UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL SCIENCES

Mathematical Sciences

Aspects of Holography beyond AdS
BMS superrotations in higher dimensions

by

Federico Capone

A thesis submitted for the degree of

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ABSTRACT

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A crucial role in the problem of four-dimensional asymptotically flat spacetime holography is long believed to be played by the Bondi-Metzner-(van der Burg)-Sachs (BMS) asymptotic symmetry group. The discovery of deep relationships between such symmetries and gravitational infrared effects - pertrubative soft theorems and gravitational wave memories - has sparkled new interest in the problem. In this context, superrotations extend the standard general relativistic definition of BMS group. While there has been some debate on the extension of BMS to higher dimensions ($d > 4$), very little has been said about superrotations. In this thesis we initiate a systematic study of such structures. We provide evidence that consistency of superrotation Killing fields with a fully non-linear $d > 4$ gravitational configuration space requires different boundary conditions than those considered in literature. The first such evidence comes from cosmic $(d-3)$-branes, which are conjectured to be related to superrotations in $d > 4$ as a natural generalization of the relationship between cosmic strings in $d = 4$ and $d = 4$ superrotations. The general boundary conditions may be taken to define *asymptotically locally Minkowski* spacetimes.

The first part of the thesis recaps fundamentals of AdS/CFT - the most well known example of holographic duality - and the second moves to asymptotically flat spacetimes. The review material marks the conceptual and technical differences between AdS holography and the attempts at flat holography. A recurring theme is that of variational principles and asymptotic charges. Their mutual consistency is pivotal in dynamically realising AdS/CFT. We take this requirement as the basis of asymptotically flat holography and the analysis of the configuration space is fundamental to the construction of well-defined variational principles and the phase space.

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Declaration of Authorship

I, Federico Capone, declare that this thesis entitled *Aspects of Holography beyond AdS, BMS superrotations in higher dimensions* and the work presented in it are my own and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted form the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:

  [1] Capone, F., Taylor, M. *Cosmic branes and asymptotic structure. JHEP* **2019**, 138 (2019). `https://doi.org/10.1007/JHEP10(2019)138`.

  [2] Capone F. (2019) *BMS Symmetries and Holography: An Introductory Overview.* In: Cacciatori S., GÃijneysu B., Pigola S. (eds) *Einstein Equations: Physical and Mathematical Aspects of General Relativity.* DOMOSCHOOL 2018. Tutorials, Schools, and Workshops in the Mathematical Sciences. BirkhÃďuser, Cham. `https://doi.org/10.1007/978-3-030-18061-4_6`

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Signed: ....

Date: ....

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Introduction

Gravity and the quantum

A famous quote circulating among physicists is the sentence spoken by Lord Kelvin at the turn of the XIX century about the two clouds affecting the brightness of the dynamical theory of heat and light. According to the most popular interpretation, Kelvin wished to stress that physics was basically complete and that those two problems - as clouds dissolving with time - would have been settled in the current framework. Perhaps instead, he foresaw that those two clouds would have generated two of the biggest storms of the history of physics. They indeed resulted in the introduction of two new universal constants, in addition to Newton's G: c - the velocity of light in vacuum - the greatest possible attainable velocity, and h - the Planck constant - the fundamental quantum of action.

The past century witnessed great advances in the understanding of the physical laws at the scales set by h and c. The introduction of the limiting velocity c required a modification of the Galilean relativity principle and brought to Einstein's relativity. Its first outcome was the modification of Newton's second law of dynamics, and - when considered together with the Equivalence Principle and the Mach principle - the change (after some ten years) of Newton's gravitational force law in favour of the theory of general relativity. Einstein provided a theory where space and time are themselves gravity and whose equations consistently contain c and G. At the time of the facts, instead, h entered the scene to account for a disordered set of experimental facts about atoms and radiation. The situation begged for more fundamental formulations. They started to be fully presented a decade after the formulation of general relativity, and triumphed in encompassing all of atomic and molecular physics, in the quantization of Maxwell's theory and in giving birth to quantum field theory, where h and c play together, and whose most notable product is the standard model of particle physics.

On the other hand, very dense clouds hide a foreseen full-fledged theory working consistently at the scale where G, c and h are relevant at the same time, the Planck scale. This is usually referred to as the problem of the marriage of the rules of quantum theories and those of gravity, where gravity is usually taken as a synonym of general relativity.

The problem of a quantum theory of gravity is so puzzling that over the years several direct or indirect approaches have been pursued. From a quantum field theory perspective, the first manifestation of the issue is the fact that general relativity is not perturbatively renormalizable. On the one hand, this does not preclude that general relativity can be consistently quantized non-perturbatively at the non-linear level, and thus retain a fundamental nature. On the other hand, the aforementioned impossibility may suggest that general relativity is not to be taken as a fundamental theory, but rather only as an effective one. While the first view is undertaken in approaches like loop quantum gravity [3], causal dynamical triangulations [4] or asymptotic safety [5], the second is pursued in string theory [6, 7] or emergent gravity scenarios [8, 9].

The various attempts at a further fundamental understanding of gravity seem to suggest that the notions of space and time and the principles on which current physics is based need to be critically rethought. In addition to model-dependent results, the strongest evidence supporting such kind of expectations stem, from generic features of quantum mechanics and general relativity. In this respect, black holes play a central role.

Black holes are amongst the most fascinating objects in nature because the spacetime singularities they hide behind the horizon, whose shadow we directly pictured very recently [10], are at the same time a fundamental prediction of general relativity and the place where the theory itself breaks down. Solving this breakdown is one of the main goals of any proposed theory of quantum gravity. However, we shall now step back from the problems of quantum gravity and move towards a semiclassical regime. This regime is where black holes pave the way toward a new understanding of gravitational interactions.

Classically, black hole horizons satisfy four laws that have a close analogy to thermodynamical laws [11]. According to this analogy, black holes have a - so-called - Bekenstein-Hawking entropy [12]

$$S_{BH} = \frac{k_B}{l_P^{d-2}} \frac{A}{4}$$

remarkably proportional to their area, rather than the volume. In the above $l_P = (\hbar G^{(d)}/c^3)^{\frac{1}{d-2}}$ is the Planck length in d dimensions, where also the Newton constant must be noted with $G^{(d)}$ as it differs according to the dimension of spacetime, and k_B is the Boltzmann constant. Using quantized fields in the classical black hole background, namely the semiclassical approximation, S.W. Hawking showed that the laws of black hole mechanics are not only analogous to thermodynamical laws, but they indeed are thermodynamical laws. In fact, Hawking discovered [13] that the entropy is associated with a temperature

$$T_H = \frac{\hbar\kappa}{2\pi k_B c}$$

κ is the surface gravity. For example, the temperature of a $d = 4$ Schwarzschild black hole is $T_H = \hbar c^3/(8\pi GM)$. This result is hard to overestimate. Black holes emit radiation at temperature T_H with a spectrum which is thermal, according to Hawking's semiclassical calculation.

To exemplify the power and limitations of semiclassical analysis of certain systems, we can consider the absorption/emission of electromagnetic radiation from atoms. When quantum theory was just a set of unjustified rules engineered to fit experimental evidence, some of the observed spectral lines of the elements were explained by Bohr via an *ad hoc* law according to which the frequency of a line is given by the energy jump between two levels in h units. As the quantum formalism developed, this rule was derived as a consequence of perturbation theory applied to a semiclassical system where the atom is quantized but the electromagnetic potential is not. In such a situation, the classical field behaves as a background field. This approximation correctly captures the induced absorption/emission probability amplitude due to the electromagnetic field, but miserably fails in describing spontaneous emission happening in absence of external electromagnetic stimulation. This fact is a limit of the semiclassical treatment of the electromagnetic interaction and a hint toward the necessity of quantizing the field. The semiclassical approximation correctly captures the influence of the external field on the atom (induced absorption and emission), but not the influence of the atom on the external field (spontaneous emission).

Similarly, semiclassical black hole radiation is an effect of the strong classical gravitational field on the quantum fields. The vacuum outside the horizon is populated by the pair production and annihilation of virtual particles. Heuristically we can say that due to the presence of the horizon there is a non-zero probability that one member of a given pair disappears in the hole while the other escapes from it and become real. These particles constitute Hawking radiation. While the formula for S_{BH} is only a first-order approximation, it is sound because subleading corrections scale as the logarithm of S_{BH} (see for example [14]). Any fundamental theory of gravity must reproduce this result.

Hawking radiation, due to its thermality, immediately leads to the famous information paradox [15]. The pair production picture outlined above implies that part of the information about the state of the hole is lost in the hole. Quantum mechanically this means that the breakdown of unitarity, namely reversibility, is unequivocal because any initial pure state will evolve in a mixed state. One of the aims of any model of a fundamental gravitational theory (or at least a new understanding of current models) is to resolve this paradox.

This thesis is not focussed on the black hole information paradox itself, but is framed within the vast literature revolving around one of the first fundamental new ideas stemming from the debate around the black hole information problem: the *Holographic Principle*.

Hawking's proposal of a breakdown of quantum mechanics was not easy to accept and indeed it was not accepted by many. Notably G. 't Hooft, willing to preserve unitarity, was brought to the first formulation of what later became known - thanks to L. Susskind - as the Holographic Principle [16, 17].

The Holographic Principle is usually taken as the statement that a holographic duality exists according to which a gravitational theory in a d-dimensional spacetime is equivalent to a non-gravitational theory in a $(d-1)$-dimensional spacetime. Its original statement only involves a bound on the number of fundamental degrees of freedom associated with spacetime regions, whereas the strong statement here is the consequence of what is considered the best understood example of holographic duality: the *Anti de Sitter/Conformal field theory* correspondence (or du-

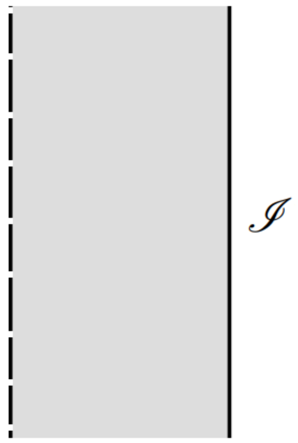

Figure 1: Penrose diagram of AdS spacetime. $\mathscr{I}$ is the timelike boundary and the discontinuous line is the symmetry axis of the diagram.

ality), AdS/CFT for short, discovered within string/M-theory independently of 't Hooft-Susskind holographic proposal.

The role of timelike asymptotics: AdS/CFT

As the label suggests, AdS/CFT is a correspondence between some gravity theory in AdS spacetime with some (relativistic) conformal field theory.

The first examples provided by J.M. Maldacena [18] are of this form, but the statement needs to be sharpened to be precise. The gravity theory is defined in spacetimes which are only asymptotically (locally) AdS. The AdS spacetime is the vacuum solution of Einstein's equations with a negative cosmological constant Λ and constant curvature. Asymptotically locally AdS spacetimes are solutions of Einstein's equations with a negative Λ that approach, broadly speaking, the AdS form at their boundary. The notion of boundary of a spacetime is well known in general relativity and formally represents the concept of "infinitely far away from the bulk of the spacetime". The CFT is naturally interpreted to be "defined on" (in the sense of associated with) the conformal boundary of AdS [19], which is a timelike hypersurface. The conformal symmetry of the field theory is reflected in the symmetries of the AdS boundary geometrical structures. The latter are called *asymptotic symmetries*.

The correspondence is dynamically realized as the equality of the partition functions of the two theories

$$Z_{grav} = Z_{CFT}.$$

Which of the two sides is more fundamental is a philosophical matter (see for example [20, 21]) which we do not address. Very interestingly, Maldacena's first example[1] - the Old Number One,

[1]Various examples were made in [18]. The first in the paper is the one to which we refer as first.

which we use to exemplify the matter - relates two (roughly speaking) known theories and does so in a strong/weak fashion: when the gravity side is weakly coupled the field theory is strongly coupled, and vice versa.

To be a bit more specific, the Old Number One, relates a maximally supersymmetric CFT, whose action is completely known, to the so-called Type IIB superstring theory in an AdS background, which is only known perturbatively. Thanks to the strong/weak property, the duality is seen as a fully non-perturbative definition of such string theory in the AdS spacetime. In fact, speaking of AdS as a background is a misnomer and the duality can be argued to be giving a background independent definition of Type IIB superstring theory on the spacetimes whose boundaries are of AdS form. The "choice" of the boundary only selects a sector of the gravity theory, which is for the rest free to be in any state and each of its states will correspond to a state in the field theory side. The reader interested in a quick account of the problem of background dependence and other conceptual points of AdS/CFT, including the relationship with the idea of emergence, is referred to [21].

Soon after Maldacena's paper, the possibility to construct other examples of strong/weak AdS/CFT dualities were realised. Indeed, the Old Number One is seen to be fundamentally based on some rules - involving the matching of symmetries of the dual pair - which can be adapted to build other holographic dualities. We can hope to appropriately lower the degree of super and conformal symmetry of the field theory and construct the bulk gravity theory with the appropriate symmetries. In this way we have a tool to analyze strongly coupled field theories of phenomenological interest to the standard model of particles or condensed matter physics. The latter further requires the field theory to be non-relativistic. We enter the domain of what is usually called "applied holography" [22, 23]. Some of these models are realized within string theory (top-down construction), some others are not (bottom-up construction) but may be. Many others, which can be found in the early review [24], do not bear any particular relevance to applied scenarios.

The upshot, twenty-three years after Maldacena's paper, is that a broad landscape of holographic dualities has been uncovered, spanning non-conformal and non-relativistic quantum field theories. They are called *gauge/gravity dualities*, a nomenclature coined to capture the generality of the correspondence. Almost all examples for which a detailed holographic dictionary has been constructed are strong/weak dualities and share a common feature: the quantum field theories are associated with timelike conformal boundaries of the bulk spacetimes, AdS being the simplest of such spacetimes.

Holography for spacetimes without timelike boundaries

As an artefact of our presentation, the last paragraph relegated the gravity side in AdS/CFT to the role of a tool. Shifting attention to gravity, as an example of holography and possibly of a complete definition of quantum gravity with AdS boundaries, AdS/CFT naturally inspires the questions: how is holography realized in de Sitter (dS) and Minkowski spacetimes? Can we explore the answers to this question to define quantum gravity in such spacetimes?

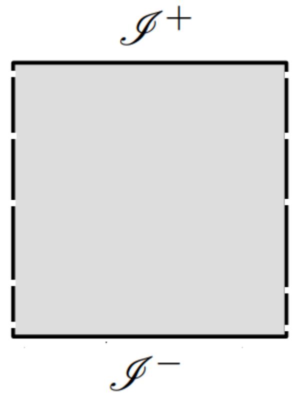

Figure 2: Penrose diagram of dS spacetime. The discontinuous lines are the north and south poles, $\mathscr{I}^+$ and $\mathscr{I}^+$ are future and past spacelike infinity.

In other words, as put by T. Banks in [25], are the boundaries of de Sitter or asymptotically flat spacetimes `appropriate` for a formulation in terms of degrees of freedom associated with such boundaries?

Such spacetimes are physically relevant. Due to the Equivalence Principle, any spacetime is locally Minkowski; the universe around us is approximately Minkowskian because the curvature radius of the observed universe is of order 10^{60} in Plank units and all fundamental non-gravitational physics is framed within Minkowski space, which is also arguably a solution of any quantum gravity theory. On the cosmological scale, instead, we know the cosmological constant is slightly positive and dS is the simplest solution of Einstein's equations with such a value and, despite not exactly describing the Universe as we know it today, several observations hint that the early and late history of the Universe are well approximated by dS [26, 27]. The role of dS is even more relevant in the inflationary epoch [28] as dS is exponentially expanding.

Insisting that the Holographic Principle is a principle of Nature, it is thus imperative to search for realizations of the Holographic Principle in such spacetimes. Because of the somewhat twofold nature of the Holographic Principle, this means both finding descriptions of known phenomena in terms of holographic degrees of freedom, but also finding appropriate complete definitions of quantum gravity in the sectors specified respectively by flat and dS boundary conditions.

As a matter of fact, string theory has not provided many (or any) clues towards these two questions up to now.

While Minkowski space is a natural background for string theory, and in fact, the first on which the theory was perturbatively defined, the only holographic model of flat gravity comes from the largely unknown M-theory[2]. Indeed, several months before Maldacena's paper, T. Banks, W.

[2] Also Little String Theory is to be mentioned in this context [29, 30]. In Section 2.1 we briefly recap the string-theoretical origin of AdS/CFT and for that purpose we need to consider D-branes and their decoupling limits. Little String Theory arises - roughly speaking - in a very similar way to what we see in Section 2.1, but D-branes must be replaced by the so-called $NS5$-branes. In AdS/CFT the AdS spacetime arises in the decoupling limit of D-branes; similarly, partial Minkowskian geometry arises from the decoupling limit of $NS5$-branes. See [30] for some more comments on this point.

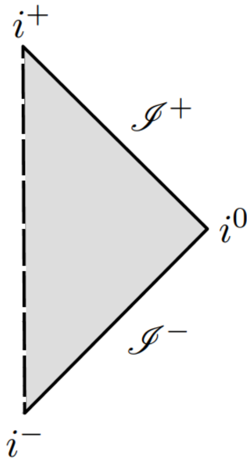

Figure 3: Penrose diagram of Minkowski spacetime. The discontinuous line is the symmetry axis of the diagram.

Fischler, S.H. Shenker and L. Susskind proposed [31] - interestingly motivated by the Holographic Principle - that M-theory in eleven-dimensional flat space can be holographically realized in terms of a quantum mechanical matrix model, now called BFSS model.

The situation with dS is subtler because there is evidence against the existence of (quasi)-dS solutions in string theory [32, 33]. Such shreds of evidence are, however, not conclusive (cfr. [34]) and dS can be obtained in appropriate supergravity and M-theory constructions [35, 36, 37].

When we look at the dS and asymptotically flat holography problem from the perspective of bottom-up extensions of AdS/CFT the situation is somewhat reversed[3]. Missing top-down guidance, the strategy is to follow the road of the asymptotic symmetries of the spacetimes under considerations. In this context, dS is conceptually less difficult to handle than asymptotically Minkowski spacetimes. Indeed, the boundaries of dS are spacelike while those of Minkowski are null.

A de Sitter/CFT (dS/CFT) correspondence was suggested by A. Strominger [39] and its relevance to standard cosmology was shown in [40]. A program of holographic cosmology for the inflationary era has been proposed in [41] and reviewed in [28]. Thanks to the peculiar causal structure of the dS boundary, some features of the AdS/CFT dictionary can be transferred under particular analytic continuations [42]. The dynamical statement of dS/CFT translates to

$$\Psi = Z_{CFT},$$

where the left-hand side is the wavefunction of the universe and the CFT on the right hand side is necessarily Euclidean. A possible concrete realization of dS holography is that in term of higher spin gauge theories.

The formulation of holography for flat spacetimes is a much subtler problem because of the causal nature of the boundary, as said. Theories associated to null manifolds (see Chapter 4) are largely

[3] Although the BFSS model realizes holography, exploring other realizations is not pointless because - as observed by Witten [19, 38] - the matrix model is not covariant and is far less understood that AdS/CFT.

unknown and uncommon - if not irrelevant - in everyday physics. The expectation is that such theories are dual to flat spacetime physics if they output the flat space S-matrix

$$??? \to \text{gravitational } S\text{-matrix.}$$

Over the past twenty years, several attempts have been done toward enlightening the left-hand-side of this relationship, but only in the past six years we have witnessed a constant steady increase in the research production on flat holography and, although still far from the end, we have some tantalizing results in four spacetime dimensions.

Asymptotically flat spacetimes. Results and organization of the material

The increased interest in such problem coincides with the discovery of a set of equivalences in the infrared physics of both asymptotically flat perturbative gravity [43], and gauge theories (an early pedagogical review is [44] and see chapter 4 for other references). For the vast majority, such chains of equivalences are triangular and relate the asymptotic symmetries of a theory with quantum soft theorems and classical memory effects. They are concisely known as infrared triangles, a name coined by A. Strominger, the first to realize such connections.

The research presented in this dissertation deals with aspects of such triangular relationships for asymptotically flat gravity in dimensions higher than four. In particular, we explore the necessary boundary conditions of gravity that allow for the extension of the infrared triangular equivalence to any asymptotically flat $d > 4$ spacetime.

An asymptotically flat spacetime is for us a d-dimensional spacetime with the same Penrose diagram of Minkowski space: the boundary consists of a future null infinity $\mathscr{I}^+$ and a past null infinity $\mathscr{I}^-$. The joint i^0 is spacelike infinity, which is a singular point in the Penrose diagram and not part of the boundary (cfr. Appendix A). Two further points i^+ and i^- represents future and past timelike infinity, which play no role in this dissertation.

In four spacetime dimensions, asymptotic flatness is taken as a synonym of spacetimes with complete null infinity, topologically equivalent to $\mathbb{R} \times S^2$, and exactly Minkowski asymptotic metric. Such spacetimes can be called asymptotically Minkowski and spacetimes with a different boundary topology or a different boundary metric may be called asymptotically locally Minkowski, borrowing the AdS/CFT jargon. The term asymptotically locally flat is also used. By extension asymptotically flat spacetimes in $d > 4$ are taken to be spacetimes with null infinity topologically equivalent to $\mathbb{R} \times S^{d-2}$, but the notion of a conformal infinity is not straightforward in $d > 4$ (cfr. Appendix A).

In pursuing our goal, we will often speak of either asymptotically Minkowski or *asymptotically locally Minkowski* boundary condition in generic number of dimensions. We will, however, use the terms flat or Minkowski and the context will clarify the meaning. We use a mostly minus signature in the thesis.

The interest in higher dimensions is mainly motivated by a theoretical conundrum. It is by now well-known that tree-level[4] scattering processes, in any theory of gravity with $d \geq 4$ flat non-compact dimensions, where one or more external gravitons become soft (vanishing energy) satisfy factorizations of the soft contribution from the finite energy (hard) particles contributions. Such factorizations occur at leading, subleading and sub-subleading order in a Taylor expansion of the amplitude in terms of the soft momentum (or momenta). The leading universal term in $d = 4$ constitutes Weinberg soft theorem [47, 48], the sub and sub-subleading universal terms were proved with explicit model-dependent computations by Cachazo and Strominger [49] in $d = 4$, whereas higher-dimensional generalisations can be found in [50], and general proof are given in [51, 52, 53].

As mentioned above, such perturbative theorems of quantum field theory have been related in $d = 4$ to the asymptotic symmetries of asymptotically flat gravity satisfying particular conditions. The leading and subleading soft theorems arise as Ward identities of the gravitational S-matrix under the so-called *supertranslations* and *superrotations*, respectively[5] (see chapter 4 and chapter 6).

Supertranslations (ST) are long known to be asymptotic symmetries of $d = 4$ asymptotically Minkowski gravity at null infinity (cfr. [55, 56]). They are an infinite-dimensional enhancement of the translational Abelian subgroup of the Poincaré group. The semidirect product of the orthochronous Lorentz group and supertanslations constitute the so-called BMS group (standing for Bondi, Metzner, (van der Burg) [57] and Sachs [58, 59])

$$BMS^{\text{glob}} = L_{+}^{\uparrow} \ltimes ST,$$

which is the asymptotic symmetry group at null infinity of radiative asymptotically flat four-dimensional spacetimes.

We can heuristically motivate the appearance of this structure with an argument by R. Geroch [55]. The Killing vectors of Minkowski spacetime have the form $\xi^\mu = \omega^\mu_\nu x^\nu + \alpha^\mu$, where α and $\omega_{\mu\nu}$ are constant and the latter is antisymmetric, x^ν is a position vector with respect to some origin. While $\omega_{\mu\nu}$ does not depend on the origin, α^μ does (just shift $x'^\mu = x^\mu + y^\mu$ to check). This is the reason for Poincaré being a semidirect product of Lorentz and translations. At large x^μ we have $\xi^\mu \sim \omega^\mu_\nu x^\nu$. A naive guess for the asymptotic Killing vector ξ^μ_{asympt} is thus that it is at least asymptotically linear in x^μ: $\xi^\mu_{asympt} = \omega^\mu_\nu x^\nu + \alpha^\mu(x)$. We cannot further demand that α^μ is constant because there is no unique way to separate the constant part of ξ^m_{asympt}. In fact the notion of "linear in position" only makes sense in flat space and then in the limit to flat. However asymptotically, if α is constant, it is dominated by the linear part of ξ^μ_{asympt} resulting in a reduction of the number of generators. The only way out is to conclude that in general we cannot give a meaning to a constant α. This is the origin of supertranslations.

Flat holography and the infrared triangle require the non-abelian factor of BMS to be enhanced to an infinite-dimensional group of transformations. These are named superrotations. Two variants have been proposed: BT-superrotations [60, 61, 62, 63] (BT standing for Barnich-Troessaert) and CL-superrotations [64, 65] (CL standing for Campiglia-Laddha). BT-superrotations form the Witt

[4]Results at loop order exists also, but results are less sharp (see the discussion in [45] and [46] as well as the references listed in the main text).

[5]Notice that at present, the sub-subleading soft theorem is not accounted for by these symmetries and we are not concerned with that here. The reader is referred to [54] for preliminary considerations.

algebra of local conformal transformations on S^2, CL-superrotations form the algebra of volume-preserving diffeomorphisms of S^2, $SDiff(S^2)$.

It is important to stress that such symmetry structures preserve some set of boundary conditions which specify the class of spacetimes under consideration. Supertranslations are essentially related to gravitational radiation and define the class of asymptotically flat (Minkowski) spacetimes in $d = 4$. Superrotations are essentially related to subleading soft theorems and require a relaxation of asymptotically flat boundary conditions. This introduces the concept of asymptotically locally flat spacetimes in our discussion.

The aforementioned higher dimensional conundrum stems from the definition of radiative asymptotically flat spacetimes [66, 67, 68, 69, 70]. Radiation-consistent boundary conditions imply a reduction of supertranslations down to translations. This automatically eliminates the infrared triangle of Weinberg's soft theorem.

In the past five years, this occurrence has been largely discussed also in relation to gravitational memory effects mostly in even spacetime dimensions [71, 72, 73, 74, 75, 76, 77] ([78] discusses memories in odd dimensions). None of such papers consider superrotations.

The paper [1] presented in chapter 7 is the first attempt to treat superrotations and supertranslations in five spacetime dimensions. The analysis of chapter 8 clarifies, completes and extends the preceding analysis to any number of dimensions.

When discussing superrotations in $d > 4$ we must decide whether to seek for higher-dimensional realizations of BT or CL. The choice is trivial from an algebraic viewpoint since no BT-algebra exists in $d - 2 > 2$.

However, the action of BT-superrotations in $d = 4$ is marked by interesting phenomenological signatures consisting of impulsive gravity waves processes accompanying the transition from Minkowski to asymptotically Minkowski spacetime. This stimulated us to conjecture that similar processes may mark higher dimensional "BT"-superrotations. We thus constructed candidate asymptotically locally Minkowski spacetimes and we built boundary conditions which are generic enough to include them in the configuration space (chapter 7). This approach is different from that usually pursued in literature. We do not engineer boundary conditions to recover a pre-determined asymptotic symmetry group.

In chapter 8 we keep on working with the asymptotically locally Minkowski conditions and we find the set of necessary conditions to obtain a configuration space which is consistent with the action of higher dimensional CL-superrotations. In this respect, we not only complete and extend the analysis initiated in chapter 7, but we also clarify some fundamental flaws that were met in [79], another recent analysis of CL-superrotations in even $d > 4$ (see also [80]). The reader interested now in knowing more about the outputs is invited to read Section 7.1 and Section 8.1.

We note, to conclude, that we have spoken about configuration spaces rather than phase spaces. Broadly speaking, the first is the set of all solutions of a theory satisfying prescribed boundary conditions. The second is the subset of physically relevant solutions, namely those characterised by finite energy and momentum or - in general - symmetry charges. As we will discuss in the course of the thesis, providing a well-defined phase space is the central goal, but it rests on the a-priori definition of the configuration space. There is a fundamental interplay between well-defined boundary conditions and phase spaces. We will comment in due course (see chapter 6) that even in four-dimensional spacetimes the situation concerning superrotation-compatible phase space is still not clear. There is ongoing work with my colleagues to explore the phase space associated to the configuration space discussed in chapter 8.

Organization of the material

To make the thesis self contained, we begin by reviewing material from both the basics of AdS/CFT and the recent advances on the flat spacetime holography problem. The review comprises Part I (on AdS/CFT), chapters 4, 5 and 6 of Part II (on flat holography) and it aims at introducing the background ideas relevant to the research explorations of the later chapters 7 and 8. The level of the review is that of introductory notes, rather than a complete rigorous account tailored to experts.

Chapters 7 and 8 are devoted respectively to [1] and some unpublished material which supposedly comprise the basis of two extended papers. Chapters 4, 5 and 6 are based on the proceeding paper [2]. Efforts have been made to organize the text in a logical and coherent flow, so the chapters corresponding to the published articles do not completely line up with the published version. Many of the relevant points just mentioned in this discussion are presented with some more detail in the main body. The order of this introductory narration is reflected in the chapters. All the ingredients needed to discuss the flat spacetime holography problem - boundaries, asymptotic symmetries, charges, among others - will be introduced in due time starting from the discussion of (mostly) the gravitational aspects of AdS/CFT in Part I. Part II can however be read independently from Part I.

In Part I, we first present the basic argument of the Holographic Principle (chapter 1) and then move to AdS/CFT (chapter 2). We motivate AdS/CFT using what we called the Old Number One (Section 2.1) and then we summarise the symmetry matching argument which is fundamental in any tentative holographic duality (Section 2.2). We then complete the dictionary of AdS/CFT with the GPKW rule (Section 2.3) and the example of holographic renormalization of the metric tensor in Fefferman-Graham gauge, with the following examples on Weyl anomalies (Section 2.3.1).

The asymptotic analysis of Einstein's equations in Fefferman-Graham gauge is to be compared and contrasted with the Bondi-Sachs analysis which we employ in asymptotically flat spacetimes.

The discussion of the holographic stress-energy tensor naturally brings us to the problem of gravitational charges (Section 3.1 and Section 3.2) and to the discussion of the covariant phase space method (Section 3.3), which is extensively used in flat holography. Along this path, the fundamental role of the variational principle in holographic renormalization and AdS/CFT is pointed out, as well as the generality of the method of holographic renormalization, so that we are able to mention the difficulties in extending the basic AdS/CFT dictionary to flat spacetimes (Section 3.1.1).

Part II is opened by a brief excursus of the various paths toward flat spacetime holography that have been undertaken over the years chapter 4. It provides more discussion on supertranslations and superrotations from a historical perspective and states in which sense the developments regarding scattering amplitudes and soft theorems points toward flat holography.

Chapter 5 lay down the basic results of the Bondi-Sachs analysis of asymptotically flat spacetimes in $d = 4$ and derives supertranslations and superrotations, motivating why the latter implies a change of boundary conditions. The orthodox analysis in $d > 4$ is also considered in parallel so to show that BMS disappear in such a case. Chapter 6 gives a brief account of supertranslation and superrotation charges in $d = 4$ and points out the issues affecting the latter. Then, the simplest and most well-posed instance of a gravitational triangle - that of supertranslations - is briefly discussed.

Chapters 7 and 8 comprise the research material, but the reader can find the proof of many results summarised in chapter 5 in chapter 8. Appendices supplement the main body.

Part I

The holographic principle and AdS/CFT

CHAPTER 1

The Holographic Principle

The Holographic Principle stated in the Introduction is based on the following argument by 't Hooft [16] and Susskind [17]. Given some quantum theory, the dimension of the Hilbert space $\mathtt{N}$ is related to the thermodynamic entropy S by

$$\mathtt{N} = e^S \tag{1.0.1}$$

In any local theory where the possible energies of the system are bounded, a reasonable expectation is that the entropy S is related to the volume of space V enclosing the system

$$S \sim V. \tag{1.0.2}$$

This is the well known extensive property of entropy as derived from statistical mechanics. It can be simply argued for in the following way. If we take a d-dimensional lattice with spacing l_p where each site hosts m orthogonal states, the dimension of the Hilbert space of the lattice in a volume V is

$$\mathtt{N}(V) = m^n, \tag{1.0.3}$$

where n is the number of lattice sites in V: $n = V/l_p^d$. Hence the entropy is

$$S = n \log m = \frac{V}{l_p^d} \log m. \tag{1.0.4}$$

A very simple example is a spin system with $m = 2$. For continuous systems the counting is more delicate but doable. One needs an entropy density, function of the energy density which should be bounded in order not to have a diverging number of states.

We can also define the total number of degrees of freedom G_{df} of the quantum system as

$$G_{df} = \log \mathtt{N} \tag{1.0.5}$$

and the number of degrees of freedom g_{df} per lattice site as

$$g_{df} = \log m \tag{1.0.6}$$

If quantum gravity behaves as a discrete local lattice theory, as may be suggested assuming that a fundamental minimal cutoff exists, then the above estimate should be valid with, say, l_p the Planck length. However, (1.0.2) is in stark contrast with an entropy bound that can be derived from black hole thermodynamics and the assumption of unitary evolution.

In a black hole spacetime the violation of the second law of thermodynamics $\delta S \geq 0$ led J. Bekenstein to suggest, before Hawking famous computation[1], that S_{BH} is a truly thermodynamic entropy and that the sum of S_{BH} with the entropy S of the outer fields satisfies a generalised second law of thermodynamics

$$\delta S_{tot} = \delta(S_{BH} + S) \geq 0. \tag{1.0.7}$$

We can now imagine that our spin-like system above evolves in such a way that gravitational collapse happens and a black hole is formed, imagining that this can be done via an adiabatic process. We can for example add energy to the system, whose initial mass is below the mass of a black hole with the same surface area. The generalised second law is not violated if and only if[2]

$$S \leq S_{BH} = \frac{A}{4G}. \tag{1.0.8}$$

This claim implies, under some assumptions[3], that a black hole is the most entropic object that can be accommodated in a volume V with area A. We started with a system whose Hilbert space has dimension $\sim e^V$ and we end up with a much smaller Hilbert space with dimension $e^{A/4G}$. The two Hilbert spaces cannot be isomorphic and so unitarity is violated. Insisting on unitary evolution, we arrive at the dramatic conclusion that the Hilbert space dimensionality was $e^{A/4G}$ to start with. This is the statement of 't Hooft-Susskind Holographic Principle: any fundamental theory of gravity (and matter) is such that this bound on the dimensionality of the Hilbert space in any space volume is manifest. In other words, the total number of degrees of freedom G_{df} in a volume bounded by an area A is

$$G_{df} = \frac{A}{4G}. \tag{1.0.9}$$

The argument we presented gives the so-called *spherical entropy bound* and rests, apart from unitarity, on several other assumptions we have glossed over. When taking them into account, it is not difficult to find situations where the bound is violated. However, there exists a more refined and fundamental version of the Holographic Principle which involves Bousso's *covariant entropy bound* [82] and is based on null surfaces. It is independent of the requirement of unitarity of quantum

[1] Which fixed the 1/4 factor in front of A in S_{BH}. Notice we here use units where $\hbar = c = \kappa_B = 1$ to make expressions more transparent.

[2] This bound is weaker than the so-called Bekenstein-Hawking bound which is instead found assuming that a black hole already exists and that matter is swallowed behind the horizon (see [81]).

[3] See the papers cited at the end of this paragraph for precise statements and counterexamples.

mechanics and has been proved to hold in great generality. However, it cannot be derived from known fundamental principles. The early review [81] contains in-depth discussions and a complete list of references. Counterexamples of the covariant entropy bound have been found more recently [83].

These entropy bounds suggest that the fundamental degrees of freedom inside a volume V can be described by a quantum mechanical theory associated with its boundary [16]. This is now considered the Holographic Principle, but it should be noted that more philosophically oriented accounts of the concept of *emergent gravity* separate holographic duality from the Holographic Principle, which is restricted to the entropy bounds. We will not enter the discussion of emergence here and we will simply take that the holographic bounds suggest that gravity may arise as a manifestation of a fundamentally different theory.

CHAPTER 2

Holography in AdS

After a cursory review of the arguments leading to the first example of Maldacena's conjecture, we draw the generic lessons which form the basis of any top-down or bottom-up extension of such example. After this, we complete the discussion with the GPKW rule and a short discussion of holographic renormalization along with some of the early checks of the duality.

2.1 The Old Number One

't Hooft-Susskind holographic principle has been first explicitly and quantitatively realised within string theory and M-theory in the so-called AdS/CFT duality put forward by J. Maldacena in 1997 [18]. A series of inspiring developments in supergravity, D-brane physics and some hints stemming from large-N gauge theories accumulated from roughly the '80s (see for example [84] for some references) led to the conjecture that the Hilbert space of various large-N field theories derived from string/M theory contains "excitations describing supergravity on various spacetimes" [18].

In his famous paper [18], Maldacena presented various incarnations of the correspondence. The "simplest" and most discussed in any introductory account is:

> Ten dimensional Type IIB string theory on $AdS_5 \times S^5$ with N units of five-form flux is dual to $\mathcal{N} = 4$ Super Yang-Mills (superconformal) theory in four spacetime dimensions with gauge group $SU(N)$.

As claimed at the beginning, we now rapidly justify it without aiming at completeness. We rather opt for a narrative exposition spiced up with some technicalities, to remind that the above statement is based on very sound arguments. The material here presented can be found in any modern book on string theory or any lecture notes on AdS/CFT.

Several names and labels, within two very different theories - a superstring theory and a non-gravitational field theory - appear in the conjecture. We depart from them to explain it.

$\mathcal{N}$: the number of multiplets F of fermionic generators to be added to the bosonic generators $B = \{P^\mu, M^{\mu\nu}\}$ of the Poincaré algebra, such that the corresponding new algebraic structure closes under (anti-) commutators to form super-Poincaré algebras labelled by $\mathcal{N}$.

IIB: is one of the two types (the other being IIA) of $\mathcal{N} = 2$ super Poincaré algebras that can be considered in $d = 10$ spacetime dimensions.

$AdS_5 \times S^5$: the space obtained in the near-horizon limit of a type IIB superstring $D3$-brane/type IIB supergravity 3-brane, which are charged extended objects.

We did not include N in this dictionary because it is somewhat derived from the ingredients introduced in the third item. The above is a checklist, given by a superior entity (i.e. any modern string theory book [6, 7]) whose elements will now be blended to produce the conjecture.

Type IIB string theory is a $\mathcal{N} = 2$ supersymmetric theory of closed strings in $d = 10$ spacetime dimensions. Having $\mathcal{N} = 2$ means that the theory has two fermionic multiplets of generators. As any fermion, they transform according to spinorial representations of the Lorentz algebra (see [6] for more details). Their dimension thus depends on the spacetime dimension. The Lorentz algebra $\mathfrak{so}(1, d-1)$ in $d = 10$ possesses two 16-dimensional irreducible spinor representations called Weyl representations. They can be further shown to be real and are called Majorana-Weyl representations. Such representations are usually denoted as $\mathbf{16}$ and $\mathbf{16'}$. When we have two distinct fermionic multiplets (as in $\mathcal{N} = 2$) we can choose each of them in either of the two representations. If they are both in the $\mathbf{16}$ or both in the $\mathbf{16'}$ we say that the super Poincaré algebra is type IIB, otherwise we say that it is type IIA. Type IIB string theory is thus the string theory whose spectrum is organised according to the representations of type IIB super Poincaré algebra. This representation theory simply boils down to the representation of the algebra of the fermionic generators F on the finite dimensional vector space V where the stabilizer subalgebra of $\mathfrak{so}(1, d-1)$ (Poincaré)[1] is represented.

Any string theory is characterised by an energy scale set by the tension T_s of the string, which is in turn related to the characteristic string length l_s scale. It is conventional to express T_s and l_s in terms of a parameter called α': $l_s^2 \sim \alpha' \sim T_s^{-1}$ (different conventions on the proportionality factors exist).

Interactions among strings are controlled by only one coupling g_s and string theory is endowed with sound rules to compute string scattering amplitudes at least perturbatively in g_s using worldsheet

[1]The representation theory of the stabilizer subalgebra is split in massive and massless states, hence also the representation theory of the supersymmetry algebra splits in massive and massless representations.

methods. In a closed string scattering each order of the perturbation series is weighted by [6]

$$g_s^{-2+2g} \tag{2.1.1}$$

where $g \geq 0$ denotes the loop contributions (genus of the Riemann surface).

Missing a complete formulation of string theory and leaving aside the possibility of using string dualities, predictions about scattering processes can be made in the weak coupling perturbative limit $g_s \ll 1$ and, on top of that, either at low energy $E \ll l_s^{-1}$ or high (eikonal approximation for example [85, 86]). For the present purposes we need to consider the low energy limit with the weak coupling limit. The fact that string theory contains Dp-branes implies that we have to consider such a pair of limits twice. Indeed in this narration, we use the first pair to discover Dp-branes, the second to describe them and finally, we combine the two

A. Low energy limit: Take 1. When $E \ll l_s^{-1}$ all the massive states in the string spectrum cannot be excited and the scattering should be equivalently described by an effective theory including only the massless modes. The finite-dimensional massless representations of $\mathfrak{so}(1,9)$ are the representations of $\mathfrak{so}(8) \subset \mathfrak{so}(1,9)$. $\mathfrak{so}(8)$ possesses three inequivalent 8-dimensional representations: a vector representation $\mathbf{8}_V$ and two inequivalent spinor representations $\mathbf{8}_S$ and $\mathbf{8}_C$.

The aforementioned $\mathbf{16}$ and $\mathbf{16}'$ decompose with respect to the action of $\mathfrak{so}(8)$ as $\mathbf{16} = \mathbf{8}_V \oplus \mathbf{8}_S$ and $\mathbf{16}' = \mathbf{8}_V \oplus \mathbf{8}_C$. In the massless case, it turns out that half of the F, F anticommutators are trivial. Thus the Weyl-Majorana representations can be assigned the $\mathbf{8}$'s. Those with trivial commutators are vectors and are assigned to $\mathbf{8}_V$ (hence the subscript) and are taken to act trivially on V. The other sixteen can be organised in two sets of eight creation and annihilation operators which are then assigned either to the $\mathbf{8}_S$ or $\mathbf{8}_C$.

Acting with the creation operators on appropriate $\mathfrak{so}(8)$ ground states we construct V and hence get all the massless states of the theory. The bosonic states are two scalars Φ and C_0 (1 component each), two 2-forms B_2 and C_2 (antisymmetric tensor fields with 28 components each), the symmetric and traceless graviton g (35 components), a self-dual four-form field C_4. Φ is called dilaton, B is called Kalb-Ramond (KR) field. The other forms do not have a proper name[2]. In addition to these, we have the fermionic states comprised of two dilatini and two gravitini, which we do not need for the later discussion.

When this spectrum is derived from the string quantization, the story is not so simple but at the end of the day, the results coincide. The important thing to notice is that the bosonic states are naturally split into two sectors (the fermionic in other two)[3]. One is called NSNS (NS stands for Neveu-Schwarz) and the other called RR (R stands for Ramond). The first contains the graviton, the KR field and the dilaton. RR contains the other forms with no name.

[2]Those with a name already appear in the spectrum of closed bosonic strings.

[3]The splitting in sectors is due to the string being closed (having thus left/right moving modes) and the boundary conditions the fermionic modes satisfy on the left/right modes.

B. Low energy, perturbative limit: Take 1. The effective low energy action of type IIB string theory, built from these modes, corresponds to the type IIB supergravity (Sugra) action. The action is basically a higher dimensional generalization of Einstein-Maxwell theory as it contains a number of form potentials. This is just an example of the well-known fact that closed string theory contains gravitons and, at the lowest orders, (higher dimensional) Einstein gravity if we turn off all the other massless fields. Schematically

$$S_{IIBSugra} = \frac{1}{2\kappa_{10}^2} \int d^d x \sqrt{-g} L[g, \Phi, B_2, C_{p+1}], \tag{2.1.2}$$

where we only list the bosonic fields, which enter the Lagrangian with their respective kinetic terms. For later purposes, we only need to recall how Newton's constant appears in the action.

In fact, to check the firmness of this result we should derive it from low-energy scattering amplitudes of string theory. In this way, we check that Sugra is really a low-energy tree-level action compared to string theory. This also implies that we assume $g_s \ll 1$ because loops are suppressed by additional powers of g_s compared to the tree-level amplitude. Indeed, from (2.1.1), the coupling constant of (2.1.2) corresponds to the tree-level coupling since $\kappa_{10}^2 = 8\pi G_{10} \sim g_s^2 l_s^8 \sim g_s^2 \alpha'^4$ (the factors of string length are to fix dimension). Thus schematically

$$IIBString \to S_{IIBSugra} + O(g_s, \alpha') \tag{2.1.3}$$

The Sugra theory contains several interesting extended object solutions which couple to either of the form fields. Those in which we are interested are p-branes (see [87] for more details). They are solutions of the supergravity action that couple to C_{p+1} when the fermionic modes, the KR field and all the $C_{p'+1}$ with $p' \neq p$ are zero. They are soliton-like objects extended in p spatial dimensions, which preserve Poincaré invariance in their $(p+1)$-dimensional worldvolume and isotropy in the other directions so that their metric is typically of the form

$$ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} (dr^2 + r^2 d\Omega_{8-p}^2) \tag{2.1.4}$$

where r is a radial coordinate distance from the brane and Greek indices run on the $(p+1)$-dimensional worldvolume of the brane. The functions A and B are determined by Einstein's equations via a harmonic function $H(r)$ behaving as

$$H(r) = 1 + \frac{\alpha}{r^{7-p}}, \quad \alpha =: l^{7-p}, \quad p < 7 \tag{2.1.5}$$

where l is a characteristic length parameter related to the energy of the brane. The metric reads as

$$ds^2 = H(r)^a \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^b (dr^2 + r^2 d\Omega_{8-p}^2) \tag{2.1.6}$$

where the powers a and b are numbers depending on the "the frame" in which the action is written[4].

Notice from (2.1.5) that despite describing an extended object, the solution resembles that of a

[4]The frame is defined by how the graviton and the dilaton are included in the action. When the Ricci scalar is not multiplied by dilaton-dependent factors we say we are in Einstein frame. Otherwise it is the string frame. For the brane under consideration, $a = -\frac{1}{2}$ and $b = \frac{1}{2}$ in the string frame, $a = \frac{p-7}{8}$ and $b = \frac{p+1}{8}$ in the Einstein frame.

point particle (0-brane) in $d = 4$ rather than that of a cosmic string (a 1-brane) which we meet in chapter 7) or a domain wall (a 2-brane) in the same number of dimensions. There is no surprise as we are in a higher dimensional theory.

The field strength F_{p+2} associated with C_{p+1} gives the flux across a surrounding sphere as

$$\int_{S^{8-p}} *F_{p+2} := N, \quad F_{p+2} = dC_{p+1}. \tag{2.1.7}$$

The flux is proportional to the charge by factors of α'. By the Dirac quantization condition, N is an integer. These p-branes can be extremal or not, namely have the $p+1$ charge density proportional - or equal in appropriate units - to the tension (i.e. the mass density) or not. Extremal p-branes preserve half of the supersymmetries of the theory (16 out of 32) and the property that the energy density is proportional to the charge means that a stack of N parallel p-branes of unit charge at an arbitrary distance among themselves is a stable system. In fact there exists a supergravity solution, called multi-centered solution, which describe this setting and is such that the single brane of charge $\propto N$ is recovered in the limit in which the N branes are placed at the same position. Sometimes the analogy is made with Reissner-Nordstrom black holes, but the analogy is not so thight since for some values of p there is no horizon. In particular, the 3-brane to which we need to focus is perfectly regular. Indeed, setting for definitness the string frame (see footnote 4), we see that the metric behaves as

$$ds^2_{AdS_5 \times S^5} \xleftarrow{r \ll l} ds^2 \xrightarrow{r \gg l} ds^2_{Mink} \tag{2.1.8}$$

where

$$ds^2_{AdS_5 \times S^5} = \frac{r^2}{l^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{l^2}{r^2}dr^2 + l^2 d\Omega^2_{S^5} \tag{2.1.9}$$

is the metric in the near-brane (near-horizon) limit and describes an $AdS_5 \times S^5$ geometry, as can be seen more readly defining $z := \frac{l^2}{r}$ so that

$$ds^2_{AdS_5 \times S^5} = \frac{l^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) + l^2 d\Omega^2_{S^5} \tag{2.1.10}$$

we recognise the typical AdS metric in Poincaré coordinates.

If Type IIB Sugra is a consistent low energy limit of Type IIB string theory, then all the Sugra solution must be present somehow in the complete theory.

Heuristically the reason is the following. Suppose you place a supergravity p-brane in a type IIB string background. It is a charged under the massless fields of string theory, so it must interact with the background. Tuning the parameter space so that strings are perturbative, the interaction is mediated by the exchange of closed strings. However, the process can be dually described via infalling closed strings on the brane which produce an excitation of the brane which later decay in a closed string. The excitation is described by open strings with the end attached to the brane. The brane is thus an object in string theory.

C. Perturbative limit: Take 2. In fact, this expectation is confirmed by the consistency of string dualities [88] which leads to the existence of heavy extended objects, called Dp-branes

[89], within string theory. Being heavy, Dp-branes are not part of the perturbative spectrum of string theory. However, at the perturbative level $g_s \ll 1$ they are described as the loci where open strings end[5], and open strings describe their excitations.

D. Perturbative, low energy limit: Take 2. Each end of the open string carries a multidimensional label (Chan-Paton factor) indicating on which Dp-brane the string end [6]. If there are N Dp-branes in the background, the label is an $N \times N$ matrix. Together with an appropriate counting of supersymmetries, analogous to the one above, this is one of the ingredients to check that the massless excitations of open strings ending on a stack of N $D3$-branes are described by a $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory on the four-dimensional worldvolume of the branes[6]. The bosonic sector of the theory consists of a gauge vector and six scalars[7].

This theory is characterised by a dimensionless coupling constant g_{YM} such that the action $S_{\mathcal{N}=4SYM}$ goes as $S_{\mathcal{N}=4SYM} \sim g_{YM}^{-2}$. This coupling does not run under the renormalization group flow, so the theory is conformal at the quantum level. So, the spacetime symmetry of the theory is a superconformal symmetry generating the supergroup $PSU(2, 2|4)$. Its maximal bosonic subgroup is $SU(2, 2) \times SU(4)_R \sim SO(4, 2) \times SO(6)_R$, where R denotes the R-symmetry under which the scalars transform in the adjoint and the gauge field as a singlet. $SO(4, 2)$ is the conformal group in four spacetime dimensions and the R-symmetry group is basically the rotation group in the transverse space of the branes as can be seen from our discussion in **B**.

The coupling constant g_{YM} can be related to the string coupling g_s in several ways. We can write down a low-energy action for the string modes ending on the brane[8] which is called Dirac-Born-Infield (DBI) (plus another term) and check that at lowest order we get the action for a gauge vector scaling as g_s^{-1} [7]. We thus obtain $g_{YM}^2 \sim g_s$.

For our purposes, it is sufficient to be comfortable with the cartoon idea that if two open strings are glued to form a closed string, the interaction can be described in terms of a single closed string coupling or twice the open string coupling.

E. Two descriptions for two regimes. We said that the open string picture of a Dp-brane excitation is valid when $g_s \ll 1$. Thus, in order for the N Dp-branes not to backreact on the geometry we need to assume $g_s N \ll 1$.

However, when we consider the stack of N branes in the sugra description we need to assume that $g_s N \gg 1$. To see this we need to explicitly compute α in (2.1.5) using (2.1.7). For the $D3$ brane

[5] This is the reason for the Dp name: D stands for Dirichlet, the kind of boundary conditions describing this situation and p is the same p as for the Sugra object.

[6] The starting point is the massless spectrum of open strings without branes: $d = 10$, $\mathcal{N} = 1$ Yang-Mills supermultiplet. As we did not enter the details of string theory and we did not discuss boundary conditions, we cannot and we do not want to be more detailed. Any introductory book on string theory will do.

[7] The scalars can be seen as the collective coordinates describing the dynamics of the transverse dimensions of the branes.

[8] In fact, this is known for a single brane.

case the result is [90]

$$\alpha = 4\pi g_s N \alpha'^2 \sim 4\pi g_s N l_s^4. \tag{2.1.11}$$

Hence $g_s N \propto \alpha/(4\pi l_s^4)$ and in order to apply the supergravity description we need large α to ensure weak curvature. Clearly we can define $l := \alpha^{1/4}$ to be the characteristic length of the brane solution.

F. Decoupling limits. Consider the N $D3$-brane picture within perturbative ($g_s N \ll 1$) string theory. As we described above, the theory describe processes mediated by closed strings propagating in the spacetime, open strings attached to the branes and the mutual interactions. In general it is not possible to decouple these processes, but schematically we can write

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}, \tag{2.1.12}$$

and we know that we can make sense of the first two terms in the low energy limit $\alpha' \to 0$: $S_{\text{bulk}} \approx S_{\text{Sugra}}$, $S_{\text{brane}} \approx S_{\mathcal{N}=4SYM}$. The only term we did not meet in the previous discussion is S_{int}. The coupling between closed and open strings is gravitational and hence it is governed by $\kappa_{10} \sim g_s \alpha'^2$. Thus at $\alpha' \to 0$ with g_s fixed this term can be neglected.

In taking this limit we must, however, ensure that the branes do not separate to keep the field theory parameters fixed[9]. If r is the separation between the branes we are thus interested in the limit: $\alpha' \to 0$, $U = r/\alpha'$ fixed. The conformal point (all masses zero) corresponds to $r \to 0$.

In this limit we thus get two decoupled systems: *i*) free type IIB Supergravity in ten-dimensional Minkowski space and *ii*) four-dimensional SYM on the flat worldvolume of the $D3$-branes.

When $g_s N \gg 1$ the supergravity description of the N $D3$ branes must be considered. There is no gauge theory dynamics in this picture. How do we recover it?

Let us notice that S_{int} in (2.1.12) is a statement of the heuristic picture drawn at the end of paragraph **B**. Namely, a closed string mode hits a D-brane and is absorbed so that the brane is excited. The absorption cross-section can be computed [84].

In a Sugra picture the brane provides, at low energy $E \ll l^{-1}$, a potential barrier separating the region $r \ll l$ and $r \gg l$ in (2.1.8). The cross-section is computed via a tunnelling probability between the two asymptotic regions [84]. The result is shown to coincide with the string computation.

Moreover, due to this potential barrier, there is a precise decoupling between low energy modes in the $r \ll l$ and $r \gg l$ regions.

According to the observer at infinity $r \to \infty$ in (2.1.6), with proper time t, the energy E_{r^*} of a

[9]The separation between two branes corresponds to giving mass to the scalars.

mode in position r^* is redshifted by[10]

$$E_{r\to\infty} = \left(\frac{g_{tt}(r^*)}{g_{tt}(\infty)}\right)^{1/2} E_{r^*} = \left(\frac{g_{tt}(r^*)}{-1}\right)^{1/2} E_{r^*} \tag{2.1.13}$$

and if we take the excitation to be near the brane $r \ll l$ we have

$$E_{r\to\infty} = \frac{r^*}{l} E_{r^*}. \tag{2.1.14}$$

So from this perspective we have *i*) free Type IIB Sugra in ten-dimensionl Minkowski space and *ii*) Type IIB Sugra on $AdS_5 \times S^5$. Such a limit must be taken so that it corresponds to the limit $\alpha' \to 0$ with U fixed (see (2.1.11)). In particular for a string excitation of energy $\sqrt{\alpha'}^{-1}$ this correspond to (2.1.14) fixed.

Since the two pairs of decoupled systems arose from two descriptions of the same object in different regimes, we reach the conclusion

Type IIB Sugra on $AdS_5 \times S^5 \equiv d = 4$ $\mathcal{N} = 4$ $SU(N)$ SYM on Minkowski space.

In the Sugra picture, the gauge theory lives in the original background. Hence, with respect to the near-brane $r \to 0$ metric (2.1.9), the field theory is related to the region $r \to \infty$ (or $z \to 0$ in the Poincaré patch).

The conformal symmetry of the field theory corresponds to the isometry of AdS_5, the field theory R-symmetry corresponds to the isometries of S^5 and, in addition to this, $AdS_5 \times S^5$ it can be shown to be a maximally supersymmetric space invariant under $PS(2,2|4)$ and hence the two sides have the same number of supersymmetries.

G. Parameters. To recap the parameters in the duality are related in this way:

$$g_{YM}^2 \sim g_s, \quad \left(\frac{l}{l_s}\right)^4 = 4\pi g_s N, \quad g_{YM}^2 N =: \lambda_t \tag{2.1.15}$$

where we also defined the 't Hooft coupling λ_t which naturally arises in large-N field theories, a limit which was suggestive of the connection between gauge theories and closed string amplitudes well before Maldacena's conjecture.

So the Supergravity approximation is valid when the gauge theory is non-perturbative and strongly coupled and vice versa. We have a strong/weak duality.

H. Field-operator map. As the global symmetries on both sides match, a map between CFT gauge-invariant operators transforming in certain irreducible representations of the superconformal algebra (or its bosonic subalgebra) and supergravity fields transforming in the same representation must be expected.

[10]To show that a low energy mode in the asymptotic region cannot penetrate the barrier we need to know the dependence of the cross section on the energy: $\sigma \propto E^3$.

Since the Sugra spacetime is a product space, such fields φ are functions of the coordinates $\{x\}$ on AdS_5 and $\{y\}$ on S^5. They can be Kaluza-Klein decomposed over S^5, namely expanded in spherical harmonics on S^5. Each resulting field $\phi(x)$ acquires a mass m which is given by the rank of the spherical harmonics, which in turn corresponds to the conformal dimension of CFT gauge invariant primary operators in the same representation of $SO(6)$.

The result of this involved analysis is tabulated in more thorough AdS/CFT references (see for example [91]). We only note that the energy-momentum tensor of the CFT ($\Delta = d$ in d dimensions) is mapped to the AdS metric fluctuations (whose mass is $m = 0$)

$$T_{ab} \longleftrightarrow h_{ab}. \tag{2.1.16}$$

Indeed, the rule for massive spin-2 fields and scalars is $m^2 l^2 = \Delta(\Delta - d)$ (hence the massless case is included) [19]. We also remark that the R-symmetry current J_a is related to gauge fields A_a in the Sugra theory, but we do not need more than this observation in the following

$$J_a \longleftrightarrow A_a. \tag{2.1.17}$$

The association between fields and operators is confirmed and put to work by the so called GPKW rule and the required procedure of holographic renormalization to be discussed from Section 2.3 onwards. Schematically we will simply write

$$\mathcal{O}_{(\Delta)} \longleftrightarrow \phi_{(m)} \tag{2.1.18}$$

with $\mathcal{O}_{(\Delta)}$ the field theory operator and $\phi_{(m)}$ the field in the gravity side, suppressing all other specifiers.

Wrap up. We have discussed the basic ideas behind the first example of Maldacena's conjecture. The strong form presented at the beginning is the extrapolation of the weak form to which we had to revert to make progress. The correspondence is strongly motivated by the symmetry arguments. We then look at such arguments again to move beyond this example.

2.2 Symmetry matching, extending the correspondence

Of all the discussion above, symmetry considerations remain. We saw that the superconformal symmetry of the field theory matches the superconformal symmetry of the gravity background solution. We take as a rule that both sides must share the same symmetries. All the other top-down realizations of AdS/CFT share this characteristic and it can, in fact, be generalised to the status of a rule to be followed when building bottom-up holographic dualities.

Looking closer at the previous example we can state that

1) The field theory conformal symmetry is realised as the isometry group of AdS,
2) The field theory R-symmetry is realised as the isometry group of the compact space.

We have written 2) mentioning a generic compact space rather than the sphere we got in the previous discussion. Indeed, there are many known examples of top-down dualities now where the sphere S^5 is substituted by a different sphere of appropriate dimensionality (i.e. as in the other examples discussed in [18]) or by a different compact space. Changing the compact space corresponds to changing the amount of supersymmetry. We can, for example, consider the so-called Sasaki-Einstein manifolds X^5 rather than the spheres S^5, or we can have other constructions involving quotients of the compact space (see for example [24] for more).

Indeed we can also extend further and formulate rule 2) without any reference to the other factor of the geometry:

2') Global symmetries of the field theory side are realised as gauge symmetries on the gravity theory in AdS, namely to each Noether current in the field theory there exists a corresponding gauge field in the AdS gravity theory.

Namely, if we are given a gravitational theory in AdS with some gauge fields and we know this rule we may guess the dual theory even if we don't know how these gauge fields appear on the gravity side. This is the spirit of bottom-up AdS/CFT. A posteriori, we may be lucky and find an appropriate description of the geometrical origin of the gauge fields. For example in our discussion above we understand (roughly speaking) the $SO(6)$ gauge fields as the metric modes of a 10-dimensional $AdS_5 \times S^5$ spacetime with an index on S^5.

However, what we need to appreciate for the rest of this thesis, is that rule 1), as stated, is misleading. Referring to the example of the previous section, the gravitational theory in $AdS_5 \times S^5$ is dynamical and the correspondence posits the identification of the Hilbert spaces on the two sides of the duality. Once conjectured the second, the first must follow. In our example, the first is evident from the construction. In particular, this implies that the gravitational system must only be asymptotically AdS. If it were not so we would miss the most interesting states in a gravity theory: black holes. As a consequence, we change rule 1) to

1') The field theory conformal symmetry is realised as the asymptotic symmetry group of asymptotically AdS spacetime.

This fact is important not only for spacetimes which evidently deviate from the AdS vacuum, such as AdS black holes but also for the vacuum itself. A pivotal example is the symmetry argument for AdS_3/CFT_2. As first discovered by Brown and Henneaux [92], the asymptotic symmetry algebra of AdS_3 is enhanced from the finite-dimensional isometry algebra $\mathfrak{so}(2,2)$ to (two copies of) the infinite-dimensional Virasorso algebra $\mathfrak{vir} \times \overline{\mathfrak{vir}}$. The latter is the conformal symmetry algebra in two dimensions, responsible for all the nice features of CFT_2 theories. The symmetries on the two sides thus match because of the symmetry enhancement phenomenon occurring in the bulk. It is called "enhancement" because the $\mathfrak{so}(2,2)$ isometries of the vacuum AdS_3 are not lost, but appear as the $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ subalgebra of the Virasoro algebra (in fact $\mathfrak{so}(2,2)$ split as stated). We will have more to say on asymptotic symmetries and their role in Section 2.3.1, in chapter 3 and they will be the focus of all of Part II.

Notice that $1')$ is a statement about the theory, not the state. Indeed, AdS/CFT provides the striking picture of black holes as thermal CFT states [93]. The scale set by the temperature breaks conformal invariance, but this is a statement about the state, not the theory. Hence, 1′) implies that a non-conformal theory cannot be dual to asymptotically AdS gravity.

For example, if we consider other Dp-branes constructions ($p \neq 3$) we land in the domain of the so-called generalised conformal structure, a term defined in [94, 95, 96]. The near-horizon geometry of such branes breaks the conformal symmetry because the solutions do not admit a constant dilaton [97, 98]. The dual theories possess a generalised conformal structure where the coupling in front of the action is dimensionful (rather than dimensionless). Precision holography for such configuration is studied in [99].

From now on when we speak of a gravity theory in AdS we always mean "asymptotically (locally) AdS".

The matching so far does not include gauge symmetries of the field theory. Except for the parameter N in the string side of the above top-down construction, there is no other gravitational manifestation that the field theory is an $SU(N)$ gauge theory. The only sensible requirement we can make is that

3) Only field theory gauge-invariant operators are associated with fields on the gravity side.

Ideally, in this journey leading to 1'), 2'), 3) we got rid of any reference to the string theory model from which we started and we are ready to state the general AdS/CFT statement considered true

Any (UV complete) quantum gravity theory in asymptotically locally AdS_{d+1} spacetimes is dual to an ordinary CFT_d without gravity

There are indeed a couple of beautiful arguments motivating AdS/CFT independently from string theory. None of them could have been suggested without knowing how the explicit string theory realization works and indeed, from the perspective of the previous section, they are rooted in the large N and large 't Hooft coupling limits.

One of the two arguments is more conceptual and the other more practical. The first is due to Polchinski and Horowitz [100] and involves avoiding the Weinberg-Witten theorem[11] by going to a dimension higher and using the renormalization group flow to identify the additional dimension with the energy scale of the field theory. The second [102, 103] proceeds first by quantizing fields in a fixed AdS background and then studying perturbative metric fluctuations around this background. Matter and gauge fields in AdS lead to the construction of boundary operators which transform conformally under the isometries of AdS. The boundary operators thus satisfy operator product expansions typical of conformal field theories. The metric fluctuations around AdS lead to a boundary stress tensor which completes the boundary CFT data.

By construction, the second approach is tied to Einstein's general relativity (large N and large 't

[11] In [100] the theorem is considered in its implication that the graviton cannot be a composite object in a Poincaré covariant theory. The theorem is proved in [101].

Hooft limit on the field theory side). As such, it helps in listing the properties a CFT must have to be dual to a quantum gravity theory with Einstein's general relativity as its semiclassical limit. It helps in motivating, but is not sufficient to prove, the following conjecture[12]

> Any strongly coupled large N CFT where all single-trace operators, except the stress tensor, have parametrically large Δ, has the stress tensor correlation functions $\langle T \dots T\rangle$ predicted by pure general relativity in AdS.

We have not met correlators in our brief account, we will move to them in the next section. For the moment let us explain this claim from our example. Recall that Δ of the operator $\mathcal{O}$ is related to the mass of the dual field ϕ by $m \approx (\Delta - d)/l$ and for the stress tensor $\Delta = d$. In our example, $\lambda^{1/4} \sim l/l_s$ but the mass m of a string state is proportional to l_s^{-1}, so we get $\lambda_t^{1/4} \sim lm$. Thus correctly, at strong coupling $\lambda_t >> 1$, the field theory operators with small scaling dimensions are dual to massless string states, which are the supergravity fields. In the large N and large 't Hooft limit, the stress-energy momentum sector is that of general relativity in AdS.

2.3 AdS/CFT at work

To move beyond the kinematical matching we need a rule relating the dynamics on both sides: a *master rule*. Such was proposed independently - roughly at the same time - by E. Witten [19] and S.S. Gubser, I.R. Klebanov, A.M. Polyakov [104], and for this known as *GPKW rule*.

This rule posits the equality of partition functions and hence is a statement of the dynamical equivalence of the two sides. Stated in this way, however, it provides a precise computational dictionary between the two sides. Before discussing the motivations for this ansatz, we note that usually, we work with Euclideanized versions of path integrals to avoid well-known issues in Lorentzian time. Here, following the historical path we do the same. Real-time formulations of the GPKW rule and all that follows has been given in [105].

The dynamics of a CFT is described by the behaviour of connected correlation functions of operators. Given an operator $\mathcal{O}$, the connected correlation function $\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n)\rangle_c$ is computed by functional differentiating the generating functional $W[\phi_0]$

$$W[\phi_0]_{cft} := \log Z[\phi_0]_{cft} := \log \langle e^{-\int d^d x \phi_0(x)\mathcal{O}(x)}\rangle \tag{2.3.1}$$

n times with respect to the arbitrary source ϕ_0 to which $\mathcal{O}$ couple.

The proposal in [104, 19] consists of computing the connected correlation functions of the CFT in the large-N and large-λ limit using the extremum of the classical on-shell action of the supposedly dual supergravity theory. The classical field equations are second-order and thus the classical solution ϕ_* must satisfy some boundary conditions.

[12]This is taken from the lectures [102] and we refer the reader there for more discussions and references.

We naturally have a boundary in our problem because in the $z \to 0$ limit the metric (2.1.10), whose AdS factor we repeat here,

$$ds^2_{AdS_5} = \frac{l^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \tag{2.3.2}$$

is singular and cannot be continued past.

In fact, the well-known conformal compactification method of Penrose allows treating this limit (see Appendix A). We can just perform a conformal rescaling of the metric by z^2/l^2

$$ds^2_{new} = \frac{z^2}{l^2}ds^2 \tag{2.3.3}$$

and the new line element defines a metric on a compact manifold with a boundary at $z = 0$. The metric at $z = 0$ is well defined and Minkowski. However, the choice above is not unique. We only need a function Ω so that

$$ds^2_{new} = \Omega^2 ds^2 \tag{2.3.4}$$

define a metric on a compact manifold. Hence, we take Ω to be positive in the compact manifold and with a first-order zero on the boundary. It is clear that any Ω' related to Ω by

$$\Omega' = e^\sigma \Omega, \tag{2.3.5}$$

with σ smooth but otherwise arbitrary, is an equivalently good compactification. Thus, the bulk metric induces a well-defined conformal class of metrics the boundary, within which a metric is picked by a choice of Ω. This is the basis of the definition of Asymptotically locally AdS spacetimes ($AlAdS$) (Appendix A).

This construction gives a precise meaning to the idea that the field theory is associated with the boundary of the spacetime [19]. Having a conformal class rather than a metric induced on the boundary is crucial for the boundary theory to be conformal.

According to this picture the source $\phi_{(0)}$ of the CFT operator $\mathcal{O}$ must be set equal to the boundary condition of the bulk classical field ϕ_* dual to $\mathcal{O}$. We thus have

$$S_{sugra}(\phi_*) = -W[\phi_0] \quad \text{provided that } \phi_*|_{\partial AdS} = \phi_{(0)}. \tag{2.3.6}$$

This is valid for any field in the correspondence, including the metric on which we mainly focus next. Thus these formulas must be understood more generally with ϕ a collection of fields indexed such that any of them correspond to a boundary operator.

The master formula (2.3.6) is the expression of Maldacena's weak conjecture but it can be soon generalised to the general form of the conjecture by simply posing

$$Z_{\mathrm{qg}}[\phi] = \int_{\phi|_{\partial AdS}=\phi_{(0)}} D\phi e^{-S_{qg}(\phi)} = Z_{cft}[\phi_{(0)}] \tag{2.3.7}$$

where we consider the full quantum gravitational (qg) theory. This statement is only formal as Z_{qg} is not known, even in the case in which $qg = string$, corresponding for example to the strong form of Maldacena's conjecture. In this case for example, (2.3.6) can be seen as the leading order in the large-N, large-λ expansion (indeed EH action scales as N^2) and the subleading corrections are captured by perturbative expansions of Z_{string}.

The holographic bound and the IR/UV connection. Are we dealing with a correspondence that manifestly satisfies the holographic bound (1.0.9)?

Witten and Susskind [106] answered positively this question using for the $AdS_5 \times S^5/\mathcal{N} = 4SYM$ a simple counting as in chapter 1 which can be easily extended to any other example of AdS/CFT. The only obstruction to the counting is that the number of degrees of freedom in a CFT is infinite as there is no ultraviolet (UV) cutoff scale. The area of the boundary of AdS is infinite as well, an infrared (IR) divergence. The two divergences are, as suggested in [106], in an exact correspondence so that a regularization procedure can be performed on both sides at the same time and the end result of the counting is a perfect agreement with the holographic bound.

The suggested UV/IR connection is the ingredient we use next to give a computational meaning to the GPKW rule and, in turn, a proper understanding of such connection.

2.3.1 Holographic renormalization

The GPKW rule is only a formal statement because both sides of the equality are infinite (so in this respect it is also trivial!). As said, the field theory side is infinite because of ultraviolet (UV) divergences. The gravity side is divergent because the rule prescribes to evaluate L_{grav} on-shell and to integrate against the infinite spacetime volume. It is an infrared (IR) divergence.

The gravity action contains all fields of the theory. In the following, we only focus on the pure gravity sector

$$L = \frac{1}{2\kappa^2}(R[g] - 2\Lambda) + \dots , \tag{2.3.8}$$

where we used dots to stress that the gravity sector may not be described only by Einstein-Hilbert theory (the only term shown above).

Holographic renormalization is a method to renormalize the divergences on the gravitational side, in a way which is consistent with the field theory renormalization. The method thus endows the GPKW rule with a computational meaning. It was developed in [107, 108] and reviewed in [42].

The method is inspired by the field theory renormalization, namely by adding local counterterms to the action that cancel the divergence. On the bulk side we have to

1. Find a convenient coordinate system to perform the asymptotic/boundary analysis, in particular, we will have a coordinate z labelling timelike hypersurfaces such that the boundary is selected by a particular z^* and there is a clear definition of boundary metric;
2. Solve asymptotically the equations of motion for any field in the theory;

3. Evaluate the on-shell action regulated on a finite volume enclosed by one of the above hypersurfaces, we denote the regulated action as S_{reg};
4. Isolate the diverging pieces in the radial coordinate: $S_{reg} = S_{div} + S_{fin}$, where S_{div} is the diverging piece when $z \to z^*$;
5. Rewrite the diverging pieces covariantly in terms of the induced metric on the regulated boundary: $S_{div}[h_{(0)}] \to S_{div}[\gamma]$;
6. Subtract these terms from the regulated action by adding a covariant counterterm action S_{ct} appropriately built from the diverging pieces. The regulator is then removed in the limit in which $z \to z^*$ and the renormalized action is defined as

$$S_{ren} = \lim_{z \to z^*} (S_{reg} + S_{ct}) = \lim_{z \to z^*} S_{fin}. \tag{2.3.9}$$

Notice that in the last step S_{reg} is written as it was first obtained, not expanded as $S_{div} + S_{fin}$. Notice also that the choice of subtracting exactly the divergent pieces corresponds to the minimal subtraction scheme in field theory. Other finite, covariant counterterms can be added, however.

The stated is the Lagrangian approach to holographic renormalization. It is conceptually straightforward, but technically challenging and tedious because of step 4. Equivalent Hamiltonian and Hamilton-Jacobi approaches have been defined that smoothen the above procedure. The reader is referred to [109, 110].

Whatever way is chosen to renormalize, one of the two independent solutions of the field equations[13] in the asymptotic expansion is associated with the source of the dual field operator and the other is associated with the exact one-point function of such operator. If $\phi_{(0)}$ represents the boundary value of (any) field ϕ sourcing the dual operator $\mathcal{O}$, the exact one-point function is computed as

$$\langle \mathcal{O}(x) \rangle_{\phi_{(0)}} = \frac{1}{\sqrt{h_{(0)}}} \frac{\delta S_{ren}}{\delta \phi_{(0)}(x)} \sim \text{other independent solution at order } p \; \phi_{(p)}(x) \tag{2.3.10}$$

where p changes according to the type of field under consideration and we have introduced the notation $h_{(0)}$ to denote the boundary metric (plus sign because of signature). We need the factor of $\sqrt{h_{(0)}}^{-1}$ whenever we have a non-trivial background geometry for the field theory. In particular, when we are interested in correlation functions of the energy momentum tensor, $\phi_{(0)}$ is the boundary metric.

When d is even, the asymptotic expansion contains a logarithmic term at order d which is associated with the field theory conformal anomaly. We exemplify these comments using pure gravity in the following section.

Notice that in (2.3.10) we do not turn off the source after computing the correlator (we can always do so) - hence the subscript in $\langle . \rangle$ - because we can now define the n-point correlator

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle_{\phi_{(0)}} \tag{2.3.11}$$

by taking $n-1$ functional derivatives of the right hand side of (2.3.10).

[13]We have in mind the field equations for scalars or Maxwell fields in a background geometry or the Einstein field equations. For actions with higher derivative terms, the equations are of higher order.

2.3.1.1 Pure gravity asymptotics

With the Einstein-Hilbert action L_{EH}, we have another problem unrelated to divergences. It does not give a well-defined variational principle for g even within a finite region of spacetime because R contains second derivatives of the field and hence any boundary term contains both variations of g and ∂g. To have a well-defined variational problem we cannot, in general, fix both the field and its derivatives on the boundary (see [111]), and we need to obtain an action with only first derivatives of g so that the variational problem will be defined with Dirichlet boundary conditions, i.e. the variation of the field vanishing at the boundary.

For a timelike boundary ∂V of a bounded region V, the correct boundary term to be added to the Einstein-Hilbert action is the well-known Gibbons-Hawking-York (GHY) term[14], so that we are led to consider

$$2\kappa^2 S = \int_V d^d x\sqrt{-g}L_{EH} + \int_{\partial V} d^{d-1}x\sqrt{\gamma}\,2K, \tag{2.3.12}$$

where K is the trace of the extrinsic curvature $K_{\mu\nu} = h^\rho_\mu \nabla_\rho n_\nu$ of the boundary, whose induced metric is γ_{ij}.

