.1.1}\\
& +\frac{1}{2} \mathcal{N}_{1 \mid 23,4,5} I_{1,23,4,5} B_{1,23,4,5}+\frac{1}{2} \mathcal{N}_{12 \mid 3,4,5} I_{12,3,4,5} B_{12,3,4,5} \\
& \left.+\frac{1}{2} \mathcal{N}_{51 \mid 2,3,4} I_{51,2,3,4} B_{51,2,3,4}+\mathcal{N}_{1 \mid 2,3,4,5}(\ell) I_{1,2,3,4,5} P_{1,2,3,4,5}+\operatorname{perm}(2,3,4,5)\right)
\end{align*}
$$

where $\mathcal{N}$ denotes the usual Berends-Giele counterpart of the $n$-gon numerator, and the $B$ and $P$ are the colour factors of the box and pentagon diagrams respectively, defined in the usual way

$$
\begin{equation*}
B_{12,3,4,5}=f^{a 12} f^{e a b} f^{b 3 c} f^{c 4 d} f^{d 5 e}, \quad P_{1,2,3,4,5}=f^{a 1 b} f^{b 2 c} f^{c 3 d} f^{d 4 e} f^{e 5 a} \tag{12.1.2}
\end{equation*}
$$

The factor of $\frac{1}{2}$ in (12.1.1) compensates the overcounting of graphs due to symmetries (for example, the 23-box diagram with ordering $A(1,2,3,4,5)$ appears in both the $\mathcal{N}_{1 \mid 23,4,5}$ and $\mathcal{N}_{1 \mid 32,4,5}$ terms). Note that the box numerators do not depend on the loop momentum.

We may see that this expression for the colour-dressed integrand (12.1.1) is BRST closed. To begin, we expand the box colour factors in terms of their pentagon constituents using the Jacobi identity [41],

$$
\begin{equation*}
B_{12,3,4,5}=P_{1,2,3,4,5}-P_{2,1,3,4,5} \tag{12.1.3}
\end{equation*}
$$

and restrict ourselves to the terms proportional to $P_{1,2,3,4,5}$. Those with other colour
factors will follow an analogous argument. These are

$$
\begin{align*}
\left.M_{5}(\ell)\right|_{P_{1,2,3,4,5}} & =\mathcal{N}_{1 \mid 2,3,4,5}(\ell) I_{1,2,3,4,5}+\frac{1}{2}\left(\mathcal{N}_{12 \mid 3,4,5} I_{12,3,4,5}-\mathcal{N}_{2| | 3,4,5} I_{1,3,4,5}\right.  \tag{12.1.4}\\
& +\left[\mathcal{N}_{1 \mid 23,4,5}-\mathcal{N}_{1 \mid 32,4,5}\right] I_{1,23,4,5}+\left[\mathcal{N}_{1 \mid 2,34,5}-\mathcal{N}_{1 \mid 2,43,5}\right] I_{1,2,34,5} \\
& \left.+\left[\mathcal{N}_{1 \mid 2,3,45}-\mathcal{N}_{1 \mid 2,3,54}\right] I_{1,2,3,45}+\mathcal{N}_{512,3,4} I_{1,2,3,4}-\mathcal{N}_{15 \mid 2,3,4} I_{15,2,3,4}\right) .
\end{align*}
$$

It is simple to see using the definition (10.3.10) that the boxes are antisymmetric in their pairs of indices,

$$
\begin{equation*}
N_{i j \mid k, l, m}=-N_{j i \mid k, l, m}, \quad N_{i \mid j k, l, m}=-N_{i \mid k j, l, m}, \tag{12.1.5}
\end{equation*}
$$

and so we may simplify the appearance of (12.1.4) to

$$
\begin{align*}
\left.M_{5}(\ell)\right|_{P_{1,2,3,4,5}} & =\mathcal{N}_{1 \mid 2,3,4,5}(\ell) I_{1,2,3,4,5}+\mathcal{N}_{1 \mid 23,4,5} I_{1,23,4,5}+\mathcal{N}_{1 \mid 2,34,5} I_{1,2,34,5} \\
& +\mathcal{N}_{1 \mid 2,3,45} I_{1,2,3,45}+\frac{1}{2} \mathcal{N}_{12 \mid 3,4,5} I_{12,3,4,5}-\frac{1}{2} \mathcal{N}_{21 \mid 3,4,5} I_{1,3,4,5}  \tag{12.1.6}\\
& \left.+\frac{1}{2} \mathcal{N}_{51 \mid 2,3,4} I_{1,2,3,4}-\frac{1}{2} \mathcal{N}_{15 \mid 2,3,4} I_{15,2,3,4}\right) .
\end{align*}
$$

We may then make a substitution $\ell^{\prime}=\ell-k_{2}$ in $I_{1,3,4,5}$, and $\ell^{\prime}=\ell+k_{5}$ in $I_{15,2,3,4}$. The result is thus the integrand of the canonically ordered amplitude $A(1,2,3,4,5 ; \ell)$ (10.3.9),

$$
\begin{align*}
\left.M_{5}(\ell)\right|_{P_{1,2,3,4,5}} & =\mathcal{N}_{1 \mid 2,3,4,5}(\ell) I_{1,2,3,4,5}+\mathcal{N}_{1 \mid 23,4,5} I_{1,23,4,5}+\mathcal{N}_{1 \mid 2,34,5} I_{1,2,34,5}  \tag{12.1.7}\\
& +\mathcal{N}_{1 \mid 2,3,45} I_{1,2,3,45}+\mathcal{N}_{12 \mid 3,4,5} I_{12,3,4,5}+\mathcal{N}_{51 \mid 3,4,5} I_{1,2,3,4}
\end{align*}
$$

We may then reintroduce the colour factor and other orderings, and the colour-dressed integrand (12.1.1) becomes

$$
\begin{equation*}
M_{5}(\ell)=A(1,2,3,4,5 ; \ell) P_{1,2,3,4,5}+\operatorname{perm}(2,3,4,5) . \tag{12.1.8}
\end{equation*}
$$

This is therefore BRST closed by the arguments related discussed for partial amplitudes. We note that reformulating the amplitude in this way agrees with the general result of [41] (see e.g. equation (3.4) of [165]).

### 12.1.2 The five-point supergravity integrand

We now construct the five-point supergravity integrand using the double-copy, and highlight the subtle requirement that the dihedral symmetries are necessary in order to have a consistent application of the double-copy. As it happens, the five point numerators have these symmetries, but the same is not true at higher points and making the observation in this simpler case will clarify future discussion.

We begin with the colour-dressed integrand (12.1.1), and as is usual with the double copy we replace the colour factors by an extra copy of the kinematic factors. This gives the expression

$$
\begin{align*}
M_{5}(\ell) & =\left(\frac{1}{2} \mathcal{N}_{1 \mid 2,3,45} \tilde{N}_{1 \mid 2,3,45} I_{1,2,3,45}+\frac{1}{2} \mathcal{N}_{1 \mid 2,34,5} \tilde{N}_{1 \mid 2,34,5} I_{1,2,34,5}\right.  \tag{12.1.9}\\
& +\frac{1}{2} \mathcal{N}_{1 \mid 23,4,5} \tilde{N}_{1 \mid 23,4,5} I_{1,23,4,5}+\frac{1}{2} \mathcal{N}_{12 \mid 3,4,5} \tilde{N}_{12 \mid 3,4,5} I_{12,3,4,5} \\
& \left.+\frac{1}{2} \mathcal{N}_{51 \mid 2,3,4} \tilde{N}_{51 \mid 2,3,4} I_{51,2,3,4}+\mathcal{N}_{1 \mid 2,3,4,5}(\ell) \tilde{N}_{1 \mid 2,3,4,5}(\ell) I_{1,2,3,4,5}+\operatorname{perm}(2,3,4,5)\right)
\end{align*}
$$

Note that the kinematic numerators on the left are in terms of Berends-Giele numerators $\mathcal{N}$, while those on the right are the local numerators $N$. This difference is purely for simplicity of notation; by extracting the Mandelstam from a $\mathcal{N}$ term the symmetry in the two numerators is made clear.

This vanishes, but only as a result of the application of cohomology identities, and not at the level of superfields. To see this, we suppose the BRST operator acts upon the left moving terms ${ }^{2}$, with the right moving terms following analogously. The variation of the canonically ordered pentagon simplifies as

$$
\begin{align*}
\left(Q N_{1 \mid 2,3,4,5}(\ell)\right) \tilde{N}_{1 \mid 2,3,4,5} I_{1,2,3,4,5} & =\frac{1}{2} V_{1} V_{2} T_{3,4,5} \tilde{N}_{1 \mid 2,3,4,5}\left[I_{1,23,4,5}-I_{12,3,4,5}\right] \\
& +\frac{1}{2} V_{1} V_{3} T_{2,4,5} \tilde{N}_{1 \mid 2,3,4,5}\left[I_{1,2,34,5}-I_{1,23,4,5}\right] \\
& +\frac{1}{2} V_{1} V_{4} T_{2,3,5} \tilde{N}_{1 \mid 2,3,4,5}\left[I_{1,2,3,45}-I_{1,2,34,5}\right]  \tag{12.1.10}\\
& +\frac{1}{2} V_{1} V_{5} T_{2,3,4} \tilde{N}_{1 \mid 2,3,4,5}\left[I_{1,2,3,4}-I_{1,2,3,45}\right],
\end{align*}
$$

where we have cancelled terms from the variation against the loop momentum integrand. Almost all of these terms are well behaved, and are cancelled by the variation of corresponding boxes accordingly. For instance, within the variation of (12.1.9) we also have the terms

$$
\begin{align*}
& \frac{1}{4 s_{23}}\left(Q N_{1 \mid 23,4,5}\right) \tilde{N}_{1 \mid 23,4,5} I_{1,23,4,5}+\frac{1}{4 s_{23}}\left(Q N_{1 \mid 32,4,5}\right) \tilde{N}_{1 \mid 32,4,5} I_{1,32,4,5}  \tag{12.1.11}\\
& \quad=-\frac{1}{2}\left(V_{1} V_{2} T_{3,4,5}-V_{1} V_{3} T_{2,4,5}\right)\left(\tilde{N}_{1 \mid 2,3,4,5}(\ell)-\tilde{N}_{1,3,2,4,5}\right) I_{1,23,4,5}
\end{align*}
$$

The terms proportional to $\tilde{N}_{1 \mid 2,3,4,5}(\ell)$ then can be seen to cancel a pair of terms in (12.1.10). However, additional complications arise with the term proportional to $I_{1,2,3,4}$. A similar approach to the above reveals that the BRST variation of (12.1.9) contains

[^22][148; 149; 150]
\[

$$
\begin{equation*}
\ldots+\frac{1}{2} V_{1} V_{5} T_{2,3,4}\left[I_{1,2,3,4} \tilde{N}_{1 \mid 2,3,4,5}(\ell)+I_{51,2,3,4}\left(\tilde{N}_{15 \mid 2,3,4}-\tilde{N}_{1 \mid 5,2,3,4}(\ell)\right)\right] \tag{12.1.12}
\end{equation*}
$$

\]

We must then perform a substitution $\ell \rightarrow \ell+k_{5}$ on the pentagon term above to put all terms over a common denominator, giving us

$$
\begin{equation*}
\frac{1}{2} V_{1} V_{5} T_{2,3,4} I_{51,2,3,4}\left[\tilde{N}_{15 \mid 2,3,4}+\tilde{N}_{1 \mid 2,3,4,5}\left(\ell-k_{5}\right)-\tilde{N}_{1 \mid 5,2,3,4}(\ell)\right] \tag{12.1.13}
\end{equation*}
$$

Therefore, if the dihedral symmetry $\tilde{N}_{1 \mid 2,3,4,5}\left(\ell-k_{5}\right)=\tilde{N}_{5 \mid 1,2,3,4}(\ell)$ was satisfied, the terms inside the bracket would vanish without the need of cohomology relations by the BCJ identity

$$
\begin{equation*}
\tilde{N}_{5 \mid 1,2,3,4}(\ell)-\tilde{N}_{1 \mid 5,2,3,4}(\ell)+\tilde{N}_{15 \mid 2,3,4}=0 \tag{12.1.14}
\end{equation*}
$$

However at the superfield level it is not true that $\tilde{N}_{1 \mid 2,3,4,5}\left(\ell-k_{5}\right)=\tilde{N}_{5 \mid 1,2,3,4}(\ell)$. Using the field-theory limit of the string correlator to generate these numerators, we find their difference to be

$$
\begin{align*}
\tilde{N}_{1 \mid 2,3,4,5}\left(\ell-k_{5}\right)-\tilde{N}_{5 \mid 1,2,3,4}(\ell)=-\tilde{V}_{1} \tilde{T}_{2,3,4,5}^{m} k_{5}^{m} & -\tilde{V}_{51} \tilde{T}_{2,3,4}  \tag{12.1.15}\\
& -\left[\tilde{V}_{1} \tilde{T}_{52,3,4}+(2 \leftrightarrow 3,4,5)\right] .
\end{align*}
$$

One must work within the pure spinor bracket in order for this to vanish,

$$
\begin{equation*}
\left\langle\tilde{N}_{1 \mid 2,3,4,5}\left(\ell-k_{5}\right)\right\rangle=\left\langle\tilde{N}_{5 \mid 1,2,3,4}(\ell)\right\rangle, \tag{12.1.16}
\end{equation*}
$$

by the identity (10.3.18).

To summarize, the five-point supergravity integrand is BRST invariant, but showing this relies upon the numerators satisfying dihedral symmetries as well as those of the colourkinematics duality. At five points, this is the case in pure spinor superspace. However, as we shall see this will not be the case at higher points.

### 12.2 Six Points

At six points, the numerators developed in this thesis fail to give consistent colour dressed SYM or supergravity integrands; both such amplitudes have non-vanishing variation. While we may change our approach to create a valid expression for the former, it is not currently known how to describe the latter. This is all because the numerators, in spite of satisfying BCJ identities, fail to satisfy the dihedral symmetries of their associated graphs.

### 12.2.1 The six-point colour-dressed integrand

The colour dressed integrand at six points will be written tentatively as [41]

$$
\begin{align*}
M_{6}(\ell) & =\left(\mathcal{N}_{1 \mid 2,3,4,5,6}^{\mathrm{col}}(\ell) I_{1,2,3,4,5,6}\right. \\
& +\frac{1}{2} \sum_{A B C D E F=23456} \mathcal{N}_{F 1 A \mid B, C, D, E}^{\mathrm{col}}(\ell) I_{1 A, B, C, D, E}  \tag{12.2.1}\\
& \left.+\frac{1}{4} \sum_{A B C D E=23456} \mathcal{N}_{E 1 A \mid B, C, D}^{\mathrm{col}} I_{1 A, B, C, D}+\operatorname{perm}(2,3,4,5,6)\right)
\end{align*}
$$

where the Berends-Giele currents $\mathcal{N}^{\mathrm{col}}$ also contain the corresponding colour factors in the natural way. For example, a box term would be

$$
\begin{equation*}
\mathcal{N}_{123 \mid 4,5,6}^{\mathrm{col}}=\frac{1}{s_{123} s_{12}} N_{123 \mid 4,5,6} B_{123 \mid 4,5,6}+\frac{1}{s_{23} s_{123}} N_{[1,23] \mid 4,5,6} B_{[1,23] \mid 4,5,6} \tag{12.2.2}
\end{equation*}
$$

Wherein the $B_{A, B, C, D}$ denotes a six point box colour factor in the natural way, and similar $P_{A, B, C, D, E}$ and $H_{A, B, C, D, E, F}$ notation will be used for six point pentagon and hexagon colour factors. The fractions are again present to deal with overcounting ${ }^{3}$.

We again use Jacobi identities to expand the colour factors in terms of those of hexagons. Focusing upon those proportional to $H_{1,2,3,4,5,6}$, there is a single hexagon numerator ${ }^{4}$ $N_{1 \mid 2,3,4,5,6}(\ell) I_{1,2,3,4,5,6}$. As the boxes are independent of the loop momentum, they will simplify analogously to at five points and become $\sum_{A B C D E=23456} \mathcal{N}_{E 1 A \mid B, C, D} I_{1 A \mid B, C, D}$. Hence the hexagon and box components of the partial amplitude $A(1,2,3,4,5,6 ; \ell)$ are obtained.

The pentagon component fails due to the absence of dihedral symmetries. The pentagons from (12.2.1) proportional to $H_{1,2,3,4,5,6}$ are

$$
\begin{align*}
& \frac{1}{2}\left(\quad \mathcal{N}_{12 \mid 3,4,5,6}(\ell) I_{12,3,4,5,6}+\mathcal{N}_{1 \mid 23,4,5,6}(\ell) I_{1,23,4,5,6}+\mathcal{N}_{1 \mid 2,34,5,6}(\ell) I_{1,2,34,5,6}\right. \\
& \quad+\mathcal{N}_{1 \mid 2,3,45,6}(\ell) I_{1,2,3,45,6}+\mathcal{N}_{1 \mid 2,3,4,56}(\ell) I_{1,2,3,4,56}+\mathcal{N}_{61 \mid 2,3,4,5}^{\prime}(\ell) I_{1,2,3,4,5}  \tag{12.2.3}\\
& - \\
& -\mathcal{N}_{21 \mid 3,4,5,6}^{\prime}(\ell) I_{1,3,4,5,6}-\mathcal{N}_{1 \mid 32,4,5,6}(\ell) I_{1,32,4,5,6}-\mathcal{N}_{1 \mid 2,43,5,6}(\ell) I_{1,2,43,5,6} \\
& - \\
& \left.-\mathcal{N}_{1 \mid 2,3,54,6}(\ell) I_{1,2,3,54,6}-\mathcal{N}_{1 \mid 2,3,4,65}(\ell) I_{1,2,3,4,65}-\mathcal{N}_{16 \mid 2,3,4,5}(\ell) I_{16,2,3,4,5}\right)
\end{align*}
$$

We would like to rewrite these terms using the numerators from $A(1,2,3,4,5,6 ; \ell)$. In most cases, this can be done immediately using relations of the form $-\mathcal{N}_{1 \mid 32,4,5,6}(\ell) I_{1,32,4,5,6}=$

[^23]$\mathcal{N}_{1 \mid 23,4,5,6}(\ell) I_{1,23,4,5,6}$. The exception are the terms
\[

$$
\begin{align*}
& -\frac{1}{2}\left(\mathcal{N}_{21 \mid 3,4,5,6}(\ell) I_{1,3,4,5,6}+\mathcal{N}_{16 \mid 2,3,4,5}(\ell) I_{16,2,3,4,5}\right) \\
& =\frac{1}{2}\left(\mathcal{N}_{12 \mid 3,4,5,6}^{a_{2}=+1}(\ell) I_{1,3,4,5,6}+\mathcal{N}_{61 \mid 2,3,4,5}^{a_{6}=-1}(\ell) I_{16,2,3,4,5}\right)  \tag{12.2.4}\\
& =\frac{1}{2}\left(\mathcal{N}_{12 \mid 3,4,5,6}^{a_{2}=+1}\left(\ell-k_{2}\right) I_{12,3,4,5,6}+\mathcal{N}_{61 \mid 2,3,4,5}^{a_{6}=-1}\left(\ell+k_{6}\right) I_{1,2,3,4,5}\right)
\end{align*}
$$
\]

where in the second line we used the antisymmetry of the pentagons discussed in section 11.1.3. If these numerators obeyed dihedral symmetries, the above would yield

$$
\begin{equation*}
\frac{1}{2}\left(\mathcal{N}_{12 \mid 3,4,5,6}(\ell) I_{12,3,4,5,6}+\mathcal{N}_{61 \mid 2,3,4,5}(\ell) I_{1,2,3,4,5}\right) \tag{12.2.5}
\end{equation*}
$$

and the terms (12.2.3) would reduce to the pentagon component of $A(1,2,3,4,5,6 ; \ell)$. Unfortunately this is not the case, and we have instead ${ }^{5}$

$$
\begin{align*}
& \mathcal{N}_{12 \mid 3,4,5,6}(\ell)-\mathcal{N}_{12 \mid 3,4,5,6}^{a_{2}=+1}\left(\ell-k_{2}\right) \approx-\frac{1}{2} V_{1} J_{2 \mid 3,4,5,6}  \tag{12.2.6}\\
& \mathcal{N}_{61 \mid 2,3,4,5}(\ell)-\mathcal{N}_{61 \mid 2,3,4,5}^{a=-1}\left(\ell+k_{6}\right) \approx-\frac{1}{2} V_{1} J_{6 \mid 2,3,4,5} \tag{12.2.7}
\end{align*}
$$

Hence the colour-dressed integrand (12.2.1) becomes

$$
\begin{equation*}
M_{6}(\ell)=\left(A(1,2,3,4,5,6 ; \ell)+C_{1,2,3,4,5,6}\right) H_{1,2,3,4,5,6}+\operatorname{perm}(2,3,4,5,6), \tag{12.2.8}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{1,2,3,4,5,6}=\frac{1}{4}\left(V_{1} J_{2 \mid 3,4,5,6} I_{12,3,4,5,6}+V_{1} J_{6 \mid 2,3,4,5} I_{16,2,3,4,5}\right) . \tag{12.2.9}
\end{equation*}
$$

This suggests that, in order to have vanishing BRST variation, we should reformulate the colour-dressed integrand as

$$
\begin{align*}
M_{6}^{\prime}(\ell) & =M_{6}(\ell)-\left[C_{1,2,3,4,5,6} H_{1,2,3,4,5,6}+\operatorname{perm}(2,3,4,5,6)\right]  \tag{12.2.10}\\
& =A(1,2,3,4,5,6 ; \ell) H_{1,2,3,4,5,6}+\operatorname{perm}(2,3,4,5,6)
\end{align*}
$$

It then follows from the partial amplitudes having this property that this is BRST invariant $Q M_{6}^{\prime}(\ell)=0$.

### 12.2.2 The Six-Point Supergravity Integrand

The six point supergravity integrand constructed using the double copy procedure with these numerators has non-vanishing variation, and therefore is invalid. To begin, we note

[^24]that the expression (12.2.10) may not be used to generate the supergravity amplitude. If such were used, and the hexagon colour factors were replaces by corresponding hexagon kinematic factors, then the resulting expression would not be symmetric in its left and right moving modes by the discussion of the previous subsection. As such, we must return to equation (12.2.1), and therein perform the double copy procedure.

To see that this then has non-vanishing variation, as at five points we focus upon a subset of terms in the left-moving BRST variation $Q M_{6}(\ell)$,

$$
\begin{align*}
& V_{1} V_{36} T_{2,4,5}\left(I_{1,2,4,5} \tilde{N}_{1 \mid 2,4,5,36}(\ell)-I_{136,2,4,5} \tilde{N}_{1 \mid 36,2,4,5}(\ell)-I_{136,2,4,5} \tilde{N}_{361 \mid 2,4,5}\right)  \tag{12.2.11}\\
& =V_{1} V_{36} T_{2,4,5} I_{136,2,4,5}\left(\tilde{N}_{1 \mid 2,4,5,36}\left(\ell-k_{36}\right)-\tilde{N}_{1 \mid 36,2,4,5}(\ell)-\tilde{N}_{361 \mid 2,4,5}\right) .
\end{align*}
$$

The missing labels in $I_{1,2,4,5}$ again arise from loop-momentum cancellations in $Q N_{1 \mid 2,4,5,36}(\ell) I_{1,2,4,5,36}$. This is compensated by the shift $\ell \rightarrow \ell-k_{36}$, which in performing we must also apply to the right-moving pentagon in the second line. If the numerators satisfied dihedral symmetries, that is if the condition

$$
\begin{equation*}
\tilde{N}_{1 \mid 2,4,5,36}\left(\ell-k_{36}\right)=\tilde{N}_{36 \mid 1,2,4,5}(\ell) \tag{12.2.12}
\end{equation*}
$$

were satisfied, then (12.2.11) would vanish identically by the BCJ relation

$$
\begin{equation*}
\tilde{N}_{36 \mid 1,2,4,5}(\ell)-\tilde{N}_{1 \mid 36,2,4,5}(\ell)-\tilde{N}_{361 \mid 2,4,5}=0, \tag{12.2.13}
\end{equation*}
$$

Unfortunately such is not the case, and unlike at five points this is not true even in the cohomology,

$$
\begin{equation*}
\left\langle\tilde{N}_{1 \mid 2,4,5,36}\left(\ell-k_{36}\right)\right\rangle \neq\left\langle\tilde{N}_{36 \mid 1,2,4,5}(\ell)\right\rangle . \tag{12.2.14}
\end{equation*}
$$

This is not a simple problem to fix, as these shifts in the loop momentum will always need to be performed when the variation of numerators described in this thesis are found. As such, a new approach is needed, and some idea of such are described in the following section.

## CHAPTER 13

## Summary and Outlook

In this part, we described a set of rules by which the field-theory limits of the KroneckerEisenstein coefficient functions in the genus-one superstring correlators derived in [20; 21; $22]$ may be taken. Using these, we found expressions for numerators in ten-dimensional SYM at one loop for five, six and seven points which satisfy BCJ identities. We therefore resolved the difficulties present in showing BCJ relations in an earlier work [1].

These field-theory limits necessarily take into account the parameterisation of the loop momentum integrands, shuffling terms between the various numerators accordingly in order to preserve BCJ identities. While BRST invariance of the overall SYM one-loop integrands is maintained in this action, the BRST properties of individual numerators changes in a non-trivial way (See the discussion around (11.1.35)). This leads to the numerators violating the dihedral symmetries one would naively expect to be present. However, without this shuffling of terms BCJ relations would be violated.

As a direct consequence of this, we learned when we attempted to apply the double copy procedure that in order for such to be successful, the numerators must satisfy both kinematic Jacobi identities, and the dihedral symmetries of the corresponding graph. Unfortunately our six-point numerators do not satisfy these symmetries and the double-copy construction initiated here remains incomplete. As such, applying the double copy procedure must be left for future work, and we outline in the following outlook section how one may go about it.

We should note here the state of the field more broadly. Supergravity integrands have been constructed using BCJ numerators in four dimensions for up to seven points in [72], and to arbitrary multiplicity in [73] using spinor helicity methods. Supergravity amplitudes were also constructed in [74], but using a partial-fraction representation of the loop momentum integrands. They have not been constructed in ten dimensions using traditional Feynman loop momentum integrands, and so if such were found it would be a new result.

We do not provide a summary of the methods described in this part here as in the previous summary, as to do so would be to repeat the discussion from the start of section 10.2 to the start of subsection 10.2 .1 . We refer the reader to such for a summary of the formulae developed in this part.

### 13.1 Outlook

There are numerous directions in which further work on this project may be performed. Here we discuss just two of them; enforcing the double copy at six points, and computing SYM amplitudes at higher points. We discuss some of the difficulties in each, and outline a potential strategy to go about each of them.

### 13.1.1 Supergravity Amplitudes at Six and Higher Points

The most immediately obvious next step for this work would be to find means by which the problems related to the double copy may be fixed. Once an approach to such has been found at six points, it is likely that the same method will hold at seven points, and for any higher point amplitudes which may be found also using these methods. Several approaches to this have been attempted, and here we discuss a few and why they have failed, as well as one route which remains unexplored.

The majority of our attention has been focused upon restoring the vanishing of the BRST variation of the colour dressed amplitude as represented in (12.2.1). One such consideration was to sum over amplitudes with different orderings. That is, rather than only considering diagrams with a single shared loop momentum structure, allow for many. Unfortunately this does not appear to work; if one replaces the numerators in (12.2.1) with their general $a_{i}$ equivalents then the same error terms are found. One may wonder about assigning different $a_{i}$ values to different numerators, however this is believed to be forced into the previous situation via the constraint of vanishing BRST variation. This last point should not be considered fully explored however.