Asymptotic solutions of the Einstein's equations with Λ can be conveniently expressed in the so called Fefferman-Graham gauge [112], where the metric near the boundary defined by $z = 0$ takes the form

$$ds^2 = \frac{l^2}{z^2}(dz^2 + h_{ij}(z,x)dx^i dx^j), \quad i,j = 0,\dots d \tag{2.3.13}$$

with $h_{ij}(z,x)$ admitting an expansion in non-negative powers of z at least to some order

$$h_{ij}(z,x) = h_{(0)ij}(x) + z h_{(1)ij} + \dots \tag{2.3.14}$$

since it must be extended to $z = 0$ by definition. Notice that the choice of z corresponds to the choice of conformal frame. The coefficients of the expansion are found by solving Einstein's equations. In general the following asymptotic forms hold[15]. For $m \in \mathbb{N}$, when the space dimension d is odd

$$h_{ij}(r,x) = h_{(0)ij}(x) + \sum_{m=1}^{<d/2} z^{2m} h_{(2m)ij}[h_{(0)}] + z^d h_{(d)ij}(x) + \dots. \tag{2.3.15}$$

For even space dimension d, the expansion is polyhomogeneous from z^d onwards

$$h_{ij}(r,x) = h_{(0)ij}(x) + \sum_{m=1}^{<d/2} z^{2m} h_{(2m)ij}[h_{(0)}] + z^d\left(h_{(d)ij}(x) + \log z^2 \mathtt{h}_{(d)ij}[h_{(0)}]\right) + \dots. \tag{2.3.16}$$

In both cases we have singled out the data $h_{(0)ij}$ and $h_{(d)ij}$ in terms of which all the others are determined. Orders greater than d depend on both, orders less than d depend only on the first. Ellipses denote higer order terms which are irrelevant for practically all purposes.

The leading order $h_{(0)ij}$ is totally free, while

in odd d: $h_{(d)ij}$ is constrained to be traceless and covariantly conserved with respect to the

[14] The opposite sign if the boundary is spacelike, but we are not interested in that case.

[15] For AdS_3 the expansion is known exactly and terminates at order z^4 without logarithmic terms.

derivative compatible with $h_{(0)}$;

in even d: the trace part of $h_{(d)ij}$ is determined by $h_{(0)ij}$, the tracefree part is undetermined; $\mathtt{h}_{(d)ij}$ is determined by $h_{(0)ij}$ and is constrained to be traceless and covariantly conserved with respect to the derivative compatible with $h_{(0)}$.

As it is instructive and useful also to compare with the Bondi-Sachs analysis of asymptotically flat spacetimes, in Appendix A.1.2 we briefly review the iterative differentiation procedure from which the above expansion is derived.

With this analysis we have performed steps 1. and 2. for the metric field. Step 3. requires evaluating (2.3.12) where ∂V is selected by $z = \epsilon$. We have

$$\frac{2\kappa^2}{l^{d-1}} S_{reg} = -\int_{z\geq\epsilon} \mathrm{d}^d x\, \mathrm{d}z \sqrt{h}(2d) z^{-(d+1)} - \int \mathrm{d}^d x \left[2z^{-d}(z\partial_z - d)\sqrt{h} \right] \Big|_{z=\epsilon} \tag{2.3.17}$$

because on-shell we have

$$R - 2\Lambda = -\frac{2d}{l^2} \tag{2.3.18}$$

and we have used[16] $\gamma_{ij} = (l/\epsilon)^2 h_{ij}$.

Done this, and moving to step 4. we isolate from $S_{fin} = O(\epsilon)$ the divergent part in (2.3.17) splitting between d even (e) and odd (o)

$$\frac{2\kappa^2}{l^{d-1}} S^e_{div} = -\int \mathrm{d}^d x \sqrt{h_{(0)}} \left[\epsilon^{-d} a_{(0)} + \epsilon^{-d+2} a_{(2)} + \cdots + \epsilon^{-2} a_{(d-2)} - \mathtt{a}_{(d)} \log \epsilon \right], \tag{2.3.19}$$

$$\frac{2\kappa^2}{l^{d-1}} S^o_{div} = -\int \mathrm{d}^d x \sqrt{h_{(0)}} \left[\epsilon^{-d} a_{(0)} + \epsilon^{-d+2} a_{(2)} + \cdots + \epsilon^{-1} a_{(d-1)} \right]. \tag{2.3.20}$$

We should now invert the metric expansion to write $S^{e/o}_{div}$ in such a way that covariance with respect to the boundary is manifest. Detailed expressions are found in [108]. The resulting counterterm Lagrangian (for space reasons) up to $d = 6$ reads

$$\sqrt{\gamma} L_{ct} = -\sqrt{\gamma} \left[\frac{2(1-d)}{l^2} + \frac{R[\gamma]}{d-2} - \frac{1}{(d-4)(d-2)^2} \left(R[\gamma]_{ij} R[\gamma]^{ij} - \frac{d R[\gamma]^2}{4(d-1)} \right) - \log \epsilon\, \mathtt{a}_{(d)} \right] \tag{2.3.21}$$

where the formula is understood as containing only divergent counterterms in each dimension. So, for even $d = 2n$ only the first n are included plus the logarithmic term, while for odd $d = 2n + 1$ the first $n + 1$ are included.

2.3.1.2 The CFT stress-tensor and the Weyl anomaly

Notice that the data at order d have all the properties a CFT energy momentum tensor should have. This confirms the general outcome stated in (2.3.10). Indeed let us summarise the steps to

[16] The signs are compatible with an Euclidean h_{ij}, but it is not so important for highlighting the general procedure.

this identification. We have

$$\langle T_{ij}\rangle = \frac{2}{\sqrt{h_{(0)}}}\frac{\delta S_{ren}}{\delta h^{ij}_{(0)}} = \lim_{\epsilon\to 0}\left(\frac{2}{\sqrt{h(\epsilon,x)}}\frac{\delta S_{fin}}{\delta h^{ij}(\epsilon,x)}\right). \tag{2.3.22}$$

We need to write everything in terms of the regulated boundary quantities, so we have

$$\langle T_{ij}\rangle = \lim_{\epsilon\to 0}\left(\frac{l^{d-2}}{\epsilon^{d-2}}T_{ij}[\gamma]\right) \tag{2.3.23}$$

where

$$T_{ij}[\gamma] = \frac{2}{\sqrt{g}}\frac{\delta S_{fin}}{\delta\gamma^{ij}} = T^{reg}_{ij}[\gamma] + T^{ct}_{ij}[\gamma]. \tag{2.3.24}$$

In the above we have split the part coming from the variation of the regulated action and that coming from the counterterms.

An explicit computation, whose details are found in [108], gives

$$\langle T_{ij}\rangle = \frac{dl^{d-1}}{2\kappa^2}h_{(d)ij} + X_{ij}[h_{(0)ij}], \quad X_{ij}\equiv 0 \text{ if } d \text{ odd}. \tag{2.3.25}$$

The tensor X_{ij} is locally constructed from the leading coefficients compared with $h_{(d)ij}$ ang hence it ultimately depends only on $h_{(0)ij}$. From the comments made in the previous section, we see that (2.3.25) is manifestly covariantly conserved with respect to $g_{(0)ij}$, as due. Indeed, this property is self-evident from construction and we will start our discussion in the next chapter from this observation.

The energy momentum tensor of a classical CFT is traceless, but upon quantization the conformal invariance can be broken. This is reflected in the non vanishing vev of the trace $T = T^i_j$ of the energy momentum tensor. General theorems state that in such a case

$$\langle T_{CFT}\rangle = \frac{1}{(4\pi)^{d/2}}\left(\sum_\iota c^\iota I^\iota_d + (-1)^{\frac{d}{2}}a_d E_d\right) \tag{2.3.26}$$

where E_d is the Euler density in d dimensions and I^ι_d are Weyl invariants, whose number depend on the dimension d and a_d, c^ι_d are theory dependent coefficients for which there exist perturbative formulas in terms of the number of each type of field in the theory.

For example in $d = 2$ there are no conformal invariants of the right dimensions and the only possibility is E_2 so that

$$\langle T_{CFT}\rangle = -\frac{c}{12}R. \tag{2.3.27}$$

In $d = 4$ we have only one Weyl invariant (the square of the Weyl tensor)

$$I_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2 \tag{2.3.28}$$

and E_4 is[17]

$$E_4 = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2. \tag{2.3.29}$$

[17]We are only considering the effect of the geometry. If the theory is coupled to gauge fields for example we also get terms depending on the gauge curvature.

We have only listed a couple of famous cases relevant for theories dual to AdS_3 and AdS_5, but in fact, the expression (2.3.26) is only valid for d even because there are no invariants constructed from the Riemann tensor and its derivatives in d odd[18]. We can see this from our holographic expression of the energy-momentum tensor. If we take the trace of (2.3.25) we get

$$d \text{ odd} : \langle T \rangle = 0 \tag{2.3.30}$$

$$d \text{ even} : \langle T \rangle = \frac{d l^{d-1}}{2\kappa^2} h_{(d)} + X. \tag{2.3.31}$$

The first result is immediately obtained as a consequence of our asymptotic analysis. We have thus another partial check that the holographic stress-energy tensor is the stress tensor of the dual CFT. To conclude the proof we need to check that (2.3.31) is non-vanishing and that it exactly reproduces the result (2.3.26). Reproducing (2.3.26) means that the bulk computation must give the same universal structure, and if the gravity theory is supposed to be dual to a known CFT (as in the first example treated in this chapter), the coefficients a_d and c_d^ι computed gravitationally must exactly match those of the CFT. As a byproduct, the robustness of the holographic dictionary we have partially discussed in the chapter implies that given a bulk gravitational theory we can perform this computation and establish the properties of the unknown dual CFT.

To check that (2.3.31) is non vanishing we need to revert to the explicit expressions given in [108]. There is however a very easy argument relating $\langle T \rangle$ with the coefficient $a_{(d)}$ (not to be confused with a_d) in (2.3.19) which we now present. The outcome is

$$\langle T \rangle = -\frac{l^{d-1}}{\kappa^2} \mathsf{a}_{(d)}, \quad d \text{ even}. \tag{2.3.32}$$

The vev of T appears into the variation of the action under a Weyl transformation. If $h_{(0)ij}$ is the metric on the d-dimensional spacetime on which the CFT is defined and $\delta_\sigma h_{(0)ij} = 2\sigma(x) h_{(0)ij}$ is a Weyl transformation, we have

$$\delta_\sigma W_{CFT} = \frac{1}{2} \int \sqrt{h_{(0)}} \, \langle T^{ij}_{CFT} \rangle \, \delta_\sigma h_{(0)ij} = \int \sqrt{h_{(0)}} \, \langle T_{CFT} \rangle \, \sigma \tag{2.3.33}$$

showing that the trace of the stress-energy tensor is related to the anomaly.

From the holographic dictionary, we know that the above is related to the variation of the supergravity action, indeed the above is roughly speaking just (2.3.22) written from the field theory point of you. However, from the bulk perspective we explicitly need a variation under a bulk diffeomorphism which reduces to a Weyl transformation on the boundary.

Since we are working in a particular gauge it is also sensible to require that the gauge is not broken by such a diffeomorphism. Observing (2.3.13) we see that this corresponds to requiring

$$\mathfrak{L}_\xi g_{zz} = 0 = \mathfrak{L}_\xi g_{zi} \tag{2.3.34}$$

[18]Strictly speaking we are considering field theories without boundaries, otherwise the boundary induce anomalies also in odd dimensions (see for example [113]).

where $\mathfrak{L}_\xi$ is the Lie derivative along ξ. The solution of these equations is

$$\xi^z(x,z) = 2z\sigma(x), \tag{2.3.35}$$

$$\xi^i(z,x) = y^i(x) - \frac{1}{2}\sigma(x)\int_0^z \mathrm{d}z' z' h^{ij}(x,z'). \tag{2.3.36}$$

Under these transformations, the remaining components of the metric transform as

$$\delta g_{ij} = \frac{l^2}{z^2}\delta h_{ij} \Rightarrow \delta h_{ij} = 2\sigma h_{(0)ij} + \mathfrak{L}_y h_{(0)ij} + O(z^2). \tag{2.3.37}$$

At the leading order, the vector field y generates diffeomorphisms of $h_{(0)}$ and σ a Weyl rescaling. As desired, we have found the bulk diffeomorphisms which preserve the gauge and correctly transform conformally the boundary metric. These diffeomorphisms are called PBH, after Penrose, Brown and Henneaux. They constitute the first example of asymptotic symmetry group we meet in this dissertation.

We are now in position to check how $S_{reg} = S_{div} + S_{fin}$ varies under a PBH transformation. Following the argument of [107], we can first consider a PBH with a constant σ, under which S_{reg} is invariant. So the variations of S_{div} and S_{fin} must compensate. Due to the structure of the counterterm $S_{ct} = -S_{div}$ it is not so difficult to convince oneself that in odd d we have $S_{ct} = 0$. Indeed each term is covariant by construction. However, for even d, S_{ct} contains the logarithmically divergent piece which transforms by a shift. Thus we have

$$\delta S_{fin} = \frac{l^{d-1}}{2\kappa^2}\int \mathrm{d}^d x\sqrt{h_{(0)}}\delta\sigma 2\mathcal{A}, \tag{2.3.38}$$

with

$$\mathcal{A} = 0 \quad d \text{ odd}, \quad \mathcal{A} = -\mathsf{a}_{(d)} \quad d \text{ even}. \tag{2.3.39}$$

and $\mathcal{A}$ evidently corresponds to $\langle T\rangle$ as we wanted to check.

The explicit computation indeed shows that the holographic anomaly can be cast in the general form (2.3.26) according to the number of dimensions and the coefficients can be read off. In the gravitational computation, they will depend upon the $(d+1)$-dimensional Newton constant and the length scale as

$$a_d \sim \frac{l^{d-1}}{G^{(d+1)}} \tag{2.3.40}$$

and similarly for c_d. Upon replacing N, α' and mapping back to the field theory parameter, we can read off the field theory coefficients.

The analysis for $d = 4$ thus provides a dynamical check of the example of AdS/CFT we have discussed because the gravitational computation gives

$$a_4 = \frac{N^2}{4} \tag{2.3.41}$$

which agree with the field theory computation

$$a_4 = \frac{N^2-1}{4} \tag{2.3.42}$$

in the large-N limit. Actually, this result is more than a check. Indeed, the anomaly coefficients in field theory are usually computed at weak coupling. The strong/weak nature of the duality suggests that this gravitational computation provides the right result at strong coupling. So this is also an example of how we can use the gravity dual to learn about properties of the field theory. For example, if we consider a higher-derivative type of gravitational action, we would generically get $a_4 \neq c_4$, so that we learn that the dual theory cannot be $\mathcal{N} = 4$ SYM, but some other superconformal theory. Another example consists of those gravitational theories with compact spaces $X \neq S^5$. Since the mapping of the anomalies on both sides requires passing from G^{d+1} to the Newton's constant of the theory compactified on X, by dividing its volume, the anomaly coefficients will differ from those obtained here, but still they will be equal each other.

In the next section, we complete this discussion emphasising the consequences of this analysis for the well-posedness of the bulk variational principle and provide an example (according to this narration) of how the field theory teaches us something about the physical properties of the gravity dual.

CHAPTER 3

Beyond AdS/CFT: asymptotics, Killings and charges

We complete our brief discussion of the fundamentals of AdS/CFT by discussing the role of holographic renormalization in the problem of gravitational charges and we introduce the covariant phase space formalism. In this way, we bridge the previous chapter to the next part.

3.1 A different look at holographic renormalization: variational principle

As we have seen in the previous chapter, the holographic computation of the expectation value of the field theory energy-momentum tensor involves a functional differentiation of the renormalized action with respect to $h_{(0)}$ which boils down to the sum of T^{reg} and T^{ct}, both written with respect to the induced metric γ_{ij} on the regulator surface.

In gravitational literature $T^{reg}_{ij}[\gamma]$, given explicitly by

$$T^{reg}_{ij} = \frac{1}{\kappa^2}(K_{ij} - K\gamma_{ij}), \tag{3.1.1}$$

is known as the Brown-York tensor, after J.D. Brown and J. W. York [114]. It gives a definition of energy-momentum of a gravitational field in a spatially bounded region and was derived adapting the Hamilton-Jacobi equation for the energy of a mechanical system,

$$H = -\partial_t \mathcal{S}, \quad p = \partial_q \mathcal{S}, \tag{3.1.2}$$

where $\mathcal{S}$ is the Hamilton's principal function.

Let us recall that the Hamilton-Jacobi formalism is based on the idea that we can solve the dynamics for a canonical pair $(q(t), p(t))$ by performing a canonical transformation in which the new canonical variables (Q, P) are time independent. This process is accomplished by $\mathcal{S}$. A solution of Hamilton-Jacobi equations is provided by the classical action giving rise to a well-defined Dirichlet variational problem [115]. Or in other words, the difference between $\mathcal{S}$ and S is a constant provided that S is the appropriate one for the variational problem. Indeed, if we perform a variation of the action S where we vary both the endpoint time t_f as well as q at this extreme of the domain, we get

$$\Delta S = \partial_t S \Delta t + \partial_q S(\dot{q}\Delta t + \delta q) = (\partial_t S + \dot{q}\partial_q S)\Delta t + \partial_q S \delta q \tag{3.1.3}$$

where $\Delta := \delta/d_\lambda|_{\lambda=0}$ denotes the derivative with respect to the parameter of the curve joining t_f with $t_f + \delta t$ evaluated at $\lambda = 0$ and δ is a standard variation where t_f is kept fixed. We also have

$$\Delta \int_{t_i}^{t_f} \mathrm{d}t L = \int_{t_i}^{t_f} \mathrm{d}t(E\delta q) + (p\delta q)|^{t_f} + L\Delta t, \quad p = \partial_{\dot{q}} L \tag{3.1.4}$$

with E denoting the equations of motion and the boundary terms different from zero only at t_f (i.e. setting to zero the variations at t_i). So, onshell we get by comparison of (8.4.90) and (8.4.91) that S satisfies the Hamilton-Jacobi equations because $H = p\dot{q} - L$.

The idea of Brown and York was to notice that the Einstein-Hilbert action with the GHY term (2.3.12) provides the Hamiltonian principal function for the gravitational system in a cylindrical box (say) with boundary $\Sigma_i \cup \Sigma_f \cup \Sigma_{if}$, the role of the endpoints of q is played by the induced metric on the past Σ_i and future Σ_f boundary, while the metric γ_{ij} induced on Σ_{if} not only encodes the proper time interval between Σ_i and Σ_f, but all the metrical information about Σ_{if}. Thus, as long as we use the GHY boundary term,

$$T_{ij}^{reg} = \frac{2}{\sqrt{\gamma}} \frac{\delta S_{reg}}{\delta \gamma_{ij}} \tag{3.1.5}$$

is the stress-energy tensor for the gravitational system in such a finite volume.

We have already claimed that this tensor is covariantly conserved and we can show it by considering a variation of S_{reg} under a boundary diffeomorphism

$$\delta S = -\int_{\partial V} \mathrm{d}^d x \sqrt{\gamma} \xi_j D_i T_{reg}^{ij} \Rightarrow D_i T_{reg}^{ij} = 0, \tag{3.1.6}$$

where here D denotes the covariant derivative with respect to γ. We have assumed that the diffeomorphism has compact support on the boundary to discard boundary terms[1].

The tensor is thus a conserved current and gives rise to a conserved charge on the boundary for any boundary Killing vector ξ. If we pick a spacelike surface $\partial\Sigma$ in ∂V and a unit future pointing

[1] This is a simplified version of Brown-York arguments. This equation is a consequence of gravitational constraints and, in general, a matter stress-tensor contribution projected onto the surface is to be included.

normal to $\partial\Sigma$ we have

$$Q^{BY}_{\xi} := \int_{\partial\Sigma} \mathrm{d}^d x \sqrt{q} T^{reg}_{ij} t^i \xi^j \tag{3.1.7}$$

where q is the determinant of the metric on[2] $\partial\Sigma$. For example, if ξ is a time translation the above expression gives the gravitational mass in V.

From the discussion in the previous chapter, we see that via holographic renormalization we can give a meaning to the Brown-York charges for the whole spacetime, namely when the boundary surface is pushed to infinity. As first realised in [116], the expectation value of the dual CFT stress tensor is also the stress tensor for $AlAdS$ spacetimes.

We can thus compute physical quantities characterising $AlAdS$ via holographic renormalization. Remarkably this produces answers which sometimes disagree with those produced by other methods and are at first sight surprising[3]. For example, the mass of AdS_5 with an $\mathbb{R} \times S^3$ boundary is not vanishing [116]. The reason is almost obvious from the CFT picture: the CFT lives on this higher dimensional cylinder and thus there is a characteristic Casimir energy, which is related to the conformal anomaly and the central charge.

This result, directly stemming from the discussion in Section 2.3.1.2, is at odds with what one would expect in a general relativistic treatment of the energy of the spacetime. Indeed, AdS would be considered the natural zero-point energy of gravity with negative cosmological constant.

In the latter statement, the underlying idea is that - to avoid infinities and ambiguities - the properties of a spacetime can be fixed in comparison to the properties of another spacetime. This approach is fundamentally different from holographic renormalization, where no mention of a reference spacetime is made.

To appreciate the difference let us consider the Brown-York charge (3.1.7). It is not unambiguously defined because arbitrary functionals $S_0[\gamma]$ can be subtracted from S_{reg}, as usual in mechanics, as long as they preserve the validity of the variational problem. So we have

$$T^{reg,sub}_{ij} = T^{reg} - \frac{2}{\sqrt{\gamma}} \frac{\delta S_{(0)}}{\delta \gamma^{ij}}, \tag{3.1.8}$$

This ambiguity was usually interpreted as a choice of zero energy and comes to help when the boundary ∂V is pushed to infinity, where the charges are bound to diverge. In this case, the choice of S_0 is made so that $S_{reg} - S_0$ leads to zero energy for an appropriate background spacetime. For example, in asymptotically flat spacetimes the zero-point energy is chosen to be Minkowski spacetime[4] and in spacetimes with $\Lambda < 0$, as we said, AdS is the natural choice. The rough idea is that if the infinities arising from the region "spacetime-V" are the same as those in the region "reference spacetime-V", then they can be cancelled.

By comparison with the GHY term, we know that - in order to preserve the variational problem -

[2]The relationship between the metric on ∂V and that on its spacelike slice is given as usual by an ADM splitting which introduces q_{ab}, a lapse and a shift.

[3]An in-depth analysis of the differences and the conditions under which all the prescriptions agree is discussed in [117].

[4]This is indeed the case considered by Brown-York.

$S_{(0)}$ must have the form

$$S_{(0)} = \frac{1}{\kappa^2} \int_{\partial V} \sqrt{\gamma} K_0 \tag{3.1.9}$$

and K_0 is the trace of the extrinsic curvature of ∂V when this surface is isometrically embedded in the reference spacetime. The problem with this background subtraction method is that it is not unambiguously defined and it requires the boundary geometries of the two spacetimes to match exactly. Many examples of spacetimes are known where these issues are manifest [118, 119].

Holographic renormalization avoids these issues because the counterterms needed to renormalize the action are intrinsic to the boundary of the given *AlAdS* spacetime. The aforementioned example of the clash between the "general relativistic" expectation and that provided by holography is rooted in what we have just discussed.

As mentioned in the previous chapter, there is another formulation of holographic renormalization via Hamiltonian methods which simplify the computational task. Indeed, more than this, it highlights the fundamental root of the consistency of the method. Despite not addressing the details, we can now appreciate the point.

The counterterm action S_{ct} has a twofold purpose. Not only it eliminates the divergences in S_{reg} when ∂V is sent to infinity, but it also provides the conditions under which the variational problem is well-defined [120].

The mechanical analogy again helps [121, 122]. The situation we encounter when $\partial V \to \infty$ is similar as $t_f \to \infty$. In such a limit, $\Delta t_f = 0$ is meaningless because infinity is preserved by the addition of any arbitrary finite δt. Furthermore, as $t_f \to \infty$ we cannot fix $q(t_f)$ to some finite value because it would over-restrict the configuration space[5]. By the same argument, imposing $\delta q|_{t_f} = 0$ is not sensible. On any regulated surface we can impose that δq is such that it remains finite in the asymptotic limit, so to preserve $q_f \to \infty$. We see that

$$\delta S = \int \mathrm{d}t E \delta q + (p\delta q)|^{t_f} \tag{3.1.10}$$

does not vanish onshell for all variations δq that satisfy the boundary condition. We thus need to add a boundary term (on a regulated boundary defined by $t = t_r$) to the action

$$L \to L + \frac{dk}{dt}, \tag{3.1.11}$$

such that on-shell the new action is stationary under the new general boundary conditions when evaluated asymptotically. Thus we need to solve the Hamilton-Jacobi equations.

The remark is made in [121] that whenever we can impose $\lim_{t_r \to t_f} \delta q(t_{reg}) = 0$ then we can find k as $k = k(q(t))$. On the other hand, if the standard Dirichlet condition cannot be imposed, k will depend explicitly on time if we insist on locality in boundary derivatives[6].

This translates to the fundamental explanation of the anomaly computation summarised in Section 2.3.1.2. To make the statements precise we should recast holographic renormalization in an

[5]In [122] this is exemplified by a particle in the potential $V(q) = q^{-2}$.
[6]In [121] we find an explicit example only of the first case. See next footnote.

ADM form where the role of time is played by the coordinate z emanating from the boundary. But for our purposes we just need to think of z as t. PBH asymptotic transformations, which are of the form $\delta q(t_{reg}) \neq 0$, are broken by the regularized boundary ∂V at $z = \epsilon$. In Fefferman-Graham gauge we needed to pick a representative of the conformal class of the (asymptotic) conformal boundary, but the variational problem as $\partial V \to \infty$ must be defined in terms of the more general variations (PBH). This gives rise to a well-defined stationarity principle for odd boundary dimensions, where the anomaly vanishes and we can take the conformal class as a boundary condition. In even boundary dimensions the local counterterm (2.3.21) explicitly depends on ϵ (the time t in the mechanical model). As suggested in [121] we can probably eliminate the ϵ dependence but the price to pay is to break locality and this feature is unwanted from the holographic viewpoint[7]. Thus, the variational principle can be defined only if we pick a representative of the boundary metric and take that fixed[8].

3.1.1 Looking ahead: variational principles and holographic dualities

The problem of gravitational charges entered very late in our presentation. In the top-down example of chapter 2 we managed not to worry about it until the Weyl anomaly computation was summarised. However, this is only an artefact. Indeed, we could have stopped to discuss how such symmetries are formally realised in the bulk spacetime when we claimed that symmetries on both sides have to match (Section 2.2). This would have automatically lead to the consideration of charges and their algebras, exactly as we do in non-gravitational theories. In fact, we also needed to introduce very early the notions of (AdS) spacetime asymptotics and their asymptotic Killing vectors. As we further motivate in the next section, these are the essential building blocks to define the notions of energy and momentum in a generally covariant gravity theory and to give it the status of a physical theory.

In searching for holography beyond AdS, we do not usually have top-down realizations that somehow guarantee (due to the internal consistency of the top theory) that the two sides work consistently. It is thus imperative to check that the supposedly dual gravitational theory - with the given boundary conditions - makes sense, i.e. if sufficiently unambiguous notions of (finite) energy are supported.

The discussion of the previous section served the scope of highlighting the relevance of holographic renormalization in this direction. A natural question arises then. How general is holographic renormalization?

The procedure of adding appropriate terms to the action with the twofold aim of cancelling divergences and providing a well-defined variational problem can be motivated on general grounds, but the requirement that counterterms be local and covariant is entirely inspired by field theory: as we said repeatedly, the way IR divergences are cancelled in the asymptotically locally AdS spacetime reflects the general procedures used to cancel field theoretical UV divergences. In AdS/CFT the

[7]In any case, physics should not change. For example, the potential $V(q) = q^{-2}$ is well-known in quantum mechanics to be anomalous. In [122] a counterterm is given which manifestly depends on t. I am not aware of any treatment of quantum mechanical anomalies from this point of view.

[8]There are other cases to be taken into account [120].

match between field theory divergences and divergences in the gravitational action is given by the UV/IR connection.

Several arguments show that locality of the counterterms is hard to extend beyond *AlAdS*. To be precise, variants of the methods we have spelt can be found (amid technical difficulties) for gravitational theories with timelike boundaries even when they are not *AlAdS* and, as we have mentioned in the Introduction, for the spacelike boundary relevant for *dS* holography. Among the non-*AlAdS* holographic dualities with timelike boundary, we shall cite the non-relativistic Schrödinger [123, 124] and Lifshitz [125, 126] holography, where the boundary field theory is a non-relativistic version of CFT based on either the Schrödinger or the Lifshitz symmetry. The bulk spacetime can be realised either as a relativistic spacetime solution of Einstein's equations with $\Lambda < 0$ (or Einstein-Maxwell [126]) or as a non-relativistic space described by a Newton-Cartan structure [127, 128]. In both cases the boundary cannot be the same as *AlAdS* because the boundary space and time must scale differently $x^i \to \lambda x^i$, $t \to \lambda^z t$, with z a dynamical exponent parametrizing the different scaling. In some cases, these holographies have been obtained in top-down realizations (see [126]).

Various problems instead arise in extending holographic renormalization to asymptotically flat spacetimes. First of all, the Fefferman-Graham gauge loses its meaning in the $l \to \infty$ limit, an occurrence which is also signalled at the level of the counterterm action. In addition to this, solution of vacuum Einstein's equations in Gaussian normal coordinates shows that the iterative asymptotic equations are differential rather than algebraic and hence each order of the metric expansion is determined non-locally by the previous orders [42]. However, see [129] for a proposal circumventing the latter problem.

Notice that by construction, these comments apply to holographic renormalizations of spacelike infinity. However, from a dynamical point of view, the interest is focussed on null infinity, to which we dedicate all of Part II. Similar issues are found.

We have by now realised the truly fundamental role of gravitational charges in *any* realization of holography. We now turn to a more specific of gravitational charges, independently of holography. We will complete the definition of asymptotic Killing fields and the related definition of asymptotic symmetries (Section 3.2) and then discuss another formalism to define charges (Section 3.3): the covariant phase space (CPS) method. This method, as the Hamiltonian or Hamilton-Jacobi method, can be applied to any theory, not only gravity. The advantage with respect to the other two is that it is covariant, as the name suggests, and hence particularly suited to gravity.

The CPS method is extensively used, but to conclude the brief list of other holographic proposals, we can mention the so-called Kerr/CFT correspondence as an example of proposed holographic duality where, missing a precise knowledge of the CFT, the main results toward the holographic interpretation are obtained from the gravitational charges, especially in the CPS formalism [130]. The Kerr/CFT correspondence [131, 132] posits a holographic duality between the near horizon limit of extremal Kerr black holes - whose geometry is similar to *AdS* in some sectors - and some sort of conformal field theories. It is an attempt inspired by the famous Strominger-Vafa (extremal, supersymmetric) black hole microstate counting [133] and to extend it to more general classes of black holes.

There are several papers in which the various approaches to defining charges are compared and contrasted; see [117] for a discussion emphasising CPS and holographic renormalization. Holographic renormalization of *AlAdS* spacetimes was discussed within the CPS formalism in [120]. We will notice that many of the formulas of the previous sections will be recast differently in Section 3.3.

3.2 The problem of gravitational charges: general notions

When we speak of charges in a dynamical (and background-independent) gravity theory we refer both to the charges associated with the matter content and those intrinsic to the gravitational field.

The Noetherian procedure to associate a charge to a matter energy-momentum tensor $T^m_{\mu\nu}$, which is *covariantly* conserved as a consequence of Bianchi identities, provides conserved charges only along the isometries ξ of the spacetime. Given $T^m_{\mu\nu}$, the charge $Q^{(m)}_\xi$ is formally the same as (3.1.7) except that it is expressed in terms of quantities on a slice Σ in the spacetime. Conservation means that the charge does not depend on the slice on which it is defined. No isometries are present in generic spacetimes. In this case, the quantity $Q^{(m)}_\xi$ depends both on the slice and ξ and is thus ambiguous and not conserved. As put by Wald [56], the reason is that we miss the local information about the purely gravitational contributions.

Such information however cannot be obtained[9]. By the equivalence principle, we can always remove gravity effects locally. The Noetherian procedure is not applicable because there is no local gravitational energy-momentum tensor.

The Brown-York tensor resembles a Noetherian definition of conserved charge for the gravitational field, but it is not a local definition. T_{BY} is known as a quasi-local energy-momentum tensor. That it is quite different from $T^m_{\mu\nu}$ is evident from (3.1.5). The metric involved in (3.1.5) is the boundary metric at the boundary of a finite volume, which is kept fixed on classical solutions. $T^m_{\mu\nu}$ is defined from variations of the full spacetime metric instead.

Giving quasilocal definitions of gravitational charges is the best we can do. Their definition is however somewhat more involved than the global ones. Indeed several quasi-local quantities are built from global definitions[10]. In either cases, the relevant physical parameters associated with any given extended region of spacetime should be determined in order to proceed with the development of a definition.

A fundamental notion in the definition of global charges is that of isolated systems (see [55]). For spacetimes with $\Lambda = 0$, appropriate definitions of isolated systems were given in the Sixties in various steps. First Arnowitt, Deser and Misner (ADM) defined asymptotic conditions in spacelike directions [137]. Then Bondi, Metzner, van der Burgh [138, 57] (based on previous explorations by Trautman mainly [139]) and Sachs [58, 59] gave an appropriate metric based definition of

[9] At least, up to now there is no local notion and hopes are really low.

[10] The reader interested in quasilocal charges is referred to the review [134] and to [135, 136] for further original works.

isolated radiative spacetime, and then Penrose [140] - followed by many others, i.e. [55, 141] - cast their analysis in an entirely gauge independent geometrical way on the basis of conformal compactification methods. These approaches have been extended to the other values of Λ, and in particular, subsumed in the conformal and metrical definition of *AlAdS* spacetimes. In the main text we will only consider the metrical approach, a summary of the conformal approach is in Appendix A.

The situation before the 1960s was somewhat similar to that of a student in her first course on general relativity. Take for example the static, spherically symmetric, Schwarzschild metric. The vacuum Einstein equations are solved up to an integration constant appearing at order $1/r$ of an appropriate coordinate system. This constant is given the meaning of the mass of the spacetime only by reverting to a Newtonian approximation. Another example is the linear analysis of gravitational waves around Minkowski spacetimes. There were no ways to give a meaning to mass and energy within the full-fledged general relativity until the works of ADM and BMS.

It is well known that in his life Einstein himself doubted that gravity waves could exist at the full non-linear level. Indeed, the works of Bondi and collaborators moved from this problem: to give a nonlinear characterization of gravity waves and disperse any doubt on their existence. This was an energetic problem: do gravity waves carry energy? Paraphrasing Bondi, can you `boil your cup of tea` with that energy?

Once generic criteria were found so that spacetimes were classified according to their asymptotic properties, one of the most active research periods in general relativity started, during which the mathematical definition of a black hole horizon and the Penrose-Hawking singularity theorems were produced (see [56, 142]).

3.2.1 The metric based approach

In principle for any kind of spacetime, the general strategy behind the metric-based approach to its asymptote is to find an appropriate coordinate system with one of the coordinates, say r, parametrising the distance from the bulk of the spacetime. For example, Bondi-Sachs wished to describe the radiation emanated by a bounded source, so they adapted a coordinate system supposing that, as in exactly Minkowski space, the null waves travelled in null directions and r represented the distance from the centre.

Then the metric tensor admits an expansion of the form

$$g_{\mu\nu}(x,r) = \tilde{g}_{\mu\nu}(x,r) + O((r)^{-m_{\mu\nu}})\,, \tag{3.2.1}$$

as long as r is pushed far away from the bulk: infinity or the boundary. Here $\tilde{g}_{\mu\nu}$ is the asymptotic metric. The coefficients $m_{\mu\nu}$ are numbers that play a central role in defining the class of spacetimes which are asymptotically $\tilde{g}$; they determine their dynamics[11] (and are determined by the dynamical equations!) and the asymptotic symmetry group. We can naturally cast this discussion as a problem of defining a *phase space* $\mathcal{F}$ for solutions of the theory with given boundary conditions

[11] For example, the peeling property of asymptotically flat spacetimes depends on such coefficients [143].

$(\tilde{g}, m_{\mu\nu})$. For example, we wish to include Schwarzschild spacetime among the spacetimes which tend asymptotically to Minkowski, because a naive limit on its metric suggests so.

Usually, the above statement is followed by the following requirements, which guarantee a well-defined phase space:

1) the fall-off conditions must be not so strong that they exclude the existence of phenomena that could otherwise occur in the deep interior of the spacetime;
2) the fall-off conditions should be sufficiently strong that useful notions that characterize the system, such as total mass and radiated energy flux, are well defined.

While the first is undoubtedly desirable, the second is not so easy to defend. In fact, the *AlAdS* case exemplifies the opposite. We first solved the asymptotic equations and after that, we were concerned with divergences. After appropriately renormalizing, we get meaningful charges. However, 2) cannot be dismissed so easily also. Indeed, for flat spacetimes at null infinity (see comments in Section 3.1.1), the situation is drastically different from *AlAdS* and holographic renormalization in the way we described in the previous chapter is missing. This does not exclude that variants could exist. We thus take 2) to be a fundamental requirement, but to be considered a posteriori.

To conclude the section, we draw a last comparison with *AlAdS*. The Fefferman-Graham expansion is clearly of the form (3.2.1), where the boundary is taken at $z = 0$. However, in our discussion of the conformal boundary of *AlAdS*, we noticed that we do not have to fix the boundary metric. This can be argued to be the case of any conformal boundary (see Appendix A). So (3.2.1) should be taken *cum grano salis*: the boundary conditions should be ideally dictated by the variational problem so that they are mutually consistent (an arduous task for null infinity[12]).

Let us notice that despite having in mind the asymptotics of a spacetime, the notion of asymptotic expansion and asymptotic symmetries are also used to move towards regions of spacetimes which are not the true asymptote, such as black hole horizons [146].

3.2.2 The asymptotic symmetry group

As said, for a general spacetime there are no isometries, namely no Killing vectors. However, when a metric asymptotically tends to another metric which enjoys some symmetries, we are led to the definition of asymptotic Killing vector fields as the vector fields ξ that satisfy

$$\mathfrak{L}_\xi g_{\mu\nu} = O((r)^{-m_{\mu\nu}})\,, \tag{3.2.2}$$

namely the diffeomorphisms which preserve the asymptotic form of the metric up to the same non trivial order of the metric expansion. PBH transformations discussed in the context of *AlAdS* spacetimes provide an example.

Such diffeomorphisms are called *allowed diffeomorphisms*. The asymptotic symmetry algebra,

[12] Only variational problems with finite (not asymptotic) null boundaries have been discussed in literature (see [144, 145]) but they present conceptual issues have not been cleared up completely.

whose exponentiation gives a group (ASG), is the quotient of allowed diffeomorphisms with *trivial diffeomorphisms*:

$$\text{ASG} = \frac{\text{allowed symmetries}}{\text{trivial symmetries}}\,. \tag{3.2.3}$$

Elements of the group are the non-trivial asymptotic transformations that move a point g in the phase space to another point of the phase space.

Symmetries are characterised by their associated charges. Any charge, to be defined so, must generate the corresponding symmetry transformation on the phase space. This is captured by the brackets[13]

$$\{g, Q_\xi\} = \delta_\xi g\,, \tag{3.2.4}$$

where $\delta_\xi g$ is used interchangeably with $\mathfrak{L}_\xi g$. The trivial diffeomorphisms are those with zero associated charge, hence the name.

For example, the PBH transformations are all part of the asymptotic symmetry group because they are the residual transformations of the Fefferman-Graham gauge. As a counter-example we can take the original analysis of the asymptotics of AdS_3 performed by Brown and Henneaux [92]. They generically stated the falloff conditions on the metric written in global coordinates t, r, ϕ as

$$\begin{aligned} g_{tt} = -\frac{r^2}{l^2} + O(r^0), \quad g_{tr} = O(r^{-3}) = g_{r\phi}, \quad g_{t\phi} = O(r^0) \\ g_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad g_{\phi\phi} = r^2 + O(r^0) \end{aligned} \tag{3.2.5}$$

where $r \to \infty$. Not all the asymptotic Killing vectors of this metric have non-zero charge because some generate trivial changes of coordinates. They are those that reduce the metric to the Fefferman-Graham form, where the metric is block diagonal with the radial coordinate forming a separate block at any order of the expansion (so we can expect the trivial diffeomorphisms to be related to subleading orders of the AKV of the Brown-Henneaux metric).

A general lesson of these two subsections is that the dimensionality of the asymptotic symmetry group depends on the strength of falloff conditions. Given the same leading terms in the expansion, slower falloff conditions usually imply a larger group of allowed diffeomorphisms than faster falloffs. Hence, in the first case, the ASG may be larger unless it collapses (due to phase space consistency requirements) to the one corresponding to the stronger fallofs. This will be apparent in the next few chapters for the case of the asymptotic symmetry group of null infinity[14].

We conclude this section with three comments. Asymptotic symmetries may generically form algebroid and groupoid structures, rather than algebras and groups, because the generators may

[13]In a canonical approach they are Dirac brackets because gravity (as any other gauge theory) is a constrained system. In the covariant formalism we can instead use the less-known Pierls brackets [147].

[14]We can here exemplify the situation implied by the standard definitions of asymptotic flatness at null $\mathscr{I}$ and spacelike infinity i^0 in $d = 4$. At i^0, the principles 1) and 2) are satisfied by a large class of asymptotic conditions so that one would end up with different ASGs, according to which condition is chosen. For sufficiently strong falloff conditions at spatial infinity i^0, compatible with 1) and 2), the ASG is the Poincaré group, whereas for weaker choices one can get enlargements of the Poincaré group. As stressed in [73], there are sufficiently good reasons to impose at i^0 the strong asymptotic conditions leading to the Poincaré group. However, at null infinity this freedom is missing and the only sensible boundary conditions allowing for gravity waves in the bulk of the spacetime are so weak that the ASG is the infinite-dimensional BMS.

depend on the field themselves. This is the case for the extensions of BMS considered in [60]. While the discussion of this section was shaped in terms of gravity, it is easily modified to accommodate any gauge theory [44]. Furthermore, the notion of asymptotic symmetries and charges can be adapted to any situation in which there is a boundary to be approached from a bulk, for example, a horizon (see for example [146, 148]).

3.3 The Covariant Phase Space method

Any Lagrangian field theory on a d-dimensional[15] manifold M can be analysed via the covariant phase space formalism. It is extremely useful in relativistic, generally covariant, theories because it manifestly preserves covariance within a canonical approach: CPS is a covariant canonical (or covariant symplectic) formalism[16].

In the covariant canonical formalism, one can build Hamiltonians and use all the powerful tools of symplectic mechanics without having to choose a time foliation. The symplectic form is built out of the boundary terms in the variations of the action in the Lagrangian formalism. The core idea on which the covariant canonical formalism is based is to consistently identify the Hamiltonian phase space with the space of on-shell classical field configurations. This implies a conceptual change in perspective[17]. While each point in the Hamiltonian phase space is a set of initial conditions (here is where the covariance is broken), each point in the covariant phase space is a solution of the theory. In the Hamiltonian phase space, the Hamiltonian flow provides the time evolution of the initial conditions. In the CPS, the Hamiltonian flow maps a solution at time t to another solution at time t' whose degrees of freedom take the same values of those of the solution at time t.

The CPS formalism, as well as other fully Hamiltonian methods - such as the well known ADM formalism - is in certain sense a much powerful tool than the Hamilton-Jacobi method at the basis of the Brwon-York approach. In fact, it gives full information about the structure of the phase space of the gravitational theory, whereas the Brown-York approach can only define “the energy function” and study its properties but does not give any clue about the underlying phase space (see the review [134]).

Probably, the most iconic result of such an approach applied to Einstein-Hilbert gravity is the derivation of the first law of thermodynamics for a stationary black hole and, correspondingly, the entropy formula [151] known as Wald entropy. This was followed by the proposal that such general entropy expression, automatically consistent with the first law, should apply to any stationary black hole[18] in any local theory of gravity, including low energy actions of string theory [152, 14].

We will use ϕ to note all possible fields in the theory, including the metric. In a generally covariant

[15] Please be aware that from here and for the rest of the thesis we will use d to denote the bulk spacetime dimension, as opposed to the previous parts.

[16] In this section we will mostly use the notation of [149], where the issues of charges at null infinity were considered within the covariant phase space approach. See, however, [150] for a clear, early account of the method in generic settings and references therein for original literature.

[17] The following explanation is taken almost verbatim from [150].

[18] The case of non-stationary black holes and the consistency of Wald entropy with the second law is still an open problem.

theory, the integrand L in an action functional is a scalar density of weight +1 which need to be integrated against a volume form $\boldsymbol{\epsilon}$ to give a scalar. The form

$$\boldsymbol{L} = L\boldsymbol{\epsilon} \tag{3.3.1}$$

is a top form (d-dimensional) over the manifold M. $\boldsymbol{L}$ is just the Hodge dual of L: $\boldsymbol{L} = \star L$.

For example in Einstein-Hilbert gravity we have $L = R/(2\kappa^2)$. The variation of the action gives

$$2\kappa^2 \delta(R\boldsymbol{\epsilon}) = \boldsymbol{\epsilon}(G^{\mu\nu}\delta g_{\mu\nu} + D_\mu \theta^\mu) \tag{3.3.2}$$

where $G_{\mu\nu}$ is the Einstein tensor and

$$\theta^\mu = g^{\nu\rho}\delta\Gamma^\mu_{\nu\rho} - g^{\mu\nu}\delta\Gamma^\rho_{\rho\nu}. \tag{3.3.3}$$

As this example shows, in the dual picture we avoid cluttering of both spacetime and field indices. We will call $\boldsymbol{L}$ the Lagrangian. Its generic variation is given by

$$\delta\boldsymbol{L}(\phi) = \boldsymbol{E}(\phi)\delta\phi + d\boldsymbol{\theta}(\phi, \delta\phi)\,, \tag{3.3.4}$$

where we suppress any field summation index appearing in the first term. The field equations are given by $\boldsymbol{E} = 0$ (for any field). The symbol "d" is an exterior derivative on M. Comparison of this expression with the above example makes clear the mapping of notation.

3.3.1 Building the phase space

Each of the above quantity is also a form of appropriate rank in the space $\mathcal{F}$ of field configurations. The space of field configuration $\mathcal{F}$ is given by the set of all fields of the theory which satisfy appropriate boundary conditions at the boundaries of M. The subspace of $\mathcal{F}$ formed by configurations satisfying the field equations is noted with $\overline{\mathcal{F}}$ and constitutes the covariant phase space[19]. The operator "δ" can be thought of as an exterior derivative on $\mathcal{F}$. The identification goes as follows. An abstract field ϕ is a point in $\mathcal{F}$. An abstract small variation $\delta\phi$ of ϕ is then a vector in the tangent space $T_\phi\mathcal{F}$ of $\mathcal{F}$ to ϕ. The symbols $\phi(x)$ and $\delta\phi(x)$ are used to denote the numerical values of the abstract fields and the value of the displacement of such fields at the point $x \in M$. A 1-form over $\mathcal{F}$ is a linear functional from $T_\phi\mathcal{F}$ to a number field: hence it is a transformation from $\delta\phi$ to the number $\delta\phi(x)$, where $x \in M$. With abuse of notation we say that $\delta\phi(x)$ is a 1-form over $\mathcal{F}$. Hence, δ acts on a zero form to give a 1-form over $\mathcal{F}$. Then, δ is an exterior derivative on $\mathcal{F}$ if i) $\delta^2 = 0$, ii) satisfies the Leibnitz rule, and iii) if acting on a k-form gives a $k+1$-form. All these properties can be verified in a more precise treatement, but we refrain from doing so and refer the reader to [150] for the details. We will use the following convention for the mapping between δ seen as an exterior derivative in $\mathcal{F}$ and δ seen as a variation over M

$$\delta f(\phi, \delta\phi) := \delta_1 f(\phi, \delta_2\phi) - \delta_2 f(\phi, \delta_1\phi) \tag{3.3.5}$$

[19] Or "pre-phase space" to take into account the possibility of degeneracies of the symplectic structure. See later.

where $\delta_1\phi$ and $\delta_2\phi$ are two different first variations (perturbations) of the fields, not to be interpreted as differential operators on $\mathcal{F}$ (in particular $\delta_i^2 \neq 0$ in general).

Introducing the notation that a (m, f)-form is an m-form over M and an f-form over $\mathcal{F}$ we have the ranks summarised in the following table (where we assume that each field ϕ_a of the collection can be a p_a-form over M.)

Form	(m, f)	Form	(m, f)
ϕ	$(p_a, 0)$	$\delta\phi$	$(p_a, 1)$
$\boldsymbol{E}$	$(d - p_a, 0)$	$\boldsymbol{E}\delta\phi$	$(d, 1)$
$\boldsymbol{\theta}$	$(d - 1, 1)$	$d\boldsymbol{\theta}$	$(d, 1)$
$\boldsymbol{L}$	$(d, 0)$	$\delta\boldsymbol{L}$	$(d, 1)$

From the 1-form $\boldsymbol{\theta}$ on $\mathcal{F}$ we can build a 2-form $\boldsymbol{\omega} = \delta\boldsymbol{\theta}$, which is automatically closed. This is the typical relation between a symplectic form and the symplectic potential in mechanics. In field theory the form $\boldsymbol{\theta}$ is called the *symplectic potential current density* and the form $\boldsymbol{\omega}$ is instead called *presymplectic current.* In the spacetime picture $\boldsymbol{\omega}$ is a $(d-1)$-form given by

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\boldsymbol{\theta}(\phi, \delta_2\phi) - \delta_2\boldsymbol{\theta}(\phi, \delta_1\phi) \tag{3.3.6}$$

The integral of $\boldsymbol{\omega}$ on a $(d-1)$-dimensional slice of M is a $(0, 2)$ form

$$\Omega_\Sigma(\phi, \delta_1\phi, \delta_2\phi) = \int_\Sigma \boldsymbol{\omega}\,, \tag{3.3.7}$$

called *presymplectic form.* Let us recall that in the symplectic formulation of mechanics, a symplectic form Ω takes two vectors v_1 and v_2 in the tangent space of a symplectic manifold and outputs a number. In the field theory case the two vectors are $(\delta_1\phi)$ and $(\delta_2\phi)$. The symplectic form is well defined if it is non degenerate and can be inverted, namely $\Omega(v_1, v_2) = 0(= \Omega_{ij}v^iv^j$ in a chosen chart) $\iff v_1 = 0$ or $v_2 = 0$. If this is not the case, then we speak of presymplectic forms. In such a case a reduction procedure removes the zero modes of the presymplectic form from the presymplectic phase space and the resulting smaller space is a proper phase space. Degeneracies are usually the case in field theories. We do not enter in many of the details of these procedures.

For the following we just need to recall that canonical transformations (or symplectic symmetries, or symplectomorphisms) are those vector fields v on the phase space such that Ω is preserved,

$$\mathfrak{L}_v\Omega = 0. \tag{3.3.8}$$

A function, called Hamiltonian H_v, is associated to each such symplectic symmetry. Indeed, from the Cartan identity

$$0 = \mathfrak{L}_v\Omega = i_v d\Omega + d(i_v\Omega) = d(i_v\Omega) \tag{3.3.9}$$

we get that $i_v\Omega$ is closed and hence, by the Poincaré Lemma, H_v exists (locally at least) such that[20]

$$dH_v = i_v \Omega. \tag{3.3.10}$$

Poisson brackets $\{.,.\}$ can also be defined to be consistent with the Lie braket of two symplectomorphisms. This boils down to

$$\{f, g\}_{PB} = \Omega^{ab} \partial_a f \partial_b g, \tag{3.3.11}$$

for any two functions on the phase space. The Hamiltonian generates the flow along the vector field via Poisson brackets

$$\{f, H_v\}_{PB} = \mathfrak{L}_v f. \tag{3.3.12}$$

Covariance. From the very definition of presymplectic form (3.3.7) it appears that, despite what promised at the beginning, we break covariance because our definitions depends on the chosen slice Σ. This is not so as long as i) ϕ is a solution, i.e. $\boldsymbol{E} = 0$, and ii) the two variations $\delta_1 \phi$ and $\delta_2 \phi$ are in $T_\phi \mathcal{F}$, namely $\delta_i \boldsymbol{E} = 0$. To check this take $0 := [\delta_1, \delta_2]\boldsymbol{L} = \delta_1(\delta_2 \boldsymbol{L}) - \delta_2(\delta_1 \boldsymbol{L})$. With the above conditions we have

$$\delta_1 \delta_2 \boldsymbol{L} \approx d\delta_1 \boldsymbol{\theta}(\phi, \delta_2 \phi), \tag{3.3.13}$$

so that

$$0 = [\delta_1, \delta_2]\boldsymbol{L} \approx d\boldsymbol{\omega}(\phi, \delta_1, \delta_2). \tag{3.3.14}$$

Here $\approx$ stands for equality on-shell.

Variational principle. In this section we have not made any statement about the variational principle. The Lagrangian may not be the one associated with an on-shell stationary action, as in the Einstein-Hilbert case. So if ∂M is a boundary of M,

$$\int_{\partial M} \boldsymbol{\theta} \tag{3.3.15}$$

may not vanish. We will see that a well-defined phase space, with a well-defined Hamiltonian, is the one for which $\boldsymbol{L}$ can be modified to

$$\boldsymbol{L} - \boldsymbol{T}, \tag{3.3.16}$$

where $\boldsymbol{T}$ is a $(d-1, 1)$ boundary term[21] such that

$$\delta \boldsymbol{T} = \overleftarrow{\boldsymbol{\theta}}, \tag{3.3.17}$$

for given boundary conditions on the field variations. The left arrow over $\boldsymbol{\theta}$ denotes its pullback to ∂M.

In other words, there is a tight relation between a well-defined phase space and a well-defined variational principle.

[20]Conversely, any function f is associated to a vector field w by $w^a = \Omega^{ab}\partial_b f$, but not all functions are Hamiltonians because not all vector fields generate symplectic symmetries. Notice also that H_v is globally defined if $\mathcal{H}^1(\mathcal{F}) = 0$ (the first cohomology group).

[21]A term $\overleftarrow{d\boldsymbol{\chi}}$, where $\boldsymbol{\chi}$ is a $(d-2, 1)$-form, is also possible. See paragraph *Ambiguities* in Section 3.3.6.

The variational principle for Einstein-Hilbert gravity in a finite region of spacetime with a timelike boundary and Dirichlet boundary conditions is well defined with the addition of the Gibbons-Hawking-York term and leads to Brown-York quasi-local charges. The same leads to a well-defined covariant phase space Hamiltonian [153]. The case of asymptotic timelike boundary was discussed in [120].

3.3.2 Definition of the Hamiltonian, existence and integrability

Neglecting the problems with degeneracies, we pretend that we are working in a proper phase space. In our case, noticing that an arbitrary vector field ξ on M naturally induces a field variation over $\mathcal{F}$ given by the Lie derivative $\delta_\xi := \mathfrak{L}_\xi$, we define the *Hamiltonian conjugate to* ξ as the function $Q_\xi : \mathcal{F} \to \mathbb{R}$ obtained as the integral in field space[22] [149]

$$Q_\xi[\phi] = \int_{\phi_0}^{\phi} \delta Q_\xi + Q_\xi^0 \,, \tag{3.3.18}$$

of

$$\delta Q_\xi = \Omega_\Sigma(\phi, \delta\phi, \mathfrak{L}_\xi \phi) = \int_\Sigma \boldsymbol{\omega}(\phi, \delta\phi, \mathfrak{L}_\xi \phi) \tag{3.3.19}$$

for all $\phi \in \overline{\mathcal{F}} \subset \mathcal{F}$ and all tangent vectors $\delta\phi$ to $\mathcal{F}$ (not necessarily to $\overline{\mathcal{F}}$). Notice that $\delta\phi \in T_\phi\mathcal{F}$ means it satisfies the linearised equations of motion: $\delta\boldsymbol{E}(\delta\phi) = 0$. We will come back to this important point later. In (3.3.18) $Q_\xi^0 = Q_\xi[\phi_0]$ is the Hamiltonian computed on a reference solution $\phi_0 \in \overline{\mathcal{F}}$, from which we move to reach the point ϕ in $\overline{\mathcal{F}}$.

The given expressions are appropriate definitions for the generating charge because we can formally easily show that Q_ξ generates the Hamiltonian flow. However, beyond formal manipulations, the above definition is subtle and strong. The Hamiltonian Q_ξ exists if $\Omega_\Sigma(\phi, \delta\phi, \mathfrak{L}_\xi\phi)$ is finite (so that δQ_ξ exists) and if it is integrable in the field space. These requirements impose constraints on the triplet ($\mathcal{F}$, ξ, Σ). In particular, the Wald-Zoupas definition [149] requires that the triplet is chosen so that Ω_Σ is finite for all $\phi \in \overline{\mathcal{F}}$ and $\delta\phi \in T_\phi\overline{\mathcal{F}}$.

The integrability condition means that (3.3.18) must be independent of the path chosen to go from a reference field configuration ϕ_0 to another ϕ, which amounts to satisfy the condition

$$(\delta_1\delta_2 - \delta_2\delta_1) Q_\xi = 0. \tag{3.3.20}$$

Assuming that Ω_Σ is finite, then the Hamiltonian does not exist unless (3.3.20) holds. In other words, for Q_ξ to exist, δQ_ξ must be an exact variation.

Once the existence of the Hamiltonian is established, another point to be analysed is the conservation. To investigate this we have to check that

$$\delta Q_\xi|_{\Sigma_2} - \delta Q_\xi|_{\Sigma_1} = 0 \tag{3.3.21}$$

[22] From now on I denote the Hamiltonian with Q_ξ rather than H_ξ because in the literature such Hamiltonians are usually called charges. Notice the dependence on the reference charge, see Section 3.1. In this section we explicitly work with transformations in field space induced by transformations on the spacetime manifold, however, the formalism is general and applies also to gauge transformations.

for any Σ_i.

Clearly, the answer to both questions is encoded in the behavior of $\boldsymbol{\omega}$. In the next section we will show, following [149], that the integrability condition (3.3.20) corresponds to

$$\int_{\partial\Sigma} i_\xi \boldsymbol{\omega}(\phi, \delta\phi_1, \delta\phi_2) = 0\,, \tag{3.3.22}$$

where ϕ and $\delta_i\phi$ are taken so that Ω_Σ is finite according to the Wald-Zoupas convention ($\phi \in \overline{\mathcal{F}}$ and $\delta\phi \in T_\phi\overline{\mathcal{F}}$). In the above, $\partial\Sigma$ is the boundary of Σ and i_ξ is the operator contracting ξ with the first index of the form on which it acts.

The conservation condition (3.3.21), instead, amounts to

$$0 = \delta Q_\xi|_{\Sigma_2} - \delta Q_\xi|_{\Sigma_1} = \delta Q_\xi|_{\partial\Sigma_2} - \delta Q_\xi|_{\partial\Sigma_1} = -\int_{B_{12}} \boldsymbol{\omega}(\phi, \delta\phi, \mathfrak{L}_\xi\phi)\,, \tag{3.3.23}$$

where B_{12} is the $(d-1)$-dimensional boundary surface enclosed by the two boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$. In discussing the conservation of charges we obviously refer to symmetry transformations, so $\mathfrak{L}_\xi\phi$ is required to be a symmetry variation, as opposed to the generic variation along any vector field ξ on the manifold M considered above.

Both the integrability and the conservation conditions boil down to surface integrals. This is perfectly consistent with the fact that in the theories under considerations the charges (3.3.19) are explicitly given as surface integrals, as we show in the next section, where we also derive (3.3.22) and (3.3.23). Before doing so we discuss the conditions on the phase space under which (3.3.22) and (3.3.23) hold.

The condition (3.3.22) is trivially satisfied if

i) ϕ satisfies appropriate boundary conditions so that $\boldsymbol{\omega} \to 0$ sufficiently rapidly that the integral of $i_\xi\boldsymbol{\omega}$ vanishes over the boundary $\partial\Sigma$, or

ii) ξ is tangent to $\partial\Sigma$, so that the pullback of $i_\xi\boldsymbol{\omega}$ to $\partial\Sigma$ vanishes.

The condition (3.3.23) is instead satisfied if the pullback of $\boldsymbol{\omega}$ to B_{12} vanishes. If instead it does not vanish, then the Hamiltonian (on $\bar{\mathcal{F}}$) does not in general exist except if ii) is satisfied. In such a case, however, it is not conserved. We stress that we are considering the Hamiltonian on $\bar{\mathcal{F}}$ because the integrability condition (3.3.22) refers only to $\bar{\mathcal{F}}$.

3.3.3 Hamiltonian in terms of Noether charge

We now show that (3.3.19) is explicitly an integral over the boundary of Σ given by

$$\delta Q_\xi = \int_{\partial\Sigma} \underbrace{[\delta \boldsymbol{q}_\xi - i_\xi \boldsymbol{\theta}(\phi, \delta\phi)]}_{\boldsymbol{k}_\xi} =: \delta q_\xi - \int_{\partial\Sigma} i_\xi \boldsymbol{\theta}(\phi, \delta\phi) \tag{3.3.24}$$

where ξ is a symmetry vector field over M, $\boldsymbol{q}_\xi$ is the so-called Noether $(d-2)$-form associated with ξ and Noether charge q_ξ. Indeed, for any ξ there exist a Noether $(d-1)$-form current $\boldsymbol{j}_\xi$

$$\boldsymbol{j}_\xi := \boldsymbol{\theta}(\phi, \mathfrak{L}_\xi\phi) - i_\xi \boldsymbol{L}\,, \tag{3.3.25}$$

which is on-shell closed $d\boldsymbol{j}_\xi \approx 0$ and hence is exact by the Poincaré lemma. It can thus be derived from the $(d-2)$-form $\boldsymbol{q}_\xi$

$$\boldsymbol{j}_\xi \approx d\boldsymbol{q}_\xi \tag{3.3.26}$$

so that

$$q_\xi = \int_\Sigma \boldsymbol{j}_\xi = \int_{\partial\Sigma} \boldsymbol{q}_\xi \tag{3.3.27}$$

The proof of (3.3.24) proceeds by first obtaining (3.3.25) and then varying it to get $\boldsymbol{\omega}$. Consider for simplicity a diffeomorphism invariant Lagrangian so that no total derivative terms are produced when we vary it under the diffeomorphism ξ:

$$\mathfrak{L}_\xi \boldsymbol{L} = i_\xi d\boldsymbol{L} + d(i_\xi \boldsymbol{L}) = d(i_\xi \boldsymbol{L}) \tag{3.3.28}$$

The intermediate step is the well known Cartan identity and in the last equality we use the fact that $d\boldsymbol{L} = 0$ identically since $\boldsymbol{L}$ is a d-dimensional form on M. From (3.3.4) we have $\mathfrak{L}_\xi \boldsymbol{L} = \boldsymbol{E}\mathfrak{L}_\xi\phi + d\boldsymbol{\theta}(\phi, \delta\phi)$. Hence

$$\mathfrak{L}_\xi \boldsymbol{L} = \boldsymbol{E}\mathfrak{L}_\xi\phi + d\boldsymbol{\theta}(\phi, \mathfrak{L}\phi) = d(i_\xi \boldsymbol{L}) \Longleftrightarrow d\left(\boldsymbol{\theta}(\phi, \mathfrak{L}\phi) - i_\xi \boldsymbol{L}\right) = -\boldsymbol{E}\mathfrak{L}_\xi\phi\,.$$

This defines the on-shell conserved Noether current[23] (3.3.25).

We now take a further generic variation of (3.3.25). Assuming that the vector field ξ does not vary under the variation of the field ϕ, so that $\delta\xi = 0$ we have

$$\delta\boldsymbol{j}_\xi = \delta\boldsymbol{\theta}(\phi, \mathfrak{L}_\xi\phi) - i_\xi\delta\boldsymbol{L} = \delta\boldsymbol{\theta}(\phi, \mathfrak{L}_\xi\phi) - i_\xi(\boldsymbol{E}\delta\phi + d\boldsymbol{\theta}(\phi, \delta\phi)). \tag{3.3.29}$$

Using again the Cartan identity we get

$$\delta\boldsymbol{j}_\xi = \underbrace{\delta\boldsymbol{\theta}(\phi, \mathfrak{L}_\xi\phi) - \mathfrak{L}_\xi\boldsymbol{\theta}(\phi, \delta\phi)}_{\boldsymbol{\omega}(\phi,\delta\phi,\mathfrak{L}_\xi\phi)} - i_\xi(\boldsymbol{E}\delta\phi) + d(i_\xi\boldsymbol{\theta}(\phi, \delta\phi)), \tag{3.3.30}$$

where we recognise the form $\boldsymbol{\omega}$ if we extend the notation $\mathfrak{L}_\xi\phi = \delta_\xi\phi$ to the covariant forms, so that $\mathfrak{L}_\xi\boldsymbol{\theta} = \delta_\xi\boldsymbol{\theta}$. Hence

$$\delta\boldsymbol{j}_\xi = \boldsymbol{\omega} + d(i_\xi\boldsymbol{\theta}(\phi, \delta\phi)) - i_\xi(\boldsymbol{E}\delta\phi).$$

On-shell $\boldsymbol{E} = 0$. Given $\boldsymbol{j}_\xi = d\boldsymbol{q}_\xi$ and the commutation property of d and δ, valid when the variations of ϕ satisfy the linearised field equations, we get

$$\boldsymbol{\omega}(\phi, \delta\phi, \mathfrak{L}_\xi\phi) \approx d(\delta\boldsymbol{q}_\xi - i_\xi\boldsymbol{\theta}(\phi, \delta\phi)) =: d\boldsymbol{k}_\xi(\phi, \delta\phi), \tag{3.3.31}$$

[23] More in general one can also construct an off-shell closed current $\boldsymbol{j}_\xi + \boldsymbol{J}$ employing the second Noether theorem. There exists a $\tilde{\boldsymbol{q}}$ such that $d\tilde{\boldsymbol{q}} = \boldsymbol{j}_\xi + \boldsymbol{J}$ (see for example [154]). In [149] the form $\boldsymbol{j}_\xi = d\boldsymbol{q}_\xi + \xi^a \boldsymbol{C}_a$ is given, with $\boldsymbol{C}_a$ the constraints of the theory vanishing on shell. Despite no reference to Noether's second theorem is made in [149], the conclusion is the same as $\boldsymbol{J}$ encodes the constraints of the theory.

which is to be substituted in (3.3.19) to finally get (3.3.24) upon integration over Σ. Notice, indeed that (3.3.14) is explicitly satisfied.

It is now easy to show the integrability condition (3.3.22). We evaluate the left hand side of (3.3.20) as

$$[\delta_1,\delta_2]Q_\xi = [\delta_1,\delta_2]q_\xi + \int_{\partial\Sigma} i_\xi \underbrace{(\delta_2\boldsymbol{\theta}(\phi,\delta_1\phi) - \delta_1\boldsymbol{\theta}(\phi,\delta_2\phi))}_{\boldsymbol{\omega}(\phi,\delta_2\phi,\delta_1\phi)}, \tag{3.3.32}$$

so that (3.3.22) follows because the first term on the right hand side of the above vanishes.

Similarly, the conservation condition (3.3.23) is trivially obtained from (3.3.24).

Notice that according to the other characterization of the integrability condition given below (3.3.20), the integrand of (3.3.24) is a total variation in the field space if there exist a $(d-1)$-form $\boldsymbol{T}(\phi,\delta\phi)$ such that

$$i_\xi\boldsymbol{\theta}(\phi,\delta\phi) = \delta\left(i_\xi\boldsymbol{T}(\phi,\delta\phi)\right). \tag{3.3.33}$$

We also notice that only integrable charges generate symplectic symmetries. Indeed

$$\delta(\delta Q_\xi) = 0 \Rightarrow \delta(\int_{\partial\Sigma} i_\xi\boldsymbol{\omega}) = \int_{\partial\Sigma}(\mathfrak{L}_\xi\boldsymbol{\omega}) = 0 \Rightarrow \mathfrak{L}_\xi\boldsymbol{\omega} = 0, \tag{3.3.34}$$

where we have switched to the phase space notation, so that δ is an exterior derivative, ξ must be understood as a tangent vector to the phase space ($\mathfrak{L}_\xi\phi$) and we have used the Cartan identity with $\delta\delta\boldsymbol{\theta} = 0$.

3.3.4 Charge algebra

The symmetry vectors ξ_1 and ξ_2 satisfy a Lie algebra under the standard Lie braket

$$[\mathfrak{L}_{\xi_1},\mathfrak{L}_{\xi_2}]\Omega = 0 \tag{3.3.35}$$

Thus $[\mathfrak{L}_{\xi_1},\mathfrak{L}_{\xi_2}] := \mathfrak{L}_{\xi_1}\xi_2$ is another symmetry vector under the standard Lie braket.

The integrable charges satisfy, under Dirac brakets, the same algebra as the vectors *up to a central extension*

$$\{Q_{\xi_1},Q_{\xi_2}\} = Q_{[\xi_1,\xi_2]} + C_{\xi_1,\xi_2}[\phi_0] \tag{3.3.36}$$

where the central term

$$C_{\xi_1,\xi_2}[\phi_0] = \int_{\partial\Sigma}\boldsymbol{k}_{\xi_1}(\phi_0,\mathfrak{L}_{\xi_2}\phi_0) \tag{3.3.37}$$

depends only on the reference configuration. The proof can be found in [154].

3.3.5 A conserved "Hamiltonian" for pathological situations

Above we gave an explicit expression of charge whose validity is tied to conditions *i*) and *ii*) at the end of section (3.3.2). They are constraints on the phase space $\mathcal{F}$. Are they innocuous? The answer is no.

If we were to define asymptotic flatness at null infinity by requiring that $\mathcal{F}$ meets the above conditions, we would miss all radiative solutions of Einstein's equations.

Intuitively, as radiation crosses null infinity, we cannot impose too strict falloff conditions on the fields. Hence $\boldsymbol{\omega}$ does not vanish sufficiently fast. In the language of conformal compactification, the smooth extension of $\boldsymbol{\omega}$ to the compactified spacetime does not have a vanishing pullback to the conformal boundary[24].

In this situation, the charges as defined above, in terms of the Noether current and satisfying integrability and conservation criteria, do not exist in general. In particular, in the subcase corresponding to *ii*), the charge is integrable (exist) but it is not conserved.

In such a case, Wald and Zoupas [149] built an appropriate definition of charges which are integrable and "conserved". To distinguish between truly conserved charges Q_ξ and the modified charges, we note the latter with $\mathcal{Q}_\xi$.

In order to connect with Section 6.1, we give a shortcut summary of the Wald-Zoupas analysis. The starting point of the construction is to recognise that the non conservation of $\delta\mathcal{Q}_\xi$ is evidently due to a flux $\boldsymbol{F}_\xi$ defined on B_{12}

$$\delta\mathcal{Q}_\xi|_{\partial\Sigma_2} - \delta\mathcal{Q}_\xi|_{\partial\Sigma_1} = -\int_{B_{12}} \delta\boldsymbol{F}_\xi\,, \tag{3.3.38}$$

where $\boldsymbol{F}_\xi$ is a $(d-1,1)$-form defined on the boundary. We require the physically motivated condition that

$$\delta\mathcal{Q}_\xi|_{\partial\Sigma_2} - \delta\mathcal{Q}_\xi|_{\partial\Sigma_1} = 0 \tag{3.3.39}$$

when the solution (the spacetime) is stationary[25], i.e. there is no radiation at the boundary for this kind of spacetime. In this case the Hamltonian reduces to δQ_ξ.

Clearly $\delta\boldsymbol{F}_\xi$ must be related to the pull-back $\overleftarrow{\boldsymbol{\omega}}$ of the presymplectic current to the boundary[26]. With the above intuition we expect

$$\boldsymbol{\omega} = \boldsymbol{\omega}_c + \overleftarrow{\boldsymbol{\omega}} \tag{3.3.40}$$

where $\boldsymbol{\omega}_c$ represents the presymplectic current corresponding to conserved charges, i.e. when the radiation is off.

[24]For spatial infinity, instead, we can sufficiently restrict $\mathcal{F}$ because no dynamical processes can occur there. A good definition of charges exists in the context of the ADM formalism and in particular the conserved charge associated with an asymptotic time translation at spatial infinity is the well known ADM mass. The formalism here summarised is consistent since it returns ADM expressions at i^0.

[25]Stationary spacetimes are those that admits a Killing vector which near the boundary is a timelike asymptotic symmetry vector.

[26]Recall that $\boldsymbol{\omega}$ is defined in the whole spacetime while the flux is defined directly on the boundary.

From (3.3.23) we see that[27]

$$\overleftarrow{\boldsymbol{\omega}} = \delta \boldsymbol{F}_\xi \,, \tag{3.3.41}$$

namely, mapping to variations over M (see (3.3.5))

$$\overleftarrow{\boldsymbol{\omega}}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1 \boldsymbol{F}_\xi(\phi, \delta_2\phi) - \delta_2 \boldsymbol{F}_\xi(\phi, \delta_1\phi) \,. \tag{3.3.42}$$

Thanks to this we can define a "conserved charge" as [149]

$$\delta \mathcal{Q}_\xi = \int_{\partial\Sigma} [\delta \boldsymbol{q} - (i_\xi \boldsymbol{\theta})] + \int_{\partial\Sigma} i_\xi \boldsymbol{F}_\xi \,. \tag{3.3.43}$$

Notice that $\boldsymbol{F}_\xi$ plays the role of a symplectic potential for $\overleftarrow{\boldsymbol{\omega}}$ and by analogy to (3.3.6) we may call $\boldsymbol{F}_\xi := \boldsymbol{\Theta}$. One can check that (3.3.43) satisfies $[\delta_1, \delta_2]\mathcal{Q}_\xi = 0$ (with ξ tangent to B).

These formulae are admittedly far from transparent and our exposition was quite quick. We will come back to these points in Section 6.1 where the charges will be motivated by BMS symmetries.

3.3.6 Ambiguities and boundaries

Basics ideas and formulas have been given, but some important points need to be mentioned.

Ambiguities. The form $\boldsymbol{\theta}$ is not unambiguously defined. Indeed

a) we can add an exact form $\boldsymbol{\theta} = \boldsymbol{\theta} + d\boldsymbol{Y}$, where $\boldsymbol{Y}$ is an arbitrary $(d-2,1)$-form,

b) if the Lagrangian changes by boundary terms $\boldsymbol{L} \to \boldsymbol{L} + d\boldsymbol{K}$, so that the equations of motion are not affected, then also $\boldsymbol{\theta} \to \boldsymbol{\theta} + \delta\boldsymbol{K}$.

Only the first one affects $\boldsymbol{\omega}$: $\boldsymbol{\omega} \to \boldsymbol{\omega} + d\delta\boldsymbol{Y}$. The arbitrariness of $\boldsymbol{Y}$ has to be fixed based on the particular problem under consideration and can be used in order to make $\boldsymbol{\theta}$ a local and covariant function of its argument, as required by the method [149].

The Noether $(d-1)$-form current $\boldsymbol{q}_\xi$ inherits the same ambiguities of $\boldsymbol{\theta}$:

- ambiguity a) implies $\boldsymbol{j}_\xi \to \boldsymbol{j}_\xi + d\boldsymbol{Y}$,
- ambiguity b) implies $\boldsymbol{j}_\xi \to \boldsymbol{j}_\xi + d(i_\xi \boldsymbol{K})$

In addition to these, the Noether $(d-2)$-form charge $\boldsymbol{q}_\xi$ is further defined only up to an exact form $d\boldsymbol{Z}$. So the ambiguities of $\boldsymbol{q}_\xi$ amounts to

$$\boldsymbol{q}_\xi \to \boldsymbol{q}_\xi + i_\xi \boldsymbol{K} + \boldsymbol{Y} + d\boldsymbol{Z} \tag{3.3.44}$$

[27] In general $\overleftarrow{\boldsymbol{\omega}} + d\boldsymbol{\chi} = \delta \boldsymbol{F}_\xi$ with $\boldsymbol{\chi}$ a $(d-2,2)$-form (see the paragraph *Ambiguities* later). As said, we are suppressing all the technical steps of the derivation [149], which was proposed the other way round: from the form $\boldsymbol{\Theta}$ defined later to $\boldsymbol{F}_\xi$.