One partial simplification of the results lies in the relations between diagrams and their reversal. That is, by expanding the colour factors in terms of structure constants, it may be found that

$$
\begin{equation*}
H_{1,6,5,4,3,2}=H_{1,2,3,4,5,6} . \tag{13.1.1}
\end{equation*}
$$

Similarly, one may make substitutions in the loop momentum to invert the ordering of the diagram they correspond with. For example, at six points one may perform the substitution $\ell^{\prime}=-\ell+k_{12}$ in $I_{12,6,5,4,3}$,

$$
\begin{align*}
\int d^{10} \ell I_{12,6,5,4,3} & =\int \frac{d^{10} \ell}{\left(\ell-k_{12}\right)^{2}\left(\ell-k_{126}\right)^{2}\left(\ell-k_{1265}\right)^{2}\left(\ell-k_{12654}\right)^{2}\left(\ell-k_{126543}\right)^{2}} \\
& =\int \frac{d^{10} \ell^{\prime}}{\left(-\ell^{\prime}\right)^{2}\left(-\ell^{\prime}-k_{6}\right)^{2}\left(-\ell^{\prime}-k_{65}\right)^{2}\left(-\ell^{\prime}-k_{654}\right)^{2}\left(-\ell^{\prime}-k_{6543}\right)^{2}}  \tag{13.1.2}\\
& =\int \frac{d^{10} \ell}{(\ell)^{2}\left(\ell-k_{12345}\right)^{2}\left(\ell-k_{1234}\right)^{2}\left(\ell-k_{123}\right)^{2}\left(\ell-k_{12}\right)^{2}} \\
& =\int d^{10} \ell I_{12,3,4,5,6}
\end{align*}
$$

Note any minus signs incurred from swapping the order of integration and in the $d^{10} \ell$ are raised to the tenth power, and so vanish. Similar is true in general, with a substitution $\ell^{\prime}=-\ell+k_{A_{1}}$ taking any $I_{A_{1}, A_{2}, \ldots, A_{n}}$ to $I_{A_{1}, A_{n}, A_{n-1}, \ldots, A_{2}}$, if the words $A_{1}, \ldots, A_{n}$ contain every particle label exactly once. This result does not cancel the error terms in the colour dressed amplitude however, it merely reduces the number of propagator structures which have to be summed over.

When expanded in terms of components, the failure of the dihedral symmetries is found to be proportional to Mandelstam variables. For instance, the difference between different representations of the 23 -pentagon in (12.2.14) is proportional to $s_{23}$ after it is expanded in components and evaluated in the pure spinor bracket;

$$
\begin{align*}
\left\langle\tilde{N}_{23 \mid 1,4,5,6}(\ell)-\tilde{N}_{1 \mid 4,5,6,23}\left(\ell-k_{23}\right)\right\rangle & =\left\langle k_{23}^{m} V_{1} T_{23,4,5,6}^{m}+V_{231} T_{4,5,6}+\left[V_{1} T_{234,5,6}+4 \leftrightarrow 5,6\right]\right\rangle \\
& \sim s_{23}(\ldots) . \tag{13.1.3}
\end{align*}
$$

This difference is the same as the terms by which the BCJ identity fails in equation (6.12) of [1].

Solving the problems related to the double copy at one loop appears to require a different approach in the pure spinor superspace context, and as the failures discussed above are purely contact terms we are drawn to the generalized double-copy prescription of [80]. In this reference, a similar situation was encountered and the double copy was successfully applied in spite of it, and as such this seems like a strong candidate for a solution. It does not seem unreasonable to speculate that the problems in constructing double copy
amplitudes encountered in this part may be more widespread, and if so the generalised double copy approach may become the standard method for generating gravity integrands from gauge-theory. Discussion of similar problems and approaches through the generalised double copy to solving them may also be found in $[166 ; 167 ; 168 ; 169 ; 170]$.

### 13.1.2 Higher point amplitudes

One alternative direction for future work on this project would be to attempt to generate higher point amplitudes. Following the approach set out in this work may lead to some success, and some preliminary work has been carried out to find a candidate for an octogan numerator at eight points based upon the candidate correlator set out in [22] and worldsheet functions at [171]. However, this approach is restrained by difficulties relating to the refined worldsheet functions. Alike at seven points, these contain double poles and partial derivatives. Here though it is not yet clear how to remove them.

To demonstrate, consider $\mathcal{Z}_{12 \mid 34,5,6,7,8}$. The partial derivative terms within this function are

$$
\begin{align*}
-g_{12}^{(1)} \partial g_{23}^{(2)} & +g_{12}^{(1)} \partial g_{24}^{(2)}+g_{23}^{(1)} \partial g_{12}^{(2)}-g_{24}^{(1)} \partial g_{12}^{(2)}+g_{34}^{(1)} \partial g_{12}^{(2)}-g_{12}^{(2)} \partial g_{23}^{(1)} \\
& +g_{12}^{(2)} \partial g_{24}^{(1)}+g_{13}^{(2)} \partial g_{23}^{(1)}-g_{14}^{(2)} \partial g_{24}^{(1)}-\partial g_{23}^{(3)}+\partial g_{24}^{(3)} . \tag{13.1.4}
\end{align*}
$$

We may modify the relation (10.3.49) for eight points, and use this to simplify several terms. Likewise, several more terms may be simplified using relations of the form

$$
\begin{equation*}
\partial\left(g_{a b}^{(1)} g_{c d}^{(2)}\right) \mathcal{I}_{8}(\ell)=-g_{a b}^{(1)} g_{c d}^{(2)} \partial \mathcal{I}_{8}(\ell)+\partial\left(g_{a b}^{(1)} g_{c d}^{(2)} \mathcal{I}_{8}(\ell)\right) . \tag{13.1.5}
\end{equation*}
$$

However, these two results will not simplify the function entirely. After applying them, we may reduce the list of derivative terms (13.1.4) to

$$
\begin{equation*}
-g_{12}^{(1)} \partial g_{23}^{(2)}+g_{12}^{(1)} \partial g_{24}^{(2)}+g_{34}^{(1)} \partial g_{12}^{(2)}+g_{13}^{(2)} \partial g_{23}^{(1)}-g_{14}^{(2)} \partial g_{24}^{(1)} . \tag{13.1.6}
\end{equation*}
$$

It is then unknown how to either rewrite these terms as those for which we know how to take field theory limits, or how to take the limits of these terms directly.

An alternative approach however, could lie in identifying patterns within the lower point amplitudes. Various patterns may be identified in the lower point amplitudes; and it may be possible to use these to generate higher point amplitudes. For example, limiting ourselves to amplitudes of the form $A(1,2, \ldots, n ; \ell)$, it seems reasonable to assume box numerators will always have the form

$$
\begin{equation*}
N_{A \mid B, C, D}=V_{A} T_{B, C, D}, \tag{13.1.7}
\end{equation*}
$$

and that at least with the $V T$ terms, the diagrams in which the $n$ and 1 legs are not part of a shared external tree follow a standard pattern. So for instance, one might expect that the 8 point $[1,[[2,3], 4]]$ pentagon has the form

$$
\begin{align*}
N_{[1,[[2,3], 4]] \mid 5,6,7,8}=V_{[1,[[2,3], 4]} T_{5,6,7,8} \ell^{m} & +\frac{1}{2}\left(V_{[[1,[[2,3], 4]], 5]} T_{6,7,8}+(5 \leftrightarrow 6,7,8)\right) \\
& +\frac{1}{2}\left(V_{[1,[[2,3], 4]]} T_{56,7,8}+(5,6 \mid 5,6,7,8)\right)  \tag{13.1.8}\\
& + \text { Possible refined terms } .
\end{align*}
$$

Similar arguments may fix the $V T$ in all such diagrams, and it may be that a similar structure holds across the diagrams with $n$ and 1 in an external tree also. Then the $n$-gon may also be partially identified by studying the form of $m$-gons at $m$ points, for $m<n$. For example, the $V_{A} T_{B, C, D, E}^{m}$ terms in the six point hexagon have a similar structure to the $V_{A} T_{B, C, D}$ terms in the five point pentagon, and similar relations hold for other terms in $n$ and ( $n-1$ )-gons.

Using results of this sort, it should be possible to fix most $V T$ terms in the eight point amplitude. The constraint of vanishing BRST variation may then be used to fix the coefficients of other possible remaining terms in the amplitude. Further, this could be done term by term in order to simplify the computation. That is, one should be able to begin with the eight point octogan, and use the patterns we anticipate being present and the constraint that the variation should be proportional to terms of the form

$$
\begin{equation*}
\left(\ell-k^{12 \ldots m}\right)^{2}, \quad m \in\{1,2, \ldots, 8\}, \tag{13.1.9}
\end{equation*}
$$

in order to fix most unknown terms. Then, the suspected patterns within numerators, the requirement that seven of the heptagon diagrams are likely given by a standard formula, and the constraint that their variation must be the negative of the relevant terms from the octogan plus terms proportional to the above, may be used to fix most terms in the eight point heptagons.

Using an approach like this, it may be possible to find the eight point one loop amplitude by brute force, without becoming computationally unfeasible. Once such is found, we would have a large data set of amplitudes, including two which include refined terms. At such a point, speculations about general formulae may become reasonable, and a nine point amplitude may be identified by patterns in numerators alone. Further, if such a general formula were known, it may be possible to prove rigorously that amplitudes it produces are always in the BRST cohomology. Finally, if one could identify an eight point amplitude, this could potentially be used to fix some of the uncertainty surrounding the eight point correlator in [22]. We stress that these speculations are just that however, and it would take considerable work to see if these approaches would work as required.

## Part IV

## Conclusion

In this thesis, we have described procedures by which scattering amplitudes at tree level and one loop may be constructed so as to satisfy BCJ identities. We will now conclude with a brief summary of these results, as well as an outline of the future directions research could take in this area.

We began in part I with a review of the literature relevant to this thesis. We outlined super Yang-Mills, supergravity, and how the two are linked through the double copy when BCJ identities are satisfied. We introduced the pure spinor formalism of string theory, and described how amplitudes are constructed within it. This construction was then simplified with the aid of multiparticle superfields, and a BCJ gauge wherein these satisfied generalised Jacobi identities was described in a number of cases. Using such, formulae for tree level and one loop amplitudes in both string and field theory were detailed, with these satisfying BCJ identities when their constituent superfields where in the eponymous gauge.

In part II, we discussed the work in [27] to generalise the concept of the BCJ gauge to higher orders. The construction of this gauge, and various other formulae related to multiparticle superfields, were reformulated in terms of the contact term map. An explicit form for the $H_{[P, Q]}$ superfields was found and conjectured to hold in general, and with this all steps of the hybrid gauge construction of the BCJ gauge were defined in general. Similarly, the procedure to move directly from the Lorenz to the BCJ gauge was described to higher orders than was previously known. It was then proven that the BCJ gauge terminology was not a misnomer; that it is indeed a gauge transformation away from the Lorenz gauge by the formulae of this part.

In part III, we detailed the work of [29], describing how the string correlators identified in $[20 ; 21 ; 22]$ may be used to generate amplitudes in field theory. Amplitudes to seven points were then identified, and these were shown to satisfy BCJ identities. However, a complication arose when we attempted to apply the double copy to these results to find amplitudes in supergravity. In order to avoid the labelling problem, extra attention had been paid to the loop momentum structure of amplitudes we constructed. While this meant we were able to then find the amplitudes and show that they satisfied BCJ identities, another unfortunate result became apparent also. Namely, the numerators we had found did not satisfy dihedral symmetries. As a result, attempts to show the vanishing of the colour dressed amplitude failed at six and higher points, and while these could be corrected for by modifying the proposed form of the amplitude, such was not also the case for the corresponding supergravity amplitudes. We were forced to conclude that, if one wishes to apply the standard double copy construction, dihedral symmetries are required in addition to the BCJ identities.

There are numerous directions the work in this thesis could be taken further. We have
described some of these in varying levels of detail, and we summarise these now. Beginning with the BCJ gauge construction, it appears that the $\tilde{H}_{A, B, C}$ terms constructed in order to describe the redefinition $H_{[P, Q]}$ terms may be used to more efficiently describe amplitudes. That is, the $\theta=0$ component of these appears to correspond exactly with the pure spinor bracket of three multiparticle integrated vertex operators. The reasoning for this is mysterious, and as it presents an opportunity to significantly increase the speed at which amplitudes are calculated it is worthy of investigation.

Then also, formulae exist for the redefinition $H_{[P, Q]}$ terms only when $P$ and $Q$ a single class of Lie monomials. While all others may be related to these by Jacobi identities, it would be interesting to find a truly general expression for these terms. Doing so would potentially reveal further details about the BCJ gauge and its origin. Some work has been done on this problem in the past, and though the results it produced are likely false they are presented in the appendix F as a starting point for any future work in this area.

There are then other areas which have been explored in less detail. Higher rank verifications of various formulae are needed, and in some cases we would like to simplify the appearance of formulae also. Further, it would be interesting to attempt to prove further relations related to the BCJ gauge construction; one initial attempt at such is provided in appendix B, and there are many other formulae which it could be beneficial to rigorously prove. It would be interesting to find a defining equation of the BCJ gauge; that is, to find a means of defining the BCJ gauge similar to how the orthogonality of superfields and momentum defines the Lorenz gauge. The construction of multiparticle superfields has been extended to the small $\alpha^{\prime}$ regime, and it may be possible to extend or formula for $H_{[P, Q]}$ to this regime also. Then finally there may be opportunities to use the formulae developed to make connections with the kinematic algebra and the $L_{\infty}$-algebra construction of Berends-Giele currents.

Moving onto the construction of one loop amplitudes using pure spinor methods, this is an area with a clear and immediate next project to be worked upon; finding a method by which supergravity amplitudes can be constructed correctly. Though we have attempted multiple approaches to rectify the problems there, it remains an open problem in need of solving. The current best candidate we have for a method by which the problem may be solved lies in the generalised double copy construction of [80]. Therein, problems comparable to those encountered in this work were found, and a method was described by which they were rectified. Performing an analogous procedure seems like a strong candidate for a method by which higher point supergravity amplitudes can be constructed.

Then there is the question of whether we may move to finding higher point one-loop amplitudes in super Yang-Mills. While the results in this thesis were derived from the string correlator, there are additional difficulties in such an approach at higher points and
so it becomes less feasible. Instead, a combination of identifying patterns in numerators at lower points and brute force is thought to be the best approach to this problem. While this would be inelegant at first, once the eight point amplitude were found the underlying structure of the amplitudes should be becoming clearer. As such, the nine point amplitude should require significantly less brute force than the eight point case, and similar for higher points.

More broadly, there are other areas of research related to the problems worked on in this thesis which remain to be investigated. Some work has been performed to at higher loops, with the two loop five point amplitude being identified in [23; 172]. These results could be extended to higher points. At three and higher loops subtleties arise when one works in the pure spinor formalism, and a more complex scheme has to be used [173]. Nevertheless, there is always the possibility that work could be performed in that direction also. Then finally, there are a range of other aspects of the double copy worthy of investigation, and many open questions in this area [12].

## Part V

## Appendix

## Additional Discussion of Notation

Here we summarise various aspects of notation used in the course of this thesis, and provide examples of such.

## A. 1 Summation Notations

## A.1.1 Permutation Sums

When we have a sum which is denoted

$$
\begin{equation*}
\text { (some function of } 1,2, \ldots, n)+(1,2, \ldots, m \mid 1,2, \ldots, n), \tag{A.1.1}
\end{equation*}
$$

this means we sum over the possible ways to select a set of $m$ letters from $1,2, \ldots, n$, maintaining their order, and substitute them in for $1,2, \ldots, m$. The unselected terms are then use the remainder to fill in any missing spots. This is much easier to explain with an example, so suppose we have

$$
\begin{equation*}
V_{1} T_{[[2,3], 4], 5,6}+(2,3,4 \mid 2,3,4,5,6) \tag{A.1.2}
\end{equation*}
$$

The above sum sums over all ways of selecting three numbers from $2,3,4,5,6$, and substituting them into the $a, b, c$ slots of $T_{[[a, b], c], d, e}$ while maintaining their order. The unselected
terms are then substituted in for $d$ and $e$. This is thus

$$
\begin{gather*}
V_{1} T_{[[2,3], 4], 5,6}+V_{1} T_{[2,3], 5], 4,6}+V_{1} T_{[[2,3], 6], 4,5}+V_{1} T_{[22,4], 5], 3,6}+V_{1} T_{[[2,4], 6], 3,5}  \tag{A.1.3}\\
+V_{1} T_{[[2,5], 6], 3,4}+V_{1} T_{[33,4], 5], 2,6}+V_{1} T_{[33,4], 6], 2,5}+V_{1} T_{[3,5], 6], 2,4}+V_{1} T_{[[4,5], 6], 2,3} .
\end{gather*}
$$

This notation may be generalised to have two or more blocks of indices being summed over. For example the summation

$$
\begin{equation*}
V_{1} T_{[2,3],[4,5], 6}+(2,3|4,5| 2,3,4,5,6) . \tag{A.1.4}
\end{equation*}
$$

This sums over all ways of selecting two distinct pairs of numbers from 2, 3, 4, 5, 6, and substituting one into the 2 and 3 slots of the $T$, and the other into the 4 and 5 slots. Note the relative ordering of the 2 and 3 , and the 4 and 5 should be maintained, but not that of the pairs ${ }^{1}$. One should be careful in these sums not to duplicate terms. That is, thanks to the symmetry of $T_{A, B, C}$ in $A, B$ and $C$, setting the first block to $[2,3]$ and the second to $[4,5]$ say should not be considered distinct from setting the first to $[4,5]$ and the second to $[2,3]$. They are the same term, and we need to be careful about including them in the above sum else we accidentally overcount terms. The expansion of (A.1.4) is

$$
\begin{align*}
& V_{1} T_{[2,3],[4,5], 6}^{m}+V_{1} T_{[2,3],[4,6], 5}^{m}+V_{1} T_{[2,3],[5,6], 4}^{m}+V_{1} T_{[2,4],[3,5], 6}^{m}+V_{1} T_{[2,4],[3,6], 5}^{m} \\
&+ V_{1} T_{[2,4],[5,6], 3}^{m}+V_{1} T_{[2,5],[3,4], 6}^{m}+V_{1} T_{[2,5],[3,6], 4}^{m}+V_{1} T_{[2,5],[4,6], 3}^{m}+V_{1} T_{[2,6],[3,4], 5}^{m}  \tag{A.1.5}\\
&+V_{1} T_{[2,6],[3,5], 4}^{m}+V_{1} T_{[2,6],[4,5], 3}^{m}+V_{1} T_{[3,4],[5,6], 2}^{m}+V_{1} T_{[3,5],[4,6], 2}^{m}+V_{1} T_{[3,6],[4,5], 2}^{m}
\end{align*}
$$

## A.1.2 Stirling Cycles

Permutations of sequences of numbers can be represented with Stirling cycles. If we were looking at for instance the permutations of the sequence 123456, then two examples of Stirling cycle representations of permutations would be (1435)(2)(6) and (1)(25)(3)(46). A bracket (...ij...) takes whatever element is in position $i$ and moves it to position $j$. So the first example sends 1 to the position of 4,4 to that of 3,3 to 5,5 to 1 , and it leaves alone 2 and 6 . So the result of acting with the first example on the sequence is 524136 . Likewise the second example only acts by swapping 2 with 5 and 4 with 6 , and so gives 153624.

We note now the structure of Stirling cycles. There is some number of brackets in each permutation. Each bracket is cyclic in its elements (so for instance, (1435) $=(4351)=$ $(3514)=(5143)$ ), but by convention we always lead each bracket with its lowest element.

[^25]Likewise, the ordering of the brackets is irrelevant (so (1435)(2)(6) = (6)(2)(1435) = $(2)(1435)(6)=\ldots)$, but by convention we always write them in order of their lowest element. With these standardisations we may construct a sum over Stirling cycles, which will appear as

$$
\begin{equation*}
\text { (some function of } \left.A_{1}, \ldots, A_{m}\right)+\left[A_{1}, \ldots, A_{m} \mid 12 \ldots n\right], \tag{A.1.6}
\end{equation*}
$$

This means that we take the sequence $12 \ldots n$, and from it construct all possible Stirling cycles with $n$ brackets. We then substitute in the first bracket for $A_{1}$, the second for $A_{2}$, and so on. So, for example

$$
\begin{align*}
V_{A_{1}} T_{A_{2}, A_{3}, A_{4}}+\left[A_{1}, A_{2}, A_{3}, A_{4} \mid 12345\right]= & +V_{12} T_{3,4,5}+V_{13} T_{2,4,5}+V_{14} T_{2,3,5}+V_{15} T_{2,3,4} \\
& +V_{1} T_{23,4,5}+V_{1} T_{24,3,5}+V_{1} T_{25,3,4}  \tag{A.1.7}\\
& +V_{1} T_{2,34,5}+V_{1} T_{2,35,4}+V_{1} T_{2,3,45}
\end{align*}
$$

To illustrate the meaning of a sum over Stirling Cycles, we give the full expansion of the six point one loop string correlator. Recall this in terms of Stirling cycle sums, equation (5.3.19) [22]

$$
\begin{align*}
\mathcal{K}_{6}(\ell) & =\frac{1}{2} V_{1} T_{2,3,4,5,6}^{m n} \mathcal{Z}_{1,2,3,4,5,6}^{m n} \\
& +V_{A} T_{B, C, D, E}^{m} \mathcal{Z}_{A, B, C, D, E}^{m}+[123456 \mid A, B, C, D, E]  \tag{A.1.8}\\
& +V_{A} T_{B, C, D} \mathcal{Z}_{A, B, C, D}+[123456 \mid A, B, C, D]
\end{align*}
$$

The first line needs no expansion, and so we begin with the second. Note we drop the $\mathcal{Z}$ worldsheet functions below for simplicity, but they follow naturally.

$$
\begin{align*}
V_{A} T_{B, C, D, E}^{m}+[123456 \mid A, B, C, D, E]= & +V_{12} T_{3,4,5,6}^{m}+V_{13} T_{2,4,5,6}^{m}+V_{14} T_{2,3,5,6}^{m} \\
& +V_{15} T_{2,3,4,6}^{m}+V_{16} T_{2,3,4,5}^{m}+V_{1} T_{23,4,5,6}^{m} \\
& +V_{1} T_{24,3,5,6}^{m}+V_{1} T_{25,3,4,6}^{m}+V_{1} T_{26,3,4,5}^{m}  \tag{A.1.9}\\
& +V_{1} T_{34,2,5,6}^{m}+V_{1} T_{35,2,4,6}^{m}+V_{1} T_{36,2,4,5}^{m} \\
& +V_{1} T_{45,2,3,6}^{m}+V_{1} T_{46,2,3,5}^{m}+V_{1} T_{56,2,3,4}^{m}
\end{align*}
$$

The third line has expansion

$$
\begin{aligned}
V_{A} T_{B, C, D}+[123456 \mid A, B, C, D]= & +V_{123} T_{4,5,6}+V_{132} T_{4,5,6}+V_{124} T_{3,5,6}+V_{142} T_{3,5,6} \\
& +V_{125} T_{3,4,6}+V_{152} T_{3,4,6}+V_{126} T_{3,4,5}+V_{162} T_{3,4,5} \\
& +V_{134} T_{2,5,6}+V_{143} T_{2,5,6}+V_{135} T_{2,4,6}+V_{153} T_{2,4,6} \\
& +V_{136} T_{2,4,5}+V_{163} T_{2,4,5}+V_{145} T_{2,3,6}+V_{154} T_{2,3,6} \\
& +V_{146} T_{2,3,5}+V_{164} T_{2,3,5}+V_{156} T_{2,3,4}+V_{165} T_{2,3,4} \\
& +V_{1} T_{234,5,6}+V_{1} T_{243,5,6}+V_{1} T_{235,4,6}+V_{1} T_{253,4,6}
\end{aligned}
$$

$$
\begin{align*}
& +V_{1} T_{236,4,5}+V_{1} T_{263,4,5}+V_{1} T_{245,3,6}+V_{1} T_{254,3,6} \\
& +V_{1} T_{246,3,5}+V_{1} T_{264,3,5}+V_{1} T_{256,3,4}+V_{1} T_{265,3,4} \\
& +V_{1} T_{345,2,6}+V_{1} T_{354,2,6}+V_{1} T_{346,2,5}+V_{1} T_{364,2,5} \\
& +V_{1} T_{356,2,4}+V_{1} T_{365,2,4}+V_{1} T_{456,2,3}+V_{1} T_{465,2,3} \\
& +V_{12} T_{34,5,6}+V_{12} T_{35,4,6}+V_{12} T_{36,4,5}+V_{12} T_{45,3,6} \\
& +V_{12} T_{46,3,5}+V_{12} T_{56,3,4}+V_{13} T_{24,5,6}+V_{13} T_{25,4,6} \\
& +V_{13} T_{26,4,5}+V_{13} T_{45,2,6}+V_{13} T_{46,2,5}+V_{13} T_{56,2,4} \\
& +V_{14} T_{23,5,6}+V_{14} T_{25,3,6}+V_{14} T_{26,3,5}+V_{14} T_{35,2,6} \\
& +V_{14} T_{36,2,5}+V_{14} T_{56,2,3}+V_{15} T_{23,4,6}+V_{15} T_{24,3,6} \\
& +V_{15} T_{26,3,4}+V_{15} T_{34,2,6}+V_{15} T_{36,2,4}+V_{15} T_{46,2,3} \\
& +V_{16} T_{23,4,5}+V_{16} T_{24,3,5}+V_{16} T_{25,3,4}+V_{16} T_{34,2,5} \\
& +V_{16} T_{35,2,4}+V_{16} T_{45,2,3}+V_{1} T_{23,45,6}+V_{1} T_{23,46,5} \\
& +V_{1} T_{23,56,4}+V_{1} T_{24,35,6}+V_{1} T_{24,36,5}+V_{1} T_{24,56,3} \\
& +V_{1} T_{25,34,6}+V_{1} T_{25,36,4}+V_{1} T_{25,46,3}+V_{1} T_{26,34,5} \\
& +V_{1} T_{26,35,4}+V_{1} T_{26,45,3}+V_{1} T_{34,56,2}+V_{1} T_{35,46,2} \\
& +V_{1} T_{36,45,2} \tag{A.1.10}
\end{align*}
$$

## A.1.3 Deshuffle Products

Frequently we encounter sums over deshuffle products, denoted $R \sqcup S=Y$, for instance in the variation of the multiparticle vertex operators with Dynkin bracket structures

$$
\begin{equation*}
Q V_{\ell(P)}=\sum_{\substack{X j Y=P \\ R \amalg S=Y}}\left(k^{X} \cdot k^{j}\right) V_{\ell(X R)} V_{\ell(j S)} \tag{A.1.11}
\end{equation*}
$$

To explain this notation, first we recall the definition of the shuffle product (2.1.18). The deshuffle product is then the sum over all words $R$ and $S$ such that their shuffle product $R \sqcup S$ contains the word $Y$. This is inherently symmetric in $R$ and $S$, and is most easily demonstrated with an example. Suppose $Y=345$, Then for $R$ and $S$ we must sum over the values

$$
R \otimes S \in\{\emptyset \otimes 345,3 \otimes 45,4 \otimes 35,5 \otimes 34,34 \otimes 5,35 \otimes 4,45 \otimes 3,345 \otimes \emptyset\}
$$

This may alternatively be thought of as the sum over the power set of the word $Y$. A power set of a word $A$ is the set of all words $B$, such that $B$ is composed exclusively of letters in $A$ and in the same order as in $A$. This is denoted $\mathcal{P}(A)$, with the example above
being

$$
\begin{equation*}
\mathcal{P}(345)=\{\emptyset, 3,4,5,34,35,45,345\} \tag{A.1.13}
\end{equation*}
$$

The sum over deshuffles then corresponds over the sum of terms in the power set paired with their complements.