Several attempts have been made at removing such ambiguities, i.e. [155], but they play a crucial role in the renormalization of superrotation charges [156, 157] (see Section 6.1) and in aspects of black hole thermodynamics.

Boundaries. We have tacitly spoken about the $(d-1)$-dimensional boundary ∂M of the d-dimensional manifold M, about the $(d-1)$-dimensional slice Σ such that $\partial\Sigma$ is a $(d-2)$-dimensional submanifold intersecting ∂M, and about the $(d-1)$-dimensional region B_{12} of ∂M.

There are several careful treatments of boundary terms in the covariant phase space approach. See for example [158] for non-null finite boundaries and [159] for some comments on null finite boundaries.

As said, a deep analysis of boundary contribution is needed to address some important questions concerning the characteristic ambiguities of the formalism. We do not address these points here, but since we have been mostly interested in the case of null infinity we point out some of the technicalities of [149].

There are further issues when we consider an asymptotic boundary. As usual in general relativistic literature, the spacetime manifold M is considered without boundary. The slice Σ is then by definition a $(d-1)$-dimensional closed, embedded submanifold without boundary. Hence, $\partial\Sigma$ does not exist. The integrals over $\partial\Sigma$, such as (3.3.22), should be understood as limiting processes where the meaningful quantities are defined on compact regions K with boundaries ∂K which approach Σ in the limit.

On the other hand, upon compactification of M we have an unphysical spacetime $\bar{M}$ with a boundary $\mathscr{I}$. Now the slices Σ of M are assumed to extend smoothly to $\mathscr{I}$ in $\bar{M}$ such that they intersect $\mathscr{I}$ in a submanifold which is $\partial\Sigma$. Assuming the compactness of such slices in $\bar{M}$ automatically ensures the convergence of the integral defining $\boldsymbol{\Omega}_\Sigma$, as already stated before. Furthermore, the form $\boldsymbol{\omega}$ is assumed to extend continuously to $\mathscr{I}$. At this point also the requirements on the validity of integrability and conservation conditions, as well as the discussion of charges at null infinity, should be rephrased in these more careful terms. For example, we have stated previously that the non-existence of a “standard” conserved Hamiltonian is typical of null infinity because of the generic presence of radiation on $\mathscr{I}$. Such a possibility depends on the given definition of $\mathcal{F}$. In turn, the choice of $\mathcal{F}$ should assure that $\boldsymbol{\omega}$ can be extended smoothly to $\mathscr{I}$, see [149, p.10] and in this light, all the discussions above have to be understood in terms of the smooth extension of $\boldsymbol{\omega}$ to the unphysical spacetime. Notice that the smoothness condition cannot be guaranteed in $d > 4$ asymptotically flat spacetimes, as we explain later.

Part II

Asymptotic flatness and holography

CHAPTER 4

Flat holography?

Minkowski spacetime is maximally symmetric and in this respect is on the same ground as AdS. The analogies with AdS, however, stop here. Physics in Minkowski backgrounds is so different from physics in AdS exactly because there are no other similarities between the two.

While light rays reach the boundary of AdS in finite proper time, thus making the boundary timelike, the boundary of Minkowski space is a more abstract concept. It is the place towards which null signals travel. But they never reach it. The boundary can always be moved far away, as the sight horizon of an observer on the Earth.

Still, there is a sensible notion of boundary via conformal compactifications (see Appendix A). It is thus seen that this far away in Minkowski space is a null surface $\mathscr{I}$, which is the union of a future $\mathscr{I}^+$ and a past $\mathscr{I}^-$ null surface through a point i^0 representing the infinity in spacelike direction. Two further points i^+ and i^- represents future and past timelike infinity. Both $\mathscr{I}^+$ and $\mathscr{I}^-$ are topologically $\mathbb{R} \times S^{d-2}$. In fact, Minkowski space is naturally foliated by lightcones emanating from the worldline of an observer at rest. There is a lightcone for each instant of time and the null boundary is the asymptotic region along these lightcones.

When the metric is allowed to fluctuate, as in any dynamical theory of gravity, Minkowski must be suitably relaxed to an appropriate notion of asymptotically Minkowski spacetime. For the moment we take it to state that the spacetime has the same asymptotic structure as Minkowski.

The different causal structure of the boundary of Minkowski spacetime is at the root of the conceptual and technical issues encountered when trying to adapt $\Lambda < 0$ holography to $\Lambda = 0$. We have mentioned in the previous chapter that a naive $\Lambda \to 0$ limit from AdS/CFT is problematic. We can see the difficulty encountered in the flat limit also from a boundary theory point of view.

From the mass-dimension formula of dual operators we see that the dimension diverges in this limit.

Indeed, some of the first attempts at the flat spacetime holography problem consist in recovering an S-matrix description *within* AdS/CFT which can then be interpreted as a controlled flat limit [160, 161].

The focus on the S-matrix - rather than the correlators, as in AdS/CFT - is motivated on very general grounds. The S-matrix

$$S = \langle p_{out}, s_{out} | p_{in}, s_{in} \rangle \,, \tag{4.0.1}$$

mapping an initial state with momentum p_{in} and quantum numbers s_{in} into a final state *out* is the only gauge/diffeomorphism invariant operator in a theory of gravity in Minkowski spacetime. In a non gravitational theory, the S-matrix is reconstructed from the correlators of local fields via the LSZ reduction formula. When dynamical gravity is present correlators such as

$$\langle \phi \dots \phi \rangle \,, \quad \langle g_{\mu\nu} \rangle \,, \quad \text{etc.}, \tag{4.0.2}$$

are easily seen to not be diffeomorphism invariant.

The S-matrix, with its analytic and unitarity properties is taken to be the output of quantum gravity in flat spacetimes [38]. In his famous talk at *Strings'98* [162], Witten thus suggested that a bulk-boundary correspondence in flat space must be based genuinely on some “structure X” whose degrees of freedom `"live" at infinity` and from which the S-matrix can be computed. Using Witten's wording, the scheme we draw in the Introduction is

$$\text{boundary structure X} \rightarrow \text{flat space } S\text{-matrix}. \tag{4.0.3}$$

This “structure X” cannot be a CFT because a CFT defines quantum gravity at $\Lambda < 0$ [162, 38].

A field theory associated with null infinity can be an example of “structure X” (interestingly Witten dubs this proposal of himself as naive [162]). The kinematics of such a theory is then to be specified by the asymptotic symmetry group at null infinity, which - up to the recent progress - was considered to be the BMS group introduced at page 10. This is yet another reason for not expecting a dual CFT.

The first extensive efforts to flat holography intrinsic to $d = 4$ asymptotically flat spacetimes date back to 2003 and are due to J. de Boer and S.N. Solodukhin [163, 164] and G. Arcioni, C. Dappiaggi [165, 166]. Both conclude, somewhat differently from Witten's suggestion, that the fundamental holographic data are stored on the spherical cross sections of $\mathscr{I}$, named celestial spheres from now onwards. The null nature of the boundary is however reflected in different ways, as we are going to see.

Compared to the work of de Boer and Solodukhin, that of Arcioni and Dappiaggi is closer in spirit to Witten's proposal as they base much of their discussions on the representation theory of BMS, which was first developed in the Seventies [167, 168, 169, 170, 171, 172, 173]. They insist the “structure X” is a BMS invariant field theory (namely, its fields carry BMS label) and moving from one cross section to another of the same component of $\mathscr{I}$ via a BMS transformation corresponds to a relabeling of fields. In this picture, the S-matrix is supposed to map the two Hilbert spaces of the two independent theories constructed on these abstract spaces ($\mathscr{I}^+$ and $\mathscr{I}^-$), but no prescription for this was given.

Using a terminology recently in vogue, BMS-invariant field theories can be called Carrollian field theories. A Carroll group is an ultrarelativistic Inönü-Wigner contraction of the Poincaré group [174, 175]. It is identified as the kinematical group of the space resulting from Minkowski when sending the speed of light to zero, a Carroll "spacetime" where the Minkowski lightcone collapses to the time axis. A conformal extension of the Carroll group (the Conformal Carroll group) was shown to be isomorphic to the BMS group [176, 177, 178]. This motivates the terminology. In Appendix A we show that the given definition of a (conformal) Carroll structure is indeed the usual definition of universal structure of an asymptotically flat spacetime, and we accordingly extend the notion of a Carroll structure to include for $Diff(S^{d-2})$ superrotations[1].

Despite being only a change of names, the latter stresses the fact that we cannot expect a usual field theory because of the null nature of the boundary. There are by now several works exploring the properties of Carrollian theories in holography and the possible relevance beyond holography, see for example [179, 180, 181, 182, 183, 184].

In fact, flat holography in terms of exotic non-relativistic theories was already discussed in the special three-dimensional case when A. Bagchi realised that the BMS-charge algebra in $d = 3$ is isomorphic to the so-called Galilean conformal algebra ($\mathfrak{gca}$), a non-relativistic ($c \to \infty$) Inonu-Wigner contraction of the relativistic conformal algebra[185, 186]. This is not in contrast with the aforementioned ultrarelativistic contraction, because in a space with only one space and one time direction (as the boundary of a $d = 3$ spacetime) the two operations are equivalent (see [176]).

Lower-dimensional gravity has always been a playground for testing and exploring the properties of gravity, although it is non-dynamical. Three-dimensional asymptotic flatness was defined in [187] and the first holographic analysis of $d = 3$ asymptotically flat boundary conditions are due to Barnich and Compère [188], following the steps of Brown-Henneaux analysis of AdS_3 asymptotic symmetries [92][2]. The configuration space of three-dimensional flat spaces was later derived as an appropriate limit of AdS_3 in Bondi-Sachs gauge[3] [191] and several checks of the proposed $\mathfrak{bms}/\mathfrak{gca}$ correspondence were performed, the most famous of which is the matching of the entropy of the cosmological horizon of flat space cosmologies via Cardy-like methods [192, 193, 194].

These developments, however, do not help in shedding light on the higher dimensional problem because there is no notion of gravitational S-matrix in $d < 4$ as gravity is topological.

Coming back to $d = 4$, de Boer and Solodukhin [163] aim at reconstructing the asymptotically flat spacetime from boundary data with a strategy which is closer in spirit to AdS/CFT. They advocate a foliation of the interior and exterior of asymptotic light cones. The interior is foliated

[1] A possibility apparently not considered in the Carroll literature.

[2] This analysis, already mentioned previously, is in fact a fundamental inspiring example for many of the proposed holographic dualities beyond AdS/CFT, including the dS/CFT, the Kerr/CFT and ultimately flat holography.

[3] The Bondi-Sachs gauge is based on a null foliation of the spacetime and hence is a more clever choice than Fefferman-Graham in taking the flat limit. The flat limit of the AdS_3 holographic stress tensor was taken in [189]. Flat limits of AdS in Bondi gauge are again in vogue after [190], concerned with $d = 4$.

via Euclidean AdS slices parametrised by Minkowski time t, while the exterior is foliated via Lorentzian dS slices parametrised by the radial coordinate r. The common boundary of such slices are the celestial spheres. The reconstruction of the bulk spacetime is done via a double asymptotic expansion along the two different slices. This process however cannot reconstruct the timelike direction t in the interior of the lightcone. Despite this, de Boer and Solodukhin further suggest that the S-matrix is obtained from correlators of a theory associated with the celestial spheres.

Since the celestial sphere S^2 is the boundary of $EAdS_3$ slices, de Boer and Solodukhin propose to extend the Lorentz part of the $\mathfrak{bms}$ algebra to include all local conformal transformations forming the Witt algebra. This was already advocated by T. Banks in a footnote of [25], arXived one week before [163]. This extension corresponds to what we named BT-superrotations in the Introduction.

The asymptotic symmetry algebra, the realization on the solution space and the charge algebra corresponding to this local version of BMS

$$\mathfrak{bms} = \mathfrak{sr} \oplus_s \mathfrak{st}, \quad \mathfrak{sr} = \mathfrak{witt} \times \overline{\mathfrak{witt}} \tag{4.0.4}$$

has been studied by G. Barnich and C. Troessaert in [61, 60]. In (4.0.4) we used the bold face notation typical of algebras because this cannot be exponentiated to a group.

The proposals summarised until now, despite focussing on different aspects of null infinity, which should be relevant in the ultimate holographic description, miss the crucial point of giving a prescription to holographically compute the S-matrix. A. Strominger realised that to make progress we must look at $\mathscr{I}^+$ and $\mathscr{I}^-$ at the same time [43] and looked to a specific sharp result feasible to be explained with the common symmetry of $\mathscr{I}^+ \cup \mathscr{I}^-$: the famous, and aforementioned, Weinberg's theorem [47]. Given an n-particle amplitude, where one is an external graviton with momentum q and vanishing energy E_q, the theorem states that the amplitude factorises as

$$\mathcal{M}_{n+1}(p_1, \dots p_n, q) = E_q^{-1} S_{(-1)} \mathcal{M}_n(p_1, \dots p_n) + O(E_q^0), \tag{4.0.5}$$

where $S_{(-1)}$ is a universal factor not depending on the details of the theory but only on the quantum numbers of the external particles participating in the scattering.

Such kind of soft factorizations are quite common and were established long ago [195, 196, 197]. It is also known that such soft behaviours are generically related to the realization of symmetries of the underlying theory [198].

Strominger and collaborators [199] derived Weinberg's soft graviton theorem as a consequence of a BMS_0-supertranslation Ward identity for the gravitational S-matrix, where BMS_0 denotes an appropriate subgroup of $BMS^+ \times BMS^-$ under a suitable identification of group elements of BMS^+ and BMS^- acting respectively on $\mathscr{I}^+$ and $\mathscr{I}^-$. This result can be seen as a first sound realization of Witten's proposal: there is a principle whose output is the expected dynamical behaviour of scattering, at least in the infrared sector.

The subleading soft-graviton theorem mentioned in the Introduction, a statement of the form

$$\mathcal{M}_{n+1}(p_1, \dots p_n, q) = \left(E_q^{-1} S_{(-1)} + S_{(0)}\right) \mathcal{M}_n(p_1, \dots p_n) + O(E_q), \tag{4.0.6}$$

where $S_{(0)}$ is also universal, was unknown until this picture emerged. It was discovered following the proposal that BT-superrotations must be included in the asymptotic symmetry algebra. Explicit field-theoretical checks have been first performed in [49], and hence the new soft theorem is named Cachazo-Strominger soft theorem(s, s as there is also a sub-subleading one).

At this point, a different extension of the BMS group enters the stage. In fact, [200] shows that the single subleading soft theorem implies BT-superrotations on the celestial sphere, but the converse is difficult to derive because of the singularities of superrotations[4]. On the contrary, M. Campiglia and A. Laddha realised that the subleading soft theorem can be derived from the asymptotic symmetries if the latter include arbitrary smooth volume preserving diffeomorphisms $SDiff(S^2)$ of the celestial sphere [64, 65]. They form what we set to call CL-superrotations.

The S-matrix arises from correlators of a -yet unknown- *celestial* (i.e. associated with the celestial sphere) conformal field theory (CCFT). Operator insertions correspond to particles entering or exiting the bulk from specified points on $\mathscr{I}^-$ or $\mathscr{I}^+$ respectively and we have some generic set of rules to relate in and out states.

The properties of the putative CCFT are currently being studied from a bulk point of view, by casting scattering amplitudes so that the boundary symmetries are manifest. Rather than using a basis of plane waves, a basis of so-called conformal primary wavefunctions is used. Elements of this basis are specified by a point on the celestial sphere and a conformal dimension Δ which is complex [202, 203, 204].

Note that in this approach, which in a sense departs from the attempt made by de Boer-Solodukhin, the null nature of infinity is reflected in the BMS transformations to which the celestial field theory is subject (quite similarly to the way proposed by Arcioni and Dappiaggi). For example, the papers just mentioned show that a supertranslation shifts Δ by one and that the two versions of superrotations are related by a shadow transform of the operators.

It is to be expected (or hoped) that the language of CCFT can be translated into the language of non-relativistic/ultrarelativistic field theories. Two papers moving in this direction are [205] and [206][5].

Are we any closer than before in uncovering Witten's structure X? A sharp answer cannot be given. Asymptotic symmetry considerations have been powerful enough to suggest, and hence giving a fundamental explanation of, generic properties of the gravitational S-matrix. At least at the perturbative level, we have uncovered the symmetries of gravitational scattering in a particular class of $d = 4$ spacetimes. The attribute "particular" will be clarified in the course of the next two chapters, where we also remind the reader that some crucial points of concern still affect the picture.

Although we may remain agnostic about the consequences for flat holography, the current developments gave a new spin to reconsider also Abelian and non-abelian gauge theories from the point of view of asymptotic symmetries. The outcome is the discovery of the vast class of triangular equivalences among infrared phenomena, mentioned in the Introduction. New and old

[4]It is to be pointed out that this is partially circumvented by double soft limits [201].

[5] Thanks to A. Bagchi for pointing this out and for related discussions.

soft theorems have been discovered or rederived for different spins [207, 208, 209, 210]. Various forms of memory effects form the manifestation of the asymptotic symmetries at a classical level. In gravity they are known since the seventies [211, 212] and stimulated an active debate on the fundamental properties of general relativity [213, 214, 72, 215, 74, 77]. There are various form of gravitational memories (see also [216, 202, 156]) and we will describe one of them in Section 6.2.1. Gauge theory memories are based on similar considerations, for a partial list of reference see [217, 218, 219, 220, 221, 222, 223, 224].

CHAPTER 5

Bondi-Sachs problem and boundary conditions

The Bondi-Sachs approach [138, 57, 58] to the study of gravitational waves in general relativity is based on the idea of using a family of outgoing null rays forming null hypersurfaces to build coordinates. First proposed by Hermann Bondi [138], it was the basis for the later developments of Penrose conformal methods. Bondi-Sachs coordinates are easily adapted to any number of spacetime dimension. Here we summarise their role in defining asymptotic flatness at null infinity in $d \geq 4$ and the resulting asymptotic symmetries.

5.1 Bondi-Sachs gauge

A d-dimensional metric in Bondi-Sachs gauge reads [57, 58, 59, 70]

$$ds^2 = -\mathcal{U}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(dx^A - W^A du)(dx^B - W^B du), \tag{5.1.1}$$

where $u = const$ picks a null surface and the other coordinates are defined by

$$g_{rr} = g_{rA} = 0\,, \quad \det(h_{AB}) = q(u, x), \tag{5.1.2}$$

with capital latin indices running over the $(d-2)$ coordinates on the cross sections of the $u = const$ null surface and q a fixed arbitrary function. The coordinate u is a properly defined retarded time coordiante because $u = const$ defines null hypersurfaces[1]. The normal vector $k^\mu = g^{\mu\nu}\partial_\nu u$ satisfies by definition $k^\mu k_\mu = 0$ $(g_{\mu\nu}g^{\nu\rho} = \delta_\mu{}^\rho)$, so that $g^{uu} = 0$. The coordinates x^A are constant

[1]It plays the role of the $u = t - r$ coordinate in Minkowski spacetimes, but in curved spacetimes it is not just $u = t - r$, consider for example the Schwarzshild case.

on the integral curves of k^μ, namely $k^\mu \partial_\mu x^A = 0$, implying that $g^{uA} = 0$. Both requirements imply $g_{rr} = 0 = g_{rA}$. The determinant condition in (5.1.2) defines r - the parameter along null geodeodesics (rays) orthogonal to k^μ - to be a luminosity distance.

Unless additional symmetry requirements are imposed, the metric functions $\mathcal{U}$, β, W^A and $g_{AB} := r^2 h_{AB}$ are functions of all the coordinates. They are determined by solving Einstein's equations. Thanks to the Bianchi identites, the strategy to solve the equations is organised in a comfortable way that holds both for vacuum and non vacuum spacetimes. Here we consider vacuum. Once the *main equations*

$$R_{rr} = R_{rA} = R_{AB} = 0\,, \tag{5.1.3}$$

are satisfied everywhere, the other components of the equations collapse to

$$R_{uA} = \partial_r\left(r^{d-2}R_{uA}\right) = 0, \quad R_{uu} = \partial_r\left(r^{d-2}R_{uu}\right) = 0, \tag{5.1.4}$$

and

$$R_{ur} \equiv 0. \tag{5.1.5}$$

Equations (5.1.4), which imply that only the order r^{2-d} of R_{uA} and R_{uu} are non trivial, are called *supplementary equations*. Equation (5.1.5) is trivially satisfied once the main equations are and it is thus called *trivial equation*. The equations $R_{AB} = 0$ are equivalently organised according to the vanishing of its trace and traceless part[2]

$$g^{BA}R_{AB} = 0\,, \qquad R_{AB} - \frac{1}{d-2}g^{AB}R_{AB} = 0, \tag{5.1.6}$$

and when the first of these is solved, the second can be written as

$$g^{DA}R_{AB} = 0. \tag{5.1.7}$$

The problem is by construction a characteristic initial value problem where suitabe initial data for $(h_{AB}, \mathcal{U}, \beta, W^A)$ are to be specified on a characteristic of the equations under considerations (a null surface) and the supplementary ad trivial equations play the role of constraints among this set of data. The solution can only be given asymptotically as $r \to \infty$.

We will explicitly see in chapter 8 that β, W^A and $\mathcal{U}$ are not independent from h_{AB}. The r-dependence of the function β is determined by h_{AB} via $R_{rr} = 0$ up to an integration constant appearing at order r^0 which we denote $\beta_{(0)}(u,x)$. The equation $R_{rA} = 0$ determines W^A up to two constants, $W^A_{(0)}(u,x)$ at order $r^{(0)}$ and $\mathcal{W}^A_{(d-1)}(u,x)$ at order r^{1-d}. The equation $g^{BA}R_{AB} = 0$ determines $\mathcal{U}$ up to the constant $\mathcal{U}_{(d-3)}(u,x)$ at order r^{3-d}. To summarise we have

$$\begin{aligned}
\beta(u,r,x) &= \beta_{(0)}(u,x) + b(u,r,x), \\
W^A(u,r,x) &= W^A_{(0)}(u,x) + \frac{\mathcal{W}^A_{(d-1)}(u,x)}{r^{d-1}} + w(u,r,x), \\
\mathcal{U}(u,r,x) &= \frac{\mathcal{U}_{(d-3)}(u,x)}{r^{d-3}} + \upsilon(u,r,x),
\end{aligned} \tag{5.1.8}$$

where the functions b, w, υ depend on h_{AB} and on the integration functions.

[2]The role of the trace part can be interchanged with R_{ur}, as can be seen from the contracted Bianchi identities. Namely, we can take $R_{ur} = 0$ as a main equation and $g^{BA}R_{AB} \equiv 0$.

There is no equation determining $\partial_r h_{AB}$. The fourth main equation only determines the radial expansion of $y(u,r,x) := \partial_u h_{AB}$ up to an integration constant at order $r^{\frac{2-d}{2}}$

$$y_{AB}(u,r,x) = \frac{y_{(\frac{d-2}{2})AB}(u,x)}{r^{\frac{d-2}{2}}} + \tilde{y}_{AB}(u,r,x) \tag{5.1.9}$$

This situation is to be contrasted with the case of negative cosmological constant spacetimes, where the radial expansion of the boundary metric was uniquely determined in Fefferman-Graham gauge[3].

The supplementary equations determine respectively $\partial_u \mathcal{W}^A_{(d-1)}$ and $\partial_u \mathcal{U}_{(d-3)}$.

5.2 Radiative asymptotically Minkowski spacetimes

The equations for the metric (5.1.1) were first solved by Bondi, Metzner, van der Burgh in the special $d = 4$ axisymmetric case [57]. Soon after, Sachs [58] solved the generic problem in $d = 4$ imposing the assumptions

1) Over the coordinate range $u_0 \leq u \leq u_1$, $r_0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ (with $\phi = \phi + 2\pi$) all the metric functions and other quantities of interest are expanded in inverse powers of r,
2) Over a characteristic, the metric functions behave asymptotically as

$$\lim_{r\to\infty} \mathcal{U} = 1\,, \quad \lim_{r\to\infty} rW^A = \lim_{r\to\infty} \beta = 0 \quad \lim_{r\to\infty} h_{AB} = \gamma_{AB}(x) \tag{5.2.1}$$

where γ_{AB} is the round metric on S^2.

The topological restrictions, as observed by Sachs, follow from the assumed range of coordinates and the form of the asymptotic metric, but nothing is implied about the topology at $r < r_0$.

These conditions were adapted by Tanabe and collaborators [70] to solve the Bondi-Sachs problem in any dimension $d > 4$. To unify the discussion and stress the role of h_{AB} we summarise the asymptotic conditions as

1') Asymptotically the base space of the null rays is S^{d-2}. The metric functions and other quantities of interest are expanded in inverse integer powers of r in even spacetime dimensions and half-integer powers in odd spacetime dimensions. The falloff of h_{AB} is taken to be

$$h_{AB} - \gamma_{AB} = O(r^{\frac{2-d}{2}}), \tag{5.2.2}$$

2') Condition 2) holds with γ_{AB} being the round sphere metric on S^{d-2}.

The necessity of half-integer powers in odd dimensions does not come as a surprise after (5.1.9):

[3] This result is in fact independent of the guage provided that the integration scheme is modified [190].

if we arbitrarily set half-integer powers to zero we miss a free integration function, which is in hindsight the most important piece of information we have. However, the behaviour (5.2.2) was imposed in [58, 69, 70] before solving the equations on the basis of two different arguments. In $d = 4$ it is consistent with *i*) Bondi orginal approach in which the decay of h_{AB} was fixed by continuity with Weyl solutions in a radiative/non radiative transition, *ii*) many exact solutions of Einstein's equations (cfr. [58] p.110), *iii*) massless fields in Minkowski space. In $d > 4$ the authors of [69, 70] were led to (5.2.2) by consistency with the asymptotic behaviour of massless fields in Minkowski[4]. Notice that this condition is consistent with (5.1.9) as long as all the terms of order r^{-p} in $\tilde{y}_{AB}$ with $p \in (0, \frac{2-d}{2})$ are zero.

With the given boundary conditions the metric behaves at infinity as

$$ds^2 \approx -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B + \text{subleading}, \tag{5.2.3}$$

and hence we will call (5.2.1) Minkowski boundary conditions and the falloff in 1') as *radiative falloff* as they are consistent with linearised radiation. Spacetimes whose metrics can be cast in Bondi-Sachs form and satisfy the above boundary conditions are called, in this thesis, asymptotically Minkowski.

In any spacetime dimensions, with the given boundary conditions the main equations are solved by [70]

$$\beta = \frac{\beta_{(d-2)}}{r^{d-2}} + \ldots, \quad \mathcal{U} = 1 - \frac{\mathcal{U}_{(d-3)}}{r^{d-3}} + \cdots, \quad W^A = \frac{W^A_{(\frac{d}{2})}}{r^{\frac{d}{2}}} + \cdots + \frac{\mathcal{W}^A_{(d-1)}}{r^{d-1}} + \cdots, \tag{5.2.4}$$

where the dots represent the orders at the powers following from those shown according to the conventions above. All the terms in the expansions are functions of (u, x^A). The function β is completely determined by h_{AB}, as well as the coefficients of the expansion of $\mathcal{U}$ and W^A up to the order where the integration functions appear. The subleading orders in such expansions are determined by the integration functions as well. It is customary to rename the free function $\mathcal{U}_{(d-2)}$

$$\mathcal{U}_{(d-3)} = \frac{2}{(d-2)\Omega_{d-2}} m \tag{5.2.5}$$

(with a factor of κ^2 to reinstate the gravitational constant) to stress that it is associated with the mass of the spacetime, Ω_{d-2} is the unit sphere solid angle. It is called *Bondi mass aspect.* The free function generically appearing in the expression of $\mathcal{W}^A_{(d-1)}$ also deserves a name: *Bondi angular momentum aspect.*

The mass and the angular momentum aspects satisfy respectivelty two evolution equations in u, obtained from the supplementary equations. Here we state the result in $d = 4$ as we only need this in the following

$$\partial_u m = \frac{1}{4} D_A D_B N^{AB} - \frac{1}{8} N_{AB} N^{AB}. \tag{5.2.6}$$

$$\partial_u \mathcal{W}_A = D_A m + \frac{1}{4}(D_B D_A D_C C^{BC} - D^2 D^C C_{CA}) + \frac{1}{4} D_B (N^{BC} C_{CA} + 2 D_B N^{BC} C_{CA}) . \tag{5.2.7}$$

[4]The details of the analysis of the Green functions for the d-dimensional wave equations can be found in the book [225]

We have defined $N_{AB} := \partial_u C_{AB}$, with C_{AB} the first correction to γ_{AB} appearing at order r^{-1}. In higher dimensions similar equations holds and C_{AB} is at order $r^{-(d-2)/2}$. This object is responsible of the non-conservation of the mass of the system. For this reason, N_{AB} is given the name of the *news tensor*. As opposed to the mass aspect, the angular momentum aspect is not constant when $N_{AB} = 0$. D_A denotes the covariant derivative on S^2 compatible with γ_{AB}.

Notice that the names here given are only justified after the full analysis of charges is completed and a physical meaning to these quantities is given. We note however that the mass aspect appears at the appropriate order for a mass parameter and that C_{AB} is traceless due to the determinant condition defining the gauge. Hence - being also symmetric - C_{AB} contains the right number of degrees of freedom of the gravitational field. When we discuss more general boundary conditions in later chapters we will see that these identifications are generically modified.

As a last comment, we point out that the name given to the falloff conditions reflects that they correspond to the linearised behaviour of massless fields in Minkowski space. Despite the hints, before having sound results on the global gravitational charges, we cannot claim that this corresponds to a physical admissible situation in the full non-linear setting. Bondi's breakthrough was to discover that the mass aspect gives rise to a mass which is correctly decreasing in time, as expected if the system is supposed to lose energy by gravitational radiation, and which remains positive [226, 227, 228, 229, 230]. Thanks to this, the approach suitably describes the physically oriented notion of asymptotic flatness as a gravitational isolated system (with zero cosmological constant). Tanabe, Kinoshita and Shiromizu (TKS from now on) [69, 70] provided in higher dimensions the most straightforward generalisation of Bondi-Sachs analysis. However, it is to be noted that already Sachs realised that the given boundary conditions are perhaps too restrictive. Quoting from Sachs [58]

> They do not constitute a set of independent assumptions; moreover, no clear distinction has been made between topological assumptions, metrical restrictions, and co-ordinate conventions.

We can notice in this regard that the definition of asymptotic flatness via conformal methods in $d = 4$ does not imply that a spacetime is asymptotically Minkowski (see Appendix A) and no a priori topological restrictions are made on the null boundary. In particular, the null boundary must be complete and topologically $\mathbb{R} \times S^2$ for an asymptotically flat spacetime to be asymptotically Minkowski. Suitable definitions of radiative asymptotically Minkowski spacetimes in higher even dimensions have also been given via conformal methods [66, 68] and they do correspond to TKS. For odd dimensions, there is no possible conformal definition of radiative asymptotically flat spacetimes because the conformal factor cannot be smooth [67].

Remarks

To be precise, we may further distinguish between *non-polyhmogeneous* and *polyhomogeneous* asymptotically Minkowski spacetimes. The requirement of an inverse power expansion is not innocuous. The field equations $R_{rA} = 0$ and $g^{AB}R_{AB} = 0$ are solved with two integrations in r, which give rise to terms of the form $r^{1-d}\log r$ in W^A and $r^{3-d}\log r$ in $\mathcal{U}$. Once these logarithmic terms are generated, they propagate down in the asymptotic expansion. An expansion containing both powers and logarithms is called polyhomogeneous.

Bondi and Sachs argued against logarithmic terms using causality arguments, which have been later dismissed as incorrect [143, 231]. Such requirements implied the absence in the expansion of h_{AB} of the terms giving rise to powers of r^{-2} in the double integrals of the aforementioned equations. In particular, in $d = 4, 5$ this implies that $h_{(2)AB}$ is set to zero as a definition of the configuration space, in $d = 6$ $h_{(4)AB}$ and so on in higher dimensions. The freedom left by the integration scheme on h_{AB} suggested the authors of [143] to directly feed the integration with a polyhomogeneous h_{AB}.

Non-polyhomogeneous four-dimensional spacetimes satisfy the so-called peeling property of the Weyl tensor (see for example [56]) while non-polyhomogeneous spacetimes do not [232, 143, 231]. The peeling property of higher dimensional asymptotically Minkowski spacetimes in the Bondi-TKS formulation has been studied in [233].

Here we stress that the general relativity community now regards the peeling property of four-dimensional spacetimes as highly restrictive and unjustified. For example, the rigorous Christodoulou-Klainerman's results on the nonlinear stability of Minkowski spacetime [232, 143] breaks this property. BMS charges for non-peeling four-dimensional spacetimes have been studied recently in [234]. In the remaining of this chapter and the next we will consider spacetimes with non-polyhomogeneous h_{AB} but possibly polyhomogeneous W^A and $\mathcal{U}$. In the last chapter we will briefly consider also a polyhomogeneous h_{AB}.

5.3 Asymptotic Killing vectors

The asymptotic Killing vectors[5] are found by solving the gauge preserving conditions

$$\mathfrak{L}_\xi g_{rr} = 0\,, \quad \mathfrak{L}_\xi g_{rA} = 0\,, \quad g^{AB}\mathfrak{L}_\xi g_{AB} = 0, \tag{5.3.1}$$

and the falloff conditions

$$\mathfrak{L}_\xi g_{uu} = O\left(\frac{1}{r}\right), \quad \mathfrak{L}_\xi g_{ur} = O\left(\frac{1}{r^2}\right), \quad \mathfrak{L}_\xi g_{uA} = O(1), \quad \mathfrak{L}_\xi g_{AB} = O\left(r\right) . \tag{5.3.2}$$

For the general d-dimensional case the gauge preserving conditions remain the same, but the falloff preserving equations become

$$\mathfrak{L}_\xi g_{uu} = O(r^{-d/2+1}), \quad \mathfrak{L}_\xi g_{ur} = O(r^{-d+2}), \quad \mathfrak{L}_\xi g_{uA} = O(r^{-d/2+2}), \tag{5.3.3}$$

$$\mathfrak{L}_\xi g_{AB} = O(r^{-d/2+3}) \tag{5.3.4}$$

[5]As usual in the litearture we also give the bulk component of them, but on $\mathscr{I}$ it is subleading.

The exact Killing equations determine the general form of ξ

$$\xi = \xi^u \partial_u + \xi^r \partial_r + \xi^A \partial_A \longrightarrow \begin{cases} \xi^u = f(u, x^A)\,, \\ \xi^r = -\dfrac{r}{n-2}\left[D_A \xi^A - W^C \partial_C f\right]\,, \\ \xi^A = Y^A(u, x^B) - \partial_B f \int_r^\infty dR e^{2\beta} g^{AB}\,, \end{cases} \tag{5.3.5}$$

in terms of two integration functions $f(u, x^A)$ and $Y^A(u, x^B)$, whose behaviour is determined by the asymptotic Killing equations (5.3.2) or (5.3.3) and crucially depends on the dimensionality of spacetime. At null infinity the Killing field reduces to $\xi_{(\alpha,Y)} = \xi^u \partial_u + \xi^A \partial_A$ and in what follows we are going to restrict to this. The notation $\xi_{(\alpha,Y)}$ is useful to recall that there is a different vector field for each choice of the pair (α, Y); sometimes only the pair is used instead of $\xi_{(\alpha,Y)}$.

The falloff preserving conditions must be solved order by order, by expanding the left hand side in (5.3.3) and setting to zero the terms of order greater than that indicated on the right hand side. In any dimension, from the first of (5.3.3) at order r and the second at order r^0 we get - respectively - the constraints

$$Y^A = Y^A(u), \quad f(u, x^A) = \alpha(x^A) + \frac{u}{n-2} F(x^A), \quad F(x^B) = D_A Y^A. \tag{5.3.6}$$

Y^A is determined to be a conformal Killing vector on S^{d-2} from the vanishing of the order r^2 term in (5.3.4), namely

$$\mathfrak{L}_Y \gamma_{AB} := D_A Y_B + D_B Y_A = \frac{1}{d-2} \gamma_{AB} D_C Y^C. \tag{5.3.7}$$

The function α is either

- an arbitrary function of the angles of S^2 when $d = 4$, so that it can be expanded in an arbitrary number of scalar spherical harmonics

  $$\alpha = t^\mu n_\mu + \sum_{l=2}^{\infty} \sum_m \alpha_l^m \mathcal{Y}_l^m\,, \tag{5.3.8}$$

  where $\mathcal{Y}_l^m$ denotes the harmonics, $t^\mu = (t^0, t^i)$ and n^μ are $l < 2$ scalar harmonics $(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \mathrm{co}$ or

- any combination of $l = 0, 1$ scalar harmonics on S^{d-2} when $d > 4$: $\alpha = t^\mu n_\mu$.

Thus, $\xi_{(\alpha,0)}$ generates translations in $d > 4$ and an infinite-dimensional extension of them (supertranslations) in $d = 4$.

These facts, are a direct consequence of the g_{AB} fall-off condition (5.2.2) and the related Killing equation, (5.3.4). Indeed, for any $d \geq 4$, (5.3.4) is of the form

$$\mathfrak{L}_\xi g_{AB} = G_{(2)AB}(u, x^A) r^2 + G_{(1)AB}(u, x^A) r + O(r^{-d/2-3}) \tag{5.3.9}$$

and $G_{(2)AB} = 0$ implies (5.3.7). As a consequence of $G_{(1)AB} = 0$ - to be imposed in $d > 4$ but not in $d = 4$ - α must satisfy

$$(d-2) D_A D_B \alpha = \gamma_{AB} D^2 \alpha, \tag{5.3.10}$$

in $d>4$ and it is free in $d=4$. The solution of (5.3.10) is as stated previously.

We now turn our attention to Y^A. From the so-called Liouville theorem we know that the algebra of conformal transformations in $d-2>2$ is finite dimensional, while when $d=2$ it gets enhanced to the infinite dimensional Witt algebra whose generators are only locally defined [235]. If we adopt stereographic coordinates $z=e^{i\phi}\cot\theta$, $\bar{z}=e^{-i\phi}\cot\theta$ on S^2 so that $\gamma_{AB}dx^Adx^B=4(1+z\bar{z})^{-2}dzd\bar{z}$, (5.3.7) splits in two equations for the components Y^z, $Y^{\bar{z}}$

$$\partial_{\bar{z}}Y^z=0, \quad \partial_z Y^{\bar{z}}=0. \tag{5.3.11}$$

Y^z and $Y^{\bar{z}}$ are respectively holomorphic and antiholomorphic and can be Laurent expanded in a basis of $l_m=-z^{m+1}\partial_z$, $m\in\mathbb{Z}$, and the complex conjugate. The algebra of conformal Killing fields on S^2 is thus the sum of two copies of the Witt algebra $\mathfrak{witt}\oplus\overline{\mathfrak{witt}}$

$$[l_m,l_n]=(m-n)l_{m+n}\,, \quad [\bar{l}_m,\bar{l}_n]=(m-n)l_{m-n}\,, \quad [l_m,\bar{l}_n]=0. \tag{5.3.12}$$

Only six of these vector fields $\{l_m,\bar{l}_m\}_{(m=0,\pm1)}$ are globally well defined and generate an invertible group of transformations, which is the well-known group of Möbius transformations isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}$ and, in turn, to the orthocronous Lorentz group $L_+^\uparrow=SO(1,3)^\uparrow$. The isomorphism between the global conformal group in $d-2$ dimensions and the Lorentz group in d dimensions holds also for $d>2$.

Bondi and Sachs, originally assumed that the Y^A were globally well-defined. With this choice their definition of the BMS algebra is

$$\mathfrak{bms}_4^{\text{glob}}=\mathfrak{l}_+^\uparrow\oplus_s\mathfrak{st}, \tag{5.3.13}$$

where $\oplus_s$ denotes the semidirect sum, $\mathfrak{st}$ the abelian normal subalgebra of supertranslations, $\mathfrak{l}_+^\uparrow$ the Lorentz algebra. This algebra is easily exponentiated to the global BMS group

$$BMS_4^{\text{glob}}=L_+^\uparrow\ltimes ST \tag{5.3.14}$$

whose elements transform u and x^A as

$$u\to u'=K^{-1}(x)(u+f(x)), \quad x^A\to x'^A(x), \tag{5.3.15}$$

where K is the conformal factor in the transformation of angular coordinates and f is the arbitrary finite supertranslation.

BT-superrotations are instead obtained by taking Y^A meromorphic and thus we get

$$\mathfrak{bms}_4^{\text{loc}}=\mathfrak{sr}\oplus_s\mathfrak{st}, \quad \mathfrak{sr}=\mathfrak{witt}\times\overline{\mathfrak{witt}} \tag{5.3.16}$$

as stated in (4.0.4). For consistency in this case also supertranslations $\alpha(z,\bar{z})$ have to be expanded as Laurent series, rather than spherical harmonics, and their expansion is $\alpha_{m,n}=(1+z\bar{z})^{-1}z^m\bar{z}^n$ (see [63] for details). The mode expanded form of the $\mathfrak{bms}_4^{\text{loc}}$ algebra is given by (5.3.12) plus the following commutators

$$[l_j,\alpha_{m,n}]=\left(\frac{j+1}{2}-m\right)\alpha_{m+j,n}\,, \quad [\bar{l}_j,\alpha_{m,n}]=\left(\frac{j+1}{2}-n\right)\alpha_{m,n+j}\,. \tag{5.3.17}$$

The subalgebra spanned by $\alpha_{m,n}, l_j, \bar{l}_k$ with the pair m, n acquiring all possible combinations of $(0,1)$ and $j, k = 0, \pm 1$ is the Poincaré algebra. Possible central extensions of the $\mathfrak{bms}_4^{\text{loc}}$ algebra have been studied [63, 236, 205]. For completeness let us state the non-mode expanded form for the commutator of the Killing vectors $\xi_{(\alpha,Y)}$ restricted to null infinity

$$[\xi_{(\alpha_1,Y_1)}, \xi_{(\alpha_2,Y_2)}] = \xi_{(\alpha_3,Y_3)}, \tag{5.3.18}$$

$$\alpha_3 = Y_1^A \partial_A \alpha_2 + \frac{1}{2}\alpha_1 D_A Y_2^A - (1 \leftrightarrow 2), \quad Y_3^A = Y_1^B D_B Y_2^A - (1 \leftrightarrow 2). \tag{5.3.19}$$

With a slight modification of the brackets the algebra can also be faithfully represented in the whole bulk [60].

In higher dimensions, instead, the BMS algebra (and group) is reduced to Poincaré

$$L_+^\uparrow \ltimes T = ISO(1, d-2), \tag{5.3.20}$$

by two effects: the choice of radiative falloff conditions (removing supertranslations) and the mathematical obstruction given by the Liouville theorem on conformal transformations.

This reduction of the symmetry group prevents Weinberg soft theorem to be found from asymptotic symmetries. In order to obtain supertranslations in higher dimensions, one can question the orthodox choice of asymptotic conditions and explore weaker conditions. Chapter 8 is mainly devoted to this aspect.

5.4 BT & CL superrotations: extending the asymptotics

How can the $Diff(S^2)$ transformations (superrotation) proposed by Campiglia and Laddha be obtained?

They cannot be accommodated within the given boundary conditions because (5.3.4) forces the Killig vector on S^2 to be conformal. Campiglia and Laddha thus argued for extending the asymptotic condition on h_{AB} to

$$\lim_{r\to\infty} h_{AB}(u,r,x) = h_{(0)AB}(x), \quad \det h_{(0)AB} = q \tag{5.4.1}$$

where $h_{(0)AB}$ is kept free to fluctuate but the determinant is fixed by the requirements of Bondi-Sachs gauge.

With such condition, (5.3.4) does not apply anymore and instead we have

$$\mathfrak{L}_\xi g_{AB} = O(r^2), \tag{5.4.2}$$

which is automatically satisfied and so does not place any constraint on $Y^A(x)$, which is now a smooth vector field.

BT-superrotations also need a non-trivial reconsideration of the boundary conditions. Indeed, such transformations are singular on S^2. (5.3.7) becomes singular at the location of the singularities of

Y (for example take $Y^z = z^{-1}$) and this singularity is reflected in the bulk expansion of the metric. It can be interpreted in terms of topological defects, as we are going to discuss in Section 7.2.

In chapter 6 we are going to see the effects of such singularities on the charges. Similarly, CL-superrotations will be affected by other issues at the level of charges.

CHAPTER 6

BMS_4 charges and the infrared triangle

We briefly discuss the supertranslation and superrotation charges in $d = 4$ and highlight the issues of the latter. We then move to build the phase space compatible with supertranslations and discuss an instance of the infrared triangle to exemplify.

6.1 Charges and fluxes

6.1.1 Heuristics

Consider the Bondi mass aspect m, on a given null hypersurface $u = u^*$. It can be integrated over the angles and the resulting function is the Bondi mass, representing the mass of the spacetime at a given u instant[1]

$$M := \frac{1}{4\pi G} \int_{\partial\Sigma} d^2\Omega m, \tag{6.1.1}$$

where we denote $d^2\Omega$ the integration measure on S^2. The mass/energy is the charge associated to time translation symmetry. Pure time translations are generated by the vector field $\xi_{(\alpha,Y)}$, given in Section 3.2.1, with $\alpha = 1$ and $Y^A = 0$. The other three spatial translations are given by $\alpha = t^i n_i$ (cfr. 5.3.8), hence we can define analogously a Bondi linear momentum P^i. Then, we are allowed to guess that for each α defining a generic supertranslation ($Y^A = 0$) we can associate a charge as

$$\frac{1}{4\pi G} \int_{\partial\Sigma} d^2\Omega \, \alpha(x) m(u, x) \, . \tag{6.1.2}$$

[1] In this chapter we explicitly write $\kappa^2 = 8\pi G$.

The charges associated to the BMS symmetry are not conserved because of fluxes of them through null infinity. Let us check this considering the Bondi mass. The Bondi mass aspect m satisfies the evolution equation (5.2.6), hence the Bondi mass satisfies

$$\partial_u M = \frac{1}{32\pi G}\int_{\partial\Sigma} d^2\Omega \left[2D_A D_B N^{AB} - N_{AB}N^{AB}\right] . \tag{6.1.3}$$

If we discard boundary terms we get the famous Bondi mass loss equation

$$\partial_u M = -\frac{1}{32\pi G}\int_{\partial\Sigma} d^2\Omega N_{AB}N^{AB} \leq 0, \tag{6.1.4}$$

stating that M is a monotonically decreasing function of time u. This eventually fully justify the interpretation of M as the mass of the spacetime (in a fully general relativistic context!) and the names given to m and N_{AB} in chapter 5. Equations (6.1.3), (6.1.4) can be integrated in the u direction between two points u_i and u_f and the resulting object is the flux of energy radiated through this section I of null infinity by gravity waves[2]

$$\Delta M = \frac{1}{32\pi G}\int_I dud^2\Omega \left[2D_A D_B N^{AB} - N_{AB}N^{AB}\right] =: -\int_I dud\Omega^2 F \tag{6.1.5}$$

with $\Delta M = M|_{I_+} - M|_{I_-}$, $I_\pm$ denote the upper/lower boundary of I along the u direction. Notice in particular that assuming that $M|_{I_+} = 0$ (i.e. all mass is radiated away) we get the remarkable equality of $M|_{I_-}$ with the flux.

For generic α the flux formula is given by (6.1.3) with M replaced by $\mathcal{Q}_\alpha$ and α inserted before the square brackets (because it is u-independent).

6.1.2 Rigour

With intuition driven by the mass aspect, we have obtained the flux integrating the putative charge along u.

In Section 3.3.5 we saw that the Wald-Zoupas formalism prescribes a flux formula for any asymptotic symmetry vector field ξ and its associated charge of the form

$$\mathcal{Q}_\xi^{I_+} - \mathcal{Q}_\xi^{I_-} := \Delta_I \mathcal{Q}_\xi = -\int_I \boldsymbol{F}\,, \tag{6.1.6}$$

where the charge for a general asymptotic Killing $\xi_{(\alpha,Y)}$ in the $\mathfrak{bms}_4^{\text{glob}}$ reads as

$$\mathcal{Q}_\xi = \frac{1}{16\pi G}\int_{\partial\Sigma} d^2\Omega \left[4\alpha m + Y^A Z_A\right] , \tag{6.1.7}$$

with $Z_A = Z_A[m, C_{AB}, N_A]$ a suitable smooth covariant vector whose explicit form is not relevant here. The explicit expressions can be found in [237].

Equation (6.1.6) can be understood as a consistency check on the charge (6.1.7), namely the

[2]Also null matter can be considered and a term proportional to T_{uu} appears.

difference in charges on two cuts must be equal to the integral of the form $\boldsymbol{F}$ on I. This works for the supertanslations ξ_α and for the Lorentz part $\xi_{Y(\text{glob})}$ of the global BMS. For example, in the case of pure supertranslations

$$\boldsymbol{F}_\alpha = \frac{1}{32\pi G} dud^2\Omega\, N^{AB}\mathfrak{L}_\alpha C_{AB}\,, \tag{6.1.8}$$

where

$$\mathfrak{L}_\alpha C_{AB} = \alpha N_{AB} - 2D_A D_B\alpha + \gamma_{AB}D^2\alpha\,, \tag{6.1.9}$$

is how C_{AB} transforms under a supertranslation.

However, Wald-Zoupas prescription does not work for generic BT superrotations $\xi_{Y(\text{BT})}$. In this case there is a discrepancy of the form [237]

$$-\int_I \boldsymbol{F} = \Delta_I \mathcal{Q}_{\xi_{Y(\text{BT})}} + \Delta_I \mathcal{F}\,, \tag{6.1.10}$$

meaning that for superrotations the charge formula (6.1.7) and the flux formula (6.1.6) prescribed by Wald and Zoupas are not consistent.

As pointed out by Flanagan and Nichols [237], this discrepancy hints that BT-superrotations are not true asymptotic symmetry vectors of asymptotically flat spacetimes. Indeed, the Wald-Zoupas prescription was explicitly tailored to vector fields in $\mathfrak{bms}_4^{\text{glob}}$. It is not possible to consider BT-superrotations to be part of the asymptotic symmetries of (the standard notion of) asymptotically flat spacetimes[3]. They act on an enlarged phase space, an expectation first exemplified in [239] and reviewed in Section 7.2.

The situation is apparently worse for CL-superrotations. The symplectic forms $\boldsymbol{\theta}$, $\boldsymbol{\omega}$ are divergent [156]. In particular

$$\theta^u = r\theta^u_{div} + \theta^u_{fin} + r^{-1}\theta^u_{sub} + O(r^{-2}), \quad \theta^u_{div} \propto \delta\sqrt{q} = 0 \tag{6.1.11}$$

$$\theta^r = r\theta^r_{div} + \theta^r_{fin} + O(r^{-1}), \tag{6.1.12}$$

where the choice of components is motivated by the fact that asymptotically the coordinates behave as $t = r + u$ and we reach $\mathscr{I}^+$ by taking $t \to \infty$ with u fixed[4]. The leading order pieces (notice that also the finite parts above are problematic since we have to integrate to get the charge) can be renormalized using the ambiguity a) of Section 3.3.6. Indeed, they can be written as a total derivative. See [156] for details.

However, as pointed out in [157], such a regularization suffers from fundamental drawbacks. The resulting symplectic current $\boldsymbol{\omega}$ cannot be renormalized by local and covariant redefinitions. Hence the resulting charges are ill-defined. The authors of [157] suggest however that a loophole in their argument is the requirement of covariance of $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ at any step of the computation (compare with the paragraph *Ambiguities* in Section 3.3.6) and that it may be relaxed without affecting the covariance of the charge, thus bypassing their no-go theorem. Such comments are relevant in the discussion of charges in higher dimensions we initiate in the last chapter of the thesis.

[3] In [238] they are named *external symmetries*.

[4] Alternatively we can select a $r = const$ surface and send it to infinity.

6.1.3 The phase space of supertranslated spacetimes

Wald-Zoupas fluxes can be integrated over all of $\mathscr{I}^+$. This is going to represent the total charge radiated off from null infinity, for example the total energy. Physically reasonable spacetimes emits only a finite amount of energy or, in general, charge. Hence a crucial requirement is the flux $\Delta_\infty \mathcal{Q}_\alpha$ to be finite. This is true as long as the news tensor satisfies

$$\lim_{|u|\to\infty} N_{AB} \approx |u|^{-\epsilon_N}\,, \quad \epsilon_N > 1\,, \tag{6.1.13}$$

namely does not develop logarithmic u terms in its evolution. The same goes for $\mathcal{Q}_Y$ provided Y is the globally well defined vector. This fall-off condition along the u direction, implies from (5.2.7), that only the electric parity piece of C_{AB} is non zero[5] at $\mathscr{I}^+_-$ ($\mathscr{I}^+_\pm$ is a standard notation to denote the past and future regions of $\mathscr{I}^+$)

$$C^e_{AB} = (D_A D_B - \frac{1}{2}\gamma_{AB} D^2)C\,, \tag{6.1.14}$$

where C is a scalar function. The solution of this equation gives the initial value of C at $u \to -\infty$ which we denote as $C|_{\mathscr{I}^+_-}$. Under these conditions, we can parametrize the space of solutions with $\{m|_{\mathscr{I}^+_-}, C|_{\mathscr{I}^+_-}, N_{AB}; \mathcal{W}^A_{(3)}|_{\mathscr{I}^+_-}\}$. We have also inserted the second order field $\mathcal{W}^A_{(3)}$ but it will not play any role here.

We can define, for later purposes,

$$\int_{-\infty}^{+\infty} \mathrm{d}u N_{AB} = \int_{-\infty}^{+\infty} \mathrm{d}u \partial_u C_{AB} = \Delta_\infty C_{AB} := (D_A D_B - \frac{1}{2}\gamma_{AB} D^2)N, \tag{6.1.15}$$

the latter following from the property (6.1.14) of C_{AB} in the non-radiative to non-radiative transition we are considering. We are defining $N := \Delta_\infty C$.

BMS supertranslations act on the fields as

$$\begin{aligned} \mathfrak{L}_\alpha N_{AB} &= \alpha \partial_u N_{AB}\,, \quad \mathfrak{L}_\alpha C_{AB} = \alpha N_{AB} - 2 D_A D_B \alpha + \gamma_{AB} D^2 \alpha\,, \\ 4\mathfrak{L}_\alpha m &= 4\alpha\partial_u m + N^{AB} D_A D_B \alpha + 2 D_A N^{AB} D_B \alpha\,, \quad \mathfrak{L}_\alpha C = \alpha. \end{aligned} \tag{6.1.16}$$

The last one is obtained by comparing (6.1.14) to $\mathfrak{L}_\alpha C_{AB}$ and we do not show the long expression for $\mathfrak{L}_\alpha N^A$ as we are not interested in that now.

These should be realised canonically on the phase space, which is to be built on the space of solutions to preserve covariance.

Despite not apparent from the discussion in Section 3.3, the natural phase space of asymptotically flat spacetimes is to be built on the whole of $\mathscr{I}^+$ (resp. $\mathscr{I}^-$). This means that the symplectic form Ω is given as an integral of $\boldsymbol{\omega}$ over $\mathscr{I}$ rather than over a slice Σ as we discussed in Section 3.3.

[5] The tensor C_{AB} on S^2, being traceless and symmetric, can be decomposed into electric and magnetic parity parts $C_{AB} = C^e_{AB} + C^m_{AB}$ where C^e_{AB} is given by (6.1.14) and $C^m_{AB} = \epsilon_{E(A} D_{B)} D^E C^m$. Notice that the $l = 0, 1$ harmonics of C^m can be set to zero because they are annihilated by $\epsilon_{E(A} D_{B)} D^E$ as they should because of symmetry. From (5.2.7) the magnetic part must satisfy $(D^2 + (D^2)^2/2)C^m = 0$ which implies that $C^m = 0$. See for example [237].

In practice, one builds it on Σ and takes a limit to $\mathscr{I}$. The construction was first provided by A. Ashtekar [141, 240, 241] in his effort toward an *asymptotic quantization* of the gravitational field, using the conformal definition of $d=4$ asymptotically flat spacetimes. We did not follow this route because Ashtekar's construction is tied to[6] $d=4$. It suffices to note, from Section 3.3.5, that the Wald-Zoupas flux integrated over the whole of $\mathscr{I}^+$ exactly provides such symplectic structure.

Wald-Zoupas fluxes are (minus) Ashtekar's Hamiltonians and with abuse of notation we can use the same symbol $\mathcal{Q}_\xi$ to denote both quantities. To motivate this consider $\mathcal{Q}_\xi^{I_+}=0$ as $I^+\to i^0$ in (6.1.6). From these comments and those after (6.1.5) we have motivated the following statement, whose proof can be found in [199]: the charge

$$\mathcal{Q}_\alpha = \frac{1}{32\pi G}\int_{\mathscr{I}^+} \mathrm{d}u\, \mathrm{d}^2\Omega\alpha(x)\left[2D_AD_BN^{AB} - N_{AB}N^{AB}\right] =: \mathcal{Q}_\alpha^\partial + \mathcal{Q}_\alpha^b. \tag{6.1.17}$$

generates the BMS transformations on the phase space

$$\{Q_\alpha, N_{AB}\} = \mathfrak{L}_\alpha N_{AB}\,, \quad \{Q_\alpha, C_{AB}\} = \mathfrak{L}_\alpha C_{AB}\,, \quad \{Q_\alpha, C\} = \mathfrak{L}_\alpha C. \tag{6.1.18}$$

to which we can also add $\{Q_\alpha, N\}=0$. We have explicitly separated in the charge a boundary $\mathcal{Q}_\alpha^\partial$ and a bulk $\mathcal{Q}_\alpha^b$ term. Using the above definitions we find

$$\mathcal{Q}_\alpha^\partial = \frac{1}{16\pi G}\int_{\mathscr{I}^+}\mathrm{d}u\,\mathrm{d}^2\Omega\alpha(x)D^AD^BN_{AB} = \int \mathrm{d}^2\Omega\alpha(x)D^AD^B\Delta_\infty C_{AB} \tag{6.1.19}$$

or[7]

$$\mathcal{Q}_\alpha^\partial = \int \mathrm{d}^2\Omega\alpha(x)\left(D^2+\frac{(D^2)^2}{2}\right)N \tag{6.1.20}$$

The boundary piece acts trivially on the classical phase space $\{Q_\alpha^\partial, N_{AB}\}=0$: it does not contribute classically to the change of solution.

Suppose having a vacuum non radiative configuration $m=C_{AB}=N_{AB}=0$. From this Minkowski configuration, a BMS supertranslation changes C_{AB} by the inhomogeneous term in $\mathfrak{L}_\alpha C_{AB}$. Since supertranslations commute with time translations, asymptotically flat gravity is characterised by a manifold of degenerate vacua ($m=N_{AB}=C_{AB}=0$), each of which characterised by a different value of C. The supertranslation invariance is broken by the fixing of a particular C. Since the term $(D_AD_B-\gamma_{AB}D^2/2)\alpha$ responsible for the breaking is annihilated by $l=0,1$ modes, pure translations are not broken.

The field C is interpreted as a Goldstone boson for the breaking of the supertranslation symmetry and upon quantization, the soft term can be shown to be the zero frequency limit of a metric perturbation, hence a *soft graviton* [43]. Schematically, working in momentum space on the celestial sphere

$$C_{AB}\sim\int\mathrm{d}\omega(a_+e^{-i\omega u}+a_-^\dagger e^{i\omega u}) \tag{6.1.21}$$

where for a given momentum p we have $\omega=p^2$ and a_+, a_- are the correspoding Fock operators.

[6] It should be possible to extend it to any even d following the conformal definitions of asymptotic flatness provided in [66], but we are not aware of any work in this direction. In any case, for the final purpose of this dissertation, this route is not viable.

[7] The expressions are somewhat more transparent in stereographic coordinates as in [43, 199].

We also have N_{AB} defined as the Fourier transform in u of (6.1.15), so that with some passages to be found in [199] we get

$$(D_A D_B - \frac{1}{2}\gamma_{AB} D^2)N \sim \lim_{\omega\to 0} N_{AB}(\omega) \tag{6.1.22}$$

In other words, the breaking of the symmetry is characterised by the insertion of soft gravitons.

We would like to point out, for historic completeness, that the asymptotic quantization program was aimed at quantizing the algebra of N_{AB}, which is named the observable algebra as only N_{AB} is directly related to i.e. the Bondi mass loss. In a formal treatment of null infinity, N_{AB} is understood as the curvature of a connection (which is nothing but the shear tensor) on the phase space [240, 241, 242]. We here motivate this by referring to the physical meaning of $N_{AB}N^{AB}$ and keeping in mind the known example of Maxwell theory, where $F_{\mu\nu}$ plays the same role. The steps summarised before, instead, correspond to the quantization of the algebra of C_{AB}, first done in [199].

All these observations are formalized by Ashtekar's asymptotic quantization, recently reviewed within this contemporary view in [242], to which we refer the reader. Here, as well as in the early lectures [241], the role of N is clarified from the representation theory point of view: a Fock representation of the observable algebra is possible if $N = 0$ but not if $N \neq 0$. If $N = 0$ the BMS algebra is realized à la Wigner, i.e. there is no spontaneous breaking. We will see further evidence of this point with the classical phenomenon of gravitational wave memories.

6.2 An instance of infrared triangle

We summarise the supertranslation phenomenology leading historically to the conjecture of the first infrared triangle.

6.2.1 Gravitational memory effects

A supertranslation can be considered responsible for a (displacement) *gravitational memory effect* [216], a long known general relativistic effect which the new gravitational interferometers may be able to measure in the next future.

The displacement memory effect is the permanent displacement of two observers after the passage of a finite burst of gravitational waves[8].

The final proper distance between two observers[9] that experience the passage of a gravity wave

[8]Historically the distinction between linear [211] and non-linear memory effect [212] was made and they were considered two separate effects: the first due to the emission process of the radiation and the second due to the full non-linearity of Einstein's gravity, with no reference to the emission process. However there may be not such a sharp distinction [213, 214, 72].

[9]In the BMS setting they are not inertial because they travel close to the boundary of spacetime along at fixed u and at two different angles.

only in the interval $u_f - u_i$ shifts, with respect to the initial value, by an amount $\Delta L = |L_f - L_i|$ depending only on ΔC_{AB} between the two null times [216, 154].

Let us take two observers travelling along timelike directions in positions (r_0, x^A) and $(r_0, x^A + \delta x^A)$ with r_0 large and with velocity v^μ approximately equal to $v^\mu = (1, \vec{0})$ and $v^2 = -1$. The separation vecotr $L^\mu = (0, \delta x^A)$ between the two geodesics satisfy the geodesic deviation equation

$$\frac{D^2 L^\mu}{d\tau^2} = -R^\mu_{\nu\rho\sigma} v^\nu v^\rho L^\sigma \Rightarrow \partial_u^2 L^B = -R^B_{uuA} v^u v^u L^A \tag{6.2.1}$$

using the solution for the metric and the Christoffel symbols one checks that

$$R_{BuuA} = \frac{r}{2} \partial_u^2 C_{AB} \tag{6.2.2}$$

so that integrating we get

$$\Delta L_A = \frac{1}{2r} \Delta C_{AB} L^B, \tag{6.2.3}$$

where Δ denotes the difference between the initial and the final values and $L_A = \gamma_{AB} L^B$. This expression can then be written in terms of ΔL.

As we have seen, ΔC_{AB} can be generated by a BMS supertranslation and hence the memory can be read off from the supertranslation charges [237]. From the Bondi mass aspect evolution equation (5.2.6) we get

$$\Delta m = \frac{1}{4} D^A D^B \Delta C_{AB} - \frac{1}{8} \int_{u_i}^{u_f} N_{AB} N^{AB} \tag{6.2.4}$$

and the change in C_{AB} can be addressed respectively to a change of mass or a passage of gravitational waves (as we supposed before). The first effect is called ordinary (or linear) memory, the second is called non-linear (or null) memory. Also null matter contributes to the latter (if we solve Einstein's equations with an energy-momentum tensor $T_{\mu\nu}$ with appropriate decay, the integrand above contains T_{uu}).

This is a beautiful way to exemplify vacuum degeneracy: two observers initially at a distance L in Minkowski space are perturbed by the passage of gravity waves. Their final distance, after the train is passed, is shifted by ΔL even if the spacetime is back to Minkowski: it is not the same Minkowski as before because of the shift in C_{AB}. In the asymptotic limit (along u) $\Delta_\infty C_{AB} = N$.

Similar considerations have led to the conjecture of the existence of a *spin memory* effect which is related to BT superrotations [202, 243, 244, 245] as well as other memories [156]. See Section 7.2 on this.

6.2.2 Christodoulou-Klainerman spacetimes and scattering

All the discussion on $\mathscr{I}^+$ can be translated in a Bondi gauge appropriate to $\mathscr{I}^-$, which is characterised by an advanced Bondi time v. Two distinct copies BMS_+ and BMS_- of the BMS group act respectively on $\mathscr{I}^+$ and $\mathscr{I}^-$. To get a unique symmetry group acting at the same time on the whole of $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ the null generators of the two disjoint components of null infinity have to be matched.

This is done in practice via an antipodal identification of points of the spheres at past and future null infinity, so to define a continuity condition between the asymptotic fields near i^0. This matching condition is always possible in Minkowski spacetime [140], because any null geodesic originating from a point p^- of coordinates (θ, ϕ) on $\mathscr{I}^-$ will end to the point q^+ of coordinates $(\theta - \pi, \phi + \pi)$ on $\mathscr{I}^+$ upon the identification $u = -v$. In stereographic coordinates on the spheres, the antipodal identification is conveniently written as $z\bar{z} = -1$.

However, for general AF spacetime, the matching is not obvious. We have already restricted the fields on $\mathscr{I}^+$ so that they do not develop logarithmic terms in u as they approach i^0. The same requirement can be imposed on the fields on $\mathscr{I}^-$.

Are there physically meaningful spacetimes corresponding to this requirement? There is at least a class: Christodoulou-Klainerman (CK) spacetimes [232]. They are characterised by $\epsilon_N = 3/2$ (see (6.1.13)) [43]. CK spacetimes are arbitrarily close to Minkowski spacetime (in a mathematically well-defined sense), are above the threshold for the formation of black holes and are characterised by $m|_{\mathscr{I}^+_+} = 0$, $M|_{\mathscr{I}^+_-} = M_{ADM}$, so that they radiate all their energy[10].

On the matched $\mathscr{I}^+$ and $\mathscr{I}^-$, the field C_{AB} at x can be given the same value of[11] C^-_{AB} evaluated at an antipodal point $\tilde{x}$ on S^2: $C_{AB}(x)|_{\mathscr{I}^+_-} = C^-_{AB}(\tilde{x})|_{\mathscr{I}^-_+}$ and the generators of BMS_0 can be identified with the subset of generators of $BMS_+ \times BMS_-$ satisfying $\alpha(x) = \alpha^-(\tilde{x})$ and $Y^A(x) = Y^A_-(\tilde{x})$. The subgroup of $BMS_+ \times BMS_-$ satisfying these conditions is the sought diagonal BMS_0.

Once the fields are antipodally identified, the charges can be antipodally identified so that we also obtain equality of fluxes along $\mathscr{I}^+$ and $\mathscr{I}^-$.

We are in the position to state the conjecture of the BMS invariance of the S-matrix. In particular, restricting to supertranslations this condition is written as

$$\langle \text{out} | \, [\mathcal{Q}^+_\alpha S - S\mathcal{Q}^-_\alpha] \, | \text{in} \rangle = 0. \tag{6.2.5}$$

To get this, one observes that supertranslation charges on $\mathscr{I}^+$ commute with those on $\mathscr{I}^-$ because the supertranslations do not affect each other: $\{\mathcal{Q}^+_\alpha, \mathcal{Q}^-_{\alpha^-}\} = 0$. Thinking of the S matrix as the exponential of the time evolution operator (the usual Hamiltonian), the claim is made that it commutes with $\mathcal{Q}_\alpha$ when $\alpha = 1$ because in such a case $\mathcal{Q}_1|_{\mathscr{I} \to i^0} = M_{ADM}$ is the "Hamiltonian" of the system [43]. From these observations, the conjecture follows that for any α we have

$$\{\mathcal{Q}_\alpha, S\} = 0 \tag{6.2.6}$$

where it is understood that the operators act on initial and final data (we use curly brakets as the same considerations apply at the classical level). This brings to (6.2.5).

[10] Spacetimes allowing for black holes with both massless and massive final states have been considered in recent BMS literature (see [154] and references therein). More in general, it seems that the only condition necessary to the matching is that the spacetimes evolve from a non-radiative ($N_{AB} = 0$) to a non-radiative configuration both at early and late advanced/retarded Bondi times, but no rigorous statements exist on these points at the moment.

[11] Quantities on $\mathscr{I}^-$ are denoted as f^-, later on we denote quantities on $\mathscr{I}^+$ with the superscript $+$, but this is not necessary.

The full equivalence of (6.2.5) to the Weinberg soft theorems is shown in [199] and it is based on the fact that the charge can be split in hard and soft parts

$$\{\mathcal{Q}_\alpha, S\} = \{\mathcal{Q}^\partial_\alpha, S\} + \{\mathcal{Q}^b_\alpha, S\} \tag{6.2.7}$$

A nice, somewhat different from [199], derivation of the soft theorems can be found in [201]. Further rigorous details on the perturbative realization of this theorem and the constraints that (6.2.6) implies can be found in [242].

Here we conclude with a couple of comments. First of all, these ideas are compatible with the statements in chapter 4 that the relevant holographic relations come from the cross sections of $\mathscr{I}$. Indeed, the first step toward the S-matrix analysis is realizing that the *in* and *out* states in (6.2.5) are represented on the cuts of $\mathscr{I}$ because they are labelled by the collection of points from which particles enter or exit the spacetime and their energies (assume no spin for simplicity): $|\text{in}\rangle = |Z, E^Z\rangle$ and $|\text{out}\rangle = |W, E^W\rangle$, where $Z = \{z_1, ..., z_k\}$ and $W = \{w_1, ..., w_k\}$ are the collection of points on $\mathscr{I}^-$, $\mathscr{I}^+$ respectively and E^Z, E^W the energies of the corresponding particles.

We also draw the attention to an early observation by Ashtekar [246], who pointed out that CK spacetimes do not, in fact, provide any evidence that the S-matrix is invariant under (diagonal) supertranslations because the associated antipodally identified charges vanish identically. For this reason and also because soft theorems are indeed established via other means, it is necessary to find other well-defined classes of spacetimes which do not trivialize the charges. Recently, K. Prabhu [247] discussed the case of spacetimes where $\epsilon_N = 1+\epsilon \neq 3/2$ and showed that the supertranslation charges are not trivialized. There is however no proof that realistic spacetimes satisfy this condition. On the other hand, other proofs of stability of Minkowski spacetimes uses $\epsilon_N = 1/2(+\epsilon)$ [248, 249] (polyhomogeneous spacetimes) but do not provide any proof that supertranslation charges are well defined. Given all these subtleties with the easiest case of supertranslations, it seems appropriate to be optimistically cautious about superrotations, which are affected by issues in the charges before the imposition of matching conditions. We will however be bolder in the next two chapters and move to higher dimensions and superrotations.

CHAPTER 7

Cosmic branes and asymptotic structure

Superrotations of asymptotically flat spacetimes in four dimensions can be interpreted in terms of including cosmic strings within the phase space of allowed solutions. In this chapter we explore the implications of the inclusion of cosmic branes on the asymptotic structure of vacuum spacetimes in dimension $d > 4$. We first show that only cosmic $(d-3)$-branes are Riemann flat in the neighbourhood of the brane, and therefore only branes of such dimension passing through the celestial sphere can respect asymptotic local flatness. We derive the asymptotically locally flat boundary conditions associated with including cosmic branes in the phase space of solutions. We find the asymptotic expansion of vacuum spacetimes in $d = 5$ with such boundary conditions; the expansion is polyhomogenous, with logarithmic terms arising at subleading orders in the expansion. The asymptotically locally flat boundary conditions identified here are associated with an extended asymptotic symmetry group, which may be relevant to soft scattering theorems and memory effects.

7.1 Introduction

As mentioned in the Introduction, BT-superrotations can be given a physical interpretation. It is similar to the memory effect interpretation of supertranslations and is due to A. Strominger and A. Zhiboedov [239]. They argue that the singularities of BT-superrotations are associated with cosmic strings piercing the celestial sphere. Indeed, local Witt transformations on the celestial sphere were already discussed in the cosmic string literature [250, 251]. A physical motivation for allowing general enough boundary conditions for the asymptotic symmetry group to include BT-superrotations (see Section 5.4) is hence to ensure that one includes in the phase space solutions such as Robinson-Trautman and their impulsive limits (i.e. snapping cosmic strings). The construction of such phase spaces was explored in [156]; see also earlier work [252].

The detailed relationship between BT-superrotations/subleading soft theorems and cosmic strings in four dimensions is not yet complete. In particular, in the derivations of [201] (see footnote 4, chapter 4) a subtle interplay between boundary terms of integrals in the complex plane and singularities of Y^A at z finite and at infinity is pointed out, but it is assumed that the only singularities of Y^A are associated with infinity. However, if more than one cosmic string pierce the celestial sphere, the corresponding punctures will correspond to singularities at finite points of the complex plane (see [251]).

This chapter is about boundary conditions and corresponding asymptotic symmetries for asymptotically (locally) flat spacetimes in dimensions higher than four.

As we stated in the Introduction, a puzzling feature is that while soft scattering theorems exist in all spacetime dimensions, the BMS symmetry is not realised within the radiative configuration space defined in [70] and while $Diff(S^{d-2})$ transformations can always be accommodated (provided they preserve the configuration space), BT-superrotations do not have any algebraic analogue in spacetime dimensions higher than four.

Following the relation between cosmic strings and superrotations discussed in [239] and summarised in Section 7.2, we use cosmic branes to define allowed boundary conditions for asymptotically (locally) flat spacetimes in $d > 4$. In $d = 4$ cosmic strings are Riemann flat except at the location of the string; their metrics are not Riemann flat in the vicinity of the string in higher dimensions. In Section 7.3 we explore cosmic branes in $d > 4$, following the original approach of Vilenkin for cosmic strings [253]. We point out that in d dimensions only cosmic $(d-3)$-branes are Riemann flat in the vicinity of the brane. (Note that there are distributional curvature singularities at the location of the brane, as for cosmic strings in four dimensions.) Therefore the direct analogue of cosmic strings in four dimensions is cosmic $(d-3)$-branes in d dimensions. Other types of cosmic branes are not locally Riemann flat near the brane; if such a brane pierces the celestial sphere, the geometry in the vicinity is not locally Riemann flat, and hence the resulting spacetime is not asymptotically locally flat.

This observation is consistent with the fact that higher dimensional generalisations of metrics describing cosmic strings snapping have never been found. For example, in [254] higher-dimensional generalisations of Robinson-Trautman spacetimes were constructed. There are no type N spherical gravitational waves in this class and because of this there is no impulsive limit; nor did [254] find an analogue of the four-dimensional C metric. It would be interesting to explore whether the class of solutions constructed in [254] could accommodate cosmic branes breaking: this seems quite likely, as the transverse spatial part of the metric is an arbitrary Riemann Einstein space, just as we find for our asymptotic solutions described below.

In Section 7.4 we analyse the asymptotic behaviour of cosmic branes, focussing for definiteness on the example of cosmic membranes in five dimensions. Following analogous discussions to those in [251, 239], we assume there are processes in which cosmic branes can snap, and infer the associated boundary conditions. In four dimensions, cosmic string snapping is consistent with asymptotically flat boundary conditions (except at the location of the punctures). In $d > 4$, the inclusion of snapping cosmic branes in the phase space requires changing the metric at leading asymptotic order and manifestly introduce asymptotically locally flat boundary conditions, which are summarised in Section 7.4.4. While for asymptotically flat spacetimes, the metric on the celestial sphere is asymptotically conformal to the round metric, the asymptotically locally flat boundary conditions allow for a general metric on the celestial sphere.

In Sections 7.5 and 7.6 we analyse the asymptotic structure of vacuum Einstein solutions with the weaker, asymptotically locally flat boundary conditions. As in [68, 69, 70, 255], the structure of the expansion depends on whether the spacetime is of odd or even dimension. For definiteness, we focus here on the case of five dimensions.

For five dimensional asymptotically locally flat spacetimes, the iterative solution of the main equations provides a minimal polyhomogeneous expansion, where each metric function is expanded as

$$f(r,u,x^A) = \sum_{i,j} f_{ij}(u,x^A)\frac{\ln^j r}{r^i} \tag{7.1.1}$$

with $f_{ij}(u,x^A)$ smooth functions of their arguments but almost all $j=0$ (this is the quality referred to as minimal, see next chapter for a specification of these terms).

There are striking analogies between the structure of the five dimensional asymptotically locally flat spacetimes we have constructed and that of asymptotically locally anti-de Sitter spacetimes in five dimensions. In both cases the coefficients of the leading logarithm terms are expressed in terms of derivatives of the non-normalizable data (the boundary conditions). In the case of anti-de Sitter, the occurrence of logarithmic terms is associated with conformal anomalies in the dual field theory.

While much of the earlier relativity literature imposed strictly anti-de Sitter boundary conditions, it is essential to relax these boundary conditions to asymptotically anti-de Sitter in the context of holography. The generalized boundary condition represents the background metric for the dual field theory. Even if one is only interested in the dual field theory in a flat background, one needs to allow the background metric source to vary to compute correlation functions of the stress energy tensor. As we discuss in sections 7.6 and sections 7.7, it would be interesting to express five-dimensional asymptotically locally anti-de Sitter solutions in Bondi gauge and take the limit of zero cosmological constant, to compare with our results here. This is an easy exercise after chapter 8 .

7.2 BT-Superrotations and cosmic strings

In section 6.1 we mentioned, following [237], that BT-superrotations are incompatible with the Wald-Zoupas formalism. This suggests that they cannot be interpreted as symmetries on the phase space of asymptotically flat spacetimes. Rather they act on a larger phase space. This interpretation was put forward by A. Strominger and A. Zhiboedov in [239].

The larger phase space contains boost-rotation symmetric solutions of Einstein's equations [256, 250, 239]. Such spacetimes possess incomplete null infinity in the sense that S^2 is missing two points, thus they are not asymptotically flat or asymptotically Minkowski. Their asymptotic structure was studied in [257, 258, 252, 256, 250, 259] to explore the isometries compatible with gravitational radiation (i.e. with asymptotic flatness)[1] In fact, BT-superrotations made their first appearance in this context [250], but they were simply called with their usual name (Witt algebra) and were not interpreted as symplectic transformations on a phase space.

The simplest example to consider is the single straight cosmic string. Other examples of such spacetimes are the C-metric and Bonnor-Swaminarayan spacetimes [260]. The line element of a straight cosmic string in $d = 4$ spacetime dimensions can be written as

$$ds^2 = -dU^2 - 2dUdR + R^2(d\Theta^2 + K^2 \sin^2\Theta^2 d\Phi^2) \tag{7.2.1}$$

where $K = 1 - \delta$ characterizes the deficit angle $2\pi\delta$ and in the weak field limit $\delta = 4G\mu$, with μ being the mass density of the string. Note that the cosmic string intersects the celestial sphere at the north and south pole i.e. $\Theta = 0, \pi$.

In order to show that this is indeed in the configuration space defined by the asymptotically Minkowski boundary conditions, we have to find an explicit map to the Bondi gauge with coordinates (u, r, θ, π). Under the assumption that it exists, the map must be of the form of a finite Bondi transformation [250, 57], namely

$$\begin{aligned} U &= U_{(0)}(u,\theta) + \frac{U_{(-1)}(u,\theta)}{r} + \cdots , & (7.2.2)\\ R &= rR_{(1)}(\theta) + R_{(0)}(u,\theta) + \cdots , & \\ \Theta &= \Theta_{(0)}(\theta) + \frac{\Theta_{(-1)}(\theta)}{r} + \cdots , & \\ \Phi &= \phi & (7.2.3) \end{aligned}$$

Here $R_{(1)}(\theta)$ and $\Theta_{(0)}(\theta)$ are necessarily independent of u to preserve the leading radial dependence of the u components of the metric and by symmetry one can identify $\Phi = \phi$.

The equations are to be solved order by order in r. At leading order, we get the match of (7.2.1) with the Minkowski metric

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.2.4}$$

on a null hypersurface at large r, namely

$$R^2(d\Theta^2 + K^2 \sin^2\Theta d\Phi^2) = r^2(d\theta^2 + \sin^2\theta d\phi^2) + \cdots , \tag{7.2.5}$$

where the ellipses denote terms that are subleading in r. Hence we obtain

$$R_{(1)}^2(\partial_\theta\Theta_{(0)})^2 = 1; \qquad K^2 R_{(1)}^2 \sin^2\Theta_{(0)} = \sin^2\theta, \tag{7.2.6}$$

[1] In the cited works, the answer is that at most two spacelike Killing vectors can exist if the spacetime is to be radiative.

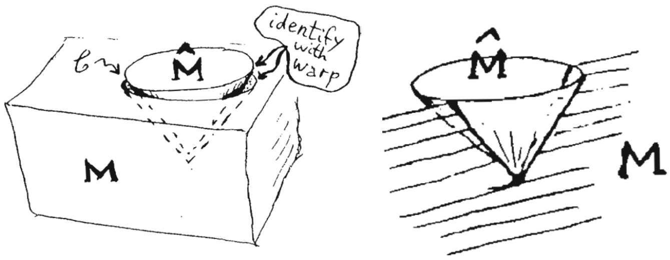

Figure 7.2.1: Penrose's cartoon of the cut and paste mapping and snapping cosmic string interpretation. Adapted from [262]. Permission granted by Prof. L. Mason, the host of the electronic copies of Twistor Newsletter.

which can be integrated to give

$$R_{(1)} = \frac{\sin\theta}{K\sin\Theta_{(0)}}; \qquad \int \operatorname{cosec}\Theta_{(0)}\, d\Theta_{(0)} = K\int \operatorname{cosec}\theta\, d\theta. \tag{7.2.7}$$

Note that the transformation is not analytic as $\theta \to 0$. The equations at subleading orders can be solved analogously. From them the cosmic string news tensor N_{AB} can be found [250] (notice here C is not the same of the previous chapter)

$$C_{AB} = \mathrm{diag}(1, -\sin^2\theta)C, \quad N_{AB} = \mathrm{diag}(1, -\sin^2\theta)\partial_u C, \quad C = -\frac{u}{\sin^2\theta}\delta\left(1 - \frac{\delta}{2}\right). \tag{7.2.8}$$

It is singular on the axis, as a consequence of the conical singularity. The string metric is thus, as we said, not asymptotically Minkowski. It can be however defined as asymptotically locally Minkowski.

Using stereographic coordinates $\zeta, \bar{\zeta}$ on S^2, (7.2.8) can be written as

$$C_{\zeta\zeta} = -u\{h,\zeta\} = \frac{-u}{2}\left[\frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2\right]. \tag{7.2.9}$$

for the transformation

$$\zeta \to h(\zeta) = \zeta^{1-\delta}. \tag{7.2.10}$$

Thanks to Penrose's cut and paste construction [261, 262], the map $(u, r, \zeta, \bar{\zeta}) \to (u, r, h, \bar{h})$ can be interpreted as the matching of the region outside a light-cone cut in Minkowski space with its interior after a warp. The complete spacetime admits a C^0 metric and describes the propagation of a spherical impulsive gravitational wave, whose generating process is the snapping/creation of a cosmic string (see Figure 7.2.1).

Strominger and Zhiboedov [239] interpreted the map $(u, r, \zeta, \bar{\zeta}) \to (u, r, h, \bar{h})$ as a finite superrotation of Minkowski space and thus concluded that superrotations map asymptotically flat to asymptotically locally flat spacetimes, in the sense appropriate to cosmic strings.

Many examples of such processes have been studied in the past [251] and it would be interesting to check if they can all be accommodated within superrotations of flat space.

We notice that [245] studies the spin memory effect - which we said was related to BT-superrtations - for compact binaries and emphasises that the spin memory effect is not related to the creation of superrotated spacetimes. Superrotated spacetimes however also generate memories because of the impulsive gravity waves accompanying the superrotation.

7.3 Defects in dimensions higher than four

In this section we will consider the behaviour of cosmic strings and branes in dimensions higher than four, following the analysis of Vilenkin in four dimensions [253]; see also discussions in [263, 264].

Let us consider a d-dimensional spacetime, with coordinates $(t, w, \mathbf{x})$. Suppose a static cosmic string is extended along the w direction, through $\mathbf{x} = \mathbf{0}$; by translation invariance the $\mathbf{x}$ position can always be chosen to be zero. Let μ be the tension of the cosmic string. Then the effective stress energy tensor sourcing the cosmic string is [253]

$$T^{\mu}_{\nu} = \mu \delta^{(d-2)}(x) \operatorname{diag}(1, 1, \mathbf{0}). \tag{7.3.1}$$

Physically, this equation states that the energy density is equal to minus the pressure along the string direction. We will discuss higher dimension defects below.

7.3.1 Linearized gravity

Now let us consider the backreaction of this stress energy tensor on the spacetime; we assume that μ is small and thus work within linearized gravity. The d-dimensional metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7.3.2}$$

where η is the Minkowski metric and h is the metric perturbation. The Einstein equations can then be expressed as

$$\partial^{\rho}\partial_{\nu}h_{\mu\rho} + \partial^{\rho}\partial_{\mu}h_{\nu\rho} - \Box h_{\mu\nu} - \partial_{\mu}\partial_{\nu}h + (\Box h - \partial^{\rho}\partial^{\sigma}h_{\rho\sigma})\eta_{\mu\nu} = 2T_{\mu\nu} \tag{7.3.3}$$

where we have set $8\pi G = 1$; $\Box$ is the d-dimensional d'Alembertian and we define

$$h = \eta^{\mu\nu}h_{\mu\nu}. \tag{7.3.4}$$

We impose the usual harmonic gauge

$$\partial^{\nu}h_{\mu\nu} = \frac{1}{2}\partial_{\mu}h. \tag{7.3.5}$$

The remaining gauge invariance is then captured by diffeomorphisms

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_{\nu}\xi_{\mu} + \partial_{\mu}\xi_{\nu} \tag{7.3.6}$$

for which

$$\Box\xi_\mu = 0. \tag{7.3.7}$$

In harmonic gauge the Einstein equations can be expressed as

$$\Box h_{\mu\nu} = -2\tilde{T}_{\mu\nu} \tag{7.3.8}$$

where $\tilde{T}_{\mu\nu}$ is the trace adjusted stress tensor

$$\tilde{T}_{\mu\nu} = (T_{\mu\nu} - \frac{1}{(d-2)}T\eta_{\mu\nu}) \tag{7.3.9}$$

with $T = \eta^{\rho\sigma}T_{\rho\sigma}$.

7.3.2 Cosmic strings in $d > 4$

We now solve the linearized Einstein equation (7.3.8) with a trace adjusted stress tensor corresponding to a cosmic string (7.3.1):

$$\tilde{T}_{\mu\nu} = \frac{\mu}{(d-2)}\delta^{(d-2)}(x)\ \mathrm{diag}\left((4-d), (d-4), -2\mathbf{1}_{(d-2)}\right). \tag{7.3.10}$$

Note that the metric backreaction should, by symmetry, be independent of the worldsheet coordinates (t, w) and should be rotationally symmetric in the transverse directions.

In $d = 4$ the solution to the linearized equations can be written as [253]

$$\begin{aligned} h_{tt} &= h_{ww} = 0; \\ h_{xx} &= h_{yy} = \tilde{\mu}\ln\left(\frac{r}{r_o}\right) \end{aligned} \tag{7.3.11}$$

where $r^2 = x^2 + y^2$ and

$$\tilde{\mu} = \frac{\mu}{\pi}. \tag{7.3.12}$$

In this solution r_o can be interpreted as the characteristic radius scale of the string. The linearized solution is valid provided that $|h| \ll 1$, and thus the linearized solution is strictly only applicable within a neighbourhood of the string.

Thus one can write the four-dimensional (linearized) cosmic string metric as

$$ds^2 = -dt^2 + dw^2 + (1-\lambda)(dr^2 + r^2 d\phi^2) \tag{7.3.13}$$

where we use (r, ϕ) as polar coordinates in the (x, y) plane and

$$\lambda = \tilde{\mu}\ln\left(\frac{r}{r_o}\right). \tag{7.3.14}$$

Introducing a new radial coordinate

$$(1-\lambda)r^2 = (1-\tilde{\mu})\tilde{r}^2 \tag{7.3.15}$$

(and working to linear order in $\tilde{\mu}$) one can change the metric into the more familiar form

$$ds^2 = -dt^2 + dw^2 + d\tilde{r}^2 + (1 - \tilde{\mu})\tilde{r}^2 d\phi^2 \tag{7.3.16}$$

i.e. the metric is locally flat in the directions transverse to the string, but there is a conical deficit proportional to $\tilde{\mu}$. Reinstating $8\pi G$ we have $\delta = 4G\mu = \tilde{\mu}/2$. Note that, even though the derivation above was at the level of the linearised equations, this metric manifestly solves the Einstein equations at non-linear order and is moreover locally flat.

Now let us turn to $d > 4$. A qualitative difference in $d \neq 4$ is that the components of (7.3.10) along the string do not vanish. Consider the equation

$$\Box f = \frac{2}{(d-2)} \mu \delta^{(d-2)}(x). \tag{7.3.17}$$

Solutions with rotational symmetry in the directions transverse to the string can be expressed as

$$f = \frac{\tilde{\mu}}{r^{(d-4)}} \tag{7.3.18}$$

for $d > 4$ with

$$\tilde{\mu} = \frac{2}{(d-2)(d-4)\Omega_{(d-2)}} \mu \tag{7.3.19}$$

where $\Omega_{(d-2)}$ is the $(d-2)-$dimensional solid angle. Then the metric near the cosmic string can be written as

$$ds^2 = \left(1 - \frac{(d-4)}{2}\frac{\tilde{\mu}}{r^{(d-4)}}\right)\left(-dt^2 + dw^2\right) + \left(1 + \frac{\tilde{\mu}}{r^{(d-4)}}\right)\left(dr^2 + r^2 d\Omega^2_{(d-3)}\right). \tag{7.3.20}$$

This solution is not locally Riemann flat close to the cosmic string, although since it satisfies the Einstein equations (with a string source) it is Ricci flat for $r \neq 0$. The metric is asymptotically locally flat for $r^{d-4} \gg \tilde{\mu}$. However, since an infinite cosmic string necessarily intersects the celestial sphere in two points, and the metric is not locally flat in the immediate neighbourhood of the string, the cosmic string metric is not asymptotically locally flat over the entire celestial sphere.

7.3.3 Cosmic branes

Let us now consider a d-dimensional spacetime, with coordinates $(t, \mathbf{w}, \mathbf{x})$, where there are p spatial coordinates $\mathbf{w}$ and correspondingly $(d-p-1)$ transverse coordinates $\mathbf{x}$. A static cosmic p-brane is extended along the $\mathbf{w}$ directions and located at $\mathbf{x} = \mathbf{0}$. (By translation invariance the $\mathbf{x}$ position can again always be chosen to be zero.) Let μ be the tension of the cosmic brane. Then the effective stress energy tensor sourcing the cosmic brane is, generalizing the cosmic string,

$$T^{\mu}_{\nu} = \mu \delta^{(d-p-1)}(x) \text{ diag}(1, 1_{(p)}, 0_{(d-p-1)}). \tag{7.3.21}$$

Physically, this equation states that the energy density is equal to minus the pressures along the brane. Note that in four dimensions a cosmic membrane would usually be referred to as a domain wall, as there is only one transverse direction, and such solutions were discussed together with cosmic strings in [253].

The corresponding trace adjusted stress tensor is then

$$\tilde{T}_{\mu\nu} = \frac{\mu}{(d-2)} \delta^{(d-p-1)}(x) \, \mathrm{diag}(-(d-p-3), (d-p-3)_{(p)}, -(p+1)_{(d-p-1)}). \tag{7.3.22}$$

In the case that $d = (p+3)$ this implies that

$$h_{tt} = h_{ww} = 0, \tag{7.3.23}$$

i.e. the metric perturbations longitudinal to the brane are zero. The transverse space to the brane then has dimension two and the corresponding form for the metric near the cosmic brane is

$$ds^2 = -dt^2 + dw \cdot dw_{(d-3)} + (1-\lambda)(dr^2 + r^2 d\phi^2) \tag{7.3.24}$$

where now

$$\lambda = \tilde{\mu} \ln \left(\frac{r}{r_o} \right) \tag{7.3.25}$$

with

$$\tilde{\mu} = \frac{(p+1)}{(d-2)} \frac{\mu}{\pi}. \tag{7.3.26}$$

Following the same logic as above, the metric (7.3.24) can be written in a form which is manifestly locally flat, namely

$$ds^2 = -dt^2 + dw \cdot dw_{(d-3)} + d\tilde{r}^2 + (1-\tilde{\mu})\tilde{r}^2 d\phi^2, \tag{7.3.27}$$

with the transverse space to the brane having a conical singularity at $\tilde{r} = 0$.

In the case that $d \neq (p+3)$ the metric perturbations longitudinal to the brane are non-zero, and the solution near the brane is Ricci flat but not locally Riemann flat, as in the case of cosmic strings in $d \neq 4$ discussed above. Thus we see that branes of codimension two play a distinguished role when we are interested in asymptotically locally flat geometries.

Notice that in terms of the codimension of the object, the result is independent of the number of dimension. Our proposal is to include codimension-two objects in the phase space of asymptotically flat gravity. In $d = 3$ this is the accepted definition [187].

Remark We should note that there has been considerable discussion in the relativity literature about distributional sources. The analysis of [265] highlighted subtleties in dealing with distributional source of codimension greater than one: the metric is inherently distributional and the curvature is constructed from products of metric derivatives. This implies that different regularisations of cosmic strings can lead to thin, static strings with different mass per unit lengths. Later work by Garfinkle [266] defined a notion of semi-regular metrics, in which the static cosmic string has a distributional stress energy; however, it is also argued in this work that such stress energy may not actually describe the physical energy content. The work of [267] explored distributional brane sources, showing that one can make sense of stress confined to codimension two surfaces in certain situations. There is also an ongoing programme of work using generalized functions to understand distributional curvature, beginning with [268, 269].

Following the earlier work of [216], the metric (7.3.27) will be the starting point for our analysis, and motivation for considering more general boundary conditions than asymptotically flat in $d > 4$. The detailed description of the distributional curvature will not be central to our analysis. Ultimately the physical interpretation of such branes may well go beyond general relativity into string theory, in which branes are valid physical objects with well understood stress energy (and where the limits of the validity of general relativity solutions are also understood).

7.3.4 Cosmic branes: general position and orientation

In the previous section we gave solutions that are longitudinal to the **w** directions and located at the origin in the transverse directions. It is clearly straightforward to generalize such solutions to arbitrary position and orientation. Let us first write the solution (7.3.27) in terms of Cartesian coordinates in the transverse directions i.e.

$$\begin{aligned} d\tilde{r}^2 + (1-\tilde{\mu})\tilde{r}^2 d\phi^2 &= \frac{1}{(x^2+y^2)}\big((x^2+K^2y^2)dx^2 + 2xy(1-K^2)dxdy \\ &\qquad +(y^2+K^2x^2)dy^2\big)\,, \\ &= dx^2+dy^2 - \frac{\tilde{\mu}}{(x^2+y^2)}\,(ydx-xdy)^2 \end{aligned} \tag{7.3.28}$$

where we use the notation $K^2 = (1-\tilde{\mu})$. Clearly when $K = 1$ this metric reduces to the standard Euclidean metric in Cartesian coordinates. Note that one can also write (7.3.27) as

$$ds^2 = dzd\bar{z} + \frac{\tilde{\mu}}{4z\bar{z}}\,(\bar{z}dz - zd\bar{z})^2 \tag{7.3.29}$$

in terms of a complex coordinate $z = (x+iy)$.

It is then straightforward to displace the brane from the origin to (x_o, y_o) by shifting

$$x \to (x-x_o) \qquad y \to (y-y_o). \tag{7.3.30}$$

Clearly for $K = 1$ this would leave the metric invariant but for general K we obtain

$$dx^2 + dy^2 - \frac{\tilde{\mu}}{((x-x_o)^2+(y-y_o)^2)}\,((y-y_o)dx-(x-x_o)dy)^2\,. \tag{7.3.31}$$

Moving back to polar coordinates, the metric takes the simple form

$$d\tilde{r}^2 + K^2(\tilde{r}-\tilde{r}_o)^2 d\phi^2 \tag{7.3.32}$$

where $\tilde{r}_o^2 = (x_o^2 + y_o^2)$.

Using the Cartesian form of the metric (7.3.28) it is also straightforward to rotate the orientation of the brane. For example, if we rotate in the (wx) plane by an angle α via

$$w \to \cos\alpha w - \sin\alpha x \qquad x \to \cos\alpha x + \sin\alpha w \tag{7.3.33}$$

we obtain

$$dw^2 + dx^2 + dy^2 - \frac{\tilde{\mu}}{((\cos\alpha x + \sin\alpha w)^2 + y^2)} \left(y(\cos\alpha dx + \sin\alpha dw) - (\cos\alpha x + \sin\alpha w)dy\right)^2 \tag{7.3.34}$$

Note that the residual rotational symmetry transverse to the brane is not manifest in this coordinate system.

7.3.5 Asymptotics

Let us now return to (7.3.27). To analyse the asymptotics we should rewrite it as

$$ds^2 = -dt^2 + dw^2 + w^2 d\Omega_{p-1}^2 + d\tilde{r}^2 + K^2\tilde{r}^2 d\phi^2 \tag{7.3.35}$$

and then let

$$w = R\cos\Theta \qquad \tilde{r} = R\sin\Theta \tag{7.3.36}$$

to obtain

$$ds^2 = -dt^2 + dR^2 + R^2\left(d\Theta^2 + \cos^2\Theta d\Omega_{p-1}^2 + K^2\sin^2\Theta d\phi^2\right). \tag{7.3.37}$$

Note that both the longitudinal $SO(p)$ rotational symmetry and the transverse $SO(2)$ rotational symmetry are manifest.

A brane which is located at $\tilde{r} = \tilde{r}_o$ (7.3.32) can be expressed as

$$\begin{aligned} ds^2 \;&=\; -dt^2 + dR^2 + R^2\left(d\Theta^2 + \cos^2\Theta d\Omega_{p-1}^2 + \sin^2\Theta d\phi^2\right) \\ &\quad -\tilde{\mu}\frac{\left(-R^2\sin^2\Theta d\phi - \tilde{r}_o\sin(\phi-\phi_o)d(R\sin\Theta) + \tilde{r}_o R\sin\Theta\cos(\phi-\phi_o)d\phi\right)^2}{(R^2\sin^2\Theta + \tilde{r}_o^2 - 2\tilde{r}_o R\sin\Theta\cos(\phi-\phi_o))} \end{aligned} \tag{7.3.38}$$

or equivalently as

$$\begin{aligned} ds^2 \;&=\; -dt^2 + dR^2 + R^2\left(d\Theta^2 + \cos^2\Theta d\Omega_{p-1}^2 + \sin^2\Theta d\phi^2\right) \\ &\quad -\tilde{\mu}\frac{\left(-R\sin^2\Theta d\phi - \sin\Theta_o\sin(\phi-\phi_o)d(R\sin\Theta) + R\sin\Theta_o\sin\Theta\cos(\phi-\phi_o)d\phi\right)^2}{(\sin^2\Theta + \sin^2\Theta_o - 2\sin\Theta_o\sin\Theta\cos(\phi-\phi_o))} \end{aligned} \tag{7.3.39}$$

where we define

$$\sin\Theta_o = \frac{\tilde{r}_o}{R}. \tag{7.3.40}$$

For Θ_o to remain finite as $R \to \infty$ we will clearly need to take $\tilde{r}_o$ to infinity with the ratio of $\tilde{r}_o/R$ fixed.

Note that the metric (7.3.39) has a hidden $U(1)$ symmetry, corresponding to rotations around $\tilde{r} = \tilde{r}_o$. The metric on a surface of constant R and t is

$$\begin{aligned} ds^2 \;&=\; R^2\left(d\Theta^2 + \cos^2\Theta d\Omega_{p-1}^2 + \sin^2\Theta d\phi^2\right) \\ &\quad -\tilde{\mu}R^2\frac{\left(\sin^2\Theta + \sin\Theta_o\sin(\phi-\phi_o-\Theta)\right)^2 d\phi^2}{(\sin^2\Theta + \sin^2\Theta_o - 2\sin\Theta_o\sin\Theta\cos(\phi-\phi_o))}\ . \end{aligned} \tag{7.3.41}$$

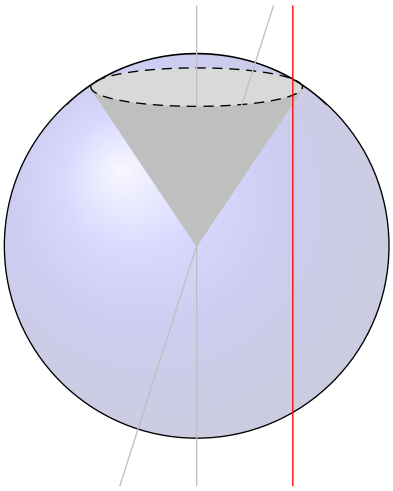

Figure 7.3.1: Three cosmic strings: a string passing though the north pole of the sphere; a string rotated with respect to this axis and a third string (red) translated with respect to the first one.

A surface of constant R and t clearly does not have such a $U(1)$ symmetry; only the $SO(p)$ symmetry along the longitudinal directions of the brane survives. This is illustrated in the case of $p = 1$ in Figure 7.3.1: the string clearly has an axial $SO(2)$ symmetry but the intersection with the celestial two sphere does not preserve this $SO(2)$ symmetry.

The asymptotics of a rotated cosmic brane can also be obtained using the radial coordinate R. In this case there is an axial $SO(2)$ symmetry which is respected by the intersection with the celestial sphere; this is however not manifest in the coordinates (Θ, ϕ). A rotated string is shown in Figure 7.3.1.

Note that much of the previous literature on 5d cosmic branes has concentrated on spacetimes with cylindrical symmetry i.e. one writes the metric for flat space as

$$ds^2 = -du^2 - 2dud\rho + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) + dz^2 \tag{7.3.42}$$

i.e. as a direct product of four-dimensional Minkowski spacetime with a line. This form of the metric is particularly convenient when one compactifies the z direction around a circle i.e. one is interested in a Kaluza-Klein spacetime or a brane world Randall-Sundrum setting. However, (7.3.42) is not expressed in a form that is natural for analysing the asymptotic structure if z is not compact; analysis of the structure close to null infinity requires the introduction of a radial coordinate $r^2 = \rho^2 + z^2$, to characterise the celestial sphere.

7.4 Cosmic branes and asymptotically locally flat spacetimes

In this section we consider the asymptotic structure of cosmic $(d-3)$-branes and show how such spacetimes can be expressed in Bondi gauge. This analysis demonstrates the boundary conditions that should be imposed on the metric functions in Bondi gauge so that cosmic $(d-3)$-branes are contained within the set of solutions of the vacuum Einstein equations.

For $d > 4$, the boundary conditions required are weaker than those imposed in earlier literature: the inclusion of cosmic $(d-3)$-branes defines boundary conditions for asymptotically locally flat spacetimes. For concreteness we focus mostly on the case of $d = 5$ but the generalization of these boundary condtions to arbitrary $d > 4$ is straightforward and is summarised at the end of this section.

7.4.1 Five dimensional cosmic brane metrics

For the metric in the vicinity of the brane to be locally flat, the brane must be a $(d-3)$-brane, i.e. a membrane in five dimensions. In five dimensions we can parameterise locally flat metrics with deficits in several ways and in this section we discuss convenient parameterisations.

Let us first consider

$$ds^2 = -dU^2 - 2dUdR + R^2(d\Theta^2 + \cos^2\Theta d\Psi^2 + K^2 \sin^2\Theta d\Phi^2). \tag{7.4.1}$$

For $K^2 = 1$ hypersurfaces of constant U are round three spheres, with a $U(1)^2$ subgroup of the $SO(4)$ isometry group made manifest. If we introduce a deficit $K^2 = 1-2\delta$, the deficit is associated with $\Theta = 0$, but extends around the entire Ψ circle i.e. there is a cosmic membrane intersecting the celestial three-sphere in a circle. To see this, it is convenient to exploit the embedding of the three sphere into R^4 i.e.

$$x = R\cos\Theta\cos\Psi;\ y = R\cos\Theta\sin\Psi;\ z = R\sin\Theta\cos\Phi;\ w = R\sin\Theta\sin\Phi. \tag{7.4.2}$$

Thus $\Theta = 0$ corresponds to the circle $x^2 + y^2 = R^2$ with $z = w = 0$.

There is an obvious generalisation of (7.4.1):

$$ds^2 = -dU^2 - 2dUdR + R^2(d\Theta^2 + K_1^2 \cos^2\Theta d\Psi^2 + K_2^2 \sin^2\Theta d\Phi^2) \tag{7.4.3}$$

in which for $K_1^2 \neq 1$ and $K_2^2 \neq 1$ there is a cosmic membrane intersecting the sphere in the circle $x^2 + y^2 = R^2$ with $z = w = 0$ and a second membrane intersecting $z^2 + w^2 = R^2$ with $x = y = 0$. This specific configuration of membranes preserves the $U(1)^2$ symmetry associated with rotations in the (x, y) and (w, z) planes.

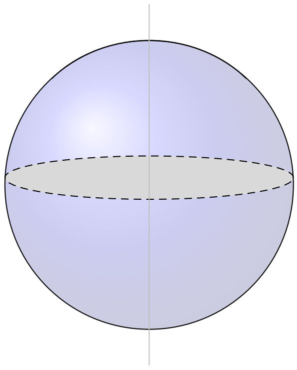

Figure 7.4.1: Cosmic string and membrane intersecting the celestial sphere.

We could alternatively study

$$ds^2 = -dU^2 - 2dUdR + R^2(d\Theta^2 + K_1^2 \sin^2\Theta dX^2 + K_2^2 \sin^2\Theta \sin^2(K_1 X) d\Phi^2) \tag{7.4.4}$$

where for $K_1^2 = K_2^2 = 1$ hypersurfaces of constant U are round three spheres, in which an $SO(3)$ subgroup of $SO(4)$ is made manifest. The metric is manifestly locally flat for $K_1 \neq 1$ and $K_2 \neq 1$: this follows from the coordinate redefinitions $\chi = K_1 X$ and $\phi = K_2\Phi$, which bring the metric into the form of a flat metric. These coordinate redefinitions are locally trivial; deficits are introduced by imposing the standard ranges on the redefined coordinates i.e. $0 \leq X \leq \pi$ and $0 \leq \Phi < 2\pi$.

When a deficit is introduced by setting $K_2^2 \neq 1$ (with $K_1^2 = 1$), the deficit is associated with $X = 0, \pi$. The interpretation is again most easily seen by embedding the (round) three sphere into R^4 as

$$x = R\sin\Theta\sin X\sin\Phi; \qquad y = R\sin\Theta\sin X\cos\Phi; \qquad z = R\sin\Theta\cos X; \qquad w = R\cos\Theta \tag{7.4.5}$$

i.e. the deficit is associated with $z^2 + w^2 = R^2$, $x = y = 0$, a great circle of the sphere. This metric thus describes the same physics as the metric shown in (7.4.1) but the parameterisation of (7.4.4) is less convenient, as it does not make manifest the second $SO(2)$ symmetry preserved by the cosmic membrane.

When $K_1^2 \neq 1$ (with $K_2^2 = 1$) the deficit is associated with geodesic incompleteness of the two spheres parameterized by (X, ϕ). For constant U, R and Θ the induced two-dimensional metric is

$$ds^2 = R^2 \sin^2\Theta K_1^2 (dX^2 + \frac{1}{K_1^2}\sin^2(K_1 X) d\Phi^2) \tag{7.4.6}$$

which describes part of a two sphere of radius $K_1 R \sin\theta$; more precisely, since $0 \leq X \leq \pi$, there is a boundary to (7.4.6) at $X = \pi$:

$$ds^2 = R^2 \sin^2\Theta \sin^2(K_1\pi) d\Phi^2 \tag{7.4.7}$$

i.e. a circle. We will not consider this case further as it does not seem to have a natural physical interpretation.

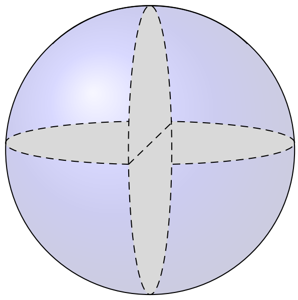

Figure 7.4.2: Cosmic string and membrane intersecting the celestial sphere.

Let us now relate these discussions to the cosmic brane solutions of the previous section. The metric (7.4.4) can be written in terms of a time coordinate

$$t = U + R \tag{7.4.8}$$

as

$$ds^2 = -dt^2 + dR^2 + R^2(d\Theta^2 + K_1^2 \cos^2 \Theta d\Psi^2 + K_2^2 \sin^2 \Theta d\Phi^2). \tag{7.4.9}$$

Now introduce coordinates

$$\tilde{r} = R\cos\Theta; \qquad w = R\sin\Theta \tag{7.4.10}$$

in terms of which the metric can be expressed as

$$ds^2 = -dt^2 + (d\tilde{r}^2 + K_1^2\tilde{r}^2 d\Phi^2) + (dw^2 + K_2^2 w^2 d\Psi^2) \tag{7.4.11}$$

Consider first the case of $K_2^2 = 1$. Comparing with (7.3.27), the cosmic membrane is located at $\tilde{r} = 0$, i.e. $\Theta = \pi/2$, and lies in the (w, Ψ) plane. This defect is visualised in Figure 7.4.1, as the plane intersecting the celestial sphere in a circle. For $K_2^2 \neq 1$, there is in addition a membrane located at $w = 0$, lying in the $(\tilde{r}, \Phi)$ plane. This second membrane intersects the first at $\tilde{r} = w = 0$, pictured in Figure 7.4.1. Note that this intersection does not take place close to the celestial sphere, so any non-linear effects at the intersection are not relevant for asymptotic analysis.

Before we move to the general asymptotic analysis, let us consider an infinite cosmic string, which as shown in Figure 7.4.1 necessarily intersects the celestial sphere at two points. We are interested in metrics which are asymptotically locally flat at infinity. However, from the discussions of the previous section, a cosmic string metric is locally flat (as opposed to Ricci flat) near the string only in four dimensions. We therefore cannot match a cosmic string metric with a flat metric on a null hypersurface, except in four dimensions; equivalently, we cannot apply coordinate transformations to a flat metric and obtain a cosmic string metric in dimensions other than four.

7.4.2 Asymptotically locally flat metrics in five dimensions

Now let us extend the discussion of Section 7.2 to five dimensions. We allow for dynamical processes in which cosmic branes are created and destroyed and, as before, we consider the matching of a cosmic brane metric to a Ricci flat metric without cosmic brane on a null hypersurface.

For computational simplicity we consider cosmic brane metrics that preserve $U(1)^2$ symmetry in the angular directions and have reflection/inversion symmetry in these directions. Such metrics can be matched to asymptotically locally flat spacetimes with corresponding symmetry which can be described using a Bondi gauge parametrisation:

$$\begin{aligned} ds^2 &= -(\mathcal{U}e^{2\beta} - r^2W^2e^{C_1})du^2 - 2e^{2\beta}dudr - 2r^2We^{C_1}dud\theta \\ &\quad + r^2(e^{C_1}d\theta^2 + e^{-(C_1+C_2)}\cos^2\theta d\psi^2 + e^{C_2}\sin^2\theta d\phi^2). \end{aligned} \tag{7.4.12}$$

Here the defining metric functions $(\mathcal{U}, W, \beta, C_1, C_2)$ depend only on (u, r, θ) due to the symmetry. We have also imposed the standard Bondi gauge conditions i.e.

$$g_{rr} = g_{rA} = 0 \tag{7.4.13}$$

and the determinant of the angular part of the metric is r^6; these conditions mirror the original four-dimensional conditions [57, 59].

The standard definition of asymptotically flat spacetimes at null infinity in five dimensions (see chapter 5 and [66, 67, 68, 69, 270]) imposes the following boundary conditions on the defining functions for solutions of the vacuum Einstein equations:

$$\begin{aligned} \mathcal{U}(u,r,\theta) &= 1 + \frac{\mathcal{U}_{(3/2)}(r,\theta)}{r^{\frac{3}{2}}} + \cdots \\ W(u,r,\theta) &= \frac{W_{(3/2)}(r,\theta)}{r^{\frac{3}{2}}} + \cdots \\ \beta(u,r,\theta) &= \frac{\beta_{(3)}(r,\theta)}{r^3} + \cdots \\ C_i(u,r,\theta) &= \frac{C_{i(3/2)}(r,\theta)}{r^{\frac{3}{2}}} + \cdots \end{aligned} \tag{7.4.14}$$

where $i = 1, 2$ and the ellipses denote terms that are subleading as $r \to \infty$. As we review below, gravitational waves are associated with the $C_{i(3/2)}$ contributions, which in turn induce subleading terms in the other metric functions. Additional integration functions arise at order $1/r^2$ and are associated with mass and angular momentum; we will discuss these later, when we derive the asymptotic expansions to all orders.

We now consider the matching between (7.4.3) and (7.4.12) on a null hypersurface as $r \to \infty$ and show that such a matching requires weaker boundary conditions than asympotically flat boundary conditions (7.4.14). By symmetry, we can identify $\Psi = \psi$ and $\Phi = \phi$. Following the four-dimensional discussion, we parameterise the coordinate transformations as

$$\begin{aligned} U &= U_{(0)}(u,\theta) + \frac{U_{(-1)}(u,\theta)}{r} + \cdots \\ R &= rR_{(1)}(\theta) + R_{(0)}(u,\theta) + \cdots \\ \Theta &= \Theta_{(0)}(\theta) + \frac{\Theta_{(-1)}(\theta)}{r} + \cdots \end{aligned} \tag{7.4.15}$$

Matching on a null hypersurface then imposes three relations at leading order:

$$\begin{aligned} (\theta\theta): \quad & R_{(1)}^2(\partial_\theta\Theta_{(0)})^2 &=& \; e^{c_1(\theta)} \\ (\psi\psi): \quad & R_{(1)}^2 \cos^2\Theta_{(0)} &=& \; e^{-(c_1(\theta)+c_2(\theta))}\cos^2\theta \\ (\phi\phi): \quad & R_{(1)}^2 \sin^2\Theta_{(0)} &=& \; \frac{1}{K^2}e^{c_2(\theta)}\sin^2\theta \end{aligned} \tag{7.4.16}$$

where we indicate the components of the induced metric being matched and we expand the defining metric functions (C_1, C_2) as

$$C_i(u,r,\theta) = c_i(\theta) + \frac{C_{i(-\lambda)}(u,\theta)}{r^\lambda} + \cdots \tag{7.4.17}$$

where the exponent $\lambda > 0$ will follow from imposing Ricci flatness. In the case that $c_i = 0$, then $\lambda = \frac{3}{2}$ as in (7.4.14) but this is no longer true when $c_i \neq 0$, as we will show below.

Before we consider the solution of (7.4.16), let us discuss the structure of the coordinate transformations in (7.4.15). As in four dimensions, the leading terms in $R(u,r,\theta)$ and $\Theta(u,r,\theta)$ are forced to be independent of u, as u dependence would induce metric components along the u direction that scale as a positive power of r, thus breaking the notion of asymptotic local flatness.

The leading order contributions to the other metric components are:

$$\begin{aligned} (r\theta) \;&:\; \mathcal{O}(r^0) \qquad g_{r\theta} = R_{(1)}\Theta_{(-1)}\partial_\theta\Theta_{(0)} + \partial_\theta U_{(0)} \\ (ur) \;&:\; \mathcal{O}(r^0) \qquad g_{ur} = R_{(1)}\partial_u U_{(0)} \\ (uu) \;&:\; \mathcal{O}(r^0) \qquad g_{uu} = -(\partial_u U_{(0)})^2 + R_{(1)}^2(\partial_u\Theta_{(-1)})^2 \\ (u\theta) \;&:\; \mathcal{O}(r) \qquad g_{u\theta} = \partial_\theta R_{(1)}\partial_u U_{(0)} - R_{(1)}^2\partial_\theta\Theta_{(0)}\partial_u\Theta_{(-1)} \\ (rr) \;&:\; \mathcal{O}(r^2) \qquad g_{rr} = 2R_{(1)}U_{(-1)} + R_{(1)}^2(\Theta_{(-1)})^2 \end{aligned} \tag{7.4.18}$$

These relations put no further conditions on $(\Theta_{(0)}, R_{(1)})$, which are determined by (7.4.16), but instead determine $(U_{(0)}, \Theta_{(-1)}, \cdots)$ in terms of these functions.

Now let us consider the solution of (7.4.16). If we impose strict asymptotic flatness as in (7.4.14), then we need to set $c_1(\theta) = c_2(\theta) = 0$. However, in this case the three conditions of (7.4.16) are clearly incompatible: the first and third relations are identical to those in four dimensions and are solved as in (7.2.7) but this solution is not consistent with the second relation in (7.4.16).

One can conceptualise why a three sphere with a ring of conical deficits cannot be mapped to a round three sphere as follows. Hypersurfaces of constant Θ are topologically tori, with the ψ and ϕ circles parameterising the independent non-contractable cycles of these tori. There is a geometric

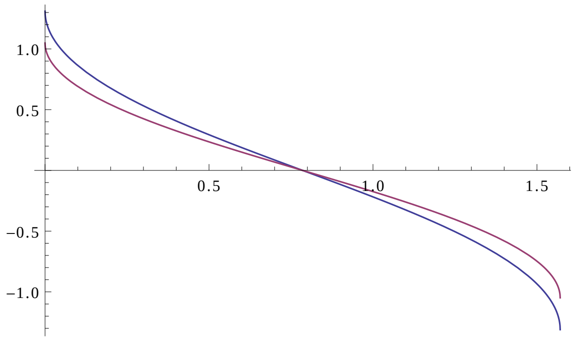

Figure 7.4.3: The blue line shows $F\left(\frac{1}{2}(\frac{\pi}{2}-2x)|2\right)$, plotted over the range $(0,\pi/2)$. The red line shows $K^{\frac{1}{2}}F\left(\frac{1}{2}(\frac{\pi}{2}-2x)|2\right)$, plotted over the same range, with $K^{\frac{1}{2}}=0.8$.

interpretation of solving the first and third relations in (7.4.16) (with $c_1(\theta)=c_2(\theta)=0$): one uses an angle dependent rescaling of the radius to remove the deficit in the ϕ circle as $\theta\to 0$. However, this same angle dependent rescaling of the radius is then incompatible with maintaining the periodicity of the ψ circle.

We thus conclude that we cannot solve (7.4.16) without allowing for non-zero $c_i(\theta)$. However, if the functions $c_i(\theta)$ are non-zero, the system of equations now seems to be under-constrained: there are only three equations for four functions $(\Theta_{(0)}(\theta), R_{(1)}(\theta), c_1(\theta), c_2(\theta))$. Combining the three equations one can obtain the following relation

$$\frac{d\Theta_{(0)}}{\sqrt{\sin 2\Theta_{(0)}}}=K^{\frac{1}{2}}\frac{e^{\frac{3c_1(\theta)}{4}}d\theta}{\sqrt{\sin 2\theta}}. \tag{7.4.19}$$

From this relation we can see that one of the four functions follows from the freedom to redefine the angular coordinate; imposing $\Theta_{(0)}=\theta$ for $K=1$ fixes $c_1(\theta)=0$. Thus $\Theta_{(0)}(\theta)$ is given by

$$\int\frac{d\Theta_{(0)}}{\sqrt{\sin 2\Theta_{(0)}}}=K^{\frac{1}{2}}\int\frac{d\theta}{\sqrt{\sin 2\theta}}, \tag{7.4.20}$$

and the other functions are determined by the relations

$$\begin{aligned} R_{(1)}(\theta) &= \frac{1}{K^{\frac{1}{2}}}\sqrt{\frac{\sin 2\theta}{\sin 2\Theta_{(0)}}} \\ e^{c_2(\theta)} &= K\frac{\tan\Theta_{(0)}}{\tan\theta}. \end{aligned} \tag{7.4.21}$$

Integrating (7.4.20) we obtain

$$F\left(\frac{1}{2}(\frac{\pi}{2}-2\Theta_{(0)})|2\right)=K^{\frac{1}{2}}F\left(\frac{1}{2}(\frac{\pi}{2}-2\theta)|2\right), \tag{7.4.22}$$

where $F(y|m)$ is the elliptic integral of the first kind. This elliptic integral is plotted in Figure 7.4.3. For K^2 just less than one, we can read off from Figure 7.4.3 the behaviour of $\Theta_{(0)}(\theta)$: given the value of $0\leq\theta\leq\frac{\pi}{2}$, we use the red curve to determine the right hand side of (7.4.22). We then map horizontally from the red curve to the blue curve to read off the value of $\Theta_{(0)}$. We note that

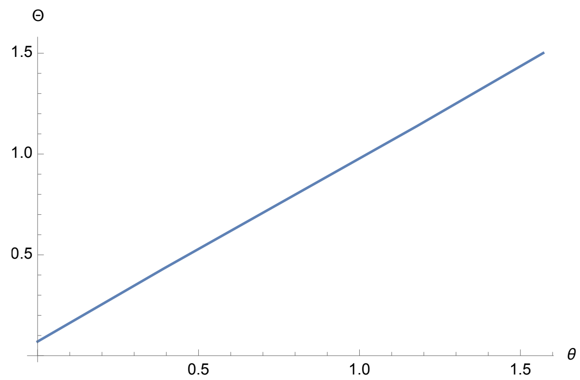

Figure 7.4.4: $\Theta_{(0)}(\theta)$ for $K^{\frac{1}{2}} = 0.8$.

by symmetry

$$\Theta_{(0)}\left(\frac{\pi}{4}\right) = \frac{\pi}{4}. \tag{7.4.23}$$

For $0 \leq \theta < \frac{\pi}{4}$, $\Theta_{(0)} > \theta$ while for $\frac{\pi}{4} < \theta \leq \frac{\pi}{2}$, $\Theta_{(0)} < \theta$. We can solve numerically for $\Theta_{(0)}(\theta)$; the plot for $K^{\frac{1}{2}} = 0.8$ is shown in Figure 7.4.4. Once $\Theta_{0)}(\theta)$ is determined, the other functions are determined using (7.4.21); the function $c_2(\theta)$ is non-trivial for $K \neq 1$.

Thus, to summarise this section, matching a cosmic membrane metric on a constant null hypersurface with a Ricci flat metric with no deficits requires a relaxation of the asymptotically flat boundary conditions (7.4.14) to weaker boundary conditions of the form (7.4.17). We will refer to Ricci flat metrics in Bondi gauge (7.4.12) satisfying (7.4.17) as *asymptotically locally flat*[2].

7.4.3 Cosmic membranes: alternative parameterisation

In the coordinate system of (7.4.4) the cosmic membrane preserves only a $U(1)$ symmetry, together with an additional inversion symmetry. To match such a metric, the required Bondi gauge parameterisation is

$$\begin{aligned} ds^2 &= -(\mathcal{U}e^{2\beta} - r^2 W^2 e^{C_1})du^2 - 2e^{2\beta}dudr - 2r^2 W e^{C_1} dud\theta \\ &\quad + r^2\left(e^{C_1}d\theta^2 + \sin^2\theta(e^{C_2}d\chi^2 + e^{-(C_1+C_2)}\sin^2\chi d\phi^2)\right), \end{aligned} \tag{7.4.24}$$

where the defining functions $(\mathcal{U}, W, \beta, C_1, C_2)$ can depend on (u, r, θ, χ) but should be even functions of χ, to respect the inversion symmetry in (7.4.4).

Now let us consider the matching between a Bondi gauge metric of the form (7.4.24) and a cosmic

[2]This notion is not to be confused with the one employed for gravitational instantons [271], although it express the similar concept that the boundary is not exactly Minkowski because of either topological or metrical differences. We stress that the name is thus to be intended in the same way that the terminology “asymptotically locally AdS” is used [42].

membrane (7.4.4) on a constant null time slice at infinity. The required coordinate maps are

$$\begin{aligned} U &= U_{(0}(u,\theta,\chi) + \frac{1}{r} U_{(-1)}(u,\theta,\chi) + \cdots \\ R &= r R_{(1)}(\theta,\chi) + R_{(0)}(u,\theta,\chi) + \cdots \\ \Theta &= \Theta_{(0)}(\theta,\chi) + \frac{1}{r}\Theta_{(-1)}(u,\theta,\chi) + \cdots \\ X &= X_{(0)}(\theta,\chi) + \frac{1}{r} X_{(-1)}(u,\theta,\chi) + \cdots \\ \Phi &= \phi \end{aligned} \tag{7.4.25}$$

Again, the leading order terms in (R, X, Θ) are forced to be independent of u to respect the asymptotic (local) flatness. Matching on a null hypersurface then imposes four relations at leading order:

$$\begin{aligned} (\theta\theta): &\quad R_{(1)}^2 \left((\partial_\theta \Theta_{(0)})^2 + \sin^2\Theta_{(0)} (\partial_\theta X_{(0)})^2 \right) = e^{c_1(\theta,\chi)} \\ (\theta\chi): &\quad (\partial_\theta \Theta_{(0)})(\partial_\chi \Theta_{(0)}) + \sin^2\Theta_{(0)} (\partial_\theta X_{(0)})(\partial_\chi X_{(0)}) = 0 \\ (\chi\chi): &\quad R_{(1)}^2 \left((\partial_\chi \Theta_{(0)})^2 + \sin^2\Theta_{(0)} (\partial_\chi X_{(0)})^2 \right) = e^{c_2(\theta,\chi)} \sin^2\theta \\ (\phi\phi): &\quad R_{(1)}^2 \sin^2\Theta_{(0)} \sin^2 X_{(0)} = \frac{1}{K^2} e^{-c_1(\theta,\chi) - c_2(\theta,\chi)} \sin^2\theta \sin^2\chi \end{aligned} \tag{7.4.26}$$

where we indicate which components of the induced metric are matched and we expand the defining functions (C_1, C_2) as

$$C_i(u,r,\theta,\chi) = c_i(\theta,\chi) + \frac{C_{i(-\lambda)}(u,\theta,\chi)}{r^\lambda} + \cdots \tag{7.4.27}$$

where the exponent $\lambda > 0$ will be determined by the Einstein equations.

The equations (7.4.26) can clearly be solved by $X_{(0)} = \chi$, i.e. the coordinate transformations depend only on θ to leading order:

$$\begin{aligned} (\theta\theta): &\quad R_{(1)}^2 (\partial_\theta \Theta_{(0)})^2 = e^{c_1(\theta)} \\ (\chi\chi): &\quad R_{(1)}^2 \sin^2\Theta_{(0)} = e^{c_2(\theta)} \sin^2\theta \\ (\phi\phi): &\quad R_{(1)}^2 \sin^2\Theta_{(0)} = \frac{1}{K^2} e^{-c_1(\theta) - c_2(\theta)} \sin^2\theta \end{aligned} \tag{7.4.28}$$

The last two equations are clearly not compatible for $K^2 \neq 1$ unless either one or both of $(c_1(\theta), c_2(\theta))$ is non-zero: combining the last two equations we obtain

$$e^{c_1 + 2c_2} = \frac{1}{K^2} \tag{7.4.29}$$

However, as in the previous discussions, these equations are under-constrained: there are three equations for four functions, and thus one can fix a linear combination of c_1 and c_2 to be zero, provided that (7.4.29) is satisfied. The latter choice represents residual gauge freedom.

The equations (7.4.28) clearly admit the solution

$$\Theta_{(0)} = \theta; \qquad R_{(1)} = \frac{1}{K^{\frac{1}{3}}}; \qquad e^{c_1} = e^{c_2} = \frac{1}{K^{\frac{2}{3}}}, \tag{7.4.30}$$

i.e. an angle independent rescaling of the radius. This solution is trivial in the sense that the

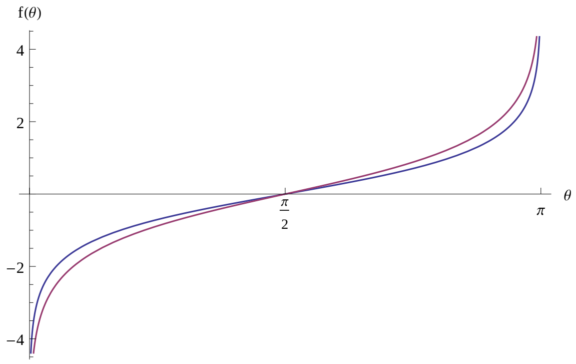

Figure 7.4.5: The red line plots $f(\theta)$ and the blue line plots $\lambda f(\theta)$ for $\lambda = 0.8$. For any value of $\lambda < 1$ the blue curve will lie closer to the horizontal axis than the red curve.

metric in coordinates (r, θ, χ, ϕ) still has a defect.

Combining the first two equations in (7.4.28), one obtains

$$\frac{\partial_\theta \Theta_{(0)}}{\sin \Theta_{(0)}} = \frac{e^{\frac{1}{2}(c_1 - c_2)}}{\sin \theta}. \tag{7.4.31}$$

Suppose we fix a gauge in which

$$e^{\frac{1}{2}(c_1 - c_2)} = \lambda, \tag{7.4.32}$$

subject to the constraint (7.4.29). Then (7.4.31) can be solved analogously to the angular equations of the previous sections.

Let us define

$$f(x) = \int \frac{dx}{\sin x} = \ln \left(\tan \left(\frac{x}{2} \right) \right). \tag{7.4.33}$$

The integrated relation (7.4.31) can hence be expressed as

$$f(\Theta_{(0)}) = \lambda f(\theta). \tag{7.4.34}$$

The function $f(\theta)$ is plotted over the range $(0, \pi)$ in Figure 7.4.5. From the same plot we can see that if $\lambda < 1$ then the relation $\Theta_{(0)}(\theta)$ has a similar form to that in the previous section, see Figure 7.4.6: for $0 < \theta \leq \pi/2$, $\Theta_{(0)} > \theta$ while for $\pi/2 \leq \theta < \pi$, $\Theta_{(0)} < \theta$.

Note that for small θ

$$\Theta_{(0)} \approx 2 \left(\frac{\theta}{2} \right)^{\lambda} \tag{7.4.35}$$

(with a corresponding expression for $\theta \sim \pi$). Furthermore, by symmetry,

$$\Theta(0) \left(\frac{\pi}{2} \right) = \frac{\pi}{2}. \tag{7.4.36}$$

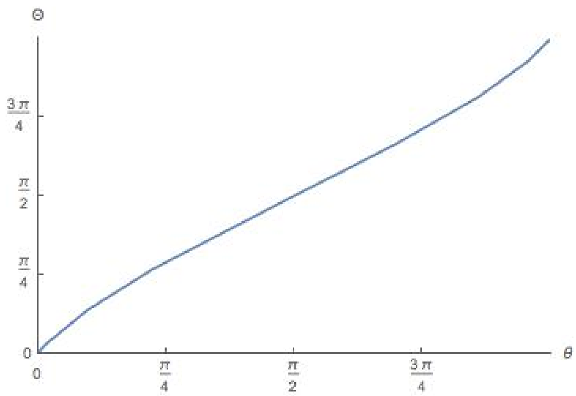

Figure 7.4.6: $\Theta_{(0)}(\theta)$ for $\lambda = 0.8$.

7.4.4 Boundary conditions in d dimensions and cosmic $(d-3)$ branes

To match the cosmic brane metric to a non-singular Bondi gauge metric on a null hypersurface at infinity. we are forced to relax asymptotic flatness to asymptotic local flatness. The general Bondi gauge parameterisation (without imposing additional symmetries) of a spacetime in arbitrary dimension d is

$$ds^2 = -\mathcal{U}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(d\Theta^A + W^A du)(d\Theta^B + W^B du), \tag{7.4.37}$$

where the coordinates θ^A run from $A = 1, \cdots (d-2)$. Here we have imposed the standard Bondi gauge conditions i.e.

$$g_{rr} = g_{rA} = 0 \tag{7.4.38}$$

and it is usual to impose the determinant condition

$$\partial_r \left(\det(h_{AB})\right) = 0. \tag{7.4.39}$$

Asymptotically flat boundary conditions require that

$$h_{AB} \to \gamma_{AB} + \frac{1}{r^{(d-2)/2}} h_{(d-2)AB} + \cdots \tag{7.4.40}$$

with γ_{AB} the metric on a unit $(d-2)$ sphere and the subleading term being associated with gravitational waves.

Such boundary conditions exclude cosmic $(d-3)$-branes passing through the celestial sphere. To allow for the latter, we need to relax the boundary conditions to asymptotically locally flat by setting

$$h_{AB} \to h_{(0)AB}(\theta^C) + \cdots \tag{7.4.41}$$

as $r \to \infty$. In the following sections we will impose such boundary conditions and consider the implications for the asymptotic structure in five dimensions.

We should note that the boundary condition (7.4.41) is manifestly more general than that obtained from cosmic branes, for which $h_{(0)}$ is a spherical metric with distributional defects. As we discuss later, the main motivation for working with the more general boundary condition is holography.

In (A)dS holography, the metric on the conformal boundary is allowed to be any non-degenerate metric, and it corresponds to the background metric for the dual quantum field theory. Even if one is only interested in the quantum field theory on a (conformally) flat background, one needs to allow for general perturbations of the boundary metric in order to compute correlation functions. If there is any holographic duality for asymptotically (locally) flat spacetimes, one would similarly expect that the spatial part of the boundary metric should be unrestricted.

If one takes a more conservative viewpoint and only wishes the boundary condition to be general enough to include distributional defects, one could regard the boundary condition (7.4.41) as encompassing all possibilities for distributional defects i.e. capturing different kinds of regularisations. One would then expect that asymptotic analysis of the field equations, combined with physical restrictions on allowed distributional curvature, will determine what additional restrictions should be placed on (7.4.41).

7.5 Bondi-Sachs problem

In this section we will use the previous discussions of cosmic branes to postulate boundary conditions for asymptotically locally flat spacetimes in Bondi-Sachs gauge in five dimensions.

The Bondi-Sachs metric in $d = 5$ can be written as

$$ds^2 = -\mathcal{U}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(d\Theta^A + W^A du)(d\Theta^B + W^B du). \tag{7.5.1}$$

with

$$h_{AB} = \begin{pmatrix} e^{C_1} & \cos\theta \sinh D_1 & \sin\theta \sinh D_2 \\ \cos\theta \sinh D_1 & e^{C_2}\sin^2\theta & \sin\theta\cos\theta \sinh D_3 \\ \sin\theta \sinh D_2 & \sin\theta\cos\theta\sinh D_3 & e^{C_3}\cos^2\theta \end{pmatrix}. \tag{7.5.2}$$

Due to the determinant constraint of the gauge

$$\partial_r \left(\det(h_{AB})\right) = 0, \tag{7.5.3}$$

only five of the six functions are independent.

As summarised in chapter 5, the vacuum Einstein equations were analysed asymptotically in [69, 70], under the assumption of asymptotic flatness, i.e. h_{AB} asymptotes to the round metric on the unit three sphere and the subleading terms in the expansions arise from integration functions on solving the Einstein equations

$$C_i \to \frac{C_{(\frac{3}{2})i}(u, x^A)}{r^{\frac{3}{2}}} \qquad D_i \to \frac{D_{(\frac{3}{2})i}(u, x^A)}{r^{\frac{3}{2}}} \tag{7.5.4}$$

as $r \to \infty$. Here $i = 1, 2, 3$ and the falloff behaviour relates to gravitational waves passing through null infinity. Without restricting to a specific choice of coordinates on the sphere, the expansion takes the form

$$h_{AB} = \gamma_{AB} + \frac{C_{(\frac{3}{2})AB}}{r^{\frac{3}{2}}} + \cdots \tag{7.5.5}$$

where γ_{AB} is a round metric on a unit three sphere.

As discussed in previous sections, if we wish to impose weaker boundary conditions that would allow for cosmic branes, we should impose

$$h_{AB} \to h_{(0)AB}(x^C) \tag{7.5.6}$$

as $r \to \infty$.

For computational simplicity we will continue to restrict to the case with $U(1)^2$ and reflection symmetry so that the functions D_i defined in (7.5.2) are zero. We can also eliminate C_3 using the determinant constraint i.e.

$$C_3 = -(C_1 + C_2). \tag{7.5.7}$$

In this case there are five main equations $(R_{rr}, R_{r\theta}, R_{\theta\theta}, R_{\psi\psi}, R_{\phi\phi})$ and three supplementary equations $(R_{uu}, R_{ur}, R_{u\theta})$, including the trivial equation. We will first write down the general form of these equations and then discuss asymptotic solutions.

The R_{rr} equation is

$$R_{rr} = \frac{6}{r}\beta_{,r} - \frac{1}{2}\left((C_{1,r})^2 + (C_{2,r})^2 + (C_{3,r})^2\right) = 0. \tag{7.5.8}$$

Here and in the subsequent Einstein equations we denote partial derivatives with commas. Clearly given (C_1, C_2) this equation can be integrated to find β, with integration functions in both β and (C_1, C_2) left undetermined. Note that this is exactly analogous to the well-known four-dimensional integration scheme: given the metric on the sphere, one can integrate to get g_{ur}.

Following the usual Bondi-Sachs integration scheme, we next use the $R_{r\theta}$ equation:

$$\begin{aligned} R_{r\theta} =& \frac{1}{2r^3}(r^5 e^{C_1 - 2\beta} W_{,r})_{,r} \\ &+ \frac{1}{r}(3\beta_{,\theta} - r\beta_{,r\theta}) + \frac{1}{2}((\cot\theta - 2\tan\theta)C_{1,r} + C_{1,r\theta}) \\ &- \frac{1}{4}(2C_{1,\theta} + C_{2,\theta})C_{1,r} - \frac{1}{4}\left(C_{1,\theta} + 2C_{2,\theta} + \frac{2}{\sin\theta\cos\theta}\right)C_{2,r} = 0. \end{aligned} \tag{7.5.9}$$

Here and from now on we use the abbreviated notation $W \equiv W^\theta$. Imposing $R_{r\theta} = 0$ allows us to integrate for W in terms of (C_i, β).

The three main equations in the sphere directions are as follows. The $R_{\theta\theta}$ equation is

$$\begin{aligned}
R_{\theta\theta} = {} & 2 - 2(\beta_{,\theta})^2 + \beta_{,\theta}C_{1,\theta} - 2\beta_{,\theta\theta} - \frac{1}{2}r^4(W_{,r})^2 e^{2C_1-4\beta} \\
& + C_{1,\theta}(\cot\theta - \tan\theta) - \frac{1}{2}\csc\theta\sec\theta(C_{1,\theta} - 2C_{2,\theta}) \\
& - \frac{1}{2}\left((C_{1,\theta})^2 + (C_{2,\theta})^2 + C_{1,\theta}C_{2,q} - C_{1,\theta\theta}\right) \\
& + e^{C_1-2\beta}(4rW_{,\theta} + rW(\cot\theta - \tan\theta) - 2\mathcal{U}) \\
& + e^{C_1-2\beta}\left(\frac{1}{2}r^2C_{1,\theta}W_{,r} - \frac{3}{2}r\mathcal{U}C_{1,r} + \frac{3}{2}rWC_{1,\theta} - rU_{,r}\right) \\
& + \frac{1}{2}r^2e^{C_1-2\beta}\left(C_{1,r}\left(W_{,\theta} - \mathcal{U}_{,r}\right) + WC_{1,r}(\cot\theta - \tan\theta)\right) \\
& + r^2e^{C_1-2\beta}\left(\frac{3C_{1,u}}{2r} - \frac{1}{2}\mathcal{U}C_{1,rr} + WC_{1,r\theta} + C_{1,ur} + W_{,r\theta}\right) = 0
\end{aligned} \tag{7.5.10}$$

The $R_{\phi\phi}$ equation is

$$\begin{aligned}
R_{\phi\phi} = {} & e^{2\beta}\left(-2\beta_{,\theta}C_{2,\theta} + C_{2,\theta}\tan\theta + C_{1,\theta}C_{2,\theta} - C_{2,\theta\theta}\right) \\
& + e^{2\beta}(4 + \cot\theta(2C_{1,\theta} - 4\beta_{,\theta} - C_{2,\theta} - 2e^{C_1}r^2W_{,r}) \\
& + r^2e^{C_1}\left(C_{2,\theta}W_{,r} + C_{2,r}W_{,\theta} + 2C_{2,ur} - C_{2,r}\mathcal{U}_{,r} - \mathcal{U}C_{2,rr}\right) \\
& + re^{C_1}\left(3C_{2,u} + 3WC_{2,\theta} - 3\mathcal{U}C_{2,r} - 2\mathcal{U}_{,r} + 2W_{,\theta}\right) \\
& - 4e^{C_1}\mathcal{U} + e^{C_1}rW((5 + rC_{2,r})(\cot\theta - \tan\theta) + 2rC_{2,r\theta}) \\
& + 3e^{C_1}rW\sec\theta\csc\theta = 0\,.
\end{aligned} \tag{7.5.11}$$

The $R_{\psi\psi}$ equation is, applying $C_3 = -(C_1 + C_2)$ to simplify,

$$\begin{aligned}
R_{\psi\psi} = {} & \sin\theta\left(4e^{2\beta}\beta_{,\theta} - 3e^{2\beta}C_{1,\theta} - e^{2\beta}C_{2,\theta} - 2e^{C_1}r^2W_{,r}\right) \\
& + \cos\theta C_{1,\theta}\left(2e^{2\beta}\beta_{,\theta} - e^{2\beta}C_{2,\theta} - e^{C_1}r^2W_{,r} + e^{2\beta}\cot\theta\right) \\
& + e^{2\beta}\cos\theta\left(2\beta_{,\theta}C_{2,\theta} + C_{2,\theta}\cot\theta - (C_{1,\theta})^2 - C_{3,\theta\theta} + 4\right) \\
& + e^{C_1}\cos\theta\left(-r^2C_{2,\theta}W_{,r} - 2r\mathcal{U}_{,r} + rW_{,\theta}\left(rC_{3,r} + 2\right)\right) \\
& + e^{C_1}rW\csc\theta\left(rC_{3,r\theta}\sin 2\theta + rC_{3,r}\cos 2\theta + 5\cos 2\theta - 3\right)) \\
& + e^{C_1}\cos\theta\left(r\left(2rC_{3,ur} + 3C_{3,u}\right) - \mathcal{U}\left(r^2C_{3,rr} + 3rC_{3,r} + 4\right)\right) \\
& + re^{C_1}\cos\theta(3WC_{3,\theta} - rC_{3,r}\mathcal{U}_{,r}) = 0.
\end{aligned} \tag{7.5.12}$$

Combining these equations to form the trace along the sphere, i.e $g^{AB}R_{AB} = 0$, one obtains an equation that determines $\mathcal{U}$ from the previously determined (β, W) and C_i

$$\begin{aligned}
g^{AB}R_{AB} = &-\frac{e^{-C_1}}{2r^2}\sec\theta\csc\theta(C_{1,\theta}+2C_{2,\theta}) \\
&+\frac{e^{-2\beta-C_1}}{r^2}(\cot\theta-\tan\theta)\left(e^{C_1}(6rW+r^2W_{,r})-2e^{2\beta}\beta_{,\theta}+\frac{5}{2}e^{2\beta}C_{1,\theta}\right) \\
&+\frac{e^{-C_1}}{2r^2}\left(12-(2\beta_{,\theta})^2+4\beta_{,\theta}C_{1,\theta}-2(C_{1,\theta})^2-C_{1,\theta}C_{2,\theta}-(C_{2,\theta})^2\right) \\
&+\frac{e^{-C_1}}{2r^2}(-4\beta_{,\theta\theta}+2C_{1,\theta\theta})-\frac{r^2}{2}e^{-4\beta+C_1}(W_{,r})^2 \\
&-3\frac{e^{-2\beta}}{r^2}\left((2+r\partial_r)\mathcal{U}-r(2+\frac{r}{3}\partial_r)W_{,\theta}\right)=0\,.
\end{aligned} \tag{7.5.13}$$

The supplementary equations are

$$\begin{aligned}
R_{uu} = &+\frac{1}{2}r^4e^{2C_1-4\beta}W^2\left(W_{,r}\right)^2+r^2e^{C_1-2\beta}W^2\left(\frac{1}{2}\mathcal{U}_{,r}C_{1,r}-2W_{,r\theta}+\frac{1}{2}\mathcal{U}C_{1,rr}\right) \\
&+r^2e^{C_1-2\beta}W\left(-C_{1,ur}W-2W_{,\theta}W_{,r}-2\mathcal{U}\beta_{,r}W_{,r}-C_{1,r,\theta}W^2+2\beta_{,\theta}W_{,r}W\right) \\
&+r^2e^{C_1-2\beta}\left(-\frac{3}{2}C_{1,\theta}W_{,r}W^2-\frac{1}{2}W_{,\theta}C_{1,r}W^2+\mathcal{U}W_{,r}C_{1,r}W+\mathcal{U}W_{,rr}W-W_{,ur}W\right) \\
&+r^2e^{C_1-2\beta}\left(+2W_{,r}\beta_{,u}W-W_{,r}C_{1,u}W+\frac{1}{2}\mathcal{U}\left(W_{,r}\right)^2\right) \\
&+re^{C_1-2\beta}\left(-\frac{3}{2}C_{1,\theta}W^3-4W_{,\theta}W^2+\mathcal{U}_{,r}W^2+\frac{3}{2}\mathcal{U}C_{1,r}W^2-\frac{3}{2}C_{1,u}W^2+5\mathcal{U}W_{,r}W\right) \\
&+2W^2(e^{C_1-2\beta}\mathcal{U}+(\beta_{,\theta})^2)-W^2\beta_{,\theta}C_{1,\theta}+(W_{,\theta})^2+\frac{1}{2}\left(C_{1,u}\right)^2+\frac{1}{2}\left(C_{2,u}\right)^2-2W\beta_{,\theta}W_{,\theta} \\
&+WW_{,\theta}C_{1,\theta}+WW_{,\theta\theta}+\frac{1}{2}W^2C_{1,\theta\theta}+2W\mathcal{U}_{,\theta}\beta_{,r}+\mathcal{U}W_{,\theta}\beta_{,r}+\frac{1}{2}W_{,\theta}\mathcal{U}_{,r}-\mathcal{U}\beta_{,r}\mathcal{U}_{,r} \\
&+\mathcal{U}\beta_{,\theta}W_{,r}-\frac{1}{2}\mathcal{U}_{,\theta}W_{,r}-W\mathcal{U}_{,\theta}C_{1,r}+2\mathcal{U}W\beta_{,r\theta}+W\mathcal{U}_{,r\theta}-\mathcal{U}^2\beta_{,rr}-\frac{1}{2}\mathcal{U}\mathcal{U}_{,rr}-2W_{,\theta}\beta_{,u} \\
&+W_{,\theta}C_{1,u}+\frac{1}{2}C_{1,u}C_{2,u}-2W\beta_{,u\theta}+W_{,u\theta}+WC_{1,u\theta}+2\mathcal{U}\beta_{,ur} \\
&+\frac{e^{-4\beta-C_1}}{2r^2}(\cot\theta-\tan\theta)\left(e^{6\beta}\left(2U\beta_{,\theta}+U_{,\theta}\right)+r^3W^2e^{2(\beta+C_1)}\left(W\left(rC_{1,r}+2\right)+2rW_{,r}\right)\right) \\
&-\frac{1}{2}(\cot\theta-\tan\theta)\left(2UW\beta_{,r}-4W\beta_{,u}+W^2C_{1,\theta}+2WC_{1,u}+WU_{,r}+2WW_{,\theta}+2W_{,u}\right) \\
&+\frac{1}{r}(-3\beta_{,r}\mathcal{U}^2+3W\beta_{,\theta}\mathcal{U}-\frac{3}{2}\mathcal{U}_{,r}\mathcal{U}+3\beta_{,u}\mathcal{U}-\frac{1}{2}W\mathcal{U}_{,\theta}-\frac{3}{2}\mathcal{U}_{,u}) \\
&+\frac{e^{2\beta-C_1}}{r^2}\left(-2\mathcal{U}\left(\beta_{,\theta}\right)^2-\mathcal{U}_{,\theta}\beta_{,\theta}+\mathcal{U}C_{1,\theta}\beta_{,\theta}+\frac{1}{2}\mathcal{U}_{,\theta}C_{1,\theta}-\mathcal{U}\beta_{,\theta\theta}-\frac{1}{2}\mathcal{U}_{,\theta\theta}\right)=0\,,
\end{aligned} \tag{7.5.14}$$

and

$$\begin{aligned}
R_{u\theta} = &\frac{1}{4}e^{-2\beta}(\cot\theta-\tan\theta)\left(W\left(4e^{2\beta}\beta_{,\theta}-2e^{C_1}r^2W_{,r}\right)+3e^{2\beta}C_{1,u}-2e^{C_1}rW^2\left(rC_{1,r}+2\right)\right) \\
&-\frac{1}{4}\left(C_{1,u}+2C_{2,u}\right)\csc\theta\sec\theta-2r^3e^{2\beta+C_1}\left(W_{,r}\left(C_{1,u}-2\beta_{,u}\right)+2W_{,\theta}W_{,r}+W_{,ur}\right) \\
&+\frac{\mathcal{U}_{,\theta}}{4r}\left(4r\beta_{,r}-2rC_{1,r}+2\right)+W\left(-\beta_{,\theta}C_{1,\theta}+2\left(\beta_{,\theta}\right)^2+\beta_{,\theta\theta}\right)+\frac{1}{2}r^4W\left(W_{,r}\right)^2e^{2C_1-4\beta} \\
&+\frac{1}{4}\left(-4\beta_{,u\theta}-C_{2,\theta}C_{1,u}-2C_{2,\theta}C_{2,u}-C_{1,\theta}\left(2C_{1,u}+C_{2,u}\right)+2C_{1,u\theta}+2\mathcal{U}_{,r\theta}\right)=0\,.
\end{aligned} \tag{7.5.15}$$

The first supplementary equations gives the u-evolution equation for the free data in the $\mathcal{U}$ expansion. The second gives the u evolution of the free data in W.

The final supplementary equation (sometimes also called the trivial equation) is:

$$\begin{aligned} R_{ur} \quad = \quad & -\frac{e^{-C_1}}{2r^2}(\cot\theta - \tan\theta)\left(-2e^{2\beta}\beta_{,\theta} + e^{C_1}r^2 W_{,r} + e^{C_1} rW\left(rC_{1,r}+2\right)\right) \\ & + r^2 e^{C_1-2\beta}\left(W\beta_{,r}W_{,r} - \frac{1}{2}\left(W_{,r}\right)^2 - \frac{1}{2}WC_{1,r}W_{,r} - \frac{1}{2}WW_{,rr} - \frac{5}{2}rWW_{,r}\right) \\ & + \frac{2}{r^2}e^{2\beta-C_1}\left((\beta_{,\theta})^2 - \beta_{,\theta}C_{1,\theta} + \beta_{,\theta,\theta}\right) + \frac{3}{r}(\mathcal{U}\beta_{,r} - 3W\beta_{,\theta} + \frac{3}{2}\mathcal{U}_{,r} - W_{,\theta}) \\ & + \beta_{,r}\mathcal{U}_{,r} + \mathcal{U}\beta_{,rr} - \beta_{,\theta}W_{,r} - W\beta_{,r,\theta} - 2\beta_{,u,r} - \frac{1}{2}C_{1,r}W_{,\theta} - \frac{1}{2}WC_{1,r\theta} \\ & - \frac{1}{2}C_{1,r}C_{1,u} - \frac{1}{4}C_{2,r}C_{1,u} - \frac{1}{4}C_{1,r}C_{2,u} - \frac{1}{2}C_{2,r}C_{2,u} + \frac{1}{2}\mathcal{U}_{,r,r} - \frac{1}{2}W_{,r\theta} = 0\,. \end{aligned} \tag{7.5.16}$$

This is automatically satisfied at each order as a consequence of Bianchi identities once the main equations are satisfied. This equation therefore does not give any new information. If we do not check Bianchi identities, then the vanishing of this equation at each order provides a check on our solution.

7.5.1 Asymptotic analysis

Having collected together the Einstein equations, let us consider asymptotic solutions of these equations. The equations in Bondi gauge are nested, and thus should be solved in the order in which they are presented above, beginning with (7.5.8).