So, an example of a variation using (A.1.11) would be

$$
\begin{align*}
Q V_{12345} & =\left(k^{1} \cdot k^{2}\right)\left(V_{1} V_{2345}+V_{13} V_{245}+V_{14} V_{235}+V_{15} V_{234}\right. \\
& \left.+V_{134} V_{25}+V_{135} V_{24}+V_{145} V_{23}+V_{1345} V_{2}\right) \\
+ & \left(k^{12} \cdot k^{3}\right)\left(V_{12} V_{345}+V_{124} V_{35}+V_{125} V_{34}+V_{1245} V_{3}\right)  \tag{A.1.14}\\
+ & \left(k^{123} \cdot k^{4}\right)\left(V_{123} V_{45}+V_{1235} V_{4}\right) \\
+ & \left(k^{1234} \cdot k^{5}\right) V_{1234} V_{5}
\end{align*}
$$

## A. 2 One Loop Amplitude Notation

We now detail the various forms of notation used to simplify the appearance of one-loop amplitudes. This is primarily based upon the notation of [1]. We begin with an object $I$ to describe propagators,

$$
\begin{equation*}
I_{B 1 A_{1}, A_{2}, \ldots, A_{n}}=\frac{1}{\left(\ell-k_{1 A_{1}}\right)^{2}\left(\ell-k_{1 A_{1} A_{2}}\right)^{2} \ldots\left(\ell-k_{1 A_{1} A_{2} \ldots A_{n}}\right)^{2}} \tag{A.2.1}
\end{equation*}
$$

So, for instance, the simplest four point denominator is given by

$$
\begin{align*}
I_{1,2,3,4} & =\frac{1}{\left(\ell-k_{1}\right)^{2}\left(\ell-k_{12}\right)^{2}\left(\ell-k_{123}\right)^{2}\left(\ell-k_{1234}\right)^{2}}  \tag{A.2.2}\\
& =\frac{1}{\left(\ell-k_{1}\right)^{2}\left(\ell-k_{12}\right)^{2}\left(\ell-k_{123}\right)^{2}\left(\ell-k_{1234}\right)^{2}}
\end{align*}
$$

where the equality follows from momentum conservation. A more complicated example of this notation would be,

$$
\begin{align*}
I_{1,2,3,456,78} & =\frac{1}{\left(\ell-k_{1}\right)^{2}\left(\ell-k_{12}\right)^{2}\left(\ell-k_{123}\right)^{2}\left(\ell-k_{123456}\right)^{2}\left(\ell-k_{12345678}\right)^{2}}  \tag{A.2.3}\\
& =\frac{1}{\ell^{2}\left(\ell-k_{1}\right)^{2}\left(\ell-k_{12}\right)^{2}\left(\ell-k_{123}\right)^{2}\left(\ell-k_{123456}\right)^{2}}
\end{align*}
$$

where we assume we are working at eight points for the equality.

As for the numerators, we will describe an $n$-gon with an object $N$ with $n$ Lie monomial

(a)

(b)

(c)

Figure A.2.1: Three examples of one loop diagrams
indices,

$$
\begin{equation*}
N_{A_{1} \mid A_{2}, \ldots, A_{n}}(\ell) \tag{A.2.4}
\end{equation*}
$$

The $A_{i}$ represent the $i^{\text {th }}$ corner as we move clockwise about the diagram, with the monomial $A_{i}$ mapping to a tree at that corner in the usual way. This is made much clearer with examples. First of all, the four point amplitude consists of a single box, and is expressed in this notation by

$$
\begin{equation*}
A^{1-l o o p}(1,2,3,4)=N_{1 \mid 2,3,4}(\ell) I_{1,2,3,4} \tag{A.2.5}
\end{equation*}
$$

Then for more complex examples, we give three one-loop diagrams in figure A.2.1. In this notation, these are represented with

$$
\begin{array}{lll}
(a) & \leftrightarrow & \frac{1}{s_{23} s_{56}} N_{1 \mid[2,3], 4,[5,6]}(\ell) I_{1,23,4,56} \\
(b) & \leftrightarrow & \frac{1}{s_{23} s_{234}} N_{1 \mid[2,3], 4], 5,6}(\ell) I_{1,234,5,6} \\
(c) & \leftrightarrow & \frac{1}{s_{23} s_{123} s_{56}} N_{[1,[2,3]| | 4,[5,6], 7,8}(\ell) I_{123,4,56,7,8} \tag{A.2.8}
\end{array}
$$

When we come to calculate amplitudes, we sum over all possible trees at each corner of the $n$-gon, and so we introduce one-loop Berends-Giele currents where we apply the $b$-map to each block of indices,

$$
\begin{equation*}
\mathcal{N}_{A_{1} \mid A_{2}, \ldots, A_{n}}(\ell)=N_{b\left(A_{1}\right) \mid b\left(A_{2}\right), \ldots, b\left(A_{n}\right)}(\ell) \tag{A.2.9}
\end{equation*}
$$

So, to illustrate, one simple and one more complex example of this notation would be

$$
\begin{align*}
\mathcal{N}_{1 \mid 2,3,4}(\ell) & =N_{1 \mid 2,3,4}(\ell), \\
\mathcal{N}_{1 \mid 234,5,67,8}(\ell) & =\frac{N_{1 \mid[[2,3], 4], 5,[6,7], 8}(\ell)}{s_{23} s_{234} s_{67}}+\frac{N_{1 \mid[[2,3], 4], 5,[6,7], 8}(\ell)}{s_{34} s_{234} s_{67}} . \tag{A.2.10}
\end{align*}
$$

## A.2.1 More Complex Loop Momentum Integrands

Frequently we will need the Feynman loop momentum integrands (A.2.16) with a general shift in the loop momentum $\ell \rightarrow \ell+a^{i} k_{i}$. This will be indicated by superscripts

$$
\begin{equation*}
I_{A_{n+1} 1 A_{1}, A_{2}, \ldots, A_{n}}^{a_{1}, a_{2}, \ldots, a_{m}}(\ell)=I_{A_{n+1} 1 A_{1}, A_{2}, \ldots, A_{n}}\left(\ell a_{1} k_{1} k_{2}\right) \tag{A.2.11}
\end{equation*}
$$

Explicitly we have

$$
\begin{equation*}
I_{A_{n+1} 1 A_{1}, A_{2}, \ldots, A_{n}}^{a_{1}, a_{2}, \ldots, a_{m}}=\frac{1}{\left(\ell+f_{a_{1} \ldots a_{m}}-k_{A_{1}}\right)^{2} \ldots\left(\ell+f_{a_{1} \ldots a_{m}}-k_{A_{1} A_{2} \ldots A_{n}}\right)^{2}}, \tag{A.2.12}
\end{equation*}
$$

where we defined for convenience

$$
\begin{equation*}
f_{a_{1}, \ldots, a_{m}}=a_{1} k_{1}+a_{2} k_{2}+\ldots+a_{m} k_{m} \tag{A.2.13}
\end{equation*}
$$

In the event of an $a_{i}$ being zero, we will omit it from the notation. Note that the words characterizing the integrands (A.2.12) are totally symmetric e.g. $I_{1,342,5,6}=I_{1,234,5,6}$.

We will sometimes simplify the notation for the loop momentum integrands by dropping all indices which are single letters, and dropping the shifts in the loop momentum. When this is done it should always be clear the colour ordering of the amplitude. For example, in the canonical ordering $A(1,2, \ldots, n ; \ell)$ we have

$$
\begin{align*}
I_{\emptyset}=I & =I_{1,2, \ldots, a_{n}}^{a_{1}, \ldots, a_{n}}, & I_{234}=I_{1,234,5,6, \ldots, n}^{a_{1} \ldots, a_{n}},  \tag{A.2.14}\\
I_{23,56} & =I_{1,23,4,56,7,8, \ldots, n}^{a_{1}, \ldots, a_{n}}, & I_{n 1,34}=I_{n 1,2,3,3,5,6, \ldots, n-1}^{a_{1}, \ldots, n_{n}} .
\end{align*}
$$

In a few instances, we may wish to use this notation when it is not immediately clear what the underlying colour ordering is. In these circumstances we will include it as a superscript in the $I$. So, for example

$$
\begin{equation*}
I_{\emptyset}^{235416}=I^{235416}=I_{2,3,5,4,1,6}, \quad I_{53}^{235416}=I_{2,35,4,1,6}, \quad I_{612}^{235416}=I_{162,3,5,4} \tag{A.2.15}
\end{equation*}
$$

In the one-loop case however, in addition to the tree-level kinematic poles in Mandelstam invariants the field-theory limit of the genus-one string correlators also yield Feynman loop momentum integrands

$$
\begin{equation*}
I_{A_{n+1} 1 A_{1}, A_{2}, \ldots, A_{n}}(\ell)=\frac{1}{\left(\ell-k_{A_{1}}\right)^{2}\left(\ell-k_{A_{1} A_{2}}\right)^{2} \cdots\left(\ell-k_{A_{1} A_{2} \ldots A_{n}}\right)^{2}} \tag{A.2.16}
\end{equation*}
$$

to be integrated over a $D$-dimensional loop momentum $\ell$ with $\int d^{D} \ell$. Note the special role played by the label 1 in the above definition; this handling fixes the freedom to shift the loop momentum and is useful in obtaining BRST-closed SYM integrands [1].

## Proof of a Relation Satisfied by Redefinition Terms

In this appendix we will show that

$$
\begin{equation*}
\mathcal{L}_{n} \circ \mathcal{L}_{n} \circ H_{12 \ldots n}=n \mathcal{L}_{n} \circ H_{12 \ldots n}, \tag{B.0.1}
\end{equation*}
$$

Consider $\mathcal{L}_{n} \circ \mathcal{L}_{n} \circ K_{12 \ldots n}^{\prime}$, where $K_{12 \ldots n}^{\prime}$ denotes $\hat{K}_{12 \ldots n}$ minus all of the redefinition terms apart from the $H_{12 \ldots n}$ term, as in (4.2.31). From (2.4.15) it follows that this can be written as

$$
\begin{align*}
& \mathcal{L}_{n} \circ\left(K_{12 \ell(34 \ldots n)}^{\prime}+K_{34 \ldots n \ell(12)}^{\prime}\right)  \tag{B.0.2}\\
& \quad=\left(\left(K_{12 \ell(\ell(34 \ldots n))}^{\prime}+K_{\ell(34 \ldots n) \ell(12)}^{\prime}\right)+\left(K_{34 \ldots n \ell(\ell(12))}^{\prime}+K_{\ell(12) \ell(34 \ldots n)}^{\prime}\right)\right)
\end{align*}
$$

Baker's identity [43] tells us that

$$
\begin{equation*}
\ell(\ell(P))=|P| \ell(P), \tag{B.0.3}
\end{equation*}
$$

This can be generalised to

$$
\begin{equation*}
\ell(\ell(P) Q)=|P| \ell(P Q), \tag{B.0.4}
\end{equation*}
$$

the proof of which follows from induction on the length of the word $Q$, and use of (B.0.3).

Note the object $K_{12 \ldots n}^{\prime}$ will satisfy all of the rank $n-1$ generalised Jacobi identities. Though we use notation to avoid writing it explicitly, there is an implicit $\ell$ symmetry
structure on the indices of a superfield satisfying such identities. Thus the object $K_{\ell(A) B c}^{\prime}$ may be thought of as behaving like $K_{\ell(\ell(A) B) c}^{\prime}$. It therefore follows from (B.0.4) that

$$
\begin{equation*}
K_{\ell(A) \ell(B)}^{\prime}=|A| K_{A \ell(B)}^{\prime} \tag{B.0.5}
\end{equation*}
$$

Two applications of (B.0.3) and two of (B.0.5) in (B.0.2) then gives us that

$$
\begin{equation*}
\mathcal{L}_{n} \circ \mathcal{L}_{n} \circ K_{12 \ldots n}^{\prime}=n \mathcal{L}_{n} \circ K_{12 \ldots n}^{\prime} \tag{B.0.6}
\end{equation*}
$$

One then substitutes in $K^{\prime}=A_{m}^{\prime}$, and uses that $\mathcal{L}_{n} \circ A_{m}^{12 \ldots n}=0$, to get that

$$
\begin{equation*}
\mathcal{L}_{n} \circ \mathcal{L}_{n} \circ H_{12 \ldots n}=n \mathcal{L}_{n} \circ H_{12 \ldots n} \tag{B.0.7}
\end{equation*}
$$

Clearly $\mathcal{L}_{n} \circ H_{12 \ldots n}=n H_{12 \ldots n}$ is a solution to the above. Showing it is the unique solution is an open problem.

## Expanded Redefinition Formulae

In this appendix we will state the non-trivial redefinitions of superfields up to rank five, and all of them at rank six, with all possible bracketing structures. This list is not wholly exhaustive, we do not include those superfields related by antisymmetry to those listed for simplicity.

## C. 1 Rank Four and Five

The redefinitions for topologies which are not Dynkin brackets at ranks four and five are given by

$$
\begin{align*}
A_{[12,34]}^{m}= & \hat{A}_{[12,34]}^{m}-k_{1234}^{m} \hat{H}_{[12,34]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[23,4]} A_{1}^{m}-\hat{H}_{[13,4]} A_{2}^{m}\right)  \tag{C.1.1}\\
& -\left(k^{3} \cdot k^{4}\right)\left(\hat{H}_{[12,3]} A_{4}^{m}-\hat{H}_{[12,4]} A_{3}^{m}\right) \\
A_{[123,45]}^{m}= & \hat{A}_{[123,45]}^{m}-k_{12345}^{m} \hat{H}_{[123,45]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[13,45]} A_{2}^{m}+\hat{H}_{[45,2]} A_{13}^{m}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(\hat{H}_{[12,45]} A_{3}^{m}+\hat{H}_{[45,3]} A_{12}^{m}\right)  \tag{C.1.2}\\
& -\left(k^{123} \cdot k^{45}\right) \hat{H}_{[12,3]} A_{45}^{m} \\
& -\left(k^{4} \cdot k^{5}\right)\left(\hat{H}_{[123,4]} A_{5}^{m}-\hat{H}_{[123,5]} A_{4}^{m}\right),
\end{align*}
$$

$$
\begin{align*}
A_{[[12,34], 5]}^{m}= & \hat{A}_{[[12,34], 5]}^{m}-k_{12345}^{m} \hat{H}_{[[12,34], 5]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[1,34]} A_{25}^{m}+\hat{H}_{[[1,34], 5]} A_{2}^{m}-(1 \leftrightarrow 2)\right) \\
& +\left(k^{3} \cdot k^{4}\right)\left(\hat{H}_{[3,12]} A_{45}^{m}+\hat{H}_{[[3,12], 5]} A_{4}^{m}-(3 \leftrightarrow 4)\right)  \tag{C.1.3}\\
& -\left(k^{12} \cdot k^{34}\right)\left(\hat{H}_{[12,5]} A_{34}^{m}-(12 \leftrightarrow 34)\right) \\
& -\left(k^{1234} \cdot k^{5}\right)\left(\hat{H}_{[12,34]} A_{5}^{m}\right)
\end{align*}
$$

Note we have not defined $H_{[[12,34], 5]}$. This is given through its relation by Jacobi identities with Dynkin brackets,

$$
\begin{equation*}
H_{[12,34], 5]}=H_{[1234,5]}-H_{[1243,5]} . \tag{C.1.4}
\end{equation*}
$$

In general, if $H_{[P, Q]}$ is such that at least one of $P$ and $Q$ is not a Dynkin bracket, we define it through its relation with $H$ terms where they are. This is justified as, by construction, $H_{[P, Q]}$ should satisfy Jacobi identities in $P$ and $Q$ separately.

## C. 2 Rank Six

The redefinitions at rank six are given by

$$
\begin{align*}
& A_{[12345,6]}^{m}=\hat{A}_{[12345,6]}^{m}-k_{123456}^{m} \hat{H}_{[12345,6]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{13456} \hat{A}_{2}^{m}+\hat{H}_{1345} \hat{A}_{26}^{m}+\hat{H}_{1346} \hat{A}_{25}^{m}+\hat{H}_{1356} \hat{A}_{24}^{m}\right. \\
& +\hat{H}_{1456} \hat{A}_{23}^{m}+\hat{H}_{134} \hat{A}_{256}^{m}+\hat{H}_{135} \hat{A}_{246}^{m}+\hat{H}_{136} \hat{A}_{245}^{m} \\
& +\hat{H}_{145} \hat{A}_{236}^{m}+\hat{H}_{146} \hat{A}_{235}^{m}+\hat{H}_{156} \hat{A}_{234}^{m} \\
& -\frac{1}{2} \hat{H}_{134} \hat{H}_{256} k_{256}^{m}-\frac{1}{2} \hat{H}_{135} \hat{H}_{246} k_{246}^{m}-\frac{1}{2} \hat{H}_{136} \hat{H}_{245} k_{245}^{m} \\
& -\frac{1}{2} \hat{H}_{145} \hat{H}_{236} k_{236}^{m}-\frac{1}{2} \hat{H}_{146} \hat{H}_{235} k_{235}^{m}-\frac{1}{2} \hat{H}_{156} \hat{H}_{234} k_{234}^{m} \\
& -(1 \leftrightarrow 2)) \\
& -\left(k^{12} \cdot k^{3}\right)\left(\hat{H}_{12456} \hat{A}_{3}^{m}+\hat{H}_{1245} \hat{A}_{36}^{m}+\hat{H}_{1246} \hat{A}_{35}^{m}+\hat{H}_{1256} \hat{A}_{34}^{m}\right. \\
& +\hat{H}_{124} \hat{A}_{356}^{m}+\hat{H}_{125} \hat{A}_{346}^{m}+\hat{H}_{126} \hat{A}_{345}^{m}  \tag{C.2.1}\\
& -\frac{1}{2} \hat{H}_{124} \hat{H}_{356} k_{356}^{m}-\frac{1}{2} \hat{H}_{125} \hat{H}_{346} k_{346}^{m}-\frac{1}{2} \hat{H}_{126} \hat{H}_{345} k_{345}^{m} \\
& -\hat{H}_{3456} \hat{A}_{12}^{m}-\hat{H}_{345} \hat{A}_{126}^{m}-\hat{H}_{346} \hat{A}_{125}^{m}-\hat{H}_{356} \hat{A}_{124}^{m} \\
& \left.+\frac{1}{2} \hat{H}_{345} \hat{H}_{126} k_{126}^{m}+\frac{1}{2} \hat{H}_{346} \hat{H}_{125} k_{125}^{m}+\frac{1}{2} \hat{H}_{356} \hat{H}_{124} k_{124}^{m}\right) \\
& -\left(k^{123} \cdot k^{4}\right)\left(\hat{H}_{12356} \hat{A}_{4}^{m}+\hat{H}_{1235} \hat{A}_{46}^{m}+\hat{H}_{1236} \hat{A}_{45}^{m}\right. \\
& +\hat{H}_{123} \hat{A}_{456}^{m}-\hat{H}_{456} \hat{A}_{123}^{m} \\
& \left.-\frac{1}{2} \hat{H}_{123} \hat{H}_{456} k_{456}^{m}+\frac{1}{2} \hat{H}_{456} \hat{H}_{123} k_{123}^{m}\right) \\
& -\left(k^{1234} \cdot k^{5}\right)\left(\hat{H}_{12346} \hat{A}_{5}^{m}+\hat{H}_{1234} \hat{A}_{56}^{m}\right) \\
& -\left(k^{12345} \cdot k^{6}\right) \hat{H}_{12345} \hat{A}_{6}^{m}
\end{align*}
$$

$$
\begin{align*}
& A_{[1234,56]}^{m}=\hat{A}_{[1234,56]}^{m}-k_{123456}^{m} \hat{H}_{[1234,56]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[1,56]} \hat{A}_{234}^{m}+\hat{H}_{[13,56]} \hat{A}_{24}^{m}+\hat{H}_{[14,56]} \hat{A}_{23}^{m}\right. \\
& +\hat{H}_{[134,56]} \hat{A}_{2}^{m}+\hat{H}_{134} \hat{A}_{[2,56]}^{m} \\
& \left.-\frac{1}{2} \hat{H}_{134} \hat{H}_{[2,56]} k_{256}^{m}-\frac{1}{2} \hat{H}_{[1,56]} \hat{H}_{234} k_{234}^{m}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(\hat{H}_{[12,56]} \hat{A}_{34}^{m}+\hat{H}_{[124,56]} \hat{A}_{3}^{m}+\hat{H}_{124} \hat{A}_{[3,56]}^{m}\right. \\
& -\hat{H}_{[3,56]} \hat{A}_{124}^{m}-\hat{H}_{[34,56]} \hat{A}_{12}^{m}  \tag{C.2.2}\\
& \left.-\frac{1}{2} \hat{H}_{124} \hat{H}_{[3,56]} k_{356}^{m}+\frac{1}{2} \hat{H}_{[3,56]} \hat{H}_{124} k_{124}^{m}\right) \\
& -\left(k^{123} \cdot k^{4}\right)\left(\hat{H}_{[123,56]} \hat{A}_{4}^{m}+\hat{H}_{123} \hat{A}_{[4,56]}^{m}-\hat{H}_{[4,56]} \hat{A}_{123}^{m}\right. \\
& \left.-\frac{1}{2} \hat{H}_{123} \hat{H}_{[4,56]} \xi_{456}^{m}+\frac{1}{2} \hat{H}_{[4,56]} \hat{H}_{123} k_{123}^{m}\right) \\
& -\left(k^{1234} \cdot k^{56}\right)\left(\hat{H}_{1234} \hat{A}_{56}^{m}\right) \\
& +\left(k^{5} \cdot k^{6}\right)\left(\hat{H}_{[5,1234]} \hat{A}_{6}^{m}-(5 \leftrightarrow 6)\right) \\
& A_{[123,456]}^{m}=\hat{A}_{[123,45,6]}^{m}-k_{123456}^{m} \hat{H}_{[123,456]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[1,456]} \hat{A}_{23}^{m}+\hat{H}_{[13,456]} \hat{A}_{2}^{m}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(\hat{H}_{[12,456]} \hat{A}_{3}^{m}-\hat{H}_{[3,456]} \hat{A}_{12}^{m}\right) \\
& -\left(k^{123} \cdot k^{456}\right)\left(\hat{H}_{123} \hat{A}_{456}^{m}-\hat{H}_{456} \hat{A}_{123}^{m}\right.  \tag{C.2.3}\\
& \left.-\frac{1}{2} \hat{H}_{123} \hat{H}_{456} k_{456}^{m}+\frac{1}{2} \hat{H}_{456} \hat{H}_{123} k_{123}^{m}\right) \\
& +\left(k^{4} \cdot k^{5}\right)\left(\hat{H}_{[4,123]} \hat{A}_{56}^{m}+\hat{H}_{[46,123]} \hat{A}_{5}^{m}-(4 \leftrightarrow 5)\right) \\
& +\left(k^{45} \cdot k^{6}\right)\left(\hat{H}_{[45,123]} \hat{A}_{6}^{m}-\hat{H}_{[6,123]} \hat{A}_{45}^{m}\right) \\
& A_{[112,34], 56]}^{m}=\hat{A}_{[12,34], 56]}^{m}-k_{123456}^{m} \hat{H}_{[12,34], 56]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[1,34]} \hat{A}_{[2,56]}^{m}+\hat{H}_{[1,56]} \hat{A}_{[2,34]}^{m}+\hat{H}_{[[1,34], 56]} \hat{A}_{2}^{m}\right. \\
& \left.-\frac{1}{2} \hat{H}_{[1,34]} \hat{H}_{[2,56]} k_{256}^{m}-\frac{1}{2} \hat{H}_{[1,56]} \hat{H}_{[2,34]} k_{234}^{m}-(1 \leftrightarrow 2)\right) \\
& +\left(k^{3} \cdot k^{4}\right)\left(\hat{H}_{[3,12]} \hat{A}_{[4,56]}^{m}+\hat{H}_{[3,56]} \hat{A}_{[4,12]}^{m}+\hat{H}_{[3,12], 56]} \hat{A}_{4}^{m}\right. \\
& \left.-\frac{1}{2} \hat{H}_{[3,12]} \hat{H}_{[4,56]} k_{456}^{m}-\frac{1}{2} \hat{H}_{[3,56]} \hat{H}_{[4,12]} k_{124}^{m}-(3 \leftrightarrow 4)\right)  \tag{C.2.4}\\
& -\left(k^{12} \cdot k^{34}\right)\left(\hat{H}_{[12,56]} \hat{A}_{34}^{m}-\hat{H}_{[34,56]} \hat{A}_{12}^{m}\right) \\
& -\left(k^{1234} \cdot k^{56}\right)\left(\hat{H}_{[12,34]} \hat{A}_{56}^{m}\right) \\
& +\left(k^{5} \cdot k^{6}\right)\left(\hat{H}_{[[12,34], 6]} \hat{A}_{5}^{m}-\hat{H}_{[12,34], 5]} \hat{A}_{6}^{m}\right)
\end{align*}
$$

$$
\begin{align*}
& A_{[[123,45], 6]}^{m}=\hat{A}_{[[123,45], 6]}^{m}-k_{123456}^{m} \hat{H}_{[[123,45], 6]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{[1,45]} \hat{A}_{236}^{m}+\hat{H}_{136} \hat{A}_{[2,45]}^{m}+\hat{H}_{[[1,45], 6]} \hat{A}_{23}^{m}\right. \\
& +\hat{H}_{[13,45]} \hat{A}_{26}^{m}+\hat{H}_{[[13,45], 6]} \hat{A}_{2}^{m} \\
& \left.-\frac{1}{2} \hat{H}_{[1,45]} \hat{H}_{236} k_{236}^{m}-\frac{1}{2} \hat{H}_{136} \hat{H}_{[2,45]} k_{245}^{m}-(1 \leftrightarrow 2)\right) \\
& -\left(k^{12} \cdot k^{3}\right)\left(\hat{H}_{126} \hat{A}_{[3,45]}^{m}+\hat{H}_{[12,45]} \hat{A}_{36}^{m}+\hat{H}_{[[12,45], 6]} \hat{A}_{3}^{m}\right. \\
& -\hat{H}_{[3,45]} \hat{A}_{126}^{m}-\hat{H}_{[3,45], 66} \hat{A}_{12}^{m}  \tag{C.2.5}\\
& \left.-\frac{1}{2} \hat{H}_{126} \hat{H}_{[3,45]} k_{345}^{m}+\frac{1}{2} \hat{H}_{[3,45]} \hat{H}_{126} k_{126}^{m}\right) \\
& +\left(k^{4} \cdot k^{5}\right)\left(\hat{H}_{[4,123]} \hat{A}_{56}^{m}+\hat{H}_{[4,123], 6]} \hat{A}_{5}^{m}-(4 \leftrightarrow 5)\right) \\
& -\left(k^{123} \cdot k^{45}\right)\left(\hat{H}_{1236} \hat{A}_{45}^{m}+\hat{H}_{123} \hat{A}_{456}^{m}-\hat{H}_{456} \hat{A}_{123}^{m}\right. \\
& \left.-\frac{1}{2} \hat{H}_{123} \hat{H}_{456} k_{456}^{m}+\frac{1}{2} \hat{H}_{456} \hat{H}_{123} k_{123}^{m}\right) \\
& -\left(k^{12345} \cdot k^{6}\right)\left(\hat{H}_{[123,45]} \hat{A}_{6}^{m}\right) \\
& A_{[[[12,34], 5], 6]}^{m}=\hat{A}_{[[12,34], 5], 6]}^{m}-k_{123456}^{m} \hat{H}_{[[12,34], 5], 6]} \\
& -\left(k^{1} \cdot k^{2}\right)\left(\hat{H}_{156} \hat{A}_{[2,34]}^{m}+\hat{H}_{[1,34]} \hat{A}_{256}^{m}+\hat{H}_{[[1,34], 6]} \hat{A}_{25}^{m}\right. \\
& +\hat{H}_{[[1,34], 5]} \hat{A}_{26}^{m}+\hat{H}_{[[[1,34], 5], 6]} \hat{A}_{2}^{m} \\
& \left.-\frac{1}{2} \hat{H}_{156} \hat{H}_{[2,34]} k_{234}^{m}-\frac{1}{2} \hat{H}_{[1,34]} \hat{H}_{256} k_{256}^{m}-(1 \leftrightarrow 2)\right) \\
& +\left(k^{3} \cdot k^{4}\right)\left(\hat{H}_{356} \hat{A}_{[4,12]}^{m}+\hat{H}_{[3,12]} \hat{A}_{456}^{m}+\hat{H}_{[[3,12], 6]} \hat{A}_{45}^{m}\right. \\
& +\hat{H}_{[[3,12], 5]} \hat{A}_{46}^{m}+\hat{H}_{[[[3,12], 5], 6]} \hat{A}_{4}^{m}  \tag{С.2.6}\\
& \left.-\frac{1}{2} \hat{H}_{356} \hat{H}_{[4,12]} k_{124}^{m}-\frac{1}{2} \hat{H}_{[3,12]} \hat{H}_{456} k_{456}^{m}-(3 \leftrightarrow 4)\right) \\
& -\left(k^{12} \cdot k^{34}\right)\left(\hat{H}_{1256} \hat{A}_{34}^{m}+\hat{H}_{126} \hat{A}_{345}^{m}+\hat{H}_{125} \hat{A}_{346}^{m}\right. \\
& \left.-\frac{1}{2} \hat{H}_{126} \hat{H}_{345} k_{345}^{m}-\frac{1}{2} \hat{H}_{125} \hat{H}_{346} k_{346}^{m}-(12 \leftrightarrow 34)\right) \\
& -\left(k^{1234} \cdot k^{5}\right)\left(\hat{H}_{[12,34]} \hat{A}_{56}^{m}+\hat{H}_{[[12,34], 6]} \hat{A}_{5}^{m}\right) \\
& -\left(k^{12345} \cdot k^{6}\right)\left(\hat{H}_{[[12,34], 5]} \hat{A}_{6}^{m}\right)
\end{align*}
$$

## APPENDIX D

## Symmetry Properties of $\tilde{H}_{A, B, C}$

The $\tilde{H}_{A, B, C}$ terms have a large number of symmetries, and I shall briefly outline them here.