As we described above, in the Bondi integration scheme we prescribe data for C_i on a null hypersurface, say $\mathcal{N}_{u0}$ in Figure 7.5.1, recursively determine the other metric coefficients using the main equations and then determine the null evolution of C_i using the final null equation. Thus we should impose boundary conditions for C_i as $r \to \infty$, and examine their consequences for the nested integration. Following the discussions of the previous section, we impose the boundary condition that $C_i \to C_{i(0)}(\theta)$ as $r \to \infty$. The u independence was established in the previous section but we will understand further below why the final main equation requires u independence of the defining data on the celestial sphere. The corresponding asymptotic expansion of C_i is therefore

$$C_i = C_{i(0)}(\theta) + \cdots \tag{7.5.17}$$

where the ellipses denote subleading terms in the radial expansion. The structure of these subleading terms will be determined below by the field equations i.e. we do not make any assumptions a priori for the form of the expansion.

We can trivially rearrange the first main equation (7.5.8) to obtain

$$\boxed{\beta_{,r} = \frac{r}{6}\left(C_{1,r}^2 + C_{1,r}C_{2,r} + C_{2,r}^2\right).} \tag{7.5.18}$$

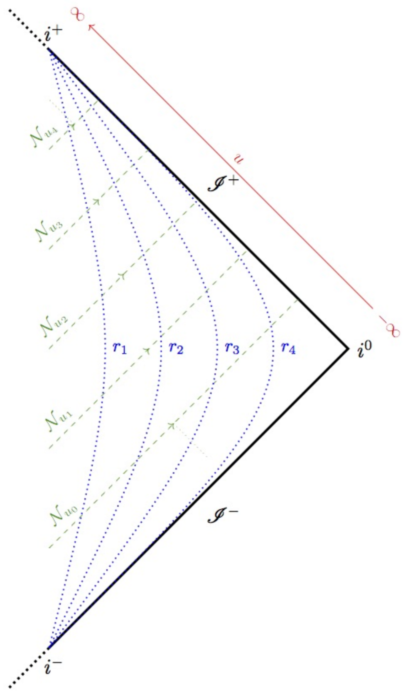

Figure 7.5.1: Penrose diagram indicating hypersurfaces of constant u and r.

Integrating this equation the leading contribution to β is an integration function

$$\beta = \beta_{(0)}(\theta, u) + \cdots \tag{7.5.19}$$

Note that this is clearly the only integration function from this equation to all orders in the radial expansion. According to the standard analysis in four and higher dimensions this integration function is set to zero to satisfy the Minkowskian boundary conditions.

Moving now to the (7.5.9) equation, we can write this in the form

$$\boxed{\frac{1}{2r^3}(r^5 e^{C_1 - 2\beta} W_{,r})_{,r} = \mathcal{G}(C_1, C_2, \beta)} \tag{7.5.20}$$

with $\mathcal{G}$ as given in (7.5.9). Integrating for W we obtain

$$W = W_{(0)} + \frac{W_{(1)}[C_{i(0)}, \beta_{(0)}]}{r} + \cdots \tag{7.5.21}$$

where $[\cdots]$ denotes the functional dependence of the coefficients. $W_{(0)}$ is again an integration function and the coefficient $W_{(1)}$ is completely determined by the $1/r$ terms in the $R_{r\theta}$ equation:

$$W_{(1)} = 2e^{2\beta_{(0)} - C_{1(0)}} \partial_\theta \beta_{(0)}. \tag{7.5.22}$$

The only way to satisfy the Einstein equation at this order is either to allow for $W_{(1)}$ or to fix $\beta_{(0)}$ to be independent of θ. However, we will see that the function $W_{(0)}$ must actually be set to zero, as a consequence of the next equation.

Next we consider the trace of the main equations along the sphere (7.5.13). One can write this in the form

$$\boxed{3\frac{e^{-2\beta}}{r^2}(2+r\partial_r)\mathcal{U} = \mathcal{F}(C_1, C_2, \beta, W),} \tag{7.5.23}$$

where the leading contribution to the functional $\mathcal{F}$ is at order[3] $1/r$

$$\mathcal{F} = 6e^{2\beta_{(0)}} = 6e^{2\beta_{(0)}}\left(\partial_\theta + (\cot\theta - \tan\theta)\right)W_{(0)}. \tag{7.5.24}$$

Then (7.5.23) implies that

$$\mathcal{U} = r\mathcal{U}_{(-1)}[W_{(0)}] + \mathcal{U}_{(0)}[C_{i(0)}, \beta_{(0)}] + \cdots \tag{7.5.25}$$

where

$$\mathcal{U}_{(-1)} = \frac{2}{3}(\partial_\theta + (\cot\theta - \tan\theta))W_{(0)}. \tag{7.5.26}$$

However, this solution implies that

$$g_{uu} = r^2 e^{C_1} W^2 - \mathcal{U}e^{2\beta} \to r^2 e^{C_{1(0)}} W_{(0)}^2 \tag{7.5.27}$$

as $r \to \infty$ i.e. ∂_u is spacelike rather than null or timelike. The requirement that ∂_u is not spacelike at infinity thus fixes $W_{(0)} = 0$.

The physical interpretation of non-zero $W_{(0)}$ can be understood using the example of Minkowski spacetime in four dimensions. Starting from

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7.5.28}$$

we can change coordinates to

$$d\phi = d\tilde{\phi} + \frac{du}{\Omega} \tag{7.5.29}$$

(where Ω characterises the angular velocity) so that

$$ds^2 = -du^2 - 2dudr + r^2\left(d\theta^2 + \sin^2\theta\left(d\tilde{\phi} + \frac{du}{\Omega}\right)^2\right) \tag{7.5.30}$$

i.e. comparing with (7.5.1) $W_{(0)}^{\tilde{\phi}} \neq 0$. Thus, physically, a non-zero $W_{(0)}$ can be interpreted as using a rotating frame at null infinity.

[3]Note that $(\partial_\theta + (\cot\theta - \tan\theta))W = D_\theta W^\theta$ where D is the covariant derivative.

Setting $W_{(0)} = 0$, the leading contribution to the function $\mathcal{F}$ in (7.5.23) is at order $1/r^2$:

$$\begin{aligned} \mathcal{F}_{(2)} &= -\frac{e^{-C_{1(0)}}}{2\sin\theta\cos\theta}(\partial_\theta C_{1(0)} + 2\partial_\theta C_{2(0)}) - e^{-C_{1(0)}}(\cot\theta - \tan\theta)(2\partial_\theta\beta - \frac{5}{2}\partial_\theta C_{1(0)}) \\ &+ e^{-C_{1(0)}}(6 - 2(\partial_\theta\beta_{(0)})^2 - 2\partial_\theta^2\beta_{(0)} + 2\partial_\theta\beta_{(0)}\partial_\theta C_{1(0)}) \\ &- \frac{e^{-C_{1(0)}}}{2}(2(\partial_\theta C_{1(0)})^2 + \partial_\theta C_{1(0)}\partial_\theta C_{2(0)} + (\partial_\theta C_{2(0)})^2 - 4\partial_\theta^2 C_{1(0)}) \end{aligned} \tag{7.5.31}$$

and thus integrating (7.5.23) we obtain

$$\mathcal{U}_{(0)} = \frac{1}{6}e^{2\beta_{(0)}}\mathcal{F}_{(2)}. \tag{7.5.32}$$

For ∂_u to be non-spacelike as $r \to \infty$ we require that

$$e^{C_{1(0)}}W_{(1)}^2 - e^{2\beta_{(0)}}\mathcal{U}_{(0)} \leq 0 \tag{7.5.33}$$

i.e.

$$\mathcal{F}_{(2)} \geq 24e^{-C_{1(0)}}(\partial_\theta\beta_{(0)})^2 \tag{7.5.34}$$

so

$$\begin{aligned} & 6 - 26(\partial_\theta\beta_{(0)})^2 - 2\partial_\theta^2\beta_{(0)} + 2\partial_\theta\beta_{(0)}\partial_\theta C_{1(0)} \\ & -\frac{1}{2\sin\theta\cos\theta}(\partial_\theta C_{1(0)} + 2\partial_\theta C_{2(0)}) - (\cot\theta - \tan\theta)(2\partial_\theta\beta - \frac{5}{2}\partial_\theta C_{1(0)}) \\ & -\frac{1}{2}\left(2(\partial_\theta C_{1(0)})^2 + \partial_\theta C_{1(0)}\partial_\theta C_{2(0)} + (\partial_\theta C_{2(0)})^2 - 4\partial_\theta^2 C_{1(0)}\right) \geq 0 \end{aligned} \tag{7.5.35}$$

This is a non-trivial constraint. In the case of cosmic membranes discussed previously the functions $(C_{i(0)}, \beta_{(0)})$ are proportional to the membrane tension; provided that the tension is much less than one, the leading term in this expression will be the first one and the constraint be satisfied. In other words, for a cosmic membrane,

$$\mathcal{U}_{(0)} \approx 1, \tag{7.5.36}$$

up to corrections of order the membrane tension.

The remaining Einstein equations do not place further constraints on this defining data. The Einstein equations along the sphere can be expressed in the form:

$$\boxed{\begin{aligned} R_{\theta\theta} = 0 \Leftrightarrow (3r + 2r^2\partial_r)\partial_u C_1 &= \mathcal{H}_1(C_i, \beta, W, \mathcal{U}); \\ R_{\phi\phi} = 0 \Leftrightarrow (3r + 2r^2\partial_r)\partial_u C_2 &= \mathcal{H}_2(C_i, \beta, W, \mathcal{U}); \\ R_{\psi\psi} = 0 \Leftrightarrow (3r + 2r^2\partial_r)\partial_u C_3 &= \mathcal{H}_3(C_i, \beta, W, \mathcal{U}), \end{aligned}} \tag{7.5.37}$$

and these determine the u evolution of the functions C_i. Here the functionals $\mathcal{H}_i$ depend on the functions $(C_i, \beta, W, \mathcal{U})$ and their (r, θ) derivatives, but not on u derivatives. The three equations are not independent: $C_3 = C_1 + C_2$.

Asymptotically, the leading contributions to $\mathcal{H}^i$ are of order one, thus determining that there are terms at order $1/r$ in the C_i expansions

$$C_i = C_{i(0)}(\theta) + \frac{C_{i(1)}(u,\theta)}{r} + \cdots \tag{7.5.38}$$

The equations (7.5.37) can immediately be integrated at leading order to give

$$C_{i(1)} = \int \mathcal{H}_i(u,\theta)du. \tag{7.5.39}$$

where $\mathcal{H}_i$ are

$$\begin{aligned}
\mathcal{H}_1 = {} & \frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left((C_{1(0),\theta})^2 + 2C_{2(0),\theta}C_{1(0),\theta} + 2(C_{2(0),\theta})^2 - C_{1(0),\theta\theta}\right) \\
& + \frac{2}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(C_{1(0),\theta}\beta_{(0),\theta} - 8(\beta_{(0),\theta})^2 - 4\beta_{(0),\theta\theta}\right) \\
& + \frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}(\tan\theta - \cot\theta)\left(C_{1(0),\theta} - 4\beta_{(0),\theta}\right) \\
& + \frac{2}{3}e^{2\beta_{(0)}-C_{1(0)}}\csc\theta\sec\theta\left(C_{1(0),\theta} + 2C_{2(0),\theta}\right)
\end{aligned} \tag{7.5.40}$$

and

$$\begin{aligned}
\mathcal{H}_2 = {} & \frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(-4\beta_{(0),\theta}C_{1(0),\theta} - 6\beta_{(0),\theta}C_{2(0),\theta}\right) \\
& + \frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(+8(\beta_{(0),\theta})^2 + 4\beta_{(0),\theta\theta} - 2C^2_{1(0),\theta}\right) \\
& - \frac{2}{3}e^{2\beta_{(0)}-C_{1(0)}}(\tan\theta + \cot\theta)C_{1(0),\theta} \\
& + \frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}(\cot\theta - 5\tan\theta)C_{2(0),\theta} \\
& - \frac{4}{3}e^{2\beta_{(0)}-C_{1(0)}}(\tan\theta + 2\cot\theta)\beta_{(0),\theta} \\
& - \frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left((C_{2(0),\theta})^2 + 4C_{1(0),\theta}C_{2(0),\theta}\right) \\
& + \frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(2C_{1(0),\theta\theta}3C_{2(0),\theta\theta}\right)
\end{aligned} \tag{7.5.41}$$

Note that $\mathcal{H}_3$ is the sum of $\mathcal{H}_1$ and $\mathcal{H}_2$.

The equations (7.5.37) demonstrate why the defining data $C_{i(0)}$ should be independent of u as $r \to \infty$: these equations cannot be solved self-consistently if $C_{i(0)}$ depends on u, without inducing an infinite series of terms in C_i that scale as positive powers of r, hence breaking the notion of local flatness.

Thus, in summary, the leading terms in the asymptotic expansions are

$$\begin{aligned}
C_i &= C_{i(0)} + \cdots \qquad \beta = \beta_{(0)} + \cdots \\
W &= \frac{W_{(1)}}{r} + \cdots \qquad \mathcal{U} = \mathcal{U}_{(0)} + \cdots
\end{aligned} \tag{7.5.42}$$

where $(C_{i(0)}(\theta), \beta_{(0)}(u,\theta))$ are the independent data and $(W_{(1)}, \mathcal{U}_{(0)})$ are determined from this data. Note that if $W_{(1)}$ is non-zero then $g_{u\theta} \sim r$. By setting $\partial_\theta \beta_{(0)} = 0$, one can set $W_{(1)} = 0$. If $\beta_{(0)}$ is a function only of u, one can then use reparameterisation freedom of the retarded time coordinate to fix $\beta_{(0)} = 0$. In this case $C_{i(0)}$ is the only remaining non-trivial data, with $\mathcal{U}_{(0)}$ determined from this data via (7.5.32).

One can use this behaviour to write down the asymptotics of the metric in the general case:

$$\begin{aligned} g_{AB} &= r^2 h_{AB} = r^2 h_{(0)AB} + \mathcal{O}(r) \\ g_{uu} &= (-\mathcal{U}e^{2\beta} + r^2 h_{AB} W^A W^B) = -\mathcal{U}_{(0)} e^{2\beta_{(0)}} + h_{(0)AB} W^A_{(1)} W^B_{(1)} + O(r^{-1}) \\ g_{ur} &= -e^{2\beta} = -e^{2\beta_0} + \mathcal{O}(r^{-2}) \\ g_{uA} &= g_{AB} W^B = r h_{(0)AB} W^B_{(1)} + \mathcal{O}(r^0), \end{aligned} \tag{7.5.43}$$

where the orders of the subleading terms follow from the next to leading contributions to the Einstein equations. Note that if one imposes the additional constraint that

$$\beta_{(0),A} = 0 \tag{7.5.44}$$

then $W^A_{(1)} = 0$ and g_{uA} is order r^0, as in the four-dimensional case.

7.6 Asymptotic expansion to all orders

In the previous section we established the leading asymptotics for the metric components, given the generalised boundary condition for the metric on the celestial sphere. In this section we will establish the asymptotic expansion for the metric in this context. It is useful to first review the structure of the expansion of an *asymptotically flat* vacuum metric, analysed in detail in [69, 270]. For direct comparison with the results above, we restrict to $U(1)^2$ symmetry and inversion symmetry, i.e. we set $D_i = 0$. The expansions for the five metric functions are then

$$\begin{aligned} C_i &= \frac{C_{i(\frac{3}{2})}}{r^{\frac{3}{2}}} + \cdots \\ \beta &= \frac{\beta_{(3)}}{r^3} + \cdots \\ W &= \frac{W_{(\frac{5}{2})}}{r^{\frac{5}{2}}} + \frac{W_{(3)}}{r^3} + \frac{W_{(\frac{7}{2})}}{r^{\frac{7}{2}}} + \frac{W_{(4)}}{r^4} + \cdots \\ \mathcal{U} &= 1 + \frac{\mathcal{U}_{(\frac{3}{2})}}{r^{\frac{3}{2}}} + \frac{\mathcal{U}_{(2)}}{r^2} + \cdots \end{aligned} \tag{7.6.1}$$

Here we have highlighted in colour the defining data for the asymptotic expansion; all other expansion coefficients can be expressed in terms of this data and its derivatives, with explicit expressions given in [69, 270]. The integration functions $C_{i(\frac{3}{2})}$ are associated with gravitational wave degrees of freedom and their u-evolution is not fixed by Einstein equations and has to be prescribed to fully determine the solution. The integration functions highlighted in red are associated with conserved charges; in particular, $\mathcal{U}_{(2)}$ is associated with the mass of the spacetime. (Note that the assumed symmetries - axisymmetry and inversion, Section 7.5 - set the angular momentum charges associated with rotations in the ϕ and ψ directions to zero.)

We now turn to the asymptotic expansions of *asymptotically locally flat* vacuum metrics. The expansions for the five metric functions are

$$\begin{aligned} C_i &= C_{i(0)} + \frac{C_{i(1)}}{r} + \frac{C_{i(\frac{3}{2})}}{r^{\frac{3}{2}}} + \cdots \\ \beta &= \beta_{(0)} + \frac{\beta_{(2)}}{r^2} + \frac{\beta_{(\frac{5}{2})}}{r^{\frac{5}{2}}} + \frac{\beta_{(3)}}{r^3} + \cdots \\ W &= \frac{W_{(1)}}{r} + \frac{W_{(2)}}{r^2} + \frac{W_{(\frac{5}{2})}}{r^{\frac{5}{2}}} + \frac{W_{(3)}}{r^3} + \frac{W_{(\frac{7}{2})}}{r^{\frac{7}{2}}} + \frac{W_{(4)}}{r^4} + \frac{\tilde{W}_{(4)}}{r^4}\ln r + \cdots \\ \mathcal{U} &= \mathcal{U}_{(0)} + \frac{\mathcal{U}_{(1)}}{r} + \frac{\mathcal{U}_{(\frac{3}{2})}}{r^{\frac{3}{2}}} + \frac{\mathcal{U}_{(2)}}{r^2} + \frac{\tilde{\mathcal{U}}_{(2)}}{r^2}\ln r + \cdots \end{aligned} \tag{7.6.2}$$

Again, the integration functions are highlighted in colours, and all other coefficients in the expansions are expressed in terms of these data. The data $(C_{i(0)}, \beta_{(0)})$ is the analogue of non-normalisable boundary data in asymptotically locally AdS spacetimes, while the integration functions at subleading orders in the expansion $(C_{i(\frac{3}{2})}, W_{(4)}, \mathcal{U}_{(2)})$ are analogous to normalisable boundary data.

Explicit expressions for the first few terms in the expansions were derived in the previous section. These are summarised together with expressions for the subleading coefficients in the Appendix B.1. Here we focus on the structure of these expansions and, in particular, how the logarithmic terms in these expansions arise.

First of all, let us note that the integration functions in C_i at order $r^{-3/2}$ are unaffected at this order by the new boundary conditions. Just as in previous works [69, 270], we therefore expect that these integration functions are associated with gravitational waves: the defining data has the correct number of degrees of freedom to represent gravitational waves, and is unconstrained. Furthermore, as shown in Appendix B.1, the u-evolution of the data $C_{i(\frac{3}{2})}$ is left unspecified by the equations (7.5.37), so that also $\partial_u C_{i(\frac{3}{2})}$ have to be given as a coordinate on the phase space, in agreement with the asymptotically flat case. Note however that the new boundary conditions do affect the explicit forms for the expansion coefficients at subleading fractional orders.

To definitely show that $C_{i(\frac{3}{2})}$ corresponds to gravitational radiation, one would need to show that spacetimes with non-vanishing radiation lose mass. We should also note that, for $C_{i(\frac{3}{2})}$ to be interpreted as the degrees of freedom corresponding directly to gravitational waves, one should show rigorously that $\partial_u C_{i(\frac{3}{2})}$ is gauge invariant, generalising the discussions of [77]. If $\partial_u C_{i(\frac{3}{2})}$ was not gauge invariant then one would need to construct a gauge invariant quantity that reduces to $\partial_u C_{i(\frac{3}{2})}$ in the asymptotically flat case.

Consider the Einstein equations (7.5.20) and (7.5.23); both these equations have associated integration functions, shown in red above. As discussed in the previous section, these equations can be viewed as inhomogeneous equations for W and $\mathcal{U}$, respectively, which determine these functions from the functions that have already been determined. The asymptotic expansions of the functionals $(\mathcal{F}, \mathcal{G})$ have the structure

$$\begin{aligned} \mathcal{G} &= \frac{\mathcal{G}_{(1)}}{r} + \frac{\mathcal{G}_{(2)}}{r^2} + \frac{\mathcal{G}_{(5/2)}}{r^{5/2}} + \frac{\mathcal{G}_{(3)}}{r^3} + \frac{\mathcal{G}_{(7/2)}}{r^{7/2}} + \frac{\mathcal{G}_{(4)}}{r^4} + \cdots \\ \mathcal{F} &= \frac{\mathcal{F}_{(2)}}{r^2} + \frac{\mathcal{F}_{(3)}}{r^3} + \frac{\mathcal{F}_{(7/2)}}{r^{7/2}} + \frac{\mathcal{F}_{(4)}}{r^4} + \cdots \end{aligned} \tag{7.6.3}$$

Then integrating (7.5.20) and (7.5.23) we obtain

$$\begin{aligned} \tilde{\mathcal{U}}_{(2)} &= \frac{1}{3}\mathcal{F}_{(4)}e^{2\beta_{(0)}} \\ \tilde{W}_4 &= -\frac{1}{2}\mathcal{G}_{(4)}e^{2\beta_{(0)}-C_{1(0)}} \end{aligned} \tag{7.6.4}$$

Consistent integration of the Einstein equations thus requires either allowing for logarithmic terms in the asymptotic expansions or imposing constraints on the defining data such that $\mathcal{F}_{(4)} = \mathcal{G}_{(4)} = 0$. In the standard case of asymptotically flat spacetimes, the defining data are chosen such that indeed these terms in the expansions of $\mathcal{F}$ and $\mathcal{G}$ vanish, so no logarithmic terms are induced. This implies overconstraining subleading terms in the expansion of h_{AB}. Note that while the terms $\tilde{\mathcal{U}}_{(2)}$ and $\tilde{W}_4$ are the leading logarithmic terms in the expansions, they clearly induce at subleading orders further logarithmic terms.

It is particularly useful to recall the case of asymptotically locally anti-de Sitter spacetimes in five dimensions (see Section 2.3.1.1 and Appendix A.1.2), which solve the Einstein equations with cosmological constant and no matter. Then, working in Fefferman-Graham coordinates, the metric expansion in the vicinity of the conformal boundary $\rho \to 0$ is[4]

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}h_{ij}(x,\rho)dx^i dx^j \tag{7.6.5}$$

where the four-dimensional metric h_{ij} is expressed as

$$h_{ij} = h_{(0)ij} + \rho^2 g_{(2)ij} + \rho^4 h_{(4)ij} + \rho^4 \log\rho\, \hat{h}_{(4)ij} + \cdots \tag{7.6.6}$$

Just as in the expansions given above, the data highlighted in colours completely determines the integration functions in solving the equations and hence the entire asymptotic expansion. All other terms in the expansion are expressed in terms of curvature tensors of $(h_{(0)ij}, h_{(4)ij})$. Recall that the interpretation of the defining data in the dual CFT is that the non-normalisable data $h_{(0)ij}$ is the background metric for the field theory, while the normalisable data $h_{(4)ij}$ determines the expectation value of the stress tensor in the CFT. The occurrence of logarithmic terms in the asymptotic expansion and in the regulated onshell action relates to Weyl anomalies in the field theory.

The coefficient of the leading log term is [107, 108]

$$\begin{aligned} \hat{h}_{(4)ij} &= \frac{1}{2}\overset{(0)}{R}_{ikjl}\overset{(0)}{R}{}^{kl} - \frac{1}{12}\nabla_i\nabla_j\overset{(0)}{R} + \frac{1}{4}\nabla^2\overset{(0)}{R}_{ij} - \frac{1}{6}\overset{(0)}{R}\overset{(0)}{R}_{ij} \\ &\quad -\frac{1}{24}\left(\nabla^2\overset{(0)}{R} - \overset{(0)}{R}{}^2 + 3R_{(0)kl}\overset{(0)}{R}{}^{kl}\right)h_{(0)ij} \end{aligned} \tag{7.6.7}$$

where ∇ is the covariant derivative associated with $h_{(0)}$ and $\overset{(0)}{R}$ denotes the curvature of $h_{(0)}$. Note that $\hat{h}_{(4)ij}$ does not depend on the normalizable data $h_{(4)ij}$: conformal anomalies depend only on the background fields of the CFT, not on the specific state of the field theory and its energy

[4]Compared to the Fefferman-Graham metric written in previous parts, we have renamed $z = \rho$ to avoid possible confusion with other z's in this part and we have automatically set to zero the terms which vanishes by the Einstein's equations.

momentum tensor.

It is interesting to compare (7.6.7) with our results for asymptotically locally flat spacetimes in five dimensions. In both cases, the coefficients of the leading log terms depend only on the non-normalizable data and its derivatives (see Appendix B.1). Furthermore, the coefficient of the leading log in W^A is not independent from the coefficient of the leading log of $\mathcal{U}$. Notice, however, that the number of independent degrees of freedom of the "normalizable data" is the same (modulo symmetry reduction) in this case as in $AlAdS$.

If one restricts either asymptotically flat or to asymptotically AdS, the coefficients of the log terms vanish.

The analysis of five-dimensional asymptotically locally anti-de Sitter spacetimes in Fefferman-Graham gauge makes manifest four-dimensional covariance, and accordingly this is the most commonly used gauge for asymptotic analysis. To make contact with our analysis of asymptotically locally flat spacetimes, one can instead use Bondi gauge for the asymptotic analysis in anti-de Sitter, see [190, 272] for the corresponding analysis in four dimensions. It would be interesting to carry out the asymptotic analysis in Bondi gauge for five dimensional asymptotically locally anti-de Sitter, and to explore the limit as the cosmological constant is taken to zero, in order to elucidate the structure found here.

In the context of asymptotically flat spacetimes in four dimensions, polyhomogeneous expansions have been discussed in a number of earlier works, see [273, 274, 275, 276, 277, 278]. In these contexts, however, the appearance of logarithmic terms is associated with non-smoothness of the boundary data; imposing suitable regularity conditions sets the logarithmic terms to zero. This fits with asymptotically locally anti-de Sitter spacetimes in four dimensions that satisfy Einstein's equations with negative cosmological constant: these also do not have logarithmic terms in their asymptotic expansions, since the Weyl anomaly of the stress tensor in a three-dimensional conformal field theory vanishes.

7.7 Summary of results

We have argued that only $(d-3)$-branes in d spacetime dimensions are flat in the vicinity of the brane, and therefore the natural generalization of cosmic strings/superrotations in four dimensions should involve $(d-3)$-branes.

We have then showed that such branes are accommodated in a configuration space defined in Bondi gauge by what we called asymptotically locally flat boundary conditions, rather than flat. The are defined in (7.4.41) in terms of a non-trivial $(d-2)$ metric, describing a $(d-2)$-manifold that is topologically a $(d-2)$-sphere. These boundary conditions include cosmic branes, but are rather general.

We have then commented on the analogies of the asymptotic solution with the well-known asymptotic solutions for asymptotically locally AdS spacetimes in Fefferman-Graham gauge and proposed a holographic interpretation for the former.

We showed that the vacuum Einstein equations can be solved consistently with these boundary conditions near the null boundary. The resulting asymptotic expansions are polyhomogeneous, with the leading logarithmic terms in the expansions are expressed in terms of derivatives of the boundary metric on the celestial sphere. Again, this seems very much analogous to the structure of asymptotically locally anti-de Sitter spacetimes in five dimensions.

CHAPTER 8

General null asymptotics and superrotation-compatible configuration spaces in $d \geq 4$

Bondi-Sachs gauge vacuum Einstein equations in spacetime dimension $d \geq 4$ are solved at the non-linear level with the most general boundary conditions preserving the null nature of infinity. When restricting to locally Minkowskian asymptotics, the analysis clarifies and gives necessary conditions for the configuration space of solutions to be acted by supertranslations and $Diff(S^{d-2})$ superrotations in $d > 4$. The interplay of radiative/non-radiative falloff behaviours with the constraints imposed on the leading asymptotic data by the former is highlighted. The minimal case in which extended supertranslations and $Diff(S^{d-2})$ superrotations form consistent allowed diffeomorphisms requires the boundary metric on cuts of $\mathscr{I}$ to be time (u) dependent. This is a new feature with respect to the four-dimensional case. We also show that an asymptotic symmetry with $Diff(S^{d-2})$ superrotations and radiative falloff behaviour (namely translations rather than supertranslations) is not allowed; hence in higher dimensions superrotations need supertranslations. We discuss both polyhomogeneous and non-polyhomogeneous expansions and show that the r^{-1} falloff required to obtain infinite-dimensional asymptotic symmetries implies that a maximal polyhomogeneous expansion is mandatory in even dimensions unless the r^{-1} term is appropriately constrained. In odd dimensions the situation can be argued to be different. We interpret these points by connecting with the asymptotic expansion of asymptotically locally AdS spacetimes and suggest a holographic motivation.

8.1 Introduction

Campiglia-Laddha's proposal of $SDiff(S^2)$ superrotations, as well as BT-superrotations in four-dimensional spacetimes, exemplifies how the realization of Strominger's triangular equivalence at subleading order requires boundary conditions on a bulk spacetime to be appropriately engineered to host enough asymptotic symmetries.

The same path can be followed to address the problem of soft theorems/asymptotic symmetries in higher dimensions [71, 279, 79, 80].

In higher dimensions, as we have said (see Section 5.3), the lack of of supertranslations is due to the choice of radiative falloff conditions in Tanabe, Kinoshita, Shiromizu analysis (TKS in the following)

$$h_{AB} - \gamma_{AB} = O(r^{\frac{2-d}{2}}). \tag{8.1.1}$$

Given that superrotations apparently only depend on the degree of freedom imposed on the boundary metric, we may wonder that we can consistently enlarge the Poincaré asymptotic symmetry in $d > 4$ to include generic $Diff(S^{d-2})$ CL-transformations, as we on the other hand know that BT-superrotations in $d > 4$ are forbidden. We leave the answer to this question at the end of this chapter.

Focussing first on leading soft theorems and supertranslations, Kapec, Lysov, Pasterski and Strominger [71] (KLPS in the following) engineered appropriate falloff conditions such that the corresponding solution space (or configuration space) hosts supertranslations. Since supertranslations are strictly related to the order r^{-1} in h_{AB}, KLPS seek to construct a configuration space where

$$h_{AB} = \gamma_{AB} + \frac{h_{(1)AB}}{r} + \dots. \tag{8.1.2}$$

This is the same falloff behaviour as in $d = 4$, but in higher dimensions it is not consistent with linearised radiation and hence we name it *non-radiative falloff* condition. They further assume that all the subleading terms are associated with integer powers. This is thus the same expansion as in $d = 4$ and a minimal overleading extension of the expansion taken by TKS in even dimension. Hence, KLPS results are valid in even dimensions. These authors furthermore restrict their analysis to the linear level[1].

We have mentioned in chapter 5 that Einstein's equations in Bondi gauge do not uniquely fix the radial expansion of h_{AB}, but only that of $\partial_u h_{AB}$. Thanks to this fact, the equations can be solved consistently for virtually any expansion of h_{AB}. Namely, set an ansatz for the expansion of h_{AB} and solve the equations with that. The solutions will constrain $\partial_u h_{AB}$ at various orders and in turn β, $\mathcal{U}$ and W^A, but not the r-dependence of h_{AB} itself. This is the reason why several forms of polyhomogeneous h_{AB} have been analysed in the $d = 4$ literature. This is also the motivation of the KLPS argument.

On the contrary, in the previous chapter we avoided assuming any expansion a priori, and instead

[1]Notice that they do not consider vacuum field equations, but impose the Ricci tensor exactly vanishes to some order.

fed the equations with generic boundary data $(h_{(0)AB}, \beta_{(0)})$. We concluded that in order not to overconstrain such data we have to allow for the order r^{-1} in h_{AB}. That is, if we do not take h_{AB} to have a r^{-1} term, then the equations imply constraints on $(h_{(0)AB}, \beta_{(0)})$. On the other hand, if we take $h_{(1)AB}$ to be non-vanishing but $h_{(0)AB} = \gamma_{AB}$ and $\beta_{(0)} = 0$, we get the constraint

$$\partial_u h_{(1)AB} = 0. \tag{8.1.3}$$

In five dimensions, the next subleading order is $r^{-3/2}$ and hosts the shear/news tensor.

Equation (8.1.3) is what characterises the configuration space built in the KLPS analysis. In any even dimension greater than four, the KLPS analysis gives the vanishing of $\partial_u h_{(p)AB}$ up to the order of the news tensor. In such a configuration space, later reconsidered via covariant phase space methods [279], pretty much all the fundamental results and the necessary conditions from $d = 4$ can be transferred [199], including (if we are brave enough) the matching conditions between future and past null infinity. The result is a linear configuration space which is consistent with the action of supertranslations, where the matter of scattering and soft theorems associated to supertranslations can be discussed. Some of the difficulties encountered at the non-linear level are briefly discussed in [279].

In the published version of the previous chapter, [1], we show that the five-dimensional configuration space we constructed supports supertranslations, thus establishing that the results of [71] and [279] can be extended to odd dimensions.

The possibility of BT-superrotations in higher dimensions is algebraically speaking impossible, as we said several times, but the extensions of CL-superrotations to $d > 4$ *only require* allowing the leading asymptotic order of h_{AB} to "fluctuate" (with fixed determinant) under the action of asymptotic Killing vectors. However, "*only require*" is a wrong statement. The analysis of this chapter moves from this point, which seems not to be appreciated in (the scarce) existing literature. Colferai and Lionetti [79] construct explicit CL-charges in higher dimensions[2] adapting the KLPS analysis to CL boundary conditions: they take the CL condition $\mathfrak{L}_\xi g_{AB} = O(r^2)$ rather than $\mathfrak{L}_\xi g_{AB} = O(r)$ (recall $g_{AB} = r^2 h_{AB}$) within the KLPS configuration space. As noted by these authors, the resulting $SDiff(S^{d-2})$ superrotations are inconsistent with the definition of the configuration space because they act on $h_{(1)AB}$ as

$$\mathfrak{L}_Y h_{(1)AB} \propto u f(x)_{AB}, \tag{8.1.4}$$

where f is a covariant combination of Y and covariant derivatives compatible with $h_{(0)}$. This transformation rule explicitly breaks the configuration space defined by (8.1.3), thus making any subsequent identification of the broader picture of Strominger's triangles and flat spacetime holography void.

We have explored in chapters 4 and 6 the necessary conditions for such pictures to hold, and we can here summarise the main points to better frame the aforementioned problem and the solution presented in this chapter. A plausible fundamental condition for the resulting pictures - including holography - to hold is the existence of a well-defined phase space. In a covariant phase space

[2] Albeit glossing over a proper derivation that takes into account the divergences and the issues of the incompatibility with the configuration space.

perspective, where the phase space is built over a configuration space by endowing the latter with a symplectic structure, the problem is schematically divided as:

1) Definition of the configuration space of fields with given boundary conditions (in this case either at $\mathscr{I}^+$ or $\mathscr{I}^-$) with consistent asymptotic Killing fields,
2) Definition of the phase space: charges associated to the asymptotic Killing fields must be finite and - in principle - integrable.

If the ultimate goal is the scattering problem, the phase space must be defined so that the conjectured symmetry of the S-matrix is consistent. This is non-trivial since BMS per se is defined separately on $\mathscr{I}^+$ and $\mathscr{I}^-$. Another condition must be met:

3) Definition of matching conditions (boundary conditions along $\mathscr{I}$) that do not trivialize charges.

This chapter addresses the first point of the previous list. We approach the problem from a general point of view which allows us to study the basic conditions underlying the consistency of the configuration space (in Bondi-Sachs gauge) with the action of asymptotic Killing fields in any dimension higher than four. We also compare and contrast the four-dimensional case with higher dimensions (Section 8.4).

We clarify in which sense, and under what conditions on the boundary data, radiative and non-radiative falloff beahviours in higher dimensions are consistent. Supertranslations and CL-like superrotations arise from subcases in an automatically consistent configuration space (section 8.5). In the next subsection we further specify the results and the organisation of the chapter.

8.1.1 Results and organization

We recognise that the root of the issue discussed around (8.1.4) is the choice of boundary conditions. To make (8.1.4) consistent with the configuration space we must provide a configuration space where

$$\partial_u h_{(1)AB} \neq 0. \tag{8.1.5}$$

Since $\partial_u h_{(1)AB}$ is determined by $(h_{(0)AB}, \beta_{(0)})$, which we called non-normalizable data in the previous chapter, in this chapter we will prove the claim

Claim: Consistency between the action of CL-like superrotations and the definition of the configuration space is obtained with the boundary data $(h_{(0)AB}, \beta_{(0)} = 0)$ with *i)* $h_{(0)AB}$ u-dependent and Einstein, or *ii)* $h_{(0)AB}$ u-independent but not Einstein, or *iii)* $h_{(0)AB}$ arbitrary[3].

We notice that already in the previous chapter (namely in [1]) we gave a $d = 5$ configuration space where CL-like superrotations are supported. That analysis corresponds to a subcase of the

[3]We can recast this claim including $\beta_{(0)}$, but since it is never considered in the literature concerning $d = 4$ (see later) we will not deeply analyse its role here.

present chapter: there $h_{(0)AB}$ is u-independent and hence, according to this Claim, it must not be Einstein to host CL-like superrotations.

The possible u-dependence of $h_{(0)AB}$ is not arbitrary, but is fixed by Einstein's equations to

$$h_{(0)AB}(u,x) = e^{2\varphi(u,x)}\hat{h}_{(0)AB}(x)\,, \quad \partial_u\varphi := \frac{l}{d-2} := \frac{\partial_u q}{(d-2)2q} \tag{8.1.6}$$

where q is the determinant of h_{AB}. See [60] for the corresponding analysis in four dimensions.

A fundamental equation in this chapter is (for $d > 4$)

$$\partial_u h_{(1)AB} - \frac{l}{d-2}h_{(1)AB} = \frac{2e^{2\beta_{(0)}}}{d-4}\left(\frac{h_{(0)AB}}{d-2}\overset{(0)}{R} - \overset{(0)}{R}_{AB} + (4-d)\mathcal{B}_{AB}[\beta_{(0)}]\right) \tag{8.1.7}$$

where

$$\mathcal{B}_{AB} = 2\left[\overset{(0)}{D}_A\partial_B\beta_{(0)} + 2\partial_A\beta_{(0)}\partial_B\beta_{(0)} - \frac{h_{(0)AB}}{d-2}\left(\overset{(0)}{D}{}^2\beta_{(0)} + 2\partial_C\beta_{(0)}\partial^C\beta_{(0)}\right)\right]. \tag{8.1.8}$$

From the above, the previous considerations arise because $\mathcal{B}_{AB} = 0$ if $\beta_{(0)} = 0$ or $\partial_A\beta_{(0)} = 0$. Notice that in the previous chapter we did not take $h_{(0)AB}$ (i.e the functions $C_{(0)i}$) to depend on u.

Equation (8.1.7) can be recast as

$$\boxed{\partial_u h_{(1)AB} - \frac{l}{d-2}h_{(1)AB} = \frac{2}{d-4}e^{2\beta_{(0)}}\left[-\hat{\mathcal{R}}_{AB} + (d-4)\left(\Phi_{AB}[\varphi] - \mathcal{B}_{AB}[\beta_{(0)}]\right)\right]}, \tag{8.1.9}$$

where $\hat{\mathcal{R}}_{AB}$ is the traceless part of the Ricci tensor $\hat{R}_{AB}$ of $\hat{h}_{(0)AB}$ and $\hat{\mathcal{R}}_{AB} - (d-4)\Phi_{AB}[\varphi] = \mathcal{R}_{AB}$ is the traceless part of the Ricci tensor $\overset{(0)}{R}_{AB}$ of $h_{(0)AB}$, so that

$$\Phi_{AB}[\varphi] = (\hat{D}_A\hat{D}_B\varphi - \hat{D}_A\varphi\hat{D}_B\varphi) - \frac{\hat{h}_{AB}}{d-2}(\hat{D}^2\varphi - (\hat{D}\varphi)^2). \tag{8.1.10}$$

Given (8.1.9) we immediately infer (see Table 8.1.1) what restrictions are imposed on the boundary data $(h_{(0)AB}(u,x), \beta_{(0)}(u,x))$ by the radiative falloff behaviour. Conversely, to have a configuration space compatible with superrotation-like asymptotic transformations we must have, as argued before $\partial_u h_{(1)AB} \neq 0$, and from (8.1.9) we get the cases summarised in Table 8.1.2

While proving these statements at the non-linear level, we observe that they also provide a solution to the concerns raised in [279] when briefly exploring supertranslations at the non-linear level in even dimension.

A different take on this problem was pursued in [80]. They partially solve the issue of [79] because they work in the linearised regime and in even dimensions and find that $h_{(1)AB}$ is pure gauge (consistently with [71, 77]) and linear in u. Our result is more general.

The summarised results are obtained as subcases of the most general choice of boundary data possible, further discussed in Section 8.2. Both $h_{(0)AB}$ and $\beta_{(0)}$ are a priori unrestricted, and

Radiative falloff $h_{(1)AB} \equiv 0$ $\Rightarrow$	$\mathcal{H}_{(2)AB} = 0$
$\partial_u h_{(0)AB} = 0$	
Einstein ($\hat{\mathcal{R}}_{AB} \equiv 0$, $\Phi_{AB} \equiv 0$)	$\mathcal{B}_{AB} = 0 \longrightarrow \partial_A \beta_{(0)} = 0$
Conformal to Einstein ($\hat{\mathcal{R}}_{AB} \equiv 0$, $\partial_u \Phi_{AB} = 0$)	$\mathcal{B}_{AB} = \Phi_{AB} \longrightarrow \varphi(x) = 2\beta_{(0)}(x)$
Not Einstein	$\hat{\mathcal{R}}_{AB} \sim \Phi_{AB} - \mathcal{B}_{AB}$
$\partial_u h_{(0)AB} = e^{2\varphi(u,x)} \hat{h}_{AB}(x)$	
$\hat{h}$ Einstein ($\hat{\mathcal{R}}_{AB} \equiv 0$)	$\mathcal{B}_{AB} = \Phi_{AB} \longrightarrow \varphi(u,x) = 2\beta_{(0)}(u,x)$
Not Einstein	$\hat{\mathcal{R}}_{AB} \sim \Phi_{AB} - \mathcal{B}_{AB}$

Table 8.1.1: Synoptic view of various constraints imposed on the boundary data $(h_{(0)AB}, \beta_{(0)})$ by the request $h_{(1)AB} \equiv 0$ in $d > 4$. The right column reports the conditions stemming from $\mathcal{H}_{(2)AB}$ under the assumptions in the left column and the simple arrow $\longrightarrow$ indicates a possible solution of the conditions.

$\partial_u h_{(1)AB} \neq 0$	$l = 0$	$\mathcal{H}_{(2)AB} \neq 0$	$\hat{\mathcal{R}}_{AB} = 0,\ \varphi(x) = 0 \Rightarrow \mathcal{B}_{AB} \neq 0$	(8.1.11)
			$\hat{\mathcal{R}}_{AB} = 0,\ \varphi(x) \neq 0 \Rightarrow \Phi_{AB} - \mathcal{B}_{AB} \neq 0$	(8.1.12)
			$\hat{\mathcal{R}}_{AB} \neq 0$	(8.1.13)
	$l \neq 0$	$\mathcal{H}_{(2)AB} = 0$	$\hat{\mathcal{R}}_{AB} = 0 \Rightarrow \Phi_{AB} = \mathcal{B}_{AB} \longrightarrow \varphi = 2\beta_{(0)}$	(8.1.14)
			$\hat{\mathcal{R}}_{AB} \neq 0$	(8.1.15)
		$\mathcal{H}_{(2)AB} \neq 0$	$\hat{\mathcal{R}}_{AB} = 0 \Rightarrow \Phi_{AB} - \mathcal{B}_{AB} \neq 0$	(8.1.16)
			$\hat{\mathcal{R}}_{AB} \neq 0$	(8.1.17)

Table 8.1.2: Various boundary data compatible with $\partial_u h_{(1)AB} \neq 0$. The cases such that $\partial_u h_{(1)AB} = 0$ can also be obtained from this under appropriate changes.

we also do not use at any point the spherical topology of the cross sections of $\mathscr{I}$, so that the conclusions reached are valid for any asymptotic null boundary with topology $\mathbb{R} \times \mathbb{B}^{d-2}$, where $\mathbb{B}^{d-2}$ is a base space. The solution space is built in section Section 8.4.

We can rephrase the conclusions of this chapter by saying that the solution space we provide is manifestly associated with a boundary generalised Carrollian structure, where "generalised" stands for the inclusion of CL-superrotations (see Appendix A.1.1).

At the end of this chapter (Section 8.6) we make some preliminary comments on the asymptotic charges and the phase space. We should expect divergences, but - at least when $\beta_{(0)} = 0$ - we believe we can consistently renormalize the charges as was done by Compère and collaborators [156] for CL-superrotation charges in $d = 4$. The criticism of [157] about the covariance of the procedure will apply also in this case.

The main points of the discussion are exemplified and summarised in Section 8.3 and Section 8.5 discusses the asymptotic Killing vectors, while Section 8.4 includes the details of the asymptotic analysis and comments on polyhomogeneous expansions.

8.2 On the role of $\beta_{(0)}$

We set out to solve vacuum Einstein's equations in Bondi gauge

$$ds^2 = -\mathcal{U}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(dx^A - W^A du)(dx^B - W^B du), \tag{8.2.1}$$

$$g_{rr} = 0 = g_{rA}, \quad \det(h_{AB}) = q(u, x), \tag{8.2.2}$$

with the boundary pair $(h_{(0)AB}(u, x), \beta_{(0)}(u, x))$.

We pause here to briefly discuss the role of a non-vanishing $\beta_{(0)}$. Indeed, up to the previous chapter we took $\beta_{(0)} = 0$ without worrying so much. $\beta_{(0)}$ is related to the definition of r (a non-affine parameter along null geodesics) such that the expansion is $\Theta = e^{2\beta}/r$. Setting $\beta_{(0)} = 0$ formally corresponds to gauge $\beta_{(0)}$ away by appropriately redefining the coordinates. This is always possible when we assume the standard notions of asymptotically flat spacetimes, namely isolated spacetimes with appropriate falloffs of the matter stress-energy tensor.

However, as argued in [143], the required transformation deforms the initial null surface in the spacetime and thus is not a natural condition from a characteristic initial value problem point of view. If we advocate a boundary perspective on the integration scheme, gauging $\beta_{(0)}$ away from the start may seem less unnatural. However, it may not be desirable if we think of a flat limit from AdS/CFT in Bondi gauge, where $\beta_{(0)}$ was given a role in the dual CFT [190, 272] From this perspective we would ideally like to keep track of the fate of the degrees of freedom included in $\beta_{(0)}$ when taking the limit.

We can still argue in favour of keeping $\beta_{(0)}$ by simply observing that, with the knowledge of the asymptotic solutions (see for example (5.2.4) or later in the chapter) the leading order metric is

$$ds^2 = e^{2\beta_{(0)}}(-g_{uu}du^2 - 2dudr) + g_{AB}dx^A dx^B \tag{8.2.3}$$

In particular we can take the boundary condition $g_{AB} = r^2\gamma_{AB}$ so that

$$ds^2 = e^{2\beta_{(0)}}(-du^2 - 2dudr) + r^2\gamma_{AB}dx^A dx^B. \tag{8.2.4}$$

Any metric in the class

$$ds^2 = e^{2\beta}(-du^2 - 2dudr) + r^2\gamma_{AB}dx^A dx^B \tag{8.2.5}$$

with $\beta = \beta(u, r, x)$ can be conformally compactified in $d = 4$ with a smooth $\Omega = r^{-1}$ and admits a smooth $\mathscr{I}$ [187], however its curvature is such that the stress-energy tensor does not satisfy the falloff requirements usually assumed in $d = 4$. This is why such metrics are not considered asymptotically flat [187]. With a non-zero $\beta_{(0)}$ we are genuinely in presence of a non-asymptotically flat spacetime (in the standard sense) which can be called asymptotically locally Minkowski by comparison with the AdS/CFT jargon.

Notice that in $d = 3$, the notion of asymptotically flat spacetimes, defined by symmetry reductions of four dimensional cylindrical waves, is necessarily of this form [187].

We have thus several good arguments to take $\beta_{(0)}$ generic, but to make contact with supertranslations and CL-superrotations we will restrict - when appropriate - to $\beta_{(0)} = 0$ and spherical cross sections. Once again we stress that the presence of $\beta_{(0)}$ will affect considerations of the well-posedness of the variational principle.

8.3 Asymptotic analysis: discussion

For ease of comparison with existing literature, we adopt conventions similar to [60]. We define the quantities (see Appendix C)

$$l_{AB} = \frac{1}{2}\partial_u g_{AB}\,, \quad k_{AB} = \frac{1}{2}\partial_r g_{AB}\,, \quad n_A = \frac{1}{2}e^{-2\beta}g_{AB}\partial_r W^B, \tag{8.3.1}$$

as well as[4]

$$\tilde{n}_A = \frac{n_A}{r^2}, \quad \tilde{K}^C_D := \frac{1}{2}h^{AC}\partial_r h_{AD}, \quad \text{so that} \quad k^A_B = \frac{\delta^A_B}{r} + \tilde{K}^A_B\,. \tag{8.3.2}$$

Einstein main equations take the form

$$R_{rr} = 0 \Rightarrow \quad \partial_r\beta = \frac{r}{2(d-2)}\tilde{K}^A_B\tilde{K}^B_A\,, \tag{8.3.3}$$

[4] Notice that [60] uses $K^A_B := r^2\tilde{K}^A_B$.

$$R_{Ar} = 0 \Rightarrow \quad \partial_r(r^d \tilde{n}_A) = \mathcal{G}_A(\beta, \tilde{K}) \tag{8.3.4}$$
$$\mathcal{G}_A(h_{AB}, \beta) = r^{d-2}\left[\left(\partial_r - \frac{d-2}{r}\right)\partial_A\beta - {}^{(d-2)}D_B\tilde{K}^B_A\right],$$

$$g^{AB}R_{AB} = 0 \Rightarrow \quad \frac{d-2}{r^2}\left[(d-3) + r\partial_r\right]\mathcal{U} = \mathcal{F}(h_{AB}, \beta, W^A)\,, \tag{8.3.5}$$
$$\mathcal{F}(h_{AB}, \beta, W^A) = e^{2\beta}\left[{}^{(d-2)}R - 2(D_AD^A\beta + \partial^A\beta\partial_A\beta + n^An_A)\right] + \left(\partial_r + 2\frac{d-2}{r}\right)D_AW^A + 2\frac{d-2}{r}l\,,$$

$$g^{DA}R_{AB} = 0 \Rightarrow \left(\partial_r + \frac{d-2}{r}\right)l^D_B + (\partial_u + l)\,k^D_B = \mathcal{H}^D_B\,, \tag{8.3.6}$$
$$\begin{aligned}\mathcal{H}^D_B = &-e^{2\beta}\left[{}^{(d-2)}R^D_B - 2\left({}^{(d-2)}D^D\partial_B\beta + \partial^D\beta\partial_B\beta + n^Dn_B\right)\right]\\ &-\left(\partial_r + \frac{d-2}{r}\right)\left(\frac{1}{2}{}^{(d-2)}D^DW_B + \frac{1}{2}{}^{(d-2)}D_BW^D - k^D_B\mathcal{U}\right)\\ &-\left[{}^{(d-2)}D_C(W^Ck^D_B) + k^D_A{}^{(d-2)}D_BW^A - k^A_B{}^{(d-2)}D_AW^D\right].\end{aligned}$$

We can stress the difference with field equations in Fefferman-Graham gauge with Λ.

In that case the terms involving the time derivative are hidden in the covariant derivatives along the timelike hypersurfaces that foliate the spacetime and the only terms that are distinguished are those analogous to k^A_B, involving derivatives along the radial direction orthogonal to the hypersurfaces: this is (appropriately normalised) the extrinsic curvature of the timelike surfaces that foliate the spacetime in the typical *AlAdS* integration scheme in terms of Gauss-Codazzi equations.

Analogously, k_{AB} is the (non-normalised) extrinsic curvature of the $r = const$ timelike surfaces. The cuts of such surfaces by $u = const$ null surfaces are spacelike and their extrinsic curvature is

$$Q_{AB} = l_{AB} + {}^{(d-2)}D_{(A}W_{B)}. \tag{8.3.7}$$

Equation (8.3.6) can be rewritten in terms of Q_{AB} and $\partial_u k_{AB}$ but not much is gained for our current purposes[5].

The solution of all the main equations can be given in a closed integral form depending on $\tilde{K}^A_B$ and encoding the asymptotic behaviour of h_{AB}. As we have commented several times, such behaviour is not uniquely determined by any equation.

The integral form of the solution in subsection 8.3.1 can be given following [60]. Then in subsection 8.3.2 we discuss, to fix the ideas, the leading order solution and the strategy pursued in Section 8.4, where full details are given and the comparison between radiative and non radiative behaviour is made and logarithmic terms are discussed.

[5]We get a shorter form, but upon performing the manipulations of the next section to remove $\partial_u k_{AB}$, the equation will almost look the same as what we already have.

8.3.1 Integral solution of the main equations

In the following, we use the somewhat imprecise convention of singling out the relevant integration functions and leaving integrals as indefinite when no chance of confusion arises. This set, we write the solution of equation (8.3.3) as

$$\beta(u,r,x) = \beta_{(0)}(u,x) + \frac{1}{2(d-2)}\int r\tilde{K}^A_B\tilde{K}^B_A. \tag{8.3.8}$$

With (**??**), (8.3.4) becomes

$$\partial_r(r^d\tilde{n}_A) = r^{d-2}\left[\left(\partial_r - \frac{d-2}{r}\right)\partial_A\beta - {}^{(d-2)}D_B\tilde{K}^B_A\right] =: \mathcal{G}(\beta,\tilde{K}) \tag{8.3.9}$$

where we have also defined for convenience

$$\tilde{n}_A = \frac{1}{2}e^{-2\beta}h_{AB}\partial_r W^B\,, \tag{8.3.10}$$

so that

$$\tilde{n}_A(u,r,x) = \frac{N_A(u,x)}{r^d} + \frac{1}{r^d}\int^r \mathcal{G}_A\,\mathrm{d}s, \tag{8.3.11}$$

and hence

$$W^A(u,r,x) = W^A_{(0)}(u,x) + 2\int^r \mathrm{d}t\, e^{2\beta(u,t,x)}h^{AB}(u,t,x)\frac{1}{t^d}\left(N_B(u,x) + \int^t \mathcal{G}_B\,\mathrm{d}s\right) \tag{8.3.12}$$

The integration of (8.3.5) gives

$$\mathcal{U}(u,r,x) = \frac{\mathcal{U}_{(d-3)}(u,x)}{r^{d-3}} + \frac{1}{r^{d-3}}\int^r \frac{\mathcal{F}}{d-2}s^{d-2}\,\mathrm{d}s. \tag{8.3.13}$$

Using

$$\partial_u k^D_B = \partial_r l^D_B - 2(l^D_A k^A_B - k^D_A l^A_B)\,, \tag{8.3.14}$$

the latter equation (8.3.6) can be conveniently rewritten as

$$\boxed{\partial_r l^D_B + \mathfrak{o}^{DA}_{CB} l^C_A = j^D_B} \tag{8.3.15}$$

where

$$\mathfrak{o}^{DA}_{CB} = \frac{d-2}{2r}\delta^D_C\delta^A_B - (\delta^D_C k^A_B - k^D_C\delta^A_B)\,, \quad j^D_B = \frac{1}{2}\left(\mathcal{H}^D_B - lk^D_B\right) \tag{8.3.16}$$

and the solution is given by Lagrange method. Indeed, it is of the form

$$\dot{y}(x) + f(x)y(x) = g(x), \tag{8.3.17}$$

whose solution is

$$y(x) = e^{-F(x)}(c + \bar{y}(x)), \tag{8.3.18}$$

where $F(x)$ is an antiderivative of $f(x)$ and $\bar{y}(x)$ is an antiderivative of $g(x)e^{F(x)}$ and c is a constant. We write the solution of (8.3.15) as

$$l^D_B = e^{-\theta^{DA}_{CB}}\left(\frac{1}{2}N^C_A + \bar{l}^D_B\right) \tag{8.3.19}$$

where the factor 1/2 is chosen to cancel later factors of 2 in the definition of N_{AB}, with

$$\theta^{DA}_{CB} = \int \mathfrak{o}^{DA}_{CB}\,, \quad \bar{l}^D_B = \int j^C_A e^{\theta^{DA}_{CB}}\,. \tag{8.3.20}$$

Notice that $\mathfrak{o}^{DA}_{CB}$ contains a term explicitly of order r^{-1}, which when integrated contributes with logarithmic terms. The integral defining θ^{DA}_{CB} is to be considered between a generic r in the bulk (where the coordinate system breaks down) and a large R to be sent to infinity. It is easy to see that the potential logarithmic divergence is absorbed in a power of r.

Equivalently, to ease comparison with [60], we can split

$$l^A_B = l^A_{(0)B} + \tilde{L}^A_B\,, \quad l^A_{(0)B} = \frac{1}{2} h^{AC}_{(0)} \partial_u h_{(0)CB}\,, \tag{8.3.21}$$

and, to remove the explicit r^{-1} piece from the operator acting on $\tilde{L}^A_B$, we can further define

$$\tilde{L}^D_B := r^{\frac{2-d}{2}} L^D_B \tag{8.3.22}$$

so that (8.3.15) becomes

$$\boxed{\partial_r L^D_B + \mathcal{O}^{DA}_{CB} L^C_A = J^D_B} \tag{8.3.23}$$

with

$$\mathcal{O}^{DA}_{CB} = -(\delta^D_C \tilde{K}^A_B - \tilde{K}^D_C \delta^A_B), \tag{8.3.24}$$

and

$$J^D_B := r^{\frac{d-2}{2}} \tilde{J}^D_B := r^{\frac{d-2}{2}} \left[j^D_B + \left(\delta^D_C \tilde{K}^A_B - \tilde{K}^D_C \delta^A_B\right) l^C_{(0)A} - \frac{(d-2)}{2r} \delta^D_C \delta^A_B l^C_{(0)A} \right]. \tag{8.3.25}$$

The solution thus read

$$\boxed{L^D_B = e^{-\Theta^{DA}_{CB}} \left(\frac{1}{2} N^C_A + \bar{L}^D_B \right)} \tag{8.3.26}$$

with

$$\Theta^{DA}_{CB} = \int \mathcal{O}^{DA}_{CB}\,, \quad \bar{L}^D_B = \int J^C_A e^{\Theta^{DA}_{CB}}\,. \tag{8.3.27}$$

To reconstruct l^D_B from L^D_B one just uses

$$l^D_B = r^{\frac{2-d}{2}} L^D_B + l^D_{(0)B}\,. \tag{8.3.28}$$

Having solved the equations, we can complete the scheme (5.1.8) as

$$\begin{aligned}
\beta(u,r,x) &= \beta_{(0)}(u,x) + b(u,r,x), && b = b[\tilde{K}] \\
W^A(u,r,x) &= \frac{\mathcal{W}^A_{(d-1)}(u,x)}{r^{d-1}} + w^A(u,r,x), && w^A = w^A[\beta_0, \tilde{K}] \\
\mathcal{U}(u,r,x) &= \frac{\mathcal{U}_{(d-3)}(u,x)}{r^{d-3}} + v(u,r,x), && v = v[\beta_0, \tilde{K}, W_{(d-1)}, \mathcal{U}_{(d-2)}]
\end{aligned} \tag{8.3.29}$$

and

$$l^A_B(u,r,x) = l^A_B[\beta_0, \tilde{K}, W_{d-1}, \mathcal{U}_{(d-2)}] \tag{8.3.30}$$

The notation here indicates that the metric functions on the left depend of the quantities in square brackets on the right hand side, $\tilde{K}$ standing for any combination of $\tilde{K}^A_B$. All the solutions are given in terms of the auxiliary quantity $\tilde{K}^A_B$ defined in (**??**). Thus, in principle, the boundary conditions should not involve h_{AB} directly but rather $\tilde{K}^A_B$ and l^A_B. Clearly the boundary condition

$$\lim_{r\to\infty} h_{AB}(u,r,x) = h_{(0)AB}(u,x) \tag{8.3.31}$$

does correspond to the boundary condition

$$\lim_{r\to\infty} l^A_B = l^A_{(0)B} \tag{8.3.32}$$

which allowed us to perform the splitting (8.3.21), in terms of which, then, the boundary condition on L^A_B is

$$\lim_{r\to\infty} L^A_B = 0. \tag{8.3.33}$$

Furthermore, it is self evident that the boundary condition (8.3.31) cannot be relaxed further.

These boundary conditions on h_{AB} or equivalently l^A_B imply a boundary condition on $\tilde{K}^A_B$

$$\lim_{r\to\infty} \tilde{K}^A_B = 0 \tag{8.3.34}$$

However, as opposed to l^A_B (or equivalently L^A_B) and remarked several times, the radial behaviour of $\tilde{K}^A_B$ is not determined by any of the equations.

We can pick virtually any form of $\tilde{K}^A_B$ and we will be able to consistently solve the equations. However, the resulting conditions at subleading order may restrict the leading data. This happens if generic boundary conditions are assumed together with an asymptotic behaviour compatible with radiation at the leading order

$$\tilde{K}^A_B \sim r^{-d/2} \Leftrightarrow h_{AB} - h_{(0)AB} \sim r^{\frac{2-d}{2}}. \tag{8.3.35}$$

That is, radiative falloff conditions necessarily restrict the asymptotically locally Minkowski boundary conditions, as we now exemplify.

The goal is to use the equation for l^A_B to infer the radial behaviour of $\tilde{K}^A_B$ without restricting the leading order data.

8.3.2 Example: leading expansion

To exemplify we take $\tilde{K}^A_B = 0$. Hence k^A_B reduces to $k^A_B = \delta^A_B/r$. We get

$$b = 0\,, \quad w^A = \frac{W^A_{(1)}}{r}\,, \quad v = r\mathcal{U}_{(-1)} + \mathcal{U}_{(0)}, \tag{8.3.36}$$

where according to the scheme above all the coefficients of this expansion depend on $\beta_{(0)}$ and $h_{(0)AB}$. In turn, $\mathfrak{o}^{DA}_{CB} = \frac{d-2}{2r}\delta^D_C\delta^A_B$ and equation (8.3.15) reduces to

$$\partial_r l^D_B + \frac{d-2}{2r}\delta^D_C\delta^A_B l^C_A = \frac{j^D_{(1)B}}{r} + \frac{j^D_{(2)B}}{r^2}\,, \tag{8.3.37}$$

where (all the expressions are taken from the next section)

$$\begin{aligned} j^D_{(1)B} &= \frac{1}{2}\left((d-2)\delta^D_B\mathcal{U}_{(-1)} - l\delta^D_B\right), \\ j^D_{(2)B} &= \frac{\mathcal{H}^D_{(2)B}}{2}, \quad \mathcal{H}^D_{(2)B} = -e^{2\beta_{(0)}}\mathcal{R}_{AB} + (4-d)\mathcal{B}_{AB}[\beta_{(0)}], \end{aligned} \tag{8.3.38}$$

where $\mathcal{R}_{AB}$ is the trace-free part of the Ricci tensor of $h_{(0)AB}$ and $\mathcal{B}_{AB}[\beta_{(0)}]$ is a trace-free tensor vanishing when $\beta_{(0)} = 0$ or $\partial_A\beta_{(0)} = 0$, see (8.4.42). The solution of this equation is

$$\begin{aligned} d > 4: \qquad l^D_B &= \frac{2}{d-2}j^D_{(1)B} + \frac{2}{d-4}\frac{j^D_{(2)B}}{r} + \frac{N^D_B}{r^{\frac{d-2}{2}}}, \\ d = 4: \qquad l^D_B &= j^D_{(1)B} + \frac{1}{r}\left(j^D_{(2)B}\log r + N^D_B\right). \end{aligned} \tag{8.3.39}$$

However, the boundary conditions imply $l^D_B = l^D_{(0)B}$, and we infer (for any d)

$$l^D_{(0)B} = \frac{2}{d-2}j^D_{(1)B} \tag{8.3.40}$$

From $l^D_B = h^{DA}\partial_u h_{AB}$ we get a condition on $\partial_u h_{(0)AB}$

$$l^D_{(0)B} = \frac{l\delta^D_B}{d-2} \Leftrightarrow \partial_u h_{(0)AB} = \frac{2l}{d-2}h_{(0)AB}\,, \tag{8.3.41}$$

which is (8.1.6).

What about the subleading parts of l^D_B depending on r? In this example they are set to zero by the boundary condition $\tilde{K}^A_B = 0$. Indeed, from (8.3.14), $\tilde{K}^A_B = 0$ imposes $\partial_r l^A_B = 0$. So, the constant of integration N^A_B is to be set to zero and the term $j^D_{(2)B}$ must vanish identically. The latter implies

$$\mathcal{H}^D_{(2)B} = 0 \tag{8.3.42}$$

with $\mathcal{H}^D_{(2)B}$ given by (8.3.38).

This condition is not necessarily valid under the more general case of $\tilde{K}^A_B \neq 0$. Indeed, suppose

$$\tilde{K}^A_B = O(r^{-a-1}) \tag{8.3.43}$$

for some a, employing the notation of next section. Then, $\tilde{K}^A_B$ not only contributes to the subleading orders, but for some values of a, the leading terms of $\tilde{K}^A_B$ may as well contribute to $j^A_{(2)B}$.

For any a not modifying the above expressions, $j^A_{(2)B} = 0$ implies the following

$$\boxed{\frac{\delta^D_B}{d-2}\overset{(0)}{R} - \overset{(0)}{R}{}^D{}_B + (4-d)e^{-2\beta_{(0)}}\mathcal{B}^D_B[\beta_{(0)}] = 0} \tag{8.3.44}$$

In $d = 4$ it reduces to

$$\frac{\delta_B^D}{2}\overset{(0)}{R} - \overset{(0)}{R}{}^D{}_B = 0, \tag{8.3.45}$$

which is automatically satisfied.

In any $d > 4$, instead, the constraint relates the pair of boundary data $(h_{(0)AB}, \beta_{(0)})$. To uncover this relationship we may proceed either by making assumptions on $\beta_{(0)}$ or on $h_{(0)AB}$.

We can assume that $h_{(0)AB}$ satisfies itself Einstein's equations

$$\overset{(0)}{R}_{AB} - \frac{1}{2}h_{(0)AB}\overset{(0)}{R} + \mathcal{C}h_{(0)AB} = 0, \tag{8.3.46}$$

so that, as in many cases of potential interest, $h_{(0)AB}$ is an Einstein metric with a curvature parameter $\mathcal{C}$

$$\mathcal{C} = \frac{d-4}{2(d-2)}\overset{(0)}{R}. \tag{8.3.47}$$

and we are left with

$$\mathcal{B}_B^D[\beta_{(0)}] = 0 \tag{8.3.48}$$

to satisfy.

On the other hand, assume that $\beta_{(0)}$ is gauged away. In this case we are left with the task of satisfying the trace-free condition of $\overset{(0)}{R}_{AB}$. A class of solution is given by Einstein metrics, but this is not the general solution in $d - 2 > 2$.

Consider for example $h_{(0)AB} = e^{2\varphi(u,x)}\gamma_{AB}$, where γ_{AB} is the round metric on S^{d-2} or any other Einstein metric, so that its Ricci tensor is trace-free. From the conformal transformation properties of the curvature tensors, we get an equation relating φ and $\beta_{(0)}$. If on the other hand, γ is not Einstein, then φ and $\beta_{(0)}$ are related by γ.

These points further suggest that the interplay of $h_{(0)}$ and $\beta_{(0)}$ is to be considered with more care in $d > 4$.

The conclusion of the toy solution presented here is the following. The metric

$$ds^2 = -e^{2\beta_{(0)}}(r\mathcal{U}_{(-1)} + \mathcal{U}_{(0)})du^2 - 2e^{2\beta_{(0)}}dudr + r^2 h_{(0)AB}\left(dx^A + \frac{W^A_{(1)}}{r}du\right)\left(dx^B + \frac{W^B_{(1)}}{r}du\right) \tag{8.3.49}$$

is a solution of the field equations if (8.3.44) holds. As $W^A_{(1)} = 0$ when $\beta_{(0)} = 0$, this means that

$$ds^2 = -(r\mathcal{U}_{(-1)} + \mathcal{U}_{(0)})du^2 - 2dudr + r^2 h_{(0)AB}dx^A dx^B \tag{8.3.50}$$

is a solution of Einstein equations in $d = 4$ but not in $d > 4$ unless (8.3.46) is satisfied.

As we will see in the next section, the situation here discussed holds for radiative falloff conditions of (8.3.43), i.e. $a = (d-2)/2$.

If instead we wish to recover an infinite dimensional asymptotic symmetry group with supertranslations and $Diff(S^{d-2})$ superrotations, then the choice $a = 1$ is necessary, and the boundary data are unconstrained. Instead, new relationships for the u-derivative of the subleading terms in h_{AB} (as for the leading (8.3.40)) are obtained.

8.4 Asymptotic solutions: details

To give general explicit solutions of the main equations we select an *ansatz* for $\tilde{K}^A_B$ or equivalently h_{AB}. As we wish to compare with previous literature, we employ

$$h_{AB}(u,r,x) = h_{(0)AB}(u,x) + \sum_p \frac{h_{(a+p)AB}(u,x)}{r^{a+p}}. \tag{8.4.1}$$

where a is the leading power that we will now discuss and $p \in \mathbb{N}_0$ if d is even and $p \in \mathbb{N}_0/2$ if d is odd. Several comments are in order before proceeding.

Radiative falloff ansatz. Linearized perturbations around Minkowski space suggest that for radiative solutions $a = \frac{d-2}{2}$, which is integer for d even and half-integer for d odd. This justifies p belonging to different sets according to d even or odd. This splitting between even and odd does not depend on the gauge.

In literature we usually find other kinds of considerations which restrict the orders at which both integer and half-integer powers coexist in odd d. We may argue that the expansion only contains half-integer powers of r up to a certain point at which integer powers starts to contribute [70, 233, 77]. Indeed, the mixture of both integer and half-integer powers can be attributed to non-linear effects, which are supposed to be negligible asymptotically.

For example, Wald and Satishchandran [77] (not working in Bondi gauge) considered the following ansatz for odd d

$$g_{\mu\nu} = \eta_{\mu\nu} + G_{\mu\nu}\,, \quad G_{\mu\nu} = \sum_{n=\frac{d-2}{2}} r^{-n} G^{(n)}_{\mu\nu} + \sum_{m=d-3} r^{-m} \tilde{G}^{(m)}_{\mu\nu}\,, \tag{8.4.2}$$

where n is half-integer and p is integer and both sums proceed with unity steps. Thus, here the integer powers enter starting from the Coulombic order. Despite sounding sensible, it is important to stress - following [233], that it is not known when exactly the nonlinearities mixing integer and half-integer expansions kicks in. It is possible that they appear before the Coulombic order.

Non-radiative falloff ansatz. Since in the following we consider slower falloff behaviour than the radiative one, and in particular $a = 1$, we are not going to restrict further $h_{(a+p)AB}$. Indeed with $a = 1$, the nonlinearities will appear at order r^{-2} already.

Mimicking (8.4.2), we could particularize (8.4.1) for $a = 1$ as

$$h_{AB} = h_{(0)AB} + \sum_{m=1} \frac{h_{(m)AB}}{r^m} + \sum_{n=\frac{d-2}{2}} \frac{\tilde{h}_{(n)AB}}{r^n} \tag{8.4.3}$$

in order to include all and only the integer terms of order $m < (d-2)/2$ (in d even the sum will not be over n). But for the problem at hand we do not find any particular reason (except simplicity) for doing so and it will break the sort of "boundary" integration scheme we have set up[6].

Furthermore we will write expressions where switching from integers to half-integers is in principle easy, although care is needed.

Polyhomogeneous expansion. Another point to be considered is the presence of logarithmic terms in the expansion. Following what is known about $d = 4$, the most general ansatz would be

$$\begin{aligned} h_{AB}(u,r,x) &= h_{(0)AB}(u,x) + \sum_p \frac{H_{(a+p)AB}(u,r,x)}{r^{a+p}}\,, \\ H_{(a+p)AB}(u,r,x) &= h_{(a+p)AB}(u,x) + \sum_j \log r^j \mathtt{h}_{(a+p,j)AB}(u,x) \end{aligned} \tag{8.4.4}$$

For a $d = 4$ radiative spacetime ($a = 1$), $j \in \mathbb{N}_0$ (see for example [280, 274]). Notice that the leading expansion of the previous section indicates the presence of the first log term at order r^{-1} in $d = 4$. So, to exemplify again what discussed above: if $H_{(1)AB}$ is imposed to be non-zero from the start, with Minkowski boundary conditions, then taking $j_{(2)} = 0$ implies that $\mathtt{h}_{(1,1)AB}$ is u-independent (consistently with [280]), while $\partial_u h_{(1)AB} = N_{AB}$.

In our analysis, we will consider logarithmic terms in h_{AB} only tangentially whenever they stem from the equations, rather than imposing a polyhomogeneous expansion to start with. We are not aware of other works along these lines. Recently BMS charges were discussed for $d = 4$ asymptotically Minkowski spacetimes with general polyhomogeneous expansion in [234].

Determinant condition constraints. With the given ansatz (8.4.1), the determinant condition $h^{AB}\partial_r h_{AB} = 0$, implies

$$h^{AB}_{(0)} h_{(a+p)AB} = 0 \quad \forall\, p < a \tag{8.4.5}$$

while the trace of the others $h_{(a+p)AB}$ ($p \geq 0$)is determined in terms of the previous orders. To exemplify consider

$$h_{AB}(u,r,x) = h_{(0)AB}(u,x) + \frac{h_{(a)AB}}{r^a} + \frac{h_{(a+p_o)AB}(u,x)}{r^{a+p_o}}, \quad p_o = \begin{cases} \frac{1}{2} & d \text{ odd} \\ 1 & d \text{ even} \end{cases}. \tag{8.4.6}$$

Its inverse is

$$h^{AB} = h^{AB}_{(0)} - \frac{h^{AB}_{(a)}}{r^a} - \frac{h^{AB}_{(a+p_o)}}{r^{a+p_o}} + \frac{h^{A}_{(a)C} h^{CB}_{(a)}}{r^{2a}}, \tag{8.4.7}$$

[6] It is explicit from the formulas of the main equations along the transverse directions that the expansions of W^A and $\mathcal{U}$ induce half-integer as well as integer powers in $\partial_u h_{AB}$.

where we understand that we have to discard all terms of order greater than $a + p_o$. We retain the order $2a$ because it may be equal to $a + p_o$ according to the cases:

R) $a = \frac{d-2}{2}$, $d \geq 4$: $2a = a + p_o$ iff $d = 4$, otherwise $a + p_o < 2a$,

NR) $a = 1$, $d > 4$: $a + p_o = 2a$ for any d even, $a + p_o < 2a$ for any d odd.

where R and NR stand for radiative and non-radiative.

The determinant condition thus implies

$$-\frac{a}{r^{a+1}} h^{AB}_{(0)} h_{(a)AB} - \frac{a+p_o}{r^{a+p_o+1}} h^{AB}_{(0)} h_{(a+p_o)AB} + \frac{a}{r^{2a+1}} h^{AB}_{(a)} h_{(a)AB} + \cdots = 0. \tag{8.4.8}$$

So we get

$$h^{AB}_{(0)} h_{(a)AB} = 0 \quad \forall a,\, d \tag{8.4.9}$$

and

$$\begin{aligned}
&d > 4 \text{ odd } \ (p_o = \frac{1}{2}),\ \ a = \frac{d-2}{2} \quad \text{or} \quad a = 1: && h^{AB}_{(0)} h_{(a+p_o)AB} = 0 \\
&d > 4 \text{ even } \ (p_o = 1),\ \ a = \frac{d-2}{2} \quad\quad\quad\quad\quad : && h^{AB}_{(0)} h_{(a+p_o)AB} = 0 \\
&d \geq 4 \text{ even } \ (p_o = 1), \quad\quad\quad\quad\quad\quad\quad a = 1: && h^{AB}_{(0)} h_{(2)AB} = \frac{1}{2} h^{AB}_{(1)} h_{(1)AB}.
\end{aligned} \tag{8.4.10}$$

As we can see from this example, the consequences of the gauge choice in even $d > 4$ with $a = 1$ is really like $d = 4$ and this is why many results can be transferred to such generic dimensions quickly. However, the situation changes with radiative falloff conditions.

Furthermore, when $d = 4$ we can also use the relationship valid for any 2×2 symmetric traceless matrices [60]

$$M^A_C M^C_B = \frac{1}{2} \delta^A_B M^C_D M^C_C, \tag{8.4.11}$$

which does not hold for higher dimensional symmetric and traceless matrices. These are subtle remark which should be taken into account when computing charges.

8.4.1 Asymptotic expansion of β, W^A and $\mathcal{U}$

With the ansatz (8.4.1) given a and the boundary condition $W^A_{(0)} = 0$ we get,

$$\boxed{\begin{aligned}
\beta &= \beta_{(0)} + \sum_{p \geq 0} \frac{\beta_{(2a+p)}}{r^{2a+p}} \\
W^A &= \frac{W^A_{(1)}}{r} + \sum_{p=0}^{d-2-a} \frac{W^A_{(a+1+p)}}{r^{a+1+p}} + \frac{1}{r^{d-1}} \left(\mathcal{W}^A_{(d-1)} + \mathtt{W}^A_{(d-1)} \log r \right) + \ldots \\
\mathcal{U} &= r\,\mathcal{U}_{(-1)} + \mathcal{U}_{(0)} + \sum_{p=0}^{a+p<d-3} \frac{\mathcal{U}_{(a+p)}}{r^{a+p}} + \frac{1}{r^{d-3}} \left(\mathcal{U}_{(d-3)} + \mathtt{U}_{(d-3)} \log r \right) + \ldots
\end{aligned}}$$

The details of the expansions can be found in Appendix C.2. Here we make some comments and list the expressions relevant in the next parts. The expansion is shown up to the order of the free functions $\mathcal{W}^A_{(d-1)}$ and $\mathcal{U}_{(d-3)}$ which are universal (i.e. their functional form does not depend on the spacetime dimension) and are given by

$$\mathcal{W}^A_{(d-1)} = -\frac{2}{d-1} e^{2\beta_{(0)}} h^{AC}_{(0)} N_C, \tag{8.4.12}$$

with N^A free, and $\mathcal{U}_{(d-3)}$ is normalized as

$$\mathcal{U}_{(d-3)} = \frac{2\kappa^2}{(d-2)\Omega_{d-2}} m. \tag{8.4.13}$$

to highlight the function m corresponding to the mass aspect in radiative asymptotically Minkowski spacetimes.

The leading terms of the expansion are universal in the sense that they take the same form for both values of a

$$\beta_{(2a)} = -\frac{a}{16(d-2)} h_{(a)AB} h^{AB}_{(a)}, \qquad W^A_{(1)} = 2e^{2\beta_{(0)}} h^{AB}_{(0)} \partial_B \beta_{(0)}, \tag{8.4.14}$$

$$W^A_{(a+1)} = \frac{e^{2\beta_{(0)}}}{a+1} \left(\frac{a}{a+2-d} h^{AC}_{(0)} \overset{(0)}{D}_B h^B_{(a)C} - e^{-2\beta_{(0)}} h^{AB}_{(a)} W_{(1)B} \right) \tag{8.4.15}$$

$$\mathcal{U}_{(-1)} = \frac{2}{d-2} l \tag{8.4.16}$$

$$\begin{aligned} \mathcal{U}_{(0)} &= \frac{e^{2\beta_{(0)}}}{(d-3)(d-2)} \left(\overset{(0)}{R} + 2(d-3) e^{-2\beta_{(0)}} \overset{(0)}{D}_A W^A_{(1)} \right), \\ \overset{(0)}{D}_A W^A_{(1)} &= 2e^{2\beta_{(0)}} \left(\overset{(0)}{D^2} \beta_{(0)} + 2\partial^A \beta_{(0)} \partial_A \beta_{(0)} \right) \end{aligned} \tag{8.4.17}$$

$$\begin{aligned} \mathcal{U}_{(a)} = \frac{1}{(d-2)(d-3-a)} \Big[& e^{2\beta_{(0)}} (\delta R)_{(a)} + (2d-5-a) \left(\overset{(0)}{D}_A W^A_{(a+1)} + \Gamma^A_{(a)AC} W^C_{(1)} \right) \\ & - (a+1) e^{-2\beta_{(0)}} W^A_{(1)} W_{(a+1)A} \Big] \end{aligned} \tag{8.4.18}$$

$$(\delta R)_{(a)} = h^{AB}_{(0)} \overset{(a)}{R}_{AB} - h^{AB}_{(a)} \overset{(0)}{R}_{AB}, \quad \overset{(a)}{R}_{AB} = \frac{1}{2} \left(\overset{(0)}{D}_C \overset{(0)}{D}_A h^C_{(a)B} + \overset{(0)}{D}_C \overset{(0)}{D}_B h^C_{(a)A} - \overset{(0)}{D^2} h_{(a)AB} \right) \tag{8.4.19}$$

In the latter, only the terms in the square brackets are universal in any dimension. Indeed when $d = 4$ ($a = 1$) and the denominator vanishes so that only the part enclosed i Notice that (8.4.18) holds for both values of a only in $d > 4$, while in $d = 4$ ($a = 1$) the term in square brackets constitute the coefficient of the logaritmic term $\mathtt{U}_{(1)}$. With the notation of Appendix C.2, $\mathcal{F}_{(a+2)}$. Hence in $d > 4$ $\mathtt{U}_{(1)} \sim \mathcal{F}_{(3)}$, while to discuss $\mathtt{U}_{(d-3)}$ in $d > 4$ we need further subleading terms as detailed in Appendix C.2 and they are discussed next. With $\beta_{(0)} = 0$, $\mathcal{F}_{(3)} = 0$ in $d = 4$ and so the logarithmic coefficient $\mathtt{U}_{(1)} = 0$. By substituting either $a = 1$ or $a = \frac{d-2}{2}$ we have

$$a = 1 \qquad \begin{cases} \mathcal{F}_{(3)} = e^{2\beta_{(0)}}\left[(\delta R)_{(1)} - \overset{(0)}{D}_A \overset{(0)}{D}_B h^{BA}_{(1)}\right] - \tilde{\mathcal{F}}_{(3)}[W_{(1)}] \\ \tilde{\mathcal{F}}_{(3)}[W_{(1)}] = -(d-2) W_{(1)B} \overset{(0)}{D}_A h^{AB}_{(1)} - (d-3) h^{AB}_{(1)} \overset{(0)}{D}_A W_{(1)B} \\ \qquad + \frac{1}{d-3} W^A_{(1)} \overset{(0)}{D}_B h^B_{(1)A} + e^{-2\beta_{(0)}} h^B_{(1)A} W^A_{(1)} W_{(1)B} \end{cases} \tag{8.4.20}$$

(8.4.21)

$$a = \frac{d-2}{2} \qquad \begin{cases} \mathcal{F}_{(\frac{d+2}{2})} = e^{2\beta_{(0)}}\left[(\delta R)_{(\frac{d-2}{2})} - \frac{8-3d}{d} \overset{(0)}{D}_A \overset{(0)}{D}_B h^{BA}_{(\frac{d-2}{2})}\right] - \tilde{\mathcal{F}}_{(\frac{d+2}{2})}[W_{(1)}] \\ \tilde{\mathcal{F}}_{(\frac{d+2}{2})}[W_{(1)}] = 2(\frac{8-3d}{d}) W_{(1)B} \overset{(0)}{D}_A h^{AB}_{(\frac{d-2}{2})} + (\frac{8-3d}{d}) h^{AB}_{(\frac{d-2}{2})} \overset{(0)}{D}_A W_{(1)B} \\ \qquad + W^A_{(1)} \overset{(0)}{D}_B h^B_{(\frac{d-2}{2})A} + e^{-2\beta_{(0)}} h^B_{(1)A} W^A_{(1)} W_{(1)B} \end{cases} \tag{8.4.22}$$

With $\beta_{(0)} = 0$, $\tilde{F}_{(a+2)}[W_{(1)}] = 0$ and the term in square brackets vanishes only when $a = 1$ if the Ricci tensor of $h_{(0)AB}$ is proportional to $h_{(0)AB}$ because (using the determinant constraint)

$$(\delta R)_{(a)} = \overset{(0)}{D}_A \overset{(0)}{D}_B h^{AB}_{(a)} - h^{AB}_{(a)} \overset{(0)}{R}_{AB} \tag{8.4.23}$$

In four dimensions the vanishing of $\mathtt{U}_{(1)}$ thus is automatic with $\beta_{(0)AB} = 0$, while in higher dimensions along with $\beta_{(0)AB} = 0$ is essential that the metric $h_{(0)AB}$ is Einstein. With radiative falloffs in higher dimensions $\mathcal{F}_{(a+2)} \neq 0 \sim \mathcal{U}_{(a)} \neq 0$ even with Minkowskian asymptotics.

At subleading orders the different expansions of h_{AB} in even and odd dimensions impact on the other expressions. To list a couple we have

$$\beta_{(2a+p)} = -\frac{a(a+p)}{8(d-2)(2a+p)} h_{(a)AB} h^{AB}_{(a+p)}. \tag{8.4.24}$$

and the next order in W^A splits into (we are expressing everything in terms of $\beta_{(0)}$ rather than $W^A_{(1)}$ here)

$$W^A_{(a+p_o+1)} = \frac{e^{2\beta_{(0)}}}{a+p_o+1}\left(\frac{a+p_o}{(a+p_o)+2-d} h^{AC}_{(0)} \overset{(0)}{D}_B h^B_{(a+p_o)C} - 2 h^{AC}_{(a+p_o)} \partial_C \beta_{(0)}\right), \tag{8.4.25}$$

$$\begin{aligned} W^A_{(2a+1)} \;=\; \frac{e^{2\beta_{(0)}}}{2a+1}\Bigg[& \frac{2h^{AC}_{(0)}}{d-2-2a}\left((d-3+2a)\partial_C \beta_{(2a)} + \frac{a}{2} h^{BD}_{(a)} D_B h_{(a)DC}\right) \\ & + \frac{a}{d-2-a} h^{AC}_{(0)} D_B h^B_{(0)C} - 4\beta_{(2a)} h^{AC}_{(0)} \partial_C \beta_{(0)}\Bigg], \end{aligned} \tag{8.4.26}$$

according to the spacetime dimension. The above terms are valid if the denominators do not vanish, otherwise they contribute to the logarithmic term in W^A. This contribution is already present in asymptotically Minkowski spacetimes ($h_{(0)AB} = \gamma_{AB}, \beta_{(0)} = 0$) in four dimensions unless the first post-radiative term in h_{AB}, namely $h_{(2)AB}$, is constrained [57, 58, 143, 60]. Similarly in higher dimensional radiative asymptotically Minkowski spacetime the first post-radiative order of h_{AB} is to be constrained to avoid the logarithmic term [69, 70].

The first logarithmic terms that appear in W^A and $\mathcal{U}$ propagate down in the expansion with various powers and unless they are consistently set to zero by constraining the leading order data they also affect h_{AB} so that the ansatz break down at some point, via h_{AB} further logarithmic terms appear in β. This gives the minimal form of polyhomogeneous expansion.

We discuss for example the log term in W^A. The logarithmic term appearing in W^A at order r^{1-d} is given by (see Appendix C.2)

$$\begin{aligned} \mathtt{W}^A_{(d-1)} &= -\frac{2}{d-1} e^{2\beta_{(0)}} h^{AB}_{(0)} \tilde{n}_{(d)B} \\ \tilde{n}_{(d)B} &= -\left[\overset{(0)}{D}_C K^C_{(d-3)B} + 2(d-2)\partial_B \beta_{(d-2)} + \sum_{p+m=d-4} \left(\cancel{\overset{(1+p)}{\Gamma}{}^B{}_{BD}} K^D_{(m)A} - \overset{(1+p)}{\Gamma}{}^D{}_{BA} K^B_{(m)D} \right) \right] \end{aligned} \tag{8.4.27}$$

With the non-radiative (NR) falloff ($d > 4$) $a = 1$ we can organise this (as any other coefficient of the metric expansion) as

$$\tilde{n}_{(d)A} = \tilde{n}^{(R)}_{(d)A} + \tilde{n}^{(NR)}_{(d)A} \tag{8.4.28}$$

with $\tilde{n}^{(NR)}_{(d)A}$ vanishing when restricting to the radiative behaviour of h (its expansion starting from $a = \frac{d-2}{2}$). In four dimensions, clearly there is no distinction. The functional dependence of $\tilde{n}^{(R)}_{(d)A}$ on the orders of h is as follows

$$\tilde{n}^{(R)}_{(d)A} = \tilde{n}^{(R)}_{(d)A}[h_{(d-2)}, h_{(\frac{d-2}{2})}]. \tag{8.4.29}$$

It is interesting to compare the explicit expressions of $\tilde{n}^{(R)}_{(d)B}$ in the cases $d = 4, 5$

$$\tilde{n}^{(R)}_{(4)A} = \overset{(0)}{D}_C h^{(tf)C}_{(2)A} \tag{8.4.30}$$

$$\tilde{n}^{(R)}_{(5)A} = \frac{3}{2} \left[\overset{(0)}{D}_C h^{(tf)C}_{(3)A} + \frac{1}{6} \overset{(0)}{D}_A (h^{CD}_{(\frac{3}{2})} h_{(\frac{3}{2})CD}) - \frac{1}{2} \overset{(0)}{D}_B (h^{BC}_{(\frac{3}{2})} h_{(\frac{3}{2})CA}) \right] \tag{8.4.31}$$

The result $\mathtt{W}_{(3)} = -\frac{2}{3} e^{2\beta_{(0)}} \overset{(0)}{D}_C h^{(tf)AC}_{(2)}$ in four dimensions is well known [60], the exponential factor being a trivial effect of allowing a generic $\beta_{(0)}$.

On the other hand, the logarithmic term in $\mathcal{U}$ is not visible in $d = 4$ with the Minkowskian boundary conditions.

8.4.2 Fourth equation: L^A_B

In order to discuss the asymptotic expansion of the general solution of the fourth main equation

$$L^D_B = e^{-\Theta^{DA}_{CB}} \left(\frac{1}{2} N^C_A + \bar{L}^C_A \right), \tag{8.4.32}$$

it is useful to collect the expansions of the intermediate quantities Θ^{DA}_{CB}

$$\Theta^{DA}_{CB} = \int \mathcal{O}^{DA}_{CB} dr = -\int (\delta^D_C \tilde{K}^A_B - \tilde{K}^D_C \delta^A_B) dr =: \frac{1}{r^a} \sum_p \frac{\Theta^{DA}_{(a+p)CB}}{r^p}, \tag{8.4.33}$$

and[7] J^D_B

$$J^D_B = r^{\frac{d-2}{2}}\left[\frac{1}{2}\left(\mathcal{H}^D_B - lk^D_B\right) + \cancel{\left(\delta^D_C\tilde{K}^A_B - \tilde{K}^D_C\delta^A_B\right)l^C_{(0)A}} - \frac{1}{r}\frac{(d-2)}{2}\delta^D_C\delta^A_B l^C_{(0)A}\right] \tag{8.4.34}$$

$$\mathcal{H}^D_B = \frac{\mathcal{H}^D_{(1)B}}{r} + \frac{\mathcal{H}^D_{(2)B}}{r^2} + \sum_{p=0}\frac{\mathcal{H}^D_{(a+1+p)B}}{r^{a+1+p}} + \frac{\log r}{r^{d-1}}\mathtt{H}_{(d-1)}[\mathtt{U}_{(d-3)}] + \frac{\log r}{r^d}\mathtt{H}_{(d)}[\mathtt{W}_{(d-1)}] + \dots \tag{8.4.35}$$

$$lk^D_B = \frac{l\delta^D_B}{r} + \frac{1}{r^{a+1}}\sum_{p=0}\frac{lK^D_{(p)B}}{r^p}. \tag{8.4.36}$$

Notice that no logarithms are generated in Θ and that the first order at which the logarithmic term appears in $\mathcal{H}^D_B$ comes from the first logarithmic term in $\mathcal{U}$ (which can be automatically vanishing without serious restrictions) and the second logarithmic term is induced from the first logarithmic term in W^A. The expansion of J^D_B is organised as

$$\boxed{\begin{aligned} J^D_B =& r^{\frac{d-4}{2}}J^D_{(-\frac{d-4}{2})B} + r^{\frac{d-6}{2}}\frac{\mathcal{H}^D_{(2)B}}{2} + \sum_{p=0} r^{\frac{d-4-2(a+p)}{2}}J^D_{(-\frac{d-4-2(a+p)}{2})B} \\ &+ \frac{\log r}{r^{\frac{d}{2}}}\frac{\mathtt{H}_{(d-1)}[\mathtt{U}_{(d-3)}]}{2} + \frac{\log r}{r^{\frac{d+2}{2}}}\mathtt{H}_{(d)}[\mathtt{W}_{(d-1)}] + \dots \end{aligned}} \tag{8.4.37}$$

where

$$J^D_{(-\frac{d-4}{2})B} = \frac{1}{2}\left(\mathcal{H}^D_{(1)B} - l\delta^D_B - (d-2)l^D_{(0)B}\right), \qquad \mathcal{H}^D_{(1)B} = (d-2)\delta^D_B\mathcal{U}_{(-1)} \tag{8.4.38}$$

$$\mathcal{H}^D_{(2)B} = (d-3)\delta^D_B\mathcal{U}_{(0)} - e^{2\beta_{(0)}}\overset{(0)}{R}{}^D{}_B + \frac{4-d}{2}\left(\overset{(0)}{D}_B W^D_{(1)} + \overset{(0)}{D}{}^D W_{(1)B}\right) - \overset{(0)}{D}_A W^A_{(1)}\delta^D_B \tag{8.4.39}$$

$$\overset{(0)}{D}{}^D W_{(1)B} = 2e^{2\beta_{(0)}}\left(\overset{(0)}{D}{}^D\partial_B\beta_{(0)} + 2\partial^D\beta_{(0)}\partial_B\beta_{(0)}\right) \tag{8.4.40}$$

$$J^D_{(-\frac{d-4-2(a+p)}{2})B} = \frac{1}{2}\left(\mathcal{H}^D_{(a+1+p)B} - lK^D_{(p)B}\right) + \cancel{\left(l^D_{(0)A}K^A_{(p)B} - K^D_{(p)C}l^C_{(0)B}\right)} \tag{8.4.41}$$

Substituting (8.4.17) in (8.4.39) and lowering indices and using that $W^A_{(1)}$ is a gradient vector, we get

$$\boxed{\mathcal{H}_{(2)AB} = -e^{2\beta_{(0)}}\mathcal{R}_{AB} + (4-d)\mathcal{B}_{AB}[\beta_{(0)}]} \tag{8.4.42}$$

where

$$\boxed{\mathcal{R}_{AB} = \overset{(0)}{R}_{AB} - \frac{h_{(0)AB}}{d-2}\overset{(0)}{R}} \tag{8.4.43}$$

$$\boxed{\mathcal{B}_{AB}[\beta_{(0)}] = \frac{1}{2}\left(\overset{(0)}{D}_A W_{(1)B} + \overset{(0)}{D}_B W_{(1)A}\right) - \frac{h_{(0)AB}}{d-2}\overset{(0)}{D}_C W^C_{(1)}} \tag{8.4.44}$$

are both traceless. $\mathcal{B}_{AB}[\beta_{(0)}]$ can equivalently be written as

$$\mathcal{B}_{AB} = 2e^{2\beta_{(0)}}\left[\overset{(0)}{D}_A\partial_B\beta_{(0)} + 2\partial_A\beta_{(0)}\partial_B\beta_{(0)} - \frac{h_{(0)AB}}{d-2}\left(\overset{(0)}{D}{}^2\beta_{(0)} + 2\partial_C\beta_{(0)}\partial^C\beta_{(0)}\right)\right] \tag{8.4.45}$$

[7] We have cancelled a term using the leading solution (8.4.67) $l^A_{(0)B} \propto \delta^A_B$, which we are going to derive next.

The first term of (8.4.41) is

$$J^D_{(-\frac{d-4-2a}{2})B} = \frac{1}{2}\left(\mathcal{H}^D_{(a+1)B} - lK^D_{(0)B}\right) + \cancel{\left(l^D_{(0)A}K^A_{(0)B} - K^D_{(0)C}l^C_{(0)B}\right)} \tag{8.4.46}$$

$$\mathcal{H}^D_{(a+1)B} = (d-2-a)K^D_{(0)B}\mathcal{U}_{(-1)}. \tag{8.4.47}$$

So that

$$\boxed{J^D_{(-\frac{d-4-2a}{2})B} = \frac{2(d-2-a)-(d-2)}{2(d-2)}lK^D_{(0)B} = \begin{cases} 0 & \text{if } a = \frac{d-2}{2} \\ \frac{d-4}{2(d-2)}lK^D_{(0)B} & \text{if } a = 1,\ d > 4 \end{cases}}. \tag{8.4.48}$$

It is also instructive to show explicitly the integral $\int J^D_B =: I^D_B$, which enters the definition of $\bar{L}^D_B$

$$a = \frac{d-2}{2}: \quad I^D_B = \int r^{\frac{d-4}{2}} J^D_{(-\frac{d-4}{2})B} + r^{\frac{d-6}{2}}\frac{\mathcal{H}^D_{(2)B}}{2} + \sum_{p=0}\frac{J^D_{(1+p)B}}{r^{1+p}} + \frac{\log r}{r^{\frac{d}{2}}}\frac{\mathtt{H}^D_{(d-1)B}}{2} + \dots \tag{8.4.49}$$

$$a = 1: \quad I^D_B = \int r^{\frac{d-4}{2}} J^D_{(-\frac{d-4}{2})B} + r^{\frac{d-6}{2}}\frac{\mathcal{H}^D_{(2)B}}{2} + \sum_{p=0} r^{\frac{d-6}{2}-p} J^D_{(-\frac{d-6}{2}+p)B} + \frac{\log r}{r^{\frac{d}{2}}}\frac{\mathtt{H}^D_{(d-1)B}}{2} + \dots \tag{8.4.50}$$

So that

$$\begin{aligned} d > 4: \quad I^D_B &= r^{\frac{d-2}{2}} I^D_{(-\frac{d-2}{2})B} + r^{\frac{d-4}{2}} I^D_{(-\frac{d-4}{2})B} + \mathtt{I}^D_{[d>4]B}\log r \\ &\quad + \begin{cases} -\sum_{p>0}\frac{1}{r^p}\frac{J^D_B}{p} & \text{if } a = \frac{d-2}{2} \\ \sum_{p>0,\neq\frac{d-4}{2}} r^{\frac{d-4}{2}-p}\frac{2J^D_B}{d-4-2p} & \text{if } a = 1 \end{cases} \\ &\quad - \frac{2}{(d-2)^2 r^{\frac{d-2}{2}}}(2+(d-2)\log r)\mathtt{H}^D_B + \dots \end{aligned} \tag{8.4.51}$$

$$d = 4: \quad I^D_B = rI^D_{(-1)B} + \mathtt{I}^D_{[d=4]B}\log r - \sum_{p>0}\frac{1}{r^p}\frac{J^D_B}{p} - \frac{(1+\log r)}{r^{\frac{d-2}{2}}}\mathtt{H}^D_B + \dots. \tag{8.4.52}$$

For brevity we have only included in J the logarithmic term corresponding to $\mathtt{U}_{(d-3)}$; the logarithmic term corresponding to $\mathtt{W}^A_{(d-1)}$ appears at order $r^{-\frac{d}{2}}$ upon integration. In any case we are not going to need these terms as in the next part we limit our considerations to the terms up to r^0. In the above we have

$$d \geq 4: \quad I^D_{(-\frac{d-2}{2})B} := \frac{2}{d-2}J^D_{(-\frac{d-4}{2})B}, \tag{8.4.53}$$

$$I^D_{(-\frac{d-4}{2})B} := \begin{cases} d > 4: \quad \frac{1}{d-4}\mathcal{H}^D_{(2)B} & \text{if } a = \frac{d-2}{2} \\ d \geq 4: \quad \frac{2}{d-4}\underbrace{\left(\frac{\mathcal{H}^D_{(2)B}}{2} + J^D_{(-\frac{d-6}{2})B}\right)}_{J^{D(tot)}_{(-\frac{d-6}{2})B}} = \frac{2}{d-4}\mathtt{I}^D_{[d=4]B} & \text{if } a = 1 \end{cases} \tag{8.4.54}$$

$$d > 4: \quad \mathtt{I}^D_{[d>4]B} := \begin{cases} J^D_{(-\frac{d-4-2(a+p)}{2})B}\big|_{p=0} & \text{if } a = \frac{d-2}{2} \\ J^D_{(-\frac{d-4-2(a+p)}{2})B}\big|_{p=\frac{d-4}{2}} & \text{if } a = 1 \end{cases} \tag{8.4.55}$$

Clearly $\mathtt{I}^D_{[d>4]B}$ and $\mathtt{I}^D_{[d=4]B}$ are just the coefficient $J^D_{(1)B}$ in (8.4.37) for the given value of a and the given dimension d. The colour assigned to the logarithmic coefficients is to signal that $\mathtt{I}^D_{[d=4]B}$ is the sum of the red and blue coefficients of the expansion with $a=(d-2)/2$ in $d=4$ (stripping off the numerical factor stemming from the integral).

The general solution of the fourth main equation

$$L^D_B = e^{-\Theta^{DA}_{CB}}\left(\frac{1}{2}N^C_A + \bar{L}^C_A\right) \tag{8.4.56}$$

can thus be expanded as

$$L^D_B = r^{\frac{d-2}{2}} L^D_{(-\frac{d-2}{2})B} + \sum_{p=0}^{(a+p)\neq(d-2)/2} r^{\frac{d-2-2(a+p)}{2}} L^D_{(-\frac{d-2-2(a+p)}{2})B} + \mathtt{L}^D_{[d\geq 4]B}\log r + L^D_{(0)B} + \dots, \tag{8.4.57}$$

For any a

$$L^D_{(-\frac{d-2}{2})B} = I^D_{(-\frac{d-2}{2})B} \tag{8.4.58}$$

$$L^D_{(0)B} = \frac{N^D_B}{2} + \sum_{k>0}([e^{-\Theta}]_{(k)}[\bar{L}]_{(-k)})^D_B \tag{8.4.59}$$

notice the form of $L_{(0)}$: $\bar{L}$ is always a sum of powers greater or less than 0 and logs, so $L_{(0)}$ is given by the free function N and the appropriate combinations of orders with $e^{-\Theta}$. The next-to-leading order is

$$d>4: \qquad L^D_{(-\frac{d-4}{2})B} := \begin{cases} I^D_{(-\frac{d-4}{2})B} & \text{if } a=\frac{d-2}{2} \\ I^D_{(-\frac{d-4}{2})B} + (\Theta_{(1)}L_{(-\frac{d-2}{2})})^D_B & \text{if } a=1 \end{cases} \tag{8.4.60}$$

$$d=4: \qquad \mathtt{L}^D_{[d=4]B} = \mathtt{I}^D_{(d=4)B} + (J_{(0)}\Theta_{(1)})^D_B \tag{8.4.61}$$

The logarithmic coefficient in $d>4$ is instead

$$\mathtt{L}^D_{[d>4]B} = \mathtt{I}^D_{[d>4]B} + \sum_{k,l}([e^{\Theta}]_{(k)}J_{(l)})|^D_{(k+l=-1)B} \tag{8.4.62}$$

In $d=5,6$ the logarithmic term appears immediately after the coefficient $L^D_{(-\frac{d-4}{2})B}$, while in $d=4$ the $r^0\log r$ term is the next-to-leading term. The next term after $r^0\log r$ is, in any dimension, r^0 where the free function N^D_B appears: the radiative order. The above expressions are thus all we need in $d=4,5,6$ to reach the radiative order. In higher dimensions the sum in (8.4.57) produces further terms between $L^D_{(-\frac{d-4}{2})B}$ and $r^0\log r$.

8.4.3 Fourth equation: $\partial_u h_{AB}$

In order to translate the results for L^D_B in terms of l^B_D and $\partial_u h_{AB}$, the expression of L^D_B obtained from integration (8.4.57) - here noted as ${}^{\text{solution}}L^D_B$ - is to be equated to the defining expression of

L^D_B ((8.3.22) and (8.3.28)), here noted as ${}^{\text{definition}}L^D_B$

$$ {}^{\text{solution}}L^D_B = {}^{\text{definition}}L^D_B . \tag{8.4.63}$$

More explicitly ${}^{\text{definition}}L^D_B$ is given by

$$\begin{aligned} {}^{\text{definition}}L^D_B &= r^{\frac{d-2}{2}} \tilde{L}^D_B = r^{\frac{d-2}{2}} \left(l^D_B - l^D_{(0)B} \right) \\ &= r^{\frac{d-2}{2}-a} l^D_{(a)B} + \sum_p r^{\frac{d-2}{2}-a-p} l^D_{(a+p)B} \\ &= \sum_{p=0} r^{\frac{d-2-2a-2p}{2}} \frac{1}{2} \left(h^{DE}_{(0)} \partial_u h_{(a+p)EB} - h^{DE}_{(a+p)} \partial_u h_{(0)EB} \right) \\ &\quad + \text{non-linear terms,} \end{aligned} \tag{8.4.64}$$

In Section 8.4.3.3 the notation notation

$$\bar{l}^D_{(a+p)B} := \frac{1}{2} \left(h^{DE}_{(0)} \partial_u h_{(a+p)EB} - h^{DE}_{(a+p)} \partial_u h_{(0)EB} \right) . \tag{8.4.65}$$

will be used. The non-linearities at each order $(a+p)$ (for appropriate a and p) come from the full inverse of h_{AB}, while $\bar{l}^D_{(a+p)B}$ is only defined with respect to $h_{(a+p)AB}$ with raised indices.

From the leading order of (8.4.63) we get, in any $d \geq 4$ and for any a

$$\boxed{L^D_{(-\frac{d-2}{2})B} = 0 \Rightarrow I^D_{(-\frac{d-2}{2})B} = 0} . \tag{8.4.66}$$

With (8.4.38) and (8.4.16) it gives

$$\boxed{l^D_{(0)B} = \frac{l \delta^D_B}{d-2} \Leftrightarrow \partial_u h_{(0)AB} = \frac{2l}{d-2} h_{(0)AB}} , \tag{8.4.67}$$

namely

$$\boxed{h_{(0)AB}(u,x) = e^{2\varphi(u,x)} \hat{h}_{(0)AB}(x) , \quad \partial_u \varphi = \frac{l}{d-2} = \frac{\partial_u q}{(d-2)2q}} . \tag{8.4.68}$$

This shows that in any dimension $h_{(0)AB}$ can depend on u only via a conformal factor, a result known from the $d=4$ analysis of [60].

As a consequence of (8.4.67) we have

$$\left(l^D_{(0)A} K^A_{(p)B} - K^D_{(p)C} l^C_{(0)B} \right) = 0 \tag{8.4.69}$$

so that the barred terms (i.e. (8.4.41)) in the expressions of the previous section are justified.

With the given l, $\mathcal{U}_{(1)}$ can also be expressed as

$$\mathcal{U}_{(-1)} = 2 \partial_u \varphi . \tag{8.4.70}$$

We now discuss the subleading solutions considering separately the radiative case $a = \frac{d-2}{2}$ and the non-radiative case $a=1$ in $d>4$. In $d=4$ there is no distinction between the two cases and we conveniently include this case in the radiative section.

8.4.3.1 Radiative falloff $a = \frac{d-2}{2}$

With radiative falloff conditions, the expansion of L^D_B up to order r^0 in $d > 4$ automatically collapses to the sum of the leading term, the next-to-leading, the $r^0 \log r$ term and the term of order r^0. In $d = 4$ we have the leading term, the $r^0 \log r$ term and r^0.

Upon using the leading solution (8.4.66), we get in $d > 4$ the following equations from (8.4.63)

$$L^D_{(-\frac{d-4}{2})B} = 0 \Rightarrow I^D_{(-\frac{d-4}{2})B} = 0, \tag{8.4.71}$$

$$\mathtt{L}^D_{[d>4]B} = 0 \Rightarrow \mathtt{I}^D_{[d>4]B} = 0, \tag{8.4.72}$$

$$N^D_B = h^{DE}_{(0)} \partial_u h_{(\frac{d-2}{2})EB} - h^{DE}_{(\frac{d-2}{2})} \partial_u h_{(0)EB}, \tag{8.4.73}$$

while, in $d = 4$

$$\mathtt{L}^D_{[d=4]B} = 0 \Rightarrow I^D_{[d=4]B} = 0, \tag{8.4.74}$$

$$N^D_B = h^{DE}_{(0)} \partial_u h_{(1)EB} - h^{DE}_{(1)} \partial_u h_{(0)EB}. \tag{8.4.75}$$

The coefficients of the logarithmic terms in ${}^{\text{solution}}L^D_B$ are equated to zero because, by our original assumption, ${}^{\text{definition}}L^D_B$ contains only powers of r. Notice, however, that (8.4.55) with (8.4.48) implies that (8.4.72) is trivially satisfied

$$I^D_{[d>4]B} \equiv 0. \tag{8.4.76}$$

This, also implies, referring to our colour convention (8.4.54), that equation (8.4.74) reduces (8.4.71) to

$$I^D_{[d=4]B} \sim \mathcal{H}^D_{(2)B} = 0 \tag{8.4.77}$$

The same condition $H^D_{(2)B} = 0$ is seen to be imposed by (8.4.71) because of (8.4.54).

At order r^0, (8.4.73) and (8.4.75) have the same structure with $d \geq 4$ and, using (8.4.67), we get

$$\boxed{N_{AB} = \partial_u h_{(\frac{d-2}{2})AB} - \frac{2l}{d-2} h_{(\frac{d-2}{2})AB}}, \tag{8.4.78}$$

which generalises the definition of the *news tensor* for u-dependent $h_{(0)AB}$ to any d (cfr. [60, 70]).

To recap, with radiative falloffs - imposing that no logarithmic terms are generated by the fourth main equation - the constraints are solved by (8.4.67) and $\mathcal{H}^D_{(2)B} = 0$. The news tensor take the usual linear form in $h_{(\frac{d-2}{2})AB}$.

In four dimensions, as already discussed, $\mathcal{H}^D_{(2)B} \equiv 0$ trivially, and this implies that the $r^0 \log r$ term is not generated by the integration. In higher dimensions, the $r^0 \log r$ is again trivially not generated, but the condition $\mathcal{H}^D_{(2)B} \equiv 0$ is to be imposed on $h_{(0)AB}$ and $\beta_{(0)}$ to ensure that no overleading powers with respect to the radiative order are generated. The discussion of the implications of $\mathcal{H}^D_{(2)B} = 0$ in $d > 4$ was presented in Section 8.3.2.

8.4.3.2 Leading logs

With a maximal polyhomogeneous expansion of h_{AB}

$$h_{AB} = h_{(0)AB} + \frac{1}{r^a}\left(h_{(a)AB} + \mathtt{h}_{(a)AB}\log r\right) + \dots, \qquad a = \frac{d-2}{2}, \tag{8.4.79}$$

equation (8.4.63) gives

$$\log r\ {}^{\text{definition}}\mathtt{L}^D_B = \log r\ {}^{\text{solution}}\mathtt{L}^D_B := \log r J^D_{(-1)B}, \tag{8.4.80}$$

where

$${}^{\text{definition}}\mathtt{L}^D_B = \frac{1}{2}\left(h^{DE}_{(0)}\partial_u \mathtt{h}_{(\frac{d-2}{2})EB} - \mathtt{h}^{DE}_{(\frac{d-2}{2})}\partial_u h_{(0)EB}\right), \tag{8.4.81}$$

thus equation (8.4.72) becomes

$$d > 4: \qquad \frac{1}{2}\left(h^{DE}_{(0)}\partial_u \mathtt{h}_{(\frac{d-2}{2})EB} - \mathtt{h}^{DE}_{(\frac{d-2}{2})}\partial_u h_{(0)EB}\right) = 0, \tag{8.4.82}$$

while (8.4.74)

$$d = 4: \qquad \frac{1}{2}\left(h^{DE}_{(0)}\partial_u \mathtt{h}_{(1)EB} - \mathtt{h}^{DE}_{(1)}\partial_u h_{(0)EB}\right) = 0. \tag{8.4.83}$$

Both are solved by

$$d \geq 4: \qquad \mathtt{h}_{(\frac{d-2}{2})AB}(u,x) = e^{2\varphi(u,x)}\hat{\mathtt{h}}_{(\frac{d-2}{2})AB}(x), \tag{8.4.84}$$

If $h_{(0)AB}$ is u-independent then $\mathtt{h}_{(\frac{d-2}{2})AB}$ is such, namely a constant of motion. This corresponds to the case summarised in [280] for $d = 4$ axisymmetric spacetimes and generalises to any dimension.

Notice that the analysis of this subsection is valid only if $\mathtt{h}_{(\frac{d-2}{2})}$ does not modify the asymptotic expansions of all the other metric functions up to the orders we need to consider to carry on this analysis. This is indeed the case.

8.4.3.3 Non-radiative falloff $a = 1$ in $d > 4$

The leading solution (8.4.66) still holds. With this, the next-to-leading order of (8.4.63) reads

$$L^D_{(-\frac{d-4}{2})B} = \frac{1}{2}\left(h^{DE}_{(0)}\partial_u h_{(1)EB} - h^{DE}_{(1)}\partial_u h_{(0)EB}\right) \tag{8.4.85}$$

where $L^D_{(-\frac{d-4}{2})B} = I^D_{(-\frac{d-4}{2})B}$ because the term $(\Theta_{(1)}L_{(-\frac{d-2}{2})})^D_B$ in the second line of (8.4.60) vanishes by the leading solution and at this order, $l^D_{(1)B} = \bar{l}^D_{(1)B}$. Using (8.4.54) and (8.4.48), (8.4.85) is

$$\frac{2}{d-4}\left(\frac{\mathcal{H}^D_{(2)B}}{2} + \frac{d-4}{2(d-2)}lK^D_{(0)B}\right) = \frac{1}{2}\left(h^{DE}_{(0)}\partial_u h_{(1)EB} - h^{DE}_{(1)}\partial_u h_{(0)EB}\right), \tag{8.4.86}$$

Using (8.4.66), we then get

$$\boxed{\partial_u h_{(1)AB} - \frac{l}{d-2}h_{(1)AB} = \frac{2}{d-4}\mathcal{H}_{(2)AB}} \tag{8.4.87}$$

which can be equivalently written in terms of $K^{D}_{(0)B}$ as

$$\partial_u K_{(0)AB} - \frac{l}{d-2} K_{(0)AB} = -\frac{1}{d-4} \mathcal{H}^{D}_{(2)B}. \tag{8.4.88}$$

The formal solution of this equation is

$$\boxed{h_{(1)AB}(u,x) = e^{\varphi(u,x)} \hat{h}_{(1)AB}(x) + e^{\varphi(u,x)} \int e^{-\varphi(u',x)} \hat{\mathcal{H}}_{(2)AB} du'}\,, \tag{8.4.89}$$

where we have absorbed the factor $2/(d-4)$ into $\hat{\mathcal{H}}_{(2)AB}$ and we have used (8.4.68). The same considerations made after (8.3.44) applies to $\hat{\mathcal{H}}_{(2)AB}$, but this time it is not forced to vanish. It is interesting to note that in this non-radiative setting $\partial_u h_{(1)AB}$ solves an equation formally analogous to those satisfied by the leading log term in Section 8.4.3.2. Differently from those cases, however, the u-dependence of $h_{(1)AB}$ is not included only in the conformal factor encoding the u-dependence of $h_{(0)AB}$.

Suppose that $h_{(0)AB}$ does not depend on u and that also $\beta_{(0)}$ is u-independent, then $h_{(1)AB}$ is linear in u. However, if $\beta_{(0)}$ is gauged away and $h_{(0)AB}$ is taken to be Einstein, then $h_{(1)AB}$ is u-independent. This is inconsistent with the action of CL-superrotations, as we have said in the introduction and as we are going to show in the next section. More general boundary conditions are then needed, as claimed. Notice also that, differently from the radiative higher dimensional case, $\beta_{(0)}$ is not related to φ.

We have now discussed the first two equations of the cascade

$$L^{D}_{(-\frac{d-2}{2})B} = 0 \tag{8.4.90}$$
$$L^{D}_{(-\frac{d-4}{2})B} = l^{D}_{(1)B} \tag{8.4.91}$$
$$\vdots$$

stemming from (8.4.63) ${}^{\text{solution}}L^{D}_{B} = {}^{\text{definition}}L^{D}_{B}$. It continues in steps of one in even dimensions and one-half in odd dimensions, so that other few are

$$L^{D}_{(-\frac{d-5}{2})B} = l^{D}_{(\frac{3}{2})B} \tag{8.4.92}$$
$$L^{D}_{(-\frac{d-6}{2})B} = l^{D}_{(2)B} \tag{8.4.93}$$
$$L^{D}_{(-\frac{d-7}{2})B} = l^{D}_{(\frac{5}{2})B} \tag{8.4.94}$$
$$\vdots$$
$$L^{D}_{(0)B} = l^{D}_{(\frac{d-2}{2})B} \tag{8.4.95}$$
$$\mathsf{L}^{D}_{[d>4]B} = 0 \tag{8.4.96}$$

where the odd figures $5, 7, \ldots$ only appear if the dimension is odd. It is understood that the equations appear iteratively up to when $L_{(-\frac{d-n}{2})} = L_{(0)}$, which is the radiative order. The equation for the logarithmic coefficient also appear at this order. Thus in $d = 5$ and $d = 6$, (8.4.90) and (8.4.91) are the only equations above the radiative order, as well as (8.4.96). In $d = 7$, for example, we need to discuss also (8.4.92) and (8.4.93) before the radiative and the log order. As said, here we do not consider equations which are more subleading than radiative in the asymptotic expansion.

We now turn to the discussion of the radiative (8.4.95) and the logarithmic order (8.4.96). For $d = 5$ and 6 the analysis is quick and exemplifies the case of higher odd and even dimensions respectively.

$d = 5$ and odd. The radiative order in five dimensions is $r^{-\frac{3}{2}}$ and hence $l_{(\frac{3}{2})}$ in (8.4.95) is simply $\bar{l}_{(\frac{3}{2})}$ with no non-linear contribution. The tensor N_{AB} satisfies the same expression as the radiative news tensor discussed above

$$d = 5: \qquad N_{AB} = \partial_u h_{(\frac{3}{2})AB} - \frac{2l}{3} h_{(\frac{3}{2})AB}. \tag{8.4.97}$$

The equation for the vanishing of the logarithmic coefficient (8.4.96) is automatically satisfied. Recall that $\mathtt{L}^D_{[d>4]B}$ is given by (8.4.62). Referring to that equation, in five dimensions only $\mathtt{I}^D_{[d=5]B}$ contributes. This is given by (8.4.55) and (8.4.41), which in five dimensions reads

$$J^D_{(1)B} = \mathcal{H}^D_{(\frac{5}{2})B} - \frac{l}{4} h^D_{(\frac{3}{2})B} \equiv 0 \tag{8.4.98}$$

because $\mathcal{H}^D_{(\frac{5}{2})B} = l \tilde{K}^D_{(\frac{5}{2})B}$.

In higher odd dimensions we can easily see that N_{AB} satisifies the same equation as in the radiative case if the half-integer powers of the expansion before the radiative order are zero, namely if

$$h_{AB} = h_{(0)AB} + \sum_{k\in\mathbb{N}}^{<\frac{d-2}{2}} \frac{h_{(1)AB}}{r} + \frac{h_{(\frac{d-2}{2})AB}}{r^{\frac{d-2}{2}}} + \dots \tag{8.4.99}$$

In such a case also the logarithmic term at radiative order automatically vanishes.

The expansion (8.4.99) is indeed very natural. As we have seen in the first example, Einstein field equations always implies a r^{-1} falloff and that the free function appears at order $r^{-\frac{d-2}{2}}$. Despite what assumed in our initial ansatz, the equations do not induce any half-integer power before the radiative order.

$d = 6$ and even. In $d = 6$ the left hand side of (8.4.95) is given by (8.4.59) with $k = 1$ and the equation reads as

$$\frac{N^D_B}{2} - [\Theta_{(1)} \bar{L}_{(-1)}]^D_B = \bar{l}^D_{(2)B} - \frac{1}{2} h^{DE}_{(1)} \left(\partial_u - \frac{l}{2}\right) h_{(1)EB} \tag{8.4.100}$$

Notice that the second term in each side of the equations is not independent from (8.4.91), which has already been analysed. In particular

$$-\frac{1}{2} h^{DE}_{(1)} \left(\partial_u - \frac{l}{2}\right) h_{(1)EB} = -\frac{1}{2} h^{DE}_{(1)} \mathcal{H}_{(2)EB} + \frac{l}{8} h^{DE}_{(1)} h_{(1)EB} = K^{DE}_{(0)} \mathcal{H}_{(2)EB} + \frac{l}{2} K^{DE}_{(0)} K_{(0)EB}. \tag{8.4.101}$$

On the other hand, since $\bar{L}^D_{(-1)B} = I^D_{(-\frac{d-4}{2})B}|_{d=6} = \frac{2}{d-2} J^{D(tot)}_{(0)B}$ we have

$$[\Theta_{(1)}\bar{L}_{(-1)}]^D_B = \frac{1}{2}\left(\mathcal{H}^D_{(2)A}K^A_{(0)B} - \mathcal{H}^C_{(2)B}K^D_{(0)C}\right). \tag{8.4.102}$$

Hence we have

$$\frac{N^D_B}{2} = \bar{l}^D_{(2)B} + \frac{1}{2}\left(K^{DE}_{(0)}\mathcal{H}_{(2)EB} + \mathcal{H}^D_{(2)A}K^A_{(0)B} + \frac{l}{2}K^{DE}_{(0)}K_{(0)EB}\right) \tag{8.4.103}$$

The trace of the equation is

$$N = lh_{(2)} \tag{8.4.104}$$

and vanishes if $l = 0$ ($h_{(0)AB}$ time-independent). Calling the "news tensor" in this non radiative situation as $N^{(NR)}_{AB}$, and the news tensor in the radiative case as $N^{(R)}_{AB}$, we get the following structure in any dimension $d \geq 6$

$$N^{(NR)}_{AB} = N^{(R)}_{AB} + \text{non linear terms depending on leading orders} \tag{8.4.105}$$

and its trace is non vanishing if $l \neq 0$ due to the non-linearities, which makes the identification of $N^{(NR)}_{AB}$ with a news tensor obscure. This is in general true also in odd dimensions unless we restrict to the (very natural) case discussed above.

Suppose further that we insist on imposing the vanishing of the non-linear terms so that $N^{(NR)}_{AB}$ reduces to a radiative news tensor despite the non-radiative expansion. Take $d = 6$ as an example, we must impose that the term in parenthesis in (8.4.103) vanishes. If $\beta_{(0)} = 0$ and $h_{(0)AB}$ is Einstein, then only $\frac{l}{2}K^{DE}_{(0)}K_{(0)EB}$ remains to be equated to zero, which again is solved by $l = 0$. In general, however, we see that with $\beta_{(0)} \neq 0$ or a non-Einstein $h_{(0)AB}$, even if time independent, the constraint is more involved.

Equation (8.4.96) in $d = 6$ is

$$J^D_{(1)B} + [\Theta_{(0)}J^{(tot)}_{(0)}]^B_D = 0, \tag{8.4.106}$$

where $[\Theta_{(0)}J_{(0)}]^B_D$ is exactly given by (8.4.102) because the numerical factor differentiating $J_{(0)}$ and $L_{(-1)}$ is 1 in $d = 6$ and $J^D_{(1)B}$ is given by (8.4.55) and (8.4.41)

$$J^D_{(1)B} = \frac{1}{2}\left(\mathcal{H}^D_{(3)B} - l\tilde{K}^D_{(3)B}\right). \tag{8.4.107}$$

Differently from $d = 5$, this is not generically zero. For the sake of clarity let us consider the case $\beta_{(0)} = 0$. This does not affect the main conclusion because, as in the previous cases (see for example $\mathcal{H}^D_{(2)B}$), $\mathcal{H}^D_{(3)B}$ is of the form

$$\mathcal{H}^D_{(3)B} = e^{2\beta_{(0)}}\bar{\mathcal{H}}^D_{(3)B}\left[\overset{(0)}{R}, h_{(1)}\right] + \mathring{\mathcal{B}}^D_{(3)B}[\beta_{(0)}] \tag{8.4.108}$$

where $\bar{\mathcal{H}}^D_{(3)B}$ does not depend on $\beta_{(0)}$ but on the Ricci curvature and scalar of $h_{(0)}$ and on covariant combinations of $h_{(1)AB}$ and $\mathring{\mathcal{B}}^D_{(3)B}[\beta_{(0)}] = 0$ whenever $\partial_A\beta_{(0)}$ or $\beta_{(0)}$ are zero (we have placed a circle on $\mathcal{B}$ to overstress the obvious fact that it is different from the analogous term appearing at the previous order). We have

$$\bar{\mathcal{H}}^D_{(3)B} = -\overset{(1)}{R}{}^D{}_B + h^{DE}_{(1)}\overset{(0)}{R}_{EB} + 2\left(\tilde{K}^D_{(3)B}\mathcal{U}_{(-1)} + \tilde{K}^D_{(2)B}\mathcal{U}_{(0)} + \delta^D_B\mathcal{U}_{(1)}\right)$$
$$-\overset{(0)}{D}_B W^D_{(2)} - \overset{(0)}{D}{}^D W_{(2)B} - \overset{(0)}{D}_C W^C_{(2)}\delta^D_B \tag{8.4.109}$$

Which, using $W^A_{(2)}$ from (8.4.15), $\mathcal{U}_{(-1)}$ from (8.4.16), $\mathcal{U}_{(0)}$ from (8.4.17) and $\mathcal{U}_{(1)} = -\frac{1}{8}h^{EF}_{(1)}\overset{(0)}{R}_{EF}$ from (8.4.18) and (8.4.23), as well as $\overset{(1)}{R}_{AB}$ from (8.4.19), reads

$$\bar{\mathcal{H}}^D_{(3)B} - l\tilde{K}^D_{(3)B} = -\frac{1}{2}\left(\overset{(0)}{D}_C\overset{(0)}{D}{}^D h^C_{(1)B} + \overset{(0)}{D}_C\overset{(0)}{D}_B h^{DC}_{(1)} - \overset{(0)}{D}{}^2 h^D_{(1)B}\right) + h^{DE}_{(1)}\overset{(0)}{R}_{EB}$$
$$-\frac{1}{12}h^D_{(1)B}\overset{(0)}{R} - \frac{1}{4}h^{EF}_{(1)}\overset{(0)}{R}_{EF}\delta^D_B \tag{8.4.110}$$
$$+\frac{1}{6}\overset{(0)}{D}_B\overset{(0)}{D}_C h^{CD}_{(1)} + \frac{1}{6}\overset{(0)}{D}{}^D\overset{(0)}{D}_C h^C_{(1)B} + \frac{1}{6}\overset{(0)}{D}_C\overset{(0)}{D}_A h^{CA}_{(1)}\delta^D_B$$

By sorting covariant derivatives we can combine the first and the last line, but they are not going to cancel each other. Also when $h_{(0)AB}$ is Einstein this term is not automatically zero.

This analysis suggests that with the non-radiative falloff r^{-1} in h, a maximal polyhomogeneous expansion is to be considered in $d \geq 6$ even. It is easy to check that if the asymptotic analysis is repeated with the logarithmic term in h_{AB} at the radiative order, the u-dependence of $h_{(0)AB}$ couples to this to give $\log^2 r$ term at the same order. This will produce a cascade of log terms si that at each order of the expansion we also have an infinite sum of logarithmic terms[8].

Conjecture on leading logs and flat limit of AlAdS. Thinking about how this asymptotic expansion compares to that of *AlAdS* spacetimes in Fefferman-Graham gauge we may conjecture that the two expansions map under a sort of dimensional transmutation:

$$AlAdS_{d-1} \longleftrightarrow AlM_d \tag{8.4.111}$$

because we know that the expansion of odd-dimensional *AlAdS* is necessarily polyhomogeneous, while that of even-dimensional *AlAdS* is not. The conjecture (8.4.111) is to be analysed by solving the equations in Bondi gauge with Λ and mapping to Fefferman-Graham gauge, as done in [190, 272] in four dimensions. These authors could not see this structure because four-dimensional AlAdS is special and does not contain logs. Clearly, we must be more careful in reaching the conclusion (8.4.111) because the metric h_{ij} (see (2.3.13))in Fefferman-Graham gauge is related to the induced metric g_{ij}

$$g_{ij} = \begin{pmatrix} -\mathcal{U}e^{2\beta} + g_{AB}W^AW^B & g_{AB}W^A \\ g_{AB}W^B & g_{AB} \end{pmatrix} \tag{8.4.112}$$

on $r = const$ surfaces in Bondi gauge, not only to h_{AB}. However, we think the result is suggestive and need to be analysed more as a path toward the flat limit of the anomaly coefficients of AdS/CFT.

[8] With a u-independent boundary metric in $d = 4$, an infinite sum of logarithmic terms was argued in [234] to reduce to powers of r. This is however not proved rigorously. Thanks to the M. Godazgar for comments on this.

8.5 Asymptotic Killing fields

The gauge preserving conditions with the determinant of g_{AB} imply (see chapter 5)

$$\mathfrak{L}_\xi g_{rr} = 0\,, \quad \mathfrak{L}_\xi g_{rA} = 0\,, \quad g^{AB}\mathfrak{L}_\xi g_{AB} = 0\,. \tag{8.5.1}$$

We can be more general and allow for conformal rescalings of the boundary metric, so that generalising [60]

$$g^{AB}\mathfrak{L}_\xi g_{AB} = 2(d-2)\omega, \tag{8.5.2}$$

for an arbitrary positive scalar function $\omega(u,x)$.

The exact Killing equations are solved by the vector field

$$\xi = \xi^u\partial_u + \xi^r\partial_r + \xi^A\partial_A \tag{8.5.3}$$

$$\begin{cases} \xi^u = f(u,x^A)\,, \\ \xi^r = -\dfrac{r}{d-2}\left[{}^{(d-2)}D_A\xi^A - W^C\partial_C f + fl - (d-2)\omega\right], \\ \xi^A = Y^A(u,x^B) - \partial_B f\int_r^\infty dRe^{2\beta}g^{AB}\,, \end{cases} \tag{8.5.4}$$

with f, Y^A and ω arbitrary. They act on the remaining metric components as

$$\mathfrak{L}_\xi g_{AB} = r^2(\delta_\xi g_{AB})_{(2)} + r(\delta_\xi g_{AB})_{(1)} + r^{2-a}(\delta_\xi g_{AB})_{(2-a)} + r^{1-a}(\delta_\xi g_{AB})_{(1-a)} + \ldots \tag{8.5.5}$$

$$\mathfrak{L}_\xi g_{uu} = r(\delta_\xi g_{uu})_{(1)} + r^{1-a}\sum_{p=0} r^{-p}(\delta_\xi g_{uu})_{(1-a-p)} \tag{8.5.6}$$

$$\mathfrak{L}_\xi g_{ur} = (\delta_\xi g_{ur})_{(0)} + r^{-a-1}\sum_{p=0} r^{-p}(\delta_\xi g_{ur})_{(-a-1-p)} \tag{8.5.7}$$

$$\mathfrak{L}_\xi g_{uA} = r^2(\delta_\xi g_{uA})_{(2)} + r(\delta_\xi g_{uA})_{(1)} + r^{2-a}\sum_{p>0} r^{-p}(\delta_\xi g_{uA})_{(2-a-p)} \tag{8.5.8}$$

Notice that all except (8.5.8) match the leading order of the metric expansion. Indeed, as $W^A_{(0)} = 0$, the leading order term on the right-hand side of (8.5.8) equates to zero and give

$$\boxed{\partial_u Y^A = 0}\,, \tag{8.5.9}$$

while the first and the third give, at leading order, the transformation laws of $h_{(0)AB}$, $\beta_{(0)}$

$$\delta_\xi h_{(0)AB} = \mathfrak{L}_Y h_{(0)AB} - \frac{2}{d-2}\left(D_C Y^C - (d-2)\omega\right)h_{(0)AB} \tag{8.5.10}$$

$$\delta_\xi \beta_{(0)} = (f\partial_u + \mathfrak{L}_Y)\beta_{(0)} + \frac{1}{2}(\partial_u - \partial_u\varphi)f - \frac{1}{2(d-2)}\left(D_A Y^A - (d-2)\omega\right), \tag{8.5.11}$$

whereas the second gives the transformation law of $\mathcal{U}_{(-1)} = 2\partial_u\varphi$, which is not independent from the above. Here D_A is the covariant derivative compatible with $h_{(0)}$, we remove the superscript (0) used in the previous sections because no confusion arise.

The boundary conditions usually considered in literature are

CL) **Campiglia-Laddha**: $\mathscr{I} = \mathbb{R} \times S^{d-2}$, $h_{(0)AB}$ u-independent and free except for its fixed determinant, $\beta_{(0)} = 0$ fixed

$$\mathfrak{L}_\xi g_{AB} = O(r^2)\,, \quad \mathfrak{L}_\xi g_{uu} = O(r^0)\,, \quad \mathfrak{L}_\xi g_{ur} = O(r^{-2a})\,, \quad \mathfrak{L}_\xi g_{uA} = O(r^{1-a}). \tag{8.5.12}$$

BS) **Bondi-Sachs**: $\mathscr{I} = \mathbb{R} \times S^{d-2}$, $h_{(0)AB} = \gamma_{AB}$ round sphere metric, u-independent and $\beta_{(0)} = 0$ fixed

$$\mathfrak{L}_\xi g_{AB} = O(r^{2-a})\,, \quad \mathfrak{L}_\xi g_{uu} = O(r^{-a})\,, \quad \mathfrak{L}_\xi g_{ur} = O(r^{-2a})\,, \quad \mathfrak{L}_\xi g_{uA} = O(r^{1-a}). \tag{8.5.13}$$

These can be taken as subcases of the following generic boundary conditions

$$\mathscr{I} = \mathbb{R} \times \mathbb{B}^{d-2}, \;\; h_{(0)AB}(u,x) \text{ free within a conformal class, } \beta_{(0)}(u,x) \text{ fixed,} \tag{8.5.14}$$

which we here take for $a = 1$ and are our generalised CL conditions. The only change with respect to (8.5.12) is

$$\mathfrak{L}_\xi g_{uu} = O(r), \tag{8.5.15}$$

which simply give the transformation law of $\mathcal{U}_{(-1)}$, namely of $\partial_u \varphi$.

If we take $\delta_\xi \beta_{(0)} = 0$ we get from (8.5.11)

$$\begin{aligned} f(u,x) &= e^{\varphi - 2\beta_{(0)}} \alpha(x) + e^{\varphi - 2\beta_{(0)}} \int^u e^{-(\varphi - 2\beta_{(0)})} \left(\frac{F}{d-2} - \omega - 2(d-2)\mathfrak{L}_Y \beta_{(0)} \right) \mathrm{d}u' \\ F(u,x) &:= D_A Y^A(x) = (d-2) Y^A \partial_A \varphi + \underbrace{\left(\frac{1}{2} \partial_A \log |\hat{q}| + \partial_A \right) Y^A}_{\hat{D}_A Y^A} \end{aligned} \tag{8.5.16}$$

The scalar α is an arbitrary function on $\mathbb{B}^{d-2}$ and we have defined F as the leading covariant divergence of Y^A, which splits in a part depending on u and a covariant divergence with respect to the u-independent factor of $h_{(0)AB}$, which we denoted with a hat and hence $\hat{q}$ is its determinant.

With (8.5.9), (8.5.10) and (8.5.16) all the leading order conditions are solved. With $a = 1$, $\delta_\xi g_{ur} = O(r^{-2})$ is now automatically satisfied because the $O(r^{-1})$ component is trivially zero. On the other hand, $(\delta_\xi g_{uA})_{(1)} = 0$ reads

$$\begin{aligned} &\frac{1}{d-2} \partial_A \left[F + fl - \omega \right] e^{2\beta_{(0)}} - \partial_u \left(\partial_C f h^{CB}_{(0)} e^{2\beta_{(0)}} \right) h_{(0)BA} - \frac{2l}{d-2} e^{2\beta_{(0)}} \partial_A f \\ &+ W_{(1)A} \left[(\partial_u + 1) f + \frac{1}{d-2} \left(F + lf - (d-2)\omega \right) \right] - Y^B \partial_B W_{(1)A} - \partial_A (Y^B) W_{(1)B} = 0 \end{aligned} \tag{8.5.17}$$

As we are ultimately interested in approaching the cases considered in literature, we consider[9] $\beta_{(0)} = 0$. The last line vanishes identically and $exp(2\beta_{(0)}) = 1$. Using the equations found previously, we check that the first line also vanishes.

[9]The result should not depend on this choice. In particular the conclusion we are going to describe holds also when $\beta_{(0)} \neq 0$ but $\partial_A \beta_{(0)} = 0$ so that $W^A_{(1)} = 0$. We have not explicitly checked the case in which $W^A_{(1)} \neq 0$, but all should be consistent as it only depends on $\beta_{(0)}$. In this latter case, if the second line does not vanish identically we get a constraint on $\beta_{(0)}$.

The asymptotic Killing fields of the given generalised set of boundary conditions thus comprise supertranslations and smooth CL-superrotations, which we should denote with $SDiff(\mathbb{B}^{d-2})$ as we have not restricted to[10] $\mathbb{B}^{d-2} = S^{d-2}$.

At subleading orders we find the transformation laws of the subleading terms of the metric expansion. To complete the goal set in the introduction of the chapter, the most relevant to address is $\delta_\xi h_{(1)AB}$. Notice that with $a = 1$, both

$$(\delta_\xi g_{AB})_{(1)} = e^{2\beta_{(0)}} \left(\frac{2}{d-2} D^2 f - 2 D_A D_B f \right) + \frac{4}{d-2} h_{(0)AB} W^E_{(1)} \partial_E f - 4 W_{(1)(A} \partial_{B)} f \tag{8.5.18}$$

and

$$(\delta_\xi g_{AB})_{(2-a)} = f \partial_u h_{(a)AB} - \frac{(2-a)}{d-2} \left(D_C Y^C + fl - (d-2)\omega \right) h_{(a)AB} + \mathfrak{L}_Y h_{(a)AB} \tag{8.5.19}$$

contribute to the end result. We get

$$\begin{aligned} \delta_\xi h_{(1)AB} = {}& 2e^{2\beta_{(0)}} \left(\frac{1}{d-2} h_{(0)AB} D^2 - D_A D_B \right) f + f \underbrace{\partial_u h_{(1)AB}}_{\substack{(8.4.87)\ d>4 \\ (8.4.78)\ d=4}} - \frac{F}{d-2} h_{(1)AB} + \mathfrak{L}_Y h_{(1)AB} \\ & - \left(\frac{fl}{d-2} + \omega \right) h_{(1)AB} + 2 \left[\frac{\partial^C f W_{(1)C}}{d-2} h_{(0)AB} - 2 \partial_{(B} f W_{(1)A)} \right] \end{aligned} \tag{8.5.20}$$

The term in square brackets vanishes for the standard boundary condition on $\beta_{(0)}$.

We now exemplify which subcases are consistent and which are not, taking $\omega = 0$ for simplicity and for comparison to the literature.

CL & $a = 1$, d > 4. This is the case in [79] for even d. The above expressions reduce to

$$f(u, x) = \alpha(x) + \frac{u}{d-2} F(x), \quad F(x) := \hat{D}_A Y^A(x) \tag{8.5.21}$$

and

$$\delta_\xi h_{(1)AB} = 2 \left(\frac{1}{d-2} h_{(0)AB} \hat{D}^2 - \hat{D}_A \hat{D}_B \right) f - f \cancel{\partial_u h_{(1)AB}} - \frac{F}{d-2} h_{(1)AB} + \mathfrak{L}_Y h_{(1)AB} \tag{8.5.22}$$

which, restricted to $(\alpha = 0, Y)$ gives[11]

$$\delta_Y h_{(1)AB} = \frac{2u}{(d-2)^2} \left(h_{(0)AB} \hat{D}^2 - (d-2) \hat{D}_A \hat{D}_B \right) F + \left(\mathfrak{L}_Y - \frac{F}{d-2} \right) h_{(1)AB} \tag{8.5.23}$$

which contains a u-dependent piece as stated in the introduction to the chapter, making it inconsistent with the configuration space.

[10] In principle, we should be cautious in using the names supertranslations and superrotations because the topology of the phase space is relevant, as shown in [281].

[11] The last term in the next equation does not appear in [79]. However, our expressions are fully consistent with [156] when restricted to $d = 4$.

BS & $a = \frac{d-2}{2}$. This corresponds to the TKS analysis [70]. From (8.5.16) we again recover [70, 60]

$$f(u,x) = \alpha(x) + \frac{u}{d-2}F(x), \quad F(x) := \hat{D}_A Y^A(x) \tag{8.5.24}$$

but now, from the vanishing of (8.5.10), we get

$$\delta_Y h_{(0)AB} = \frac{2}{d-2} F h_{(0)AB}, \tag{8.5.25}$$

so that Y is a conformal Killing vector.

With this value of a, $(\delta_\xi g^{(1)}_{AB})$ is of the same order of $(\delta_\xi g^{(2-a)}_{AB})$ only in $d = 4$ and hence

$$\delta_\xi h_{(1)AB} = 2\left(\frac{1}{d-2} h_{(0)AB}\hat{D}^2 - \hat{D}_A\hat{D}_B\right) f - f\,\underbrace{\partial_u h_{(1)AB}}_{N_{AB}} - \frac{F}{d-2} h_{(1)AB} + \mathfrak{L}_Y h_{(1)AB} \tag{8.5.26}$$

only in $d = 4$. The function α remains arbitrary. We get the $\mathfrak{bms}$ algebra (either global or local).

When $d > 4$, we have a constraint on f coming from $(\delta_\xi g^{(1)}_{AB}) = 0$

$$\frac{2}{d-2} h_{(0)AB}\hat{D}^2 f - 2\hat{D}_A\hat{D}_B f = 0, \tag{8.5.27}$$

from which we get

$$\frac{2}{d-2} h_{(0)AB}\hat{D}^2 \alpha - 2\hat{D}_A\hat{D}_B \alpha = 0, \tag{8.5.28}$$

as F can be proved to satisfy

$$\hat{D}_A\hat{D}_B F - \frac{1}{d-2}\hat{D}^2 F h_{(0)AB} = 0 \tag{8.5.29}$$

from the properties of the Riemann tensor of a maximally symmetric space [70]. F is given by the $l = 1$ modes of the scalar harmonics on the hypersphere and α by $l = 0, 1$ modes. Given this, the subleading conditions

$$(\delta_\xi g_{ur})_{(-a-1-p)} = 0 \quad \forall p < a-1, \quad (\delta_\xi g_{uA})_{(2-a-p)} \quad \forall p < 1 \tag{8.5.30}$$

are automatically satisfied as in [70]. Let us take for example $(\delta_\xi g_{ur})_{(-a-1-p)} = 0$. For each $p < a-1$ it is of the form

$$h_{(a+p)AB}\hat{D}^A\hat{D}^B f = 0. \tag{8.5.31}$$

Hence it holds using (8.5.27) and the fact that $h_{(a+p)AB}$ is traceless for any $p < a$.

The resulting algebra of asymptotic symmetries is Poincaré. The action of the asymptotic Killing on the radiative data follow from $(\delta_\xi g^{(2-a)}_{AB})$ as

$$\delta_\xi h_{(\frac{d-2}{2})AB} = -f\,\underbrace{\partial_u h_{(\frac{d-2}{2})AB}}_{N_{AB}} - \frac{F}{d-2} h_{(\frac{d-2}{2})AB} + \mathfrak{L}_Y h_{(\frac{d-2}{2})AB} \tag{8.5.32}$$

For $\mathbb{B}^{d-2} \neq S^{d-2}$ in $d = 4$ the asymptotic symmetries have been studied in [281].

BS in $d > 4$ & $a = 1$. In even spacetime dimensions this is the case considered by KLPS [71]. The analysis is the same as the above except that (8.5.27) does not apply. Thus α is free and $h_{(1)AB}$ transforms formally as in $d = 4$

$$\delta_\xi h_{(1)AB} = 2\left(\frac{1}{d-2}h_{(0)AB}D^2 - D_A D_B\right) f - f\cancel{\partial_u h_{(1)AB}} - \frac{F}{d-2}h_{(1)AB} + \mathfrak{L}_Y h_{(1)AB} \qquad (8.5.33)$$

Notice that since Y generates Lorentz transformations in this case, the remark after (8.5.23) does not apply because the u-dependent piece of (8.5.23) automatically vanishes. The action of the Killing field on the radiative data is read by pushing the expansion of (8.5.5) up to $p = \frac{d-4}{2}$. So for example in $d = 5$ it is found at $p = 1/2$, as it should as $a(=1) + p(=1/2) = 3/2$. As we have commented in Section **??**, the news tensor in this case is linear only in $d = 5$.

CL & $a = \frac{d-2}{2}$. In all our discussion we have never mentioned the possibility that we can have superrotations without supertranslations in $d > 4$. This may seem plausible since we have repeatedly stated that superrotations only depend on the boundary conditions, while supertranslations depend on the falloff conditions. So can

$$Diff(S^{d-2}) \ltimes T \qquad (8.5.34)$$

be an asymptotic symmetry group?

Given the CL conditions, (8.5.27) does not apply. However, the conditions (8.5.30) must be considered. While they were automatically satisfied by the solutions of (8.5.27), they are not so now. For example consider again (8.5.31). Since α exponentiates to translations (and hence satisfies (8.5.28)), (8.5.31) becomes

$$h_{(a+p)AB}\hat{D}^A\hat{D}^B F = 0, \qquad (8.5.35)$$

constraining Y^A. As we know a solution of this equation is given by (8.5.29), and we come back to the standard Bondi-Sachs case. We have not investigated other solutions.

Thus, (8.5.34) cannot be an asymptotic symmetry group. We are either forced to the Poincaré group or to some other group where the diffeomorphisms generated by Y^A are restricted. It would have been puzzling otherwise, because we would have been able, in principle, to recover subleading soft theorems from the $Diff(S^{d-2})$ invariance but there is not enough symmetry for the leading theorems.

8.6 Comments on the asymptotic charges

To analyse the charges and the variational principle, we have to use (see Section 3.3)

$$\frac{1}{2\kappa^2}\theta^\mu = \frac{1}{2\kappa^2}\left(g^{\nu\rho}\delta\Gamma^\mu_{\nu\rho} - g^{\mu\nu}\delta\Gamma^\rho_{\rho\nu}\right), \qquad (8.6.1)$$

which is the boundary term obtained in the variation of Einstein-Hilbert action ($2\kappa^2 = 16\pi G$). In the following we absorb the numerical factor in the definition of θ^μ.

The relevant components for the boundary analysis are θ^r and θ^u

$$
\begin{aligned}
\theta^r =& 2r^{d-2}\partial_u\left(\sqrt{q}\delta\beta\right) + \sqrt{q}\partial_r\left(r^{d-2}\delta\mathcal{U}\right) - 2\sqrt{q}\partial_r\left(\delta\beta r^{d-2}\mathcal{U}\right) \\
& - r^{d-2}\sqrt{q}^{(d-2)}D_A\left(\delta W^A\right) + 2r^{d-2}\sqrt{q}^{(d-2)}D_A\left(\delta\beta W^A\right) \\
& + 2r^{d-2}\sqrt{q}\delta\beta\partial_r\mathcal{U} - 2r^{d-2}\sqrt{q}\delta W^A(n_A + \partial_A\beta) + 2r^{d-2}\sqrt{q}\delta\mathcal{U}\partial_r\beta \\
& - r^{d-4}\sqrt{q}\delta h^{AB}\left({}^{(d-2)}D_A W_B + l_{AB} - \mathcal{U}k_{AB}\right) - 4r^{d-2}\sqrt{q}\partial_u\left(\delta\beta\right) \\
& + r^{d-2}\sqrt{q}\partial_u\left(h_{AB}\delta h^{AB}\right) + 4r^{d-2}\sqrt{q}\mathcal{U}\partial_r\left(\delta\beta\right) - 4r^{d-2}\sqrt{q}W^A\partial_A\left(\delta\beta\right),
\end{aligned}
\tag{8.6.2}
$$

$$
\theta^u = \frac{1}{2}r^{d-2}\sqrt{q}\left[g^{AB}\delta\partial_r g_{AB} - \frac{4\delta\beta(d-2)}{r} + 4\delta\partial_r\beta\right].
\tag{8.6.3}
$$

Some considerations on the variational principle with null boundaries at finite positions have appeared in literature (see for example [144]), but they are all affected by unsolved fundamental issues. For example, the strategy pursued in [144] to obtain the "Gibbons-Hawking-York" term for a null boundary imposes too strong restrictions on the variations, which are in principle unwanted. Anyway, we do not have to follow the same conceptual route used in AdS/CFT, because the on-shell Einstein-Hilbert action in our case is automatically zero. So we can directly discuss the variational principle at infinity by analysing the above two expressions.

For example, the leading order diverging piece of θ^r is

$$
\begin{aligned}
O\left(r^{d-2}\right) :& \, 2\partial_u\left(\sqrt{q}\delta\beta_{(0)}\right) + \sqrt{q}(d-1)\delta\mathcal{U}_{(-1)} - 2\sqrt{q}(d-2)\mathcal{U}_{(-1)}\delta\beta_{(0)} \\
& - \frac{\sqrt{q}}{2}\delta h^{AB}_{(0)}\partial_u h^{(0)}_{AB} - 2\mathcal{U}_{(-1)}\delta\left(\sqrt{q}\right) - 4\sqrt{q}\partial_u\left(\delta\beta_{(0)}\right) - 2\sqrt{q}\partial_u\left(\frac{\delta\sqrt{q}}{\sqrt{q}}\right).
\end{aligned}
\tag{8.6.4}
$$

As we are working on-shell we can use our solutions and see that this term is of the form

$$
\partial_u(\sqrt{q}\delta\beta_{(0)}) + A\partial_u\delta\sqrt{q}.
\tag{8.6.5}
$$

In literature ($d = 4$), the first term does not appear since $\beta_{(0)} = 0$ fixed from the start and, since $h_{(0)AB}$ is taken to be u-independent, the latter term automatically vanishes. If we insist on the u-independence of $h_{(0)AB}$, then we have to look at $\delta\beta_{(0)}$. However it enters a total derivative, so it only contributes as a corner term. Furthermore, with $h_{(0)AB}$ depending on u via the conformal factor φ, it can be shown that the variation in the total derivative term becomes of the form $\delta(\beta_{(0)} - \varphi)$. According to the cases considered thus only one among $\beta_{(0)}$ and φ is independent and if we if we require, consistently with CL superrotations that the volume of the boundary metric is kept fixed, then this diverging piece cancel.

All these matters are under investigation at the time of writing. To conclude we notice that the expansion of θ^r and θ^u correctly generalises the one known in $d = 4$ (see (6.1.11)) to any d. Indeed

$$
\theta^r = r^{d-2}\theta^r_{(d-2)} + r^{d-3}\theta^r_{(d-3)} + r^{d-2-a}\theta^r_{(d-2-a)} + \dots
\tag{8.6.6}
$$

$$
\theta^u = r^{d-3}\theta^u_{(d-3)} + r^{d-3-a}\theta^u_{(d-3-a)} + \dots
\tag{8.6.7}
$$

and each coefficient reduce to the expressions in [156] under those conditions.

Conclusions and Outlook

The problem of holography in asymptotically flat spacetimes is being uncovered in recent years. In this thesis, we presented some preliminary results which should be taken into account to extend the four-dimensional picture to higher dimensions. We conclude the journey of this dissertation by summarising the picture motivating it, the results obtained and future directions.