They are antisymmetric in $A, B$, and $C$, i.e.

$$
\begin{equation*}
\tilde{H}_{A, B, C}=-\tilde{H}_{B, A, C}=\tilde{H}_{B, C, A}=-\tilde{H}_{C, B, A}=\tilde{H}_{C, A, B}=-\tilde{H}_{A, C, B} \tag{D.0.1}
\end{equation*}
$$

Each of the three sets of indices satisfies generalised Jacobi identities within it, so for example

$$
\begin{align*}
\tilde{H}_{123, B, C}+\tilde{H}_{213, B, C} & =0,  \tag{D.0.2}\\
\tilde{H}_{123, B, C}+\tilde{H}_{231, B, C}+\tilde{H}_{312, B, C} & =0 . \tag{D.0.3}
\end{align*}
$$

There are then also a number of other more complex relations between some $\tilde{H}$ s. These have a very non-obvious origin, and had to be found by brute force initially. Now that (7.3.3) is known though, these can be identified from the condition that $H_{[A, B]}$ satisfies generalised Jacobi identities in each of $A$ and $B$. For example, we must have that $\mathcal{L}_{3} \circ$ $H_{[123,4]}=0, \mathcal{L}_{3} \circ H_{[1234,5]}=0$, and $\mathcal{L}_{4} \circ H_{[1234,5]}=0$, and so expanding these $H$ s in terms
of $\tilde{H} \mathrm{~s}$ we see that we must have

$$
\begin{array}{r}
\mathcal{L}_{3} \circ\left(\tilde{H}_{12,3,4}+\tilde{H}_{34,1,2}\right)=0, \\
\mathcal{L}_{3} \circ\left(\tilde{H}_{123,4,5}-\tilde{H}_{543,2,1}+\tilde{H}_{54,3,12}\right)=0, \\
\mathcal{L}_{4} \circ\left(\tilde{H}_{123,4,5}-\tilde{H}_{543,2,1}+\tilde{H}_{54,3,12}\right)=0 . \tag{D.0.6}
\end{array}
$$

These identities can be described in general by considering the formula for $H_{[A, B]}$ found in this report, (7.3.4). Consider $\mathcal{L}_{n} \circ H_{[A, B]}$, with $n<|A|$. One half of (7.3.4) will disappear under the action of the $\mathcal{L}$, as

$$
\begin{equation*}
\mathcal{L}_{n} \circ\left(\sum_{X j Y=a B^{T}}(-1)^{|Y|} \tilde{H}_{Y^{T}, j, X}\right)=\mathcal{L}_{n} \circ\left(\sum_{X j Y=B^{T}}(-1)^{|Y|} \tilde{H}_{Y^{T}, j, a X}\right)=0, \tag{D.0.7}
\end{equation*}
$$

where in the second sum $X$ is not constrained to be non-empty. The final equality then just comes from the fact that $\tilde{H}_{A, B, C}$ is constructed so as to satisfy generalised Jacobi identities in each of $A, B$, and $C$. Using this and (7.3.4) it then just follows that, if $\mathcal{L}_{n} \circ H_{[A, B]}=0$ for $n<|A|$, then

$$
\begin{equation*}
\mathcal{L}_{n} \circ\left(\sum_{X j Y=b A^{T}}(-1)^{|Y|} \tilde{H}_{Y^{T}, j, X}\right)=0, \quad n<|A| \tag{D.0.8}
\end{equation*}
$$

for any word $A$ and letterification $b$.

## Example applications of the $C$ and $\tilde{C}$ maps

In this appendix we display some example applications of the $C$ and $\tilde{C}$ maps acting over some simple Lie polynomials. These examples help to elucidate how the algorithms are used, and can be used to verify that the redefinition formulas arising from the general formulas match the formulas for the simplest cases that were previously known.

## E. 1 Examples of the $C$ map

To demonstrate the contact term algorithm (6.1.1), the first few expansions generated from it are

$$
\begin{align*}
C \circ 1= & 0  \tag{E.1.1}\\
C \circ[1,2]= & \left(k_{1} \cdot k_{2}\right)(1 \otimes 2-2 \otimes 1) \\
C \circ[[1,2], 3]= & \left(k_{1} \cdot k_{2}\right)([1,3] \otimes 2+1 \otimes[2,3]-[2,3] \otimes 1-2 \otimes[1,3]) \\
& +\left(k_{12} \cdot k_{3}\right)([1,2] \otimes 3-3 \otimes[1,2]) \\
C \circ[1,[2,3]]= & \left(k_{2} \cdot k_{3}\right)([1,2] \otimes 3+2 \otimes[1,3]-[1,3] \otimes 2-3 \otimes[1,2]) \\
& +\left(k_{1} \cdot k_{23}\right)(1 \otimes[2,3]-[2,3] \otimes 1) \\
C \circ[[[1,2], 3], 4]= & \left(k_{1} \cdot k_{2}\right)([[1,3], 4] \otimes 2+[1,3] \otimes[2,4]+[1,4] \otimes[2,3]+1 \otimes[[2,3], 4] \\
& -[[2,3], 4] \otimes 1-[2,3] \otimes[1,4]-[2,4] \otimes[1,3]-2 \otimes[1,3], 4])
\end{align*}
$$

$$
\begin{aligned}
& +\left(k_{12} \cdot k_{3}\right)([[1,2], 4] \otimes 3+[1,2] \otimes[3,4]-[3,4] \otimes[1,2]-3 \otimes[[1,2], 4]) \\
& +\left(k_{123} \cdot k_{4}\right)([[1,2], 3] \otimes 4-4 \otimes[[1,2], 3]) \\
C \circ[[1,[2,3]], 4]= & \left(k_{2} \cdot k_{3}\right)([[1,2], 4] \otimes 3+[1,2] \otimes[3,4]+[2,4] \otimes[1,3]+2 \otimes[[1,3], 4] \\
& -[[1,3], 4] \otimes 2-[1,3] \otimes[2,4]-[3,4] \otimes[1,2]-3 \otimes[[1,2], 4]) \\
& +\left(k_{1} \cdot k_{23}\right)([1,4] \otimes[2,3]+1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1-[2,3] \otimes[1,4]) \\
& +\left(k_{123} \cdot k_{4}\right)([1,[2,3]] \otimes 4-4 \otimes[1,[2,3]]) \\
C \circ[[1,2],[3,4]]= & \left(k_{1} \cdot k_{2}\right)([1,[3,4]] \otimes 2+1 \otimes[2,[3,4]]-[2,[3,4]] \otimes 1-2 \otimes[1,[3,4]]) \\
& +\left(k_{3} \cdot k_{4}\right)([[1,2], 3] \otimes 4+3 \otimes[[1,2], 4]-[[1,2], 4] \otimes 3-4 \otimes[[1,2], 3]) \\
& +\left(k_{12} \cdot k_{34}\right)([1,2] \otimes[3,4]-[3,4] \otimes[1,2]) \\
C \circ[1,[2,[3,4]]]= & \left(k_{3} \cdot k_{4}\right)([1,[2,3]] \otimes 4+[2,3] \otimes[1,4]+[1,3] \otimes[2,4]+3 \otimes[1,[2,4]] \\
& -[1,[2,4]] \otimes 3-[2,4] \otimes[1,3]-[1,4] \otimes[2,3]-4 \otimes[1,[2,3]]) \\
& +\left(k_{2} \cdot k_{34}\right)([1,2] \otimes[3,4]+2 \otimes[1,[3,4]]-[1,[3,4]] \otimes 2-[3,4] \otimes[1,2]) \\
& +\left(k_{1} \cdot k_{234}\right)(1 \otimes[2,[3,4]]-[2,[3,4]] \otimes 1) \\
C \circ[1,[[2,3], 4]]= & \left(k_{2} \cdot k_{3}\right)([1,[2,4]] \otimes 3+[1,2] \otimes[3,4]+[2,4] \otimes[1,3]+2 \otimes[1,[3,4]] \\
& -[1,[3,4]] \otimes 2-[1,3] \otimes[2,4]-[3,4] \otimes[1,2]-3 \otimes[1,[2,4]]) \\
& +\left(k_{23} \cdot k_{4}\right)([1,[2,3]] \otimes 4+[2,3] \otimes[1,4]-[1,4] \otimes[2,3]-4 \otimes[1,[2,3]]) \\
& +\left(k_{1} \cdot k_{234}\right)(1 \otimes[[2,3], 4]-[[2,3], 4] \otimes 1) .
\end{aligned}
$$

One application at multiplicity five is given by

$$
\begin{align*}
C \circ[[[1,2], 3],[4,5]]= & \left(k_{1} \cdot k_{2}\right)(1 \otimes[[2,3],[4,5]]+[1,3] \otimes[2,[4,5]]  \tag{E.1.2}\\
& +[1,[4,5]] \otimes[2,3]+[[1,3],[4,5]] \otimes 2-(1 \leftrightarrow 2)) \\
+ & \left(k_{12} \cdot k_{3}\right)([1,2] \otimes[3,[4,5]]+[[1,2],[4,5]] \otimes 3-([1,2] \leftrightarrow 3)) \\
+ & \left(k_{123} \cdot k_{45}\right)([[1,2], 3] \otimes[4,5]-([[1,2], 3] \leftrightarrow[4,5])) \\
+ & \left(k_{4} \cdot k_{5}\right)(4 \otimes[[[1,2], 3], 5]+[[[1,2], 3], 4] \otimes 5-(4 \leftrightarrow 5)),
\end{align*}
$$

## E. 2 Examples of the $\tilde{C}$ map

As an illustration of the $\tilde{C}$ map, we get

$$
\begin{align*}
\tilde{C} \circ 1 & =0  \tag{E.2.1}\\
\tilde{C} \circ[1,2] & =0 \\
\tilde{C} \circ[[1,2], 3] & =\left(k_{1} \cdot k_{2}\right)([1,3] \otimes 2-[2,3] \otimes 1) \\
\tilde{C} \circ[1,[2,3]] & =\left(k_{2} \cdot k_{3}\right)([1,2] \otimes 3-[1,3] \otimes 2) \\
\tilde{C} \circ[[[1,2], 3], 4] & =\left(k_{1} \cdot k_{2}\right)([[1,3], 4] \otimes 2+[1,4] \otimes[2,3]-[[2,3], 4] \otimes 1-[2,4] \otimes[1,3])
\end{align*}
$$

$$
\begin{aligned}
& +\left(k_{12} \cdot k_{3}\right)([[1,2], 4] \otimes 3-[3,4] \otimes[1,2]) \\
\tilde{C} \circ[[1,[2,3]], 4]= & \left.\left(k_{2} \cdot k_{3}\right)([[1,2], 4] \otimes 3+[2,4] \otimes[1,3]-[[1,3], 4]] \otimes 2-[3,4] \otimes[1,2]\right) \\
& +\left(k_{1} \cdot k_{23}\right)([1,4] \otimes[2,3]-[[2,3], 4] \otimes 1) \\
\tilde{C} \circ[[1,2],[3,4]]= & \left(k_{1} \cdot k_{2}\right)([1,[3,4]] \otimes 2-[2,[3,4]] \otimes 1) \\
& +\left(k_{3} \cdot k_{4}\right)([[1,2], 3] \otimes 4-[[1,2], 4] \otimes 3) \\
\tilde{C} \circ[1,[2,[3,4]]]= & \left(k_{3} \cdot k_{4}\right)([1,[2,3]] \otimes 4+[1,3] \otimes[2,4]-[1,[2,4]] \otimes 3-[1,4] \otimes[2,3]) \\
& +\left(k_{2} \cdot k_{34}\right)([1,2] \otimes[3,4]-[1,[3,4]] \otimes 2) \\
\tilde{C} \circ[1,[[2,3], 4]]= & \left(k_{2} \cdot k_{3}\right)([1,[2,4]] \otimes 3+[1,2] \otimes[3,4]-[1,[3,4]] \otimes 2-[1,3] \otimes[2,4]) \\
& +\left(k_{23} \cdot k_{4}\right)([1,[2,3]] \otimes 4-[1,4] \otimes[2,3])
\end{aligned}
$$

One application at multiplicity six is given by

$$
\begin{aligned}
\tilde{C} \circ[[[[1,2], 3],[4,5]], 6] & =\left(k_{1} \cdot k_{2}\right)([[[1,3],[4,5]], 6] \otimes 2+[[1,3], 6] \otimes[2,[4,5]] \\
& +[[1,[4,5]], 6] \otimes[2,3]+[1,6] \otimes[[2,3],[4,5]]-(1 \leftrightarrow 2)) \\
& +\left(k_{12} \cdot k_{3}\right)([[[1,2],[4,5]], 6] \otimes 3+[[1,2], 6] \otimes[3,[4,5]]-([1,2] \leftrightarrow 3)) \\
& +\left(k_{4} \cdot k_{5}\right)([[[[1,2], 3], 4], 6] \otimes 5+[4,6] \otimes[[[1,2], 3], 5]-(4 \leftrightarrow 5)) \\
& +\left(k_{123} \cdot k_{45}\right)([[[1,2], 3], 6] \otimes[4,5]-([[1,2], 3] \leftrightarrow[4,5])) .
\end{aligned}
$$

## appendix F

## Further Generalisation of $H_{[P, Q]}$

In the discussion of potential directions for future research at the end of part II, some speculations were made about the possibility of a further generalisation of the formula for $H_{[P, Q]}$ to $P$ and $Q$ arbitrary Lie monomials. Some work was performed on this, and the following formulae were found with the aid of FORM [148; 149; 150] for all more complex topologies of $H_{[P, Q]}$ to rank seven:

$$
\begin{align*}
H_{[[12,34], 5]} & =\frac{1}{5}\left(H_{12,34,5}^{\prime}-H_{125,3,4}^{\prime}+H_{345,1,2}^{\prime}\right)  \tag{F.0.1}\\
H_{[112,34], 56]} & =\frac{1}{6}\left(4 H_{[12,34], 6,5}^{\prime}-2 H_{[12,56], 3,4}^{\prime}+2 H_{[34,56], 1,2}^{\prime}+2 H_{12,34,56}^{\prime}\right) \\
H_{[[123,45], 6]} & =\frac{1}{6}\left(H_{123,45,6}^{\prime}+H_{12,3,456}^{\prime}-H_{1236,4,5}^{\prime}-H_{4563,1,2}^{\prime}\right) \\
H_{[[12,34], 5], 6]} & =\frac{1}{6}\left(H_{12,34,56}^{\prime}+H_{[12,34], 5,6}^{\prime}-H_{[12,56], 3,4}^{\prime}+H_{[34,56], 1,2}^{\prime}\right) \\
H_{[[1234,56], 7]} & =\frac{1}{7}\left(H_{1234,56,7}^{\prime}-H_{12347,5,6}^{\prime}+H_{567,123,4}^{\prime}-H_{5674,12,3}^{\prime}+H_{56743,1,2}^{\prime}\right) \\
H_{[[123,456], 7]} & =\frac{1}{7}\left(-H_{123,7,456}^{\prime}+H_{1237,6,45}^{\prime}-H_{12376,5,4}^{\prime}-H_{4567,3,12}^{\prime}+H_{45673,2,1}^{\prime}\right) \\
H_{[[123,45], 67]} & =\frac{1}{7}\left(5 H_{[123,45], 7,6}^{\prime}+2 H_{123,45,67}^{\prime}-2 H_{[123,67], 4,5}^{\prime}+2 H_{[45,67], 12,3}^{\prime}-2 H_{[[45,67], 3], 1,2}^{\prime}\right) \\
H_{[[12,34], 567]} & =\frac{1}{7}\left(4 H_{[12,34], 7,56}^{\prime}-4 H_{[[12,34], 7], 6,5}^{\prime}+3 H_{12,34,567}^{\prime}-3 H_{[12,567], 3,4}^{\prime}+3 H_{[34,567], 1,2}^{\prime}\right) \\
H_{[[[123,45], 6], 7]} & =\frac{1}{7}\left(H_{123,45,67}^{\prime}+H_{[123,45], 6,7}^{\prime}-H_{[123,67], 4,5}^{\prime}+H_{[45,67], 12,3}^{\prime}-H_{[[45,67], 3], 1,2}^{\prime}\right) \\
H_{[[[12,34], 5], 67]} & =\frac{1}{7}\left(5 H_{[[12,34], 5], 7,6}^{\prime}+2 H_{12,34,[5,67]}^{\prime}+2 H_{[12,34], 5,67}^{\prime}-2 H_{[12,[5,67], 3,4}^{\prime}+2 H_{[34,[5,67]], 1,2}^{\prime}\right)
\end{align*}
$$

$$
\begin{aligned}
H_{[[12,34], 56], 7]} & =\frac{1}{7}\left(H_{12,34,567}^{\prime}+H_{[12,34], 56,7}^{\prime}-H_{[[12,34], 7], 5,6}^{\prime}-H_{[12,567], 3,4}^{\prime}+H_{[34,567], 1,2}^{\prime}\right) \\
H_{[[[12,34], 5], 6], 7]} & =\frac{1}{7}\left(-H_{12,34,675}^{\prime}-H_{[12,34], 67,5}^{\prime}+H_{[[12,34], 5], 6,7}^{\prime}+H_{[12,675], 3,4}^{\prime}-H_{[34,675], 1,2}^{\prime}\right)
\end{aligned}
$$

Unfortunately, these are now believed to be wrong. To see why, take for instance the simplest case of $H_{[[12,34], 5]}$. This should be antisymmetric in 1 and 2. However, the form above would suggest it is not;

$$
\begin{align*}
5 H_{[[12,34], 5]}+5 H_{[[21,34], 5]} & =H_{12,34,5}^{\prime}-H_{125,3,4}^{\prime}+H_{345,1,2}^{\prime}+H_{21,34,5}^{\prime}-H_{215,3,4}^{\prime}+H_{345,2,1}^{\prime} \\
& =2 H_{345,1,2}^{\prime} \neq 0 . \tag{F.0.2}
\end{align*}
$$

Such a significant error does not bode well for the validity of these expressions. We state them here merely as a starting point, from which a more correct set of values may be found.

Further, a set of maps to describe (F.0.1) were found. These were then shown to reproduce (7.3.4) in the case of Dynkin brackets. We outline these results here, and again stress that these maps likely produce something incorrect and should serve only as a starting point for developing a more correct algorithm.

We define a pair of maps on Dynkin brackets, $H_{1}$ and $H$,

$$
\begin{gather*}
H_{1} \circ[P, Q]=\frac{|P|}{|P|+|Q|} P \vee(H \circ Q)+\frac{|Q|}{|P|+|Q|}(H \circ P) \vee Q+D \circ[P, Q]  \tag{F.0.3}\\
H \circ[P, Q]=P \vee(H \circ Q)+(H \circ P) \vee Q+D \circ[P, Q],  \tag{F.0.4}\\
H \circ i=H \circ[i, j]=0 .
\end{gather*}
$$

Note that $H_{1}$ is a function of $H$ and not $H_{1}$. Further, $H_{1}$ is identical to $H$ up to some weighting. The function $D$ in the above was then defined by

$$
\begin{equation*}
D\left(\left[\left[P_{1}, P_{2}\right],\left[Q_{1}, Q_{2}\right]\right]\right)=\frac{1}{|P|+|Q|}\left(|P|\left[P_{1}, P_{2}\right] \otimes Q_{1} \otimes Q_{2}-|Q| P_{1} \otimes P_{2} \otimes\left[Q_{1}, Q_{2}\right]\right) \tag{F.0.5}
\end{equation*}
$$

Note we implicitly assume $\left|P_{1}\right|<\left|P_{2}\right|,\left|Q_{1}\right|<\left|Q_{2}\right|$, and $\left|P_{1}\right|+\left|P_{2}\right|<\left|Q_{1}\right|+\left|Q_{2}\right|$ in the above. This map violates the required antisymmetry in a number of places. The $H_{[P, Q]}$ superfields were then believed to be given by applying the $H_{1}$ map to $[P, Q]$

$$
\begin{equation*}
H_{[P, Q]}=H_{1}\left[\left[H^{\prime}\right]\right] \circ[P, Q]:=\sum_{A \otimes B \otimes C \in H_{1} \circ[P, Q]} H_{A, B, C}^{\prime} \tag{F.0.6}
\end{equation*}
$$

The $\vee$ map was then defined recursively, working from the out-most such inwards. When

2 or more such maps were present, this was defined by

$$
\begin{align*}
A \vee(B \vee P) & =[A, B] \vee P, \\
A \vee(P \vee B) & =P \vee[A, B],  \tag{F.0.7}\\
(B \vee P) \vee A & =[B, A] \vee P, \\
(P \vee B) \vee A & =P \vee[B, A] .
\end{align*}
$$

Then the final instance of this was defined by

$$
\begin{align*}
& A \otimes B \otimes C \vee D=A \otimes B \otimes[C, D]  \tag{F.0.8}\\
& D \vee A \otimes B \otimes C=A \otimes B \otimes[D, C]
\end{align*}
$$

So, to give a single example,

$$
\begin{align*}
(((1 \otimes 2 \otimes 3 \vee 4) \vee 5) \vee[6,7]) \vee 8 & =((1 \otimes 2 \otimes 3 \vee 4) \vee 5) \vee[[6,7], 8] \\
& =(1 \otimes 2 \otimes 3 \vee 4) \vee[5,[[6,7], 8]]  \tag{F.0.9}\\
& =1 \otimes 2 \otimes 3 \vee[4,[5,[[6,7], 8]]] \\
& =1 \otimes 2 \otimes[3,[4,[5,[[6,7], 8]]]]
\end{align*}
$$

It may be verified that $H_{[P, Q]}$ defined as such is antisymmetric in $P$ and $Q$,

$$
\begin{align*}
H \circ[Q, P] & =(H \circ Q) \vee P+Q \vee(H \circ P)+(D \circ Q) \otimes P-(D \circ P) \otimes Q \\
& =-P \vee(H \circ Q)-(H \circ P) \vee Q-(D \circ P) \otimes Q+(D \circ Q) \otimes P  \tag{F.0.10}\\
& =-H \circ[P, Q] .
\end{align*}
$$

We may also show that the results of part II of this thesis is reproduced. To show such, we begin with the identity

$$
\begin{align*}
H \circ \ell\left(a_{1} \ldots a_{n}\right) & =\sum_{i=2}^{n-1} \ell\left(a_{1} \ldots a_{i-1}\right) \otimes a_{i} \otimes a_{i+1} \vee a_{i+2} \vee \ldots \vee a_{n}  \tag{F.0.11}\\
& =-(-1)^{n} \sum_{X j Y=a_{n} \ldots a_{1}}(-1)^{|Y|} \ell(\tilde{Y}) \otimes j \otimes \ell(X) \tag{F.0.12}
\end{align*}
$$

where the $i=n-1$ case in the first sum above is understood to denote $\ell\left(a_{1} \ldots a_{n-2}\right) \otimes$ $a_{n-1} \otimes a_{n}$.

The first line follows as the $H$ and $D$ maps applied to a letter vanish. The second line requires the intermediate result, that for $A, B, C$, and $D_{i}$ Lie monomials,

$$
\begin{equation*}
A \otimes B \otimes C \vee D_{1} \vee D_{2} \vee \ldots \vee D_{n}=(-1)^{n-1} A \otimes B \otimes C \vee \ell\left(d_{n} d_{n-1} \ldots d_{1}\right) \tag{F.0.13}
\end{equation*}
$$

for $d_{i}$ the letterifications of the brackets $D_{i}$ within the $\ell$, defined so that for example if we have $\ell(12 d 6)$, where $d$ is the letterification of $[[3,4], 5]$, then this is the Lie bracket
$\ell(12 d 6)=[[[1,2], d], 6]=[[[1,2],[[3,4], 5]], 6]$. This follows from an inductive argument. From this, (F.0.12) follows

$$
\begin{align*}
H \circ \ell\left(a_{1} \ldots a_{m}\right) & =\sum_{i=2}^{m-1} \ell\left(a_{1} \ldots a_{i-1}\right) \otimes a_{i} \otimes a_{i+1} \vee a_{i+2} \vee \ldots \vee a_{m-1} \vee a_{m} \\
& =\sum_{i=2}^{m-1}(-1)^{m-i-2} \ell\left(a_{1} \ldots a_{i-1}\right) \otimes a_{i} \otimes a_{i+1} \vee \ell\left(a_{m} a_{m-1} \ldots a_{i+2}\right)  \tag{F.0.14}\\
& =\sum_{i=2}^{m-1}(-1)^{m-i-1} \ell\left(a_{1} \ldots a_{i-1}\right) \otimes a_{i} \otimes \ell\left(a_{m} a_{m-1} \ldots a_{i+1}\right) \\
& =-(-1)^{m} \sum_{X j Y=a_{m} \ldots a_{1}}(-1)^{|Y|} \ell(\tilde{Y}) \otimes j \otimes \ell(X)
\end{align*}
$$

As a consequence of this it follows that

$$
\begin{aligned}
& H_{1} \circ[\ell(P), \ell(Q)]=\frac{|Q|}{|P|+|Q|} \sum_{X j Y=\dot{q} \tilde{P}}(-1)^{|P|+|Y|} \ell(\tilde{Y}) \otimes j \otimes \ell(X) \\
&-\frac{|P|}{|P|+|Q|} \sum_{X j Y=\dot{p} \tilde{Q}}(-1)^{|Q|+|Y|} \ell(\tilde{Y}) \otimes j \otimes \ell(X)
\end{aligned}
$$

where $\dot{p}$ denotes the letterification $P$. This would then suggest consistency with the results of part II, and would be a strong piece of evidence in favour of these methods, were it not for the fact that their likely invalidity is already known.

## APPENDIX G

## The Six Point Amplitude With Arbitrary Loop Momentum Structure

In this appendix we give formulae for numerators of the six point amplitude with arbitrary loop momentum structure $[148 ; 149 ; 150]$. We assume that we are looking at the amplitude with colour ordering $A(1,2,3,4,5,6)$, and the loop momentum structure is such that the momentum going from leg 6 to leg 1 in the hexagon is $\ell+a_{1} k_{1}+a_{2} k_{2}+a_{3} k_{3}+a_{4} k_{4}+$ $a_{5} k_{5}+a_{6} k_{6}$. Also, we will not give formulae for the box numerators, as these do not depend upon the loop momentum. Instead, to find them one may just extract them from this amplitude with a more standard loop momentum structure.

We begin with the pentagon terms. The majority of these have structures similar to the following. We stress that in what follows, with any terms of the form (... + combinations), the sum over combinations applies only within the brackets.

$$
\begin{align*}
N_{1 \mid 23,4,5,6}^{(5) \text { General }}(\ell)= & +V_{1} T_{2,3,4,5,6}^{m n} k_{2}^{m} k_{3}^{n}\left(a_{3}-a_{2}\right) \\
& +\left(a_{3}-a_{2}\right)\left(V_{1} T_{23,4,5,6}^{m}+(3 \leftrightarrow 4,5,6)\right) k_{3}^{m} \\
& +\left(a_{3}-a_{2}\right) V_{1} T_{23,4,5,6}^{m} k_{2}^{m}+\left(a_{3}-a_{2}\right)\left(V_{1} T_{2,34,5,6}^{m}+(4 \leftrightarrow 5,6)\right) k_{2}^{m} \\
& +V_{12} T_{3,4,5,6}^{m}\left(\ell^{m}-k_{1}^{m}\left(a_{2}-a_{1}\right)\right) \\
& +V_{12} T_{3,4,5,6}^{m} k_{3}^{m}\left(a_{2}-a_{3}\right)+V_{13} T_{2,4,5,6}^{m} k_{2}^{m}\left(a_{2}-a_{3}\right) \\
& +\left(V_{1} T_{[23,4], 5,6}\left(\frac{1}{2}-a_{2}+a_{4}\right)+(23,4 \mid 23,4,5,6)\right)  \tag{G.0.1}\\
& +\left(a_{3}-a_{2}\right)\left(V_{1} T_{24,35,6}+(4,5 \mid 4,5,6)\right) \\
& +\left(a_{2}-a_{3}\right)\left(V_{1} T_{243,5,6}+(4 \leftrightarrow 5,6)\right) \\
& +\left(a_{2}-a_{3}\right)\left(\left(V_{12} T_{34,5,6}+(4 \leftrightarrow 5,6)\right)+(2 \leftrightarrow 3)\right) \\
& +\left(V_{14} T_{23,5,6}\left(\frac{1}{2} \ell-a_{1}+a_{4}\right)+(4 \leftrightarrow 5,6)\right) \\
& +V_{123} T_{4,5,6}\left(\frac{1}{2} \ell-a_{1}+a_{3}\right)-V_{132} T_{4,5,6}\left(\frac{1}{2} \ell-a_{1}+a_{2}\right)
\end{align*}
$$

Two terms have a slightly different structure to the above, with one of them being is

$$
\begin{align*}
N_{12 \mid 3,4,5,6}^{(5) \text { General }}(\ell)= & +V_{1} T_{2,3,4,5,6}^{m n} k_{1}^{m} k_{2}^{n}\left(a_{2}-a_{1}\right) \\
& +\left(a_{2}-a_{1}\right)\left(V_{1} T_{23,4,5,6}^{m} k_{1}^{m}+(3 \leftrightarrow 4,5,6)\right) \\
& +V_{12} T_{3,4,5,6}^{m}\left(\ell^{m}-k_{1}^{m}\left(a_{2}-a_{1}\right)\right) \\
& +k_{2}^{m}\left(a_{2}-a_{1}\right)\left(V_{12} T_{3,4,5,6}^{m}+(2 \leftrightarrow 3,4,5,6)\right) \\
& +\left(V_{12} T_{34,5,6}\left(\frac{1}{2}-a_{3}+a_{4}\right)+(3,4 \mid 3,4,5,6)\right)  \tag{G.0.2}\\
& +\left(a_{2}-a_{1}\right)\left(V_{13} T_{24,5,6}+(3|4| 3,4,5,6)\right) \\
& +\left(V_{123} T_{4,5,6}\left(\frac{1}{2}+a_{3}-a_{1}\right)+(3 \leftrightarrow 4,5,6)\right) \\
& +\left(V_{132} T_{4,5,6}\left(a_{1}-a_{2}\right)+(3 \leftrightarrow 4,5,6)\right)
\end{align*}
$$

These terms both reduce to a common formula in the $a_{i}=0 \forall i$ case though, and it may be possible to find such a formula here. Then, the term which doesn't fit this pattern in
the simpler case is again an exception here, and is given by

$$
\begin{align*}
N_{61 \mid 2,3,4,5}^{(5) \text { General }}(\ell)= & +V_{1} T_{2,3,4,5,6}^{m n} k_{1}^{m} k_{2}^{n}\left(-1+a_{1}-a_{6}\right) \\
& +\left(1+a_{6}-a_{1}\right)\left(V_{1} T_{26,3,4,5}^{m} k_{1}^{m}+(2 \leftrightarrow 3,4,5)\right) \\
& +k_{m}^{6}\left(V_{12} T_{3,4,5,6}^{m}\left(-1-a_{6}+a_{1}\right)+(2 \leftrightarrow 3,4,5)\right) \\
& +V_{16} T_{2,3,4,5}^{m}\left(-\ell^{m}-k_{m}^{6}\left(1-a_{6}+a_{1}\right)+k_{m}^{1}\left(1-a_{6}+a_{1}\right)\right) \\
& +\left(1-a_{1}+a_{6}\right)\left(V_{12} T_{36,4,5}+(2|3| 2,3,4,5)\right)  \tag{G.0.3}\\
& +\left(V_{16} T_{23,4,5}\left(-\frac{1}{2}+a_{2}-a_{3}\right)+(2,3 \mid 2,3,4,5)\right. \\
& +\left(1-a_{1}+a_{6}\right)\left(V_{126} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right) \\
& +\left(\left(-\frac{1}{2}+a_{1}-a_{2}\right) V_{162} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right)
\end{align*}
$$

Then, finally, the six point hexagon in this general case is given by

$$
\begin{align*}
N_{1 \mid 2,3,4,5,6}^{(6) \text { General }}(\ell)= & +V_{1} T_{2,3,4,5,6}^{m n}\left(\frac{1}{2} \ell^{m} \ell^{n}+\left(k_{1}^{m} k_{2}^{n}\left(\frac{1}{12}+\frac{\left(a_{2}-a_{1}\right)\left(a_{2}-a_{1}+1\right)}{2}\right)+(1,2 \mid 1,2,3,4,5,6)\right)\right) \\
+ & \left(V _ { 1 } T _ { 2 3 , 4 , 5 , 6 } ^ { m } \left(\ell^{m}\left(\frac{1}{2}-a_{2}+a_{3}\right)+\left(k_{3}^{m}-k_{2}^{m}\right)\left(\frac{1}{12}+\frac{\left(a_{3}-a_{2}\right)\left(a_{3}-a_{2}+1\right)}{2}\right)\right.\right. \\
& \left.+\left(k_{1}^{m}\left(\frac{\left(a_{2}-a_{1}\right)\left(a_{2}-a_{1}+1\right)}{2}-\frac{\left(a_{3}-a_{1}\right)\left(a_{3}-a_{1}+1\right)}{2}\right)+(1 \leftrightarrow 4,5,6)\right)\right) \\
& \quad+(2,3 \mid 2,3,4,5,6)) \\
& +\left(V _ { 1 2 } T _ { 3 , 4 , 5 , 6 } ^ { m } \left(\ell^{m}\left(\frac{1}{2}-a_{1}+a_{2}\right)+\frac{1}{12}\left(k_{2}^{m}-k_{1}^{m}\right)+k_{1}^{m}\left(a_{1}\left(a_{2}-a_{1}\right)\right)\right.\right. \\
& \left.\left.\quad+\left(k_{2}^{m}\left(a_{2}-a_{1}\right) a_{2}+(2 \leftrightarrow 3,4,5,6)\right)\right)+(2 \leftrightarrow 3,4,5,6)\right) \\
+ & \quad V_{12} T_{3,4,5,6}^{m}\left(k_{1}^{m} a_{1}+k_{2}^{m} a_{2}+\ldots+k_{6}^{m} a_{2}\right)  \tag{G.0.4}\\
& \quad+V_{13} T_{2,4,5,6}^{m}\left(k_{1}^{m} a_{1}+k_{2}^{m} a_{2}+k_{3}^{m} a_{3}+\ldots+k_{6}^{m} a_{3}\right) \\
+ & \left(\left(\frac{1}{3}-\frac{1}{2} a_{2}+\frac{1}{2} a_{3}\right) V_{1} T_{234,5,6}+(2,3,4 \mid 2,3,4,5,6)\right) \\
+ & \left(-\left(\frac{1}{6}+\frac{1}{2} a_{2}-\frac{1}{2} a_{3}\right) V_{1} T_{243,5,6}+(2,3,4 \mid 2,3,4,5,6)\right) \\
+ & \left(\left(\frac{1}{4}+\frac{1}{2}\left(a_{5}-a_{4}+a_{3}-a_{2}\right)\right) V_{1} T_{23,45,6}^{m}+(2,3|4,5| 2,3,4,5,6)\right) \\
+ & \left(\left(\frac{1}{3}-\frac{1}{2} a_{1}+\frac{1}{2} a_{3}\right) V_{123} T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right) \\
+ & \left(-\left(\frac{1}{6}+\frac{1}{2} a_{1}-\frac{1}{2} a_{2}\right) V_{132} T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right) \\
+ & \left(\left(\frac{1}{4}+\frac{1}{2}\left(a_{4}-a_{3}+a_{2}-a_{1}\right)\right) V_{12} T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right)
\end{align*}
$$

Given the significant increase in complexity in these objects when we move to this general case, it should be clear why we chose to present the BRST variation of a specific case in more detail. We should also note at this point, that by choosing the appropriate values
for $a_{i}$ we can reduce the above to the previously discussed cases.
We will not discuss in detail the finding of the BRST variation of this object, but the relevant calculation has been completed with the aid of FORM [148; 149; 150]. The resulting variation is then purely anomalous,

$$
\begin{align*}
Q A^{\text {general }}(1,2,3,4,5,6)= & V_{1} Y_{2,3,4,5,6}\left(I _ { 1 , 2 , 3 , 4 , 5 , 6 } \left(-\frac{1}{2} \ell^{2}-\left(\sum_{i=2}^{5} a_{i}\left(k_{1} \cdot k_{2 \ldots i . i}\right)\right)\right.\right. \\
& \left.+\left(\sum_{i=1}^{5} \sum_{j=i+1}^{6} a_{i} a_{j}\left(k^{i} \cdot k^{j}\right)\right)\right) \\
& +\frac{1}{2} I_{1,2,3,4,5}\left(1+a_{6}-a_{1}\right)+\frac{1}{2} I_{1,2,3,4,56}\left(a_{5}-a_{6}\right)  \tag{G.0.5}\\
& +\frac{1}{2} I_{1,2,3,45,6}\left(a_{4}-a_{5}\right)+\frac{1}{2} I_{1,2,34,5,6}\left(a_{3}-a_{4}\right) \\
& \left.+\frac{1}{2} I_{1,23,4,5,6}\left(a_{2}-a_{3}\right)+\frac{1}{2} I_{12,3,4,5,6}\left(a_{1}-a_{2}\right)\right)
\end{align*}
$$

This looks much more complicated than what was previously discussed in simpler cases, but it is completely analogous. It can be shown that

$$
\begin{gather*}
-\frac{1}{2} \ell^{2}-\left(\sum_{i=2}^{5} a_{i}\left(k_{1} \cdot k_{2 \ldots i}\right)\right)+\left(\sum_{i=1}^{5} \sum_{j=i+1}^{6} a_{i} a_{j}\left(k^{i} \cdot k^{j}\right)\right) \\
=-\left(a_{1}-a_{2}\right)\left(\ell-k_{1}+h\left(k_{1}, \ldots, k_{6}\right)\right)^{2}-\ldots-\left(a_{5}-a_{6}\right)\left(\ell-k_{12345}+h\left(k_{1}, \ldots, k_{6}\right)\right)^{2}  \tag{G.0.6}\\
-\left(1+a_{6}-a_{1}\right)\left(\ell+h\left(k_{1}, \ldots, k_{6}\right)\right)^{2} .
\end{gather*}
$$

Hence, the leftover terms in the variation (G.0.5) would appear to cancel. A similar discussion to that of the anomaly mentioned earlier should then follow.

## appendix H

## Construction and Variation of a Seven Point Numerator

In this appendix we identify the full expression for the $[5,[6,7]]$-pentagon in the amplitude $A\left(1,2,3,4,5,6,7 ; \ell+4 k_{4}-6 k_{5}\right)$, and confirm that its variation has the desired form. We begin by finding the coefficient of one term contributing to the numerator in detail, namely $V_{1} T_{2576,3,4}$. Within the string correlator this is associated with the worldsheet function

$$
\begin{align*}
\mathcal{Z}_{1,2576,3,4} & =g_{25}^{(1)} g_{57}^{(1)} g_{76}^{(1)}+g_{25}^{(3)}+g_{57}^{(3)}+g_{76}^{(3)}-2 g_{62}^{(3)}+g_{25}^{(1)}\left(g_{57}^{(2)}+g_{76}^{(2)}-g_{62}^{(2)}\right) \\
& +g_{57}^{(1)}\left(g_{25}^{(2)}+g_{76}^{(2)}-g_{62}^{(2)}\right)+g_{76}^{(1)}\left(g_{25}^{(2)}+g_{57}^{(2)}-g_{62}^{(2)}\right) . \tag{H.0.1}
\end{align*}
$$

Only two of these terms contain the $s_{67} s_{567}$ pole structure, $g_{25}^{(1)} g_{57}^{(1)} g_{76}^{(1)}$ and $g_{76}^{(1)} g_{57}^{(2)}$. The contribution of the former was identified in (10.2.25), and the latter follows from (10.2.6),

$$
\begin{equation*}
c_{76}^{(1)} c_{57}^{(2)}=\frac{1}{2} \cdot \frac{6}{2}=\frac{3}{2} . \tag{H.0.2}
\end{equation*}
$$

Summing these together, the $V_{1} T_{2576,3,4}$ contribution to the [5, [6, 7]]-pentagon is

$$
\begin{equation*}
\left(-\frac{11}{8}+\frac{3}{2}\right) V_{1} T_{2576,3,4} \hat{\phi}(1234567 \mid 576) I_{567}=-\frac{1}{8 s_{67} s_{567}} V_{1} T_{2576,3,4} I_{1,2,3,4,567} \tag{H.0.3}
\end{equation*}
$$

Similar calculations for all other terms in the correlator yield the numerator expression [148; 149; 150]

$$
N_{1 \mid 2,3,4,[5,[6,7]]}^{a_{4}=4, a_{5}=-6}(\ell)=6 V_{1} T_{2,3,4,5,67}^{m n} k_{5}^{m} k_{67}^{n}+V_{1} T_{2,3,4,[5,67]}^{m}\left(\ell^{m}-6 k_{5}^{m}+6 k_{67}^{m}\right)
$$

$$
\begin{align*}
& -6\left(\left(V_{1} T_{25,3,4,67}^{m} k_{67}^{m}+(2 \leftrightarrow 3,4)\right)+V_{15} T_{2,3,4,67}^{m} k_{67}^{m}+(5 \leftrightarrow[6,7])\right) \\
& +\frac{1}{2}\left(V_{12} T_{3,4,[5,67]}+(2 \leftrightarrow 3,4,[5,67])\right) \\
& +\frac{1}{2}\left(V_{1} T_{23,4,[5,67]}+(2,3 \mid 2,3,4,[5,67])\right)  \tag{H.0.4}\\
& +6\left(\left(V_{1} T_{25,[3,67], 4}+(2,3 \mid 2,3,4)\right)+(2 \leftrightarrow 3)\right) \\
& +6\left(\left(V_{15} T_{[2,67], 3,4}+(2 \leftrightarrow 3,4)\right)+(5 \leftrightarrow[6,7])\right) \\
& +6\left(\left(V_{1} T_{2675,3,4}+(2 \leftrightarrow 3,4)\right)-(6 \leftrightarrow 7)\right) \\
& +6\left(V_{1675} T_{2,3,4}-(6 \leftrightarrow 7)\right)+4\left(V_{1} T_{24,3,[5,67]}+(2 \leftrightarrow 3)\right) \\
& +4 V_{14} T_{2,3,[5,67]}-4 V_{1} T_{2,3,[4,[5,67]]}+6 V_{1} J_{5 \mid 2,3,4,6,7}^{m}\left(k_{6}^{m}-k_{7}^{m}\right) \\
& +6 s_{67}\left(\left(V_{1} J_{5 \mid 27,3,4,6}+(2 \leftrightarrow 3,4,6)\right)+V_{17} J_{5 \mid 2,3,4,5,6}-(6 \leftrightarrow 7)\right) .
\end{align*}
$$

The $V J$ terms above are those which arise naively by looking to the $s_{67} s_{567}$ poles in the correlator. As discussed previously it may be that they require some rearrangement to be in a BCJ representation, but for illustrating the field theory limit methods we give the numerator in the above form. A lengthy calculation yields the variation [148; 149; 150]

$$
\begin{align*}
& Q N_{1 \mid 2,3,4,5,[6,7]]}^{a_{4}=4, a_{5}=-6}(\ell)=\frac{1}{2} V_{1} V_{2} T_{3,4,[5,67]}\left(\left(\ell-k_{12}+4 k_{4}-6 k_{5}\right)^{2}-\left(\ell-k_{1}+4 k_{4}-6 k_{5}\right)^{2}\right) \\
&+\frac{1}{2} V_{1} V_{3} T_{2,4,[5,67]}\left(\left(\ell-k_{123}+4 k_{4}-6 k_{5}\right)^{2}-\left(\ell-k_{12}+4 k_{4}-6 k_{5}\right)^{2}\right) \\
&+\frac{1}{2} V_{1} V_{4} T_{2,3,[5,67]}\left(\left(\ell-k_{1234}+4 k_{4}-6 k_{5}\right)^{2}-\left(\ell-k_{123}+4 k_{4}-6 k_{5}\right)^{2}\right) \\
&+\frac{1}{2} V_{1} V_{[5,67]} T_{2,3,4}\left(\left(\ell-k_{1234567}+4 k_{4}-6 k_{5}\right)^{2}-\left(\ell-k_{1234}+4 k_{4}-6 k_{5}\right)^{2}\right) \\
&+\left(k^{6} \cdot k^{7}\right)\left(\left(6 V_{1} V_{26} T_{3,4,5,7}^{m} k_{5}^{m}+(2 \leftrightarrow 3,4,5)\right)+V_{1} V_{57} T_{2,3,4,6}^{m}\left(\ell^{m}+6 k_{57}^{m}\right)\right. \\
&+6 V_{1} V_{7} T_{2,3,4,5,6}^{m n} k_{5}^{m} k_{67}^{n}+6 k_{5}^{m}\left(V_{1} V_{6} T_{27,3,4,5}^{m}+(2 \leftrightarrow 3,4,5)\right)  \tag{H.0.5}\\
&+\left(\ell^{m}+6 k_{67}^{m} V_{1} V_{7} T_{2,3,4,56}^{m}+6\left(V_{1} V_{6} T_{25,3,4,7}^{m} k_{67}^{m}+(2 \leftrightarrow 3,4)\right)\right. \\
&+6 V_{15} V_{6} T_{2,3,4,7}^{m} k_{67}^{m}+6 V_{17} V_{6} T_{2,3,4,5}^{m} k_{5}^{m} \\
&+6 V_{1} V_{5} T_{2,3,4,6,7}^{m} k_{5}^{m} k_{7}^{n}+V_{16} V_{5} T_{2,3,4,7}^{m} k_{5}^{m} \\
&+\left(6 V_{1} V_{25} T_{3,4,6,7}^{m} k_{6}^{m}+(2 \leftrightarrow 3,4)\right)+\left(6 V_{1} V_{5} T_{26,3,4,7}^{m} k_{5}^{m}+(2 \leftrightarrow 3,4,7)\right) \\
&+\frac{1}{2}\left(V_{1} V_{[2,57]} T_{3,4,6}+(2 \leftrightarrow 3,4)\right) \\
&-\frac{1}{2}\left(V_{1} V_{26} T_{3,4,57}+(2 \leftrightarrow 3,4)\right)-\frac{1}{2}\left(V_{1} V_{56} T_{23,4,7}+(2,3 \mid 2,3,4,7)\right) \\
&+\frac{1}{2}\left(V_{1} V_{7} T_{[2,3], 4,56}+(2,3 \mid 2,3,4,56)\right)+\frac{1}{2}\left(V_{12} V_{57} T_{3,4,6}+(2 \leftrightarrow 3,4)\right) \\
&+\frac{1}{2}\left(V_{12} V_{7} T_{3,4,56}+(2 \leftrightarrow 3,4,56)\right)-\frac{1}{2} V_{17} V_{56} T_{2,3,4}+\frac{1}{2} V_{175} V_{6} T_{2,3,4} \\
&+6\left(\left(V_{1} V_{27} T_{[3,5], 4,6}+(3 \leftrightarrow 4,6)\right)+(2 \leftrightarrow 3,4)\right)+6\left(V_{1} V_{7} T_{[26,5], 3,4}+(2 \leftrightarrow 3,4)\right) \\
&+6\left(V_{1} V_{7} T_{25,36,4}+V_{1} V_{7} T_{26,35,4}+(2,3 \mid 2,3,4)\right)+6\left(V_{15} V_{27} T_{3,4,6}+(2 \leftrightarrow 3,4)\right) \\
&+6\left(V_{15} V_{7} T_{26,3,4}+(2 \leftrightarrow 3,4)\right)+6\left(V_{16} V_{7} T_{25,3,4}+(2 \leftrightarrow 3,4)\right)
\end{align*}
$$

$$
\begin{aligned}
&+6\left(\left(V_{1} V_{25} T_{37,4,6}+(3 \leftrightarrow 4,6)\right)+V_{17} V_{25} T_{3,4,6}+(2 \leftrightarrow 3,4)\right) \\
&+6 V_{165} V_{7} T_{2,3,4}+6\left(V_{1}\left(V_{257}+V_{275}\right) T_{3,4,6}+(2 \leftrightarrow 3)\right)+6 V_{1} V_{576} T_{2,3,4} \\
&+4\left(V_{1} V_{57} T_{24,3,6}+(2 \leftrightarrow 3,6)\right)+4 V_{14} V_{57} T_{2,3,6}+4\left(V_{1} V_{7} T_{24,3,56}+(2 \leftrightarrow 3,56)\right) \\
&+4 V_{14} V_{7} T_{2,3,56}+4 V_{1} V_{46} T_{2,3,57}+2 V_{1} V_{457} T_{2,3,6}+20 V_{1} V_{475} T_{2,3,6} \\
&+6 V_{1} Y_{2,3,4,5,6,7}^{m} m_{7}^{m}+6\left(V_{1} Y_{26,3,4,5,7}+(2 \leftrightarrow 3,4,5,7)\right)+6 V_{16} Y_{2,3,4,5,7} \\
&-(6 \leftrightarrow 7)) \\
&+\left(k^{5} \cdot k^{67}\right)\left(\left(\begin{array}{l}
\frac{1}{2} \\
(
\end{array} V_{1} V_{[2,67]} T_{3,4,5}+V_{12} V_{67} T_{3,4,5}+(2 \leftrightarrow 3,4)\right)+4 V_{1} V_{5} T_{2,3,67}+4 V_{14} V_{67} T_{2,3,5}\right. \\
&\left.+\frac{1}{2}\left(V_{1} V_{67} T_{23,4,5}+(2,3 \mid 2,3,4,5)\right)+4\left(V_{1} V_{67} T_{24,3,5}+(2 \leftrightarrow 3,5)\right)-(5 \leftrightarrow 67)\right) \\
&-\frac{1}{2}\left(V_{15} V_{67} T_{2,3,4}+(5 \leftrightarrow 6,7)\right)+6\left(\left(V_{1} V_{2} 5 T_{3,4,67}-(25 \leftrightarrow 67)\right)+(2 \leftrightarrow 3,4)\right) \\
&\left.-6 V_{15} V_{67} T_{2,3,4}-6 V_{1} Y_{2,3,4,5,67}\right) \\
&+6\left(k^{6} \cdot k^{7}\right)\left(k^{5} \cdot k^{6}\right) V_{1} V_{5}\left(J_{7 \mid 2,3,4,6}+J_{6 \mid 2,3,4,7}\right)-6\left(k^{5} \cdot k^{67}\right)\left(k^{6} \cdot k^{7}\right) V_{1} V_{5} J_{7 \mid 2,33,4,6}
\end{aligned}
$$

This has intentionally been expressed with factors ( $\ell \cdot k$ ) reformulated in terms of propagators. For an $n$-point amplitude in the canonical ordering with arbitrary loop momentum structure, this is done with

$$
\begin{align*}
\left(\ell \cdot k_{i(i+1) \ldots j}\right)=-\frac{1}{2}(\ell & \left.+\sum_{m=1}^{n} a_{m} k_{m}-k_{12 \ldots j}\right)^{2}+\frac{1}{2}\left(\ell+\sum_{m=1}^{n} a_{m} k_{m}-k_{12 \ldots(i-1)}\right)^{2} \\
& -k_{i(i+1) \ldots j} \cdot\left(\sum_{m=1}^{n} a_{m} k_{m}-\frac{1}{2} k_{i(i+1) \ldots j}\right) \tag{H.0.6}
\end{align*}
$$

We may then be reassured of the validity of this numerator expression, as those terms in the variation proportional to propagators cancel terms from other box numerators. For example, one such set of terms is

$$
\begin{gather*}
V_{1} V_{3} T_{2,4,[5,67]}\left(\left(\ell-k_{123}+4 k_{4}-6 k_{5}\right)^{2}-\left(\ell-k_{12}+4 k_{4}-6 k_{5}\right)^{2}\right) I_{1,2,3,4,567}^{a_{4}=4, a_{5}=-6} \\
=V_{1} V_{3} T_{2,4,[5,67]}\left(I_{1,2,34,567}^{a_{4}=4, a_{5}=-6}-I_{1,23,4,567}^{a_{4}=4, a_{5}=-6}\right) \tag{H.0.7}
\end{gather*}
$$

This then cancels one term in the variation of the $[3,4],[5,[6,7]]$-box, and one from the $[2,3],[5,[6,7]]$ box. Similar holds true for all other terms in the variation, and the remaining terms in (H.0.5) are canceled themselves by analogous results in the variation of hexagons.

## APPENDIX

## Discussion of the amplitude $A(1,2,3,4,5,6,7 ; \ell)$

In this appendix we present in detail the ingredients in the canonically ordered amplitude with the standard loop momentum assignment, $A(1,2,3,4,5,6,7 ; \ell)$. All of these were found using the field theory limit rules detailed in part III. While every effort has been made to avoid typos, in expressions of this scale such are inevitable, and expressions generated using FORM $[148 ; 149 ; 150]$ which are less compact but free of typos are available from [28].

Note, there is inherent choice in which numerators to assign the refined $V J$ terms, as is discussed in section 11.2. Here, we choose to present them with their loop momentum cancelled against the propagators as discussed there. In such a representation, all numerators should satisfy BCJ relations, apart from the 71-hexagon and $[[6,7], 1],[6,[7,1]],[[7,1], 2]$, and $[7,[1,2]]$-pentagons, which require further manipulation as discussed in that section.