Bird's eye view of the motivations. Soft theorems characterise scattering processes in any theory of gravity with $d \geq 4$ flat non-compact dimensions [51, 52]. Extensions of the BMS group of four-dimensional asymptotically flat spacetimes have been conjectured to be symmetries of semiclassical scattering, at least at a perturbative level, because the action of the associated generating charges on the S-matrix can be argued to imply gravitational leading (Weinberg [47]) and subleading (Cachazo-Strominger [49]) soft theorems [43, 199, 200, 64]. The soft graviton factor is also understood as a Fourier transform of classical gravitational memory formulae [216, 202]. These chains of equivalences pictorially forms triangles with asymptotic symmetries, soft theorems and memories at each vertex and hold analogously in gauge theories [44].

The BMS group is defined as preserving the universal structure of asymptotically flat spacetimes at either future $\mathscr{I}^+$ or past $\mathscr{I}^-$ null infinity, where $\mathscr{I}^\pm$ have $\mathbb{R} \times S^2$ topology and the universal structure is the pair formed by a null normal to $\mathscr{I}$ and the round sphere metric on S^2. Such spacetimes can be easily called asymptotically Minkowski [55]. Under such conditions, the BMS group is the semidirect product of supertranslations (the Abelian factor), acting geometrically by arbitrarily shifting each point of S^2 along $\mathbb{R}$, and the proper orthocronous Lorentz group (the non-Abelian part), acting on S^2 as global conformal transformations. Supertranslations are intrinsically related to gravitational waves leaking through null infinity and the displacement memory effect. The space of asymptotically Minkowski spacetimes can be given the structure of a covariant phase space where Hamiltonian charges generating supertranslations act mapping a solution to a different solution.

Extensions of BMS concern its non-Abelian part. The global conformal transformations of S^2 can be relaxed to either local conformal transformations (generated by two copies of Witt algebra) or to arbitrary smooth diffeomorphisms of S^2. Local conformal transformations are usually called superrotations in this context. We will refer to them as BT-superrotations - BT standing for Barnich-Troessaert, the authors that first studied the associated phase space [60, 63] - because we will use CL-superrotations, CL standing for Campiglia-Laddha [64, 65], for $Diff(S^2)$.

Both extensions of BMS require relaxing the above asymptotic Minkowski conditions. Campiglia-Laddha superrotations need the universal structure to be defined as a pair involving a normal and a volume form [64, 157] over S^2. As transformations over a phase space, CL-superrotations map spacetimes with different metrics on the cross sections of $\mathscr{I}$ but with the same volume. This process can also be associated to memory effects [272].

The Witt algebra naturally arises in boost-rotation symmetric spacetimes [250, 256] because these spacetimes possess incomplete null infinity. They are usually called asymptotically locally flat. BT-superrotations are interpreted as inducing transitions in the larger phase space of asyptotically locally flat solutions, as crystallised by the creation or snapping of a cosmic string [239] and the production of impulsive memory.

The uncovered triangular relations revived the interest in the asymptotically flat spacetime holography problem because it fuels a program long believed to hint to holography [163]: casting scattering in Minkowski spacetime in terms of correlators of operators associated with the boundaries [282, 203, 204, 205, 206]. This is supposed to relate to holography in flat spacetime in the same way that quantum fields in AdS written in terms of boundary operators relate to AdS/CFT in hindsight [102]. However, AdS/CFT is greatly more general than this [18, 24]. Analogously, the dynamical principles such that a boundary structure X (using Witten's terminology [162, 38]) outputs the S-matrix are probably hidden beyond perturbative physics around Minkowski spacetime. We are thus led to approach the problem from more general points of view. One of various ways toward the more general principle is looking for (asymptotically) flat (Minkowski) holography as Ricci flat hlography [163] and its relations with AdS holography [42, 283, 185, 191, 193, 182, 284, 285]. This is the underlying spirit of the thesis.

A plausible fundamental condition for the resulting pictures - including holography - to hold is the existence of a well-defined phase space. In a covariant phase space perspective, where the phase space is built over a configuration space by endowing the latter with a symplectic structure. The analysis of such configuration space is the original contribution of the thesis in chapters 7 and 8.

Summary of results and further directions. Motivated by the relation between cosmic strings and superrotations in four dimensions, we began chapter 7 by exploring cosmic branes in higher dimensions. We argued that only $(d-3)$-branes in d spacetime dimensions are flat in the vicinity of the brane, and therefore the natural generalization of cosmic strings/superrotations in four dimensions should involve $(d-3)$-branes. We then showed that, if one wishes to allow cosmic $(d-3)$-branes to penetrate the celestial sphere, one needs to relax the boundary conditions from asymptotically flat to asymptotically locally flat.

The proposed generalized boundary conditions are defined in (7.4.41) in terms of a non-trivial $(d-2)$ metric, describing a $(d-2)$-manifold that is topologically a $(d-2)$-sphere. These boundary conditions include cosmic branes, but the rather general form is primarily motivated by the analogy with asymptotically locally anti-de Sitter spacetimes. The generalization of d-dimensional asymptotically anti-de Sitter spacetimes (for which the metric on the conformal boundary is $\mathcal{R}_t \times S^{d-2}$) to asymptotically locally anti-de Sitter spacetimes is obtained by allowing the metric on the conformal boundary is a generic smooth (non-degenerate) metric.

The analysis of the generic boundary conditions has been extended beyond five dimensions in chapter 8 and consistent configuration spaces supporting the action of supertranslations and superrotations in higher dimensions have been constructed there. We are confident that the analysis can produce a consistently renormalized phase space in any number of dimensions at the non-linear level. This will encompass all the cases previously considered in even spacetime dimensions, and will potentially answer the questions on the past/future matching conditions that previous works left unanswered. A fundamental ingredient we need, which we have not included in this account yet, is the solution of the supplementary equations, giving the u-evolution of the (analogous of)

Bondi mass and angular momentum aspects.

Although we should acknowledge that - at the time of writing - there are interesting fundamental issues in the four-dimensional picture which have not been tamed yet, several reasons motivate the interest in the higher dimensional explorations we have pursued. We have listed them in the Introduction, but here we can add one more. AdS/CFT realises the Holographic Principle irrespectively of the number of spacetime dimensions. We may thus expect the same for flat holography. However, the most interesting thing that can happen in studying physics in diverse dimensions is the discovery that some laws and principles only work in specific number of dimensions, perhaps four.

Perhaps the connection between asymptotic symmetries and soft theorems does not hold in higher dimensions because of the impossibility (fundamental, not technical) to consistently renormalize the asymptotic charges and/or the lack of relevant phase spaces with appropriate matching conditions between $\mathscr{I}^+$ and $\mathscr{I}^-$. While it is too early to speculate on the latter point - as it can only be answered after the radial divergences to each $\mathscr{I}$ are separately (roughly speaking) removed - we are confident that a regularization prescription along the lines of those used in four dimensions can be found. However, there are subtle differences. We hope that our analysis enlightens such points.

On the gravitational side, the first aim is to address the problem of Bondi mass and its evolution, as well as the angular momentum, for the spacetimes with the more general boundary conditions and non-radiative falloffs. However, we also note that a more modest goal, missing in literature, is the analysis of covariant phase space charges for radiative falloff conditions.

To get the connection with soft theorems, appropriate matching conditions have to be found. A preliminary comment on the issues of the non-linear analysis of supertranslations in even dimensions was given in [279] and we noted that the general boundary conditions may resolve this issue. It remains to check what additional subtleties are brought in by the more general boundary conditions and the non-linearities.

A tantalizing direction to pursue, given the results of chapter 7 and chapter 8, is to perform the integration of Einstein's equations in Bondi gauge for AlAdS spacetimes in arbitrary dimensions and take the flat space limit. This was done in $d = 4$ in [272] along the lines of [190]. Odd-dimensional bulk spacetimes are more interesting in this respect because of the AdS/CFT conformal anomaly[12]. Providing the solution of Einstein's equations with Λ is going to be an easy exercise after the analysis performed in chapter 8 because the structure of the equations is unchanged. In this way we can check the conjecture made in the last chapter that the logarithmic terms in h_{AB} in a d-dimensional spacetime with null asymptotics is related to the logarithmic term in g_{ij} of a $(d-1)$-dimensional $AlAdS$ spacetime. This analysis may open a new important window on the properties of a holographic dual of flat spacetimes. This point is related to the completion of the analysis of the polyhomogeneous expansion of asymptotically flat spacetimes.

One would also like to prove that the general boundary conditions are stable and that the class so defined is physical. Rigorous proofs and derivations may be challenging. In the case of AlAdS

[12] Only recently we became aware of [189, 286]. A stress tensor for asymptotically flat gravity is defined by a flat limit of AdS_3 and Robinson-Trautmann AdS_4 in Bondi gauge. In such cases no log terms arise in the AdS solution. The first non-trivial case is thus AdS_5.

spacetimes, rigorous proofs of existence and uniqueness in Euclidean signature were given in the original mathematics literature [112, 287], but many outstanding issues still remain in Lorentzian signature. In the case of zero cosmological constant, the analysis is inherently Lorentzian.

Other possible interesting directions, which did not find any mention in this dissertation, for which generalised BMS symmetries and our constructions may be relevant, are the asymptotic symmetries of string theories (see [288]) and the near-horizon symmetries of generic null horizons, including the dS cosmological horizon. The role of BMS and extended BMS symmetries in relation to the information paradox was discussed in [289, 290, 291] and various BMS-like near-horizon symmetries have been defined [146, 292, 293, 294] also in three [295] and arbitrary dimensions [148], while some BMS-like transformations may be relevant for the evolution of the entropy current of dynamical horizons [296][13].

As a last comment and warning, we mention that physically reasonable matching rules of $\mathscr{I}^+$ and $\mathscr{I}^-$ are unknown in the presence of black holes. Flat holography is still in her infancy, if not in an embryonic state.

[13]Thanks to N. Kundu for sharing opinions on this.

APPENDIX A

Asymptotics in the conformal language

A.1 Spacetime asymptotics

Definition. An asymptote of a D-dimensional spacetime $(\bar{M}, \bar{g})$ is a triplet (M, g, Ω) plus a diffeomorphism $\psi : \bar{M} \to M \backslash \mathscr{I}$, where M is a manifold with boundary $\mathscr{I}$, $\bar{g}$ is a smooth metric on M, ψ identifies M with the interior $M \backslash \mathscr{I}$ of M, and $\Omega : M \to \mathbb{R}$ is a smooth function which is strictly positive in the interior $M \backslash \mathscr{I}$ and such that

i) $g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}$ on $\bar{M}$,

ii) $\Omega = 0$, $n := d\Omega \neq 0$ at $\mathscr{I}$.

This is a slight adaptation of Geroch's definition of asymptote of a spacetime [55], where we take D generic rather than $D = 4$. The smoothness assumption is to be commented momentarily. We use D rather than d or $d + 1$ so as to facilitate maps of formulas according to the various conventions used.

The last condition implies that g is finite at infinity, Ω can be used as a coordinate on M and defines the one form n which is associated with the normal $n^\mu = g^{\mu\nu} \bar{D}_\nu \Omega$ to the boundary.

Manifestly, the given definition does not totally determine topology of $\mathscr{I}$. We will see however, that when $\Lambda = 0$ the boundary topology is restricted.

Conformal freedom. Given a spacetime $(\bar{M}, \bar{g})$ and an asymptote (M, g, Ω) and any smooth positive scalar function ω on M, $(M, \omega^2 g, \omega\Omega)$ is an equivalent asymptote. Under the conformal transformation

$$\Omega \to \Omega' := \omega\Omega, \quad g_{\mu\nu} \to g'_{\mu\nu} = \omega^2 g_{\mu\nu} \tag{A.1.1}$$

the normal n^μ transforms as

$$n^\mu \to n'^\mu = \omega^{-1} n^\mu + \omega^{-2}\Omega D^\mu \omega. \tag{A.1.2}$$

The function Ω is called "defining function" in mathematics literature [112] and the gauge/gravity duality literature [19, 42].

Causal structure of the boundary. Einstein's equations have to be imposed to infer the causal nature of $\mathscr{I}$. The boundary $\mathscr{I}$ is timelike if the spacetime solves Einstein's equations with $\Lambda < 0$, null if $\Lambda = 0$ and spacelike if $\Lambda > 0$, because

$$|d\Omega|^2 = -\frac{2\Lambda}{(D-1)(D-2)} = \mp\frac{1}{l^2}, \quad |d\Omega|^2 = g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega \tag{A.1.3}$$

where we have used the relation between Λ and the characteristic length scale l and the sign is $-$ is for $\Lambda > 0$ and $+$ for $\Lambda < 0$.

The given relationship is true in vacuum or as long as the stress-energy tensor falloffs sufficiently fast at infinity. The definitions of asymptotically flat, AdS or dS spacetimes require such conditions.

The Riemann tensor $\bar{R}_{\mu\nu\rho\sigma}$ of $(\bar{M}, \bar{g})$ behaves in the limit $\Omega \to 0$ as

$$\bar{R}_{\mu\nu\rho\sigma} \approx -|d\Omega|^2(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}), \tag{A.1.4}$$

and the Ricci tensor as

$$\bar{R}_{\mu\nu} \approx -(D-1)|d\Omega|^2\bar{g}_{\mu\nu} \tag{A.1.5}$$

This result is seemingly in tension with the previous consideration. If the spacetime is asymptotically Ricci-flat, this equation is solved by either admitting $d\Omega = 0$ or that the boundary is null. The first condition is not allowed by the definition of asymptote because it implies that the boundary surface collapse to a point. Indeed, these are the points i^0, $i^\pm$ in the conformal compactification of Minkowski spacetime. They are considered part of the boundary but must be treated separately.

Universal structure. The smoothness assumption is necessary to provide the necessary analytical tools to do tensor analysis on $\mathscr{I}$ as induced from M, but taking $\mathscr{I}$ abstractly as "detached" from M: i.e. we can safely define a pullback operation from M to $\mathscr{I}$. We denote the pulled-back quantities with an over arrow pointing left, i.e. $\overleftarrow{g}_{\mu\nu}$, $\overleftarrow{n}^\mu$.

The pulled-back fields which are shared by all spacetimes in the same class (i.e. asymptotically flat or asymptotically $(A)dS$) define the *universal geometry.* Asymptotic symmetries preserve the universal geometry.

As discussed in the main text, the smoothness assumption overrestrict the space of solutions of both asymptotically flat and asymptotically AdS spacetimes and cannot be extended to odd $D > 4$ asymptotically flat radiative spacetimes [66]. Cases with lower regularity $\mathscr{I}$ have been studied in literature.

A.1.1 Asymptotic flatness, null infinity and Carroll manifolds

Topology of null infinity. Since $\mathscr{I}$ is null, its normal $\overleftarrow{n}^{\mu}$ is both null and tangent and $\overleftarrow{g}_{\mu\nu}\overleftarrow{n}^{\nu} = 0$, meaning that $\overleftarrow{g}_{\mu\nu}$ is degenerate.

The set $\mathbb{B}$ of all maximally extended integral curves of $\overleftarrow{n}^{\mu}$ can be given the structure of a manifold provided that for any given point p along one such curve, the curve itself does not reenters sufficiently small neighborhood of p. This is accomplished by the mapping $\Pi : \mathscr{I} \to \mathbb{B}$ sending each $p \in \mathscr{I}$ to the integral curve to which it lies.

The manifold $\mathbb{B}$ is the base space of $\mathscr{I}$ and by abuse of terminology $\mathbb{B}$ is a cross section of $\mathscr{I}$. The topology of $\mathscr{I}$ is

$$\mathscr{I} \sim \mathbb{R} \times \mathbb{B} \tag{A.1.6}$$

The usual definition of asymptotic flatness in $D = 4$ and by extension $D > 4$ takes $\mathbb{B} = S^{D-2}$ and the null generators of $\mathscr{I}$ to be complete. This is the *asymptotically Minkowski* case.

Bondi condition. Due to the conformal freedom and Einstein's equations, $\overleftarrow{g}$ and $\overleftarrow{n}$ are related by $\mathfrak{L}_{\overleftarrow{n}}\overleftarrow{g}_{\mu\nu} = b\overleftarrow{g}_{\mu\nu}$for a positive function b. It is always possible to find b locally such that $\mathfrak{L}_{\overleftarrow{n}}\overleftarrow{g}_{\mu\nu} = 0$. This defines Ω as

$$D_\mu n_\nu = 0 \Leftrightarrow D_\mu D_\nu \Omega = 0 \tag{A.1.7}$$

on $\mathscr{I}$ and defines the so-called Bondi frame. In this frame there is a residual conformal freedom given by

$$\mathfrak{L}_n \overleftarrow{\omega} = 0, \quad \overleftarrow{\omega} > 0 \tag{A.1.8}$$

This is sufficient to show that in $D = 4$ all asymptotically flat spacetimes have locally the same conformally flat boundary metric [55] and when $\mathbb{B} = S^2$ a natural choice is the standard round sphere metric.

The phase space of asymptotically flat spacetimes is usually defined by such condtions. This immediately lead to BMS without CL-superrotation. Apart from the spherical case, the group of conformal motions of the other possible simply connected $\mathbb{B}^2$ has been studied in [281], but the analysis of asymptotic symmetries and charges in such cases lack. An explicit example of a spacetime with a non simply connected null boundary was found [297] as an A-metric with toroidal $\mathbb{B}^2$.

Abstract $\mathscr{I}$ as a Carroll structure. A Carroll manifold is defined in [177] as a triple (C, q, χ), where C is a smooth $(D-1)$-dimensional manifold endowed with a twice-symmetric covariant positive tensor field q whose kernel is generated by the nowhere vanishing, complete vector field ξ.

A generic Carroll structure is given by $C^{D-1} = \mathbb{B}^{D-2} \times \mathbb{R}$, $\chi = \partial_s$ where s is the Dth coordinate known as Carrollian time. The standard Carroll manifold is defined by $\mathbb{B}^{D-2} = \mathbb{R}^{D-2}$ and $q_{\mu\nu} = \delta_{\mu\nu}$ (notice we use the same indices as before for brevity).

The isometry group of the Carroll manifold is the infinite dimensional group of transformations $x'^A = x^A$, $s' = s + \alpha(x)$. A conformal Carroll transformation of level N is defined as the group of transformations preserving the tensor

$$\Gamma_{(N)} = q \otimes \chi^{\otimes N} = q_{\mu\nu}\chi^{\rho_1} \dots \chi^{\rho_N}, \tag{A.1.9}$$

The conformal Carroll group transforms q and χ as

$$q_{\mu\nu} \to a^2 q_{\mu\nu}\,, \quad \chi^\mu \to a^{-2/N}\chi^\mu\,. \tag{A.1.10}$$

When $N = 2$, (A.1.9) with the identification of $C \equiv \mathscr{I}$ and $q \equiv \overleftarrow{g}$, $\chi \equiv \overleftarrow{n}$ $(s \equiv u)$, $a \equiv \overleftarrow{\omega}$ is the universal geometry of null infinity as defined by Geroch [55].

There is thus no surprise in the claim that Conformal Carroll transformations of level two are (standard) BMS transformations. The interesting insight provided by the Carrollian language is that BMS arise as (a conformal extension) of a Inönu-Wigner contraction of the Poincaré group.

The point to be stressed, however, is that null infinity in the conformal sense is only well defined for $D = 4$ (or even). The identifications we made here between the abstract fields on the Carroll manifold (i.e. abstract $\mathscr{I}$) and the pull-backs of bulk fields are only allowed when the pull-back operation can be given a meaning.

Extended Carroll structures and CL-superrotations. The Carrollian picture can be easily extended to explicitly include CL-superrotations. We define the *extended conformal Carroll group* of level P as the group of transformations preserving the tensor

$$\tilde{\Gamma}_{(P)} = \epsilon \otimes \chi^{\otimes P} = \epsilon_{\mu_1 \dots \mu_{D-1}}\chi^{\rho_1} \dots \chi^{\rho_P} \tag{A.1.11}$$

where ϵ is the volume element on C. The infinitesimal transformation acts on ϵ and χ as

$$\mathfrak{L}_\xi \epsilon_{\mu_1 \dots \mu_{D-1}} = \lambda \epsilon_{\mu_1 \dots \mu_{D-1}}, \quad \mathfrak{L}_\xi \chi^\mu = k\chi^\mu, \quad k = -\frac{\lambda}{P} \tag{A.1.12}$$

If the Carroll manifold is the null boundary of a spacetime $C \equiv \mathscr{I}$ and we take $P = D - 1$ with χ identified with the normal to $\mathscr{I}$, and ϵ taken as the pullback of the $(D-1)$-form induced by the spacetime volume element $\epsilon_{\mu_1,\dots\mu_D} = D\epsilon_{[\mu_1 \dots \mu_{D-1}} n_{\mu_D]}$, (A.1.12) constitute BMS extended with CL-superrotations. Indeed the above identifications correspond to choosing the normal and the induced volume form to $\mathcal{I}$ as universal structure of asymptotically flat spacetimes [157]. Again, here the spacetime picture is only valid when the Carroll manifold can be consistently “attached” to the bulk.

A.1.2 Asymptotic analysis of AlAdS in Fefferman-Graham gauge

Asymptotically locally AdS spacetimes are usually named conformally compact Einstein metrics of negative cosmological constant $\Lambda < 0$. This is a synonim of the previous definition.

In the vicinity of the boundary they admit the Fefferman-Graham metric (2.3.13)

$$ds^2 = \frac{l^2}{z^2}(dz^2 + h_{ij}(z,x)dx^i dx^j), \quad i,j = 0, \dots d. \tag{A.1.13}$$

The nomenclature "asymptotically locally" indicate that the leading order metric of the expansion is not necessarily the standard AdS metric in Poincaré coordinates. Also the topology of the boundary is not fixed.

We solve vacuum Einstein's equations in any $D = d + 1$

$$R_{\mu\nu} = -\frac{d}{l^2} g_{\mu\nu}, \quad \Lambda = -\frac{d(d-1)}{2l^2} \tag{A.1.14}$$

N.B.: here unbarred quantities refer to the physical spacetime, as in chapter 2. The ij components of Einstein's equations are

$$z\partial_z^2 h_{ij} + (1-d)\partial_z h_{ij} - 2kh_{ij} - 2zk_i^l \partial_z h_{lj} + zk\partial_z h_{ij} - 2z\,{}^{(d)}R_{ij} = 0, \tag{A.1.15}$$

and the other two are

$$zi: \quad 2\,{}^{(d)}D_i k - {}^{(d)}D^j(\partial_z g_{ij}) = 0, \tag{A.1.16}$$

$$zz: \quad zh^{ij}\partial_z^2 h_{ij} - 2k - 2zk_j^i k_i^j = 0. \tag{A.1.17}$$

We have defined ${}^{(d)}R_{ij}$ the Ricci tensor of h_{ij} and

$$k_{ij} = \frac{1}{2}\partial_z h_{ij}, \quad k_j^i = \frac{1}{2}h^{il}\partial_z h_{lj}, \quad k = h^{ij}k_{ij} \tag{A.1.18}$$

to shorten the equations. It is interesting to note that if we perform a splitting as in the Bondi-Sachs strategy (cfr. Section 5.1 and Section 8.3) the equations can be entirely written in terms of k_{ij} [60].

They can be solved iteratively starting from (A.1.15). At step p the equation is differentiated p times and z is set to zero. The resulting equation is solved algebrically giving $h_{(p+1)ij}$ in terms of $h_{(0)}$, up to the order $p + 1 = d$, where it is necessary to distinguish between d even and odd.

Indeed, the p^{th} derivative of (A.1.15), after setting $z = 0$, reads as

$$(p+1-d)\partial_z^{p+1}h_{ij} - h_{ij}h^{lm}\partial_z^{p+1}h_{lm} = \mathcal{F}_{ij}[\partial_z^q h_{ij}|_{q<p+1}] \tag{A.1.19}$$

where $\mathcal{F}_{ij}$ is a sum of products of terms involving $\partial_z^q h_{ij}$ with $q < p + 1$ such that the sum of the derivative orders of each term of the product is $p + 1$. This is due to the fact that the sum of powers of z and derivatives with respect to z in each term of (A.1.15) has a definite parity, odd in particular. As a consequence each term in (A.1.19) must have the same number of derivatives of the first one $(p+1)$, because z has been set to zero.

To solve the equation we first take its trace and determine $h^{ij}\partial_z^{p+1}h_{ij}$, then we substitute back to get the (traceless part of)[1] $\partial_z^{p+1}h_{ij}$, so that each $h_{(p)ij}$ is determined. This can be continued as long as $p+1<d$.

For example, at order $p=0$ we set $z=0$ in (A.1.15)/(A.1.19) and, by evaluating its trace and substituting it back, we get $h_{(1)ij}=0$. At order $p=1$, repeating the procedure of taking the trace and putting back we get

$$h_{(2)ij} = -\frac{1}{d-2}\left(\overset{(0)}{R}_{ij} - \frac{1}{2(d-1)}h_{(0)ij}\overset{(0)}{R}\right). \tag{A.1.20}$$

where the superscript (0) is for quantities referred to $h_{(0)ij}$.

When $p+1=d$ odd, (A.1.19) collapse to

$$h_{ij}h^{lm}\partial_z^d h_{lm} = 0 \tag{A.1.21}$$

because of the parity argument. The trace of $\partial_z^d h_{lm}$ vanishes and the traceless part is left undetermined by this equation. By similar arguments as those seen, the equations for the zi components constrain it to be covariantly conserved with respect to the derivative compatible with $h_{(0)ij}$

$$^{(0)}D^i(\partial_z^d h_{(p)ij}) = 0 \tag{A.1.22}$$

When $p+1=d$ even, (A.1.19) becomes

$$h_{ij}h^{lm}\partial_z^d h_{lm} = -\mathcal{F}_{ij}[\partial_z^q h_{ij}|_{q<d}] \tag{A.1.23}$$

where the terms in $\mathcal{F}_{ij}$ are locally determined by $h_{(0)ij}$. The trace of this equation determines $h^{lm}\partial_z^d h_{lm}$ locally in terms of $h_{(0)ij}$. By substituting back we get

$$\mathcal{F}^T_{ij} = 0 \tag{A.1.24}$$

where T stands for trace-free. This is a constraint on the previous coefficients of the expansion and signals that the power-law assumption breaks down. This implies that the expansion contains a term

$$z^d \log z \mathtt{h}_{(d)ij} \tag{A.1.25}$$

where $\mathtt{h}_{(d)ij}$ is traceless, $h^{ij}_{(0)}\mathtt{h}_{(d)ij}=0$, and determined by $\mathcal{F}^T_{ij}$. Furthermore, it is also covariantly conserved with respect to $^{(0)}D$ as a consequence of the zi equation. Indeed, the $(d-1)$th derivative of that equation gives $\log z\, ^{(0)}D^i\mathtt{h}_{(d)ij}$ which must be set to zero in order to have a well-defined $z\to 0$ limit.

[1]Notice that this exactly correspond to what we will do in Bondi-Sachs gauge (cfr. Section 5.1 and Section 8.3), where however we will split the equations for the (degenerate) boundary metric in trace and traceless part from the start.

APPENDIX B

Appendix to chapter 7

B.1 Solutions of the main equations and supplementary equations

In this appendix we collect the solutions of the main equations (7.5.8), (7.5.9), (7.5.10) and (7.5.13) as well as the supplementary equations (7.5.14) and (7.5.15). In writing the appendix a logistic problem concerning the typesetting of the equations arose: whether to write all equations in terms of the initial and free data or implicitly in terms of the previously determined data. We have used one form or the other according to space constraints; shorter equations are usually written in the fully expanded form while the longest ones are not.

B.1.1 β coefficients

$$\beta_{(2)} = -\frac{1}{24}(C_{1(1)}^2 - C_{1(1)}C_{2(1)} - C_{2(1)}) \tag{B.1.1}$$

$$\beta_{(5/2)} = -\frac{1}{20}(2C_{1(1)}C_{1(3/2)} + C_{2(1)}C_{1(3/2)} + C_{1(1)}C_{2(3/2)} + 2C_{2(1)}C_{2(3/2)}) \tag{B.1.2}$$

$$\begin{aligned}\beta_{(3)} \quad = \quad & -\frac{1}{16}\left(C^2_{1(\frac{3}{2})} + C_{2(\frac{3}{2})}C_{1(\frac{3}{2})} + C^2_{2(\frac{3}{2})}\right) \\ & -\frac{1}{9}\left(C_{1(1)}C_{1(2)} + C_{2(1)}C_{2(2)}\right) \\ & +\frac{1}{18}\left(C_{1(2)}C_{2(1)} + C_{1(1)}C_{2(2)}\right)\end{aligned} \tag{B.1.3}$$

B.1.2 W coefficients

$$W_{(1)} = 2e^{2\beta_{(0)} - C_{1(0)}} \partial_\theta \beta_{(0)} \tag{B.1.4}$$

$$\begin{aligned}
8e^{-2\beta_{(0)}+C_{1(0)}} W_{(2)} &= 2C_{1(0),\theta}C_{1(1)} + C_{2(0)\theta}C_{1(1)} + C_{1(0),\theta}C_{2(1)} \\
&+ 2C_{2(0),\theta}C_{2(1)} - 2C_{1(1),\theta} + 2\csc\theta\sec\theta C_{2(1)} \\
&+ 2(2\tan\theta - \cot\theta)C_{1(1)} - 4C_{1(1)}W_{(1)}
\end{aligned} \tag{B.1.5}$$

$$\begin{aligned}
5e^{-2\beta_{(0)}+C_{1(0)}} W_{(5/2)} &= 2C_{1\left(\frac{3}{2}\right)}\left(-e^{-2\beta_{(0)}+C_{1(0)}}W_{(1)} + C_{1(0),\theta}C_{2(0),\theta}\right) \\
&+ 4C_{1\left(\frac{3}{2}\right)}(2\tan\theta - \cot\theta) - 2C_{1\left(\frac{3}{2}\right),\theta} \\
&+ C_{2\left(\frac{3}{2}\right)}\left(\left(C_{1(0),\theta} + 2C_{2(0),\theta}\right) + 2\csc\theta\sec\theta\right)
\end{aligned} \tag{B.1.6}$$

$$\begin{aligned}
3e^{-2\beta_{(0)}+C_{1(0)}} W_{(3)} &= 10\beta^{(0,1)}_{(2)} - 2C_{1(2),\theta} + C_{1(1)}C_{1(1),\theta} \\
&+ \frac{1}{2}C_{1(1)}C_{2(1),\theta} + \frac{1}{2}C_{2(1)}C_{1(1),\theta} + C_{2(1)}C_{2(1),\theta} \\
&+ 2C_{1(2)}C_{1(0),\theta} + C_{2(2)}C_{1(0),\theta} + C_{1(2)}C_{2(0),\theta} \\
&+ 2C_{2(2)}C_{2(0),\theta} - 2C_{1(2)}(\cot\theta - 2\tan\theta) + 2C_{2(2)}\csc\theta\sec\theta \\
&+ e^{C_{1(0)}-2\beta_{(0)}}\left(2\beta_{(2)}W_{(1)} - \frac{1}{2}(C_{1(1)})^2W_{(1)} - 2C_{1(1)}W_{(2)} - C_{1(2)}W_{(1)}\right)
\end{aligned} \tag{B.1.7}$$

$$\begin{aligned}
7e^{-2\beta+C_{(1)}} W_{(7/2)} &= 44\beta_{\left(\frac{5}{2}\right),\theta} - 10C_{1\left(\frac{5}{2}\right),\theta} + 4C_{1(1)}C_{1\left(\frac{3}{2}\right),\theta} + 2C_{1(1)}C_{2\left(\frac{3}{2}\right),\theta} + 6C_{1\left(\frac{3}{2}\right)}C_{1(1),\theta} \\
&+ 3C_{1\left(\frac{3}{2}\right)}C_{2(1),\theta} + 2C_{2(1)}C_{1\left(\frac{3}{2}\right),\theta} + 4C_{2(1)}C_{2\left(\frac{3}{2}\right),\theta} + 3C_{2\left(\frac{3}{2}\right)}C_{1(1),\theta} \\
&+ 6C_{2\left(\frac{3}{2}\right)}C_{2(1),\theta} + 10C_{1\left(\frac{5}{2}\right)}C_{1(0),\theta} + 5C_{2\left(\frac{5}{2}\right)}C_{1(0),\theta} + 5C_{1\left(\frac{5}{2}\right)}C_{2(0),\theta} \\
&+ 10C_{2\left(\frac{5}{2}\right)}C_{2(0),\theta} - 10C_{1\left(\frac{5}{2}\right)}(\cot\theta - 2\tan\theta) + 10C_{2\left(\frac{5}{2}\right)}\csc\theta\sec\theta \\
&+ e^{C_1(0)-2\beta(0)}\left(4\beta_{\left(\frac{5}{2}\right)}W_{(1)} - 2C_{1(1)}C_{1\left(\frac{3}{2}\right)}W_{(1)} - 2C_{1\left(\frac{5}{2}\right)}W_{(1)}\right) \\
&- e^{C_1(0)-2\beta(0)}\left(5C_{1(1)}W_{\left(\frac{5}{2}\right)} + 4C_{1\left(\frac{3}{2}\right)}W_{(2)}\right)
\end{aligned} \tag{B.1.8}$$

At order r^{-4} the equation determine the coefficient $\tilde{W}_{(4)}$ of the log term

$$\begin{aligned}
16e^{-2\beta+C_{1(0)}} \tilde{W}_{(4)} &= 48\beta_{(3),\theta} + 12C_{1(0),\theta}C_{1(3)} + 6C_{2(0),\theta}C_{1(3)} + 6C_{1(0),\theta}C_{2(3)} \\
&+ 12C_{2(0),\theta}C_{2(3)} - 12C_{1(3),\theta} + 4C_{1(2),\theta}C_{1(1)} + 2C_{2(2),\theta}C_{1(1)} \\
&+ 6C_{1\left(\frac{3}{2}\right),\theta}C_{1\left(\frac{3}{2}\right)} + 3C_{2\left(\frac{3}{2}\right),\theta}C_{1\left(\frac{3}{2}\right)} + 8C_{1(1),\theta}C_{1(2)} + 4C_{2(1),\theta}C_{1(2)} \\
&+ 2C_{1(2),\theta}C_{2(1)} + 4C_{2(2),\theta}C_{2(1)} + 3C_{1\left(\frac{3}{2}\right),\theta}C_{2\left(\frac{3}{2}\right)} + 6C_{2\left(\frac{3}{2}\right),\theta}C_{2\left(\frac{3}{2}\right)} \\
&+ 4C_{1(1),\theta}C_{2(2)} + 8C_{2(1),\theta}C_{2(2)} - 12(2\tan\theta + \cot\theta)C_{1(3)} \\
&+ 12\csc\theta\sec\theta C_{2(3)}
\end{aligned} \tag{B.1.9}$$

Here, substituting for $\beta_{(3)}$ results in

$$\begin{aligned}24e^{2\beta-C_{1(0)}}\tilde{W}_{(4)} \quad = \quad & -18C_{1(3),\theta}-2C_{1(1)}C_{1(2),\theta}-C_{1(1)}C_{2(2),\theta}+4C_{1(2)}C_{1(1),\theta} \\ & +2C_{1(2)}C_{2(1),\theta}-C_{2(1)}C_{1(2),\theta}-2C_{2(1)}C_{2(2),\theta}+2C_{2(2)}C_{1(1),\theta} \\ & +4C_{2(2)}C_{2(1),\theta}+18C_{1(3)}C_{1(0),\theta}+9C_{2(3)}C_{1(0),\theta}+9C_{1(3)}C_{2(0),\theta} \\ & +18C_{2(3)}C_{2(0),\theta}+36C_{1(3)}\tan\theta+18C_{2(3)}\csc\theta\sec\theta-18C_{1(3)} \quad \text{(B.1.10)}\end{aligned}$$

The subleading terms $W_{(k)}$ with $k>4$ can all be determined in terms of the previous ones as was the case up to $W_{(7/2)}$.

B.1.3 $\mathcal{U}$ coefficients

to be consistent with notation are better written in a non-expanded form, but this implies a rweriting of all equations

$$\begin{aligned}12e^{2\beta_{(0)}+C_{1(0)}}\mathcal{U}_{(0)} = {} & 4\beta_{(0),\theta}e^{4\beta_{(0)}}\left(C_{1(0),\theta}-(\cot\theta-\tan\theta)-\beta_{(0),\theta}\right) \\ & +e^{4\beta_{(0)}}\left(\beta_{(0),\theta\theta}-2(C_{1(0),\theta})^2-(C_{2(0),\theta})^2-C_{1(0),\theta}C_{2(0),\theta}+2C_{1(0),\theta\theta}\right) \\ & +e^{4\beta_{(0)}}\left(5(\cot\theta-\tan\theta)C_{1(0),\theta}-\csc\theta\sec\theta\left(C_{1(0),\theta}+2C_{2(0),\theta}\right)+12\right) \\ & +10W_{(1)}(\cot\theta-\tan\theta)e^{C_{1(0)}+2\beta_{(0)}}-e^{2C_{1(0)}}W_{(1)}^2 \qquad \text{(B.1.11)}\end{aligned}$$

$$\begin{aligned}6e^{2\beta_{(0)}+C_{1(0)}}\mathcal{U}_{(1)} = {} & -e^{4\beta_{(0)}}(4C_{1(1),\theta}C_{1(0),\theta}+C_{2(1),\theta}C_{1(0),\theta}+4C_{1(1)}\beta_{(0),\theta}C_{1(0),\theta}+) \\ & -e^{4\beta_{(0)}}C_{1(1),\theta}C_{2(0),\theta}-2e^{4\beta_{(0)}}C_{2(1),\theta}C_{2(0),\theta}+8W_{(2),\theta}e^{C_{1(0)}+2\beta_{(0)}} \\ & +e^{4\beta_{(0)}}(10C_{1(1),\theta}+8\beta_{(0),\theta}C_{1(1)})\cot 2\theta-e^{4\beta_{(0)}}C_{1(1),\theta}\csc\theta\sec\theta \\ & -2e^{4\beta_{(0)}}C_{2(1),\theta}\csc\theta\sec\theta+4e^{4\beta_{(0)}}\beta_{(0),\theta}C_{1(1),\theta}+2e^{4\beta(0)}C_{1(1),\theta\theta} \\ & +4e^{4\beta_{(0)}}C_{1(1)}(\beta_{(0),\theta})^2+4e^{4\beta(0)}C_{1(1)}\beta_{(0),\theta\theta}+2e^{4\beta_{(0)}}C_{1(1)} \\ & +(C_{1(0),\theta})^2+e^{4\beta_{(0)}}C_{1(1)}(C_{2(0),\theta})^2+e^{4\beta_{(0)}}C_{1(1)}C_{1(0),\theta}C_{2(0),\theta} \\ & -10e^{4\beta_{(0)}}C_{1(1)}\cot 2\theta C_{1(0),\theta}+e^{4\beta_{(0)}}C_{1(1)}\csc\theta\sec\theta C_{1(0),\theta} \\ & +2e^{4\beta_{(0)}}C_{1(1)}\csc\theta\sec\theta C_{2(0),\theta}-2e^{4\beta_{(0)}}C_{1(1)}C_{1(0),\theta\theta} \\ & -4W_{(2)}e^{C_{1(0)}}\left(W_{(1)}e^{C_{1(0)}}-4e^{2\beta_{(0)}}\cot 2\theta\right) \\ & -C_{1(1)}W_{(1)}^2e^{2C_{1(0)}}-12e^{4\beta_{(0)}}C_{1(1)} \qquad \text{(B.1.12)}\end{aligned}$$

$$
\begin{aligned}
3e^{-2\beta_{(0)}+C_{1(0)}}\mathcal{U}_{(3/2)} = {} & (-4C_{1(0),\theta}C_{1(\frac{3}{2}),\theta} - C_{2(0),\theta}C_{1(\frac{3}{2}),\theta} - C_{1(0),\theta}C_{2(\frac{3}{2}),\theta}) \\
& + (-2C_{2(0),\theta}C_{2(\frac{3}{2}),\theta} + 2(C_{1(0),\theta})^2C_{1(\frac{3}{2})} + (C_{2(0),\theta})^2C_{1(\frac{3}{2})}) \\
& + (C_{1(0),\theta}C_{2(0),\theta}C_{1(\frac{3}{2})} - 4C_{1(0),\theta}\beta_{(0),\theta}C_{1(\frac{3}{2})}) \\
& + \csc 2\theta(2C_{1(0),\theta}C_{1(\frac{3}{2})} + 4C_{2(0),\theta}C_{1(\frac{3}{2})} - 2C_{1(\frac{3}{2}),\theta} - 4C_{2(\frac{3}{2}),\theta}) \\
& (-2C_{1(0),\theta\theta}C_{1(\frac{3}{2})} + 4\beta_{(0),\theta}C_{1(\frac{3}{2}),\theta} + 2C_{1(\frac{3}{2}),\theta\theta} - 12e^{4\beta_{(0)}}C_{1(\frac{3}{2})}) \\
& + 4((\beta_{(0),\theta})^2C_{1(\frac{3}{2})} + 4\beta_{(0),\theta\theta}C_{1(\frac{3}{2})}) \\
& + \cot 2\theta(-10C_{1(0),\theta}C_{1(\frac{3}{2})} + 10C_{1(\frac{3}{2}),\theta} + 8\beta_{(0),\theta}C_{1(\frac{3}{2})}) \\
& + e^{-2\beta_{(0)}+C_{1(0)}}(14W_{(\frac{5}{2})} + 7W_{(\frac{5}{2}),\theta}) \\
& - e^{-2\beta_{(0)}+2C_{1(0)}}(C_{1(\frac{3}{2})}(W_{(1)})^2 - 5W_{(\frac{5}{2})}W_{(1)})
\end{aligned} \tag{B.1.13}
$$

$$
\begin{aligned}
12e^{2\beta_{(0)}+C_{1(0)}}\tilde{\mathcal{U}}_{(2)} = {} & e^{2\beta_{(0)}+C_{1(0)}}(48\beta_{(2)}U_{(0)} - \cot 2\theta(80\beta_{(2)}W_{(1)} + 24W_{(3)})) \\
& + e^{2\beta_{(0)}+C_{1(0)}}(-40\beta_{(2)}W_{(1),\theta} + 12W_{(3),\theta}) \\
& + e^{2C_{1(0)}}(-C_{1(1)}^2W_{(1)}^2 + 8\beta_{(2)}W_{(1)}^2 - 2C_{1(2)}W_{(1)}^2) \\
& + e^{2C_{1(0)}}(-12W_{(3)}W_{(1)} - 8W_{(2)}C_{1(1)}W_{(1)} - 8W_{(2)}^2) \\
& + e^{4\beta_{(0)}}(-4C_{1(1)}^2(\beta_{(0),\theta})^2 + 8C_{1(2)}(\beta_{(0),\theta})^2 - 16\beta_{(2),\theta}\beta_{(0),\theta}) \\
& + e^{4\beta_{(0)}}(8C_{1(2),\theta}\beta_{(0),\theta} - 8C_{1(1),\theta}C_{1(1)}\beta_{(0),\theta} + 4C_{1(1)}^2C_{1(0),\theta}\beta_{(0),\theta}) \\
& + e^{4\beta_{(0)}}(-8C_{1(2)}C_{1(0),\theta}\beta_{(0),\theta} - 4(C_{1(1),\theta})^2 - 2(C_{2(1),\theta})^2 - 4\beta_{(0),\theta\theta}C_{1(1)}^2) \\
& + e^{4\beta_{(0)}}(12C_{1(1)}^2 - 2C_{1(1)}^2(C_{1(0),\theta})^2 + 4C_{1(2)}(C_{1(0),\theta})^2 - C_{1(1)}^2(C_{2(0),\theta})^2) \\
& + e^{4\beta_{(0)}}(2C_{1(2)}(C_{2(0),\theta})^2 + \csc 2\theta(-4C_{1(2),\theta} - 8C_{2(2),\theta} + 4C_{1(1),\theta}C_{1(1)})) \\
& + e^{4\beta_{(0)}}(-2C_{1(1),\theta}C_{2(1),\theta} - 8\beta_{(2),\theta\theta} + 4C_{1(2)},\theta\theta + 8\csc 2\theta C_{2(1),\theta}C_{1(1)}) \\
& + e^{4\beta_{(0)}}(-4C_{1(1)},\theta\theta C_{1(1)} + 8\beta_{(0),\theta\theta}C_{1(2)} - 24C_{1(2)} - 2\csc 2\theta C_{1(1)}^2C_{1(0),\theta}) \\
& + e^{4\beta_{(0)}}(8\beta_{(2),\theta}C_{1(0),\theta} - 8C_{1(2),\theta}C_{1(0),\theta} - 2C_{2(2),\theta}C_{1(0),\theta} + 8C_{1(1),\theta}C_{1(1)}C_{1(0),\theta}) \\
& + e^{4\beta_{(0)}}(2C_{2(1),\theta}C_{1(1)}C_{1(0),\theta} + 4\csc 2\theta C_{1(2)}C_{1(0),\theta}) \\
& + e^{4\beta_{(0)}}\cot 2\theta(-8\beta_{(0),\theta}C_{1(1)}^2 + 10C_{1(0),\theta}C_{1(1)}^2 - 20C_{1(1),\theta}C_{1(1)}) \\
& + e^{4\beta_{(0)}}(-16\beta_{(2),\theta} + 20C_{1(2),\theta} + 16\beta_{(0),\theta}C_{1(2)} - 20C_{1(2)}C_{1(0),\theta}) \\
& + e^{4\beta_{(0)}}(-4\csc 2\theta C_{1(1)}^2C_{2(0),\theta} - 2C_{1(2),\theta}C_{2(0),\theta} - 4C_{2(2),\theta}C_{2(0),\theta}) \\
& e^{4\beta_{(0)}}(+2C_{1(1),\theta}C_{1(1)}C_{2(0),\theta} + 4C_{2(1),\theta}C_{1(1)}C_{2(0),\theta} + 8\csc 2\theta C_{1(2)}C_{2(0),\theta}) \\
& e^{4\beta_{(0)}}(-C_{1(1)}^2C_{1(0),\theta}C_{2(0),\theta} + 2C_{1(2)}C_{1(0),\theta}C_{2(0),\theta} + 2C_{1(1)}^2C_{1(0),\theta\theta} - 4C_{1(2)}C_{1(0),\theta\theta})
\end{aligned} \tag{B.1.14}
$$

The next equations determine $\mathcal{U}_{(i)}$ with $i \geq 7/2$ and as said in the main text $\mathcal{U}_{(2)}$ remains free.

B.1.4 $C_{i,u}$ coefficients

We collect here the derivatives with respect to u of $C_i(n)$ determined by the equations (7.5.37).

At order r^0:

$$\begin{aligned}C_{1(1),u} = \mathcal{H}_1 = &\frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left((C_{1(0),\theta})^2 + 2C_{2(0),\theta}C_{1(0),\theta} + 2(C_{2(0),\theta})^2 - C_{1(0),\theta\theta}\right)\\
&+\frac{2}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(C_{1(0),\theta}\beta_{(0),\theta} - 8(\beta_{(0),\theta})^2 - 4\beta_{(0),\theta\theta}\right)\\
&+\frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}(\tan\theta - \cot\theta)\left(C_{1(0),\theta} - 4\beta_{(0),\theta}\right)\\
&+\frac{2}{3}e^{2\beta_{(0)}-C_{1(0)}}\csc\theta\sec\theta\left(C_{1(0),\theta} + 2C_{2(0),\theta}\right)\end{aligned}\tag{B.1.15}$$

and

$$\begin{aligned}C_{2(1),u} = \mathcal{H}_2 = &\frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(-4\beta_{(0),\theta}C_{1(0),\theta} - 6\beta_{(0),\theta}C_{2(0),\theta}\right)\\
&+\frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(+8(\beta_{(0),\theta})^2 + 4\beta_{(0),\theta\theta} - 2C^2_{1(0),\theta}\right)\\
&-\frac{2}{3}e^{2\beta_{(0)}-C_{1(0)}}(\tan\theta + \cot\theta)C_{1(0),\theta}\\
&+\frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}(\cot\theta - 5\tan\theta)C_{2(0),\theta}\\
&-\frac{4}{3}e^{2\beta_{(0)}-C_{1(0)}}(\tan\theta + 2\cot\theta)\beta_{(0),\theta}\\
&-\frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left((C_{2(0),\theta})^2 + 4C_{1(0),\theta}C_{2(0),\theta}\right)\\
&+\frac{1}{3}e^{2\beta_{(0)}-C_{1(0)}}\left(2C_{1(0),\theta\theta}3C_{2(0),\theta\theta}\right)\end{aligned}\tag{B.1.16}$$

There are no equations at order $r^{-1/2}$. In particular no equations constrain $C_{i(\frac{3}{2}),u}$ and the next derivative to be determined is $C_{1(2),u}$ from the order r^{-1}:

$$\begin{aligned}C_{1(2),u} = &-e^{2\beta_{(0)}-C_{1(0)}}\left(2C_{1(1),\theta}C_{1(0),\theta} + C_{2(1),\theta}C_{1(0),\theta} + C_{1(1),\theta}C_{2(0),\theta} + 2C_{2(1),\theta}C_{2(0),\theta}\right)\\
&+e^{2\beta_{(0)}-C_{1(0)}}\left(2\beta_{(0),\theta}C_{1(1),\theta} + C_{1(1),\theta\theta} + 2C_{1(1),\theta}(\cot\theta - \tan\theta) - 2C_{1(1)}W^2_{(1)}\right)\\
&-e^{2\beta_{(0)}-C_{1(0)}}\csc\theta\sec\theta\left(C_{1(1),\theta} + 2C_{2(1),\theta}\right)\\
&+5C_{1(1)}W_{(1),\theta} + C_{1(1)}C_{1(1),u} + 4W_{(2),\theta}\\
&+W_{(2)}\left(-4W_{(1)}e^{C_{1(0)}-2\beta_{(0)}} + C_{1(0),\theta} + 2(\cot\theta - \tan\theta)\right)\\
&+2C_{1(1)}W_{(1)}C_{1(0),\theta} - 3C_{1(1)}\mathcal{U}_{(0)} + C_{1(1)}W_{(1)}(\cot\theta - \tan\theta) - 2\mathcal{U}_{(1)}\end{aligned}\tag{B.1.17}$$

$$
\begin{aligned}
C_{2(2),u} &= e^{2\beta_{(0)}-C_{1(0)}}(C_{2(1),\theta}C_{1(0),\theta} + C_{1(1),\theta}C_{2(0),\theta} + 2C_{1(1)}\beta_{(0),\theta}C_{2(0),\theta} - 2C_{2(1)}\beta_{(0),\theta}C_{2(0),\theta}) \\
&+ \tan\theta\left(C_{2(1),\theta}e^{2\beta_{(0)}-C_{1(0)}} - C_{1(1)}e^{2\beta_{(0)}-C_{1(0)}}C_{2(0),\theta} + C_{2(1)}e^{2\beta_{(0)}-C_{1(0)}}C_{2(0),\theta}\right) \\
&+ \cot\theta e^{2\beta_{(0)}-C_{1(0)}}\left(4C_{1(1)}\beta_{(0),\theta} - 4C_{2(1)}\beta_{(0),\theta} + 2C_{1(1),\theta} - C_{2(1),\theta} - 2C_{1(1)}C_{1(0),\theta}\right) \\
&+ \cot\theta e^{2\beta_{(0)}-C_{1(0)}}\left(+2C_{2(1)}C_{1(0),\theta} + C_{1(1)}C_{2(0),\theta} - C_{2(1)}C_{2(0),\theta}\right) \\
&+ e^{2\beta_{(0)}-C_{1(0)}}(-2\beta_{(0),\theta}C_{2(1),\theta} - C_{2(1),\theta\theta}) + C_{2(1)}W_{(1),\theta} + C_{2(1)}C_{2(1),u} \\
&+ 2W_{(2),\theta} + e^{2\beta_{(0)}-C_{1(0)}}(-C_{1(1)}C_{1(0),\theta}C_{2(0),\theta} + C_{2(1)}C_{1(0),\theta}C_{2(0),\theta} + C_{1(1)}C_{2(0),\theta\theta}) \\
&+ 2W_{(2),\theta} + e^{2\beta_{(0)}-C_{1(0)}}(-C_{2(1)}C_{2(0),\theta\theta} - 4C_{1(1)} + 4C_{2(1)}) \\
&+ \left(-4C_{2(1)}W_{(1)} - 3W_{(2)}\right)\tan\theta + \left(2C_{2(1)}W_{(1)} + 3W_{(2)}\right)\cot\theta \\
&+ W_{(2)}C_{2(0),\theta} + 2C_{2(1)}W_{(1)}C_{2(0),\theta} - 3C_{2(1)}\mathcal{U}_{(0)} \\
&+ \left(3C_{2(1)}W_{(1)} + W_{(2)}\right)\csc\theta\sec\theta - 2\mathcal{U}_{(1)}
\end{aligned} \tag{B.1.18}
$$

The equations at subleading orders determine the u-derivatives of $C_{i(n/2)}$ with $n > 3$. In general, the equation for $C_{i(n/2),u}$ with $n \in \mathbb{N} \setminus 3$ comes from the order $r^{n/2-1}$.

B.1.5 Supplementary equations

In order for the Bondi procedure to be consistent, once the main equations are solved the supplementary equations turns out to be automatically solved except at the order in which the free integration functions $\mathcal{U}_{(2)}$ and $W_{(4)}$ enter, in which case they give their evolution equation. These are the following.

At order r^{-3} in $R_{u\theta} = 0$

$$
\begin{aligned}
3(\mathcal{U}_{(2)} - 4\log r\tilde{\mathcal{U}}_{(2)})_{,u} \quad = \quad & 24\log r\tilde{\mathcal{U}}_{(2)}\beta_{(0),u} + W_{(1)} + (C_{1(\frac{3}{2}),u})^2 + (C_{2(\frac{3}{2}),u})^2 + \mathcal{U}_{(0)}\mathcal{U}_{(1)} \\
& +2\mathcal{U}_{(0)}W_{(2)}\beta_{(0),\theta} + W_{(2)}\mathcal{U}_{(0),\theta} - \mathcal{U}_{(1)}W_{(1),\theta} - 4W_{(2)}\beta_{(0),\theta}W_{(1),\theta} \\
& +4W_{(1),\theta}W_{(2),\theta} + 2W_{(2)}W_{(1),\theta\theta} + 6U_{(2)}\beta_{(0),u} - 4W_{(3),\theta}\beta_{(0),u} \\
& -2\mathcal{U}_{(0)}\beta_{(2),u} - 4W_{(1),\theta}\beta_{(2),u} + 2W_{(2),\theta}C_{1(1),u} + 2W_{(1),\theta}C_{1(2),u} \\
& +2C_{1(1),u}C_{1(2),u} + C_{1(2),u}C_{2(1),u} + C_{1(\frac{3}{2}),u}C_{2(\frac{3}{2}),u} + C_{1(1),u}C_{2(2),u} \\
& +2C_{2(1),u}C_{2(2),u} - 4W(3)\beta_{(0),u\theta} + 2W(3)^{(1,1)} + 2W_{(2)}C_{1(1),u\theta} \\
& +e^{C_{1(0)}-2\beta_{(0)}}(\tan\theta - \cot\theta)(C_{1(1)}W_{(1)}^3 + 2W_{(2)}W_{(1)}^2) \\
& +e^{2C_{1(0)}-4\beta_{(0)}}(2C_{1(1)}W_{(1)}^4 + 6W_{(2)}W_{(1)}^3) + 2W_{(2)}W_{(1),\theta}C_{1(0),\theta} \\
& +e^{2\beta_{(0)}-C_{1(0)}}(\cot\theta - \tan\theta)(2\mathcal{U}_{(1)}\beta_{(0),\theta} - 2\mathcal{U}_{(0)}C_{1(1)}\beta_{(0),\theta}) \\
& +e^{2\beta_{(0)}-C_{1(0)}}(\cot\theta - \tan\theta)(\mathcal{U}_{(1),\theta} - \mathcal{U}_{(0),\theta}C_{1(1)}) \\
& +(\cot\theta - \tan\theta)(-C_{1(1),\theta}W_{(1)}^2 + \mathcal{U}_{(1)}W_{(1)} - 2W_{(2),\theta}W_{(1)} + 4\beta_{(2),u}W_{(1)}) \\
& -2(\cot\theta - \tan\theta)(C_{1(2),u}W_{(1)} + W_{(2)}C_{1(0),\theta}W_{(1)} + W_{(2)}W_{(1),\theta}) \\
& +2(\cot\theta - \tan\theta)(2W_{(3)}\beta_{(0),u} - W_{(3),u} - W_{(2)}C_{1(1),u}) \\
& +e^{C_{1(0)}-2\beta_{(0)}}(2C_{1(1),\theta}W_{(1)}^3 - 4\beta_{(0),\theta}C_{1(1)}W_{(1)}^3 - 2\beta_{(0),u}{C_{1(1)}}^2W_{(1)}^2) \\
& +e^{C_{1(0)}-2\beta_{(0)}}(-3\mathcal{U}_{(1)}W_{(1)}^2 - 16W_{(2)}\beta_{(0),\theta}W_{(1)}^2 + 4W_{(2),\theta}W_{(1)}^2) \\
& +e^{C_{1(0)}-2\beta_{(0)}}(+8\beta_{(2)}\beta_{(0),u}W_{(1)}^2 - 4\beta_{(2),u}W_{(1)}^2 + 3C_{1(2),u}W_{(1)}^2) \\
& +e^{C_{1(0)}-2\beta_{(0)}}(W_{(1),\theta}C_{1(1)}W_{(1)}^2 + C_{1(1),u}C_{1(1)}W_{(1)}^2 - 4\beta_{(0),u}C_{1(2)}W_{(1)}^2)
\end{aligned}
$$

$$
\begin{aligned}
&+e^{C_{1(0)}-2\beta_{(0)}}(3W_{(2)}C_{1(0),\theta}W_{(1)}^2+W_{(1),u}{C_{1(1)}}^2W_{(1)}-2\mathcal{U}_{(0)}W_{(2)}W_{(1)})\\
&+e^{C_{1(0)}-2\beta_{(0)}}(+4W_{(2)}W_{(1),\theta}W_{(1)}-16W(3)\beta_{(0),u}W_{(1)}-4\beta_{(2)}W_{(1),u}W_{(1)})\\
&+e^{C_{1(0)}-2\beta_{(0)}}(6W_{(3),u}W_{(1)}+4W_{(2)}C_{1(1),u}W_{(1)}-12W_{(2)}\beta_{(0),u}C_{1(1)}W_{(1)})\\
&+e^{C_{1(0)}-2\beta_{(0)}}(4W_{(2),u}C_{1(1)}W_{(1)}+2W_{(1),u}C_{1(2)}W_{(1)}-8W_{(2)}^2\beta_{(0),u})\\
&+e^{C_{1(0)}-2\beta_{(0)}}(2W(3)W_{(1),u}+4W_{(2)}W_{(2),u}+2W_{(2)}W_{(1),u}C_{1(1)})\\
&+e^{2\beta_{(0)}-C_{1(0)}}(-4\mathcal{U}_{(1)}(\beta_{(0),\theta})^2+4\mathcal{U}_{(0)}C_{1(1)}(\beta_{(0),\theta})^2)\\
&+e^{2\beta_{(0)}-C_{1(0)}}(-2\mathcal{U}_{(1),\theta}\beta_{(0),\theta}+2\mathcal{U}_{(0)}C_{1(1),\theta}\beta_{(0),\theta}+2\mathcal{U}_{(0),\theta}C_{1(1)}\beta_{(0),\theta})\\
&+e^{2\beta_{(0)}-C_{1(0)}}(2\mathcal{U}_{(1)}C_{1(0),\theta}\beta_{(0),\theta}-2\mathcal{U}_{(0)}C_{1(1)}C_{1(0),\theta}\beta_{(0),\theta}+\mathcal{U}_{(0),\theta}C_{1(1),\theta})\\
&+e^{2\beta_{(0)}-C_{1(0)}}(-2\mathcal{U}_{(1)}\beta_{(0),\theta\theta}-\mathcal{U}_{(1),\theta\theta}+2\mathcal{U}_{(0)}\beta_{(0),\theta\theta}C_{1(1)})\\
&+e^{2\beta_{(0)}-C_{1(0)}}(\mathcal{U}_{(0),\theta\theta}C_{1(1)}+\mathcal{U}_{(1),\theta}C_{1(0),\theta}-\mathcal{U}_{(0),\theta}C_{1(1)}C_{1(0),\theta})\,, \qquad \text{(B.1.19)}
\end{aligned}
$$

and at order $r^{-7/2}$ in $R_{u\theta}=0$

$$
\begin{aligned}
24W_{(4),u} = &-2\log r\left(-36\tilde{\mathcal{U}}_{(2)}W_{(1)}-12e^{2\beta_{(0)}-C_{1(0)}}\tilde{\mathcal{U}}_{(2),\theta}+48\tilde{W}_{(4),u}-96\tilde{W}_{(4)}\beta_{(0),u}\right)\\
&-3\cot\theta\left({C_{1(1)}}^2W_{(1)}^2+4\beta_{(2)}W_{(1)}^2+2C_{1(2)}W_{(1)}^2+4e^{2\beta_{(0)}-C_{1(0)}}\beta_{(2),\theta}W_{(1)}\right)\\
&-3\cot\theta\left(2W_{(2)}C_{1(1)}W_{(1)}+4e^{2\beta_{(0)}-C_{1(0)}}W_{(3)}\beta_{(0),\theta}+3e^{2\beta_{(0)}-C_{1(0)}}C_{1(3),u}\right)\\
&+3\tan\theta\left(C_{1(1)}^2W_{(1)}^2+4\beta_{(2)}W_{(1)}^2+2C_{1(2)}W_{(1)}^2+4e^{2\beta_{(0)}-C_{1(0)}}\beta_{(2),\theta}W_{(1)}\right)\\
&+3\tan\theta\left(2W_{(2)}C_{1(1)}W_{(1)}+4e^{2\beta(0)-C_{1(0)}}W_{(3)}\beta_{(0),\theta}+3e^{2\beta_{(0)}-C_{1(0)}}C_{1(3),u}\right)\\
&-3e^{C_{1(0)}-2\beta_{(0)}}\left(4C_{1(1)}^2W_{(1)}^3-8\beta_{(2)}W_{(1)}^3+4C_{1(2)}W_{(1)}^3+14W_{(3)}W_{(1)}^2\right)\\
&-3e^{C_{1(0)}-2\beta_{(0)}}\left(20W_{(2)}C_{1(1)}W_{(1)}^2+16W_{(2)}^2W_{(1)}\right)\\
&-12e^{2\beta_{(0)}-C_{1(0)}}\left(W_{(3)}\left(\beta_{(0),\theta}\right)^2+4W_{(1)}\beta_{(2),\theta}\beta_{(0),\theta}-W_{(2)}C_{1(1),\theta}\beta_{(0),\theta}\right)\\
&+6e^{2\beta_{(0)}-C_{1(0)}}\left(2W_{(1)}C_{1(2),\theta}\beta_{(0),\theta}+2W_{(3)}C_{1(0),\theta}\beta_{(0),\theta}-\tilde{\mathcal{U}}_{(2),\theta}+4\beta_{(2)}\mathcal{U}_{(0),\theta}\right)\\
&+3e^{2\beta_{(0)}-C_{1(0)}}\left(2\mathcal{U}_{(2),\theta}-4W_{(3)}\beta_{(0),\theta\theta}-4W_{(1)}\beta_{(2),\theta\theta}+2C_{1(2),\theta}C_{1(1),u}\right)\\
&+3e^{2\beta_{(0)}-C_{1(0)}}\left(C_{2(2),\theta}C_{1(1),u}+2C_{1\left(\frac{3}{2}\right),\theta}C_{1\left(\frac{3}{2}\right),u}+C_{2\left(\frac{3}{2}\right),\theta}C_{1\left(\frac{3}{2}\right),u}\right)\\
&+3e^{2\beta_{(0)}-C_{1(0)}}\left(2C_{1(1),\theta}C_{1(2),u}+C_{2(1),\theta}C_{1(2),u}+C_{1(2),\theta}C_{2(1),u}\right)\\
&+3e^{2\beta_{(0)}-C_{1(0)}}\left(2C_{2(2),\theta}C_{2(1),u}+C_{1\left(\frac{3}{2}\right),\theta}C_{2\left(\frac{3}{2}\right),u}+2C_{2\left(\frac{3}{2}\right),\theta}C_{2\left(\frac{3}{2}\right),u}\right)\\
&+3e^{2\beta_{(0)}-C_{1(0)}}\left(C_{1(1),\theta}C_{2(2),u}+2C_{2(1),\theta}C_{2(2),u}+\csc\theta\sec\theta\left(C_{1(3),u}+2C_{2(3),u}\right)\right)\\
&+6e^{2\beta_{(0)}-C_{1(0)}}\left(2\beta_{(3),\theta\theta}-6C_{1(3),\theta\theta}-\mathcal{U}_{(1),\theta}C_{1(1)}-2\mathcal{U}_{(0),\theta}C_{1(2)}+2W_{(1)}\beta_{(2),\theta}C_{1(0),\theta}\right)\\
&+3e^{2\beta_{(0)}-C_{1(0)}}\left(2C_{1(3),u}C_{1(0),\theta}+C_{2(3),u}C_{1(0),\theta}+C_{1(3),u}C_{2(0),\theta}+2C_{2(3),u}C_{2(0),\theta}\right)\\
&-W_{(1),u}C_{1(1)}^3+2W_{(1)}\beta_{(0),u}C_{1(1)}^3-3U_{(0)}W_{(1)}C_{1(1)}^2+3W_{(1)}W_{(1),\theta}C_{1(1)}^2\\
&+6W_{(1)}^2\beta_{(0),\theta}C_{1(1)}^2-6W_{(2),u}C_{1(1)}^2+12W_{(2)}\beta_{(0),u}C_{1(1)}^2+3W_{(1)}^2C_{1(0),\theta}C_{1(1)}^2\\
&-6\mathcal{U}_{(0)}W_{(2)}C_{1(1)}-6W_{(1)}W_{(2),\theta}C_{1(1)}+36W_{(1)}W_{(2)}\beta_{(0),\theta}C_{1(1)}-6W_{(1)}^2C_{1(1),\theta}C_{1(1)}\\
&+12\beta_{(2)}W_{(1),u}C_{1(1)}-18W_{(3),u}C_{1(1)}+36W_{(3)}\beta_{(0),u}C_{1(1)}-24W_{(1)}\beta_{(2)}\beta_{(0),u}C_{1(1)}\\
&+12W_{(1)}\beta_{(2),u}C_{1(1)}-6W_{(2)}C_{1(1),u}C_{1(1)}-12W_{(1)}C_{1(2),u}C_{1(1)}-6W_{(1),u}C_{1(2)}C_{1(1)}\\
&+12W_{(1)}\beta_{(0),u}C_{1(2)}C_{1(1)}-3W_{(1),u}{C_{1\left(\frac{3}{2}\right)}}^2+6W_{(1)}\beta_{(0),u}{C_{1\left(\frac{3}{2}\right)}}^2-12\tilde{\mathcal{U}}_{(2)}W_{(1)}\\
&+18\mathcal{U}_{(2)}W_{(1)}+12\mathcal{U}_{(1)}W_{(2)}-6U_{(0)}W_{(3)}+36U_{(0)}W_{(1)}\beta_{(2)}-6W_{(3)}W_{(1),\theta}
\end{aligned}
$$

$$
\begin{aligned}
&- 36W_{(1)}\beta_{(2)}W_{(1),\theta} - 12W_{(2)}W_{(2),\theta} - 18W_{(1)}W_{(3),\theta} + 24W_{(2)}^2\beta_{(0),\theta} \\
&+ 48W_{(1)}W_{(3)}\beta_{(0),\theta} - 24W_{(1)}^2\beta_{(2)}\beta_{(0),\theta} + 12W_{(1)}^2\beta_{(2),\theta} - 24W_{(1)}W_{(2)}C_{1(1),\theta} \\
&- 18W_{(1)}^2C_{1(2),\theta} + 6\tilde{W}_{(4),u} + 12\beta_{(3)}W_{(1),u} + 24\beta_{(2)}W_{(2),u} - 12\tilde{W}_{(4)}\beta_{(0),u} \\
&+ 48W_{(4)}\beta_{(0),u} - 48W_{(2)}\beta_{(2)}\beta_{(0),u} - 24W_{(1)}\beta_{(3)}\beta_{(0),u} + 24W_{(2)}\beta_{(2),u} \\
&+ 12W_{(1)}\beta_{(3),u} - 12W_{(3)}C_{1(1),u} - 15W_{(\frac{5}{2})}C_{1(\frac{3}{2}),u} - 18W_{(2)}C_{1(2),u} \\
&- 24W_{(1)}C_{1(3),u} - 15W_{(\frac{5}{2}),u}C_{1(\frac{3}{2})} + 30W_{(\frac{5}{2})}\beta_{(0),u}C_{1(\frac{3}{2})} - 6W_{(1)}C_{1(\frac{3}{2}),u}C_{1(\frac{3}{2})} \\
&- 18U_{(0)}W_{(1)}C_{1(2)} + 6W_{(1)}W_{(1),\theta}C_{1(2)} + 12W_{(1)}^2\beta_{(0),\theta}C_{1(2)} - 12W_{(2),u}C_{1(2)} \\
&+ 24W_{(2)}\beta_{(0),\theta}C_{1(2)} - 6W_{(1),u}C_{1(3)} + 12W_{(1)}\beta_{(0),u}C_{1(3)} - 6W_{(2)}^2C_{1(0),\theta} \\
&- 12W_{(1)}W_{(3)}C_{1(0),\theta} - 12W_{(1)}^2\beta_{(2)}C_{1(0),\theta} + 6W_{(1)}^2C_{1(2)}C_{1(0),\theta}
\end{aligned}
\tag{B.1.20}
$$

B.2 Iterative differentiation approach

In analysing equations asymptotically as $r \to \infty$, it is more elegant to change the radial variable so that the boundary is at $z \to 0$. One may then proceed to determine the asymptotic solution via an iterative differentiation procedure as in AdS [108, 190].

To illustrate this, let us implement the change of variables $r = 1/z^2$ in the first three main equations (7.5.8),(7.5.9) and (7.5.13)

$$
\beta_{,z} = -\frac{z}{24}\left((C_{1,z})^2 + (C_{2,z})^2 + (C_{3,z})^2\right) \tag{B.2.1}
$$

$$
\begin{aligned}
\frac{z^9}{4}\partial_z\left(\frac{1}{2z^7}e^{C_1-2\beta}W_{,z}\right) = {}& \frac{ze^{C_1-2\beta}}{4}\left(zW_{,zz} + zC_{1,z}W_{,z} - 2z\beta_{,z}W_{,z} - 7W_{,z}\right) \\
&- \frac{z^2}{2}\left(6\beta_{,\theta} + z\beta_{,\theta z}\right) + \frac{z^3}{4}\left((\cot\theta - 2\tan\theta)C_{1,z} + C_{1,z\theta}\right) \\
&- \frac{z^3C_{1,z}}{8}(2C_{1,\theta} + C_{2,\theta}) - \frac{z^3C_{2,z}}{8}\left(C_{1,\theta} + 2C_{2,\theta} + \frac{2}{\sin\theta\cos\theta}\right)
\end{aligned}
\tag{B.2.2}
$$

$$
\begin{aligned}
3z^4(2 - z\partial_z)\mathcal{U} = {}& -\frac{z^4e^{2\beta-C_1}}{2}\sec\theta\csc\theta(C_{1,\theta} + 2C_{2,\theta}) \\
&+ z^4e^{-C_1}(\cot\theta - \tan\theta)\left(\frac{e^{C_1}}{2}(12z^{-2} - z^{-1}\partial_z)W - 2e^{2\beta}\beta_{,\theta} + \frac{5}{2}e^{2\beta}C_{1,\theta}\right) \\
&+ \frac{z^4e^{2\beta-C_1}}{2}\left(12 - (2\beta_{,\theta})^2 + 4\beta_{,\theta}C_{1,\theta} - 2(C_{1,\theta})^2 - C_{1,\theta}C_{2,\theta} - (C_{2,\theta})^2\right) \\
&+ \frac{z^4e^{2\beta-C_1}}{2}\left(-4\beta_{,\theta\theta} + 2C_{1,\theta\theta}\right) - \frac{z^2}{8}e^{-4\beta+C_1}(W_{,z})^2 \\
&- 3z^2\left(2 - \frac{z}{6}\partial_z\right)W_{,\theta}
\end{aligned}
\tag{B.2.3}
$$

The reason for the specific choice of variable z is that the powers in the resulting asymptotic series will then be integer.

Let us now explain the iterative differentiation approach, beginning with equation (B.2.1). Taking the limit of (B.2.1) as $z \to 0$ we obtain

$$[\beta_{,z}]_{z=0} = 0 \tag{B.2.4}$$

provided that $z^{\frac{1}{2}} C_{i,z} \to 0$ as $z \to 0$; this limit can be justified using the last of the main equations, applying the given boundary conditions. Clearly (B.2.1) does not impose any restrictions on $[\beta]_{z=0}$, while the equation above implies that the term in the asymptotic expansion of β at order z vanishes, as we found before. Differentiating (B.2.1) and taking the limit as $z \to 0$ then gives

$$[\beta_{,zz}]_{z=0} = -\frac{1}{24}\left[(C_{1,z})^2 + (C_{2,z})^2 + (C_{3,z})^2\right]_{z=0} \tag{B.2.5}$$

Continuing the process, we will clearly determine the derivatives of β as $z \to 0$ to all orders.

Just as in the standard Bondi analysis, we solve the differential equations in the nested order, substituting the solution of (B.2.1) into the righthandside of (B.2.2), and then using the solutions of both (B.2.1) and (B.2.2) in (B.2.3). To understand how logarithms arise in the asymptotic expansion it is useful to rewrite (B.2.3) in the form

$$(2 - z\partial_z)\mathcal{U} = \mathcal{P}[\beta, C_i, W], \tag{B.2.6}$$

where $\mathcal{P}$ is implicitly a functional of the functions (β, C_i, W) and their derivatives. Using the solutions of the other main equations one can show that

$$[\mathcal{P}]_{z=0} \qquad [\mathcal{P}_{,z}]_{z=0} \qquad [\mathcal{P}_{,zz}]_{z=0} \tag{B.2.7}$$

are all non-vanishing for generic boundary data $([\beta]_{z=0}, [C_i]_{z=0})$. From (B.2.6) and its first derivative one obtains

$$[\mathcal{U}]_{z=0} = \frac{1}{2}[\mathcal{P}]_{z=0} \qquad [\mathcal{U}_{,z}]_{z=0} = [\mathcal{P}_{,z}]_{z=0} \tag{B.2.8}$$

but differentiating again one obtains

$$[z\partial_z^3\mathcal{U}]_{z=0} = -[\mathcal{P}_{,zz}]_{z=0} \tag{B.2.9}$$

i.e. $[\mathcal{U}_{,zz}]_{z=0}$ is unconstrained, while $\partial_z^3\mathcal{U}$ has a first order pole at $z \to 0$. The latter induces the logarithmic term in the asymptotic expansion at order z^2.