We begin with the heptagon numerator. This is given by ${ }^{1}$

$$
\begin{align*}
N_{1 \mid 2,3,4,5,6,7}(\ell)= & -\frac{1}{24}\left(s_{12} \Delta_{1|2| 3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right)  \tag{I.0.1}\\
& +\frac{1}{24} V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{2}^{n}\left[\left(k_{m}^{3} k_{n}^{3}+(3 \leftrightarrow 4,5,6,7)\right)-k_{m}^{1} k_{n}^{1}\right] \\
& +\frac{1}{24} V_{1} J_{3 \mid 2,4,5,6,7}^{m} k_{3}^{n}\left[\left(k_{m}^{4} k_{n}^{4}+(4 \leftrightarrow 5,6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2)\right)\right]
\end{align*}
$$

[^26]\[

$$
\begin{aligned}
& +\frac{1}{24} V_{1} J_{4 \mid 2,3,5,6,7}^{m} k_{4}^{n}\left[\left(k_{m}^{5} k_{n}^{5}+(5 \leftrightarrow 6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3)\right)\right] \\
& +\frac{1}{24} V_{1} J_{5 \mid 2,3,4,6,7}^{m} k_{5}^{n}\left[\left(k_{m}^{6} k_{n}^{6}+(6 \leftrightarrow 7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4)\right)\right] \\
& +\frac{1}{24} V_{1} J_{6 \mid 2,3,4,5,7}^{m} k_{6}^{n}\left[k_{m}^{7} k_{n}^{7}-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4,5)\right)\right] \\
& -\frac{1}{24} V_{1} J_{7 \mid 2,3,4,5,6}^{m}\left[s_{17} k_{1}^{m}+(1 \leftrightarrow 2,3,4,5,6)\right] \\
& +\frac{1}{24}\left[V_{1} J_{2 \mid 34,5,6,7}\left(s_{23}-s_{24}\right)+(3,4 \mid 3,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{1} J_{3 \mid 2,45,6,7}\left(s_{34}-s_{35}\right)+(4,5 \mid 4,5,6,7)\right] \\
& -\frac{1}{24}\left[V_{1} J_{3 \mid 24,5,6,7}\left(s_{23}+s_{34}\right)+(4 \leftrightarrow 5,6,7)\right] \\
& +\frac{1}{24}\left[V_{1} J_{4 \mid 2,3,56,7}\left(s_{45}-s_{46}\right)+(56 \leftrightarrow 23,57,67)\right] \\
& -\frac{1}{24}\left[\left[V_{1} J_{4 \mid 25,5,6,7}\left(s_{24}+s_{45}\right)+(5 \leftrightarrow 6,7)\right]+(2 \leftrightarrow 3)\right] \\
& +\frac{1}{24}\left[V_{1} J_{5 \mid 23,4,6,7}\left(s_{25}-s_{35}\right)+(23 \leftrightarrow 24,34,67)\right] \\
& -\frac{1}{24}\left[\left[V_{1} J_{5 \mid 26,4,6,7}\left(s_{25}+s_{56}\right)+(2 \leftrightarrow 3,4)\right]+(6 \leftrightarrow 7)\right] \\
& +\frac{1}{24}\left[V_{1} J_{6 \mid 23,4,5,7}\left(s_{36}-s_{26}\right)+(2,3 \mid 2,3,4,5)\right] \\
& -\frac{1}{24}\left[V_{1} J_{6 \mid 27,3,4,5}\left(s_{26}+s_{67}\right)+(2 \leftrightarrow 3,4,5)\right] \\
& +\frac{1}{24}\left[V_{1} J_{7 \mid 23,4,5,6}\left(s_{37}-s_{27}\right)+(2,3 \mid 2,3,4,5,6)\right] \\
& -\frac{1}{24}\left[s_{23} V_{1} J_{23 \mid 4,5,6,7}+(2,3 \mid 2,3,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{12} J_{3 \mid 4,5,6,7}\left(s_{23}-s_{13}\right)+(2,3 \mid 2,3,4,5,6,7)\right] \\
& -\frac{1}{24}\left[V_{13} J_{2 \mid 4,5,6,7}\left(s_{12}+s_{23}\right)+(2,3 \mid 2,3,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{1} T_{2,3,4,5,6,7}^{m n n}\left(4 \ell^{m} \ell^{n} \ell^{p}-\ell^{m}\left(k_{1}^{n} k_{1}^{p}+k_{2}^{n} k_{2}^{p}+\ldots+k_{7}^{n} k_{7}^{p}\right)\right]\right. \\
& +\frac{1}{48}\left[V_{1} T_{23,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{2}^{n}+4 \ell^{m} k_{3}^{n}-2 k_{2}^{m} k_{3}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right)\right. \\
& +(2,3 \mid 2,3,4,5,6,7)] \\
& +\frac{1}{48}\left[V_{12} T_{3,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{1}^{n}+4 \ell^{m} k_{2}^{n}+2 k_{1}^{m} k_{2}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right)\right. \\
& +(2 \leftrightarrow 3,4,5,6,7)] \\
& +\frac{1}{12}\left[V_{1} T_{234,5,6,7}^{m}\left(4 \ell^{m}-k_{2}^{m}+k_{4}^{m}\right)+(2,3,4 \mid 2,3,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{1} T_{243,5,6,7}^{m}\left(2 \ell^{m}+k_{4}^{m}\right)+(2,3,4 \mid 2,3,4,5,6,7)\right] \\
& +\frac{1}{12}\left[V_{123} T_{4,5,6,7}^{m}\left(4 \ell^{m}-k_{1}^{m}+k_{3}^{m}\right)+(2,3,4 \mid 2,3,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{132} T_{4,5,6,7}^{m}\left(2 \ell^{m}+k_{3}^{m}\right)+(2,3 \mid 2,3,4,5,6,7)\right]
\end{aligned}
$$
\]

$$
\begin{aligned}
& +\frac{1}{24}\left[V_{1} T_{23,45,6,7}^{m}\left(6 \ell^{m}-k_{24}^{m}+k_{35}^{m}\right)+(2,3|4,5| 2,3,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{12} T_{34,5,6,7}^{m}\left(6 \ell^{m}-k_{13}^{m}+k_{24}^{m}\right)+(2|3,4| 2,3,4,5,6,7)\right] \\
& +\frac{1}{12} V_{1}\left[3 T_{2345,6,7}-T_{2354,6,7}-T_{2435,6,7}-T_{2453,6,7}-T_{2534,6,7}+T_{2543,6,7}\right. \\
& +(2,3,4,5 \mid 2,3,4,5,6,7)] \\
& +\frac{1}{12}\left[\left(3 V_{1234}-V_{1243}-V_{1324}-V_{1342}-V_{1423}+V_{1432}\right) T_{5,6,7}\right. \\
& +(2,3,4 \mid 2,3,4,5,6,7)] \\
& +\frac{1}{6}\left[V_{1} T_{234,56,7}+(2,3,4|5,6| 2,3,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{1} T_{243,56,7}+(2,3,4|5,6| 2,3,4,5,6,7)\right] \\
& +\frac{1}{8}\left[V_{1} T_{23,45,67}+(2,3|4,5| 6,7 \mid 2,3,4,5,6,7)\right] \\
& +\frac{1}{6}\left[V_{123} T_{45,6,7}+(2,3|4,5| 2,3,4,5,6,7)\right] \\
& +\frac{1}{6}\left[V_{12} T_{345,6,7}+(2|3,4,5| 2,3,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{132} T_{45,6,7}+(2,3|4,5| 2,3,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{12} T_{354,6,7}+(2|3,4,5| 2,3,4,5,6,7)\right] \\
& +\frac{1}{8}\left[V_{12} T_{34,56,7}+(2|3,4| 5,6 \mid 2,3,4,5,6,7)\right] .
\end{aligned}
$$

The variation of this may be reexpressed entirely, up to anomaly terms, as superfields multiplied by Feynman loop propagators,

$$
\begin{align*}
Q N_{1 \mid 2,3,4,5,6,7}(\ell)= & -\frac{1}{2} V_{1} Y_{2,3,4,5,6,7}^{m} \ell^{m} \ell^{2}  \tag{I.0.2}\\
& -\frac{1}{4} \ell^{2}\left[V_{1} Y_{23,4,5,6,7}+(2,3 \mid 2,3,4,5,6,7)\right] \\
& -\frac{1}{4} \ell^{2}\left[V_{12} Y_{3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right] \\
+\left(\ell-k_{1}\right)^{2}( & +\frac{1}{24} V_{1} Y_{2,3,4,5,6,7}^{m}\left(k_{2}^{m}-k_{1}^{m}\right) \\
& +\frac{1}{24} V_{1}\left[Y_{23,4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right] \\
& -\frac{1}{24}\left[2 V_{12} Y_{3,4,5,6,7}+\left(V_{13} Y_{2,4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right)\right] \\
& +V_{1} V_{2} T_{3,4,5,6,7}^{m n}\left[-\frac{1}{4} \ell^{m} \ell^{n}+\frac{1}{48}\left(k_{1}^{m} k_{1}^{n}+\ldots+k_{7}^{m} k_{7}^{n}\right)\right] \\
& +V_{1} V_{2}\left[T_{34,5,6,7}^{m}\left(-\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{3}^{m}-k_{4}^{m}\right)\right)+(3,4 \mid 3,4,5,6,7)\right] \\
& +V_{1}\left[V_{23} T_{4,5,6,7}^{m}\left(-\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{2}^{m}-k_{3}^{m}\right)\right)+(3 \leftrightarrow 4,5,6,7)\right] \\
& +\left[V_{13} V_{2} T_{4,5,6,7}^{m}\left(-\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{1}^{m}-k_{3}^{m}\right)\right)+(3 \leftrightarrow 4,5,6,7)\right]
\end{align*}
$$

$$
\begin{aligned}
& +\frac{1}{12} V_{1} V_{2}\left[T_{354,6,7}-2 T_{345,6,7}+(3,4,5 \mid 3,4,5,6,7)\right] \\
& -\frac{1}{8} V_{1} V_{2}\left[T_{34,56,7}+(3,4|5,6| 3,4,5,6,7)\right] \\
& -\frac{1}{8} V_{1}\left[V_{23} T_{45,6,7}+(3|4,5| 3,4,5,6,7)\right] \\
& +\frac{1}{12} V_{1}\left[\left(V_{243}-2 V_{234}\right) T_{5,6,7}+(3,4 \mid 3,4,5,6,7)\right] \\
& -\frac{1}{8}\left[V_{13} V_{2} T_{45,6,7}+(3|4,5| 3,4,5,6,7)\right] \\
& -\frac{1}{8}\left[V_{13} V_{24} T_{5,6,7}+(3|4| 3,4,5,6,7)\right] \\
& \left.+\frac{1}{12}\left[\left(V_{143}-2 V_{134}\right) V_{2} T_{5,6,7}+(3,4 \mid 3,4,5,6,7)\right]\right) \\
& +\left(\ell-k_{12}\right)^{2}\left(+\frac{1}{24} V_{1} Y_{2,3,4,5,6,7}^{m} k_{3}^{m}+\frac{1}{24} V_{12} Y_{3,4,5,6,7}\right. \\
& +\frac{1}{24} V_{1}\left[Y_{2,34,5,6,7}+(4 \leftrightarrow 2,5,6,7)\right] \\
& +V_{1} V_{2} T_{3,4,5,6,7}^{m n}\left(\frac{1}{4} \ell^{m} \ell^{n}-\frac{1}{48}\left(k_{1}^{m} k_{1}^{n}+\ldots+k_{7}^{m} k_{7}^{n}\right)\right) \\
& +V_{1}\left[V_{2} T_{34,5,6,7}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{4}^{m}-k_{3}^{m}\right)\right)+(3,4 \mid 3,4,5,6,7)\right] \\
& +V_{1}\left[V_{24} T_{3,5,6,7}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{4}^{m}-k_{2}^{m}\right)\right)+(4 \leftrightarrow 5,6,7)\right] \\
& +\left[V_{13} V_{2} T_{4,5,6,7}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{3}^{m}-k_{1}^{m}\right)\right)+(3 \leftrightarrow 4,5,6,7)\right] \\
& +\frac{1}{12} V_{1}\left[V_{2}\left(2 T_{345,6,7}-T_{354,6,7}\right)+(3,4,5 \mid 3,4,5,6,7)\right] \\
& +\frac{1}{8} V_{1}\left[V_{2} T_{34,56,7}+(3,4|5,6| 3,4,5,6,7)\right] \\
& +\frac{1}{8} V_{1}\left[\left(V_{24} T_{35,6,7}+(3,5 \mid 3,5,6,7)\right)+(4 \leftrightarrow 5,6,7)\right] \\
& +\frac{1}{12} V_{1}\left[\left(2 V_{245}-V_{254}\right) T_{5,6,7}+(4,5 \mid 4,5,6,7)\right] \\
& -\frac{1}{8}\left[V_{12} V_{3} T_{45,6,7}+(2|4,5| 2,4,5,6,7)\right] \\
& -\frac{1}{8}\left[V_{12} V_{34} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right)+\left(V_{14} V_{35} T_{2,6,7}+(4|5| 4,5,6,7)\right] \\
& +\frac{1}{12}\left[\left(V_{142}-2 V_{124}\right) V_{3} T_{5,6,7}+(2,4 \mid 2,4,5,6,7)\right] \\
& -(2 \leftrightarrow 3)) \\
& +\left(\ell-k_{123}\right)^{2}\left(+\frac{1}{24} V_{1} Y_{2,3,4,5,6,7}^{m} k_{4}^{m}+\frac{1}{24} V_{13} Y_{2,4,5,6,7}\right. \\
& +\frac{1}{24} V_{1}\left[Y_{42,3,5,6,7}+(2 \leftrightarrow 3,5,6,7)\right] \\
& +V_{1} V_{3} T_{2,4,5,6,7}^{m n}\left(\frac{1}{4} \ell^{m} \ell^{n}-\frac{1}{48}\left(k_{1}^{m} k_{1}^{n}+\ldots+k_{7}^{m} k_{7}^{n}\right)\right) \\
& +V_{1}\left[V_{3} T_{24,5,6,7}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{4}^{m}-k_{2}^{m}\right)\right)+(2,4 \mid 2,4,5,6,7)\right] \\
& +V_{1}\left[V_{23} T_{4,5,6,7}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{3}^{m}-k_{2}^{m}\right)\right)+(23 \leftrightarrow 35,36,37)\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\left[V_{12} V_{3} T_{4,5,6,7}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{2}^{m}-k_{1}^{m}\right)\right)+(2 \leftrightarrow 4,5,6,7)\right] \\
& +\frac{1}{12} V_{1}\left[V_{3}\left(2 T_{245,6,7}-T_{254,6,7}\right)+(2,4,5 \mid 2,4,5,6,7)\right] \\
& +\frac{1}{8} V_{1}\left[V_{3} T_{24,56,7}+(2,4|5,6| 2,4,5,6,7)\right] \\
& -\frac{1}{8} V_{1}\left[V_{\operatorname{Can}(24)} T_{35,6,7}+(2|3,5| 2,3,5,6,7)\right] \\
& +\frac{1}{12} V_{1}\left[\left(2 V_{\operatorname{Can}(235)}-V_{\operatorname{Can}(23)}(253)\right) T_{5,6,7}+(2,5 \mid 2,5,6,7)\right] \\
& -\frac{1}{8}\left[V_{12} V_{4} T_{35,6,7}+(2|3,5| 2,4,5,6,7)\right] \\
& \left.-\frac{1}{8}\left[V_{13} V_{24} T_{5,6,7}+(24 \leftrightarrow 45,46,47)\right)+\left(V_{12} V_{\operatorname{Can}(45)} T_{3,6,7}+(2|5| 2,5,6,7)\right)\right] \\
& +\frac{1}{12}\left[\left(V_{132}-2 V_{123}\right) V_{4} T_{5,6,7}+(2,3 \mid 2,3,5,6,7)\right]
\end{aligned}
$$

$$
\begin{array}{r}
+\left(\ell-k_{1234}\right)^{2}(\ldots) \\
+\left(\ell-k_{12345}\right)^{2}(\ldots) \\
+\left(\ell-k_{123456}\right)^{2}(\ldots)
\end{array}
$$

$$
+\left(\ell-k_{1234567}\right)^{2}\left(+\frac{1}{24} V_{1} Y_{2,3,4,5,6,7}^{m}\left(k_{1}^{m}-k_{7}^{m}\right)\right.
$$

$$
+\frac{1}{24} V_{1}\left[Y_{27,3,4,5,6}+(2 \leftrightarrow 3,4,5,6)\right]
$$

$$
+\frac{1}{24}\left[2 V_{17} Y_{2,3,4,5,6}+\left(V_{12} Y_{3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right)\right]
$$

$$
+V_{1} V_{7} T_{2,3,4,5,6}^{m n}\left[+\frac{1}{4} \ell^{m} \ell^{n}-\frac{1}{48}\left(k_{1}^{m} k_{1}^{n}+\ldots+k_{7}^{m} k_{7}^{n}\right)\right]
$$

$$
+V_{1} V_{7}\left[T_{23,4,5,6}^{m}\left(\frac{1}{4} \ell^{m}-\frac{1}{24}\left(k_{2}^{m}-k_{3}^{m}\right)\right)+(2,3 \mid 2,3,4,5,6)\right]
$$

$$
+V_{1}\left[V_{27} T_{3.4,5,6}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{7}^{m}-k_{2}^{m}\right)\right)+(2 \leftrightarrow 3,4,5,6)\right]
$$

$$
+\left[V_{12} V_{7} T_{3,4,5,6}^{m}\left(\frac{1}{4} \ell^{m}+\frac{1}{24}\left(k_{2}^{m}-k_{1}^{m}\right)\right)+(2 \leftrightarrow 3,4,5,6)\right]
$$

$$
+\frac{1}{12} V_{1} V_{7}\left[2 T_{234,6,7}-T_{243,6,7}+(2,3,4 \mid 2,3,4,5,6)\right]
$$

$$
+\frac{1}{8} V_{1} V_{7}\left[T_{23,45,6}+(2,3|4,5| 2,3,4,5,6)\right]
$$

$$
+\frac{1}{8} V_{1}\left[V_{27} T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right]
$$

$$
+\frac{1}{12} V_{1}\left[\left(2 V_{237}-V_{273}\right) T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right]
$$

$$
+\frac{1}{8}\left[V_{12} V_{7} T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right]
$$

$$
+\frac{1}{8}\left[V_{12} V_{37} T_{4,5,6}+(2|3| 2,3,4,5,6)\right]
$$

$$
\left.+\frac{1}{12}\left[\left(2 V_{123}-V_{132}\right) V_{7} T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right]\right)
$$

Note we have introduced the notation $\operatorname{Can}(A)$ denotes the word $A$ written in the canonical ordering, and $C a n_{\sigma}(A)$ denotes the canonical ordering of $A$ with permutation $\sigma$ applied to the resulting expression. This is done as a means of simplifying expressions, and some examples of these maps would be

$$
\begin{array}{rlrl}
\operatorname{Can}(123)=123, & \operatorname{Can}(132) & =123, &  \tag{I.0.3}\\
C a n(2436)=2346, \\
\operatorname{Can}_{(12)}(123)=213, & \operatorname{Can}_{(23)}(132) & =132, & \\
\operatorname{Can}_{(234)}(2436)=2634 .
\end{array}
$$

The terms proportional to $\left(\ell-k_{1234}\right),\left(\ell-k_{12345}\right)$, and $\left(\ell-k_{123456}\right)$ in (I.0.2) have not been included, as they follow a similar pattern to those proportional to $\left(\ell-k_{12}\right)$ and $\left(\ell-k_{123}\right)$, and so do not represent any new information.

We then cancel these propagator terms against the $I_{1,2,3,4,5,6,7}$ denominator, and then each set of terms in the above cancels against those from the hexagon term with the same denominator. So for instance, by using the relation

$$
\begin{equation*}
\left(\ell-k^{1}\right)^{2} I_{1,2,3,4,5,6,7}=I_{12,3,4,5,6,7}, \tag{I.0.4}
\end{equation*}
$$

we see that the terms proportional to $\left(\ell-k^{1}\right)^{2}$ in (I.0.2) should cancel against terms from the 12 -hexagon. This hexagon is found to be given by ${ }^{2}$

$$
\begin{align*}
N_{12 \mid 3,4,5,6,7}(\ell)= & -\frac{1}{12} s_{12} V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{1}^{m}  \tag{I.0.5}\\
& -\frac{1}{12} s_{12}\left[V_{13} J_{2 \mid 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right] \\
& -\frac{1}{12} s_{12} \Delta_{1|2| 3,4,5,6,7} \\
& +\frac{1}{24} V_{12} T_{3,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}+2 k_{1}^{m} k_{2}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right) \\
& +\frac{1}{12}\left[V_{12} T_{34,5,6,7}^{m}\left(6 \ell^{m}-k_{3}^{m}+k_{4}^{m}\right)+(3,4 \mid 3,4,5,6,7)\right] \\
& +\frac{1}{12}\left[V_{123} T_{4,5,6,7}^{m}\left(6 \ell^{m}-k_{12}^{m}+k_{3}^{m}\right)+(3 \leftrightarrow 4,5,6,7)\right] \\
& +\frac{1}{6}\left[2 V_{12} T_{345,6,7}-V_{12} T_{354,6,7}+(3,4,5 \mid 3,4,5,6,7)\right] \\
& +\frac{1}{4}\left[V_{12} T_{34,56,7}+(3,4|5,6| 3,4,5,6,7)\right] \\
& +\frac{1}{6}\left[2 V_{1234} T_{5,6,7}-V_{1243} T_{5,6,7}+(3,4 \mid 3,4,5,6,7)\right] \\
& +\frac{1}{4}\left[V_{123} T_{45,6,7}+(3|4,5| 3,4,5,6,7)\right]
\end{align*}
$$

This then has variation

$$
\begin{equation*}
Q N_{12 \mid 3,4,5,6,7}(\ell)=\left[\left(\ell-k_{123}\right)^{2}-\left(\ell-k_{12}\right)^{2}\right]\left(\frac{1}{2} V_{12} V_{3} T_{4,5,6,7}^{m} 7^{m}\right. \tag{I.0.6}
\end{equation*}
$$

[^27]\[

$$
\begin{aligned}
& +\frac{1}{4}\left[V_{12} V_{3} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{12} V_{34}+V_{124} V_{3}\right) T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]\right) \\
& +\left[\left(\ell-k_{1234}\right)^{2}-\left(\ell-k_{123}\right)^{2}\right]\left(\frac{1}{2} V_{12} V_{4} T_{3,5,6,7}^{m} \ell^{m}\right. \\
& +\frac{1}{4}\left[V_{12} V_{4} T_{35,6,7}+(3,5 \mid 3,5,6,7)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{12} V_{\operatorname{Can}(34)}+V_{123} V_{4}\right) T_{5,6,7}+(3 \leftrightarrow 5,6,7)\right]\right) \\
& +\left[\left(\ell-k_{12345}\right)^{2}-\left(\ell-k_{1234}\right)^{2}\right]\left(\frac{1}{2} V_{12} V_{5} T_{3,4,6,7}^{m} \ell^{m}\right. \\
& +\frac{1}{4}\left[V_{12} V_{5} T_{34,6,7}+(3,4 \mid 3,4,6,7)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{12} V_{\operatorname{Can}(35)}+V_{123} V_{5}\right) T_{4,6,7}+(3 \leftrightarrow 4,6,7)\right]\right) \\
& +\left[\left(\ell-k_{123456}\right)^{2}-\left(\ell-k_{12345}\right)^{2}\right]\left(\frac{1}{2} V_{12} V_{6} T_{3,4,5,7}^{m} \ell^{m}\right. \\
& +\frac{1}{4}\left[V_{12} V_{6} T_{34,5,7}+(3,4 \mid 3,4,5,7)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{12} V_{C a n(36)}+V_{123} V_{6}\right) T_{4,5,7}+(3 \leftrightarrow 4,5,7)\right]\right) \\
& +\left[\left(\ell-k_{1234567}\right)^{2}-\left(\ell-k_{123456}\right)^{2}\right]\left(\frac{1}{2} V_{12} V_{7} T_{3,4,5,6}^{m} \ell^{m}\right. \\
& +\frac{1}{4}\left[V_{12} V_{7} T_{34,5,6}+(3,4 \mid 3,4,5,6)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{12} V_{\operatorname{Can}(37)}+V_{123} V_{7}\right) T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right]\right) \\
& -\frac{1}{2} \ell^{2} V_{12} Y_{3,4,5,6,7} \\
& +s_{12}\left(-\frac{1}{12} V_{1} Y_{2,3,4,5,6,7}^{m}\left(k_{2}^{m}-k_{1}^{m}\right)\right. \\
& -\frac{1}{12} V_{1}\left[Y_{23,4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right] \\
& +\frac{1}{12}\left[2 V_{12} Y_{3,4,5,6,7}+\left(V_{13} Y_{2,4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right)\right] \\
& -V_{1} V_{2} T_{3,4,5,6,7}^{m n}\left[-\frac{1}{2} \ell^{m} \ell^{n}+\frac{1}{24}\left(k_{1}^{m} k_{1}^{n}+\ldots+k_{7}^{m} k_{7}^{n}\right)\right] \\
& -V_{1} V_{2}\left[T_{34,5,6,7}^{m}\left(-\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{3}^{m}-k_{4}^{m}\right)\right)+(3,4 \mid 3,4,5,6,7)\right] \\
& -V_{1}\left[V_{23} T_{4,5,6,7}^{m}\left(-\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{2}^{m}-k_{3}^{m}\right)\right)+(3 \leftrightarrow 4,5,6,7)\right] \\
& -\left[V_{13} V_{2} T_{4,5,6,7}^{m}\left(-\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{1}^{m}-k_{3}^{m}\right)\right)+(3 \leftrightarrow 4,5,6,7)\right] \\
& -\frac{1}{6} V_{1} V_{2}\left[T_{354,6,7}-2 T_{345,6,7}+(3,4,5 \mid 3,4,5,6,7)\right] \\
& +\frac{1}{4} V_{1} V_{2}\left[T_{34,56,7}+(3,4|5,6| 3,4,5,6,7)\right] \\
& +\frac{1}{4} V_{1}\left[V_{23} T_{45,6,7}+(3|4,5| 3,4,5,6,7)\right] \\
& -\frac{1}{6} V_{1}\left[\left(V_{243}-2 V_{234}\right) T_{5,6,7}+(3,4 \mid 3,4,5,6,7)\right]
\end{aligned}
$$
\]

$$
\begin{aligned}
& +\frac{1}{4}\left[V_{13} V_{2} T_{45,6,7}+(3|4,5| 3,4,5,6,7)\right] \\
& +\frac{1}{4}\left[V_{13} V_{24} T_{5,6,7}+(3|4| 3,4,5,6,7)\right] \\
& \left.-\frac{1}{6}\left[\left(V_{143}-2 V_{134}\right) V_{2} T_{5,6,7}+(3,4 \mid 3,4,5,6,7)\right]\right)
\end{aligned}
$$

Up to anomaly terms, the part of the above not proportional to Feynman loop propagators is then the negative of the relevant terms from (I.0.2 $)^{3}$ and so cancels them exactly.

Similar holds with the other hexagon terms. The 23 -hexagon is given by ${ }^{4}$

$$
\begin{align*}
N_{1 \mid 23,4,5,6,7}(\ell)= & +\frac{1}{12} s_{23}\left[V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{3}^{m}-(2 \leftrightarrow 3)\right]-\frac{1}{12} s_{23} V_{1} J_{23 \mid 4,5,6,7}  \tag{I.0.7}\\
& +\frac{1}{12} s_{23}\left[\left(V_{1} J_{2 \mid 34,5,6,7}+(4 \leftrightarrow 5,6,7)\right)+V_{12} J_{3 \mid 4,5,6,7}-(2 \leftrightarrow 3)\right] \\
& +\frac{1}{24} V_{1} T_{23,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}-2 k_{2}^{m} k_{3}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right) \\
& +\frac{1}{12}\left[V_{1} T_{[23,4], 5,6,7}^{m}\left(6 \ell^{m}-k_{23}^{m}+k_{4}^{m}\right)+(23,4 \mid 23,4,5,6,7)\right] \\
& +\frac{1}{12}\left[V_{[1,23]} T_{4,5,6,7}^{m}\left(6 \ell^{m}-k_{1}^{m}+k_{23}^{m}\right)+(23 \leftrightarrow 4,5,6,7)\right] \\
& +\frac{1}{6}\left[2 V_{1} T_{[[23,4], 5], 6,7}-V_{1} T_{[[23,5], 4], 6,7}+(23,4,5 \mid 23,4,5,6,7)\right] \\
& +\frac{1}{4}\left[V_{1} T_{[23,4],[5,6], 7}+(23,4|5,6| 23,4,5,6,7)\right] \\
& +\frac{1}{6}\left[2 V_{[[1,23], 4]} T_{5,6,7}-V_{[[1,4], 23]} T_{5,6,7}+(23,4 \mid 23,4,5,6,7)\right] \\
& +\frac{1}{4}\left[V_{[1,23]} T_{[4,5], 6,7}+(23|4,5| 23,4,5,6,7)\right] .
\end{align*}
$$

This then has variation

$$
\begin{align*}
Q N_{1 \mid 23,4,5,6,7}(\ell)= & {\left[\left(\ell-k_{123}\right)^{2}-\left(\ell-k_{1}\right)^{2}\right]\left(\frac{1}{2} V_{1} V_{23} T_{4,5,6,7}^{m} \ell^{m}\right.}  \tag{I.0.8}\\
& +\frac{1}{4}\left[V_{1} V_{23} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{1} V_{234}+V_{14} V_{23}\right) T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]\right) \\
+ & {\left[\left(\ell-k_{1234}\right)^{2}-\left(\ell-k_{123}\right)^{2}\right]\left(\frac{1}{2} V_{1} V_{4} T_{23,5,6,7}^{m} \ell^{m}\right.} \\
& +\frac{1}{4}\left[V_{1} V_{4} T_{[23,5], 6,7}+(23,5 \mid 23,5,6,7)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{1} V_{C a n(234)}+V_{[1,23]} V_{4}\right) T_{5,6,7}+(23 \leftrightarrow 5,6,7)\right]\right) \\
+ & {\left[\left(\ell-k_{12345}\right)^{2}-\left(\ell-k_{1234}\right)^{2}\right]\left(\frac{1}{2} V_{1} V_{5} T_{23,4,6,7}^{m} \ell^{m}\right.}
\end{align*}
$$

[^28]\[

$$
\begin{aligned}
&+\frac{1}{4}\left[V_{1} V_{5} T_{[23,4], 6,7}+(23,4 \mid 23,4,6,7)\right] \\
&\left.+\frac{1}{4}\left[\left(V_{1} V_{C a n(235)}+V_{[1,23]} V_{5}\right) T_{4,6,7}+(23 \leftrightarrow 4,6,7)\right]\right) \\
&+ {\left[\left(\ell-k_{123456}\right)^{2}-\left(\ell-k_{12345}\right)^{2}\right]\left(\frac{1}{2} V_{1} V_{6} T_{23,4,5,7}^{m} \ell^{m}\right.} \\
&+\frac{1}{4}\left[V_{1} V_{6} T_{[23,4], 5,7}+(23,4 \mid 23,4,5,7)\right] \\
&\left.+\frac{1}{4}\left[\left(V_{1} V_{C a n(236)}+V_{[1,23]} V_{6}\right) T_{4,5,7}+(23 \leftrightarrow 4,5,7)\right]\right) \\
&+ {\left[\left(\ell-k_{1234567}\right)^{2}-\left(\ell-k_{123456}\right)^{2}\right]\left(\frac{1}{2} V_{1} V_{7} T_{23,4,5,6}^{m}{ }^{m}\right.} \\
&+\frac{1}{4}\left[V_{1} V_{7} T_{[23,4], 5,6}+(23,4 \mid 23,4,5,6)\right] \\
&\left.+\frac{1}{4}\left[\left(V_{1} V_{C a n(237)}+V_{[1,23]} V_{7}\right) T_{4,5,6}+(23 \leftrightarrow 4,5,6)\right]\right) \\
& s_{23}( -\frac{1}{12} V_{1} Y_{2,3,4,5,6,7}^{m} k_{3}^{m}-\frac{1}{12} V_{12} Y_{3,4,5,6,7} \\
&-\frac{1}{12} V_{1}\left[Y_{2,34,5,6,7}+(4 \leftrightarrow 2,5,6,7)\right] \\
&-V_{1} V_{2} T_{3,4,5,6,7}^{m n}\left(\frac{1}{2} \ell^{m} \ell^{n}-\frac{1}{24}\left(k_{1}^{m} k_{1}^{n}+\ldots+k_{7}^{m} k_{7}^{n}\right)\right) \\
&-V_{1}\left[V_{2} T_{34,5,6,7}^{m}\left(\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{4}^{m}-k_{3}^{m}\right)\right)+(3,4 \mid 3,4,5,6,7)\right] \\
&-V_{1}\left[V_{24} T_{3,5,6,7}^{m}\left(\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{4}^{m}-k_{2}^{m}\right)\right)+(4 \leftrightarrow 5,6,7)\right] \\
&-\left[V_{13} V_{2} T_{4,5,6,7}^{m}\left(\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{3}^{m}-k_{1}^{m}\right)\right)+(3 \leftrightarrow 4,5,6,7)\right] \\
&-\frac{1}{6} V_{1}\left[V_{2}\left(2 T_{345,6,7}-T_{354,6,7}\right)+(3,4,5 \mid 3,4,5,6,7)\right] \\
&-\frac{1}{4} V_{1}\left[V_{2} T_{34,56,7}+(3,4|5,6| 3,4,5,6,7)\right] \\
&-\frac{1}{4} V_{1}\left[\left(V_{24} T_{35,6,7}+(3,5 \mid 3,5,6,7)\right)+(4 \leftrightarrow 5,6,7)\right] \\
&-\frac{1}{6} V_{1}\left[\left(2 V_{245}-V_{254}\right) T_{5,6,7}+(4,5 \mid 4,5,6,7)\right] \\
&+\frac{1}{4}\left[V_{12} V_{3} T_{45,6,7}+(2|4,5| 2,4,5,6,7)\right] \\
&+\frac{1}{4}\left[V_{12} V_{34} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right)+\left(V_{14} V_{35} T_{2,6,7}+(4|5| 4,5,6,7)\right] \\
&-\frac{1}{6}\left[\left(V_{142}-2 V_{124}\right) V_{3} T_{5,6,7}+(2,4 \mid 2,4,5,6,7)\right] \\
&-\frac{1}{2} \ell^{2} V_{1} Y_{23,4,5,6,7}-(2 \leftrightarrow 3)) \\
& \\
&
\end{aligned}
$$
\]

Again, up to anomaly terms this cancels the relevant terms from (I.0.2), in this case those proportional to $\left(\ell-k_{12}\right)^{2}$.

All remaining hexagons will have a form similar to the above, and cancel terms from the
heptagon variation accordingly. The exception to this is the 71-hexagon, which has a differing form

$$
\left.\begin{array}{rl}
N_{71 \mid 2,3,4,5,6}(\ell)= & -\left(\frac{1}{4} \ell^{2}+\frac{11}{24} s_{17}\right) \Delta_{1|7| 2,3,4,5,6}  \tag{I.0.9}\\
& -V_{1} J_{7 \mid 2,3,4,5,6}^{m}\left(\frac{1}{4} k_{1}^{m} \ell^{2}+\frac{11}{24} k_{1}^{m} s_{17}\right) \\
& -\left[V_{12} J_{7 \mid 3,4,5,6}+(2 \leftrightarrow 3,4,5,6)\right]\left(\frac{1}{4} \ell^{2}+\frac{11}{24} s_{17}\right) \\
& +V_{1} T_{2,3,4,5,6,7}^{m n p}\left(-\frac{1}{2} \ell^{m} k_{1}^{n} k_{7}^{p}+\frac{1}{4} k_{1}^{m} k_{1}^{n} k_{7}^{p}-\frac{1}{4} k_{1}^{m} k_{7}^{n} k_{7}^{p}\right) \\
& -\frac{1}{4} k_{1}^{m} k_{7}^{n}\left[V_{1} T_{23,4,5,6,7}^{m n}+(2,3 \mid 2,3,4,5,6)\right] \\
& +\frac{1}{4}\left[V_{1} T_{27,3,4,5,6}^{m n}\left(2 \ell^{m} k_{1}^{n}-\frac{1}{4} k_{1}^{m} k_{1}^{n}+\frac{1}{4} k_{1}^{m} k_{7}^{n}\right)+(2 \leftrightarrow 3,4,5,6,7)\right] \\
& +\frac{1}{4}\left[V_{12} T_{3,4,5,6,7}^{m n}\left(-2 \ell^{m} k_{7}^{n}+k_{1}^{m} k_{7}^{n}-k_{7}^{m} k_{7}^{n}\right)+(2 \leftrightarrow 3,4,5,6)\right] \\
& +\left[V _ { 1 7 } T _ { 2 , 3 , 4 , 5 , 6 } ^ { m n } \left(-\frac{1}{4} \ell^{m} \ell^{n}+\frac{1}{2} \ell^{m} k_{1}^{n}-\frac{1}{2} \ell^{m} k_{7}^{n}-\frac{5}{24}\left(k_{m}^{2} k_{n}^{2}+k_{m}^{3} k_{n}^{3}+k_{m}^{4} k_{n}^{4}+k_{m}^{5} k_{n}^{5}+k_{m}^{6} k_{n}^{6}\right)\right.\right. \\
& \left.\left.+\frac{11}{4}\left[V_{1} T_{237,4,5,6}^{m} k_{2}^{m} k_{3456}^{n}+k_{3}^{m} k_{456}^{n}+k_{4}^{m} k_{56}^{n}+k_{5}^{m} k_{6}^{n}\right)+\frac{3}{4} k_{1}^{m} k_{7}^{n}\right)\right] \\
& +\frac{1}{4}\left[V_{1} T_{23,47,5,6}^{m} k_{1}^{m}+(2,3|4| 2,3,4,4,5,6)\right] \\
& +\frac{1}{4}\left[V_{12} T_{37,4,5,6}^{m}\left(2 \ell^{m}-k_{1}^{m}+k_{7}^{m}\right)+(2|3| 2,3,4,5,6)\right] \\
& -\frac{1}{4}\left[V_{12} T_{34,5,6,7}^{m} k_{7}^{m}+(2|3,4| 3,4,5,6)\right] \\
& +\frac{1}{4}\left[V_{127} T_{3,4,5,6}^{m}\left(2 \ell^{m}-k_{1}^{m}+k_{7}^{m}\right)+(2 \leftrightarrow 3,4,5,6)\right] \\
& +\frac{1}{4}\left[\left(V_{132}-V_{123}\right) T_{4,5,6,7}^{m} k_{7}^{m}+(2,3 \mid 2,3,4,5,6)\right] \\
& +\left[V_{17} T_{23,4,5,6}^{m}\left(\frac{1}{4}\left(k_{1}^{m}-\ell^{m}-k_{7}^{m}\right)+\frac{1}{24}\left(k_{2}^{m}-k_{3}^{m}\right)\right)+(2,3 \mid 2,3,4,5,6)\right] \\
& +\left[V_{172} T_{3,4,5,6}^{m}\left(\frac{1}{4}\left(-k_{7}^{m}-\ell^{m}\right)+\frac{1}{24}\left(k_{17}^{m}-k_{2}^{m}\right)\right)+(2 \leftrightarrow 3,4,5,6)\right] \\
& +\frac{1}{4}\left[\left(V_{1237}-V_{1327}\right) T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right] \\
& +\frac{1}{4}\left[V_{12} T_{347,5,6}+(2|3,4| 2,3,4,5,6)\right] \\
& +\frac{1}{4}\left[V_{12} T_{34,57,6}+(2|34| 5 \mid 2,3,4,5,6)\right] \\
& +\frac{1}{4}\left[\left(V_{123}-V_{132}\right) T_{47,5,6}+(2,3|4| 2,3,4,5,6)\right] \\
& +\frac{1}{8}\left[\left(2 V_{127}-V_{172}\right) T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right] \\
& +\frac{1}{12}\left[V_{17}\left(-2 T_{234,5,6}+T_{243,5,6}\right)+(2,3,4 \mid 2,3,4,5,6)\right] \\
8 & \left.\frac{1}{8} V_{17} T_{23,45,6}+(2,3|4,5| 2,3,4,5,6)\right] \\
\hline
\end{array}\right)
$$

$$
+\frac{1}{12}\left[\left(-2 V_{1723}+V_{1732}\right) T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right]
$$

This may then be found to have a variation with similar properties however, namely

$$
\begin{align*}
& Q N_{71 \mid 2,3,4,5,6}(\ell)=s_{17}\left(-\frac{1}{12} V_{1} Y_{2,3,4,5,6,7}^{m}\left(k_{1}^{m}-k_{7}^{m}\right)\right.  \tag{I.0.10}\\
& -\frac{1}{12} V_{1}\left[Y_{27,3,4,5,6}+(2 \leftrightarrow 3,4,5,6)\right] \\
& +V_{1} Y_{2,3,4,5,6,7}^{m} \ell^{m} \\
& +V_{1}\left[Y_{23,4,5,6,7}+(2,3 \mid 2,3,4,5,6,7)\right] \\
& +\frac{5}{12}\left[V_{12} Y_{3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right]-\frac{1}{12} V_{17} Y_{2,3,4,5,6} \\
& -V_{1} V_{7} T_{2,3,4,5,6}^{m n}\left[+\frac{1}{2} \ell^{m} \ell^{n}-\frac{1}{24}\left(k_{1}^{m} k_{1}^{n}+\ldots+k_{7}^{m} k_{7}^{n}\right)\right] \\
& -V_{1} V_{7}\left[T_{23,4,5,6}^{m}\left(\frac{1}{2} \ell^{m}-\frac{1}{12}\left(k_{2}^{m}-k_{3}^{m}\right)\right)+(2,3 \mid 2,3,4,5,6)\right] \\
& -V_{1}\left[V_{27} T_{3.4,5,6}^{m}\left(\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{7}^{m}-k_{2}^{m}\right)\right)+(2 \leftrightarrow 3,4,5,6)\right] \\
& -\left[V_{12} V_{7} T_{3,4,5,6}^{m}\left(\frac{1}{2} \ell^{m}+\frac{1}{12}\left(k_{2}^{m}-k_{1}^{m}\right)\right)+(2 \leftrightarrow 3,4,5,6)\right] \\
& -\frac{1}{6} V_{1} V_{7}\left[2 T_{234,6,7}-T_{243,6,7}+(2,3,4 \mid 2,3,4,5,6)\right] \\
& -\frac{1}{4} V_{1} V_{7}\left[T_{23,45,6}+(2,3|4,5| 2,3,4,5,6)\right] \\
& -\frac{1}{4} V_{1}\left[V_{27} T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right] \\
& -\frac{1}{6} V_{1}\left[\left(2 V_{237}-V_{273}\right) T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right] \\
& -\frac{1}{4}\left[V_{12} V_{7} T_{34,5,6}+(2|3,4| 2,3,4,5,6)\right] \\
& -\frac{1}{4}\left[V_{12} V_{37} T_{4,5,6}+(2|3| 2,3,4,5,6)\right] \\
& \left.-\frac{1}{6}\left[\left(2 V_{123}-V_{132}\right) V_{7} T_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right]\right) \\
& +\left(\ell-k_{1}\right)^{2}\left(-\frac{1}{2} V_{1} Y_{2,3,4,5,6,7}^{m} k_{7}^{m}+\frac{1}{2} V_{1}\left[V_{1} Y_{27,3,4,5,6}+(2 \leftrightarrow 3,4,5,6,7)\right]+\frac{1}{2} V_{17} Y_{2,3,4,5,6}\right. \\
& +\frac{1}{2} V_{1} V_{2} T_{3,4,5,6,7}^{m n} k_{1}^{m} k_{7}^{n} \\
& -\frac{1}{2}\left[V_{1} V_{2} T_{37,4,5,6}^{m} k_{1}^{m}+(3 \leftrightarrow 4,5,6)\right]-\frac{1}{2} V_{1} V_{27} T_{3,4,5,6}^{m} k_{1}^{m} \\
& +\frac{1}{2}\left[V_{13} V_{2} T_{4,5,6,7}^{m} k_{7}^{m}+(3 \leftrightarrow 4,5,6,7)\right]+\frac{1}{2} V_{17} V_{2} T_{3,4,5,6}^{m}\left(\ell^{m}-k_{1}^{m}\right) \\
& -\frac{1}{2}\left[V_{13} V_{2} T_{47,5,6}+(3|4| 3,4,5,6)\right] \\
& -\frac{1}{2}\left[\left(V_{13} V_{27}-\frac{1}{2} V_{17} V_{23}\right) T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right] \\
& +\frac{1}{4} V_{17} V_{2}\left[T_{34,5,6}+(3,4 \mid 3,4,5,6)\right] \\
& \left.+\frac{1}{4}\left[\left(V_{173}-2 V_{137}\right) T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right]\right)
\end{align*}
$$

$$
+\left(\ell-k_{12}\right)^{2}(\ldots)+\left(\ell-k_{123}\right)^{2}(\ldots)+\left(\ell-k_{1234}\right)^{2}(\ldots)+\left(\ell-k_{12345}\right)^{2}(\ldots)+\left(\ell-k_{123456}\right)^{2}(\ldots) .
$$

The terms not proportional to loop propagators in this expression then cancel those proportional to $\left(\ell-k_{1234567}\right)^{2}=\ell^{2}$ in the heptagon variation as required, up to anomaly terms. We have not included all of the propagator terms in the above, but the case provided should demonstrate the broad structure such terms have.

We must then cancel the propagator terms in these hexagon variations, and this is done using the variation of pentagon terms similarly. The majority of the pentagons have the standard form first identified in [1]. For example the $[[1,2], 3]$ pentagon is given by

$$
\begin{align*}
N_{[11,2], 3] \mid 4,5,6,7}(\ell)=V_{123} T_{4,5,6,7}^{m} \ell^{m} & +\frac{1}{2}\left[V_{123} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right]  \tag{I.0.11}\\
& +\frac{1}{2}\left[V_{1234} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right],
\end{align*}
$$

and likewise the $[1,[2,3]]$ pentagon is given by

$$
\begin{align*}
N_{[1,[2,3]] \mid 4,5,6,7}(\ell)=V_{[1,[2,3]]} T_{4,5,6,7}^{m} 7^{m} & +\frac{1}{2}\left[V_{[1,[2,3]]} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right]  \tag{I.0.12}\\
& +\frac{1}{2}\left[V_{[11,[2,3]], 4]} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right] .
\end{align*}
$$

These then have variation

$$
\begin{align*}
& Q N_{[11,2], 3][4,5,6,7}(\ell)=+\frac{1}{2}\left[\left(\ell-k_{1234}\right)^{2}-\left(\ell-k_{123}\right)^{2}\right] V_{123} V_{4} T_{5,6,7}  \tag{I.0.13}\\
&+\frac{1}{2}\left[\left(\ell-k_{12345}\right)^{2}-\left(\ell-k_{1234}\right)^{2}\right] V_{123} V_{5} T_{4,6,7} \\
&+\frac{1}{2}\left[\left(\ell-k_{123456}\right)^{2}-\left(\ell-k_{12345}\right)^{2}\right] V_{123} V_{6} T_{4,5,7} \\
&+\frac{1}{2}\left[\left(\ell-k_{1234567}\right)^{2}-\left(\ell-k_{123456}\right)^{2}\right] V_{123} V_{7} T_{4,5,6} \\
&+\left(k^{1} \cdot k^{2}\right)\left(V_{1} V_{23} T_{4,5,6,7}^{m} \ell^{m}+\frac{1}{2}\left[V_{1} V_{23} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right]\right. \\
&\left.+\frac{1}{2}\left[V_{1} V_{234} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]+\frac{1}{2}\left[V_{14} V_{23} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]-(1 \leftrightarrow 2)\right) \\
&+\left(k^{12} \cdot k^{3}\right)\left(V_{12} V_{3} T_{4,5,6,7}^{m} \ell^{m}+\frac{1}{2}\left[V_{12} V_{3} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right]\right. \\
&\left.+\frac{1}{2}\left[V_{12} V_{34} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]+\frac{1}{2}\left[V_{124} V_{3} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]\right) \\
& Q N_{[1,[2,3]] \mid 4,5,6,7}(\ell)=+\frac{1}{2}\left[\left(\ell-k_{1234}\right)^{2}-\left(\ell-k_{123}\right)^{2}\right] V_{[1,23]} V_{4} T_{5,6,7}  \tag{I.0.14}\\
&+\frac{1}{2}\left[\left(\ell-k_{12345}\right)^{2}-\left(\ell-k_{1234}\right)^{2}\right] V_{[1,23]} V_{5} T_{4,6,7} \\
&+\frac{1}{2}\left[\left(\ell-k_{123456}\right)^{2}-\left(\ell-k_{12345}\right)^{2}\right] V_{[1,23]} V_{6} T_{4,5,7} \\
&+\frac{1}{2}\left[\left(\ell-k_{1234567}\right)^{2}-\left(\ell-k_{123456}\right)^{2}\right] V_{[1,23]} V_{7} T_{4,5,6}
\end{align*}
$$

$$
\begin{aligned}
+\left(k^{2} \cdot k^{3}\right)( & V_{12} V_{3} T_{4,5,6,7}^{m} \ell^{m}+\frac{1}{2}\left[V_{12} V_{3} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right] \\
& \left.+\frac{1}{2}\left[V_{12} V_{34} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]+\frac{1}{2}\left[V_{124} V_{3} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]-(2 \leftrightarrow 3)\right) \\
+\left(k^{23} \cdot k^{1}\right)( & V_{1} V_{23} T_{4,5,6,7}^{m} \ell^{m}+\frac{1}{2}\left[V_{1} V_{23} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right] \\
& \left.+\frac{1}{2}\left[V_{1} V_{234} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]+\frac{1}{2}\left[V_{14} V_{23} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]\right)
\end{aligned}
$$

The parts of the pentagon variations not proportional to loop propagators then combine and cancel corresponding terms from the variation of hexagons. For example, bringing in the mandelstam terms associated with each numerator, these terms in the above sum to

$$
\begin{align*}
\frac{1}{s_{12}}\left(V_{12} V_{3} T_{4,5,6,7}^{m} \ell^{m}\right. & +\frac{1}{2}\left[V_{12} V_{3} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right]  \tag{I.0.15}\\
+ & \left.\frac{1}{2}\left[\left(V_{12} V_{34}+V_{124} V_{3}\right) T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]\right) \\
+\frac{1}{s_{23}}\left(V_{1} V_{23} T_{4,5,6,7}^{m} \ell^{m}\right. & +\frac{1}{2}\left[V_{1} V_{23} T_{45,6,7}+(4,5 \mid 4,5,6,7)\right] \\
+ & \left.\frac{1}{2}\left[\left(V_{1} V_{234}+V_{14} V_{23}\right) T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]\right)
\end{align*}
$$

These then cancel the $\left(\ell-k_{12}\right)^{2}$ and $\left(\ell-k_{1}\right)^{2}$ terms of $Q N_{12 \mid 3,4,5,6,7}(\ell)$ and $Q N_{1 \mid 23,4,5,6,7}(\ell)$ respectively, as can be seen in the form of these variations (I.0.6) and (I.0.8).

The exceptions to this structure are those numerators corresponding with diagrams in which the 7 and 1 external particles are on a shared external tree. Three examples of such are the 17,23 -pentagon ${ }^{5}$

$$
\begin{align*}
N_{71 \mid 23,4,5,6}(\ell)= & +\frac{1}{2}\left[\left(V_{[[1,23], 7]}+V_{[1,[23,7]]}\right) T_{4,5,6}+(23 \leftrightarrow 4,5,6)\right]  \tag{I.0.16}\\
& +\left[V_{[1,23]} T_{[4,7], 3,4}+(23|4| 23,4,5,6)\right] \\
& -\frac{1}{2}\left[V_{[1,7]} T_{[23,4], 5,6}+(23,4 \mid 23,4,5,6)\right] \\
& -\left[V_{[1,23]} T_{4,5,6,7}^{m} k_{7}^{m}+(23 \leftrightarrow 4,5,6)\right] \\
& +\left[V_{1} T_{[23,7], 4,5,6}^{m} k_{1}^{m}+(23 \leftrightarrow 4,5,6)\right] \\
& -V_{[1,7]} T_{23,4,5,6}^{m}\left(\ell^{m}+k_{7}^{m}-k_{1}^{m}\right) \\
& -V_{1} T_{23,4,5,6,7}^{m n} k_{1}^{m} k_{7}^{n}
\end{align*}
$$

the $[[7,1], 2]$ pentagon,

$$
\begin{align*}
N_{[[7,1], 2] \mid 3,4,5,6}\left(\ell^{m}\right)= & +s_{17} \Delta_{1|7| 2,3,4,5,6}  \tag{I.0.17}\\
& -s_{17} V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{7}^{m} \\
& +\left(k^{12} \cdot k^{7}\right) V_{1} J_{7 \mid 2,3,4,5,6}^{m} k_{1}^{m}
\end{align*}
$$

[^29]\[

$$
\begin{aligned}
& +s_{17}\left[V_{1} J_{2 \mid 37,4,5,6}+(3 \leftrightarrow 4,5,6)\right] \\
& +s_{17} V_{1} J_{27 \mid 3,4,5,6} \\
& +\left(k^{12} \cdot k^{7}\right)\left[V_{12} J_{7 \mid 3,4,5,6}+(2 \leftrightarrow 3,4,5,6)\right] \\
& +s_{17} V_{17} J_{2 \mid 3,4,5,6} \\
& -V_{1} T_{2,3,4,5,6,7}^{m n p} k_{1}^{m} k_{2}^{n} k_{7}^{p} \\
& -\left[V_{1} T_{23,4,5,6,7}^{m n}+(2 \leftrightarrow 3,4,5,6)\right] k_{1}^{m} k_{7}^{n} \\
& +k_{1}^{m} k_{2}^{n}\left[V_{1} T_{27,4,5,6}^{m n}+(2 \leftrightarrow 3,4,5,6)\right] \\
& -k_{2}^{m} k_{7}^{n}\left[V_{12} T_{3,4,5,6,7}^{m n}+(2 \leftrightarrow 3,4,5,6)\right] \\
& +V_{17} T_{2,3,4,5,6}^{m n} k_{1}^{m} k_{2}^{n} \\
& +\left[V_{1} T_{23,47,5,6} k_{1}^{m}+(2,3|4| 2,3,4,5,6)\right] \\
& +k_{2}^{m}\left[V_{12} T_{37,4,5,6}^{m}+(2|3| 2,3,4,5,6)\right] \\
& +\left[V_{1} T_{237,4,5,6} k_{1}^{m}+(3 \leftrightarrow 4,5,6)\right] \\
& +k_{7}^{m}\left[\left(V_{132}-V_{123}\right) T_{4,5,6,7}^{m}+(3 \leftrightarrow 4,5,6)\right] \\
& +k_{2}^{m}\left[V_{127} T_{3,4,5,6}^{m}+(2 \leftrightarrow 3,4,5,6)\right] \\
& -k_{7}^{m}\left[V_{13} T_{24,5,6,7}+(3|4| 3,4,5,6)\right] \\
& +\left[V_{17} T_{23,4,5,6}^{m} k_{1}^{m}+(3 \leftrightarrow 4,5,6)\right] \\
& -V_{172} T_{3,4,5,6}^{m}\left(\ell^{m}+k_{7}^{m}\right) \\
& +\left[\left(V_{1237}-V_{1327}-\frac{1}{2} V_{1723}\right) T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right] \\
& +\left[\left(V_{123}-V_{132}\right) T_{47,5,6}+(3|4| 2,3,4,5,6)\right] \\
& +\left[V_{13} T_{24,57,6}+(3|4| 5 \mid 3,4,5,6)\right] \\
& +\left[V_{13} T_{247,5,6}+(3|4| 3,4,5,6)\right] \\
& +\left[V_{137} T_{24,5,6}+(3|4| 3,4,5,6)\right] \\
& -\frac{1}{2}\left[V_{172} T_{34,5,6}+(3,4 \mid 3,4,5,6)\right] \\
& +
\end{aligned}
$$
\]

and the $[7,[1,2]]$ pentagon ${ }^{6}$,

$$
\begin{align*}
N_{[7,[1,2]] \mid 3,4,5,6}\left(\ell^{m}\right)= & +s_{12} V_{1} J_{7 \mid 2,3,4,5,6}^{m}\left(k_{2}^{m}-k_{1}^{m}\right)  \tag{I.0.18}\\
& +s_{12}\left[V_{1} J_{7 \mid 23,4,5,6}+(3 \leftrightarrow 4,5,6)\right] \\
& -2 s_{12} V_{12} J_{7 \mid 3,4,5,6} s_{12}-\left[s_{12} V_{13} J_{7 \mid 2,4,5,6}+(3 \leftrightarrow 4,5,6)\right] \\
& +\frac{1}{2}\left[\left(V_{[[12,3], 7]}+V_{[12,[3,7]]}\right) T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right] \\
& +\left[V_{[12,3]} T_{[4,7], 5,6}+(3|4| 3,4,5,6)\right] \\
& -\frac{1}{2}\left[V_{[12,7]} T_{[3,4], 5,6}+(3,4 \mid 3,4,5,6)\right]
\end{align*}
$$

[^30]\[

$$
\begin{aligned}
& -\left[V_{[12,3]} T_{4,5,6,7}^{m} k_{7}^{m}+(3 \leftrightarrow 4,5,6)\right] \\
& +\left[V_{12} T_{[3,7], 4,5,6}^{m} k_{1}^{m}+(3 \leftrightarrow 4,5,6)\right] \\
& -V_{[12,7]} T_{3,4,5,6}^{m}\left(\ell^{m}+k_{7}^{m}-k_{1}^{m}\right) \\
& -V_{12} T_{3,4,5,6,7}^{m n} k_{12}^{m} k_{7}^{n}
\end{aligned}
$$
\]

However these again have variation of the correct form, cancelling terms from hexagon diagrams and leaving only those proportional to loop propagators. These loop propagator terms from the pentagons then cancel against the variation of the boxes, which all follow the structure found in [1],

$$
\begin{equation*}
N_{A \mid B, C, D}=V_{A} T_{B, C, D} \tag{I.0.19}
\end{equation*}
$$

These will then combine in their variation and cancel the loop propagator terms from the variation of the pentagons.