APPENDIX C

Appendix to chapter 8

C.1 Derivation of Einstein's equations

The Ricci tensor

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \partial_\mu \Gamma^\rho_{\rho\nu} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu} \tag{C.1.1}$$

is conveniently computed using

$$\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{|{}^{(d)}g|}}\partial_\mu \sqrt{|{}^{(d)}g|}\,, \quad |{}^{(d)}g| = e^{4\beta}|{}^{(d-2)}g| = e^{4\beta}r^{2(d-2)}\underbrace{|h_{(0)}|}_{q} \tag{C.1.2}$$

such that

$$\Gamma^\rho_{\rho\mu} = \partial_\mu\left(2\beta + \frac{1}{2}\log|{}^{(d-2)}g|\right) =: \partial_\mu \mathcal{O}_2 \tag{C.1.3}$$

and

$$R_{\mu\nu} = \underbrace{\left[\partial_\rho + \partial_\rho\left(2\beta + \frac{1}{2}\log|{}^{(d-2)}g|\right)\right]}_{\mathcal{O}_\rho}\Gamma^\rho_{\mu\nu} - \partial_\mu\partial_\nu\left(2\beta + \frac{1}{2}\log|{}^{(d-2)}g|\right) - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\rho\nu}. \tag{C.1.4}$$

Some useful relationships involving $\det g_{AB} = |{}^{(d-2)}g|$ are found using

$$g^{AB}\partial_\mu g_{AB} = \frac{\partial_\mu(\det g_{AB})}{\det g_{AB}} \tag{C.1.5}$$

so that[1]

$$g^{AB}\partial_r g_{AB} = \tfrac{2(d-2)}{r} \qquad \text{and} \quad h^{AB}\partial_r h_{AB} = 0$$
$$g^{AB}\partial_u g_{AB} = \tfrac{\partial_u q}{q} \qquad \text{and} \quad h^{AB}\partial_u h_{AB} = \frac{\partial_u q}{q} \tag{C.1.6}$$
$$g^{AB}\partial_C g_{AB} = \frac{\partial_C q}{q} \qquad \text{and} \quad h^{AB}\partial_C h_{AB} = \frac{\partial_C q}{q} \tag{C.1.7}$$

To make contact with [60] we define

$$l_{AB} = \frac{1}{2}\partial_u g_{AB}\,, \quad k_{AB} = \frac{1}{2}\partial_r g_{AB}\,, \quad n_A = \frac{1}{2}e^{-2\beta}g_{AB}\partial_r W^B \tag{C.1.8}$$

whose indices are raised and lowered with g_{AB} and such that

$$l^{AB} = -\frac{1}{2}\partial_u g^{AB}\,, \quad k^{AB} = -\frac{1}{2}\partial_r g^{AB} \tag{C.1.9}$$

The Christoffel symbols are given in the following as

$$\begin{bmatrix} \Gamma^\mu_{uu} & \Gamma^\mu_{ur} & \Gamma^\mu_{uC} \\ \Gamma^\mu_{ru} & \Gamma^\mu_{rr} & \Gamma^\mu_{rC} \\ \Gamma^\mu_{Au} & \Gamma^\mu_{Ar} & \Gamma^\mu_{BC} \end{bmatrix}\,, \quad \mu = u, r, A \tag{C.1.10}$$

$$\begin{aligned} \Gamma^u_{uu} &= 2\partial_u\beta - \tfrac{1}{2}(2\partial_r\beta + \partial_r)\mathcal{U} + 2n_A W^A + e^{-2\beta}k_{CD}W^C W^D\,, \\ \Gamma^u_{ur} &= 0\,, \quad \Gamma^u_{uA} = \partial_A\beta - n_A - e^{-2\beta}k_{AB}W^B\,, \\ \Gamma^u_{rr} &= 0\,, \quad \Gamma^u_{rA} = 0\,, \quad \Gamma^u_{AB} = e^{-2\beta}k_{AB}\,, \end{aligned}$$

$$\begin{aligned} \Gamma^r_{uu} &= \tfrac{1}{2}(\partial_u - 2\partial_u\beta)\mathcal{U} + \tfrac{1}{2}\mathcal{U}(\partial_r + 2\partial_r\beta)\mathcal{U} - \tfrac{1}{2}W^A(\partial_A + 2\partial_A\beta)\mathcal{U} - 2\mathcal{U}\,n_A W^A \\ &\quad + e^{-2\beta}\left(\mathcal{U}k_{AB}W^A W^B + l_{AB}W^A W^B + W^A W^{B\,(d-2)}D_A W_B\right) \\ \Gamma^r_{ur} &= \tfrac{1}{2}\left(2\partial_r\beta + \partial_r\right)\mathcal{U} - \left(\partial_A\beta + n_A\right)W^A\,, \\ \Gamma^r_{uA} &= \tfrac{1}{2}(\partial_A + 2n_A)\mathcal{U} - \tfrac{1}{2}e^{-2\beta}W^B\left({}^{(d-2)}D_A W_B + {}^{(d-2)}D_B W_A + 2l_{AB} - 2k_{AB}\mathcal{U}\right) \\ \Gamma^r_{rr} &= 2\partial_r\beta\,, \quad \Gamma^r_{rA} = \tfrac{1}{2}e^{-2\beta}g_{AC}\partial_r W^C + \partial_A\beta\,, \\ \Gamma^r_{AB} &= \tfrac{e^{-2\beta}}{2}\left({}^{(d-2)}D_B W_A + {}^{(d-2)}D_A W_B + 2l_{AB} - 2\mathcal{U}k_{AB}\right) \end{aligned} \tag{C.1.11}$$

$$\begin{aligned} \Gamma^A_{uu} &= 2W^A\partial_u\beta - \tfrac{1}{2}W^A(\partial_r + 2\partial_r\beta)\mathcal{U} + 2n_B W^B W^A - \partial_u W^A - 2l^A_C W^C \\ &\quad + e^{-2\beta}k_{BC}W^A W^B W^C + \tfrac{1}{2}e^{2\beta}(2\partial^A\beta + \partial^A)\mathcal{U} + \tfrac{1}{2}{}^{(d-2)}D^A(W^2) \\ \Gamma^A_{ur} &= -k^A_C W^C + e^{2\beta}\partial^A\beta - \tfrac{1}{2}\delta^A_C\partial_r W^C = -k^A_C W^C + e^{2\beta}(\partial^A\beta - n^A)\,, \\ \Gamma^A_{uB} &= W^A(\partial_B\beta - n_B) - e^{-2\beta}k_{BC}W^A W^C + l^A_B + \tfrac{1}{2}{}^{(d-2)}D^A W_B - \tfrac{1}{2}{}^{(d-2)}D_B W^A \\ \Gamma^A_{rr} &= 0\,, \quad \Gamma^A_{rB} = k^A_B\,, \quad \Gamma^A_{BC} = e^{-2\beta}W^A k_{BC} + {}^{(d-2)}\Gamma^A_{BC}\,, \end{aligned}$$

[1] I.e.: $\underbrace{\partial_r|g_{AB}|}_{2(d-2)r^{2(d-2)-1}q} = \underbrace{|g_{AB}|}_{r^{2(d-2)}q}\, g^{AB}\partial_r g_{AB}$, $g^{AB}\partial_r g_{AB} = \frac{h^{AB}}{r^2}\partial_r\left(r^2 h_{AB}\right) = \frac{2(d-2)}{r} + h^{AB}\partial_r h_{AB}$

The relevant Ricci tensor components giving rise to the main equations are

$$R_{rr} = \mathcal{O}_r \Gamma^r_{rr} - \partial^2_r \mathcal{O}_2 - \Gamma^r_{rr}\Gamma^r_{rr} - \Gamma^r_{rA}\Gamma^A_{rr} + \Gamma^B_{rA}\Gamma^A_{Br} \tag{C.1.12}$$

$$R_{rA} = \mathcal{O}_r \Gamma^r_{rA} + \mathcal{O}_B \Gamma^B_{rA} - \partial_r \partial_A \mathcal{O}_2 - \Gamma^r_{rr}\Gamma^r_{rA} - \Gamma^r_{rB}\Gamma^B_{rA} + \Gamma^C_{rB}\Gamma^B_{CA} + \Gamma^C_{ru}\Gamma^u_{CA} \tag{C.1.13}$$

$$\begin{aligned} R_{AB} &= \mathcal{O}_u \Gamma^u_{AB} + \mathcal{O}_r \Gamma^r_{AB} + \mathcal{O}_C \Gamma^C_{AB} - \partial_A \partial_B \mathcal{O}_2 \\ &- \Gamma^u_{Au}\Gamma^u_{uB} - \Gamma^u_{AC}\Gamma^C_{uB} - \Gamma^r_{AC}\Gamma^C_{rB} - \Gamma^r_{Ar}\Gamma^r_{rB} - \Gamma^C_{Au}\Gamma^u_{CB} - \Gamma^C_{Ar}\Gamma^r_{CB} - \Gamma^C_{AD}\Gamma^D_{CB} \end{aligned} \tag{C.1.14}$$

The components (C.1.12),(C.1.13) are easily evaulated. The symbol K^A_B can be further defined as $K^A_B = \frac{r^2}{2} h^{AC}\partial_r h_{CB}$ so that $k^A_B = \frac{1}{r}\delta^A_B + \frac{1}{r^2}K^A_B$ and $R_{rr} = \frac{1}{2(d-2)r^3}K^A_B K^B_A$ can be directly compared with (4.33) of [60].

In C.1.14, the third, fourth and latter terms can be arranged to

$$\begin{aligned} \mathcal{O}_C \Gamma^C_{AB} - \partial_A \partial_B \mathcal{O}_2 - \Gamma^C_{AD}\Gamma^D_{CB} &= {}^{(d-2)}R_{AB} + \partial_C\left(e^{-2\beta} W^C k_{AB}\right) \\ &+ 2e^{-2\beta}\partial_C \beta W^C k_{AB} + {}^{(d-2)}\Gamma^D_{DC} e^{-2\beta} W^C k_{AB} \\ &+ 2^{(d-2)}\Gamma^C_{AB}\partial_C\beta - 2\partial_A\partial_B\beta - e^{-4\beta}W^C W^D k_{AD} k_{CB} \\ &- e^{-2\beta}U^C k_{AD}{}^{(d-2)}\Gamma^D_{CB} - e^{-2\beta}U^C k_{DB}{}^{(d-2)}\Gamma^D_{AC}. \end{aligned} \tag{C.1.15}$$

so that

$$\begin{aligned} R_{AB} &= \left(\partial_r + 2\partial_r\beta + \frac{d-2}{r}\right)\Gamma^r_{AB} + \left(\partial_u + 2\partial_u\beta + l\right)\Gamma^u_{AB} - 2^{(d-2)}D_A\partial_B\beta + {}^{(d-2)}R_{AB} \\ &+ {}^{(d-2)}D_C\left(e^{-2\beta}U^C k_{AB}\right) + 2e^{-2\beta}\partial_C\beta U^C k_{AB} - e^{-4\beta}U^C U^E k_{AE} k_{CB} \\ &- \Gamma^u_{Au}\Gamma^u_{uB} - \Gamma^u_{AC}\Gamma^C_{uB} - \Gamma^r_{AC}\Gamma^C_{rB} - \Gamma^r_{Ar}\Gamma^r_{rB} - \Gamma^C_{Au}\Gamma^u_{CB} - \Gamma^C_{Ar}\Gamma^r_{CB} \end{aligned} \tag{C.1.16}$$

and by contraction

$$\begin{aligned} g^{DA}R_{AB} &= {}^{(d-2)}R^D_B - 2\left({}^{(d-2)}D^D\partial_B\beta + \partial^D\beta\partial_B\beta + n^D n_B\right) \\ &+ e^{-2\beta}\left(\partial_r + \frac{d-2}{r}\right)\left(\frac{1}{2}{}^{(d-2)}D^D W_B + \frac{1}{2}{}^{(d-2)}D_B W^D + l^D_B - k^D_B\mathcal{U}\right) \\ &+ e^{-2\beta}\left[\left(\partial_u + l\right)k^D_B + {}^{(d-2)}D_C(W^C k^D_B) + k^{D}_{A}{}^{(d-2)}D_B W^A - k^{A}_{B}{}^{(d-2)}D_A W^B\right] \end{aligned} \tag{C.1.17}$$

leading to (8.3.6).

C.2 Recursive formulae

Given h_{AB} as (8.4.1)

$$h_{AB}(u,r,x) = h_{(0)AB}(u,x) + \sum_p \frac{h_{(a+p)AB}(u,x)}{r^{a+p}}. \tag{C.2.1}$$

its inverse is recursively given by

$$\begin{aligned} h^{-1} &= h^{-1}_{(0)} + \sum_{n=1}^{\lfloor a+p_o \rfloor} (-1)^n \left(\sum_{p}^{p_o} \frac{h^{-1}_{(0)} h_{(a+p)}}{r^{a+p}} \right)^n h^{-1}_{(0)} \\ &= h^{-1}_{(0)} + \sum_{n}^{\lfloor a+p_o \rfloor} \frac{(-1)^n}{r^{an}} \sum_{\oplus j_t = n} \binom{n}{j_{[0,p_o]}} \prod_{t=0}^{p_o} \left(\frac{h^{-1}_{(0)} h_{(a+t)}}{r^t} \right)^{j_t} h^{-1}_{(0)} \end{aligned} \tag{C.2.2}$$

where n is integer and $\lfloor a + p_o \rfloor$ is the floor of the maximal power $a + p_o$ we keep in (C.2.1). Eventually discard from h^{-1} terms of order higher than $a + p_o$. In the second line we have used the multinomial theorem and defined

$$\binom{n}{j_{[0,p_o]}} := \binom{n}{j_0, \ldots, j_t, \ldots, j_{p_o}} := \frac{n!}{\prod_{t=0}^{p_o} (j_t!)} \tag{C.2.3}$$

where $j_{[0,p_o]}$ is the collection of non negative integer indices j_p associated to the object $r^{-p} h^{-1}_{(0)} h_{(a+p)}$ for each p. The sum $\sum_{\oplus j_p = n}$ is taken on any combination of j_p such that their total sum $\oplus j_p$ is equal to n. The tensor indices must match in the contraction and each inverse is taken with respect to $h_{(0)}$, so that $h_{(0)} h^{-1}_{(0)} = \delta$, $h^{-1}_{(0)} h_{(a+p)} h^{-1}_{(0)} = h^{-1}_{(a+p)}$.

These conventions translates the practice. To find the explicit expressions of the metric functions we have to fix a and expand up to the relevant order, so up to a $p = p_o$. For example in $d = 4$ we have $a = 1$ and it is appropriate to take the maximal p to be $p_o = 2$ (but for many purposes $p_o = 1$ suffices in $d = 4$).

The same conventions apply to the expansion of any other object, so in the following we simplify notation where no confusion arise.

$\tilde{K}^A_B = \frac{1}{2} h^{AC} \partial_r h_{CB}$ now follows as

$$\begin{aligned} \tilde{K}^A_B &= -\sum_p \frac{a+p}{2r^{a+p+1}} h^A_{(a+p)B} - \frac{1}{2} \sum_n (-1)^n \left(\sum_p \frac{h^{-1}_{(0)} h_{(a+p)}}{r^{a+p}} \right)^n h^{-1}_{(0)} \sum_q \frac{a+q}{r^{a+q+1}} h_{(a+q)} \\ &= \frac{1}{r^{a+1}} \sum_p \frac{\tilde{K}^E_{(a+1+p)F}}{r^p} \left[\delta^A_E \delta^F_B + \sum_n \frac{(-1)^n}{r^{an}} \left(\sum_q \frac{h^A_{(a+q)E}}{r^q} \right)^n \delta^F_B \right] \\ &= \frac{1}{r^{a+1}} \sum_p^{p_o} \frac{\tilde{K}^E_{(a+1+p)F}}{r^p} \left[\delta^A_E \delta^F_B + \sum_{n=1}^{\sim a+p_o} \frac{(-1)^n}{r^{an}} \sum_{\oplus j_t = n} \binom{n}{j_{[0,q_o]}} \prod_{t=0}^{p_o} \left(\frac{h^A_{(a+q)E}}{r^t} \right)^{j_t} \delta^F_B \right] \\ &= \frac{1}{r^{a+1}} \sum_p^{p_o} \frac{\tilde{K}^E_{(a+1+p)F}}{r^p} \left[\delta^A_E \delta^F_B + \sum_{n=1}^{\sim a+p_o} \frac{(-1)^n}{r^{an}} \sum_{\oplus j_t = n} \binom{n}{j_{[0,q_o]}} r^{-\sum t j_t} \prod_{t=0}^{p_o} (h^A_{(a+q)E})^{j_t} \delta^F_B \right] \end{aligned} \tag{C.2.4}$$

where

$$\tilde{K}^E_{(a+1+p)F} = -\frac{a+p}{2} h^A_{(a+p)B} \tag{C.2.5}$$

at convenience we reshuffle indices and write

$$\tilde{K}^A_B = \frac{1}{r^{a+1}} \sum_{p=0} \frac{K^A_{(p)B}}{r^p} \tag{C.2.6}$$

Christoffel symbols are expanded as

$$
\begin{aligned}
{}^{(d-2)}\Gamma^A_{BC} &= \overset{(0)}{\Gamma}{}^A_{BC} + \frac{1}{2}\sum_p \frac{1}{r^{a+p}} h^{AD}_{(0)} \left(\partial_B h_{(a+p)DC} + \partial_C h_{(a+p)BD} - \partial_D h_{(a+p)BC}\right) \\
&+ \frac{1}{2}\sum_{n=1}(-1)^n \left(\sum_p \frac{h^{AE}_{(0)} h_{(a+p)EF}}{r^{a+p}}\right)^n h^{FD}_{(0)} \left(\partial_B h_{(0)DC} + \partial_C h_{(0)BD} - \partial_D h_{(0)BC}\right) \\
&+\frac{1}{2}\sum_{n=1}(-1)^n \left(\sum_p \frac{h^{AE}_{(0)} h_{(a+p)EF}}{r^{a+p}}\right)^n h^{FD}_{(0)} \sum_{q=1} \frac{1}{r^{a+q}} \left(\partial_B h_{(a+q)DC} + \partial_C h_{(a+q)BD} - \partial_D h_{(a+q)BC}\right)
\end{aligned}
\tag{C.2.7}
$$

and appropriately rewrites each order in terms of the quantites at the previous orders, so for example

$$
{}^{(a)}\Gamma^A_{BC} = \frac{1}{2}\left(\overset{(0)}{D}_B h^A_{(a)C} + \overset{(0)}{D}_C h^A_{(a)B} - \overset{(0)}{D}{}^A h_{(a)BC}\right). \tag{C.2.8}
$$

Curvature tensors and scalar are expanded as

$$
{}^{(d-2)}R = r^{-2}\overset{(0)}{R} + r^{-(2+a)}\sum_p r^{-p}\overset{(a+p)}{R}. \tag{C.2.9}
$$

C.2.1 First equation: β

The integrand of (8.3.8) is

$$
\begin{aligned}
r\tilde{K}^A_B\tilde{K}^B_A &= \frac{1}{r^{2a+1}}\sum_{p,q}\frac{\tilde{K}^{EG}_{(p+q)BA}}{r^{p+q}}\left[\delta^A_E\delta^B_G + 2\delta^A_E\sum_m \frac{(-1)^m}{r^{am}}\sum_{\oplus l_s=m}\binom{m}{l_{[0,p_o]}} r^{-\sum s l_s}\prod_{t=0}^{p_o}(h^B_{(a+q)G})^{l_s}\right. \\
&+ \left.\sum_{n,m}\frac{(-1)^{n+m}}{r^{a(n+m)}}\sum_{\oplus j_t=n}\binom{n}{j_{[0,p_o]}} r^{-\sum t j_t}\prod_{t=0}^{p_o}(h^A_{(a+t)E})^{j_t}\sum_{\oplus l_s=m}\binom{m}{l_{[0,p_o]}} r^{-\sum s l_s}\prod_{s=0}^{p_o}(h^B_{(a+s)G})^{l_s}\right]
\end{aligned}
\tag{C.2.10}
$$

So that we can list the orders in the expansion of $\beta(u,r,x)$ with an expression which is not so enlightening, but turns out to be useful

$$
\begin{aligned}
\beta = \beta_{(0)} + \sum_{p,q}^{p_o}\Bigg[\frac{\breve{\beta}_{(2a+p+q)}}{r^{2a+p+q}} &+ \sum_{m=1}^{\lfloor a+p_o\rfloor}\Bigg(\sum_{\oplus l_s=m}\frac{\mathring{\beta}_{((2+m)a+p+q+sl_s)}}{r^{(2+m)a+p+q+sl_s}} \\
&+ \sum_{n=1}^{\lfloor a+p_o\rfloor}\sum_{\oplus j_t=n}\frac{\breve{\beta}_{((2+m+n)a+p+q+sl_s+tj_t)}}{r^{(2+m+n)a+p+q+sl_s+tj_t}}\Bigg)\Bigg]
\end{aligned}
\tag{C.2.11}
$$

For example, at order $2a$ we have only

$$
\beta_{(2a)} = \breve{\beta}_{(2a)} = -\frac{a}{16(d-2)}\frac{1}{r^{2a}}h_{(a)AB}h^{AB}_{(a)} \tag{C.2.12}
$$

while at order $3a$ we have

$$
\beta_{(3a)} = \mathring{\beta}_{(3a)} + \left.\frac{\breve{\beta}_{(2a+p+q)}}{r^{2a+p+q}}\right|_{2a+p+q=3a} \tag{C.2.13}
$$

and so on for all orders.

As a is fixed, all the terms can be reorganised as

$$\beta = \beta_{(0)} + \sum_{k>0} \frac{\beta_{(2a+k)}}{r^{2a+k}} \tag{C.2.14}$$

where k, as p moves forward by half integer steps if d is odd and by integer steps if d is even. We will such expressions with the understanding that each order $2a+k$ has been computed with a procedure like the one exemplified above. So for example

$$\begin{aligned} \beta_{(2a)} &= \breve{\beta}_{(2a)} \\ &\vdots \\ \beta_{(3a)} &= \mathring{\beta}_{(3a)} + \left.\frac{\breve{\beta}_{(2a+p+q)}}{r^{2a+p+q}}\right|_{2a+p+q=3a} . \end{aligned} \tag{C.2.15}$$

In particular for $d=4$ and $a=1$ and p moves in integer steps, so $\beta_{(3a)}$ is the next order after $\beta_{(2a)}$ and we get

To produce the two terms given in the main text we take the expansion with $p \in [0, p_o = 1\text{or}1/2]$ and hence $n, m \in [1, \lfloor a+1 \rfloor]$.

$$\beta_{(2a)} = -\frac{a}{16(d-2)} \frac{1}{r^{2a}} h_{(a)AB} h^{AB}_{(a)} \tag{C.2.16}$$

$$\beta_{(2a+p)} = -\frac{a(a+p)}{8(d-2)(2a+p)} h_{(a)AB} h^{AB}_{(a+p)} \tag{C.2.17}$$

C.2.2 Second equation: W^A

The two terms collected in $\mathcal{G}_A$ in (8.3.9) are expanded as

$$\begin{aligned} r^{d-2}\left(\partial_r - \frac{d-2}{r}\right)\partial_A \beta &= -(d-2) r^{d-3} \partial_A \beta_{(0)} + \sum_{k=0} r^{d-3-2a-k} \mathcal{G}^{[1]}_{(k)A}, \\ r^{d-2(d-2)} D_B \tilde{K}^B_A &\sim r^{d-3-a} \left(\overset{(0)}{D}_B + r^{-a} \sum_p r^{-p} \overset{(\text{a+p})}{\Gamma}_B \right) \sum_m r^{-m} K^B_{(m)A} \\ &= r^{d-3-a} \overset{(0)}{D}_B \sum_m r^{-m} K^B_{(m)A} + r^{d-3-2a} \sum_{p,m} r^{-p-m} \overset{(\text{a+p})}{\Gamma}_B K^B_{(m)A} \end{aligned} \tag{C.2.18}$$

where

$$\mathcal{G}^{[1]}_{(k)A} = (2-d-2a-k)\partial_A \beta_{(2a+k)}, \tag{C.2.19}$$

and in the second expansion we noted the subleading terms in the covariant derivatives with a Γ for the subleading Christoffel symbols and we assume implicit that the indices must be organised to respect covariant differentiation rules. We can again formally reorganise the sums in the second term so to have only one index to sum over, we define

$$\sum_q r^{d-3-a-q} \mathcal{G}^{[2]}_{(q)A} = r^{d-3-a} \overset{(0)}{D}_B \sum_m r^{-m} K^B_{(m)A} + r^{d-3-2a} \sum_{p,m} r^{-p-m} \overset{(\text{a+p})}{\Gamma}_B K^B_{(m)A} \tag{C.2.20}$$

So we have

$$r^d\tilde{n}_A = \int dr\mathcal{G}_A = -\frac{\partial_A\beta_{(0)}}{r^{2-d}} + \int dr\left[\sum_k r^{d-3-2a-k}\mathcal{G}^{[1]}_{(k)A} - \sum_m r^{d-3-a-q}\mathcal{G}^{[2]}_{(q)A}\right] \tag{C.2.21}$$

When $k = d-2-2a$ and $q = d-2-a$ the integral give a $\log r$ term. This happens when

R) $a = \frac{d-2}{2}$, $d \geq 4$: $k = 0$, $q = \frac{d-2}{2}$,

NR) $a = 1$, $d > 4$: $k = d-4$, $q = d-3$.

Thus under the case R) we get

$$\tilde{n}_A = -\frac{\partial_A\beta_{(0)}}{r^2} - \sum_{k>0}\frac{\mathcal{G}^{[1]}_{(k)A}}{kr^{d+k}} - \sum_{\substack{m=0,\\ \neq\frac{d-2}{2}}}\frac{\mathcal{G}^{[2]}_{(m)A}}{(\frac{d-2}{2}-m)r^{\frac{d+2}{2}+m}} + \frac{\log r}{r^d}\left(\mathcal{G}^{[1]}_{(0)A} - \mathcal{G}^{[2]}_{(\frac{d-2}{2})A}\right) \tag{C.2.22}$$

In the case NR)

$$\tilde{n}_A = -\frac{\partial_A\beta_{(0)}}{r^2} + \sum_{\substack{k=0,\\ \neq d-4}}\frac{\mathcal{G}^{[1]}_{(k)A}}{(d-4-k)r^{k+4}} - \sum_{\substack{m=0,\\ \neq d-3}}\frac{\mathcal{G}^{[2]}_{(m)A}}{(d-3-m)r^{m+3}} + \frac{\log r}{r^d}\left(\mathcal{G}^{[1]}_{(d-4)A} - \mathcal{G}^{[2]}_{(d-3)A}\right) \tag{C.2.23}$$

Notice that the notation is a bit misleading because $\mathcal{G}^{[1]}_A$ enters the asymptotic expansion only after $\mathcal{G}^{[2]}_A$ Then W^A follows from

$$W^A = 2\int dre^{2\beta}h^{AB}\tilde{n}_B \tag{C.2.24}$$

so that in general we can write

$$W^A = W^A_{(0)} + \frac{W^A_{(1)}}{r} + \sum_{p=0}^{d-2-a}\frac{W^A_{(a+1+p)}}{r^{a+1+p}} + \frac{1}{r^{d-1}}\left(\mathcal{W}^A_{(d-1)} + \mathtt{W}^A_{(d-1+p)}\log r\right) + \dots \tag{C.2.25}$$

where the exact expression of each coefficient depends on whether we are dealing with R) or NR).

C.2.3 Third equation: $\mathcal{U}$

Now move to the equation for $\mathcal{U}$ (8.3.5). Using the above results, $\mathcal{F}$ is expanded

$$\mathcal{F} = \frac{\mathcal{F}_{(1)}}{r} + \frac{\mathcal{F}_{(2)}}{r^2} + \sum_p\left(\frac{\mathcal{F}_{(a+p+2)}}{r^{a+p+2}} + \frac{1}{r^{d+p}}\left(\mathcal{F}_{(d+p)} + \log r\tilde{\mathcal{F}}_{(d+p)}\right)\right) + \dots, \tag{C.2.26}$$

hence the integrand in (8.3.13) is

$$r^{d-2}\mathcal{F} = r^{d-3}\mathcal{F}_{(1)} + r^{d-4}\mathcal{F}_{(2)} + \sum_p r^{d-4-a-p}\mathcal{F}_{(a+p+2)} + r^{-2-p}\left(\mathcal{F}_{(d+p)} + \log r\mathtt{F}_{(d+p)}\right) + \dots . \tag{C.2.27}$$

Notice that $d-4-a-p > -2-p$ for both values of a we are considering, while

$$d-4-a-p=-1 \Leftrightarrow p=d-a-3 \Rightarrow p=\begin{cases} d-\frac{d-2}{2}-3=\frac{d-4}{2} \\ d-1-3=d-4 \end{cases} \tag{C.2.28}$$

So the solution (8.3.13) of (8.3.5) can be organised as

$$\mathcal{U}=r\mathcal{U}_{(-1)}+\mathcal{U}_{(0)}+\sum_{p=0}^{a+p<d-3}\frac{\mathcal{U}_{(a+p)}}{r^{a+p}}+\frac{1}{r^{d-3}}\left(\mathcal{U}_{(d-3)}+\mathtt{U}_{(d-3)}\log r\right)+\dots \tag{C.2.29}$$

where the sum in (C.2.27) integrates to the first sum in (C.2.29) and in the r^{d-3}, $r^{d-3}\log r$ terms, as well as in the further subleading parts noted wit $\dots$, to which also $\mathcal{F}_{(d+p)}$ and $\mathtt{F}_{(d+p)}$ contribute.

Bibliography

[1] F. Capone and M. Taylor, *Cosmic branes and asymptotic structure*, *JHEP* **10** (2019) 138, [1904.04265].

[2] F. Capone, *Bms symmetries and holography: an introductory overview*, in *Einstein Equations: Physical and Mathematical Aspects of General Relativity* (S. Cacciatori, B. Günesiu and S. Pigola, eds.), Tutorials, Schools, and Workshops in the Mathematical Sciences, Birkhäuser, 2019.

[3] C. Rovelli, *Loop quantum gravity*, *Living Rev. Rel.* **1** (1998) 1, [gr-qc/9710008].

[4] R. Loll, *Quantum Gravity from Causal Dynamical Triangulations: A Review*, *Class. Quant. Grav.* **37** (2020) 013002, [1905.08669].

[5] R. Percacci, *Asymptotic Safety*, 0709.3851.

[6] J. H. S. Michael B. Green and E. Witten, *Superstring Theory*, vol. I of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, 25th anniversary edition ed., 2012.

[7] J. Polchinski, *String Theory*, vol. I of *Cambridge Monographs on Mathematical Physics*. Cambridge University Press, 1998.

[8] T. Padmanabhan, *Emergent Gravity Paradigm: Recent Progress*, *Mod. Phys. Lett. A* **30** (2015) 1540007, [1410.6285].

[9] E. P. Verlinde, *Emergent Gravity and the Dark Universe*, *SciPost Phys.* **2** (2017) 016, [1611.02269].

[10] Event Horizon Telescope collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, *Astrophys. J. Lett.* **875** (2019) L1, [1906.11238].

[11] J. M. Bardeen, B. Carter and S. W. Hawking, *The four laws of black hole mechanics*, *Communications in Mathematical Physics* **31** (June, 1973) 161–170.

[12] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev.* **D7** (1973) 2333–2346.

[13] S. W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199–220.

[14] A. Dabholkar and S. Nampuri, *Quantum black holes*, *Lect. Notes Phys.* **851** (2012) 165–232, [`1208.4814`].

[15] S. W. Hawking, *Breakdown of Predictability in Gravitational Collapse*, *Phys. Rev.* **D14** (1976) 2460–2473.

[16] G. 't Hooft, *Dimensional reduction in quantum gravity*, in *Salamfestschrift: a collection of talk* (J. E. A. Ali and S. Randjbar-Daemi, eds.), vol. 4 of *World Scientific Series in 20th Century Physics*, World Scientific, 1993. `gr-qc/9310026`.

[17] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377–6396, [`hep-th/9409089`].

[18] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [`hep-th/9711200`].

[19] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [`hep-th/9802150`].

[20] D. Dieks, J. van Dongen and S. de Haro, *Emergence in holographic scenarios for gravity*, *Studies in the History and Philosophy of Modern Physics* **52** (2015) 203–216, [`1501.04278`].

[21] S. De Haro, D. R. Mayerson and J. N. Butterfield, *Conceptual Aspects of Gauge/Gravity Duality*, *Found. Phys.* **46** (2016) 1381–1425, [`1509.09231`].

[22] J. Zaanen, Y. Liu, Y.-W. Sun and K. Schalm, *Holographic Duality in Condensed Matter Physics*. Cambridge University Press, 2015, 10.1017/CBO9781139942492.

[23] Y. Kim, I. J. Shin and T. Tsukioka, *Holographic QCD: Past, Present, and Future*, *Prog. Part. Nucl. Phys.* **68** (2013) 55–112, [`1205.4852`].

[24] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386, [`hep-th/9905111`].

[25] T. Banks, *A Critique of pure string theory: Heterodox opinions of diverse dimensions*, `hep-th/0306074`.

[26] S. Perlmutter, G. Aldering, G. Goldhaber, R. A. Knop, P. Nugent, P. G. Castro et al., *Measurements of Ω and Λ from 42 High-Redshift Supernovae*, *ApJ* **517** (1999) 565–586, [`astro-ph/9812133`].

[27] Planck Collaboration, N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi et al., *Planck 2018 results. VI. Cosmological parameters*, *A&A* **641** (2020) A6, [`1807.06209`].

[28] H. Nastase and K. Skenderis, *Holography for the very early universe and the classic puzzles of hot big bang cosmology*, *Phys. Rev. D* **101** (2020) 021901, [`1904.05821`].

[29] D. Kutasov, "Introduction to little string theory." `https://users.ictp.it/~pub_off/lectures/lns007/Kutasov/Kutasov.pdf`, 2001.

[30] E. Martinec, "Overview of little string theory." `https://www.youtube.com/watch?v=vj_OiRpGnYo`, 2019.

[31] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *M theory as a matrix model: A conjecture*, *Phys. Rev. D* **55** (1997) 5112–5128, [`hep-th/9610043`].

[32] G. Obied, H. Ooguri, L. Spodyneiko and C. Vafa, *De Sitter Space and the Swampland*, 1806.08362.

[33] M. Dine, J. A. P. Law-Smith, S. Sun, D. Wood and Y. Yu, *Obstacles to Constructing de Sitter Space in String Theory*, *JHEP* **02** (2021) 050, [2008.12399].

[34] D. Andriot, *Open Problems on Classical de Sitter Solutions*, *Fortsch. Phys.* **67** (2019) 1900026, [1902.10093].

[35] R. Kallosh, *N=2 supersymmetry and de Sitter space*, hep-th/0109168.

[36] C. M. Hull, *De Sitter space in supergravity and M theory*, *JHEP* **11** (2001) 012, [hep-th/0109213].

[37] A. Chamblin and N. D. Lambert, *de Sitter space from M theory*, *Phys. Lett. B* **508** (2001) 369–374, [hep-th/0102159].

[38] E. Witten, *Quantum gravity in de Sitter space*, in *Strings 2001: International Conference*, 6, 2001. hep-th/0106109.

[39] A. Strominger, *The dS / CFT correspondence*, *JHEP* **10** (2001) 034, [hep-th/0106113].

[40] A. Strominger, *Inflation and the dS / CFT correspondence*, *JHEP* **11** (2001) 049, [hep-th/0110087].

[41] P. McFadden and K. Skenderis, *Holography for cosmology*, *Phys. Rev. D* **81** (2010) 021301, [0907.5542].

[42] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849–5876, [hep-th/0209067].

[43] A. Strominger, *On BMS invariance of gravitational scattering*, *JHEP* **2014** (2014) 152, [1312.2229].

[44] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, 1703.05448.

[45] Z. Bern, S. Davies and J. Nohle, *On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons*, *Phys. Rev. D* **90** (2014) 085015, [1405.1015].

[46] F. Cachazo and E. Y. Yuan, *Are Soft Theorems Renormalized?*, 1405.3413.

[47] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516–B524.

[48] S. Weinberg, *The Quantum Theory of Fields*. No. v. 1 in The Quantum Theory of Fields 3 Volume Hardback Set. Cambridge University Press, 1995.

[49] F. Cachazo and A. Strominger, *Evidence for a New Soft Graviton Theorem*, *ArXiv e-prints* (Apr., 2014) , [1404.4091].

[50] B. U. W. Schwab and A. Volovich, *Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations*, *Phys. Rev. Lett.* **113** (2014) 101601, [1404.7749].

[51] A. Laddha and A. Sen, *Sub-subleading Soft Graviton Theorem in Generic Theories of Quantum Gravity*, *JHEP* **10** (2017) 065, [1706.00759].

[52] S. Chakrabarti, S. P. Kashyap, B. Sahoo, A. Sen and M. Verma, *Subleading Soft Theorem for Multiple Soft Gravitons*, *JHEP* **12** (2017) 150, [1707.06803].

[53] A. Laddha and A. Sen, *Logarithmic Terms in the Soft Expansion in Four Dimensions*, *JHEP* **10** (2018) 056, [1804.09193].

[54] M. Campiglia and A. Laddha, *Sub-subleading soft gravitons and large diffeomorphisms*, *JHEP* **01** (2017) 036, [1608.00685].

[55] R. Geroch, *Asymptotic structure of space-time*, in *Asymptotic Structure of Space-Time* (F. P. Esposito and L. Witten, eds.), pp. 1–105. Springer US, 1977. DOI.

[56] R. Wald, *General Relativity*. University of Chicago Press, 1984.

[57] H. Bondi, M. van der Burg and A. Metzner, *Gravitational waves in general relativity. VII. waves from axi-symmetric isolated systems*, *Proc. R. Soc. Lond. A* **269** (1962) 21–52.

[58] R. Sachs, *Gravitational waves in general relativity. VIII. waves in asymptotically flat space-time*, *Proc. R. Soc. Lond. A* **270** (1962) 103–126.

[59] R. Sachs, *Asymptotic symmetries in gravitational theory*, *Phys. Rev.* **128** (1962) 2851–2864.

[60] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, *JHEP* **5** (2010) , [1001.1541].

[61] G. Barnich and C. Troessaert, *Symmetries of asymptotically flat four-dimensional spacetimes at null infinity revisited*, *Phys. Rev. Lett.* **105** (2010) , [0909.2617].

[62] G. Barnich and C. Troessaert, *Supertranslations call for superrotations*, *PoS* (2010) 10, [1102.4632].

[63] G. Barnich and C. Troessaert, *BMS charge algebra*, *JHEP* **12** (2011) 105, [1106.0213].

[64] M. Campiglia and A. Laddha, *Asymptotic symmetries and subleading soft graviton theorem*, *Phys. Rev. D* **90** (Dec, 2014) 124028, [1408.2228v3].

[65] M. Campiglia and A. Laddha, *New symmetries for the Gravitational S-matrix*, *JHEP* **04** (2015) 076, [1502.02318].

[66] S. Hollands and A. Ishibashi, *Asymptotic flatness at null infinity in higher dimensional gravity*, in *Proceedings, 7th Hungarian Relativity Workshop (RW 2003)*, pp. 51–61, 2004. hep-th/0311178.

[67] S. Hollands and R. Wald, *Conformal null infinity does not exist for radiating solutions in odd spacetime dimensions*, *Class. Quantum Grav.* **21** (2004) 5139–5145, [gr-qc/0407014].

[68] S. Hollands and A. Ishibashi, *Asymptotic flatness and Bondi energy in higher dimensional gravity*, *J. Math. Phys.* **46** (2005) , [gr-qc/0304054].

[69] K. Tanabe, N. Tanahashi and T. Shiromizu, *On asymptotic structure at null infinity in five dimensions*, *J. Math. Phys.* **51** (2010) 062502, [0909.0426].

[70] K. Tanabe, S. Kinoshita and T. Shiromizu, *Asymptotic flatness at null infinity in arbitrary dimensions*, *Phys. Rev. D* **84** (2011) 044055, [1104.0303v2].

[71] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, *Higher-dimensional supertranslations and Weinberg's soft graviton theorem*, *Ann. Math. Sci. Appl.* **02** (2017) 69–94, [1502.07644].

[72] A. Tolish and R. Wald, *Retarded fields of null particles and the memory effect*, *Phys. Rev. D* **89** (2014) 064008, [1401.5831].

[73] S. Hollands, A. Ishibashi and R. M. Wald, *BMS supertranslations and memory in four and higher dimensions*, *Class. Quantum Grav.* **34** (2017) , [1612.03290].

[74] D. Garfinkle, S. Hollands, A. Ishibashi, A. Tolish and R. M. Wald, *The Memory Effect for Particle Scattering in Even Spacetime Dimensions*, *Class. Quant. Grav.* **34** (2017) 145015, [1702.00095].

[75] P. Mao and H. Ouyang, *Note on soft theorems and memories in even dimensions*, *Phys. Lett. B* **774** (2017) 715–722, [1707.07118].

[76] M. Pate, A.-M. Raclariu and A. Strominger, *Gravitational Memory in Higher Dimensions*, *JHEP* **06** (2018) 138, [1712.01204].

[77] G. Satishchandran and R. M. Wald, *Asymptotic behavior of massless fields and the memory effect*, *Phys. Rev. D* **99** (2019) 084007, [1901.05942].

[78] G. Satishchandran and R. M. Wald, *Memory effect for particle scattering in odd spacetime dimensions*, *Phys. Rev. D* **97** (2018) 024036, [1712.00873].

[79] D. Colferai and S. Lionetti, *Asymptotic symmetries and subleading soft graviton theorem in higher dimensions*, 2005.03439.

[80] A. Campoleoni, D. Francia and C. Heissenberg, *On asymptotic symmetries in higher dimensions for any spin*, *JHEP* **2020** (2020) 129, [2011.04420].

[81] R. Bousso, *The holographic principle*, *Rev. Mod. Phys.* **74** (Aug, 2002) 825–874, [hep-th/0203101].

[82] R. Bousso, *A Covariant entropy conjecture*, *JHEP* **07** (1999) 004, [hep-th/9905177].

[83] A. Masoumi and S. D. Mathur, *A violation of the covariant entropy bound?*, *Phys. Rev. D* **91** (2015) 084058, [1412.2618].

[84] I. R. Klebanov, *TASI Lectures:. Introduction to the AdS/CFT Correspondence*, in *Strings, Branes and Gravity TASI 99*, pp. 615–650, 2001. hep-th/0009139. DOI.

[85] D. Amati, M. Ciafaloni and G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, *Int. J. Mod. Phys. A* **3** (1988) 1615–1661.

[86] S. B. Giddings, D. J. Gross and A. Maharana, *Gravitational effects in ultrahigh-energy string scattering*, *Phys. Rev. D* **77** (2008) 046001, [0705.1816].

[87] K. S. Stelle, *Lectures on supergravity p-branes*, in *ICTP Summer School in High-energy Physics and Cosmology*, 6, 1996. hep-th/9701088.

[88] J. Dai, R. G. Leigh and J. Polchinski, *New Connections Between String Theories*, *Mod. Phys. Lett. A* **4** (1989) 2073–2083.

[89] J. Polchinski, *Dirichlet Branes and Ramond-Ramond Charges*, *Phys. Rev. Lett.* **75** (1995) 4724–4727, [hep-th/9510017].

[90] J. M. Maldacena, *Black holes in string theory.* PhD thesis, Princeton U., 1996. hep-th/9607235.

[91] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS / CFT correspondence*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and Extra Dimensions*, pp. 3–158, 1, 2002. hep-th/0201253.

[92] J. D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: An example from three dimensional gravity*, *Commun. Math. Phys* **104** (1986) 207–226.

[93] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [`hep-th/9803131`].

[94] A. Jevicki and T. Yoneya, *Space-time uncertainty principle and conformal symmetry in D particle dynamics*, *Nucl. Phys. B* **535** (1998) 335–348, [`hep-th/9805069`].

[95] A. Jevicki, Y. Kazama and T. Yoneya, *Quantum metamorphosis of conformal transformation in D3-brane Yang-Mills theory*, *Phys. Rev. Lett.* **81** (1998) 5072–5075, [`hep-th/9808039`].

[96] A. Jevicki, Y. Kazama and T. Yoneya, *Generalized conformal symmetry in D-brane matrix models*, *Phys. Rev. D* **59** (1999) 066001, [`hep-th/9810146`].

[97] H. J. Boonstra, K. Skenderis and P. K. Townsend, *The domain wall / QFT correspondence*, *JHEP* **01** (1999) 003, [`hep-th/9807137`].

[98] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and the large N limit of theories with sixteen supercharges*, *Phys. Rev. D* **58** (1998) 046004, [`hep-th/9802042`].

[99] I. Kanitscheider, K. Skenderis and M. Taylor, *Precision holography for non-conformal branes*, *JHEP* **09** (2008) 094, [`0807.3324`].

[100] G. T. Horowitz and J. Polchinski, *Gauge/gravity duality*, `gr-qc/0602037`.

[101] S. Weinberg and E. Witten, *Limits on Massless Particles*, *Phys. Lett. B* **96** (1980) 59–62.

[102] J. Penedones, *TASI Lectures on AdS/CFT*, in *New Frontiers in Fields and Strings (TASI 2015* (J. Polchinski and et al., eds.), pp. 75–136, 2017. `1608.04948`. DOI.

[103] J. Kaplan, "Lectures on the ads/cft from the bottom up." `https://sites.krieger.jhu.edu/jared-kaplan/files/2016/05/AdSCFTCourseNotesCurrentPublic.pdf`.

[104] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [`hep-th/9802109`].

[105] K. Skenderis and B. C. van Rees, *Real-time gauge/gravity duality: Prescription, Renormalization and Examples*, *JHEP* **05** (2009) 085, [`0812.2909`].

[106] L. Susskind and E. Witten, *The Holographic Bound in Anti-de Sitter Space*, `hep-th/9805114`.

[107] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **07** (1998) 023, [`hep-th/9806087`].

[108] S. de Haro, K. Skenderis and S. N. Solodukhin, *Holographic reconstruction of spacetime and renormalization in the ads/cft correspondence*, *Comm. Math. Phys.* **217** (2001) 595–622, [`hep-th/0002230`].

[109] J. de Boer, E. P. Verlinde and H. L. Verlinde, *On the holographic renormalization group*, *JHEP* **08** (2000) 003, [`hep-th/9912012`].

[110] I. Papadimitriou and K. Skenderis, *AdS / CFT correspondence and geometry*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 73–101, [`hep-th/0404176`].

[111] E. Dyer and K. Hinterbichler, *Boundary terms, variational principles, and higher derivative modified gravity*, *Phys. Rev. D* **79** (2009) , [0809.4033].

[112] C. Fefferman and C. R. Graham, *Conformal invariants*, in *Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984*, no. S131 in Astérisque. Société mathématique de France, 1985.

[113] D. V. Fursaev and S. N. Solodukhin, *Anomalies, entropy and boundaries*, *Phys. Rev. D* **93** (2016) 084021, [1601.06418].

[114] J. D. Brown and J. York, James W., *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev. D* **47** (1993) 1407–1419, [gr-qc/9209012].

[115] J. V. José and E. J. Saletan, *Classical Dynamics A Contemporary Approach.* Cambridge University Press, 1998.

[116] V. Balasubramanian and P. Kraus, *A Stress Tensor for Anti-de Sitter Gravity*, *Comm. Math. Phys.* **208** (1999) 413–428, [hep-th/9902121].

[117] S. Hollands, A. Ishibashi and D. Marolf, *Comparison between various notions of conserved charges in asymptotically AdS spacetimes*, *Class. Quantum Grav.* **22** (2005) 2881–2920, [hep-th/0503045].

[118] G. W. Gibbons and S. W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Phys. Rev. D* **15** (1977) 2752–2756.

[119] S. W. Hawking and G. T. Horowitz, *The Gravitational Hamiltonian, action, entropy and surface terms*, *Class. Quant. Grav.* **13** (1996) 1487–1498, [gr-qc/9501014].

[120] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) , [hep-th/0505190].

[121] I. Papadimitriou, *Holographic renormalization as a canonical transformation*, *JHEP* **11** (2010) , [1007.4592].

[122] D. Grumiller, *Path Integral for Half-Binding Potentials as Quantum Mechanical Analog for Black Hole Partition Functions*, in *Path Integrals - New Trends and Perspectives*, pp. 236–241, 2008. 0711.4115. DOI.

[123] D. T. Son, *Toward an AdS/cold atoms correspondence: A geometric realization of the Schrödinger symmetry*, *Phys. Rev. D* **78** (2008) 046003, [0804.3972].

[124] K. Balasubramanian and J. McGreevy, *Gravity Duals for Nonrelativistic Conformal Field Theories*, *Phys. Rev. Lett.* **101** (2008) 061601, [0804.4053].

[125] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, *Phys. Rev. D* **78** (2008) 106005, [0808.1725].

[126] M. Taylor, *Lifshitz holography*, *Class. Quantum Grav.* **33** (2016) 033001, [1512.03554].

[127] M. H. Christensen, J. Hartong, N. A. Obers and B. Rollier, *Boundary stress-energy tensor and Newton-Cartan geometry in Lifshitz holography*, *Journal of High Energy Physics* **2014** (Jan., 2014) 57, [1311.6471].

[128] J. Hartong, E. Kiritsis and N. A. Obers, *Lifshitz space-times for Schrödinger holography*, *Phys. Lett. B* **746** (2015) 318–324, [1409.1519].

[129] R. B. Mann and D. Marolf, *Holographic renormalization of asymptotically flat spacetimes*, *Class. Quantum Grav.* **23** (2006) 2927–2950, [hep-th/0511096].

[130] G. Compère, *The Kerr/CFT correspondence and its extensions*, *Living Rev. Rel.* **20** (2017) 1, [1203.3561].

[131] M. Guica, T. Hartman, W. Song and A. Strominger, *The Kerr/CFT correspondence*, *Phys. Rev. D* **80** (2009) 124008, [0809.4266].

[132] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, *Lectures on the Kerr/CFT Correspondence*, *Nuclear Physics B Proceedings Supplements* **216** (2011) 194–210, [1103.2355].

[133] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99–104, [hep-th/9601029].

[134] L. B. Szabados, *Quasi-Local Energy-Momentum and Angular Momentum in General Relativity*, *Living Rev. Rel.* **12** (2009) 4.

[135] M.-T. Wang and S.-T. Yau, *Quasilocal mass in general relativity*, *Phys. Rev. Lett.* **102** (2009) 021101, [0804.1174].

[136] M.-T. Wang, *Four lectures on quasi-local mass*, 1510.02931.

[137] S. Arnowitt, R. Deser and C. Misner, *Republication of: The dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997–2027, [gr-qc/0405109].

[138] H. Bondi, *Gravitational waves in general relativity*, *Nature* **186** (1960) 535.

[139] A. Trautman, *Lectures on general relativity*, *Gen. Rel. Grav.* **34** (2002) 721–762.

[140] R. Penrose, *Asymptotic properties of fields and space-times*, *Phys. Rev. Lett.* **10** (Jan, 1963) 66–68.

[141] A. Ashtekar, *Asymptotic quantization of the gravitational field*, *Phys. Rev. Lett.* **46** (1981) 573–576.

[142] S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathem. Cambridge University Press, 1973.

[143] P. T. Chruściel, M. A. H. MacCallum and D. B. Singleton, *Gravitational waves in general relativity XIV. bondi expansions and the 'polyhomogeneity' of scri*, *Phil. Trans. R. Soc. Lond. A* **350** (1995) 113–141.

[144] K. Parattu, S. Chakraborty, B. R. Majhi and T. Padmanabhan, *A Boundary Term for the Gravitational Action with Null Boundaries*, *Gen. Rel. Grav.* **48** (2016) 94, [1501.01053].

[145] L. Lehner, R. C. Myers, E. Poisson and R. D. Sorkin, *Gravitational action with null boundaries*, *Phys. Rev. D* **94** (2016) 084046, [1609.00207].

[146] L. Donnay, G. Giribet, H. A. González and M. Pino, *Extended Symmetries at the Black Hole Horizon*, *JHEP* **09** (2016) 100, [1607.05703].

[147] G. Esposito, G. Marmo and C. Stornaiolo, *Peierls brackets in theoretical physics*, hep-th/0209147.

[148] C. Shi and J. Mei, *Extended Symmetries at Black Hole Horizons in Generic Dimensions*, *Phys. Rev.* **D95** (2017) 104053, [1611.09491].

[149] R. Wald and A. Zoupas, *General definition of "conserved quantities" in general relativity and other theories of gravity*, *Phys. Rev. D* **61** (2000) 084027, [gr-qc/9911095].

[150] C. Crnkovic, *Symplectic geometry of the convariant phase space*, *Class. Quantum Grav.* **5** (1988) 1557–1575.

[151] R. M. Wald, *Black hole entropy is the noether charge*, *Phys. Rev. D* **48** (1993) R3427–R3431, [gr-qc/9307038].

[152] V. Iyer and R. M. Wald, *Some properties of the noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846–864, [gr-qc/9403028].

[153] V. Iyer and R. M. Wald, *A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes*, *Phys. Rev. D* **52** (1995) 4430–4439, [gr-qc/9503052].

[154] G. Compère and A. Fiorucci, *Advanced Lectures in General Relativity*, 1801.07064.

[155] B. Julia and S. Silva, *On covariant phase space methods*, hep-th/0205072.

[156] G. Compère, A. Fiorucci and R. Ruzziconi, *Superboost transitions, refraction memory and super-Lorentz charge algebra*, *JHEP* **11** (2018) 200, [1810.00377].

[157] E. E. Flanagan, K. Prabhu and I. Shehzad, *Extensions of the asymptotic symmetry algebra of general relativity*, *JHEP* **01** (2020) 002, [1910.04557].

[158] D. Harlow and J.-Q. Wu, *Covariant phase space with boundaries*, *JHEP* **10** (2020) 146, [1906.08616].

[159] K. Shi, X. Wang, Y. Xiu and H. Zhang, *Covariant phase space with null boundaries*, 2008.10551.

[160] J. Polchinski, *S matrices from AdS space-time*, hep-th/9901076.

[161] L. Susskind, *Holography in the flat space limit*, *AIP Conf. Proc.* **493** (1999) 98–112, [hep-th/9901079].

[162] E. Witten, *Baryons and branes in anti de sitter space*, in *Strings '98*, 1998. http://online.itp.ucsb.edu/online/strings98/witten/.

[163] J. de Boer and S. N. Solodukhin, *A Holographic reduction of Minkowski space-time*, *Nucl. Phys. B* **665** (2003) 545–593, [hep-th/0303006].

[164] S. N. Solodukhin, *Reconstructing Minkowski space-time*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 123–163, [hep-th/0405252].

[165] G. Arcioni and C. Dappiaggi, *Exploring the holographic principle in asymptotically flat space-times via the BMS group*, *Nucl. Phys. B* **674** (2003) 553–592, [hep-th/0306142].

[166] G. Arcioni and C. Dappiaggi, *Holography in asymptotically flat spacetimes and the BMS group*, *Class. Quant. Grav.* **21** (2004) 5655–5674, [hep-th/0312186].

[167] P. J. McCarthy, *Structure of the bondi-metzner-sachs group*, *J. Math. Phys.* **13** (1972) 1837–1842.

[168] P. J. McCarthy, *Asymptotically flat space-times and elementary particles*, *Phys. Rev. Lett.* **29** (1972) 817–819.

[169] P. J. McCarthy and M. Crampin, *Representations of the bondi-metzner—sachs group iii. poincaré spin multiplicities and irreducibility*, *Proc. R. Soc. of London A* **335** (1973) 301–311.

[170] P. J. McCarthy, *Representations of the bondi-metzner-sachs group - ii. properties and classification of the representations*, *Proc. R. Soc. of London A* **333** (1973) 317–336.

[171] M. Crampin and P. J. McCarthy, *Physical significance of the topology of the bondi-metzner-sachs group*, *Phys. Rev. Lett.* **33** (Aug, 1974) 547–550.

[172] P. McCarthy, *The bondi-metzner-sachs group in the nuclear topology*, *Proc. R. Soc. of London A* **343** (1975) 489–523.

[173] M. Crampin and P. J. McCarthy, *Representations of the bondi-metzner-sachs group iv. cantoni representations are induced*, *Proc. R. Soc. of London A* **351** (1976) 55–70.

[174] E. Inonu and E. P. Wigner, *On the contraction of groups and their representations*, *Proceedings of the National Academy of Sciences* **39** (1953) 510–524.

[175] J.-M. Lévy-Leblond, *Une nouvelle limite non-relativiste du groupe de Poincaré*, *Annales de l'I.H.P. Physique théorique* **3** (1965) 1–12.

[176] C. Duval, G. W. Gibbons and P. A. Horvathy, *Conformal Carroll groups and BMS symmetry*, *Class. Quant. Grav.* **31** (2014) 092001, [1402.5894].

[177] C. Duval, G. W. Gibbons and P. A. Horvathy, *Conformal Carroll groups*, *J. Phys. A* **47** (2014) 335204, [1403.4213].

[178] C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, *Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time*, *Class. Quant. Grav.* **31** (2014) 085016, [1402.0657].

[179] J. Hartong, *Gauging the Carroll algebra and ultra-relativistic gravity*, *JHEP* **08** (2015) 69.

[180] A. Barducci, R. Casalbuoni and J. Gomis, *Confined dynamical systems with Carroll and Galilei symmetries*, *Phys. Rev. D* **98** (2018) 085018.

[181] L. Ciambelli and C. Marteau, *Carrollian conservation laws and Ricci-flat gravity*, *Class. Quant. Grav.* **36** (2019) 085004, [1810.11037].

[182] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos and K. Siampos, *Flat holography and Carrollian fluids*, *JHEP* **07** (2018) 165, [1802.06809].

[183] L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos and K. Siampos, *Covariant Galilean versus Carrollian hydrodynamics from relativistic fluids*, *Class. Quant. Grav.* **35** (2018) 165001, [1802.05286].

[184] L. Ciambelli, R. G. Leigh, C. Marteau and P. M. Petropoulos, *Carroll Structures, Null Geometry and Conformal Isometries*, *Phys. Rev. D* **100** (2019) 046010, [1905.02221].

[185] A. Bagchi, *Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories*, *Phys. Rev. Lett.* **105** (2010) 171601, [1006.3354].

[186] A. Bagchi and R. Fareghbal, *BMS/GCA redux: towards flatspace holography from non-relativistic symmetries*, *JHEP* **10** (2012) 92, [1203.5795].

[187] A. Ashtekar, J. Bicak and B. G. Schmidt, *Asymptotic structure of symmetry reduced general relativity*, *Phys. Rev. D* **55** (1997) 669–686, [gr-qc/9608042].

[188] G. Barnich and G. Compere, *Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions*, *Class. Quant. Grav.* **24** (2007) F15–F23, [gr-qc/0610130].

[189] R. Fareghbal and A. Naseh, *Flat-Space Energy-Momentum Tensor from BMS/GCA Correspondence*, *JHEP* **03** (2014) 005, [1312.2109].

[190] A. Poole, K. Skenderis and M. Taylor, *(A)dS$_4$ in Bondi gauge*, *Class. Quant. Grav.* **36** (2019) 095005, [1812.05369].

[191] G. Barnich, A. Gomberoff and H. A. Gonzalez, *The Flat limit of three dimensional asymptotically anti-de Sitter spacetimes*, *Phys. Rev. D* **86** (2012) 024020, [1204.3288].

[192] G. Barnich, *Entropy of three-dimensional asymptotically flat cosmological solutions*, *JHEP* **10** (2012) 095, [1208.4371].

[193] A. Bagchi, S. Detournay, R. Fareghbal and J. Simón, *Holography of 3D Flat Cosmological Horizons*, *Phys. Rev. Lett.* **110** (2013) 141302, [1208.4372].

[194] A. Bagchi and R. Basu, *3D Flat Holography: Entropy and Logarithmic Corrections*, *JHEP* **03** (2014) 020, [1312.5748].

[195] F. Bloch and A. Nordsieck, *Note on the radiation field of the electron*, *Phys. Rev.* **52** (1937) 54–59.

[196] F. E. Low, *Scattering of light of very low frequency by systems of spin ½*, *Phys. Rev.* **96** (1954) 1428–1432.

[197] M. Gell-Mann and M. L. Goldberger, *Scattering of low-energy photons by particles of spin ½*, *Phys. Rev.* **96** (1954) 1433–1438.

[198] S. L. Adler, *Consistency conditions on the strong interactions implied by a partially conserved axial-vector current*, *Phys. Rev.* **137** (1965) B1022–B1033.

[199] T. He, V. Lysov, P. Mitra and A. Strominger, *BMS supertranslations and weinberg's soft graviton theorem*, *JHEP* **5** (2015) 151, [1401.7026].

[200] D. Kapec, V. Lysov, S. Pasterski and A. Strominger, *Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix*, *JHEP* **08** (2014) , [1406.3312].

[201] J. Distler, R. Flauger and B. Horn, *Double-soft graviton amplitudes and the extended BMS charge algebra*, *arXiv e-prints* (Aug., 2018) , [1808.09965].

[202] S. Pasterski, A. Strominger and A. Zhiboedov, *New gravitational memories*, *JHEP* **12** (2016) 53, [1502.06120].

[203] S. Pasterski and S.-H. Shao, *Conformal basis for flat space amplitudes*, *Phys. Rev. D* **96** (2017) 065022, [1705.01027].

[204] L. Donnay, S. Pasterski and A. Puhm, *Asymptotic Symmetries and Celestial CFT*, *JHEP* **09** (2020) 176, [2005.08990].

[205] A. Bagchi, R. Basu, A. Kakkar and A. Mehra, *Flat holography: aspects of the dual field theory*, *JHEP* **12** (2016) 147, [1609.06203].

[206] S. Banerjee, S. Ghosh and R. Gonzo, *BMS symmetry of celestial OPE*, *JHEP* **04** (2020) 130, [2002.00975].

[207] V. Lysov, S. Pasterski and A. Strominger, *Low's Subleading Soft Theorem as a Symmetry of QED*, *Phys. Rev. Lett.* **113** (2014) 111601, [1407.3814].

[208] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, *Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance*, *Phys. Rev. D* **90** (2014) 084035, [1406.6987].

[209] M. Campiglia and A. Laddha, *Asymptotic symmetries of QED and Weinberg's soft photon theorem*, *JHEP* **07** (2015) 115, [1505.05346].

[210] A. Campoleoni, D. Francia and C. Heissenberg, *On higher-spin supertranslations and superrotations*, *JHEP* **05** (2017) 120, [1703.01351].

[211] Y. Zel'dovich and A. Polnarev, *Radiation of gravitational waves by a cluster of superdense stars*, *Sov.Ast.* **18** (1974) 17.

[212] D. Christodoulou, *Nonlinear nature of gravitation and gravitational-wave experiments*, *Phys. Rev. Lett.* **67** (1991) 1486–1489.

[213] K. Thorne, *Gravitational-wave bursts with memory: The christodoulou effect*, *Phys. Rev. D* **45** (1992) 520–524.

[214] A. Wiseman and C. Will, *Christodoulou's nonlinear gravitational-wave memory: Evaluation in the quadrupole approximation*, *Phys. Rev. D* **44** (1991) R2945–R2949.

[215] L. Bieri and D. Garfinkle, *Perturbative and gauge invariant treatment of gravitational wave memory*, *Phys. Rev. D* **89** (2014) 084039, [1312.6871].

[216] A. Strominger and A. Zhiboedov, *Gravitational memory, BMS supertranslations and soft theorems*, *JHEP* **01** (2016) 86, [1411.5745].

[217] L. Bieri and D. Garfinkle, *An electromagnetic analogue of gravitational wave memory*, *Class. Quant. Grav.* **30** (2013) 195009, [1307.5098].

[218] S. Pasterski, *Asymptotic Symmetries and Electromagnetic Memory*, *JHEP* **09** (2017) 154, [1505.00716].

[219] L. Susskind, *Electromagnetic Memory*, 1507.02584.

[220] M. Pate, A.-M. Raclariu and A. Strominger, *Color Memory: A Yang-Mills Analog of Gravitational Wave Memory*, *Phys. Rev. Lett.* **119** (2017) 261602, [1707.08016].

[221] P. Mao, H. Ouyang, J.-B. Wu and X. Wu, *New electromagnetic memories and soft photon theorems*, *Phys. Rev. D* **95** (2017) 125011, [1703.06588].

[222] P. Mao and J.-B. Wu, *Note on asymptotic symmetries and soft gluon theorems*, *Phys. Rev. D* **96** (2017) 065023, [1704.05740].

[223] Y. Hamada and S. Sugishita, *Soft pion theorem, asymptotic symmetry and new memory effect*, *JHEP* **11** (2017) 203, [1709.05018].

[224] H. Afshar, E. Esmaeili and M. M. Sheikh-Jabbari, *String Memory Effect*, *JHEP* **02** (2019) 053, [1811.07368].

[225] S. Hassani, *Mathematical Physics: A Modern Introduction to Its Foundations.* Springer-Verlag, 1st ed. ed., 2002.

[226] R. Schoen and S.-T. Yau, *On the Proof of the positive mass conjecture in general relativity*, *Commun. Math. Phys.* **65** (1979) 45–76.

[227] R. Schoen and S. T. Yau, *Proof that the bondi mass is positive*, *Phys. Rev. Lett.* **48** (1982) 369–371.

[228] E. Witten, *A new proof of the positive energy theorem*, *Comm. Math. Phys.* **80** (1981) 381–402.

[229] M. Ludvigsen and J. A. G. Vickers, *The positivity of the bondi mass*, *Journal of Physics A: Mathematical and General* **14** (1981) L389–L391.

[230] G. T. Horowitz and M. J. Perry, *Gravitational energy cannot become negative*, *Phys. Rev. Lett.* **48** (1982) 371–374.

[231] J. A. V. Kroon, *A Comment on the Outgoing Radiation Condition for the Gravitational Field and the Peeling Theorem*, *Gen. Rel. Grav.* **31** (Aug., 1999) 1219, [gr-qc/9811034].

[232] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the minkowski space*, *Princeton Math.* **41** (1993) .

[233] M. Godazgar and H. S. Reall, *Peeling of the Weyl tensor and gravitational radiation in higher dimensions*, *Phys. Rev. D* **85** (2012) 084021, [1201.4373].

[234] M. Godazgar and G. Long, *BMS charges in polyhomogeneous spacetimes*, *Phys. Rev. D* **102** (2020) 064036, [2007.15672].

[235] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory*, vol. 759 of *Lecture Notes in Physics*. Springer-Verlag Berlin Heidelberg, 2 ed., 2008, 10.1007/978-3-540-68628-6.

[236] B. Oblak, *BMS Particles in Three Dimensions*. PhD thesis, Brussels U., 2016. 1610.08526. 10.1007/978-3-319-61878-4.

[237] E. Flanagan and D. Nichols, *Conserved charges of the extended Bondi-Metzner-Sachs algebra*, *Phys. Rev. D* **95** (2015) , [1510.03386].

[238] G. Compère and J. Long, *Vacua of the gravitational field*, *Journal of High Energy Physics* **2016** (Jul, 2016) 137, [1601.04958v3].

[239] A. Strominger and A. Zhiboedov, *Superrotations and black hole pair creation*, *Class. Quantum Grav.* **34** (2017) 064002, [1610.00639].

[240] A. Ashtekar and M. Streubel, *Symplectic geometry of radiative modes and conserved quantities at null infinity*, *Proc. Roy. Soc. Lond. A* **376** (1981) 585–607.

[241] A. Ashtekar, *Asymptotic quantization: based on 1984 Naples Lectures*. 1987.

[242] A. Ashtekar, M. Campiglia and A. Laddha, *Null infinity, the BMS group and infrared issues*, *General Relativity and Gravitation* **50** (Nov., 2018) 140, [1808.07093].

[243] E. E. Flanagan and D. A. Nichols, *Observer dependence of angular momentum in general relativity and its relationship to the gravitational-wave memory effect*, *Phys. Rev. D* **92** (2015) 084057.

[244] E. E. Flanagan, D. A. Nichols, L. C. Stein and J. Vines, *Prescriptions for measuring and transporting local angular momenta in general relativity*, *Phys. Rev. D* **93** (2016) 104007, [1602.01847].

[245] D. A. Nichols, *Spin memory effect for compact binaries in the post-Newtonian approximation*, *Phys. Rev.* **D95** (2017) 084048, [1702.03300].

[246] A. Ashtekar, "The bms group, conservation laws, and soft gravitons." http://pirsa.org/16080055, Talk at the Perimeter Institute (2016).

[247] K. Prabhu, *Conservation of asymptotic charges from past to future null infinity: Supermomentum in general relativity*, *JHEP* **03** (2019) 148, [1902.08200].

[248] L. Bieri, *An Extension of the Stability Theorem of the Minkowski Space in General Relativity*, *J. Diff. Geom.* **86** (2010) 17–70, [0904.0620].

[249] L. Bieri and P. T. Chruściel, *Future-complete null hypersurfaces, interior gluings, and the Trautman-Bondi mass*, 1612.04359.

[250] J. Bicak and B. Schmidt, *On the asymptotic structure of axisymmetric radiative spacetimes*, *Class. Quantum Grav.* **6** (1989) 1547.

[251] J. Podolský and J. B. Griffiths, *The collision and snapping of cosmic strings generating spherical impulsive gravitational waves*, *Class. Quant. Grav.* **17** (2000) 1401, [gr-qc/0001049].

[252] A. Ashtekar and T. Dray, *On the Existence of Solutions to Einstein's Equation With Nonzero Bondi News*, *Commun. Math. Phys.* **79** (1981) 581–589.

[253] A. Vilenkin, *Gravitational Field of Vacuum Domain Walls and Strings*, *Phys. Rev.* **D23** (1981) 852–857.

[254] J. Podolsky and M. Ortaggio, *Robinson-Trautman spacetimes in higher dimensions*, *Class. Quant. Grav.* **23** (2006) 5785–5797, [gr-qc/0605136].

[255] K. Tanabe, S. Kinoshita and T. Shiromizu, *Angular momentum at null infinity in higher dimensions*, *Phys. Rev. D* **85** (2012) , [1203.0452].

[256] J. Bičák and B. G. Schmidt, *Isometries compatible with gravitational radiation*, *J. Math. Phys.* **25** (1984) 600–606.

[257] A. Ashtekar and B. C. Xanthopoulos, *Isometries compatible with asymptotic flatness at null infinity: A complete description*, *J. Math. Phys.* **19** (1978) 2216–2222.

[258] A. Ashtekar and B. G. Schmidt, *Null infinity and killing fields*, *J. Math. Phys.* **21** (1980) 862–867.

[259] J. Bicák and A. Pravdová, *Symmetries of asymptotically flat electrovacuum space–times and radiation*, *J. Math. Phys* **39** (1998) 6011–6039, [gr-qc/9808068].

[260] J. P. Jerry B. Griffiths, *Exact Space-Times in Einstein's General Relativity.* Cambridge University Press, 2009.

[261] R. Penrose, *The geometry of impulsive gravitational waves*, in *General relativity (papers in honour of J. L. Synge)*, p. 101–115. Clarendon Press, Oxford, 1972.

[262] Y. Nutku and R. Penrose, *On impulsive gravitational waves*, *Twistor Newsletter* **34** (1992) .

[263] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects.* Cambridge University Press, 2000.

[264] M. Anderson, *The Mathematical Theory of Cosmic Strings: Cosmic Strings in the Wire Approximation*. Series in High Energy Physics, Cosmology and Gravitation. Taylor & Francis, 2002.

[265] R. P. Geroch and J. H. Traschen, *Strings and Other Distributional Sources in General Relativity*, *Phys. Rev. D* **36** (1987) 1017.

[266] D. Garfinkle, *Metrics with distributional curvature*, *Class. Quant. Grav.* **16** (1999) 4101–4109, [gr-qc/9906053].

[267] J. Traschen, *Codimension Two Branes and Distributional Curvature*, *Class. Quant. Grav.* **26** (2009) 075002, [0809.2923].

[268] C. J. S. Clarke, J. A. Vickers and J. P. Wilson, *Generalized functions and distributional curvature of cosmic strings*, *Class. Quant. Grav.* **13** (1996) 2485–2498, [gr-qc/9605060].

[269] J. P. Wilson, *Distributional curvature of time dependent cosmic strings*, *Class. Quant. Grav.* **14** (1997) 3337–3351, [gr-qc/9705032].

[270] K. Tanabe, S. Kinoshita and T. Shiromizu, *Angular momentum at null infinity in five dimensions*, *J. Math. Phys.* **52** (2011) , [1010.1664].

[271] T. Eguchi, P. B. Gilkey and A. J. Hanson, *Gravitation, Gauge Theories and Differential Geometry*, *Phys. Rept.* **66** (1980) 213.

[272] G. Compère, A. Fiorucci and R. Ruzziconi, *The Λ-BMS_4 charge algebra*, *JHEP* **10** (2020) 205, [2004.10769].

[273] Chrusciel, Piotr and Friedrich, Helmut, *The Einstein equations and the large scale behavior of gravitational fields*. Springer, 2004.

[274] J. A. V. Kroon, *Conserved quantities for polyhomogeneous space-times*, *Class. Quant. Grav.* **15** (1998) 2479–2491, [gr-qc/9805094].

[275] J. A. Valiente Kroon, *Logarithmic Newman-Penrose constants for arbitrary polyhomogeneous space-times*, *Class. Quant. Grav.* **16** (1999) 1653–1665, [gr-qc/9812004].

[276] J. A. Valiente-Kroon, *Polyhomogeneity and zero rest mass fields with applications to Newman-Penrose constants*, *Class. Quant. Grav.* **17** (2000) 605–621, [gr-qc/9907097].

[277] J. A. Valiente-Kroon, *On the existence and convergence of polyhomogeneous expansions of zero rest mass fields*, *Class. Quant. Grav.* **17** (2000) 4365–4376, [gr-qc/0005087].

[278] J. A. Valiente Kroon, *Polyhomogeneous expansions close to null and spatial infinity*, *Lect. Notes Phys.* **604** (2002) 135–160, [gr-qc/0202001].

[279] A. Aggarwal, *Supertranslations in Higher Dimensions Revisited*, *Phys. Rev. D* **99** (2019) 026015, [1811.00093].

[280] J. A. Valiente Kroon, *Can one detect a nonsmooth null infinity?*, *Class. Quant. Grav.* **18** (2001) 4311–4316, [gr-qc/0108049].

[281] J. Foster, *Asymptotic Symmetry and the Global Structure of Future Null Infinity*, *Int. J. Th. Phys.* **26** (1987) 1107–1124.

[282] C. Cheung, A. de la Fuente and R. Sundrum, *4D scattering amplitudes and asymptotic symmetries from 2D CFT*, *JHEP* **01** (2017) 112, [1609.00732].

[283] M. M. Caldarelli, J. Camps, B. Goutéraux and K. Skenderis, *AdS/Ricci-flat correspondence*, *JHEP* **04** (2014) 071, [1312.7874].

[284] R. N. C. Costa, *Holographic Reconstruction and Renormalization in Asymptotically Ricci-flat Spacetimes*, *JHEP* **11** (2012) 046, [1206.3142].

[285] R. N. Caldeira Costa, *Aspects of the zero Λ limit in the AdS/CFT correspondence*, *Phys. Rev. D* **90** (2014) 104018, [1311.7339].

[286] R. Fareghbal and I. Mohammadi, *Flat-space holography and correlators of Robinson-Trautman stress tensor*, *Annals Phys.* **411** (2019) 167960, [1802.05445].

[287] C. Graham and J. M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, *Advances in Mathematics* **87** (1991) 186–225.

[288] S. G. Avery and B. U. W. Schwab, *Burg-metzner-sachs symmetry, string theory, and soft theorems*, *Phys. Rev. D* **93** (Jan, 2016) 026003.

[289] S. Hawking, M. Perry and A. Strominger, *Soft hair on black holes*, *Phys. Rev. Lett.* **116** (2016) 231301, [1601.00921].

[290] S. W. Hawking, M. J. Perry and A. Strominger, *Superrotation Charge and Supertranslation Hair on Black Holes*, *JHEP* **05** (2017) 161, [1611.09175].

[291] R. Bousso and M. Porrati, *Soft Hair as a Soft Wig*, *Class. Quant. Grav.* **34** (2017) 204001, [1706.00436].

[292] D. Grumiller, A. Pérez, M. M. Sheikh-Jabbari, R. Troncoso and C. Zwikel, *Spacetime structure near generic horizons and soft hair*, *Phys. Rev. Lett.* **124** (2020) 041601, [1908.09833].

[293] R. F. Penna, *BMS invariance and the membrane paradigm*, *JHEP* **03** (2016) 023, [1508.06577].

[294] R. F. Penna, *Near-horizon BMS symmetries as fluid symmetries*, *JHEP* **2017** (2017) 49, [1703.07382].

[295] S. Carlip, *Near-horizon Bondi-Metzner-Sachs symmetry, dimensional reduction, and black hole entropy*, *Phys. Rev. D* **101** (2020) 046002, [1910.01762].

[296] J. Bhattacharya, S. Bhattacharyya, A. Dinda and N. Kundu, *An entropy current for dynamical black holes in four-derivative theories of gravity*, *JHEP* **06** (2020) 017, [1912.11030].

[297] B. G. Schmidt, *Vacuum spacetimes with toroidal null infinities*, *Class. Quant. Grav.* **13** (1996) 2811.