Thus the variation of the amplitude cancels up to anomaly terms,

$$
\begin{align*}
Q A(1,2,3,4,5,6,7 ; \ell)= & \left(I_{1,2,3,4,5,6}-I_{1,2,3,4,5,6,7}\right) \ell^{2}\left(\frac{1}{2} V_{1} Y_{2,3,4,5,6,7}^{m} \ell^{m}\right.  \tag{I.0.20}\\
& +\frac{1}{4}\left[V_{1} Y_{23,4,5,6,7}+(2,3 \mid 2,3,4,5,6,7)\right] \\
& \left.+\frac{1}{4}\left[V_{12} Y_{3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right]\right) \\
& +\frac{1}{4 s_{12}} V_{12} Y_{3,4,5,6,7}\left(I_{12,3,4,5,6}-\ell^{2} I_{12,3,4,5,6,7}\right) \\
& +\frac{1}{4 s_{23}} V_{1} Y_{23,4,5,6,7}\left(I_{1,23,4,5,6}-\ell^{2} I_{1,23,4,5,6,7}\right) \\
& +\frac{1}{4 s_{34}} V_{1} Y_{2,34,5,6,7}\left(I_{1,2,34,5,6}-\ell^{2} I_{1,2,34,5,6,7}\right) \\
& +\frac{1}{4 s_{45}} V_{1} Y_{2,3,45,6,7}\left(I_{1,2,3,45,6}-\ell^{2} I_{1,2,3,45,6,7}\right) \\
& +\frac{1}{4 s_{56}} V_{1} Y_{2,3,4,56,7}\left(I_{1,2,3,4,56}-\ell^{2} I_{1,2,3,4,56,7}\right) \\
& +\frac{1}{4 s_{67}} V_{1} Y_{2,3,4,5,67}\left(I_{1,2,3,4,5}-\ell^{2} I_{1,2,3,4,5,67}\right)
\end{align*}
$$

These would naively appear to vanish, but subtleties relating to dimensional regularisation will arise and a more careful analysis is needed, similar to the six point anomaly discussion of section 4.5 of [1]. This we leave to future work.

## I. 1 Details of a Seven Point BCJ identity

Now that we have stated these complete seven point formulae, we may use them to verify BCJ relations. In this appendix we will discuss specifically a relation between two heptagons and a hexagon,

$$
\begin{equation*}
N_{1 \mid 2,3,4,5,6,7}(\ell)-N_{1 \mid 3,2,4,5,6,7}(\ell)-N_{1 \mid 23,4,5,6,7}(\ell)=0 \tag{I.1.1}
\end{equation*}
$$

The first of these numerators is given by (I.0.1), the third by (I.0.7). The second we have not yet stated, but this follows similarly from the field theory limit rules and is given by

$$
\begin{align*}
N_{1 \mid 3,2,4,5,6,7}(\ell)= & -\frac{1}{24}\left(s_{12} \Delta_{1|2| 3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right)  \tag{I.1.2}\\
& +\frac{1}{24} V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{2}^{n}\left[\left(k_{m}^{4} k_{n}^{4}+(4 \leftrightarrow 5,6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 3)\right)\right] \\
& +\frac{1}{24} V_{1} J_{3 \mid 2,4,5,6,7}^{m} k_{3}^{n}\left[\left(k_{m}^{2} k_{n}^{2}+(2 \leftrightarrow 4,5,6,7)\right)-k_{m}^{1} k_{n}^{1}\right] \\
& +\frac{1}{24} V_{1} J_{4 \mid 2,3,5,6,7}^{m} k_{4}^{n}\left[\left(k_{m}^{5} k_{n}^{5}+(5 \leftrightarrow 6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3)\right)\right] \\
& +\frac{1}{24} V_{1} J_{5 \mid 2,3,4,6,7}^{m} k_{5}^{n}\left[\left(k_{m}^{6} k_{n}^{6}+(6 \leftrightarrow 7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4)\right)\right] \\
& +\frac{1}{24} V_{1} J_{6 \mid 2,3,4,5,7}^{m} k_{6}^{n}\left[k_{m}^{7} k_{n}^{7}-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4,5)\right)\right] \\
& -\frac{1}{24} V_{1} J_{7 \mid 2,3,4,5,6}^{m}\left[s_{17} k_{1}^{m}+(1 \leftrightarrow 2,3,4,5,6)\right] \\
& +\frac{1}{24}\left[V_{1} J_{2 \mid 3,45,6,7}\left(s_{24}-s_{25}\right)+(4,5 \mid 4,5,6,7)\right] \\
& -\frac{1}{24}\left[V_{1} J_{2 \mid 34,5,6,7}\left(s_{23}+s_{24}\right)+(4 \leftrightarrow 5,6,7)\right] \\
& +\frac{1}{24}\left[V_{1} J_{3 \mid 24,5,6,7}\left(s_{23}-s_{34}\right)+(2,4 \mid 2,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{1} J_{4 \mid 2,3,56,7}\left(s_{45}-s_{46}\right)+(56 \leftrightarrow 32,57,67)\right] \\
& -\frac{1}{24}\left[\left[V_{1} J_{4 \mid 35,5,6,7}\left(s_{34}+s_{45}\right)+(5 \leftrightarrow 6,7)\right]+(2 \leftrightarrow 3)\right] \\
& +\frac{1}{24}\left[V_{1} J_{5 \mid 32,4,6,7}\left(s_{35}-s_{25}\right)+(32 \leftrightarrow 34,24,67)\right] \\
& -\frac{1}{24}\left[\left[V_{1} J_{5 \mid 36,4,6,7}\left(s_{35}+s_{56}\right)+(3 \leftrightarrow 2,4)\right]+(6 \leftrightarrow 7)\right] \\
& +\frac{1}{24}\left[V_{1} J_{6 \mid 32,4,5,7}\left(s_{26}-s_{36}\right)+(3,2 \mid 3,2,4,5)\right] \\
& -\frac{1}{24}\left[V_{1} J_{6 \mid 37,2,4,5}\left(s_{36}+s_{67}\right)+(3 \leftrightarrow 2,4,5)\right] \\
& +\frac{1}{24}\left[V_{1} J_{7 \mid 32,4,5,6}\left(s_{27}-s_{37}\right)+(3,2 \mid 3,2,4,5,6)\right] \\
& -\frac{1}{24}\left[s_{23} V_{1} J_{32 \mid 4,5,6,7}+(3,2 \mid 3,2,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{13} J_{2 \mid 4,5,6,7}\left(s_{23}-s_{12}\right)+(3,2 \mid 3,2,4,5,6,7)\right]
\end{align*}
$$

$$
\begin{aligned}
& -\frac{1}{24}\left[V_{12} J_{3 \mid 4,5,6,7}\left(s_{13}+s_{23}\right)+(3,2 \mid 3,2,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{1} T_{2,3,4,5,6,7}^{m m p}\left(4 \ell^{m} \ell^{n} \ell^{p}-\ell^{m}\left(k_{1}^{n} k_{1}^{p}+k_{2}^{n} k_{2}^{p}+\ldots+k_{7}^{n} k_{7}^{p}\right)\right]\right. \\
& +\frac{1}{48}\left[V_{1} T_{32,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{3}^{n}+4 \ell^{m} k_{2}^{n}-2 k_{2}^{m} k_{3}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right)\right. \\
& +(3,2 \mid 3,2,4,5,6,7)] \\
& +\frac{1}{48}\left[V_{13} T_{2,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{1}^{n}+4 \ell^{m} k_{3}^{n}+2 k_{1}^{m} k_{3}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right)\right. \\
& +(3 \leftrightarrow 2,4,5,6,7)] \\
& +\frac{1}{12}\left[V_{1} T_{324,5,6,7}^{m}\left(4 \ell^{m}-k_{3}^{m}+k_{4}^{m}\right)+(3,2,4 \mid 3,2,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{1} T_{342,5,6,7}^{m}\left(2 \ell^{m}+k_{4}^{m}\right)+(3,2,4 \mid 3,2,4,5,6,7)\right] \\
& +\frac{1}{12}\left[V_{132} T_{4,5,6,7}^{m}\left(4 \ell^{m}-k_{1}^{m}+k_{2}^{m}\right)+(3,2 \mid 3,2,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{123} T_{4,5,6,7}^{m}\left(2 \ell^{m}+k_{2}^{m}\right)+(3,2 \mid 3,2,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{1} T_{32,45,6,7}^{m}\left(6 \ell^{m}-k_{34}^{m}+k_{25}^{m}\right)+(3,2|4,5| 3,2,4,5,6,7)\right] \\
& +\frac{1}{24}\left[V_{13} T_{24,5,6,7}^{m}\left(6 \ell^{m}-k_{12}^{m}+k_{34}^{m}\right)+(3|2,4| 3,2,4,5,6,7)\right] \\
& +\frac{1}{12} V_{1}\left[3 T_{3245,6,7}-T_{3254,6,7}-T_{3425,6,7}-T_{3452,6,7}-T_{3524,6,7}+T_{3542,6,7}\right. \\
& +(3,2,4,5 \mid 3,2,4,5,6,7)] \\
& +\frac{1}{12}\left[\left(3 V_{1324}-V_{1342}-V_{1234}-V_{1243}-V_{1432}+V_{1423}\right) T_{5,6,7}\right. \\
& +(3,2,4 \mid 3,2,4,5,6,7)] \\
& +\frac{1}{6}\left[V_{1} T_{324,56,7}+(3,2,4|5,6| 3,2,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{1} T_{342,56,7}+(3,2,4|5,6| 3,2,4,5,6,7)\right] \\
& +\frac{1}{8}\left[V_{1} T_{32,45,67}+(3,2|4,5| 6,7 \mid 3,2,4,5,6,7)\right] \\
& +\frac{1}{6}\left[V_{132} T_{45,6,7}+(3,2|4,5| 3,2,4,5,6,7)\right] \\
& +\frac{1}{6}\left[V_{13} T_{245,6,7}+(3|2,4,5| 3,2,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{123} T_{45,6,7}+(3,2|4,5| 3,2,4,5,6,7)\right] \\
& -\frac{1}{12}\left[V_{13} T_{254,6,7}+(3|2,4,5| 3,2,4,5,6,7)\right] \\
& +\frac{1}{8}\left[V_{13} T_{24,56,7}+(3|2,4| 5,6 \mid 3,2,4,5,6,7)\right] .
\end{aligned}
$$

If we begin with the anomalous $\Delta$ terms, these exist in the first and second numerators
only, and contribute to (I.1.1) as

$$
\begin{align*}
& -\frac{1}{24}\left(s_{12} \Delta_{1|2| 3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right)  \tag{I.1.3}\\
& +\frac{1}{24}\left(s_{13} \Delta_{1|3| 2,4,5,6,7}+(3 \leftrightarrow 2,4,5,6,7)\right)=0 .
\end{align*}
$$

We may then move onto the refined terms in each numerator. Looking in particular at those with a vector index, these contribute to (I.1.1) as

$$
\begin{align*}
& \left(\frac{1}{24} V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{2}^{n}\left[\left(k_{m}^{3} k_{n}^{3}+(3 \leftrightarrow 4,5,6,7)\right)-k_{m}^{1} k_{n}^{1}\right]\right.  \tag{I.1.4}\\
& +\frac{1}{24} V_{1} J_{3 \mid 2,4,5,6,7}^{m} k_{3}^{n}\left[\left(k_{m}^{4} k_{n}^{4}+(4 \leftrightarrow 5,6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2)\right)\right] \\
& +\frac{1}{24} V_{1} J_{4 \mid 2,3,5,6,7}^{m} k_{4}^{n}\left[\left(k_{m}^{5} k_{n}^{5}+(5 \leftrightarrow 6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3)\right)\right] \\
& +\frac{1}{24} V_{1} J_{5 \mid 2,3,4,6,7}^{m} k_{5}^{n}\left[\left(k_{m}^{6} k_{n}^{6}+(6 \leftrightarrow 7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4)\right)\right] \\
& +\frac{1}{24} V_{1} J_{6 \mid 2,3,4,5,7}^{m} k_{6}^{n}\left[k_{m}^{7} k_{n}^{7}-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 2,3,4,5)\right)\right] \\
& \left.-\frac{1}{24} V_{1} J_{7 \mid 2,3,4,5,6}^{m}\left[s_{17} k_{1}^{m}+(1 \leftrightarrow 2,3,4,5,6)\right]\right) \\
& -\left(\frac{1}{24} V_{1} J_{3 \mid 2,4,5,6,7}^{m} k_{3}^{n}\left[\left(k_{m}^{2} k_{n}^{2}+(2 \leftrightarrow 4,5,6,7)\right)-k_{m}^{1} k_{n}^{1}\right]\right. \\
& +\frac{1}{24} V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{2}^{n}\left[\left(k_{m}^{4} k_{n}^{4}+(4 \leftrightarrow 5,6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 3)\right)\right] \\
& +\frac{1}{24} V_{1} J_{4 \mid 3,2,5,6,7}^{m} k_{4}^{n}\left[\left(k_{m}^{5} k_{n}^{5}+(5 \leftrightarrow 6,7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 3,2)\right)\right] \\
& +\frac{1}{24} V_{1} J_{5 \mid 3,2,4,6,7}^{m} k_{5}^{n}\left[\left(k_{m}^{6} k_{n}^{6}+(6 \leftrightarrow 7)\right)-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 3,2,4)\right)\right] \\
& +\frac{1}{24} V_{1} J_{6 \mid 3,2,4,5,7}^{m} k_{6}^{n}\left[k_{m}^{7} k_{n}^{7}-\left(k_{m}^{1} k_{n}^{1}+(1 \leftrightarrow 3,2,4,5)\right)\right] \\
& \left.-\frac{1}{24} V_{1} J_{7 \mid 3,2,4,5,6}^{m}\left[s_{17} k_{1}^{m}+(1 \leftrightarrow 3,2,4,5,6)\right]\right) \\
& -\frac{1}{12} s_{23}\left[V_{1} J_{2 \mid 3,4,5,6,7}^{m} k_{3}^{m}-(2 \leftrightarrow 3)\right]=0 .
\end{align*}
$$

The vanishing of the above is immediate as a result of the symmetry of $J_{A \mid B, C, D, E, F}^{m}$ in $B, C, D, E, F$. Similar results hold for the refined terms without a vector index.

Looking then at the $V T$ terms, those with two vector indices in the BCJ relation (I.1.1) are

$$
\begin{align*}
&\left(\frac { 1 } { 4 8 } \left[V_{1} T_{23,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{2}^{n}+4 \ell^{m} k_{3}^{n}-2 k_{2}^{m} k_{3}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right)\right.\right. \\
&+(2,3 \mid 2,3,4,5,6,7)]  \tag{I.1.5}\\
&+\frac{1}{48}\left[V _ { 1 2 } T _ { 3 , 4 , 5 , 6 , 7 } ^ { m n } \left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{1}^{n}+4 \ell^{m} k_{2}^{n}\right.\right.\left.+2 k_{1}^{m} k_{2}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right) \\
&+(2 \leftrightarrow 3,4,5,6,7)])
\end{align*}
$$

$$
\begin{aligned}
-\left(\frac { 1 } { 4 8 } \left[V _ { 1 } T _ { 3 2 , 4 , 5 , 6 , 7 } ^ { m n } \left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{3}^{n}+4 \ell^{m} k_{2}^{n}\right.\right.\right. & \left.-2 k_{3}^{m} k_{2}^{n}-k_{1}^{m} k_{1}^{n}-k_{3}^{m} k_{3}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right) \\
& +(3,2 \mid 3,2,4,5,6,7)]
\end{aligned} \begin{aligned}
&+\frac{1}{48}\left[V _ { 1 3 } T _ { 2 , 4 , 5 , 6 , 7 } ^ { m n } \left(12 \ell^{m} \ell^{n}-4 \ell^{m} k_{1}^{n}+4 \ell^{m} k_{3}^{n}+\right.\right.\left.2 k_{1}^{m} k_{3}^{n}-k_{1}^{m} k_{1}^{n}-k_{3}^{m} k_{3}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right) \\
&+(3 \leftrightarrow 2,4,5,6,7)]) \\
&- \frac{1}{24} V_{1} T_{23,4,5,6,7}^{m n}\left(12 \ell^{m} \ell^{n}-2 k_{2}^{m} k_{3}^{n}-k_{1}^{m} k_{1}^{n}-k_{2}^{m} k_{2}^{n}-\ldots-k_{7}^{m} k_{7}^{n}\right. \\
&=0
\end{aligned}
$$

This requires a property of the BCJ gauge to show, namely $T_{32,4,5,6,7}^{m n}=-T_{23,4,5,6,7}^{m n}$.

One final example would be the terms in which the $V$ vertex operator contains particle labels $1,2,3$, and 4 . These terms in (I.1.1) are

$$
\begin{align*}
& \left(\frac{1}{12}\left(3 V_{1234}-V_{1243}-V_{1324}-V_{1342}-V_{1423}+V_{1432}\right) T_{5,6,7}\right) \\
- & \left(\frac{1}{12}\left(3 V_{1324}-V_{1342}-V_{1234}-V_{1243}-V_{1432}+V_{1423}\right) T_{5,6,7}\right) \\
- & \frac{1}{6}\left(2 V_{[[1,23], 4]} T_{5,6,7}-V_{[[1,4], 23]} T_{5,6,7}\right)=0 . \tag{I.1.6}
\end{align*}
$$

By expanding the hexagon contributions as

$$
\begin{align*}
& V_{[1,23], 4]}=V_{1234}-V_{1324},  \tag{I.1.7}\\
& V_{[1,4], 23]}=V_{1423}-V_{1432}, \tag{I.1.8}
\end{align*}
$$

the vanishing of these terms becomes clear.

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[^0]:    ${ }^{1}$ This formula is for the $(p+1)$-particle amplitude with particle $(p+1)$ off-shell. Its origin may be understood as a sum over all possible diagrams with 3 and 4 -point vertices, with the $[\cdot, \cdot]$ and $\{\cdot, \cdot, \cdot\}$ brackets in the current definition (2.1.23) corresponding with each.
    ${ }^{2}$ We note that this amplitude calculation has been detailed to demonstrate the methods of BerendsGiele currents, not because of any particular significance of the three point amplitude itself. In fact, such an amplitude vanishes in certain circumstances, as can be seen in for instance [46].

[^1]:    ${ }^{3}$ A gauge may be regarded as a fixing of some freedom in the relations specifying a system. For example, if an object $y$ was completely specified by the constraint $\frac{d y}{d x}=0$, then there would be a freedom in selecting the value of the constant we set $y$ to. Choosing a specific gauge does not change the physical system we are describing, only the mathematics involved.

[^2]:    ${ }^{4}$ Note that where SYM amplitudes are denoted with the letter $A$, it is standard convention to denote amplitudes in supergravity with the letter $M$ in a similar way.
    ${ }^{5}$ At this level, it seems reasonable to assign 23 inputs as relating to the first row/column of a matrix, and 32 as the second row/column. However, if the input words had length 3 we would have input words $234,243,324,342,423$ and 432 , and it is far less clear to which row/column one would assign say 423.

[^3]:    ${ }^{6}$ An MHV amplitude is a maximum helicity violating amplitude. That is, one in which two external particles share one helicity, and all others share the other. These are particularly well understood, with for example the Parke-Taylor expression for their tree level amplitudes known since the 80's [71]

[^4]:    ${ }^{1}$ We note that, while this fully defines the gauge, there is still a residual gauge freedom by the addition of terms which vanish under the d'Alembertian operator (3.1.16). Components of other superfields are constrained by this gauge definition by their relation with $\mathbb{A}^{m}$.

[^5]:    ${ }^{2} \mathrm{~A}$ few notes on this operator. This charge is not defined through gauge fixing, but rather is built up by identifying a sequence of operators with cohomology equivalent to each other. For details of this, see [99]. As such, the operator is in effect already gauge-fixed. A form of $Q$ in which such is not the case may be beneficial for connecting with the RNS formalism, and for such [103] should be consulted.

[^6]:    ${ }^{3}$ A vertex operator may be regarded as an operator inserted into calculations to describe a physical state of the string.

[^7]:    ${ }^{4}$ A zero mode here means a point at which the operator is zero. Precisely the right number of these must be contained in amplitude integrals in order for them not to vanish, as is discussed in section 5.3 of [3] and in [107]. In particular, on a genus 1 surface we must make sure there are $16 d_{\alpha}$ zero modes, which follows from the Riemann-Roch theorem $[107 ; 39]$. There are 11 such in the picture changing operators, and so a further five are needed from the $b$-ghost and vertex operators. A similar reasoning is why $5 \theta^{\alpha}$ terms are needed, and likewise with the 3 pure spinors and $0 N_{m n}$ terms [39].

[^8]:    ${ }^{1}$ For the purposes of this thesis, a Lie polynomial is a linear combination of Lie monomials

[^9]:    ${ }^{1}$ Additional complications arise when the integrand contains a loop momentum $\ell$ term in the numerator. We do not detail this here, for such the reader should consult the appendix of [1].
    ${ }^{2}$ These values can be formally derived, see [1] appendix A. 2 for details. Additionally, details of the sign associated with the limits have been ignored, and will be introduced later.

[^10]:    ${ }^{1}$ It is interesting to note here that, if one does not take $\mathcal{L}_{6} \circ H_{[12345,6]}=6 H_{[12345,6]}$, then we can miss the solution by terms in the kernel of $\mathcal{L}_{6}$. The value given here is equivalent, up to terms in this kernel, to $H_{[12345,6]}=\frac{1}{3}\left(H_{123,45,6}^{\prime}+H_{1234,5,6}^{\prime}\right)$. This seems far more appetising than the correct value but, unfortunately, it is wrong. This does not satisfy lower order Jacobi identities, as can be seen in for instance the relation $\mathcal{L}_{5} \circ\left(H_{123,45,6}^{\prime}+H_{1234,5,6}^{\prime}\right) \neq 0$

[^11]:    ${ }^{1}$ Note there are no $L_{3}$ terms in the below. These have been omitted intentionally as any such terms would be of the form $\sum_{X Y Z=12 \ldots 6} H_{X} H_{Y} A_{Z}$, and since each $H$ requires at least three indices to be non-zero all terms of this form will be zero.
    ${ }^{2}$ That is, if one takes (8.3.8) and expands the superfields as series expansions in Lie algebra generators as in (4.2.59), and equates the terms with coefficient $T^{p_{1}} T^{p_{2}} T^{p_{3}} T^{p_{4}} T^{p_{5}} T^{p_{6}}$, then one finds (8.3.7).

[^12]:    ${ }^{1}$ Note this is not the most efficient way to perform this step, we merely chose to describe it in this way here in order to avoid introducing more material which will not be used elsewhere in this thesis. In brief, while one does need the above formulae for the $H$ terms as a function of the standard $\left\{A_{\alpha}^{P}, A_{P}^{m}, W_{P}^{\alpha}, F_{m n}^{P}\right\}$ multiparticle superfields, these do not then need to be expanded down to single particle superfields. Instead, one may take advantage of the Harnad-Shnider gauge [152], defined by the constraint $\theta^{\alpha} \mathbb{A}_{\alpha}=0$. A multiparticle version of this was developed in [88], and this was combined with the BCJ gauge therein also. In this gauge theta expansions for multiparticle superfields are then known [88], reducing the number of calculations needed considerably.

[^13]:    ${ }^{1}$ For simplicity we will consider only the planar topology.

[^14]:    ${ }^{2}$ The $I$ notation below is detailed in appendix A. 2

[^15]:    ${ }^{3}$ Note in this reference the relation differs in appearance, as they use an alternative representation of the Bernoulli numbers which differs only in that $B_{1}=-\frac{1}{2}$ instead of the $+\frac{1}{2}$ we use.

[^16]:    ${ }^{4}$ Note the expression for this is also available from [28].

[^17]:    ${ }^{5}$ Note this term is not explicit in the numerator, as we chose instead to combine it with the $V_{1} T_{234,5,6}$ term to simplify the appearance.

[^18]:    ${ }^{6}$ See the discussion of [132], and its summary in section 4.5 of [1], to understand why (10.3.40) this does not trivially vanish due to the cancellation of propagators in the integrand.

[^19]:    ${ }^{1}$ Note in order to find amplitudes which are not in the canonical ordering when represented in their 1 -leading form, we exploit the total symmetry of the six-point correlator (5.3.19) in 2, 3, 4, 5, 6. That is, in order to derive an amplitude which in its 1 -leading form is $A\left(1, \sigma ; \ell+a_{i} k^{i}\right)$, we start with the alternative expression for (5.3.19),

    $$
    \begin{align*}
    \mathcal{K}_{6}(\ell) & =\frac{1}{2} V_{A} T_{B, C, D, E, F}^{m n} \mathcal{Z}_{A, B, C, D, E, F}^{m n}+[1 \sigma \mid A, B, C, D, E, F] \\
    & +V_{A} T_{B, C, D, E}^{m} \mathcal{Z}_{A, B, C, D, E}^{m}+[1 \sigma \mid A, B, C, D, E]  \tag{11.1.11}\\
    & +V_{A} T_{B, C, D} \mathcal{Z}_{A, B, C, D}+[1 \sigma \mid A, B, C, D] .
    \end{align*}
    $$

[^20]:    ${ }^{2}$ We have verified that the general $a_{i}$ six point amplitude discussed in appendix G has vanishing BRST variation in fact $[148 ; 149 ; 150]$. That these amplitudes have vanishing variation is a consequence of this

[^21]:    ${ }^{1}$ The dihedral group $D_{n}$ associated with an $n$-gon is the group of size $2 n$ consisting of symmetries associated with that $n$-gon. For example, for a pentagon this consists of an identity, four rotations, and five reflections along lines between corners and midpoints of edges. Likewise for a hexagon there is an identity, five rotations, three reflections along lines between corners, and three reflections along lines between midpoints of edges. One would naively expect that a numerator associated with an $n$ gon should be invariant under the action of all elements of this group upon it. So for example, that $N_{1 \mid 2,3,4,5,6}(\ell)=N_{2 \mid 3,4,5,6,1}\left(\ell-k_{1}\right)$, and that $N_{1 \mid 2,3,4,5,6}(\ell)=N_{6 \mid 5,4,3,2,1}(-\ell)$. See [164] for further discussion of these symmetries in an amplitudes context.

[^22]:    ${ }^{2}$ Recall in the double copy construction, the numerators with a tilde are referred to as right moving, and those without one are left moving

[^23]:    ${ }^{3}$ The $\frac{1}{2}$ is needed to deal with the equivalence of say $N_{1 \mid 23,4,5,6}(\ell)$ and $N_{1 \mid 32,4,5,6}$. The $\frac{1}{4}$ is needed to deal with the equivalence of say $N_{1 \mid[2,3], 4], 5,6}, N_{1 \mid[[3,2], 4], 5,6}, N_{1 \mid[4,[2,3]], 5,6}$ and $N_{1 \mid[4,[3,2]], 5,6}$. The relative minus signs which these may appear to differ by are cancelled out by likewise appearing in the corresponding colour factors.
    ${ }^{4}$ There is second occurring when the relation $H_{1,6,5,4,3,2}=H_{1,2,3,4,5,6}$ is used. However the equivalent of this was not needed at five points, and we shall discuss this more shortly.

[^24]:    ${ }^{5}$ We describe this difference using the refined building block $J_{2 \mid 3,4,5,6}$ only to simplify notation. The true difference is a much larger expression, depending only upon non-refined building blocks. However, this longer expression is BRST equivalent to (12.2.6) by using the identities from [20].

[^25]:    ${ }^{1}$ That is, for example, if one had say $V_{1} T_{[[2,3], 4],[5,6], 7,8}^{m}+(2,3,4|5,6| 2,3,4,5,6,7,8)$, this sum would not contain a term $V_{1} T_{[[2,4], 3],[5,6], 7,8}^{m}$ as the order of the 3 and 4 has been swapped. It may contain a term $V_{1} T_{[44,5], 6],[2,3], 7,8}^{m}$ though, as the overall ordering of the sets of numbers we select is irrelevant

[^26]:    ${ }^{1}$ Note this, and all numerators in this appendix, are not necessarily in their optimal representation. It may be possible to find a general algorithm to describe these without the need for such lengthy expressions.

[^27]:    ${ }^{2}$ Note the $V T$ terms in this expression have the form suggested by equation (7.2) in [1], up to a single term $\frac{1}{12} V_{12} T_{3,4,5,6,7}^{m n} k_{1}^{m} k_{2}^{n}$.

[^28]:    ${ }^{3}$ There is a relative factor of $\frac{1}{2 s_{12}}$ between these two sets of terms. This however is not a concern, as such a factor appears in the denominator of the hexagon terms relative to the heptagon terms and so they cancel
    ${ }^{4}$ Note the $V T$ terms in this expression have the form suggested by equation (7.2) in [1], up to a single term $\frac{1}{12} V_{1} T_{23,4,5,6,7}^{m n} k_{2}^{m} k_{3}^{n}$.

[^29]:    ${ }^{5}$ Note this matches the formula for the six point pentagon $N_{61 \mid 2,3,4,5}(\ell)$, equation (11.1.25), once the natural replacement of particle labels is made.

[^30]:    ${ }^{6}$ Note the $V T$ terms of this expression match with the six point pentagon $N_{612,3,4,5}(\ell)$, equation (11.1.25), once the natural replacement of particle labels is made.

